Completeness Properties in Function Spaces with the Compact-Open Topology

by

Glenn Hughes

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Approved by

Gary Gruenhage, Professor of Mathematics
Stuart Baldwin, Professor of Mathematics
Michel Smith, Professor of Mathematics

Abstract

It is an open problem to characterize those spaces X for which $C_k(X)$, the space of real-valued continuous functions on X with the compact-open topology, has various completeness properties, in particular, the Baire property. It has been conjectured that the $C_k(X)$ is Baire if and only if X has the moving off property. We show that this is the case for a special class of fans with topologies intermediate to the sequential and metric fans. Furthermore we show that the moving off property on X characterizes Baireness of $C_k(X)$ on other classes of spaces, including: closed images of locally compact paracompact spaces, Lašnev spaces, and special types of collapsed spaces. We also introduce a new completeness property motivated by the Čech complete property.

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Chapter 1

Introduction

1.1 Basic Definitions

If X is a topological space, we will use $C(X)$ to denote the set of all continuous real-valued functions from X into $\mathbb{R}$. There are several natural topologies that can be placed on $C(X)$. The primary focus of this paper will be on the Compact-Open Topology.

Suppose X is a topological space. For every compact set $K \subseteq X$ and every open set $U \subseteq \mathbb{R}$ define the set $[K, U] = \{f \in C(X) : f(K) \subseteq U\}$. Recall that the **Compact-Open Topology**, denoted $C_k(X)$, is the topology on $C(X)$ generated by $\{[K, U] : K \subseteq X$ compact, $U \subseteq \mathbb{R}\}$ as a subbase. In many cases, instead of using this subbase it is more convenient to use the following well known basis for $C_k(X)$.

Proposition 1.1. *Suppose X is a topological space. For each $f \in C_k(X)$, $K \subseteq X$ compact, and $\epsilon > 0$ let $B(f, K, \epsilon) = \{g \in C_k(X) :$ for all $x \in K$ $|g(x) - f(x)| < \epsilon\}$. Then the collection $\mathcal{B} = \{B(f, K, \epsilon) : f \in C_k(X), K \subseteq X$ compact, $\epsilon > 0\}$ forms a basis for $C_k(X)$.*

It is natural to be interested in the following type of questions when we are given a method to generate a new topological space from an old one, such as the case with the Compact-Open Topology. For example, we will look at questions as follows:

a) Given a property Q on $C_k(X)$ is there a property P such that X has property P if and only if $C_k(X)$ has property Q?

b) Which properties are equivalent in $C_k(X)$?

c) What are examples of Compact-Open topologies which distinguish between topological properties?

Our particular focus in this paper will be on a certain class of properties known as completeness properties. The definitions of several of these properties will be given in Chapter 2. However we now give the definitions of the strongest and weakest completeness properties which will be interested in.

Suppose (X, d) is a metric space. Recall that a **Cauchy sequence** $(s_i : i \in \omega)$ in X is a sequence of points in X such that for every $\epsilon > 0$ there exists an $N \in \omega$ such that if $m, n \in \omega$, $m > N$ and $n > N$ then $d(s_n, s_m) < \epsilon$. A metric $d : X \times X \to \mathbb{R}$ on X is said to be a **complete metric**, or simply **complete**, if every Cauchy sequence in X converges. A topological space X is **completely metrizable** if there exists a complete metric d on X such that d generates the same topology on X, i.e. the collection of all open balls $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ forms a base for the topology on X.

Complete metrizability is the strongest of the completeness properties; if X is completely metrizable then X will have every other completeness property. The Baire Property is the weakest of the completeness properties.

A topological X is said to be **second category** if whenever $\{U_i : i \in \omega\}$ is a sequence of dense open sets in X then $\bigcap\{U_i : i \in \omega\} \neq \emptyset$, and X is said to be **first category** if it is not second category. X is a **Baire space**, or simply **Baire**, if whenever $\{U_i : i \in \omega\}$ is a family of dense open sets in X then $\bigcap\{U_i : i \in \omega\}$ is dense. And X is **hereditarily Baire** if every closed subset of X is Baire. Clearly if X is a Baire space then X is also second category. The following is a well-known result.

Theorem 1.2. *Suppose X is a homogeneous topological space. Then X is a Baire space iff X is second category.*

Corollary 1.3. *Suppose X is a topological space. If $C_k(X)$ is second category then $C_k(X)$ is a Baire space.*

Proof. The result follows immediately since $C_k(X)$ is homogeneous. $\square$

1.2 Further Definitions and Known Results

There are many known results pertaining to completeness properties and function spaces that will be useful for our investigation. Since the construction of the subbase for $C_k(X)$ depends on the compact subsets of X, it natural to expect that many of the properties of $C_k(X)$ are inherited from the structure of the family of compact subsets of X. Before we get to some main results we will review a few such structures.

Recall that a family $\mathcal{K}$ of compact subsets of a a space X is said to **dominate the compact subsets of X**, or to be a **dominating family of compact sets**, if given any compact set $H \subseteq X$ there exists a $K \in \mathcal{K}$ such that $H \subseteq K$. A space X is said to be **hemicompact** if there is a countable dominating family of compact subsets of X.

Note that hemicompactness implies σ-compactness, since if $\{K_i : i \in \omega\}$ were a dominating family of compact sets then for any $x \in X$ the singleton $\{x\}$ would necessarily be a subset of at least one K_j, and thus $X = \bigcup\{K_i : i \in \omega\}$. However the converse is not true. While any countable space, being the countable union of its singletons, is σ-compact, countable spaces need not be hemicompact. In particular, the family of rationals $\mathbb{Q}$ is an example of a σ-compact space that is not hemicompact. The metric fan M (see Chapter 3) is another example of a countable non-hemicompact space.

Another common notion is that of a k-space. Recall a space X is said to be a **k-space** or **compactly generated** if whenever a subset $A \subseteq X$ has the property that $A \cap K$ is closed in K for all compact K, then A is closed in X. The k-space property simultaneously generalizes local compactness and first countability. So a first countable space (or a locally compact space) is necessarily a k-space. However the converse is not necessarily true. The sequential fan S_ω defined in Chapter 3 is a k-space which is neither locally compact nor first countable. The following well-known theorem is another characterization of k-spaces.

Theorem 1.4. *X is a k-space if and only if X is the quotient image of a locally compact space.*

The following two results are due to Arens and Warner, respectively. They illustrate how structures on the compact sets, such as hemicompactness, influence the structure on the compact-open topology.[1]

Theorem 1.5. *[Ar] In a completely regular space X the following are equivalent:*

a) X is a hemicompact

b) $C_k(X)$ is metrizable

c) $C_k(X)$ is first countable

Theorem 1.6. *[Wa] A completely regular topological space X is a hemicompact k-space if and only if $C_k(X)$ is completely metrizable.*

It is still an open question if there is a property P such that X has property P if and only if $C_k(X)$ is Baire. To motivate the definition of a candidate property we will look at the analogous case in the function space with the point-open topology.

Suppose X is a topological space. For each $x \in X$, $f \in C(X)$ and $\epsilon > 0$ let $\langle f, x, \epsilon \rangle = \{g \in C(X) : |g(x) - f(x)| < \epsilon\}$. Then the topology generated by the subbase $\{\langle f, x, \epsilon \rangle : f \in C(X), x \in X, \epsilon > 0\}$ is called the **point-open topology on C(X)** and is denoted $C_p(X)$. The point-open topology is another well studied topology on the collection of continuous real-valued functions on a space. To see how our knowledge of the point-open topology might help us with the compact-open topology, a few more definitions are convenient.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are collections of non-empty subsets of a topological space X. It is said that $\mathcal{A}$ **moves off** $\mathcal{B}$ if for any $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $A \cap B = \emptyset$. A collection of subsets $\mathcal{A}$ of a topological space X is said to be **discrete** if for any $x \in X$ there exists an open set U such that $x \in U$ and $U \cap A \neq \emptyset$ for at most one $A \in \mathcal{A}$. The collection $\mathcal{A}$ is said to be **strongly discrete** if there exists a discrete collection $\{U_A : A \in \mathcal{A}\}$ of open sets such that $A \subseteq U_A$ for each $A \in \mathcal{A}$.

[1] For a survey paper on many similar results see [MN]

A necessary and sufficient condition on X for $C_p(X)$ to be Baire was was given independently by E. K. van Douwen (unpublished; see page 34 in [vD]), E.G. Pytkeev [Py], and V.V. Tkachuk [Tk1, Tk2]. It is equivalent to the following theorem.

Theorem 1.7. *Suppose X is completely regular. Then $C_p(X)$ is Baire if and only if every collection of finite subsets $\mathcal{F}$ of X that moves off the set of all finite subsets of X has a strongly discrete infinite subcollection $\mathcal{F}' \subseteq \mathcal{F}$.*

The above characterization discusses a structure on the finite subsets of X. This is reasonable since the topology on $C_p(X)$ uses basic open sets defined using finite subsets of X. To determine a candidate property on X which is a necessary and sufficient condition for $C_k(X)$ to be Baire, it is natural to consider an equivalent structure on the compact subsets of X. The following, due to Gruenhage and Ma in [GM], is such a generalization. It is said that a completely regular space X has the **moving off property** (or **MOP**, for short) if every collection $\mathcal{K}$ of compact sets that move off the set of all compact sets has an infinite strongly discrete subcollection.

Recall that a space is a **q-space** if each $x \in X$ has a sequence of neighborhoods $U_0, U_1, \ldots$ such that if $x_n \in U_n$ for all $n \in \omega$ then the set $\{x_n : n \in \omega\}$ has a cluster point. Note, every first countable, and therefore every metric space is a q-space.

Theorem 1.8. *[GM]*

(i) If $C_k(X)$ is Baire then X has the moving off property.

(ii) Suppose X is a q-space then X has moving off property if and only if $C_k(X)$ is Baire.

(iii) If X is a q-space which has the moving off property, then X must be locally compact.

It was also conjectured that this result holds for all completely regular spaces.

Conjecture 1.9. *Suppose X is a completely regular space. X has the moving off property if and only if $C_k(X)$ is Baire.*

In Chapters 3 and 4 we will show the that the conjecture holds for other specific classes of topological spaces.

1.3 Topological Games and Notation

Since the compact-open topology is often difficult to study directly, we commonly use other techniques to investigate specific properties of $C_k(X)$. Topological games are sometimes a very useful tool for this purpose. In particular the existence (or non-existence) of winning strategies for players in topological games can characterize common topological properties on the space. A game theoretic characterization of the Baire property is well known and discussed below.

The topological game $Ch(X)$, known as the **Choquet game**[2], on a space X has two players E and NE. On move 0, E plays a non-empty open subset $U_0 \subseteq X$. NE responds with a non-empty open subset $V_0 \subseteq U_0$. If $U_0, V_0, \ldots, U_{n-1}, V_{n-1}$ have been defined and are a partial play of $Ch(X)$, then on move n, E plays a non-empty open set $U_n \subseteq V_{n-1}$ and NE responds with a non-empty open set $V_n \subseteq U_n$. E wins if $\bigcap\{V_i : i \in \omega\} = \emptyset$.

The following result is due to Oxtoby in [Ox], we include a proof to illustrate how topological games can be used to characterize topological properties.

Theorem 1.10. *A space X is a Baire space iff E does not have a winning strategy in $Ch(X)$.*

Proof. Suppose X is not Baire. Let $\{W_i : i \in \omega\}$ be a family of dense open sets such that $\bigcap\{W_i : i \in \omega\}$ is not dense. Say $\mathbf{U}$ is an open set such that $\mathbf{U} \cap \bigcap\{W_i : i \in \omega\} = \emptyset$. Then define the strategy σ for E in $Ch(X)$ as follows. Let $U_0 = \sigma(\emptyset) = W_0 \cap \mathbf{U}$. And if $U_0, V_0, \ldots, U_{n-1}, V_{n-1}$ is a partial play of $Ch(X)$, then define $U_n = \sigma(U_0, V_0, \ldots, U_{n-1}, V_{n-1}) = W_n \cap V_{n-1}$. This intersection is a non-empty since W_n is open dense and V_{n-1} is open. Then $\bigcap\{V_i : i \in \omega\} \subseteq \bigcap\{W_i : i \in \omega\} \cap \mathbf{U} = \emptyset$. Hence σ is a winning strategy for E.

[2]This is also known as the Banach-Mazur game with players β and α taking the roles of E and NE, respectively.

On the other hand, suppose X is Baire, and σ is a strategy for E in $Ch(X)$. We will show that σ is not a winning strategy. Let $U_\emptyset = \sigma(\emptyset)$. Let $\{U_\alpha : \alpha \in \kappa_0\}$ be a maximal pairwise disjoint family of open sets such that for each $\alpha \in \kappa_0$ there is an open set $V_\alpha \subseteq U_\emptyset$ such that $\sigma(U_\emptyset, V_\alpha) = U_\alpha$. Suppose for every finite sequence s such that $|s| \leq n$ we have defined $\kappa_{|s|}$, U_s and V_s. Let $\{U_{s\frown\alpha} : \alpha \in \kappa_{n+1}\}$ be a maximal pairwise disjoint family of open sets such that for each $\alpha \in \kappa_{n+1}$ there exists an open set $V_{s\frown\alpha} \subseteq U_s$ such that $\sigma(U_\emptyset, V_{s\restriction 0}, \ldots, U_s, V_{s\frown\alpha}) = U_{s\frown\alpha}$. This inductively defines U_τ and V_τ for all sequences $\tau : \omega \to \bigcup\{\kappa_i : i \in \omega\}$, where $\tau(i) \in \kappa_i$. For each $i > 0$ let $W_i = \bigcup\{U_s : |s| = i\}$, which is a disjoint union of open sets.

We claim that each W_i is dense in $U_\emptyset$. Suppose not. Let $n \in \omega$ be the smallest positive integer such that W_n is not dense and let $V \subseteq U_\emptyset$ be open such that $V \cap W_n = \emptyset$. Then there exists a U_s for $|s| = n-1$ such that $U_s \cap V \neq \emptyset$ since $W_{n-1} = \bigcup\{U_s : |s| = n-1\}$ is dense in $U_\emptyset$. Let $U = \sigma(U_\emptyset, V_{s\restriction 0}, \ldots, U_{s\restriction n-1}, V \cap U_{s\restriction n-1})$. Note $U \subseteq V$ and thus $U \cap U_{s\frown\alpha} = \emptyset$ for all $\alpha \in \kappa_n$. But this is a contradiction to the maximality of $\{U_{s\frown\alpha} : \alpha \in \kappa_n\}$. Hence W_i is dense in $U_\emptyset$ for all $i \in \omega$.

Since X is Baire, and $U_\emptyset$ is an open subset of X it follows that $U_\emptyset$ is Baire, and hence of second category. Let $x \in \bigcap\{W_i : i \in \omega\}$. Let α_0 be the (unique) element in κ_0 such that $x \in U_{\alpha_0}$. If α_i has been defined for all $i < n$ such that $x \in U_{\alpha_0\alpha_1\cdots\alpha_{n-1}}$ let α_n be the (unique) element in κ_n such that $x \in U_{\alpha_0\alpha_1\cdots\alpha_n}$. This defines a sequence α such that $x \in U_{\alpha\restriction n}$ for each $n \in \omega$. Consequently the play of the game $U_\emptyset, V_{\alpha\restriction 0}, U_{\alpha\restriction 0}, V_{\alpha\restriction 1}, \ldots$ is not winning for E. Hence σ is not a winning strategy. Consequently E has no winning strategy in $Ch(X)$. $\square$

This game-theoretic characterization of the Baire space property will be used extensively during our investigation of the compact-open topology. We trade in the absoluteness of an internal characterization involving dense open sets for the ability to work with simpler subsets in a game environment. The difficulty that naturally comes in working with proofs involving the existences of specific strategies is often preferable to considering the more complicated subsets in $C_k(X)$, such as dense open sets. For example, in Chapter 3, we consider countable spaces with only one non-isolated point. However, even in this relatively simple scenario, the compact-open topology on such spaces can still be difficult to conceptualize. But in such spaces, individual continuous functions and basic open subsets of the compact-open topology, which are used game-theoretic characterization of the Baire property, are relatively trivial to consider.

There is also a topological game which characterizes the moving off property discussed by Gruenhage and Ma. The game $\mathbf{G}^{\circ}_{\mathbf{K,L}}(\mathbf{X})$ on the space X, is a topological game with players K and L, defined as follows. On move 0, K chooses a compact set K_0. L responds with a compact set L_0 such that $L_0 \cap K_0 = \emptyset$. On move 1, K chooses compact set K_1 and L responds with compact set L_1 such that $L_1 \cap (K_0 \cup K_1) = \emptyset$. Suppose $K_0, L_0, \ldots, K_{n-1}, L_{n-1}$ have been chosen such that $L_i \cap \bigcap\{K_j : j \leq i\} = \emptyset$ for each $i \leq n-1$. On move n, K plays compact K_n and L responds with compact L_n such that $L_n \cap \bigcap\{K_i : i \leq n\} = \emptyset$. L wins if $\{L_i : i \in \omega\}$ is not strongly discrete. The game $G_{K,L}(X)$ is defined similarly with the exception that L wins if $\{L_i : i \in \omega\}$ is not discrete.

Theorem 1.11. *[GM] X has the moving off property if and only if L does not have a winning strategy in $G^{\circ}_{K,L}(X)$.*

With the game-theoretic equivalent conditions defined, it is possible to reword Conjecture 1.9 in the follow way.

Conjecture 1.12. *Suppose X is a completely regular topological space. Then L has a winning strategy in $G^{\circ}_{K,L}(X)$ if and only if E has a winning strategy in $Ch(C_k(X))$.*

1.4 Chapter Overviews

In Chapter 2 we will recall several more important and well-studied completeness properties, as well as define a few completeness properties through topological games based off of a characterization of Čech completeness. In the second half of the chapter we investigate the implications of these properties such as determining which completeness properties are implied by them, and how these properties are inherited under different kinds of subsets and products.

In Chapter 3 we will define a class of fans with intermediate topologies to the well-known metric and sequential fans, utilizing filters on ω. We will show that the compact-open topology on such fans can not have certain completeness properties such as the Choquet property or hereditarily Bairness. We will also show that $C_k(X)$ for a fan X is only Baire if it is also metrizable. Furthermore we give internal filter-based characterizations for various completeness and metrizability properties on the compact-open topology on these fans. We will conclude the section by demonstrating that Conjecture 1.9 holds for this class of spaces.

In Chapter 4 we consider a variety of different classes of spaces. Most notably we will show that if X is the finite product of closed images of locally compact paracompact spaces then $C_k(X)$ is neccesarily Choquet. We will show that Conjecture 1.9 holds for the class of Lǎsnev spaces and for quotient images obtained by collapsing the non-isolated points of a suitably nice topological space to a single point.

Chapter 2

Completeness Properties

Completeness properties are those properties which simultaneously generalize both compactness and complete metrizability. In section 2.1 we will define some well known completeness properties, namely: Čech completeness, the Choquet property, psuedocompleteness, and subcompactness. In particular, an internal characterization of Čech completness will motivate us to define the topological games $\Gamma(X)$ and $\widehat{\Gamma}(X)$. In secion 2.2 we investigate the notion of γ-completeness and $\widehat{\gamma}$-completeness, as well as weak versions of these propeties, which are defined based off of the existence or non-existence of winning strategies for players in the games $\Gamma(X)$ and $\widehat{\Gamma}(X)$.

2.1 Overview of Completeness Properties

We will begin by reviewing completeness properties.

2.1.1 Čech completeness

Recall that a completely regular space X is **Čech Complete** if X is G_δ in any completely regular space in which it is densely embedded. In particular, X is Čech complete if it is G_δ in its Stone-Čech compactification.

An immediate consequence of the above definition is that if X is a Hausdorff noncompact locally compact space then X is Čech complete, since X is G_δ in its one-point compactification. Also, any completely metrizable space is Čech complete. One of the primary motivations behind the definition of Čech completeness is that for metric spaces Čech completeness is equivalent to complete metrizability.

Theorem 2.1. *Suppose X is a metric space. Then X is completely metrizable if and only if X is Čech complete.*

Recall that a collection $\mathcal{A}$ of subsets of a space X is said to have the **finite intersection property** if $\mathcal{A} \neq \emptyset$ and the intersection of any finite number of sets in $\mathcal{A}$ is non-empty, i.e if $|\mathcal{A}'| < \omega$ and $\mathcal{A}' \subseteq \mathcal{A}$ then $\bigcap \mathcal{A}' \neq \emptyset$. The following is an important internal characterization of Čech completeness given independently by Frolik [Fr] and Archangelskii [Arh3].

Theorem 2.2. *A space X is Čech complete iff there exists a sequence $\{\mathcal{U}_i : i \in \omega\}$ of open covers of X with the property that if $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property and for each $i \in \omega$ there exists a $U \in \mathcal{U}_i$ and $F \in \mathcal{F}$ with $F \subseteq U$, then $\bigcap \mathcal{F} \neq \emptyset$.*

Recall that a space X is compact if and only if whenever $\mathcal{F}$ is a collection of closed subsets with the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$. The above characterization shows that for Čech complete spaces we have to consider a more restricted family of collection of closed sets with the finite intersection property for the intersection of a such a collection to necessarily be non-empty. We are motivated by this characterization of Čech completeness to define the following two topological games, which spread out the creation of the family of open covers and closed sets between plays in the game:

Definition 2.3. *Define a game $\Gamma(X)$ (resp. $\widehat{\Gamma}(X)$) on the topological space X with two players, P1 and P2 as follows. On move 0, P1 chooses an open cover $\mathcal{U}_0$, and P2 chooses closed set F_0 (resp. a regular closed set F_0) such that $F_0 \subseteq U_0$ for some $U_0 \in \mathcal{U}_0$. On move 1, P1 plays an open cover $\mathcal{U}_1$ of U_0, and P2 chooses a closed set F_1 (resp. regular closed set F_1) such that $F_1 \subseteq U_1$ for some $U_1 \in \mathcal{U}_1$ and $F_1 \subseteq F_0$. On move n suppose the open covers $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$, and the sets $F_0, U_0, F_1, U_1, \ldots, F_{n-1}, U_{n-1}$ have been defined such that $\mathcal{U}_i$ is an open cover of U_{i-1}, F_i is a closed subset (resp. regular closed subset) of U_i, $U_i \in \mathcal{U}_i$, and $(F_i : i < k)$ is a decreasing sequence. Then P1 will choose an open cover $\mathcal{U}_k$ of U_{k-1} and P2* *will choose a closed subset (resp. regular closed subset) $F_k \subseteq U_k$ for some $U_k \in \mathcal{U}_k$. P1 wins if $\bigcap\{F_i : i \in \omega\} \neq \emptyset$. P2 wins otherwise.*

The next result is immediate

Theorem 2.4. *If X is Čech complete then $P1$ has a winning strategy in $\Gamma(X)$.*

Proof. Suppose X is Čech complete. Let $\{\mathcal{U}_i : i \in \omega\}$ be a sequence of open covers of X that witness the internal characterization of Čech completeness for X. Then we define the strategy σ for P1 as follows: $\sigma(\emptyset) = \mathcal{U}_0$. If $\sigma(s)$ has been defined for the partial play s, and P2 responds to $\sigma(s)$ by choosing a closed set $F_k \subseteq U_k \in \sigma(s)$. Then define P1 response to be $\{V \cap U_k : V \in \mathcal{U}_{k+1}\}$. Then the sets $\{F_i : i \in \omega\}$ have the finite intersection property and $F_i \subseteq U_i$ for some $U_i \in \mathcal{U}_i$, hence $\cap\{F_i : i \in \omega\} \neq \emptyset$ and consequently σ is a winning strategy for P1 in $\Gamma(X)$. $\square$

We will return to investigate the implications of various winning strategies in these two games after the introduction of other completeness properties.

2.1.2 Choquet Spaces

We introduced the Choquet game $Ch(X)$ in Chapter 1. Recall that player E does not having a winning strategy in $Ch(X)$ if and only if X is Baire. We will also consider spaces X, such that NE has a winning strategy. A space X is said to be a **Choquet space**[1] if NE has a winning strategy in $Ch(X)$. Clearly if NE has a winning strategy in $Ch(X)$ then E has no winning strategy in $Ch(X)$. It follows that every Choquet space is a Baire space. Recall that Theorem 1.11 states that if $C_k(X)$ is Baire then L has no winning strategy in $G^{\circ}_{K,L}(X)$. We can relate $C_k(X)$ being Choquet to the game $G^{\circ}_{K,L}(X)$ in the following theorem.

Theorem 2.5. *[GM] If $C_k(X)$ is Choquet then K has a winning strategy in $G^{\circ}_{K,L}(X)$.*

Much in the same nature as our main conjecture (Conjecture 1.9), it is unknown if the converse of the above statement holds.

[1]This property is also known as weakly α-favorable.

There is also well studied variation on the Choquet game, which gives the first player E more control of the open sets NE can respond with. The **Strong Choquet Game**, $Ch^*(X)$, is a game with two players E and NE. On move 0, E picks an open set U_0 and a point $x_0 \in U_0$ and NE responds with an open set V_0 such that $x_0 \in V_0 \subseteq U_0$. On move 1, E picks and open set $U_1 \subseteq V_0$ and a point $x_1 \in U_1$ and NE responds with an open set V_1 such that $x_1 \in V_1 \subseteq U_1$. On move n, after x_i, V_i, and U_i have been defined for all $i \leq n$ such that $x_i \in V_i \subseteq U_i \subseteq V_{i-1}$, E picks an open set U_{n+1} and a point $x_{n+1} \in U_{n+1}$ and NE responds with an open set V_{n+1} such that $x_{n+1} \in V_{n+1} \subseteq U_{n+1}$. NE wins if $\bigcap\{V_i : i \in \omega\} = \emptyset$. E wins otherwise. A space X is said to be **strongly Choquet** if NE has a winning strategy in $Ch^*(X)$[2].

Clearly, if a space X is strongly Choquet then X is also Choquet. In metric spaces strongly Choquet is equivalent to Čech completeness. The following result is due to Choquet.

Theorem 2.6. *[Ch] If X is a metric space, then X is strongly Choquet if and only if X is completely metrizable (resp. Čech complete).*

2.1.3 Psuedocompleteness

Recall that a **π-base** for a topological space X is a collection of non-empty open sets $\mathcal{U}$ such that if V is a non-empty open set in X then there exists a $U \in \mathcal{U}$ such that $U \subseteq V$ The space X is **quasi-regular** if there exists a π-base $\mathcal{U}$ such that if V is a non-empty open set then there exists a $U \in \mathcal{U}$ such that $\overline{U} \subseteq V$. Recall that a quasi-regular space X is **psuedocomplete** if there exits a sequence $\{\mathcal{B}_i : i \in \omega\}$ of π-bases such that if $\{U_i : i \in \omega\}$ is a sequence such that for all $i \in \omega$ we have $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subseteq U_i$, then $\bigcap\{U_i : i \in \omega\} \neq \emptyset$.

In metric spaces this notion is closely linked to the Čech complete property. The following result is due to J.M. Aarts and David Lutzer.

[2] If NE has a winning tactic in $Ch^*(X)$, that is a winning strategy that only relies on E's last move, then X is said to be strongly α-favorable. In metric spaces, strongly Choquet is equivalent to strongly α-favorable (see [Ch]).

Theorem 2.7. *[AL] Suppose X is a metric space. Then X is psuedocomplete if and only if it contains a dense completely metrizable subspace (resp. contains a dense Čech complete subspace[3]).*

2.1.4 Subcompactness

De Groot in [dG] introduced a class of completeness properties which are stronger than psuedocompleteness. These properties were later termed the Amsterdam Properties. We will focus mostly on one of these properties know as subcompactness.

A collection $\mathcal{F}$ of non-empty open subsets of a topological space X is called a **regular filter base** if for any $W, V \in \mathcal{F}$ there exists $U \in \mathcal{F}$ such that $\overline{U} \subseteq W \cap V$. A regular space X is said to be **subcompact** (resp. **countably subcompact**) if there exists a base $\mathcal{B}$ such that if $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base (resp. $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base such that $|\mathcal{F}| \leq \omega$) then $\bigcap \mathcal{F} \neq \emptyset$.

The following is a basic well known characterization of countable subcompactness. We include a proof.

Theorem 2.8. *A regular space X is countably subcompact if and only if there exists a base $\mathcal{B}$ such that if $(U_i : i \in \omega)$ is a strongly decreasing sequence of non-empty open sets from $\mathcal{B}$ then $\bigcap\{U_i : i \in \omega\} \neq \emptyset$.*

Proof. Suppose X is countably subcompact. Let $\mathcal{B}$ be a base that witnesses X is countably subcompact. Suppose $(U_i : i \in \omega)$ is a strongly decreasing sequence of non-empty open sets in $\mathcal{B}$. Set $\mathcal{F} = \{U_i : i \in \omega\}$. We claim that this is a countable regular filter base. Suppose $U_k, U_j \in \mathcal{F}$. Without loss we may assume that $j > k$. Since the sequence is strongly decreasing, it follows that $\overline{U_{j+1}} \subseteq U_j = U_j \cap U_k$. Hence $\mathcal{F}$ is indeed a countable regular filter base. Thus $\bigcap \mathcal{F} \neq \emptyset$.

On the other hand, suppose X satisfies the strongly decreasing sequence condition, with $\mathcal{B}$ being a base that witnesses it. Suppose $\mathcal{F}$ is a countable filter base of elements in $\mathcal{B}$.

[3]Spaces X for which X contains a dense Čech complete subspace are known as almost Čech complete

Enumerate $\mathcal{F} = \{V_0, V_1, \ldots\}$. We will recursively define U_i for each $i \in \omega$. Let $U_0 = V_0$. Let $U_1 \in \mathcal{F}$ such that $\overline{U_1} \subseteq U_0 \cap V_1$. Let $U_2 \in \mathcal{F}$ such that $\overline{U_2} \subseteq U_1 \cap V_2$. If U_i has been defined for all $i < k$ such that $\overline{U_i} \subseteq U_{i-1} \cap V_i$, then choose $U_k \in \mathcal{F}$ such that $\overline{U_k} \subseteq U_{k-1} \cap V_k$. Then $(U_i : i \in \omega)$ is a strongly decreasing collection of non-empty sets such of elements from $\mathcal{B}$. Hence $\emptyset \neq \bigcap\{U_i : i \in \omega\} \subseteq \bigcap\{V_i : i \in \omega\} = \bigcap \mathcal{F}$. It follows that $\mathcal{B}$ witnesses that X is countably subcompact. $\square$

2.2 $\Gamma(X)$ and $\widehat{\Gamma}(X)$

We now investigate the properties of the previously defined games $\Gamma(X)$ and $\widehat{\Gamma}(X)$. For convenience, we will use the following terminology when referring to the existence or non-existence of winning strategies for the games.

Definition 2.9. *If P1 has a winning strategy in $\Gamma(X)$ then we say X is* **γ-complete***. If P2 does not have a winning strategy in $\Gamma(X)$ then we say X is* **weakly γ-complete***. If there exists a base $\mathcal{B}$ of X such that P1 has a winning strategy in $\widehat{\Gamma}(X)$ when P2's moves are restricted to closures of sets in $\mathcal{B}$ then we say that X is* **$\widehat{\gamma}$-complete***. If there exists a base $\mathcal{B}$ such that P2 doesn't have a winning strategy in $\widehat{\Gamma}(X)$ when P2 is restricted to playing closures of sets in $\mathcal{B}$, then we say that X is* **weakly $\widehat{\gamma}$-complete***.*

We have already seen that if X is Čech complete then X is γ-complete. However the converse isn't true.

Proposition 2.10. *The Sorgenfrey Line is γ-complete but not Čech-complete.*

Proof. Let X be the Sorgenfrey Line. Throughout this proof we will use $\mathrm{cl}_{\mathbb{R}}F$ to denote the Euclidean closure of F as opposed to the closure in X. We will define a winning strategy σ for P1 in the game $\Gamma(X)$. Let $\mathcal{U}_0 = \sigma(\emptyset) = \{[n, n+1) : n \in \mathbb{Z}\}$. Suppose P2 chooses an open set $[a_0, b_0) \in \mathcal{U}_0$ and a closed set $F_0 \subseteq [a_0, b_0)$. Let $L_0 = \mathrm{cl}_{\mathbb{R}}F_0 \setminus F_0$.

We claim that no point of L_0 is a limit point of L_0, in X. Suppose, to the contrary that $y \in L_0$ were a limit point of L_0. Let $\epsilon > 0$ and consider the basic open set $[y, y+\epsilon)$. Since y

is a limit point of L_0 it follows that there exists $x \in L_0 \cap (y, y+\epsilon)$. And since x is a Euclidean limit point of F_0 it follows that there exists a $z \in F_0 \cap (y, y+\epsilon) \subseteq [y, y+\epsilon)$. Therefore every open set that contains y contains a point from F_0. Hence $y \in F_0 \cap L_0$, contrary to the fact that $F_0 \cap L_0 = \emptyset$.

L_0 is Lindelöf since X is hereditarily Lindelöf. Suppose towards a contradiction that L_0 were uncountable, say $L_0 = \{x_\alpha : i \in \kappa\}$ where $\kappa > \omega$. Then since L_0 has no limit points in itself, we can choose pairwise disjoint basic open sets $\{[x_\alpha, y_\alpha) : \alpha \in \kappa\}$, which will produce an uncountable open cover of L_0 with no countable subcover, contrary to the fact that L_0 is Lindelöf. Hence L_0 is countable.

Let $L_0 = \{x_i^0 : i \in \omega\}$. Let $A_0 = \{x_0^0\}$. Let $(s_i)_{i\in\omega}$ be an increasing sequence in $[a_0, b_0)$ such that $s_0 = a_0$ and $(s_i)_{i\in\omega}$ converges to x_0^0 in the Euclidean topology. Define $\mathcal{U}_1 = \sigma(\mathcal{U}_0, ([a_0, b_0), F_0)) = \{[s_i, s_{i+1}) : i \in \omega\} \cup \{[x_0^0, b_0)\}$

Suppose P2 picks $[a_1, b_1) \in \mathcal{U}_1$ and a closed set $F_1 \subseteq F_0 \cap [a_1, b_1)$. By our definition of $\mathcal{U}_1$, it will follow that x_0^0 is not in the Euclidian closure of F_1; i.e. no point in A_0 is in the Euclidean closure of F_1. Let $L_1 = \mathrm{cl}_{\mathbb{R}} F_1 \setminus F_1$. By the same argument as above L_1 is countable. Let $L_1 = \{x_i^1 : i \in \omega\}$. Let $A_1 = \{x_0^1, x_1^0\}$. Without loss of generality assume $x_0^1 < x_1^0$. Let $(s_i^0)_{i\in\omega}$ and $(s_i^1)_{i\in\omega}$ be increasing sequences such that $s_j^1 < s_k^0$ for all $j, k \in \omega$, $s_0^1 = a_1$, $s_0^0 = x_0^1$, $(s_i^1)_{i\in\omega}$ converges to x_0^1 and $(s_i^0)_{i\in\omega}$ converges to x_1^0 in the Euclidean topology. If $x_1^0 \notin [a_1, b_1]$ define $\mathcal{U}_2 = \sigma(\mathcal{U}_0, ([a_0, b_0), F_0), \mathcal{U}_1, ([a_1, b_1), F_1)) = \{[s_i^1, s_{i+1}^1) : i \in \omega\} \cup \{[x_0^1, b_1)\}$ otherwise define $\mathcal{U}_2 = \sigma(\mathcal{U}_0, ([a_0, b_0), F_0), \mathcal{U}_1, ([a_1, b_1), F_1)) = \{[s_i^1, s_{i+1}^1) : i \in \omega\} \cup \{[s_i^0, s_{i+1}^0) : i \in \omega\} \cup \{[x_1^0, b_1)\}$. Note if P2 responds with an open set $[a_2, b_2) \in \mathcal{U}_2$ and $F_2 \subseteq F_1 \cap [a_2, b_2)$ then no point in A_1 is in the Euclidean closure of F_2.

Suppose $S = \mathcal{U}_0, ([a_0, b_0), F_0), \ldots, \mathcal{U}_k$ is a legal partial play of the game and for all $n < k$, $L_n = \{x_i^n : i \in \omega\}$ defined as above, $A_n = \{x_i^j : i + j = n\}$. Also assume that for each $i < k - 1$ and each $p \in A_i$ that p is not in the Euclidean closure of F_{i+1}. Suppose P2 responds with $[a_k, b_k) \in \mathcal{U}_k$ and closed $F_k \subseteq F_{k-1} \cap [a_k, b_k)$. Let $L_k = \mathrm{cl}_{\mathbb{R}} F_k \setminus F_k$. Denumerate $L_k = \{x_i^k : i \in \omega\}$. Let $A_k = \{x_i^j : i + j = k\}$. Let $\{c_0, c_1, \ldots, c_m\}$ denote points in $A_k \cap [a_k, b_k]$, such that $c_0 < c_1 < \cdots < c_m$. Let (s_i^0) be an increasing sequence such that $s_0^0 = a_k$ and (s_i^o) converges to c_0. For $j > 0$ let (s_i^j) be an increasing sequence such that $s_0^j = c_{j-1}$ and (s_i^j) converges to c_j in the Euclidean topology. Define $\mathcal{U}_{k+1} = \sigma(S, ([a_k, b_k), F_k)) = \bigcup\{\{[s_i^j, s_{i+1}^j) : i \in \omega\} : j \leq m\} \cup \{[c_m, b_k)\}$. Then if F_{k+1} is the closed set in a response by P2, then no point in A_k is in the Euclidean closure of F_{k+1}.

We will show that $\bigcap\{F_k : k \in \omega\} \neq \emptyset$. Suppose it is empty. Note $\bigcap\{\mathrm{cl}_{\mathbb{R}} F_k : k \in \omega\} \neq \emptyset$ since it is the intersection of compact sets. Let $p \in \bigcap\{\mathrm{cl}_{\mathbb{R}} F_k : k \in \omega\}$. Let $n \in \omega$ be the smallest such that $p \notin F_n$. Then $p \in L_n$, say $p = x_j^n$. Then p is not $\mathrm{cl}_{\mathbb{R}} F_{j+n+1}$. But this is contrary to the the way p was chosen. It follows that $\bigcap\{F_k : k \in \omega\} \neq \emptyset$ and hence σ is a winning strategy for P1 in $\Gamma(X)$. Thus the Sorgenfrey line is γ-complete. $\square$

It is helpful to consider strategies which have additional properties. The next definition and following lemma indicate that existence of a winning strategy for P_1 in $\Gamma(X)$ guarantees that there is winning strategy for P_1 with properties which will be helpful to prove later results.

Definition 2.11. *Suppose σ is a strategy for P_1 in $\Gamma(X)$. Then a* **regular refinement** *of σ is a strategy $\widehat{\sigma}$ with the following properties:*

1. *For every $U \in \widehat{\sigma}(\emptyset)$ there exists a $U' \in \sigma(\emptyset)$ such that $cl_X(U) \subseteq U'$.*

2. *If $S = \mathcal{U}_0, (C_0, U_0), \mathcal{U}_1, (C_1, U_1), \ldots, \mathcal{U}_n, (C_n, U_n)$ is a partial play using $\widehat{\sigma}$ with corrosponding play $S' = \mathcal{U}_0', (C_0, U_0'), \mathcal{U}_1', (C_1, U_1'), \ldots, \mathcal{U}_n', (C_n, U_n')$. Then $\mathcal{U}_{n+1} = \widehat{\sigma}(S)$ is an open cover of U_n with the property that if $U \in \mathcal{U}_{n+1}$ then there exists a $U' \in \mathcal{U}_{n+1}' = \sigma(S')$ such that $cl_X(U) \subseteq U'$.*

Lemma 2.12. *Suppose X is regular and σ is a winning strategy for P_1 in the game $\Gamma(X)$. There exists a regular refinement $\widehat{\sigma}$ of σ which is a winning strategy for P_1*

Proof. Define $\mathcal{V}_0 = \widehat{\sigma}(\emptyset)$ to be any regular refinement of the open cover $\sigma(\emptyset) = \mathcal{U}_0$. Suppose P_2 plays $C_0 \subseteq U_0 \in \mathcal{V}_0$ in response to P_1 who is using $\widehat{\sigma}$. Then there is a $U_0' \in \sigma(\emptyset)$ such that $\mathrm{cl}(U_0) \subseteq U_0'$. Therefore (C_0, U_0') is a legal move for P_2 in response to $\sigma(\emptyset)$. We have that $\mathcal{U}_1 = \sigma(\mathcal{U}_0, (C_0, U_0'))$ is an open cover of U_0' and consequently an open cover of U_0. We can define $\mathcal{V}_1 = \widehat{\sigma}(\mathcal{V}_0, (C_0, U_0))$ to be any open refinement of $\mathcal{U}_1$.

Continue in this process to define $\widehat{\sigma}$. Since every closed set C_i in a repones by P_2 to a partial play S in the game with P_1 using $\widehat{\sigma}$ is also a legal move using the closed set C_i to the partial play S in the game with P_1 using σ, it follows that $\bigcap C_i \neq \emptyset$ and therefore $\widehat{\sigma}$ is a winning strategy for P_1. $\square$

A regular refinement $\widehat{\sigma}$ of a strategy σ ensures that the closures of an open set used in $\widehat{\sigma}$ is a subset of an open set in the corresponding play using σ. It will be useful to extend this idea further by ensuring that the closures of open sets used in subsequent open covers are also subsets of a fixed sequence of open sets.

Definition 2.13. *We say that the strategy σ for P_1* **refines the decreasing sequence of open sets** $\{W_i : i \in \omega\}$ *if whenever S is a partial play of $\Gamma(X)$ such that $|S| = 2n$ and $S(2n-1) = (C_n, U_n)$ with $C_n \subseteq U_n \cap W_n$ then $\sigma(S)$ has the property that if $U \in \sigma(S)$ then $U \cap C = \emptyset$ or $cl_X(U) \subseteq W_n \cap U_n$.*

Lemma 2.14. *Suppose X is a regular space, σ is a winning strategy of P_1 in $\Gamma(X)$, and $\{W_i : i \in \omega\}$ is a decreasing sequence of open subsets of X. Then there is a regular refinement $\widehat{\sigma}$ of σ which refines $\{W_i : i \in \omega\}$ and is a winning strategy for P_1.*

Proof. Proceed as in Lemma 2.12 to define a regular refinement $\widehat{\sigma}$, with the following additional condition. If (C_n, U_n) is a response to $\widehat{\sigma}(S)$ with $\overline{U}_n \subseteq U_n' \in \sigma(S')$ (where S' is the corresponding play by using σ) and if $C_n \subseteq W_n$, then let $\widehat{\sigma}(S \frown (C_n, U_n))$ be an open refinement of $\sigma(S' \frown (C_n, U_n'))$ such that if $V \in \widehat{\sigma}(S \frown (C_n, U_n))$ then either $\overline{V} \subseteq W_n \cap U_n'$ or $\overline{V} \subseteq (X \setminus C_n) \cap U_n'$. $\square$

We will look at the relationship with the other completeness properties, as well as how γ-completeness is preserved under various subsets, unions, and mappings.

Theorem 2.15. *The property of being γ-complete is hereditary under closed sets.*

Proof. Suppose σ witnesses γ-completeness for X. Let $\mathbf{C}$ be a closed subset of X. We will define a winning strategy τ for P1 in $\Gamma(\mathbf{C})$ using the strategy σ. Let $\mathcal{U}_0 = \sigma(\emptyset)$. Define

$$\tau(\emptyset) = \{V \cap \mathbf{C} : V \in \mathcal{U}_0\} = \mathcal{U}'_0.$$

Suppose P2 responds in $\Gamma(U)$ with $(V_0 \cap \mathbf{C}, C_0)$ where $V_0 \cap \mathbf{C} \in \mathcal{U}'_0$, $V_0 \in \mathcal{U}_0$, and $C_0 \subseteq V_0 \cap \mathbf{C}$ is closed in $\mathbf{C}$. It follows that C_0 is closed in X. Hence (V_0, C_0) is a legal response by P2 in $\Gamma(X)$ to the partial play $\mathcal{U}_0$. Let $\mathcal{U}_1 = \sigma(\mathcal{U}_0, (V_0, C_0))$ which is an open cover of V_0. Define

$$\tau(\mathcal{U}'_0, (V_0 \cap \mathbf{C}, C_0)) = \{V \cap (V_0 \cap \mathbf{C}) : V \in \mathcal{U}_1\} = \mathcal{U}'_1.$$

Since $\mathcal{U}'_1$ is an open cover of $V_0 \cap \mathbf{C}$, it is therefore a legal play by P1. Suppose P2 responds with $(V_1 \cap (V_0 \cap \mathbf{C}), C_1)$ where $V_1 \in \mathcal{U}_1$ (thus $V_1 \cap (V_0 \cap \mathbf{C}) \in \mathcal{U}'_1$), and $C_1 \subseteq V_1 \cap V_0 \cap \mathbf{C}$ is closed. Then C_1 is a closed subset of V_1 in X, hence (V_1, C_1) is a legal response by P2 in the game $\Gamma(X)$ in response to the partial play: $\mathcal{U}_0, (V_0, C_0), \mathcal{U}_1$. Let $\mathcal{U}_2 = \sigma(\mathcal{U}_0, (V_0, C_0), \mathcal{U}_1, (V_1, C_1),$ which is an open cover of V_1. Define

$$\tau(\mathcal{U}'_0, (V_0 \cap \mathbf{C}, C_0), \mathcal{U}'_1, (V_0 \cap V_1 \cap U, C_1)) = \{V \cap (V_0 \cap V_1 \cap \mathbf{C}) : V \in \mathcal{U}_2\} = \mathcal{U}'_2.$$

Suppose for a $k > 0$ that the partial play $\mathcal{U}_0, (V_0, C_0), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, C_{k-1}), \mathcal{U}_k$ is a legal partial play in the game $\Gamma(X)$ using the strategy σ, and suppose the associated partial play $\mathcal{U}'_0, (V_0 \cap \mathbf{C}, C_0), \ldots, \mathcal{U}'_{k-1}, (V_0 \cap V_1 \cap \cdots \cap V_{k-1} \cap \mathbf{C}, C_{k-1}), \mathcal{U}'_k$ using the partially defined strategy τ.

Note $\mathcal{U}'_k$ is an open cover for $V_0 \cap V_1 \cap \cdots \cap V_{k-1} \cap \mathbf{C}$. Suppose P2 responds with $(V_0 \cap V_1 \cap \cdots V_{k-1} \cap V_k \cap \mathbf{C}, C_k)$ where $V_k \in \mathcal{U}_k$ and C_k is a closed subset of $V_0 \cap V_1 \cap \cdots V_k$ and hence a closed subset of V_k in X. Hence (V_k, C_k) is a legal response by P2 in $\Gamma(X)$ to the partial play: $\mathcal{U}_0, (V_0, C_0), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, C_{k-1}), \mathcal{U}_k$. Let

$$\mathcal{U}_{k+1} = \sigma(\mathcal{U}_0, (V_0, C_0), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, C_{k-1}), \mathcal{U}_k, (V_k, C_k))$$

and define

$$\tau(\mathcal{U}'_0, (V_0 \cap \mathbf{C}, C_0), \ldots, \mathcal{U}'_k, (V_0 \cap \cdots V_k \cap \mathbf{C}, C_k)) = \mathcal{U}'_{k+1}.$$

This inductively defines a strategy τ for P1 in the game $\Gamma(\mathbf{C})$. Since σ is a winning strategy for P1 in $\Gamma(X)$ it follows that $\bigcap\{C_i : i \in \omega\} \neq \emptyset$, and hence τ is a winning strategy for P1 in $\Gamma(\mathbf{C})$. Therefore $\mathbf{C}$ is γ-complete.

□

Theorem 2.16. *For quasi-regular X, weak γ-completeness is hereditary under closed sets.*

Proof. Suppose P2 has a winning strategy in $\widehat{\Gamma}(C)$ for some closed subspace C of quasi-regular X, let τ be such a strategy. We will use τ to construct a winning strategy σ for P2 in $\Gamma(X)$. Suppose P1 plays $\mathcal{U}_0$ in the game $\Gamma(X)$. Then $\mathcal{U}'_0 = \{U \cap C : U \in \mathcal{U}_0\}$ is an open cover of C. Let $(U_0 \cap C, F_0)$ be P2's response to $\mathcal{U}'_0$, using τ. Note: F_0 is a closed subset of C, hence F_0 is closed in X. Furthermore, $F_0 \subseteq U_0 \in \mathcal{U}_0$. Define $\sigma(\mathcal{U}'_0) = (U_0, F_0)$. Suppose P1 responds with $\mathcal{U}_1$, then $\mathcal{U}'_1 = \{U \cap C : U \in \mathcal{U}_1\}$ is an open cover of C. Let $(U_1 \cap C, F_1)$ be P2's response using τ to the moves: $\mathcal{U}'_0, (U_0 \cap C, F_0), \mathcal{U}'_1$. Then $F_1 \subseteq U_1 \in \mathcal{U}_1$ and F_1 is also closed in X. Define $\sigma(\mathcal{U}_0, (U_0, F_0), \mathcal{U}_1) = (U_1, F_1)$. Continue in this process. Since τ is a winning strategy for P2 in $\Gamma(C)$ it follows that $\bigcap\{F_i : i \in \omega\} = \emptyset$. Hence σ is a winning strategy for P2 in $\Gamma(X)$. □

Theorem 2.17. *For regular X the property of being γ-complete is hereditary under G_δ sets.*

Proof. Suppose $A = \bigcap W_i$ is a G_δ set where $W_{n+1} \subseteq W_n$, and σ witnesses γ-completeness for a regular space X. By Lemma 2.14 we may assume that σ refines $\{W_i : i \in \omega\}$. Let

$\widehat{\sigma}(X)$ be a regular refinement of $\sigma(X)$. Define a strategy τ for P_1 in $\Gamma(A)$ as follows:

$\Gamma(X)$	$\Gamma(A)$
$\mathcal{U}_0 = \sigma(\emptyset)$	$\mathcal{V}_0 = \tau(\emptyset) = \widehat{\sigma}(\emptyset) \restriction A$
$(\mathrm{cl}_X(C_0), U_0')$	$(C_0, U_0 \cap A)$
$\mathcal{U}_1 = \sigma(\mathcal{U}_0, (\mathrm{cl}_X(C_0), U_0'))$	$\mathcal{V}_0 = \tau(\mathcal{V}_0, (C_0, U_0 \cap A)) = \widehat{\sigma}(\mathcal{U}_0, (\mathrm{cl}_X(C_0), U_0')) \restriction A \cap U_0$
$\vdots$	$(C_1, U_1 \cap A)$

An explanation: P_1 plays $\tau(\emptyset) = \widehat{\sigma}(\emptyset) \restriction A = \{U \cap A : U \in \widehat{\sigma}(\emptyset)\}$. Suppose P_2 responds with $(C_0, U_0 \cap A)$ where C_0 is closed in $A \subseteq W_0$. Let $U_0' \in \mathcal{U}_0$ such that $\mathrm{cl}_X(U_0) \subseteq U_0'$. Thus $\mathrm{cl}_X(C) \subseteq \mathrm{cl}_X(U_0) \subset U_0'$. Therefore $(\mathrm{cl}_X(C_0), U_0')$ is a legal play by P_2 in repsonse to $\mathcal{U}_0$ in $\Gamma(X)$. Let $\mathcal{U}_1$ be the response by P_1 in $\Gamma(X)$ to the partial play: $\mathcal{U}_0, (\mathrm{cl}_X(C_0), U_0)$. Also, $\mathrm{cl}_X(C_0) \subseteq W_0$ since σ refines $\{W_i : i \in \omega\}$. Define the response by P_1 in $\Gamma(A)$ to the partial play: $\mathcal{V}_0, (C_0, U_0 \cap A)$ by the open cover $\widehat{\sigma}(\mathcal{U}_0, (\mathrm{cl}_X(C_0), U_0')) \restriction A \cap U_0$.

Continuing this process defines the strategy τ. Since σ is a winning strategy for P_1 in $\Gamma(X)$ it follows that $\bigcap\{\mathrm{cl}_X(C_i) : i \in \omega\} \neq \emptyset$. Let $x \in \bigcap\{\mathrm{cl}_X(C_i) : i \in \omega\}$. Since $\mathrm{cl}_X(C_n) \subseteq W_n$ for each $n \in \omega$ it follows that $x \in A$. Therefore $x \in \bigcap\{C_i : i \in \omega\}$. Hence τ is a winning strategy for P_1 in $\Gamma(A)$, and A is γ-complete. $\square$

Theorem 2.18. *For regular X, if X is $\widehat{\gamma}$-complete then X is strongly Choquet.*

Proof. Suppose σ witnesses $\widehat{\gamma}$-completeness with NE restricted to closures of open sets in the base $\mathcal{B}$. We will define a strategy τ for NE in the game $Ch^*(X)$. Suppose E plays (U_0, x_0) on move 0. Define $\mathcal{U}_0 = \sigma(\emptyset)$, which is an open cover of X. Let $V_0 \in \mathcal{U}_0$ such that $x_0 \in V_0$. Let $W_0 \in \mathcal{B}$ such that $x_0 \in W_0 \subseteq \overline{W}_0 \subseteq U_0 \cap V_0 \subseteq V_0 \in \mathcal{U}_0$. Hence $(V_0, \overline{W}_0)$ is a legal move for P2 in response to $\mathcal{U}_0$ in the game $\widehat{\Gamma}(X)$. Define $\tau((U_0, x_0)) = W_0$.

Let $\mathcal{U}_1 = \sigma(\mathcal{U}_0, (V_0, \overline{W}_0))$, which is an open cover of V_0. Suppose E plays (U_1, x_1) in $Ch^*(X)$ in response to the partial play $(U_0, x_0), W_0$. Let $V_1 \in \mathcal{U}_1$ such that $x_1 \in V_1$. let

$W_1 \in \mathcal{B}$ such that $x_1 \in W_1 \subseteq \overline{W}_1 \subseteq V_1 \cap U_1 \subseteq V_1 \in \mathcal{U}_1$. Hence $(V_1, \overline{W}_1)$ is a legal play by P2 in response to $\mathcal{U}_0, \overline{W}_0, \mathcal{U}_1$. Define $\tau((U_0, x_0), W_0, (U_1, x_1)) = W_1$.

Suppose for $k > 0$ that $\mathcal{U}_0, (V_0, \overline{W}_0), \mathcal{U}_1, (V_1, \overline{W}_1), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, \overline{W}_{k-1})$ is a legal partial play of $\widehat{\Gamma}(X)$ using the strategy σ, and suppose $(U_0, x_0), W_0, (U_1, x_1), W_1, \ldots, (U_{k-1}, x_{k-1}), W_{k-1}$ is the associated partial play in the game $Ch^*(X)$ using τ, i.e. $x_i \in W_i \subseteq \overline{W}_i \subseteq U_i \cap V_i \subseteq V_i \in \mathcal{U}_i$ and $W_i \in \mathcal{B}$ for all $i \leq k-1$. Suppose E responds to the partial play with (U_k, x_k). Let

$$\mathcal{U}_k = \sigma(\mathcal{U}_0, (V_0, \overline{W}_0), \mathcal{U}_1, (V_1, \overline{W}_1), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, \overline{W}_{k-1}))$$

which is an open cover of V_{k-1}. Let $V_k \in \mathcal{U}_k$ such that $x_k \in V_k$. Choose $W_k \in \mathcal{B}$ such that $x_k \in W_k \subseteq \overline{W}_k \subseteq V_k \cap U_k \subseteq V_k \in \mathcal{U}_k$. It follows that $(V_k, \overline{W}_k)$ is a legal response by P2 to the partial play: $\mathcal{U}_0, (V_0, \overline{W}_0), \mathcal{U}_1, (V_1, \overline{W}_1), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, \overline{W}_{k-1}), \mathcal{U}_k$. Define

$$\tau((U_0, x_0), W_0, (U_1, x_1), W_1, \ldots, (U_{k-1}, x_{k-1}), W_{k-1}, (U_k, x_k)) = W_k$$

This defines a strategy τ for NE in $Ch^*(X)$. Note: $\overline{W}_{i+1} \subseteq W_i$ for all $i \in \omega$. Hence $\bigcap\{\overline{W}_i : i \geq 1\} \subseteq \bigcap\{W_i : i \in \omega\}$. Since σ is a winning strategy for P1 in $\widehat{\Gamma}(X)$ it follows that $\bigcap\{\overline{W}_i : i \in \omega\} \neq \emptyset$. Hence $\bigcap\{W_i : i \in \omega\} \neq \emptyset$. It follows that τ is a winning strategy for NE in $Ch^*(X)$. Thus X is strongly Choquet. □

Theorem 2.19. *For quasi-regular X, if X is $\widehat{\gamma}$-complete then X is Choquet.*

Proof. Suppose σ witnesses $\widehat{\gamma}$-completeness for X when P2 is restricted to using closures of open sets in the base $\mathcal{B}$. We will proceed similarly to the previous result by defining a winning strategy for NE in $Ch(X)$. Suppose E plays U_0 on move 0 in $Ch(X)$. Let $\mathcal{U}_0 = \sigma(\emptyset)$, and let $V_0 \in \mathcal{U}_0$ such that $V_0 \cap U_0 \neq \emptyset$. Let $W_0 \in \mathcal{B}$ such that $\overline{W_0} \subseteq V_0 \cap U_0$. Then $(V_0, \overline{W_0})$ is a legal response to $\mathcal{U}_0$ by P2 in $\widehat{\Gamma}(X)$. Define $\tau(U_0) = W_0$.

Suppose for $k > 0$ that $\mathcal{U}_0, (V_0, \overline{W}_0), \mathcal{U}_1, (V_1, \overline{W}_1), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, \overline{W}_{k-1})$ is a legal partial play of $\widehat{\Gamma}(X)$ using the strategy σ, and suppose $U_0, W_0, U_1, W_1, \ldots, U_{k-1}, W_{k-1}$ is the associated partial play in the game $Ch(X)$ using τ, i.e. $W_i \subseteq \overline{W}_i \subseteq U_i \cap V_i \subseteq V_i \in \mathcal{U}_i$ and $W_i \in \mathcal{B}$ for all $i \leq k-1$. Suppose E responds with the to this partial play in $Ch(X)$ with U_k. Let

$$\mathcal{U}_k = \sigma(\mathcal{U}_0, (V_0, \overline{W}_0), \mathcal{U}_1, (V_1, \overline{W}_1), \ldots, \mathcal{U}_{k-1}, (V_{k-1}, \overline{W}_{k-1}))$$

which is necessarily an open cover of V_{k-1}. Let $V_k \in \mathcal{U}_k$ such that $V_k \cap U_k \neq \emptyset$. Let $W_k \in \mathcal{B}$ such that $\overline{W}_k \subseteq V_k \cap U_k$. Then $(V_k, \overline{W}_k)$ is a legal response for P2. Define

$$\tau(U_0, V_0, \ldots, U_{k-1}, W_{k-1}, U_k) = W_k.$$

Then this defines a strategy τ for NE. For all $i \in \omega$ we have $\overline{W}_{i+1} \subseteq U_i \subseteq W_i$. Therefore $\bigcap\{\overline{W}_i : i \in \omega\} = \bigcap\{W_i : i \in \omega\}$. Since σ witnessed $\widehat{\gamma}$-completeness it follows that $\bigcap\{\overline{W}_i : i \in \omega\} \neq \emptyset$. Hence τ is a winning strategy for NE in $Ch(X)$.

$\square$

Lemma 2.20. *If a quasi-regular space X is weakly γ-complete then X is Baire.*

Proof. We will show the contrapositive. Suppose σ is a winning strategy for E in $Ch(X)$. We will define a strategy winning strategy τ for P2 in the game $\Gamma(X)$. Suppose P1 plays $\mathcal{U}_0$. Let $V_0' = \sigma(\emptyset)$. Let $U_0 \in \mathcal{U}_0$ and V_0 open such that $\overline{V_0} \subseteq U_0 \cap V_0'$. Define

$$\tau(\mathcal{U}_0) = (U_0, \overline{V_0}).$$

Suppose P1 responds with $\mathcal{U}_1$. Let $U_1 \in \mathcal{U}_1$ such that $U_1 \cap V_0 \neq \emptyset$. Then U_1 is a valid play by NE in $Ch(X)$ to the move: V_0'. Let $V_1' = \sigma(V_0', U_1)$. Let V_1 open such that $\overline{V_1} \subseteq V_1' \subseteq V_0'$. Define:

$$\tau(\mathcal{U}_0, (U_0, \overline{V_0}), \mathcal{U}_1) = (U_1, \overline{V_1})$$

Continue this process. Note that for all $i \in \omega$ we have $V_{i+1} \subseteq \overline{V}_i \subseteq V_i'$. It follows from the fact that σ is a winning strategy for E in $Ch(X)$ that $\bigcap\{V_i' : i \in \omega\} = \emptyset$. Hence $\bigcap\{\overline{V}_i : i \in \omega\} = \emptyset$. Therefore τ is a winning strategy for P2 in $\Gamma(X)$.

□

Corollary 2.21. *If quasi-regular X is weakly γ-complete then X is hereditarily Baire.*

Proof. Suppose X is weakly γ-complete. Let $C \subseteq X$ be closed. Then C is weakly γ-complete and hence Baire. Thus X is hereditarily Baire. □

Proposition 2.22. *The space $X = (\omega_1 + 1) \times [0,1] \setminus (\{\omega_1\} \times \mathbb{P})$, where $\mathbb{P}$ are the irrationals, is a space that is $\widehat{\gamma}$-complete but not hereditarily Baire.*

Proof. Since $\{\omega_1\} \times (\mathbb{Q} \cap [0,1])$ is a closed copy of the rationals, it follows that X is not hereditarily Baire. Let σ be the strategy for P1 in $\widehat{\Gamma}(X)$ defined by $\sigma(S) = \{X\}$ for all partial plays S. We will show that S is a winning strategy. Let $(\overline{V}_i : i \in \omega)$ be a sequence that represents a play by P2 in response to P1 using σ.

Suppose that there exists a $j \in \omega$ such that $\pi_1(\overline{V}_j)$ is bounded in ω_1. Then $\overline{V}_j$ is a compact set. And it follows that $\bigcap\{\overline{V}_i : i \geq j\} \neq \emptyset$ since it would be the intersection of a nested sequence of compact sets. Since $\{\overline{V}_i : i \in \omega\}$ has the finite intersection property, it would follow that $\bigcap\{\overline{V}_i : i \in \omega\} \neq \emptyset$.

On the other hand suppose that for all $j \in \omega$ that $\pi_1(\overline{V}_j)$ is unbounded in ω_1. We will show that $\pi_1(\overline{V}_i)$ is closed for all i. Towards a contradiction, suppose $\pi_1(\overline{V}_k)$ is not closed. Let $\alpha \in \overline{\pi_1(\overline{V}_k)} \setminus \pi_1(\overline{V}_k)$. Let $(\alpha_i : i \in \omega)$ be a increasing sequence of points in $\pi_1(\overline{V}_k)$ that converges to α. For each $i \in \omega$ let $x_i \in [0,1]$ such that $(\alpha_i, x_i) \in \overline{V}_j$. Let $x \in [0,1]$ be a limit point of $\{x_i : i \in \omega\}$. We claim that $(\alpha, x) \in \overline{V}_i$. Let $U \times W$ be a basic open set in X that contains (α, x). Let $n_0 \in \omega$ such that if $i > n_0$ then $\alpha_i \in U$. Let $n_1 \in \omega$ such that if $i > n_1$ then $x_i \in W$. Let $n = \max\{n_0, n_1\}$. Note $(\alpha_{n+1}, x_{n+1}) \in U \times W$. Hence $(\alpha, x) \in \overline{V}_k$. But that would imply that $\alpha \in \pi_1(\overline{V}_k)$, contrary to the way we picked α. It follows that $\pi_1(\overline{V}_i)$ is closed and unbounded in ω_1.

Since the intersection of countably many closed unbounded subsets of ω_1 is non-empty, we can choose a $\beta \in \omega_1$ such that $\beta \in \bigcap\{\pi_1(\overline{V}_i) : i \in \omega\}$. Then the sequence $\{\overline{V}_i \cap \pi_1^{-1}(\beta)\}$:

$i \in \omega$ is a nested sequence of compact sets. Hence $\emptyset \neq \bigcap\{\overline{V}_i \cap \pi_1^{-1}(\beta) : i \in \omega\} \subseteq \bigcap\{\overline{V}_i : i \in \omega\}$.

It follows that σ is a winning strategy for P1 in $\widehat{\Gamma}(X)$. Hence X is $\widehat{\gamma}$-complete. $\square$

Corollary 2.23. *$\widehat{\gamma}$-completeness does not imply γ-completeness.*

Chapter 3

Fan Spaces

Throughout this section we will look at the set $(\omega \times \omega) \cup \{\infty\}$ where points in $\omega \times \omega$ are isolated and for each $a \in \omega$ the set $(\{a\} \times \omega) \cup \{\infty\}$ is homeomorphic to a convergent sequence. We will call such a space a **fan**. Suppose $A \subseteq \omega$, we will denote the set $P_A = \{(i, j) : i \in A\}$. When there is no confusion we will use the notation P_n for the set $P_{\{n\}}$.

Two well known fans are the metric fan M and sequential fan S_ω, defined in the next section. In this chapter we will be particularly interested in fans with intermediate topologies between the metric and sequential fans. To get such topologies, given a free filter $\mathfrak{u}$ on ω we can define a fan $S_\mathfrak{u}$ which we will call a filter-fan.

Fans are relatively simple spaces as they only have one non-isolated point. However the class of fans is diverse enough to isolate the different completeness properties on the compact-open topologies. We will show that for filter fans $S_\mathfrak{u}$, if $C_k(S_\mathfrak{u})$ is Baire then it is metrizable. Also $C_k(S_\mathfrak{u})$ is never hereditarily Baire or Choquet.

We will also classify different completeness properties on $C_k(S_\mathfrak{u})$ by properties of the filter $\mathfrak{u}$, and show that the class of fans do not serve as a counter example to the main conjecture.

3.1 The Metric Fan and the Sequential Fan

Recall that the **metric fan** is the space $M = (\omega \times \omega) \cup \{\infty\}$ with the following topology Points in $(\omega \times \omega)$ are isolated and a basic open set around ∞ is of the form $U(n) = \{(a, b) : b \geq n\} \cup \{\infty\}$. The metric fan has the coarsest topology that we will consider in this chapter. The following is a well known result.

Theorem 3.1. *M is not hemicompact.*

Proof. Suppose $\{K_i : i \in \omega\}$ is a countable family of compact sets. For each $n \in \omega$ let $R_n = \{(i, n) : i \in \omega\}$. For each $i \in \omega$ let $x_i \in R_i$ such that $x_i \notin K_i$; this is possible because R_i is a collection of isolated points, hence $R_i \cap K_i$ must be finite. Let $K = \{x_i : i \in \omega\} \cup \{\infty\}$. Then K is not a subset of K_i for any $i \in \omega$. However K is compact since any open set containing ∞ will contain all but a finite subset of K. It follows that $\{K_i : i \in \omega\}$ is not a dominating family of compact sets. Hence the metric fan is not hemicompact. $\square$

Corollary 3.2. *The space $C_k(M)$ is not metrizable.*

It will be shown, in a more general context that M doesn't have the Moving Off Property. This will show, in particular, that $C_k(M)$ is neither Baire nor metrizable. However, there is another common fan whose compact open topology is completely metrizable.

Recall that the **Sequential Fan** is the fan $S_\omega = (\omega \times \omega) \cup \{\infty\}$ with the following topology. For every $f \in \omega^\omega$ let $U(f) = \{(i, j) \in \omega \times \omega : j > f(i)\} \cup \{\infty\}$. A local basis for ∞ is $\{U(f) : f \in \omega^\omega\}$.

There is a more common and natural way to view the sequential fan in terms of a quotient space.

Theorem 3.3. *For each $i \in \omega$ let $S_i = (s^i_n : n \in \omega) \cup \{s^i_*\}$ be a convergent sequence with limit point s^i_*. Let $X = \bigoplus\{S_i : i \in \omega\}$. Consider the map $q : X \to (\omega \times \omega) \cup \{\infty\}$ defined by $q(s^i_j) = (i, j)$ and $q(s^i_*) = \infty$. Then the quotient topology generated by q is homeomorphic to S_ω.*

That is to say, S_ω can be generated by taking a countable collection of pairwise disjoint sequences with their limit points and identifying the limit points. The following is a well known and simple result.

Theorem 3.4. *The sequential fan is a hemicompact k-space.*

Proof. By the above theorem we know that S_ω is the quotient image of a locally compact space, hence it is a k-space. We claim that $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ witnesses hemicompactness. Suppose $A \subseteq S_\omega$ such that $A \nsubseteq P_F$ for any finite $F \subseteq \omega$. We will show that A is not compact. For each $i \in \omega$ let $(a_i, b_i) \in A$ such that $a_i > i$; note if no such (a_i, b_i) existed then $A \subseteq K_{\{}0, 1, \ldots, i\}$. Let $B = \{(a_i, b_i) : i \in \omega\} \cup \{\infty\}$. B is closed since $\infty \in B$ and ∞ is the only non isolated point in S_ω. We claim that B is a noncompact closed subset of A, and hence A is not compact. Consider the following open cover of B: $\mathcal{U} = \{\{(a_i, b_i)\} : i \in \omega\} \cup \{U(f)\}$ where $f : \omega \to \omega$ is defined by $f(i) = b_i + 1$. Then $U(f) \cap \{(a_i, b_i) : i \in \omega\} = \emptyset$. So $\mathcal{U}$ is an open cover of B with no finite subcover. It follows that A is not compact and $\mathcal{K}$ witnesses that S_ω is hemicompact. $\square$

Corollary 3.5. *$C_k(S_\omega)$ is completely metrizable.*

3.2 Filter Fans

We will now look at fans whose topologies are finer than the metric fan but coarser than the sequential fan. But first we will recall a couple definitions.

Suppose X is a set. A **filter** $\mathcal{F}$ on X is a collection of subsets of X such that:

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
2. If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$
3. $\emptyset \notin \mathcal{F}$.

A filter which is maximal with respect to inclusion is called an **ultrafilter**. If $\mathcal{F}$ is a filter on X and $\bigcap \mathcal{F} = \emptyset$ then $\mathcal{F}$ is said to be a **free** filter.

We will use the concept of a filter construct a topology on a fan.

Definition 3.6. *Suppose $\mathfrak{u}$ is a free filter on ω. For each $f \in \omega^\omega$ and $A \in \mathfrak{u}$ and $n \in \omega$ let $\langle f, A, n\rangle = \{\infty\} \cup \{(i,j) : i \notin A, j \geq f(i)\} \cup \{(i,j) : i \in A, j \geq n\}$. We will show below that $\mathcal{B} = \{\langle f, A, n\rangle : f \in \omega^\omega, n \in \omega, A \in \sqcap\}$ is a local base for ∞. Let $S_\sqcap$ be the fan with $\mathcal{B}$ as a local base for ∞. We call $S_\mathfrak{u}$ the **$\mathfrak{u}$-fan** or simply a filter-fan. If $\mathcal{F}$ is a free ultrafilter on ω then we call the fan $S_\mathcal{F}$ an **ultrafilter fan**.*

It is worth noting that the sequential fan and the metric fan are special cases of filter-fans. If $\mathfrak{u}$ is the co-finite filter on ω then $S_{\mathfrak{u}}$ is the metric fan, and if $\mathfrak{u}$ is a fixed filter then $S_{\mathfrak{u}} = S_\omega$.

3.2.1 Hemicompactness of $C_k(S_{\mathfrak{u}})$

We have noted that the metric fan is not hemicompact while S_ω is hemicompact, and that the family $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ is a dominating family of compact subsets of S_ω. The result below shows that a filter-fan is hemicompact if and only if it doesn't contain a copy of the metric fan, and furthermore, if a filter-fan is hemicompact then the above set $\mathcal{K}$ of compact sets is a dominating family. Hemicompactness of the filter-fan $S_{\mathfrak{u}}$ is also characterized by an internal property of the filter $\mathfrak{u}$, and by the space $\omega \cup \{\mathfrak{u}\}$, where ω is the set of isolated points and a neighborhood of $\mathfrak{u}$ has the form $F \cup \{\mathfrak{u}\}$, where $F \in \mathfrak{u}$.

Recall that, given a filter $\mathfrak{u}$ on ω, a subset $A \subseteq \omega$ is called **$\mathfrak{u}$-positive** if $A \cap F \neq \emptyset$ for all $F \in \mathfrak{u}$. Equivalently, A is $\mathfrak{u}$-positive if and only if $\omega \setminus A \notin \mathfrak{u}$.

Proposition 3.7. *Let $\mathfrak{u}$ be a free filter on ω. Then the following are equivalent*

(i) *$S_{\mathfrak{u}}$ is hemicompact.*

(ii) *$S_{\mathfrak{u}}$ doesn't contain a copy of the metric fan.*

(iii) *There is no infinite $A \subseteq \omega$ such that A is almost contained in every filter member; i.e. there is no infinite $A \subseteq \omega$ such that $|A \setminus F| < \omega$ for all $F \in \mathfrak{u}$.*

(iv) *For all infinite $J \subseteq \omega$ there is an infinite subset $A \subseteq J$ such that A is not $\mathfrak{u}$-positive.*

(v) *The space $\omega \cup \{\mathfrak{u}\}$ has no non-trivial convergent sequences.*

(vi) *The family $\{P_F : F \subseteq \omega$ finite$\}$ is a dominating family of compact sets.*

(vii) *$C_k(S_{\mathfrak{u}})$ is metrizable.*

Proof. Statement (vi) immediately implies (i). According to Theorem 1.5 statements (i) and (vii) are equivalent.

We show (i) implies (ii). Suppose $S_{\mathfrak{u}}$ is hemicompact but contains a copy Y of the metric fan. Since $\infty \in Y$ it follows that Y is closed and therefore Y is hemicompact, contrary to the fact that the metric fan is not hemicompact. Therefore (i) implies (ii).

To show that (ii) implies (iii), suppose that there is an infinite $J \subseteq \omega$ such that J is almost contained in every filter member. Consider the set $Y = P_J$ with the subspace topology. We claim that Y is a copy of the metric fan. Suppose $\langle f, A, n\rangle \cap P_J$ is a basic open set around ∞ in Y. Then $J \setminus A = J'$ is finite. Let $m = \max\{f(a) : a \in J'\} \cup \{n\}$. Then $\langle f, A, n\rangle \cap S_J \subseteq \{(j,k) \in J \times \omega : k \geq m\}$. It follows that each open set in Y contains a metric-fan open set. The converse is clear.

To show (iii) implies (iv). Suppose the negation of (iv), i.e. assume that there is an infinite $J \subseteq \omega$ such that all infinite $A \subseteq J$ are $\mathfrak{u}$-positive. We will show that J is almost contained in every filter element. Suppose towards a contradiction that $F \in \mathfrak{u}$ such that $|J \setminus F| < \omega$. Then $A = J \cap F$ is infinite and therefore is $\mathfrak{u}$-positive. However A misses the filter element F, i.e. $A \cap F = \emptyset$. This is a contradiction. Hence J is almost contained in every filter element.

To show (iv) implies (v). Suppose the negation of (v). Let $S = (s_i : i \in \omega)$ be a non-trivial sequence converging to $\mathfrak{u}$ in the single-filter space $\omega \cup \{u\}$. Define $J = \{s_i : i \in \omega\}$. We claim that every infinite subset of J is $\mathfrak{u}$-positive. Let $A \subseteq J$ be infinite. Then A is cofinal in ω and corresponds to a subseteq of S, i.e. $A = \{s_{n_i} : i \in \omega\}$ which as a sequence converges to $\mathfrak{u}$. Therefore if $F \in \mathfrak{u}$ it follows that F contains a tail of the sequence A. In particular $F \cap A \neq \emptyset$. Consequently A is $\mathfrak{u}$-positive.

To show (v) implies (vi). Suppose $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ is not a dominating family of compact sets. Let K be a compact set which is not contained in P_F for any finite $F \subseteq \omega$. Let $K' = \{(x_n, y_n) \in K : n \in \omega\}$ be a subset of K such that $x_n > n$ for all n. Since

K' is an infinite subset of K and K is compact, it follows that K' is not discrete. Hence $\infty \in \mathrm{cl}(K')$. This implies that $(x_n : n \in \omega)$ is a non-trivial convergent sequence in $\omega \cup \{\mathfrak{u}\}$.

□

3.2.2 Fréchetness of $C_k(S_{\mathfrak{u}})$

Recall that a space X is **Fréchet** if given any $A \subseteq X$ we have that if $x \in \mathrm{cl}(A) \setminus A$ then there is a sequence of points in A converging to X. A space is said to be **Strongly Fréchet** if for any sequence $A_0 \supseteq A_1 \supseteq \cdots$ and for any $x \in \bigcap\{\mathrm{cl}(A_i) : i \in \omega\}$ there exists a sequence $(a_i : i \in \omega)$ converging to x such that $a_i \in A_i$ for each $i \in \omega$.

We will give a internal characterization on the filter of when $S_{\mathfrak{u}}$ is a Fréchet (or equivalently a k-space). This will lead us to conclude that $C_k(S_{\mathfrak{u}})$ is not completely metrizable for any free filter $\mathfrak{u}$.

Proposition 3.8. *Suppose $\mathfrak{u}$ is a free filter on ω. $S_{\mathfrak{u}}$ is Fréchet (or a k-space) if and only if $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet.*

Proof. Suppose $S_{\mathfrak{u}}$ is Fréchet. We need to check that $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet at $\mathfrak{u}$. Let $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be a sequence such that $\mathfrak{u} \in \bigcap\{\overline{A}_i : i \in \omega\}$. This implies that each A_i is $\mathfrak{u}$-positive. For each $i \in \omega$ let $D_i = A_i \times \{i\}$. Let $D = \bigcup\{D_i : i \in \omega\}$. Then $\infty \in \overline{D}$. Let $((n_i, m_i))$ be a sequence of points in D that converge to ∞. We may assume $m_i < m_{i+1}$. For each $j \in \omega$ let $a_j = \min\{n_i : m_i \geq j\}$. It is easy to chech that $a_j \in A_j$ for each $j \in \omega$ and (a_j) limits to $\mathfrak{u}$. Therefore $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet.

To show the converse suppose that $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet. Aiming to show that $S_{\mathfrak{u}}$ is a Fréchet, suppose $D \subseteq S_{\mathfrak{u}}$ and $\infty \in \overline{D}$. Let $A_0 = \pi(D)$. For all $i > 0$ let $A_i = \pi(D \setminus \omega \times \{0, 1, \cdots, i-1\})$. Then $A_0, A_1, A_2, \cdots$ is a decreasing sequence of $\mathfrak{u}$-positive sets. Let (a_i) be a sequence that converges to $\mathfrak{u}$ such that $a_i \in A_i$ for each $i \in \omega$. For each $i \in \omega$ let $b_i \in \omega$ such that $b_i \geq i$ and $(a_i, b_i) \in D$. Then the sequence $((a_i, b_i))$ converges to ∞, hence $S_{\mathfrak{u}}$ is Fréchet. □

Corollary 3.9. *Suppose $\mathfrak{u}$ is a free filter on ω. If $S_\mathfrak{u}$ is a k-space then $S_\mathfrak{u}$ is not hemicompact and $C_k(S_\mathfrak{u})$ is not metrizable.*

Corollary 3.10. *Suppose $\mathfrak{u}$ is a free filter on ω. Then $C_k(S_\mathfrak{u})$ is not completely metrizable.*

Proof. This follows immediately by Theorem 1.5 and from the fact that $S_\mathfrak{u}$ can not be both a k-space and hemicompact. □

3.2.3 Baireness of $C_k(S_\mathfrak{u})$

We will make use of the following game to show an equivalence of when $C_k(S_\mathfrak{u})$ is Baire.

Definition 3.11. *If $\mathfrak{u}$ is a filter on ω define a game $G(\mathfrak{u})$ with two players P_1 and P_2 as follows. P_1 chooses a finite subset $A_0 \subseteq \omega$ and then P_2 chooses a finite subset $B_0 \subseteq \omega$. On play $n > 0$, P_1 choose a finite subset A_n such that $A_n \cap A_i = \emptyset$ and $A_n \cap B_i = \emptyset$ for all $i < n$, and P_2 chooses a finite subset $B_n \subseteq \omega$ (with no restrictions). P_1 wins if $\bigcup\{A_i : i \in \omega\}$ is $\mathfrak{u}$-positive. P_2 wins otherwise.*

Proposition 3.12. *Let $\mathfrak{u}$ be a free filter on ω. Then P_2 has no winning strategy in $G(\mathfrak{u})$, and if $\mathfrak{u}$ is an ultrafilter, then $G(\mathfrak{u})$ is undetermined.*

Proof. It is easy to show that $G(\mathfrak{u})$ is equivalent, in terms of the existence of winning strategies, to the game $G'(\mathfrak{u})$ in which P_2 has to play by the same rules as P_1, i.e., $B_n \cap A_i = \emptyset$ for all $i \leq n$, and $B_n \cap B_i = \emptyset$ for all $i < n$. Then it follows that neither player can have a winning strategy in which the union of his chosen sets lies in the filter $\mathfrak{u}$; for if he did, his opponent could essentially employ the same strategy to also force the union of his chosen sets to also be in $\mathfrak{u}$, yielding a pair of disjoint filter members. Also, in either game, P_1 or P_2 can always guarantee that $\bigcup_{n\in\omega} A_n \cup B_n = \omega$ by adding $\{n\}$ to his/her play in round n if $\{n\}$ hasn't already been covered.

Now to prove the proposition. By the above comment, a winning strategy for P_2 in $G'(\mathfrak{u})$ would give one in which the union of P_2's sets are in $\mathfrak{u}$; hence P_2 has no winning strategy in either game. For ultrafilters, $\mathfrak{u}$-positive sets are in $\mathfrak{u}$, so we get a similar contradiction if we assume a winning strategy for P_1. $\square$

Lemma 3.13. *Suppose $\mathfrak{u}$ is a free filter. If P_1 has no winning strategy in $G(\mathfrak{u})$ then $S_\mathfrak{u}$ is hemicompact.*

Proof. Suppose $S_\mathfrak{u}$ is not hemicompact. We will show that P_1 has a winning strategy in $G(\mathfrak{u})$. By Proposition 3.7 there is an infinite set $A \subseteq \omega$ which is almost contained in every filter element. If P_1 chooses points from A, then P_1 will win the game $G(\mathfrak{u})$. $\square$

Lemma 3.14. *Suppose $\mathfrak{u}$ is a free filter on ω. If $S_\mathfrak{u}$ has the Moving Off Property then $S_\mathfrak{u}$ is hemicompact.*

Proof. Suppose $S_\mathfrak{u}$ is not hemicompact. Then by Proposition 3.7, $S_\mathfrak{u}$ contains a closed copy M of the metric fan. Since the metric fan is a non-locally compact metric space, it follows that M doesn't have the Moving Off Property by Theorem 1.8(iii). Since the Moving Off Property is hereditary under closed sets it follows that $S_\mathfrak{u}$ doesn't have the Moving Off Property. $\square$

Corollary 3.15. *Suppose $\mathfrak{u}$ is a free filter on ω. If $C_k(S_\mathfrak{u})$ is Baire, then $S_\mathfrak{u}$ is hemicompact and $C_k(S_\mathfrak{u})$ is metrizable.*

Proof. Suppose $C_k(S_\mathfrak{u})$ is Baire. It follows from Theorem 1.8(a) that $S_\mathfrak{u}$ has the Moving Off Property. By Lemma 3.14 $S_\mathfrak{u}$ is hemicompact. Therefore by Theorem 1.5, $C_k(S_\mathfrak{u})$ is metrizable. $\square$

The converse of the above corollary is not true, as is shown by Example 3.24 in the next section.

We now show that there is a strong connection between the games $G(\mathfrak{u})$ and $G_{K,L}(S_\mathfrak{u})$.

Proposition 3.16. *Suppose $\mathfrak{u}$ is a free filter.*

(i) P_1 has a winning strategy in $G(\mathfrak{u})$ if and only if L has a winning strategy in $G_{K,L}(S_\mathfrak{u})$.

(ii) P_2 *has a winning strategy in* $G(\mathfrak{u})$ *if and only if* K *has a winnings strategy in* $G_{K,L}(S_\mathfrak{u})$.

Proof. We will begin by showing (i). Suppose σ is a winning strategy for P_1 in $G(\mathfrak{u})$ but L doesn't have a winning strategy in $G_{K,L}(S_\mathfrak{u})$. This implies by Lemma 3.14 that $S_\mathfrak{u}$ is hemicompact. Define a strategy τ for L in $G_{K,L}(S_\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_\mathfrak{u})$
$A_0 = \sigma(\emptyset)$	K_0
$B_0 = \pi(K_0)$	$L_0 = \tau(K_0) = A_1 \times \{0\}$
$A_1 = \sigma(B_0)$	K_1
$B_1 = \pi(K_1)$	$L_1 = \tau(K_0, K_1) = A_2 \times \{1\}$
$A_2 = \sigma(B_0, B_1)$	$\vdots$

To interpret the chart: A_0 is P_1's first play using σ in $G(\mathfrak{u})$, and K_0 is K's first play in $G_{K,L}(S_\mathfrak{u})$. Then we let $B_0 = \pi(K_0)$ be P_2's response, which is finite since $S_\mathfrak{u}$ is hemicompact, and consider P_1's reply A_1 to this play. Then let $\tau(K_0) = L_0 = A_1 \times \{0\}$ be L's response to K_0, etc.

Since σ is a winning strategy for P_1 it follows that $\bigcup\{A_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Hence if $F \in \mathfrak{u}$ then $\{i \in \omega : A_i \cap F \neq \emptyset\}$ is infinite. Suppose $U = \langle f, F, n\rangle$ is a basic open set around ∞ in $S_\mathfrak{u}$. By the previous observation $\{i > n : A_i \cap F \neq \emptyset\}$ is infinite. Therefore $\{i : L_i \cap U \neq \emptyset\}$ is infinite. It follows that $\{L_i : i \in \omega\}$ is not a strongly discrete family. Therefore τ is a winning strategy for L in $G_{K,L}(S_\mathfrak{u})$, a contradiction.

On the other hand suppose σ is a winning strategy for L in $G_{K,L}(S_{\mathfrak{u}})$. By Lemma 3.13, $S_{\mathfrak{u}}$ is hemicompact. Define a strategy τ for P_1 in $G(\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_{\mathfrak{u}})$
$A_0 = \tau(\emptyset) = \{0\}$	$K_0 = P_{B_0 \cup A_0}$
B_0	$L_0 = \sigma(K_0)$
$A_1 = \tau(B_0) = \pi(L_0)$	$K_1 = P_{B_1 \cup A_1}$
B_1	$L_1 = \sigma(K_0, K_1)$
$A_2 = \tau(B_0, B_1) = \pi(L_1)$	$K_2 = P_{B_2 \cup A_2}$
B_2	$\vdots$

That is, start with $\tau(\emptyset) = \{0\} = A_0$, let B_0 be P_2's response, then let $K_0 = P_{B_0 \cup A_0}$ be K's first play in $G_{K,L}(S_{\mathfrak{u}})$, and if L responds with L_0, let $\tau(B_0)$ be $A_1 = \pi(L_0)$, etc.

We claim that τ is a winning strategy. Assume towards a contradiction that $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. Let $F \in \mathfrak{u}$ such that $F \cap A_i = \emptyset$ for all $i \in \omega$. Since $\infty \notin L_i$ for any $i \in \omega$, it follows that each L_i is a finite subset of $\omega \times \omega$. Furthermore $P_{\pi(L_i)} \cap P_{\pi(L_j)} = \emptyset$ if $i \neq j$. Therefore we can pick a function $f : \omega \to \omega$ that dominates $\bigcup\{L_i : i \in \omega\}$ in the sense that, for any $k \in \omega$, $f(k) > \max\{j : (k,j) \in \bigcup_{i\in\omega} L_i\}$. Then the open set $U = \langle f, F, 0\rangle$ is a basic open set around ∞ that misses each L_i. It follows that $\{L_i : i \in \omega\}$ is a (strongly) discrete family, contrary to the fact that σ is winning for L. This completes the proof of statement (i).

We will now show (ii). Suppose K has a winning strategy σ in $G_{K,L}(S_{\mathfrak{u}})$. We may assume $\infty \in \sigma(\emptyset)$. Since L doesn't have a winning strategy in $G_{K,L}(S_{\mathfrak{u}})$ it follows that $S_{\mathfrak{u}}$ has the Moving Off Property and is hemicompact by Lemma 3.14. Construct a winning strategy τ for P_2 in $G(\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_\mathfrak{u})$
A_0	$K_0 = \sigma(\emptyset)$
$B_0 = \tau(A_0) = \pi(K_0)$	$L_0 = A_1 \times \{0\}$
A_1	$K_1 = \sigma(K_0, L_0)$
$B_1 = \tau(A_0, A_1) = \sigma(K_1)$	$A_2 \times \{1\}$
A_2	$\vdots$

Since σ is a winning strategy for K in $G_{K,L}(S_\mathfrak{u})$ it follows that there is a basic open set $U = \langle f, F, k\rangle$ around ∞ such that $U \cap L_i = \emptyset$ for all $i \in \omega$. Therefore $\{i \in \omega : F \cap A_i \neq \emptyset\} \subseteq \{0, 1, \ldots, k\}$ is finite. So $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive.

On the other hand suppose σ is a winning strategy for P_2. Since P_1 doesn't have a winning strategy it follows that $S_\mathfrak{u}$ is hemicompact by Lemma 3.13. Define a strategy τ for K in $G_{K,L}(S_\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_\mathfrak{u})$
$A_0 = \pi(L_0)$	$K_0 = \tau(\emptyset) = P_0$
$B_0 = \sigma(A_0)$	L_0
$A_1 = \pi(L_1)$	$K_1 = \tau(L_0) = P_{B_0 \cup A_0}$
$B_0 = \sigma(A_0, A_1)$	L_1
$A_2 = \pi(L_2)$	$K_2 = \tau(L_0, L_1) = P_{B_1 \cup A_1}$
$B_1 = \sigma(A_0, A_1, A_2)$	L_2
$\vdots$	$\tau(L_0, L_1, L_2) = P_{B_2 \cup A_2}$

Since σ is a winning strategy for P_2 in $G(\mathfrak{u})$ it follows that $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. Let $F \in \mathfrak{u}$ such that $F \cap A_i = \emptyset$ for all $i \in \omega$. By similar observations as above we can find a function $f : \omega \to \omega$ that dominates $\bigcup\{L_i : i \in \omega\}$. Let $U = \langle f, F, 0\rangle$. Then

$U \cap L_i = \emptyset$ for all $i \in \omega$. It follows that $\{L_i : i \in \omega\}$ is strongly discrete. Hence τ is a winning strategy for K in $G_{K,L}(S_{\mathfrak{u}})$. $\square$

Corollary 3.17. *Suppose $\mathfrak{u}$ is a free filter on ω. Then $C_k(S_{\mathfrak{u}})$ is not Choquet.*

Proof. By Proposition 3.12, P_2 does not have a winning strategy in $G(\mathfrak{u})$. Consequently by Proposition 3.16, K does not have a winning strategy in $G^{\circ}_{K,L}(S_{\mathfrak{u}})$, and so by Theorem 2.5, $C_k(S_{\mathfrak{u}})$ is not Choquet. $\square$

The next corollary, which is immediate from Corollaries 3.15 and 3.17, shows that S_ω is the only space among those we are considering whose function space is completely metrizable.

Corollary 3.18. *Let $\mathfrak{u}$ be a free filter. If $C_k(S_{\mathfrak{u}})$ is Baire, then $C_k(S_{\mathfrak{u}})$ is metrizable but not Choquet.*

We proceed to characterize when $C_k(S_{\mathfrak{u}})$ is Baire. First, we give a characterization on the filter $\mathfrak{u}$ for P_1 not having a winning strategy in $G(\mathfrak{u})$. By Lemma 3.13 and Proposition 3.7, we know if P_1 has no winning strategy in $G(\mathfrak{u})$ then for all infinite $J \subseteq \omega$ there is an infinite subset $A \subseteq J$ such that A is not $\mathfrak{u}$-positive. A strengthening of this will give us our characterization.

Proposition 3.19. *Suppose $\mathfrak{u}$ is a free filter on ω. The following are equivalent.*

(i) *P_1 has no winning strategy in $G(\mathfrak{u})$.*

(ii) *If $\mathcal{F}$ is a collection of finite subsets of ω that moves off the finite sets, then there exists an infinite $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}'$ is not $\mathfrak{u}$-positive.*

Proof. We will show $(ii) \to (i)$ by contrapositive. Suppose σ is a winning strategy for P_1 in $G(\mathfrak{u})$. Let $\{B_0, B_1, \ldots\}$ denote the finite subsets of ω. Let $A_\emptyset = \sigma(\emptyset)$. For each $i \in \omega$ let $A_i = \sigma(A_\emptyset, B_i)$. If A_s has been defined for all $s \in \omega^{<\omega}$ such that $|s| = n$ then for each $i \in \omega$ let:

$$A_{s^\frown i} = \sigma(A_\emptyset, B_{s(0)}, A_{s(0)}, B_{s(1)}, A_{(s(0),s(1))}, \ldots, B_{s(i)}, A_{s\restriction i+1}, \ldots, B_{s(n-1)}, A_s, B_i)$$

This defines a tree whose branches correspond to plays of the game $G(\mathfrak{u})$. Note if $s \in \omega^\omega$ then $\bigcup\{A_{s\restriction n} : n \in \omega\}$ is $\mathfrak{u}$-positive. We will create a collection $\mathcal{F}$ of finite sets which moves off the finite sets such that if $\mathcal{F}' \subseteq \mathcal{F}$ is infinite, then $\bigcup \mathcal{F}'$ contains the union of $\{A_s \restriction n : n \in \omega\}$ for some sequence $s \in \omega^\omega$; consequently $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive.

Let $K_0 = \{(0)\}$ and $F_0 = \bigcup\{A_0\}$. Let $K_1 = \{(1), (0, 1)\}$ and $F_1 = \bigcup\{A_1, A_{(0,1)}\}$. In general let $K_n = \{s :$ s is a finite increasing sequence in ω whose last term is $n\}$ and let $F_n = \bigcup\{A_s : s \in K_n\}$. Note if $s \in K_n$ then since $s(n-1) = n$ it follows that A_s is a play in response to P_2 playing B_n, hence $A_s \cap B_n = \emptyset$. Therefore $B_n \cap F_n = \emptyset$. Thus $\mathcal{F} = \{F_i : i \in \omega\}$ moves off the finite sets.

Suppose $\mathcal{F}' \subseteq \mathcal{F}$ is infinite. Then write $\mathcal{F}' = \{F_{n_0}, F_{n_1}, \ldots\}$ where $s = (n_i)_{i\in\omega}$ is an increasing sequence. The sequence $s \restriction (i+1) \in K_{n_i}$ since $s \restriction (i+1)$ is increasing and $s(i) = n_i$. Hence $A_{s\restriction(i+1)} \subseteq F_{n_i}$ for all $i \in \omega$. Therefore $\bigcup\{A_{s\restriction i} : i \in \omega\} \subseteq \bigcup \mathcal{F}'$, and consequently $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive.

Thus we have shown that there exists a collection $\mathcal{F}$ of finite sets that move off the finite sets that has the property that if $\mathcal{F}' \subseteq \mathcal{F}$ is infinite then $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive. This is the negation of statement (ii).

On the other hand, to show $(i) \to (ii)$, suppose P_1 does not have a winning strategy in $G(\mathfrak{u})$, and suppose $\mathcal{F}$ is a collection of finite subsets of ω that move off the finite subsets of ω. Clearly there is a strategy σ for P_1 such that P_1 always plays a member of $\mathcal{F}$. Since σ can't be winning, there must be a sequence $\mathcal{F}' = \{F_0, F_1, \ldots\}$ of members of $\mathcal{F}$ corresponding to plays by P_1 using σ whose union is not $\mathfrak{u}$-positive. $\square$

Lemma 3.20. *Suppose* $\mathfrak{u}$ *is a free filter on* ω*. If* P_1 *has no winning strategy in* $G(\mathfrak{u})$*, then* $C_k(S_\mathfrak{u})$ *is Baire.*

Proof. Suppose P_1 has no winning strategy in $G(\mathfrak{u})$. Then $S_\mathfrak{u}$ is hemicompact by Lemma 3.13. By Proposition 3.7 we have that $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ dominates the compact subsets of $S_\mathfrak{u}$. We will show that E has no winning strategy in $\mathrm{Ch}(C_k(S_\mathfrak{u}))$.

Recall that a basic open set in $C_k(X)$ has the form $B(f, K, \epsilon) = \{g \in C_k(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in K\}$, where $f \in C_k(X)$ and K is compact. This is still a base if the compact sets are restricted to members of a dominating family. So in the play of the game, we may assume the players are restricted to choosing basic open sets $B(f, K, \epsilon)$ where $K \in \mathcal{K}$.

Aiming towards a contradiction, suppose that $C_k(S_\mathfrak{u})$ is not Baire. Then E has a winning strategy σ in $\mathrm{Ch}(C_k(S_\mathfrak{u}))$. Let NE choose finite sets $G_0, G_1, \ldots$ as follows. Suppose $A_0 = \sigma(\emptyset) = B(f_0, P_{F_0}, \epsilon_0)$ is E's first play in the Choquet game. Let G_0 be any finite set such that $G_0 \cap F_0 = \emptyset$ and $\{0\} \subseteq G_0 \cup F_0$. Define $B_0 = B(g_0, P_{F_0 \cup G_0}, \epsilon_0/4)$ as NE's first play, where $g_0 \restriction P_{F_0} = f_0$ and $g_0 \restriction S_{F_0^c} = f_0(\infty)$. Then $B_0 \subseteq A_0$, and it therefore legal play by NE. Suppose $A_0, F_0, f_0, B_0, G_0, g_0, \ldots, A_{n-1}, F_{n-1}, f_{n-1}, B_{n-1}, G_{n-1}, g_{n-1}$ have been defined as above, then let

$$A_n = \sigma(A_0, B_0, \ldots, A_{n-1}, B_{n-1}) = B(f_n, P_{F_0 \cup G_0 \cup \cdots \cup F_{n-1} \cup G_{n-1} \cup F_n}, \epsilon_n)$$

where F_n is disjoint from all previous G_i's and F_i's. Let NE pick a finite set G_n disjoint from all previous F_i's and G_i's such that $\{0, \ldots, n\} \subseteq \bigcup\{G_i \cup F_i : i \leq n\}$ and define NE's play at round n as:

$$B_n = B(g_n, P_{F_0 \cup G_0 \cup \cdots \cup F_n \cup G_n}, \epsilon_n/4)$$

where $g_n \restriction P_{F_0 \cup G_0 \cup \cdots \cup F_n} = f_n$ and $g_n \restriction (P_{F_0 \cup G_0 \cup \cdots \cup F_n})^c = f_n(\infty)$.

We claim that if σ is winning strategy, then $\bigcup\{F_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Suppose $\bigcup\{F_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. We will show that there is a continuous function in $\bigcap\{B_i : i \in \omega\}$. Let $F = \bigcup\{F_i : i \in \omega\}$ and $H = \omega \setminus F$. Note $H \in \mathcal{F}$. Define a function $g : S_\mathcal{F} \to \mathbb{R}$ by $g(x) = \lim_{i \to \infty} g_i(x)$. We will show that g is continuous.

Let $\epsilon > 0$. Let $n \in \omega$ such that $\epsilon_n < \epsilon/3$. Then for all $x \in S_H$ we have

$$|g_n(x) - g(x)| \leq \sum_{i=0}^{\infty} |g_{n+i}(x) - g_{n+i+1}(x)| < \sum_{i=0}^{\infty} \epsilon_n/2^i = \epsilon_n < \epsilon/3.$$

Let $A = \bigcup\{G_i : i \leq n\}$ which is finite. By definition, for all $b \in H \setminus A$ and for all $i \in \omega$ we have $g_n(b,i) = f_n(\infty)$. For each $a \in A$ let $N_a \in \omega$ such that if $i > N_a$ then $|g_n(a,i) - g_n(\infty)| < \epsilon/3$. Let $N = \max\{N_a : a \in A\}$. Then for all $h \in H$ and all $i > N$ it follows that $|g_n(h,i) - g_n(\infty)| < \epsilon/3$. Consequently for all $h \in H$ and all $i > N$ we have

$$|g(h,i) - g(\infty)| \leq |g(h,i) - g_n(h,i)| + |g_n(h,i) - g_n(\infty)| + |(g_n(\infty) - g(\infty)| < \epsilon$$

For all $i \in F$ we have that $g \restriction P_i$ is continuous, since P_i is compact and the g_i's uniformly converge to g. For each $i \in F$ let $n_i \in \omega$ such that if $m > n_i$ then $|g(i,m) - g(\infty)| < \epsilon$. Define a function $f : \omega \to \omega$ by $f(i) = n_i$ if $i \in F$ and $f(i) = 0$ otherwise. Then for all $x \in \langle f, H, N\rangle$ we have $|g(x) - g(\infty)| < \epsilon$.

It follows that g is continuous and $g \in \bigcap\{B_i : i \in \omega\}$. Therefore σ isn't a winning strategy. In summary, if σ is a winning strategy for E then $\bigcup\{F_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Therefore if σ is a winning strategy there will be a corresponding winning strategy for P_1 in $G(\mathfrak{u})$, which is a contradiction. Hence E has no winning strategy. It follows that $C_k(S_\mathfrak{u})$ is Baire. □

We can summarize the above results by proving the following equivalent conditions for $C_k(S_\mathfrak{u})$ being Baire.

Theorem 3.21. *Suppose $\mathfrak{u}$ is a free filter on ω. The following are equivalent.*

(i) $S_\mathfrak{u}$ has the Moving Off Property.

(ii) L has no winning strategy in $G_{K,L}(S_\mathfrak{u})$.

(iii) P_1 has no winning strategy in $G(\mathfrak{u})$.

(iv) For any collection $\mathcal{F}$ of finite subsets of ω that moves off the finite sets, there exists an infinite $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}'$ is not $\mathfrak{u}$-positive.

(v) $C_k(S_\mathfrak{u})$ is Baire.

Proof. (i) implies (ii) by Theorem 1.11. (ii) implies (iii) by Proposition 3.16. Proposition 3.19 shows (iii) and (iv) are equivalent. Proposition 3.20 shows (iii) implies (v). And Theorem 1.8 shows (v) implies (i). □

We will now show for a free filter $\mathfrak{u}$ that $C_k(S_{\mathfrak{u}})$ is not hereditarily Baire. While the Choquet game $\mathrm{Ch}(X)$ characterizes Baireness of X, Debs shows in [De] the strong Choquet game $\mathrm{Ch}^*(X)$ characterizes hereditary Baireness for many spaces.

Theorem 3.22. *[De] Let X be a regular first-countable space in which every closed set is a G_δ-set. Then the following are equivalent:*

(i) X is hereditarily Baire;

(ii) E has no winning strategy in $\mathrm{Ch}^*(X)$.

Proposition 3.23. *Suppose $\mathfrak{u}$ is a free filter on ω. $C_k(S_{\mathfrak{u}})$ is not hereditarily Baire.*

Proof. If $C_k(S_{\mathfrak{u}})$ is not Baire, then of course it is not hereditarily so, thus we may suppose $C_k(S_{\mathfrak{u}})$ is Baire. Then by Corollary 3.15, $C_k(S_{\mathfrak{u}})$ is metrizable.

We will show that E has a winning strategy in $\mathrm{Ch}^*(C_k(S_{\mathfrak{u}}))$. By Corollary 3.15 and Proposition 3.7, $\{P_F : |F| < \omega\}$ is a dominating family of compact subsets of $S_{\mathfrak{u}}$. Thus if E plays the non-empty open set U and the point $f \in U$, we may assume U is of the form $B(f, P_F, \epsilon)$. If NE responds with the basic open set V, then $f \in V$ so we may assume V has the form $B(f, P_{F\cup G}, \delta)$, where $G \cap F = \emptyset$ and $\delta \leq \epsilon$.

Define the strategy σ for E in $\mathrm{Ch}^*(X)$ as follows. Let $F_0 = \{0\}, \epsilon_0 = 1$, and

$$f_0(\mathbf{x}) = \begin{cases} 1 & : \mathbf{x} = (i,j), i \in \omega, j = 0 \\ 0 & : \text{otherwise} \end{cases}$$

Let $\sigma(\emptyset) = U_0 = B(f_0, P_{F_0}, \epsilon_0\rangle)$. Suppose NE responds with $V_0 = \langle f_0, P_{F_0 \cup G_0}, \delta_0\rangle$. Let $F_1 = \{\min\{i \in \omega : i \notin F_0 \cup G_0\}\}$, $\epsilon_1 = \delta_0/2$, and

$$f_1(\mathbf{x}) = \begin{cases} f_0(\mathbf{x}) & : \mathbf{x} \in P_{F_0 \cup G_0} \\ 1 & : \mathbf{x} = (i,j), \mathbf{x} \notin P_{F_0 \cup G_0}, i \in \omega, j \leq 1 \\ 0 & : \text{ otherwise} \end{cases}$$

Let $\sigma(U_0, V_0) = U_1 = B(f_1, P_{F_0 \cup G_0 \cup F_1}, \epsilon_1)$.

Suppose $U_i, V_i, f_i, F_i, G_i, \epsilon_i$, and δ_i have been defined for all $i \leq k$. Let $F_{k+1} = \{\min\{i \in \omega : i \notin F_0 \cup G_0 \cup \cdots \cup F_k \cup G_k\}\}$, $\epsilon_{k+1} = \delta_k/2$ and

$$f_{k+1}(\mathbf{x}) = \begin{cases} f_k(\mathbf{x}) & : \mathbf{x} \in P_{F_0 \cup G_0 \cup \cdots \cup F_k \cup G_k} \\ 1 & : \mathbf{x} = (i,j), \mathbf{x} \notin P_{F_0 \cup G_0 \cdots \cup F_k \cup G_k}, i \in \omega, j \leq k+1 \\ 0 & : \text{ otherwise} \end{cases}$$

Let $\sigma(U_0, V_0, U_1, V_1, \cdots, U_k, V_k) = B(f_{k+1}, P_{F_0 \cup G_0 \cup \cdots \cup F_k \cup G_k \cup F_{k+1}}, \epsilon_{k+1})$.

We will show that $\bigcap U_i = \emptyset$. Let $f = \lim f_i$, where the limit is the pointwise limit. Since $\lim \epsilon_i = 0$ and $\omega = \bigcup (F_i \cup G_i)$ it follows that f is the only candidate for a point in $\bigcap U_i$. We will show that f is not continuous at ∞. Note that $f(\infty) = 0$.

Let $\langle g, A, n\rangle$ be a basic open set around ∞. Since $\mathfrak{u}$ is free, the filter element A is infinite. Let $k \in A$ such that $k \notin F_0 \cup G_0 \cup \cdots \cup F_n \cup G_n$. Then $(k,n) \in \langle g, A, n\rangle$ and $|f(\infty) - f(k,n)| = |0 - 1| = 1$. It follows that f is not continuous and $\bigcap U_i = \emptyset$. We have constructed a winning strategy σ for E in $\mathrm{Ch}^*(C_k(S_\mathfrak{u}))$. Since $C_k(S_\mathfrak{u})$ is metrizable, it follows from Theorem 3.22 that $C_k(S_\mathfrak{u})$ is not hereditarily Baire. $\square$

3.2.4 Examples of Fans

We will now look at several examples of fan spaces whose compact-open topology have various completeness properties. As we have seen, for the metric fan M the space $C_k(M)$ is not Baire nor metrizable. And for the sequential fan S_ω the space $C_k(S_\omega)$ is completely metrizable. We have show that if $C_k(S_\mathfrak{u})$ is Baire then it is metrizable. However, the next example shows that the converse is not true.

Example 3.24. *Suppose $\mathfrak{u}$ is isomorphic to the co-nowhere-dense filter $\mathfrak{n}$ on the rationals $\mathbb{Q}$; i.e. $A \in \mathfrak{n}$ iff $\mathbb{Q} \setminus A$ is nowhere-dense. Then $S_\mathfrak{u}$ is hemicompact and $C_k(S_\mathfrak{u})$ is metrizable, but $C_k(S_\mathfrak{u})$ is not Baire.*

Proof. It is easy to see that every infinite subset A of $\mathbb{Q}$ contains an infinite nowhere-sense subset. E.g., if no point of A is a limit point of A, then A is nowhere-dense, while if some point x of A is a limit point of A, then any sequence in A converging to x is nowhere-dense. Thus $S_\mathfrak{u}$ is hemicompact and $C_k(S_\mathfrak{u})$ is metrizable by Proposition 3.7.

It is also easy to see that P_1 has a winning strategy in the game $G(\mathfrak{n})$: in round n, he simply has to choose a rational within $1/2^n$ of q_n, where $\mathbb{Q} = \{q_i\}_{i\in\omega}$. It now follows from Theorem 3.21((v)$\Rightarrow$(ii)) that $C_k(S_u)$ is not Baire. $\square$

Example 3.25. *Suppose $\mathfrak{u}$ is a free ultrafilter on ω. Then $C_k(S_\mathfrak{u})$ is Baire and metrizable but not hereditarily Baire or Choquet, hence not completely metrizable.*

Proof. By Proposition 3.12, P_1 has no winning strategy in $G(\mathfrak{u})$, hence $C_k(S_\mathfrak{u})$ is Baire by Theorem 3.21((i) $\iff$ (iii)). The rest is immediate from Corollary 3.18 and Proposition 3.23. $\square$

Our final example shows that a free filter $\mathfrak{u}$ need not be an ultrafilter for $C_k(S_\mathfrak{u})$ to be Baire.

Example 3.26. *There is a free filter $\mathfrak{v}$ on ω which is not an ultrafilter such that $C_k(S_\mathfrak{v})$ is Baire.*

Proof. Let $\mathfrak{u}$ be a free ultrafilter on ω. Let $\mathfrak{u}_2$ be the filter on two disjoint copies of ω such that $F \in \mathfrak{u}_2$ iff F meets each copy in a member of $\mathfrak{u}$. Then $\mathfrak{u}_2$ is not an ultrafilter because the disjoint copies of ω are both $\mathfrak{u}_2$-positive. It is easy to see that if P_1 had a winning strategy in $G(\mathfrak{u}_2)$, then he would have one in $G(\mathfrak{u})$ too, a contradiction. (The idea is that if F is a play of P_1 in $G(\mathfrak{u}_2)$ using a winning strategy, then the union of the traces of F on the two copies should win for P_1 in $G(\mathfrak{u})$.) So P_1 has no winning strategy in $G(\mathfrak{u}_2)$. Hence if $\mathfrak{v}$ is a filter on ω isomorphic to $\mathfrak{u}_2$, then $\mathfrak{v}$ is not an ultrafilter and $C_k(S_\mathfrak{v})$ is Baire by Theorem 3.21. $\square$

Chapter 4

Special Classes of Spaces

In this chapter we will consider a few more special classes of spaces, and show that Conjecture 1.9 holds in each of these cases. In section 4.1 we consider another class of spaces with only one non-isolated point. In particular we investigate the quotient image of collapsing all non-isolated points of a certain topological spaces to a single point. In section 4.2 we will consider closed images of locally compact paracompact spaces. Daniel Ma in [Ma] showed that such spaces have the moving off property. We will show that the compact-open topology on these spaces are not only Baire, but necessarily Choquet. In section 4.3, we consider the family of closed images of first countable paracompact spaces. We will show that the closed image Y of a first countable paracompact space X has the moving off property only in the case that X is also locally compact. A corollary to this result, and the results from section 4.2, shows that the class of Lašnev spaces satisfy Conjecture 1.9. In section 4.4, we define another topological game which characterizes a structure of the compact subsets of a space, after a few preliminary results, we conclude that Conjecture 1.9 holds for spaces which are the finite product of closed images of locally compact paracompact spaces.

4.1 Collapsed Spaces

We will see that under certain weak conditions if we consider the quotient space obtained by collapsing all non-isolated points of a space to a point, then the resultant space would have the moving off property if and only if its compact open topology is Baire.

The following property was defined by Vaughan in [Va]. A space X is said to be a **wD-space** if given any collection $\mathcal{D}$ of discrete points in X there exists an infinite strongly discrete subset $\mathcal{D}' \subseteq \mathcal{D}$. This is a large class of topological spaces. Any submetrizable space and any realcompact space is a wD space. In the realm of T_3 spaces any paracompact or countably compact space is wD. Furthermore, any T_4 space is wD.

Throughout the rest of this section let X be a locally compact completely regular wD-space and $A \subseteq X$ be the collection of non-isolated points, and let $B = X \setminus A$. Suppose A is paracompact. Let $Y = B \cup \{\infty\}$ and $q : X \to Y$ be the map defined by $q(x) = x$ if $x \in B$ and $q(x) = \infty$ if $x \in A$. Give Y the topology generated by q.

For each $x \in A$ let K_x be a compact neighborhood of x. For each finite subset $F \subseteq B$ and finite set $G \subseteq A$ let $K(G, F) = \bigcup\{q(K_x) : x \in G\} \cup F$.

Lemma 4.1. *The family $\mathcal{K} = \{K(G, F) : G \subseteq A, F \subseteq B, |G| < \omega, |F| < \omega\}$ dominates the compact subsets of Y.*

Proof. Suppose $K \subseteq Y$ is compact. Let $A' = \mathrm{cl}_X(q^{-1}(K)) \cap A$ and $B' = q^{-1}(K) \cap B$. Note if B' is finite then we can choose any $a \in A$ and it would follow that $K \subseteq K(\{a\}, B')$. So assume B' is infinite.

In the case that A' is compact, there would exist a finite subset $G \subseteq A'$ such that $U = \bigcup\{K_x : x \in G\}$ would cover A'. Note that $B' \setminus U$ must be finite, otherwise $\{\{x\} : x \in B' \setminus U\}$ is a infinite discrete subset of $q^{-1}(K)$ and since q is closed it would follow that K contains an infinite discrete subset, contrary to the fact that it is compact. Hence $B' \setminus U$ is finite, call it F. It would follow then that $K \subseteq K(G, F)$.

On the other hand if A' is not compact, then it isn't countably compact. Let D be an infinite discrete set of points in A'. Since X is wD and D is discrete in X it follows that there exists and infinite subset $D' \subseteq D$ with a discrete open expansion $\{U_d : d \in D'\}$. For each $d \in D'$ let $x_d \in B' \cap U_d$. Note: $U_d \cap B' = \{x_d\}$ for each $d \in D'$. And $\{x_d : d \in D'\}$ is therefore an infinite discrete subset of $q^{-1}(K)$, and since q is closed it would follow that K has an infinite discrete subset contrary to the fact that it is compact. It follows that A' has to be compact and hence $K \subseteq K(G, F)$ for some $K(G, F) \in \mathcal{K}$.

$\square$

Proposition 4.2. *If Y has the moving off property then $C_k(Y)$ is Baire.*

Proof. Suppose $C_k(Y)$ is not Baire. Let σ be a winning strategy for E in the game $\mathrm{Ch}(C_k(Y))$. We will create a winning strategy τ for L in the game $G_{K,L}(Y)$. We may assume all players are using basic open sets constructed from $\mathcal{K}$.

Let $U_0 = \sigma(\emptyset) = \langle f_0, A_0 = K(G_0, F_0), \epsilon_0 \rangle$. Suppose K plays the compact set $K_0 = K(H_0, J_0)$. Let NE play the move $V_0 = \langle g_0, A_0 \cup K_0, \min\{\epsilon_0, \frac{1}{2^0}\}\rangle$ where $g_0 \restriction A_0 = f_0 \restriction A_0$ and $g_0(x) = f_0(\infty)$ for all $x \notin A_0$. Let $U_1 = \langle f_1, A_1 = K(G_1, F_1), \epsilon_1 \rangle$. Define $L_0 = \tau(K_0) = K(\emptyset, F_1) \setminus (K(G_1, \emptyset) \cup A_0 \cup K_0)$.

Continue in this fashion to define τ.

Claim 1: $\bigcup\{A_i : i \in \omega\} = \bigcup\{K(G_i, 0) : i \in \omega\} \cup \bigcup\{L_i : i \in \omega\} \cup \bigcup\{K_i \setminus \bigcup\{A_j : j \leq i\} : i \in \omega\} \cup A_0$

Proof of Claim 1: We only need to show $\subseteq$. Suppose the left hand side is not a subset of the right hand side. Let $x \in \bigcup\{A_i : i \in \omega\}$ but $x \notin \bigcup\{K(F_i, 0) : i \in \omega\} \cup \bigcup\{L_i : i \in \omega\} \cup \bigcup\{K_i \setminus \bigcup\{A_j : j \leq i\} : i \in \omega\} \cup A_0$. Let j be the smallest natural number such that $x \in A_j$. (Note: $j > 0$). So $x \in K(G_j, F_j)$ but $x \notin K(G_j, \emptyset)$, thus $x \in K(\emptyset, F_j)$. However, since $x \notin L_{j-1} = K(\emptyset, F_j) \setminus (K(G_1, \emptyset) \cup A_{j-1} \cup K_{j-1})$ it would have to follow that $x \in K_{j-1}$. But this would imply that $x \in K_{j-1} \setminus \bigcup\{A_i : i \leq j-1\}$ contrary to our assumption. The above equality holds.

Let $f : \bigcup\{A_i : i \in \omega\} \to \mathbb{R}$ be defined by $f = \lim\left(f_i \restriction \bigcup\{A_i : i \in \omega\}\right)$. Note: since σ is a winning strategy the function f can't be continuous (otherwise it could be simply extended to a function $\overline{f} \in \bigcap\{U_i : i \in \omega\}$).

Claim 2: If $\{L_i : i \in \omega\}$ is a discrete family then f is continuous.

Proof of Claim 2: Suppose $\{L_i : i \in \omega\}$ is discrete. Let $\epsilon > 0$. Let U be an open set around ∞ such that $U \cap L_i = \emptyset$ for all $i \in \omega$. Since for each $i \in \omega$ the sequence $(f_n \restriction K(F_i, \emptyset) : n \in \omega)$ converges uniformally to the function $f \restriction K(F_i, \emptyset)$ we can find for any $x \in F_i$ an open set V_x such that $V_x \subseteq U \cap K(F_i, \emptyset)$ and if $y \in V_x$ then $|f(y) - f(\infty)| < \epsilon$. Let $W = \bigcup\{V_x : x \in F_i, i \in \omega\}$. So W is open in $\bigcup\{A_i : i \in \omega\}$.

Suppose $x \in W \cap \bigcup\{A_i : i \in \omega\}$. Then $x \in U$ and thus $x \notin \bigcup\{L_i : i \in \omega\}$. If $x \in K(F_i, \emptyset)$ for some $i \in \omega$ then $|f(x) - f(\infty)| < \epsilon$. If $x \notin \bigcup\{L_i : i \in \omega\} \cup \bigcup\{K(F_i, \emptyset) : i \in \omega\}$ then $x \in K_j \setminus \bigcup\{A_i : i \leq j\}$ by Claim 1. Let $N > i$ such that $1/2^N < \epsilon/2$. Note $g_N(x) = f_N(\infty)$ and $g_k(x) = f_k(x)$ for all $k > N$. Hence $|f(x) - f(\infty)| \leq |f(x) - g_N(x)| + |g_N(x) - f_N(\infty)| + |f_N(\infty) - f(\infty)| < \epsilon$. Hence for all $x \in W \cap \bigcup\{A_i : i \in \omega\}$ we have $|f(x) - f(\infty)| < \epsilon$. Hence f is continuous.

It follows that $\{L_i : i \in \omega\}$ is not discrete. Hence τ is a winning strategy for L in $G_{K,L}(Y)$. $\square$

4.2 Closed Images of Locally Compact Paracompact Spaces

We will show that the the compact open topology of closed images of locally compact paracompact spaces are Choquet, and hence Baire. Daniel Ma showed that all such spaces have the Moving Off Property. The current result will show that none of these spaces can serve as a counter example to the main conjecture: if X has the moving off property then $C_k(X)$ is Baire. Recall that any locally compact paracompact space can be written as a topological sum of locally compact σ-compact spaces.

Lemma 4.3. *If X is a locally compact Lindelöf space, then X is hemicompact.*

Proof. Suppose X is a locally compact and Lindelöf. For each $x \in X$ let $K_x \subseteq X$ be a compact subset such that $x \in \text{int}(K_x)$. Then $\mathcal{K} = \{\text{int}(K_x) : x \in X\}$ is an open cover of X. Let $x_1, x_2, \ldots \in X$ such that $\mathcal{K}' = \{\text{int}(K_{x_i}) : i \in \omega\}$ covers X. For each $F \subseteq \omega$ with F finite, let $K_F = \bigcup\{K_{x_i} : i \in F\}$. Let $\mathcal{H} = \{K_F : F \subseteq \omega, |F| < \omega\}$. We will show that $\mathcal{H}$ witnesses hemicompactness. Let $C \subseteq X$ be compact. Then $\mathcal{K}'$ is an open cover of C. Let $F \subseteq \omega$ be finite such that $C \subseteq \{\text{int}(H_{x_i}) : i \in F\}$ covers C. It follows that $C \subseteq K_F \in \mathcal{H}$. $\square$

Proposition 4.4. *Suppose κ is an ordinal, and for each $i \in \kappa$ that H_i is a locally compact σ-compact space. Let $X = \bigoplus\{H_\alpha : \alpha \in \kappa\}$ and $f : X \to Y$ a closed continuous surjective map. Then $C_k(Y)$ is Choquet.*

Proof. By the above proposition each H_α is a hemicompact k-space. For each $\alpha \in \kappa$ let $\{K(\alpha, j) : j \in \omega\}$ be an increasing family of compact spaces which witness hemicompactness for H_α. For each $\sigma \in Fn(\kappa, \omega)$ let $K_\sigma = \bigcup\{f(K(\alpha, j)) : (\alpha, j) \in \sigma\}$. Then $\mathcal{K} = \{K_\sigma : \sigma \in Fn(\kappa, \omega)\}$ is a family of $|\kappa|$ many compact sets which dominate the compact subsets of Y. Consider the Banach-Mazur game on $C_k(Y)$ using these compact sets.

We will construct a winning strategy for NE. Two things to note:

(1) Once a compact set K_σ has been played as part of a basic open set in $C_k(Y)$, the next play of the game must include a compact set, $K_{\sigma'}$ which is a superset K_σ. We can assume then, without loss of generality, that σ' extends σ.

(2) In the way we construct the strategy for NE, it is important that NE forces the game to be played using basic open sets which consist of certain compact sets. In particular, once $\alpha \in \mathrm{dom}(\sigma)$, for some K_σ used in a basic open set in the game, NE wants to force the game to be played on all of $f(K_\alpha)$.

Towards satisfying (2) above, for each $\sigma \in Fn(\kappa, \omega)$ let $\sigma + 1$ denote the function defined by $(\sigma+1)(\alpha) = \sigma(\alpha)+1$. Suppose E plays $U_0 = B(f_0, K_{\sigma_0}, \epsilon_0)$, then NE should play $V_0 = B(f_0, K_{\sigma_0+1}, \epsilon_0/2)$. If on the n^{th} play E plays $U_n = B(f_n, K_{\sigma_n}, \epsilon_n)$, then NE should play $V_n = B(f_n, K_{\sigma_n+1}, \epsilon_n/2)$. This defines a stationary strategy for NE. We claim that this strategy is winning. Let $D = \{\alpha \in \kappa : (\exists i \in \omega)(\alpha \in \mathrm{dom}(\sigma_i)\}$. Note: $|D| \leq \omega$.

Let $A = \bigcup\{K_{\sigma_i} : i \in \omega\} = \bigcup\{f(K_\alpha) : \alpha \in D\} = f\left(\bigcup_{\alpha\in D} K_\alpha\right)$. A is the closed image of a hemicompact k-space, so it is a hemicompact k-space. Since A is a k-space it follows that A is a $k_{\mathbb{R}}$ space. Furthermore, A is closed, since it's the image of a closed set.

Define a function $f : A \to \mathbb{R}$ as follows:

$$f(x) = \lim_{n\to\infty} f_n \restriction A$$

To see that this is continuous, suppose $H \subseteq A$ is compact, and let $m \in \omega$ such that $H \subseteq K_{\sigma_m}$. Then $\{f_n \restriction H : n \in \omega\}$ converges uniformly to $f \restriction H$. Consequently $f \restriction H$ is continuous.

Since A is a $k_{\mathbb{R}}$ space, this is enough to show that f is continuous on A. By the Tietze Extension Theorem we can do the following: Let $\hat{f} : Y \to \mathbb{R}$ be a continuous extension of f. Such an $\hat{f}$ is in $\bigcap\{V_i : i \in \omega\}$.

Thus NE has a winning strategy, and $C_k(Y)$ is Choquet. $\square$

Corollary 4.5. *If Y is the closed image of a locally compact paracompact space, then $C_k(Y)$ is Baire.*

4.3 Lašnev Spaces and Closed Images of First Countable Paracompact Spaces

We use results from the previous section to investigate closed images of first countable paracompact spaces. We will include that in the class of Lašnev spaces, X has the moving off property if and only if $C_k(X)$ is Choquet.

A closed surjective map $f : X \to Y$ is said to be **irreducible** if there does not exist a closed proper subset C of X such that $f \restriction C$ is onto. A closed surjective map $f : X \to Y$ is said to be **inductively irreducible** if there exists a closed set $C \subseteq X$ such that $f : C \to Y$ is irreducible.

It has been shown that in many types of spaces every closed map is inductively irreducible. We will make use of the following result by Lašnev.

Lemma 4.6. *[La] If X is a first countable paracompact space and $f : X \to Y$ is a closed surjection, then f is inductively irreducible.*

Fibers of irreducible maps have some well known properties which we will use in the upcoming result.

Lemma 4.7. *If $f : X \to Y$ is a closed irreducible surjection and $y \in Y$, then the following are equivalent:*

a.) The fiber $f^{-1}(\{y\})$ is open

b.) The fiber $f^{-1}(\{y\})$ is an isolated point.

c.) y is an isolated point.

Proof. Suppose $f^{-1}(\{y\})$ is open, but not an isolated point. Suppose $x \in f^{-1}(\{y\})$. Then $A = (X \setminus f^{-1}(\{y\})) \cup \{x\}$ is a proper closed subset of X and $f \upharpoonright A$ is a surjection, contrary to the fact that f is irreducible. Therefore if $f^{-1}(\{y\})$ is open then it is an isolated point. The converse is immediate.

Since f is a quotient map we also have that $\{y\}$ is open, and therefore an isolated point, if and only if $f^{-1}(\{y\})$ is open. □

Lemma 4.8. *If $f : X \to Y$ is a closed irreducible surjection and the fiber $f^{-1}(\{y\})$ is not an isolated point, then $f^{-1}(\{y\})$ has non-empty interior.*

Proof. Suppose the fiber $f^{-1}(\{y\})$ is not an isolated point but has non-empty interior. Then by the previous lemma there would be a proper open set contained in $f^{-1}(\{y\})$, say $U \subseteq f^{-1}(\{y\})$. Then $A = X \setminus U$ is a proper closed subset of X and $f \upharpoonright A$ would be a surjection onto Y, contrary to the fact that f is irreducible. We can conclude that if a fiber is not an isolated point then it has non-empty interior. □

We now turn to the main result in this section.

Proposition 4.9. *Suppose X is a first countable paracompact space and $f : X \to Y$ is a closed surjective map. If Y has the moving off property then X is locally compact.*

Proof. Since closed surjections on first countable paracompact spaces are inductively irreducible and being first countable paracompact is hereditary under closed sets, we may assume that f is irreducible. Suppose X is not locally compact. We will show that Y contains a copy of the metric fan and therefore doesn't have the moving off property. Let $x^* \in X$ have no compact neighborhood. Let $\{W_i : i \in \omega\}$ be a strongly decreasing local base at x^*, let $y^* = f(x^*)$, for each $x \in X$ let $C(x) = f^{-1}(\{f(x)\})$ be the fiber of f that contains x.

Define $V_0 = W_0$. Since $\mathrm{cl}_X(V_0)$ is a neighborhood of x^* it is not compact. Furthermore, since in paracompact spaces all countably compact subsets are compact, it follows that

$\mathrm{cl}_X(V_0)$ is not countably compact. We can choose a discrete collection $\{x_i : i \in \omega\} \subseteq \mathrm{cl}_X(V_0)$. We may assume that $\{x_i : i \in \omega\}$ is strongly discrete since X is paracompact and therefore collectionwise normal. Let $\{U_i : i \in \omega\}$ be a discrete open expansion of $\{x_i : i \in \omega\}$ such that $x_i \in U_i$ for each $i \in \omega$. Since f is irreducible by the previous lemmas, it follows that the fibers of f have empty interiors or they are isolated points. We may define a new discrete collection of points as follows. Let $p_0^0 \in U_0 \setminus C(x^*)$. Let $p_1^0 \in U_1 \setminus (C(x^*) \cup C(p_0^0))$. Suppose p_i^0 has been defined for each $i \leq k$ such that $p_i^0 \in U_i$ and p_i^0 is not in a fiber with x^* or any p_0^j for $j < i$. Let $p_{k+1}^0 \in U_{k+1} \setminus (C(x^*) \cup \bigcup\{C(p_i^0) : i \leq k\})$. This inductively defines the set $P_0 = \{p_i^0 : i \in \omega\}$. Then P_0 is closed discrete and since each element in P_0 is in a different fiber it follows that $f(P_0)$ is an infinite closed discrete set with $y^* \notin f(P_0)$.

Let $n \in \omega$ be the smallest such that $\mathrm{cl}_X(W_n) \subseteq f^{-1}(Y \setminus f(P_0))$. Define V_1 be an open set such that $x^* \in V_k \subseteq \mathrm{cl}_X(V_1) \subseteq W_n$. Since $\mathrm{cl}_X(V_1)$ is a neighborhood of x^* we can find a discrete collection $\{x_i : i \in \omega\} \subseteq \mathrm{cl}_X(V_1)$, which we may assume is strongly discrete. Let $\{U_i : i \in \omega\}$ be a discrete open expansion of $\{x_i : i \in \omega\}$ such that $U_i \subseteq W_k$ for all $i \in \omega$. We may proceed as in the previous case to create a discrete family $P_1 = \{p_i^1 : i \in \omega\}$, such that for all $i \in \omega$ the point p_i^1 is not in a fiber with x^* or any p_j^1 for $j \neq i$. Then $f(P_1)$ will be a closed infinite discete set.

Suppose that V_i and $P_i = \{p_j^i : j \in \omega\}$ have been defined for all $i < k$, such $f(P_i)$ is an infinite closed discrete set, $y^* \notin f(P_i)$ and $f(P_j) \cap f(P_i) = \emptyset$ for $j \neq i$. Let $n \in \omega$ be the smallest such that $\mathrm{cl}_X(W_n) \subseteq f^{-1}(P_{k-1})$. Let V_k be an open set such that $x^* \in V_k \subseteq \mathrm{cl}_X(V_k) \subseteq W_n$. Since $\mathrm{cl}_X(V_k)$ is a neighborhood of x^*, it is not compact and hence we can find a closed discrete set $\{x_i : i \in \omega\} \subseteq \mathrm{cl}_X(V_k)$. We may assume that $\{x_i : i \in \omega\}$ is strongly discrete with the family of open sets $\{U_i : i \in \omega\}$ being an discrete open expansion. By selecting points in U_i as we did previously we can define a discrete family $P_k = \{p_i^k : i \in \omega\}$ such that each point is in a unique fiber and no point is in a fiber with x^*.

This inductively defines the family $\{P_i : i \in \omega\}$. Let $M = \{y^*\} \cup \bigcup\{f(P_i) : i \in \omega\}$. We claim that M is a copy of the metric fan where y^* is the non-isolated point and each

$f(P_i)$ is a ring in the fan. If $n \in \omega$ then the set $\{y^*\} \cup \bigcup\{f(P_i) : i \geq n\}$ is open in M since $\bigcup\{f(P_i : i < n\}$ is either empty or is the finite union of closed discrete sets and is thus closed. □

Corollary 4.10. *Suppose Y is the closed image of a first countable paracompact space X under the map f. The following are equivallent:*

a. Y has the moving off property.

b. $C_k(Y)$ is Choquet

c. $C_k(Y)$ is Baire

d. K has a winning strategy in $G^{\circ}_{K,L}(Y)$

e. L has no winning strategy in $G^{\circ}_{K,L}(Y)$

Proof. $b \Rightarrow c \Rightarrow a$ and $b \Rightarrow d \Rightarrow e$ and $a \Leftrightarrow e$ follow directly from previous results. Suppose Y has the moving off property. Then by the above lemma X is locally compact. Hence $C_k(Y)$ is Choquet since Y is the closed image of a locally compact paracompact space. Thus $a \Rightarrow b$. □

There is a well-known better class of spaces that is a subclass of the closed images of first countable paracompact spaces. Recall that a **Lašnev** space is a space that is the closed image of a metric space.

Since each metric space is first countable and paracompact it follows that the above corollary will hold for the class of Lašnev spaces as well. In particular, a weakening of the above result is the following corollary.

Corollary 4.11. *Suppose Y is a Lašnev space. Then Y has the moving off property if and only if $C_k(Y)$ is Baire.*

4.4 Normal $k_{\mathbb{R}}$-spaces and a game

The definition for the following games are motivated by the above proof to proposition 4.5. Recall that each basic open set in $C_k(Y)$ has an underlying compact subset of Y. A critical component of the proof was the ability of NE in the game $Ch(C_k(Y))$ could create a play of basic open sets in $C_k(Y)$ whose underlying compact sets would union to a closed hemicompact k-space.

Definition 4.12. *Define the game $\mathbf{G}_C(X)$ on a topological space X, with two players P_1 and P_2, as follows: On move 0, P_1 plays compact set K_0 and P_2 responds with compact $H_0 \supseteq K_0$. On move n, P_1 plays compact $K_n \supseteq H_{n-1}$ and P_2 plays $H_n \supseteq K_n$. Player P_2 wins if $\bigcup\{H_i : i \in \omega\}$ is closed and $\{H_i : i \in \omega\}$ is a dominating family of compact subsets of H, otherwise P_1. Let $\mathbf{G}_{CK}$ denote the similar game, where P_2 must also have H as a k-space.*

Proposition 4.13. *If Y is the closed image of a locally compact paracompact space then P_2 has a winning strategy in $\mathbf{G}_{CK}(Y)$.*

Proposition 4.14. *Suppose X is a normal $k_{\mathbb{R}}$ space. Then if P_2 has a winning strategy in $\mathbf{G}_{CK}$. Then $C_k(X)$ is Choquet*

Proof. Suppose P_2 has a winning strategy σ in the above defined game. Define a strategy NE in the Choquet game on $C_k(X)$ as follows: If E's first play is $E_0 = B(f_0, K_0, \epsilon_0)$ then NE plays $N_0 = B(f_0, H_0, \epsilon_0/2)$, where $H_0 = \sigma(K_0)$. If N_i and E_i have been defined for all $i \leq n \in \omega$, and E plays $E_n = B(f_n, K_n, \epsilon_n)$ then NE plays $N_n = B(f_n, H_n, \epsilon_n/2)$, where $N_n = \sigma(K_0, H_0, K_1, H_1, \ldots, K_{n-1}, H_{n-1}, K_n)$. Since σ is a winning strategy, it follows that $H := \bigcup\{H_i : i \in \omega\}$ is closed.

Define $f : H \to \mathbb{R}$ as follows:

$$f(x) = \lim_{n \to \infty} f_n(x)$$

Let $K \subseteq H$ be any compact set. Since H is closed it follows that K is a compact subset of X. Since $f \upharpoonright K$ is a uniform limit of continuous functions on a compact set it is continuous on K. Since being a $k_{\mathbb{R}}$ space is hereditary it follows that f is continuous on H. By the Tietze Extension Theorem we can extend f to a continuous function $\widehat{f} : X \to \mathbb{R}$, and $\widehat{f} \in \bigcap\{B(f, H_i, \epsilon_i/2) : i \in \omega\}$. Thus NE has a winning strategy in the Choquet game on $C_k(X)$. Hence $C_k(X)$ is Choquet. $\square$

Proposition 4.15. *Suppose X is a normal and P_1 doesn't have a winning strategy in $\mathbf{G}_{CK}(X)$. Then $C_k(X)$ is Baire.*

Proof. Suppose P_1 doesn't have a winning strategy in $G(X)$. Let σ be a strategy for E in the game $Ch(C_k(X))$. We will show that σ is not a winning strategy. Define a strategy τ for P_1 in $G(X)$ as follows. Suppose $U_0 = \sigma(\emptyset) = B(f_0, K_0, \epsilon_0)$. Define $\tau(\emptyset) = K_0$. Suppose $H_0 \supseteq K_0$ is compact. Let $U_1 = \sigma(U_0, V_0 = B(f_0, H_0, \epsilon_0/2)) = B(f_1, K_1, \epsilon_1)$. Define $\tau(K_0, H_0) = K_1$.

Suppose $\tau(K_0, H_0, K_1, H_1, \ldots, K_i, H_i) = K_{i+1}$, U_i, f_i and ϵ_i have been defined for all $i < n$, and V_i has been defined for all $i < n-1$. Let $H_n \supseteq K_n$. Let $U_k = \sigma(U_0, V_0, U_1, \ldots, U_{n-1}, V_{n-1} = B(f_{n-1}, H_n, \epsilon_{n-1}/2)) = B(f_{n+1}, K_{n+1}, \epsilon_{n+1})$. Define $\tau(K_0, H_0, K_1, H_1, \ldots, K_n, H_n) = K_{n+1}$.

This inductively defines the strategy τ for P_1 in $G(X)$. Since τ is not a winning strategy, P_2 can pick a sequence of compact sets $(H_i : i \in \omega)$ which will witness that τ is not winning. Hence $H = \bigcup\{H_i : i \in \omega\}$ is closed. We show that the corrosponding moves $(V_i : i \in \omega)$ for P_2 in $Ch(C_k(X))$ will witness that σ is not winning for E.

Define $f : H \to \mathbb{R}$ as follows:

$$f(x) = \lim_{n \to \infty} f_n(x)$$

As in the above proposition, f is continuous on H which is closed and hence can be extended to a continuous function $\widehat{f}$ on all of X. And $\widehat{f} \in \bigcap\{V_i : i \in \omega\}$. Hence σ is not a winning strategy.

$\square$

We will use the following simple results, to conclude that Player 2 having a winning strategy in the above defined game $\mathbf{G}_{CK}(X)$ is finitely productive.

Lemma 4.16. *If $\mathcal{K}_0$ and $\mathcal{K}_1$ are dominating families of compact subsets of X_0 and X_1, respectively, then $\mathcal{K} = \{K_0 \times K_1 : K_0 \in \mathcal{K}_0, K_1 \in \mathcal{K}_1\}$ is a dominating family of compact subsets of $X_0 \times X_1$.*

Lemma 4.17. *If $\{H_i : i \in \omega\}$ witnesses hemicompactness in X_0 and $\{K_i : i \in \omega\}$ witnesses hemicompactness in X_1, then $\{H_i \times K_i : i \in \omega\}$ witnesses hemicompactness in $X_0 \times X_1$.*

Proposition 4.18. *P_2 having a winning strategy in $\mathbf{G}_{CK}(X)$ is finitely productive.*

Proof. Suppose P_2 has a winning strategy σ_0 in X_0 and a winning strategy σ_1 in X_1. Define a strategy τ for P_2 in the game $\mathbf{G}_C(X_0 \times X_1)$ in the following manner. Define $\tau(K_0) = H_0 \times J_0$ where $H_0 = \sigma_0(\pi_0(K_0))$ and $J_0 = \sigma_1(\pi_1(K_0))$. Continue in this fashion. Then $H = \bigcup\{H_i \times J_i : i \in \omega\}$ is closed k-space and $\{H_i \times J_i : i \in \omega\}$ witnesses hemicompactnes for H. $\square$

Corollary 4.19. *Suppose $X_0, X_1, \ldots, X_n$ are closed images of locally compact paracompact spaces. Then $C_k\left(\prod\{X_i : i \leq n\}\right)$ is Choquet.*

Proof. Since P_2 has a winning strategy in $\mathbf{G}_{CK}(X)$ for each X_i, by the above lemma it follows that P_2 has a winning strategy in $\prod\{X_i : i \in \omega\}$. Therefore $C_k\left(\prod\{X_i : i \in \omega\}\right)$ is Choquet. $\square$

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