Topics in Edge Regular Graphs

by

Kelly Bragan

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 2, 2014

Keywords: Edge-Regular, Friendship Graph, Clique, Clique Graph, Regular Clique Assembly

Approved by

Peter D. Johnson, Chair, Professor of Mathematics and Statistics
Dean G. Hoffman, Professor of Mathematics and Statistics
Charles C. Lindner, Distinguished University Professor of Mathematics and Statistics
Jessica M. McDonald, Professor of Mathematics and Statistics
George T. Flowers, Graduate School Dean

Abstract

A graph G is edge-regular with parameters (n, d, λ) if G is regular of degree d on n vertices and for all $u, v \in V(G)$ such that $uv \in E(G)$, $|N(u) \cap N(v)| = \lambda$, where $N(v)$ denotes the open neighborhood of a vertex $v \in V(G)$. We explore the structure of edge-regular graphs with particular emphasis on the case $\lambda = 1$.

Acknowledgments

I would like to thank several people for their help, directly or indirectly. First and foremost, I want to thank my advisor, Pete Johnson, for all of his patience, encouragement, and guidance. I would also like to thank James Hammer and Ken Roblee for their insights and collaboration and my committee for their support and suggestions. Finally, I want to thank my family for their endless love and support and Sarah and Mark, my most significant others, for being there for the best and the worst.

Table of Contents

List of Figures

Chapter 1

Introduction

The goal of this dissertation is to add to the existing body of work regarding the existence and structure of edge-regular graphs. Our primary results, which appear in Chapter 3, address the question: For which orders, n, and degrees, d, do there exist edge-regular graphs with $\lambda = 1$? Our efforts to classify those edge-regular graphs for which $\lambda = 1$ led to a particular focus on edge-regular graphs in which each vertex has as its open neighborhood the disjoint union of complete graphs, and this class of edge-regular graphs is described in the second chapter.

A graph $G = (V, E)$ has vertex set $V(G)$ and edge set $E(G)$. If an edge exists between vertices $u, v \in V(G)$, we say $uv \in E(G)$. Distinct vertices u and v are said to be *adjacent* if $uv \in E(G)$. All graphs discussed herein will be finite and simple, meaning that the vertex set is finite and there are no multiple edges between vertices and no loops from any vertex to itself. We denote the *open neighbor set* or *open neighborhood* of a vertex $v \in V(G)$ by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *degree* of v by $d_G(v) = |N_G(v)|$. When the graph being referred to is clear from context, the subscript G may be omitted. The *closed neighborhood* in G of $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, $S \neq \emptyset$, the subgraph of G induced by S has vertex set S and edge set $\{uv \mid u, v \in S \text{ and } uv \in E(G)\}$ and will be denoted $G[S]$. The complete graph on n vertices will be denoted K_n. If $E(G) = \{uv \mid u, v \in V(G)\}$, we call G a complete graph. If $E(G) = \emptyset$, we say G is *empty*.

An *edge-regular graph* is a regular graph G for which there exists an integer λ such that if $uv \in E(G)$ then $|N(u) \cap N(v)| = \lambda$. That is, every pair of adjacent vertices in G have exactly λ common neighbors. If G is an edge-regular graph with parameter λ, regular of degree d on n vertices such that $0 < d \leq n - 1$, we write $G \in ER(n, d, \lambda)$.

A *strongly regular graph* is an edge-regular graph $G \in ER(n, d, \lambda)$ for some n, d, and λ, $0 < d < n-1$, for which there exists an integer μ such that for $u, v \in V(G)$, $u \neq v$, if $uv \notin E(G)$ then $|N(u) \cap N(v)| = \mu$. That is, each pair of distinct non-adjacent vertices in G have exactly μ common neighbors. If G is such a graph, we write $G \in SR(n, d, \lambda, \mu)$.

Strongly regular graphs have been described as lying "somewhere between the highly structured and the apparently random [5]." They remain rare enough that the discovery of new ones is always of interest, yet numerous and varied enough to defy easy classification. So it stands to reason that the more abundant class of edge-regular graphs will be unlikely to fall into tidy subclasses. Nonetheless some interesting results have been obtained for edge-regular graphs satisfying various structural or extremal conditions.

1. In [9] all $G \in ER(n, d, \lambda)$ satisfying $d - \lambda \leq 3$ are described.

2. If $G \in ER(n, d, \lambda)$ and $\lambda > 0$ then $n \geq 3(d - \lambda)$ ([10],[12]). In [12] the edge-regular graphs with $\lambda = 2$ and $n = 3(d - \lambda)$ are completely characterized, and in [17] the main result in [12] is extended to a characterization of all edge-regular graphs satisfying $n = 3(d - \lambda)$ with $\lambda > 0$ even and d sufficiently large (depending on λ).

3. Edge-regular graphs with $n = 3(d - \lambda) + 1$, $\lambda > 0$, satisfying certain local structural requirements are considered in [6] and [11]. The main result of [11] is of interest here: For every d, $ER(3d - 2, d, 1) = \emptyset$.

If G and H are graphs, the *join* of G and H, formed by taking disjoint copies of G and H and putting in all edges with one end in $V(G)$ and the other in $V(H)$, will be denoted $G \vee H$. The *disjoint union*, or *sum*, of G and H, formed by taking disjoint copies of G and H and putting in no edges at all will be denoted $G + H$. If m is a positive integer, $mG = G + \cdots + G$, with G appearing m times in the sum.

A *friendship graph* is a graph of the form $K_1 \vee mK_2$ for some positive integer m, where K_n indicates the complete graph on n vertices. A graph is *clique friendly* if and only if for each $v \in V(G)$, the graph induced by $N[v]$ is $K_1 \vee mK_p$ for some $m, p \in \mathbb{N}$.

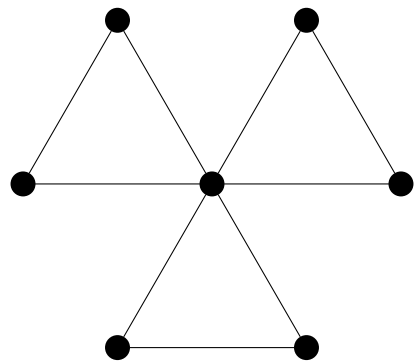


Figure 1.1: A friendship graph $K_1 \vee 3K_2$

The friendship graphs are, famously, the only finite simple graphs in which each pair of distinct vertices has exactly one common neighbor [4]. A graph in which each pair of distinct vertices has exactly k common neighbors is called *k-friendly.* The 0-friendly graphs are $mK_1 + pK_2$, which are regular only if $m = 0$ or $p = 0$. For $k > 1$, any k-friendly graph must be regular [1], and therfore also strongly regular with parameters (n, d, k, k) for some $d > 0$. Thus, the friendship graphs, $K_1 \vee mK_2$ with $m > 1$ are the only k-friendly graphs with $k > 0$ which are not regular.

In Chapter 2 we focus on edge-regular graphs which are clique-friendly, which we call "regular clique assemblies" and describe the correspondence between regular clique assemblies and the geometric structures known as configurations. Among the regular clique assemblies are all edge regular graphs for which $\lambda = 1$, and this case is examined at length in Chapter 3. Also in Chapter 3, the following questions are addressed extensively for $\lambda = 1$ and some observations are made for $\lambda > 1$:

1. For which triples (n, d, λ) does $ER(n, d, \lambda) \neq \emptyset$?

2. For which triples (n, d, λ) does $ER(n, d, \lambda)$ contain a connected graph?

Chapter 4 describes the classes of graphs that result from relaxing certain requirements in the definition of regular clique assemblies, and the final chapter enumerates some of the remaining open problems.

Chapter 2

Regular clique assemblies and corresponding structures

We shall begin by considering a subset of the edge-regular graphs known as *regular clique assemblies* (RCAs), which we shall see are precisely those edge-regular graphs which are also clique-friendly.

2.1 RCAs

The *line graph* of a graph G, denoted $L(G)$, is formed by representing each edge in G with a vertex in $L(G)$. Two vertices in $L(G)$ are adjacent if the associated edges in G are incident with a common vertex in G.

The *clique number* of a graph G, denoted $\omega(G)$, is the maximum order of a clique in G.

The *clique graph* of G, denoted $CL(G)$, is the graph whose vertices are the maximal cliques of G, in which any two distinct maximal cliques of G are adjacent if and only if they have at least one vertex in common.

If G has no isolated vertices and $\omega(G) = 2$, then $CL(G) = L(G)$, the line graph of G.

G is a *regular clique assembly* if G is regular, $\omega(G) \geq 2$, and

(1) every maximal clique of G is maximum;

(2) each edge of G is in exactly one maximum clique of G.

If G is a regular clique assembly on n vertices, regular of degree d, with $k = \omega(G)$, we write $G \in RCA(n, d, k)$. Certainly $d \geq k - 1$. If $d = k - 1$, then the graph in question will be of the form $\frac{n}{k}K_k$, so for all that follows we will assume $n > d > k - 1 \geq 1$.

Lemma 2.1. *If G is a regular clique assembly, then any two different maximum cliques in G have at most one vertex in common. Further, if H_1, H_2, and H_3 are maximum cliques in G, $V(H_1) \cap V(H_2) = \{u\}$, $V(H_1) \cap V(H_3) = \{v\}$ and $u \neq v$, then $V(H_2) \cap V(H_3) = \emptyset$.*

Proof. If distinct maximum cliques in G had two vertices in common, then condition (2) in the RCA definition would be violated. Suppose H_1, H_2, H_3, u, and v are as described above. Then H_1, H_2, and H_3 are distinct maximum cliques. Suppose $w \in V(H_2) \cap V(H_3)$. If $w \in \{u, v\}$ then H_1 and one of H_2, H_3 have two vertices in common, and condition (2) is violated. Therefore $w \notin \{u, v\}$. Then u, v, w induce a K_3 in G, which is contained in a maximal, and therefore maximum, clique H_4 in G which is none of H_1, H_2, and H_3. Then uv is in both H_1 and H_4, violating (2). □

Proposition 2.1. *If $G \in RCA(n, d, k)$ then $k - 1$ divides d, and for each $v \in V(G)$, $G[N_G(v)] \simeq \frac{d}{k-1} K_{k-1}$. Conversely, if G is a graph on n vertices such that, for some m, $p \geq 1$, $G[N_G(v)] \simeq mK_p$ for all $v \in V(G)$, then $G \in RCA(n, mp, p+1)$*

Proof. Suppose $G \in RCA(n, d, k)$. Suppose that $v \in V(G)$. A neighbor u of v is in the unique maximum clique $\simeq K_k$ containing the edge uv. Any two of the maximum cliques of G containing v have only v in common, by Lemma 2.1; thus $G[N_G(v)] \simeq mK_{k-1}$ for some m. Since G is d-regular, $d = m(k-1)$.

Now suppose that G is a finite simple graph such that for every $v \in V(G)$, $G[N_G(v)] \simeq mK_p$ for some positive integers m, p. Then G is regular of degree mp. G can contain no K_{p+2}, and any K_r in G, $r \leq p+1$, must be contained in one of the K_{p+1}'s comprising the closed neighbor set of any one of its vertices. (For all $v \in V(G)$, $G[N_G[v]] \simeq K_1 \vee mK_p$.) Thus $\omega(G) = p + 1$, and (1) in the RCA definition holds; (2) is obvious. □

Corollary 2.1. *$RCA(n, d, k) \subseteq ER(n, d, k-2)$, with equality when $k \in \{2, 3\}$.*

Proof. If $G \in RCA(n, d, k)$, then, since every edge uv of G is contained in one K_k in G, vertices outside which cannot be adjacent to both u and v, it follows that G is edge-regular with $\lambda = k - 2$. If $G \in ER(n, d, 0)$, then G is triangle-free and d-regular; clearly $G \in$

$RCA(n,d,2)$. Suppose that $G \in ER(n,d,1)$. Since, for any $uv \in V(G)$, $|N_G(u) \cap N_G(v)| = 1$, G can contain no K_4, and no K_1 nor K_2 in G is a maximal clique ($d > 0$). Thus $\omega(G) = 3$ and (1) and (2) in the definition of RCAs hold. Therefore, $G \in RCA(n,d,3)$ $\square$

Theorem 2.1. *If $G \in RCA(n,d,k)$, ($d > k-1$), then $CL(G) \in RCA\left(\frac{nd}{k(k-1)}, \frac{k(d-k+1)}{k-1}, \frac{d}{k-1}\right)$. Further, $CL(CL(G)) \simeq G$.*

Proof. The vertices of $CL(G)$ are the maximum cliques of G. Counting the ordered pairs (v, K), K a maximum clique in G and $v \in V(K)$, in two different ways, we find that the number of maximum cliques in G is given by

$$\frac{(|V(G)|)(\text{number of maximum cliques containing each vertex})}{\text{number of vertices in each clique}} = \frac{(n)(\frac{d}{k-1})}{k} = \frac{nd}{k(k-1)}.$$

By Lemma 2.1, two maximum cliques in G are adjacent as vertices in $CL(G)$ if and only if they have exactly one vertex in common. Let K be a maximum clique in G and $v \in V(K)$. In view of Proposition 2.1, K is adjacent in $CL(G)$ to each of $\frac{d}{k-1} - 1 = \frac{d-k+1}{k-1}$ other maximum cliques containing v — indeed, in $CL(G)$ these cliques induce, with K, a clique of order $\frac{d}{k-1}$. By Lemma 2.1, the maximum cliques "adjacent to K at v" are distinct from the maximum cliques adjacent to K at any other vertex of K. Then $CL(G)$ is regular of degree $\frac{k(d-k+1)}{k-1}$. The maximum cliques adjacent to K at v are also not adjacent to the maximum cliques adjacent to K at any other vertex. Suppose a clique H_1 shares v with K and a clique H_2 shares a vertex $u \neq v$ with K. Then by Lemma 2.1 H_1 and H_2 have no vertices in common, and thus the corresponding vertices in $CL(G)$ are not adjacent.

It follows that $CL(G)\left[N_{CL(G)}(K)\right] \simeq kK_{\frac{d}{k-1}-1}$. Since this holds for every vertex K of $CL(G)$, by Proposition 2.1 we conclude that $CL(G) \in RCA(\frac{nd}{k(k-1)}, \frac{k(d-k+1)}{k-1}, \frac{d}{k-1})$. Applying this result with $CL(G)$ replacing G, we find that

$$CL(CL(G)) \in RCA\left(\frac{\frac{nd}{k(k-1)} \cdot \frac{k(d-k+1)}{k-1}}{\frac{d}{k-1}\left(\frac{d}{k-1} - 1\right)}, \frac{\frac{d}{k-1}\left(\frac{k(d-k+1)}{k-1} - \frac{d}{k-1} + 1\right)}{\frac{d}{k-1} - 1}, \frac{\frac{k(d-k+1)}{k-1}}{\frac{d}{k-1} - 1}\right) = RCA(n,d,k).$$

From this we take that $CL(CL(G))$ has the same number of vertices as G and is d-regular.

For $v \in V(G)$, let $S(v)$ denote the $\frac{d}{k-1}$-clique induced in $CL(G)$ by the k-cliques in G that contain v; $S : V(G) \to V(CL(CL(G)))$ is clearly injective, and is therefore surjective. If u and v are adjacent in G then $S(u)$ and $S(v)$ have a vertex in common in $CL(G)$, namely, the unique maximum clique in G containing the edge uv; therefore, $S(u)$ and $S(v)$ are adjacent in $CL(CL(G))$. Since G and $CL(CL(G))$ are both d-regular, and S preserves adjacency, it must also preserve non-adjacency. Therefore S is a graph isomorphism; so G and $CL(CL(G))$ are isomorphic. □

Corollary 2.2. *$G \in RCA(n, 2k-2, k)$ for some n and $k > 2$ if and only if G is the line graph of a triangle-free k-regular graph.*

Proof. If $G \in RCA(n, 2k-2, k)$ and $k > 2$, then by Theorem 2.1, $CL(G) \in RCA(\frac{2n}{k}, k, 2)$, so $CL(G)$ is triangle-free and k-regular and $G \simeq CL(CL(G)) = L(CL(G))$. On the other hand, if $G = L(H)$, H triangle-free and k-regular, then $H \in RCA(t, k, 2)$, for $t = |V(H)|$, so $G = L(H) = CL(H) \in RCA(\frac{tk}{2}, 2(k-1), k)$, again by Theorem 2.1. □

By Corollary 2.1, regular clique assemblies with clique number $k = 2$ are precisely the triangle-free regular graphs. For $k = 3$ they are the edge regular graphs with $\lambda = 1$; we shall see that there are quite a few of these, although they are not quite as easy to find as the triangle-free regular graphs.

The *cartesian product* of two disjoint graphs, G and H, denoted $G\Box H$, has vertex set $V(G) \times V(H)$, and two vertices (g, h) and (g', h') in the product are adjacent if and only if either $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G.

Taking powers using the cartesian product of graphs, $(K_k)^t \in RCA(k^t, t(k-1), k)$ for all integers $k, t \geq 2$, so there are non-trivial RCAs with clique number k for all $k > 3$. By Theorem 2.1, $CL((K_k)^t) \in RCA(k^{t-1}t, k(t-1), t)$, which enlarges the supply of these assemblies somewhat. We can also produce RCAs by applying Theorem 2.1 with $k \in \{2, 3\}$. If $G \in RCA(n, d, 2) = ER(n, d, 0)$ and $d > 1$ then $CL(G) = L(G)$, the line graph of G, and, by Theorem 2.1, $L(G) \in RCA\left(\frac{nd}{2}, 2(d-1), d\right)$. If $G \in RCA(n,d,3) = ER(n,d,1)$ and $d > 2$, then $CL(G) \in RCA(\frac{nd}{6}, \frac{3(d-2)}{2}, \frac{d}{2})$.

By Theorem 2.1, if $G \in RCA(n, k(k-1), k)$, then $CL(G) \in RCA(n, k(k-1), k)$, which naturally generates the question: is G necessarily isomorphic to $CL(G)$? And, if not necessarily, then for which n and k does $G \in RCA(n, k(k-1), k)$ exist such that $G \simeq CL(G)$? The answer to the first question is, generally: no, as we will see later. We know very little about the second question, but we do know this: there is exactly one graph in $ER(15,6,1) = RCA(15,6,3)$, and therefore it is isomorphic to its clique graph.

2.2 Configurations

An *incidence structure* $\mathcal{S}$ is a triple $(\mathcal{P}, \mathcal{B}, \mathcal{J})$ where $\mathcal{P}$ is a set of points, $\mathcal{B}$ is a set of lines (or blocks), and $\mathcal{J} \subseteq \mathcal{P} \times \mathcal{B}$ is the incidence relation of $\mathcal{S}$. If $(p, B) \in \mathcal{J}$ we say that the point p lies on (or is contained in) the line B. A *configuration* (v_r, b_k) is an incidence structure of v points and b lines such that each line contains k points, each point lies on r lines, and two different points are connected by at most one line. Counting ordered pairs $(p, B) \in \mathcal{P} \times \mathcal{B}$ where $p \in B$, we see that $bk = vr$ for any configuration (v_r, b_k). If $v = b$ (equivalently, $r = k$), the configuration is *symmetric* and is denoted v_k. A *triangle* or *trilateral* in a configuration is a set of three points which are pairwise collinear but not all three contained in a single line.

Proposition 2.2. *Taking the vertices of a graph as points and maximal cliques as lines, an element of $RCA(n,d,k)$ corresponds to a trilateral-free configuration $\left(n_{\left(\frac{d}{k-1}\right)}, \left(\frac{nd}{k(k-1)}\right)_k\right)$, and a trilateral-free configuration (v_r, b_k) corresponds to an element of $RCA(v, r(k-1), k)$.*

Proof. Suppose $G \in RCA(n,d,k)$ and the incidence structure $\mathcal{S}$ is defined by $\mathcal{P} = V(G)$, $\mathcal{B} = \{\{v_1, \ldots, v_k\} \mid \{v_1, \ldots, v_k\}$ induces a maximal clique in $G\}$. By definition, each edge of G is contained in exactly one maximum clique, so a pair of points in $\mathcal{P}$ will be connected by at most one line. Every maximal clique in G is maximum, so G cannot contain an induced

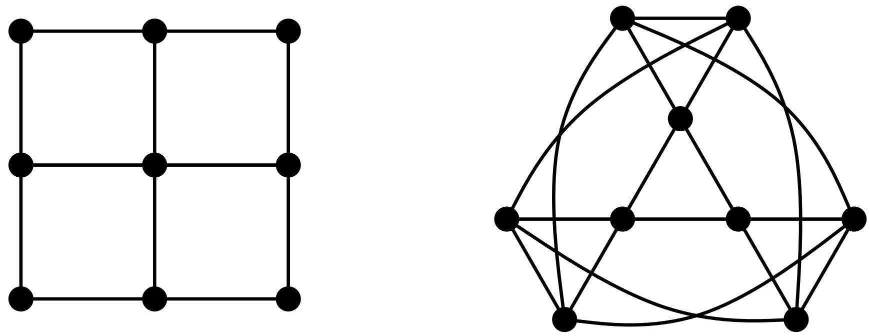


Figure 2.1: A configuration $(9_2, 6_3)$ and corresponding graph in $RCA(9,4,3)$

K_3 which is not contained in a maximum clique. Then $\mathcal{S}$ must be trilateral-free. Clearly $\mathcal{P}$ contains n points and the number of points on each line in $\mathcal{B}$ is k. The number of maximum cliques containing a vertex $v \in V(G)$ is $\frac{d}{k-1}$, so the corresponding v in $\mathcal{P}$ lies on $\frac{d}{k-1}$ lines. The number of maximum cliques in G is determined as in the proof of Theorem 2.1. We count $\frac{d}{k-1}$ maximum cliques for each of the n vertices. Each clique is counted k times, once for each of its vertices, so the total number of maximum cliques in G, and thus the number of lines in $\mathcal{B}$, is $\frac{nd}{k(k-1)}$. Then the incidence structure $\mathcal{S}$ is a configuration $\left(n_{\left(\frac{d}{k-1}\right)}, \left(\frac{nd}{k(k-1)}\right)_k\right)$.

Suppose $(\mathcal{P}, \mathcal{B})$ is a trilateral-free (v_r, b_k) configuration and G is defined by $V(G) = \mathcal{P}$ and $u, v \in V(G)$, $u \neq v$, are adjacent in G if and only if u and v are in the same $B \in \mathcal{B}$. By definition G is a graph on v vertices. For $u \in V(G)$, the corresponding point u in $\mathcal{P}$ is contained in r lines, each containing k points and each pair of which intersect only at u. So the degree of u in G is $r(k-1)$. A maximal clique in G corresponds to a set of pairwise collinear points in $\mathcal{P}$. We have supposed $(\mathcal{P}, \mathcal{B})$ to be trilateral-free, so any set of pairwise collinear points must all lie on a common line. Then a maximal clique in G corresponds to a line in $\mathcal{B}$. Thus the clique number of G is k and a maximal clique in G is maximum. An edge in G corresponds to a pair of points in $\mathcal{P}$, which, by definition, are connected by at most one line. Then an edge in G must be contained in exactly on maximum clique. Therefore $G \in RCA(v, r(k-1), k)$. $\square$

If $(\mathcal{P}, \mathcal{B})$ is a (v_r, b_k) configuration, then its dual $(\mathcal{B}, \mathcal{P})$ is a (b_k, v_r) configuration [8]. (In the dual, $B \in \mathcal{B}$ is in the block associated with $p \in \mathcal{P}$ if and only if p is in B in the configuration $(\mathcal{P}, \mathcal{B})$. In $(\mathcal{P}, \mathcal{B})$ two different points are contained in at most one common line. Consequently, lines in $(\mathcal{P}, \mathcal{B})$ intersect in at most one point, so two points in the dual $(\mathcal{B}, \mathcal{P})$ are incident to at most one common line.) It is clear that the dual of the dual is the original configuration, and this observation provides an elegant alternate proof of Theorem 2.1, when it is realized that if $(\mathcal{P}, \mathcal{B})$ is trilateral-free, and G is the corresponding regular clique assembly, then $CL(G)$ is the graph corresponding to the dual $(\mathcal{B}, \mathcal{P})$ (which is also trilateral-free, either by virtue of its correspondence to $CL(G)$ or by direct proof). The thrashing around in the proof of Theorem 2.1 has, in this proof, been absorbed into the verification of the correspondence between trilateral free configurations and regular clique assemblies.

A symmetric (n_k) trilateral-free configuration corresponds to a graph in $RCA(n, k(k-1), k)$, one of the classes of regular clique assemblies closed under taking clique graphs. Obviously $G \in RCA(n, k(k-1), k)$ is isomorphic to $CL(G)$ if and only if the configuration corresponding to G is isomorphic to its dual; so these are cases in which a graph isomorphism question is interchangeable with a geometric isomorphism question.

By Proposition 2.2, a regular clique assembly with parameters $(n, 6, 3)$ corresponds to a trilateral-free symmetric configuration (n_3). In Section 3.1.3 we consider $\bigcup\limits_{n} ER(n, 6, 1) = \bigcup\limits_{n} RCA(n, 6, 3)$ at some length, so the primary result of [16] (Theorem 1.2) is of particular interest:

Theorem 2.2 (Raney, 2013)**.** *For every $n \geq 15$ except $n = 16$, there are trilateral-free (n_3) congurations.*

Corollary 2.3. *For every $n \geq 15$ except $n = 16$, $RCA(n, 6, 3) = ER(n, 6, 1)$ is non-empty.*

In Section 3.1.3 we provide constructions for graphs in $ER(n, 6, 1)$ for infinitely many values of n and give an alternate proof that $ER(n, 6, 1)$ is non-empty for all but finitely many values of n.

Chapter 3

Edge-regular graphs

We shall focus most of our attention on the edge-regular graphs with $\lambda = 1$ with some observations about edge-regular graphs with $\lambda = 2$ which are not regular clique assemblies.

3.1 $\lambda = 1$

By Corollary 2.1, $ER(n, d, 1) = RCA(n, d, 3)$. We sum up the conclusions of Chapter 2 for $ER(n, d, 1)$ in the following.

Proposition 3.1. *Suppose $ER(n, d, 1) \neq \emptyset$. Then*

1. *d is even;*

2. *$3 \mid nd$*

3. *for each $G \in ER(n, d, 1)$ and $v \in V(G)$, $N_G[v]$ induces in G a friendship graph, $\{v\} \vee \frac{d}{2}K_2$;*

4. *if $d > 2$, each $G \in ER(n, d, 1)$ is the clique graph of its clique graph, $CL(G) \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$.*

Conversely,

3.′ *If G is a graph such that for some positive integer m, for each $v \in V(G)$, $G\left[N_G[v]\right] \simeq \{v\} \vee mK_2$, then $G \in ER(n, 2m, 1)$, $n = |V(G)|$; and*

4.′ *if G is the clique graph of some $H \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$, for some integers n and $d > 2$, then $G \in ER(n, d, 1)$.*

For non-negative integers d and λ, let $S_\lambda(d) = \{n \mid ER(n,d,\lambda) \neq \emptyset\}$ and $S^c_\lambda(d) = \{n \mid ER(n,d,\lambda)$ contains a connected graph $\}$. Observe that $S_\lambda(d)$ is closed under addition, since if $G_i \in ER(n_i,d,\lambda)$, $i = 1,2$, then $G_1 + G_2 \in ER(n_1+n_2,d,\lambda)$. Thus, to find $S_\lambda(d)$ it suffices to find $S^c_\lambda(d)$.

3.1.1 $d = 2$

Clearly the only edge regular graphs with $d = 2$ and $\lambda = 1$ are the graphs mK_3, $m = 1,2,\ldots$. Therefore $S_1(2) = \{3,6,9,\ldots\}$. Obviously $S^c_1(2) = \{3\}$.

3.1.2 $d = 4$

Corollary 3.1. *$G \in ER(n,4,1)$, $n = |V(G)|$ if and only if G is the line graph of a triangle-free 3-regular graph.*

Proof. By Corollary 2.1, $ER(n,4,1) = RCA(n,4,3)$; the conclusion follows from Corollary 2.2. □

As a side note, in Section 4 of [15] the authors consider a 4-regular K_4-free graph G with the property that for every $u \in V(G)$, $G[N[u]] = K_1 \vee 2K_2$. If H is the graph whose vertices correspond to triangles in G and if vertices of H are adjacent if and only if the associated triangles in G have a common vertex, then the authors conclude that G is the line graph of H and H is 3-regular and make further remarks from which it follows that H is K_3-free. Thus Corollary 3.1 could also be drawn almost entirely from this observation.

By Proposition 3.1, if $3 \nmid d$ and $ER(n,d,1) \neq \emptyset$, then $3|n$. Therefore, if $3 \nmid d$, $S_1(d)$ is contained in $\{3,6,9,\ldots\}$. By Remark 2 at the end of Chapter 1, plus a little work, the unique smallest edge-regular graph with $d = 4$, $\lambda = 1$, is $L(K_{3,3})$ with $9 = 3(4-1)$ vertices. For $m > 3$ it is easy to obtain a connected bipartite — and therefore triangle-free — 3-regular graph H on $2m$ vertices. Then $L(H) \in ER(3m,4,1)$. Thus $S^c_1(4) = S_1(4) = \{9,12,15,...\}$.

Corollary 3.2. *There are exactly two graphs in $ER(12,4,1)$, the line graphs of $K_{4,4} - M$, where M is a perfect matching in $K_{4,4}$, and of the graph in Figure 3.1.*

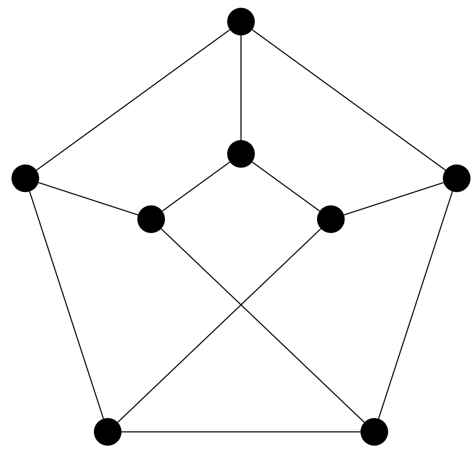


Figure 3.1: A graph whose line graph is in $ER(12, 4, 1)$

Proof. Both $G_1 = K_{4,4} - M$ and the other graph, G_2, are 3-regular and triangle-free with 12 edges. Therefore, their line graphs are in $ER(12, 4, 1)$. Since, by Theorem 2.1, each G_i is the clique graph of its line graph, their line graphs are distinct.

Now suppose that $G \in ER(12, 4, 1)$. By Corollary 3.1, $G = L(H)$ for some $H \in ER(8, 3, 0)$. If H is bipartite, then, because H is bipartite and regular, H is 1-factorizable and so H must be $K_{4,4} - M$, for some perfect matching M.

If H is not bipartite, then, since H is K_3-free on 8 vertices, H must contain either a C_5 or a C_7 or both. If H contains a C_5, it must be induced in H, because H is triangle-free. Each vertex on the C_5 must therefore be adjacent to exactly one of the 3 vertices not on the C_5. If one of those vertices were adjacent to 3 vertices on the C_5, there would be a triangle in H. Therefore 2 of the 3 vertices off the C_5 are adjacent to 2 vertices each, on the C_5, and the third is adjacent to one vertex on the cycle and both of the other off-cycle vertices. From there it is easy to see that H must be G_2, the graph depicted above.

If H contains a C_7 then, because the one vertex off the cycle is adjacent to only 3 vertices on the cycle, H must contain two chords of the cycle. Any chord of a C_7 which does not create a K_3 must create a C_5, so H contains a C_5. Therefore $H \simeq G_2$. $\square$

Corollary 3.1 shows that $ER(n, 4, 1)$ contains a connected graph for infinitely many n, and we shall soon see that $ER(n, 6, 1)$ contains a connected graph for infinitely many n. In passing, we note that these facts point to a powerful difference between the class of all edge regular graphs and the class of strongly regular graphs. An elementary necessary condition for $SR(n, d, \lambda, \mu)$ to be non-empty is that $d(d - \lambda - 1) = \mu(n - d - 1)$ [2]. It follows that for given d, λ satisfying $d > \lambda + 1$ there can be only finitely many pairs (n, μ) such that $SR(n, d, \lambda, \mu) \neq \emptyset$. If $\mu > 0$, any graph in $SR(n, d, \lambda, \mu)$ is connected. Corollary 3.1 and the construction to come show that for $(d, \lambda) \in \{(4, 1), (6, 1)\}$, there are infinitely many n such that $ER(n, d, \lambda)$ contains a connected graph. It is an open question whether or not $ER(n, d, 1)$ contains a connected graph for infinitely many n, for $d > 6$, d even, except for $d = 10$, as we shall see.

3.1.3 $d = 6$

By a remark in the Introduction, if $ER(n, 6, 1) \neq \emptyset$ then $n \geq 3(6 - 1) = 15$. We shall see that $ER(15, 6, 1)$ contains exactly one graph and then use that graph to construct connected graphs in $ER(n, 6, 1)$ for infinitely many values of n. By Corollary 2.3, $S_1(6) = \{15\} \cup \{17, 18, 19, ...\}$. We shall give an alternate proof for many of these values and consider $S_1^c(6)$ as well.

Suppose m and k are positive integers. Let $[m] = \{1, \ldots, m\}$ and let $\binom{[m]}{k}$ denote the set of all k-subsets of [m]. If $1 \leq k \leq \frac{m}{2}$, the *Kneser graph* $K(m, k)$ has vertex set $\binom{[m]}{k}$, with $u, v \in \binom{[m]}{k}$ adjacent if and only if $u \cap v = \emptyset$

Lemma 3.1. *If m and k are integers satisfying $1 \leq k \leq \frac{m}{2}$, then $K(m, k) \in ER(\binom{m}{k}, \binom{m-k}{k}, \binom{m-2k}{k})$. If $m \geq 4$, $K(m, 2) \in SR(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2})$.*

Proof. The verification is straightforward. □

Corollary 3.3. *If $k \geq 1$, $K(3k, k) \in ER(\binom{3k}{k}, \binom{2k}{k}, 1)$.*

Theorem 3.1. *$K(6, 2)$ is the unique graph in $ER(15, 6, 1)$.*

Proof. For any graph $G \in ER(n, 6, 1) = RCA(n, 6, 3)$, for any n, if $u, v, w \in V(G)$ induce a K_3 in G then, by Lemma 2.1 and its corollaries, the subgraph of G induced by $N[\{u, v, w\}] = N[u] \cup N[v] \cup N[w]$ has a spanning subgraph as depicted in Figure 3.2. By Corollary 3.3, $K(6, 2) \in ER(15, 6, 1)$. For any $G \in ER(15, 6, 1)$, for any $u, v, w \in V(G)$ inducing K_3 in G, all 15 of G's vertices are on display in Figure 3.2. The edges of G not depicted are among the 12 vertices of $V(G)\backslash\{u, v, w\}$. Consider x_1. All 4 vertices to which x_1 is adjacent besides x_2 and v are among the z_j and the y_j. But x_1 cannot be adjacent to both z_1 and z_2, for instance, because the unique common neighbor of z_1 and z_2 is u. Therefore x_1 is adjacent to at most one of z_1, z_2, to at most one of z_3, z_4, to at most one of y_1, y_2, and to at most one of y_3, y_4. Therefore, x_1 is adjacent to *exactly* one of z_1, z_2, to exactly one of z_3, z_4, etc., because x_1 must have 4 neighbors among the 8 vertices.

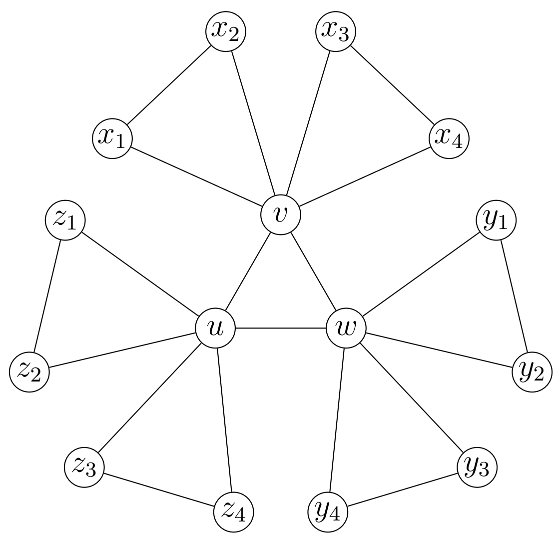


Figure 3.2: Spanning subgraph of $G[N[\{u, v, w\}]]$ for any $K_3 = G[\{u, v, w\}]$ in $G \in ER(n, 6, 1)$, for some n

Therefore, u and x_1 have exactly 3 common neighbors, v and two among $z_1, \ldots, z_4$. But, because the diagram in Figure 3.2 will be the same (except for the vertex names), no matter which K_3 you start with, u and x_1 could be any two non-adjacent vertices in G. Therefore G is strongly regular: $G \in SR(15, 6, 1, 3)$. According to [11], $K(6, 2)$ is the only graph in $SR(15, 6, 1, 3)$. $\square$

For those who don't care for proof by appeal to websites, a more laborious proof can be given which provides an independent corroboration of the fact that $K(6, 2)$ is the unique member of $SR(15, 6, 1, 3)$. The full structure of the graph induced by the edges of G among the 12 vertices of $G - \{u, v, w\}$, excluding the edges shown in Figure 3.2 (x_1x_2, x_3x_4, etc.), can be deduced from the assumption that $G \in ER(15, 6, 1)$. For a somewhat shorter proof, note that that graph on 12 vertices must be in $ER(12, 4, 1)$; of the two possibilities given in Corollary 3.2, $L(G_2)$, where G_2 is the non-bipartite graph depicted, can be ruled out as follows.

Let H be a graph in $ER(12, 4, 1)$ on vertices x_i, y_i, z_i, $i = 1, 2, 3, 4$, which completes the graph in Figure 3.2 to a graph $G \in ER(15, 6, 1)$. Observe that the x_i, the y_i, and the z_i are 3 independent sets of 4 vertices each, in H, and that x_1 and x_2 can have no common neighbor in H. The same holds for x_3 and x_4, for y_1 and y_2, for y_3 and y_4, for z_1 and z_2, and for z_3 and z_4. Also, by what has been noted already, if $i \in \{1, 2\}$, $j \in \{3, 4\}$, and $w \in \{x, y, z\}$, then w_i and w_j are not adjacent in G, and therefore have 3 neighbors in common in G — and therefore have two neighbors in common in H.

Keeping in mind that vertices of H adjacent in $G - E(H)$ can have no common neighbor in H, we see that if $H = L(Q)$ where Q is one of the 3-regular graphs mentioned in Corollary 3.2, $K_{4,4} - M$ or G_2, then Q must have a proper edge-coloring with 3 colors such that each color class can be partitioned into two matchings of two edges each, with pairs of edges from different 2-edge matchings within the color class commonly adjacent to exactly two other edges not in that color class. It can be verified directly that G_2 has no such edge-coloring, and that $K_{4,4} - M$ has essentially only one. (To see this, in each case color the 3 edges incident to a single vertex with the colors x_1, y_1, z_1. Then ask: which edges can or must be colored x_3, x_4? And then: where does the color x_2 go? By this time it will be clear that G_2 has been eliminated, and that the edge coloring of $K_{4,4} - M$ is essentially unique.) Thus G is unique.

3.1.3.1 Construction of connected graphs in $ER(n, 6, 1)$ for infinitely many n

Start with the graph shown in Figure 3.2; we call this the *primary scaffold*. Each vertex in it has degree 2 or 6, any two vertices adjacent in the scaffold have a unique common neighbor in the scaffold, and non-adjacent vertices in the scaffold have at most one common neighbor. We can build new scaffolds with these properties from the primary scaffold in a number of ways. We shall describe the most straightforward construction method, leading to graphs in $ER(15+16k, 6, 1)$, $k = 1, 2, \ldots$, and then mention variations of the method that can produce graphs in $ER(n, 6, 1)$ for many other n, including all $n \geq 47$.

In a scaffold, each vertex of degree 6 is *finished*, and each vertex of degree 2 is *unfinished*. Produce a new scaffold by joining an unfinished vertex to the vertices of a $2K_2$ whose vertices are new to the scene. The 4 new vertices are unfinished in the new scaffold, and the formerly unfinished vertex to which they are joined is finished. The number of vertices has increased by 4 and the number of unfinished vertices has increased by 3.

The primary scaffold has 15 vertices, 12 of them unfinished. Therefore, after t iterations of the new-scaffold-generating process, the resulting scaffold will have $15+4t$ vertices, $12+3t$ of them unfinished. When $t = 4k$ for some integer k, we have a scaffold on $15+16k$ vertices, with $12(k+1)$ of them unfinished.

At such a point we can stop building scaffolds and attempt to complete the scaffold we have to a graph in $ER(15+16k, 6, 1)$ by executing the following plan: partition the set of unfinished vertices in the scaffold into $k+1$ sets $P_1, \ldots, P_{k+1}$ of 12 vertices each and then put edges among the vertices of P_j, for each j, so that the graph on those vertices, with those edges, is one of the two graphs in $ER(12, 4, 1)$ mentioned in Corollary 3.2.

For any choice of the P_j, and any insertion of the edges of one of the graphs in $ER(12, 4, 1)$ on the vertices of the P_j, $j = 1, \ldots, k+1$, the resulting graph on $15+16k$ will be regular of degree 6, any two adjacent vertices will be joined by one or two edges (possibly one from the scaffold and one inserted) and will have one or two neighbors in common (possibly one common neighbor in the scaffold and one in the imposed graph from $ER(12,4,1)$). We need to make arrangements so that there are no doubled edges in the completed graph and no two adjacent vertices in the completed graph have two common neighbors in that graph.

We posit the following requirements on $P_1, \ldots, P_{k+1}$ and on the graphs $H_j \in ER(12,4,1)$ obtained by inserting edges among the vertices of P_j, $j = 1, \ldots, k+1$:

Each P_j must be partitionable into 3 sets Q_{1j}, Q_{2j}, Q_{3j} of 4 vertices each such that if $u \in Q_{ij}$, $v \in Q_{tj}$, $1 \leq i < t \leq 3$, then u, v are distant at least 3 from each other in the scaffold. Explanation: Each graph in $ER(12,4,1)$ has chromatic number 3 and vertex independence number 4. The Q_{ij} will be independent sets of vertices in H_j, so pairs of vertices adjacent in H_j will be from different Q_{ij}. Therefore, because vertices in Q_{ij} and Q_{tj} for $t \neq i$ are distant at least 3 from each other in the scaffold, there will be no chance that an edge of the imposed H_j will double an edge of the scaffold. It is now sufficient to take care that no two vertices adjacent in H_j have a common neighbor in the scaffold and that no two vertices in P_j adjacent in the scaffold have a common neighbor in H_j.

Since two vertices adjacent in H_j are in Q_{ij} for different values of i, they are distant at least 3 from each other in the scaffold, and therefore have no common neighbor in the scaffold. Now suppose that $u, v \in P_j$ are adjacent in the scaffold. Then they must belong to the same Q_{ij}, since no two vertices in different Q_{ij} can be adjacent in the scaffold.

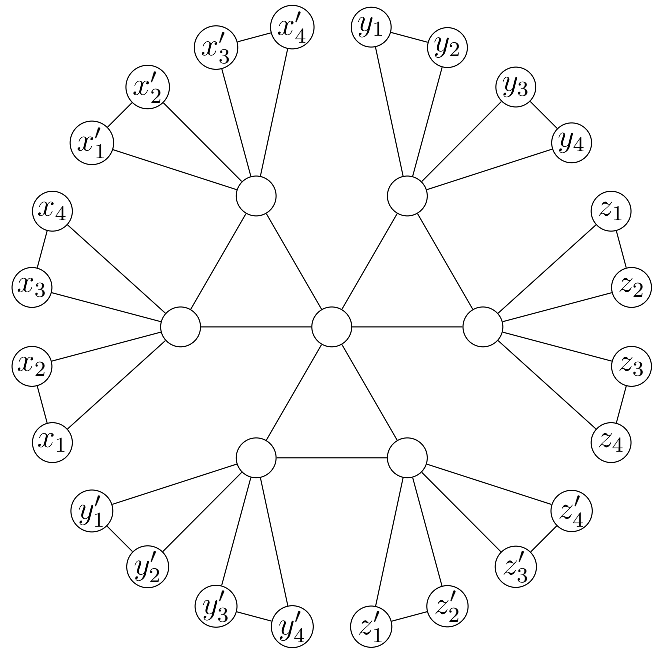
Since any 4 unfinished vertices in the scaffold induce one of $4K_1$, $2K_1 + K_2$, or $2K_2$ in the scaffold, we can require that each Q_{ij} be partitioned into two 2-element sets, R_{1ij} and R_{2ij}, such that no vertex in R_{1ij} is adjacent to any vertex in R_{2ij} in the scaffold. Then form $H_j \simeq L(K_{4,4}) - M$ with $R_{1,1,j}$, $R_{2,1,j}$,…, $R_{1,3,j}$, $R_{2,3,j}$ playing the roles of $\{x_1, x_2\}$, $\{x_3, x_4\}$,…, $\{z_1, z_2\}$, $\{z_3, z_4\}$, respectively, in a copy of $L(K_{4,4} - M)$ which completes the primary scaffold depicted in Figure 3.2 to $K(6,2)$.

Observe that the recommendation above for taking $H_j \simeq L(K_{4,4} - M)$ under certain circumstances is not an iron-clad requirement. It may well be that $L(K_{4,4} - M)$ may be successfully imposed upon P_j in other ways than the recommended way, or that $L(G_2)$ (see Corollary 3.2) may be successfully imposed, even if some Q_{ij}, $i \in \{1, 2, 3\}$, contains a pair of vertices adjacent in the scaffold. $L(G_2)$ may certainly be used if P_j is an independent set of vertices in the scaffold.

In Figure 3.3 are depicted two very different scaffolds of order 31, each with 24 unfinished vertices partitioned into two sets of 12 vertices, each of which is partitioned into 3 sets of 4 vertices each, satisfying the requirement that two vertices from different partition sets of 4 within either set of 12 are distant at least 3 from each other in the scaffold. In neither circumstance is the partition into 12-vertex subsets unique. In the top example, if the partition of the unfinished vertices into 12-vertex sets is as given, then the partition within each into 4-element sets is forced. This is not true in the other example. In each case, we intend $L(K_{4,4}-M)$ to be the graph imposed on each 12-vertex partition set for the completion of the given scaffold to a graph in $ER(31, 6, 1)$. We are certain that this is the only possible choice of an imposed graph from $ER(12, 4, 1)$ no matter what the partition choices for the top scaffold, and we are pretty sure that the same holds for the bottom scaffold. As k goes up, the $12(k + 1)$ unfinished vertices in scaffolds of order $15 + 16k$ become more numerous and "spaced away" from each other, offering many more choices for admissible partitions into 12-vertex sets. It becomes easier to make arrangements so that $L(G_2)$ (see Cor. 4) can be used in the construction.

It is clear that $ER(15+16k, 6, 1) \neq \emptyset$ for $k = 0, 1, 2, \ldots$, by the preceding. This disproves the conjecture that $ER(n, 6, 1) \neq \emptyset$ implies that $3|n$.

It is not clear that different choices made in building the scaffold and then completing it to a graph in $ER(15 + 16k, 6, 1)$ will result in non-isomorphic graphs. There is only one graph in $ER(15, 6, 1)$ (Theorem 3.1); it would be interesting to know how many isomorphism classes of graphs are represented in $ER(31, 6, 1)$.



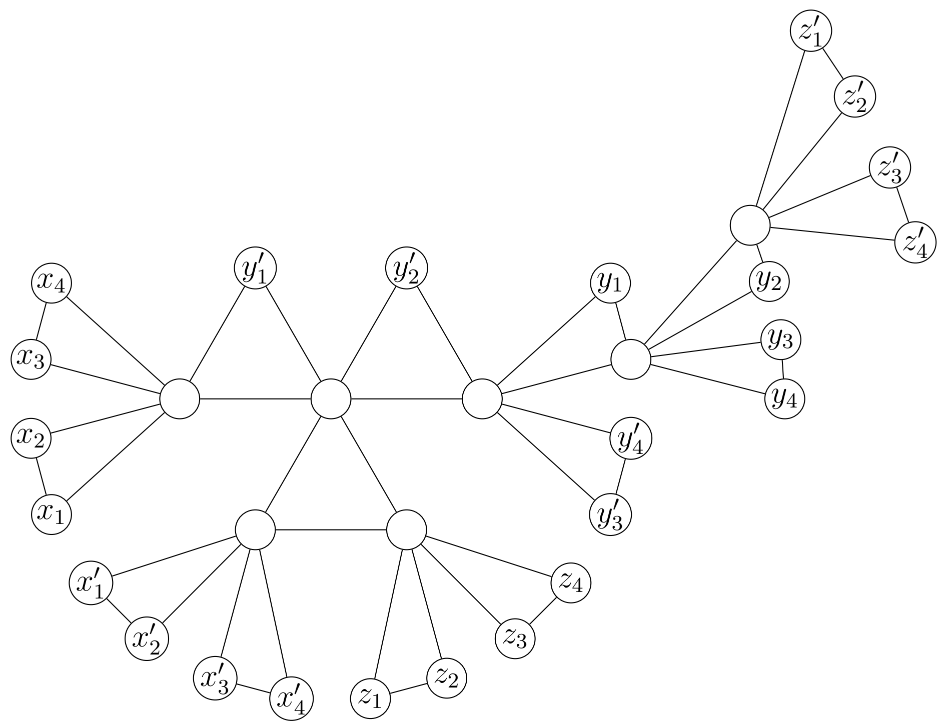


Figure 3.3: Two different scaffolds of order 31, with admissible partitions of the unfinished vertices into two 12-vertex sets, admissibly: $\{x_1, x_2, \ldots, z_3, z_4\}$, $\{x_1', x_2', \ldots, z_3', z_4'\}$

3.1.3.2 Orders of (connected) edge-regular graphs with $d = 6$ and $\lambda = 1$

In this subsection we are going to find all but a finite subset of $S_1^c(6)$.

Let the scaffold-building operation described previously, in which an unfinished vertex is joined to a new $2K_2$, be called Method 1, or M1 for short. Here are two other scaffold-building operations.

M2: Take two unfinished vertices, a distance ≥ 3 from each other in the current scaffold; join them and join each to a new vertex. Finish each by joining it to a K_2 — the K_2s being disjoint and formed from new vertices.

Note that the number of vertices has increased by 5 and the number of unfinished vertices has increased by 3.

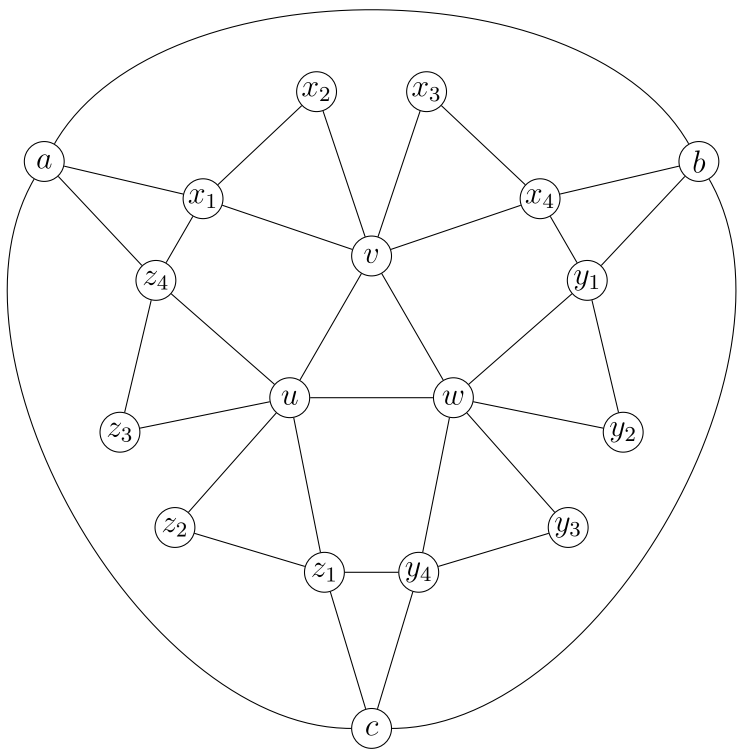
M3: Take 3 unfinished vertices, any two distant at least 3 from each other in the current scaffold. Make them the vertices of a K_3, and then join each up to its own K_2, whose vertices are new and unfinished in the new scaffold.

The number of vertices has increased by 6, and the number of unfinished vertices has increased by 3.

Proposition 3.2. $\{15, 18, 27\} \cup (\{n \mid n \geq 31\} \backslash \{40, 41, 42, 43, 44, 46\}) \subseteq S_1^c(6) \subseteq S_1(6)$. *Further,* $\{30, 42, 46\} \in S_1(6)$ *and* $16 \notin S_1(6)$.

Proof. Starting with the primary scaffold (Figure 3.2), we perform M1 a times, M2 b times, and M3 c times to obtain a scaffold with $15 + 4a + 5b + 6c$ vertices and $12 + 3(a + b + c)$ unfinished vertices. By previous remarks, it is clear that $Z = \{15 + 4a + 5b + 6c \mid a, b, c$ are non-negative integers and $a + b + c \equiv 0 \mod 4\} \subseteq S_1^c(6)$. It is straightforward to see that $Z = \{15\} \cup \{31, 32, \ldots\} \backslash \{40, \ldots, 46\}$. For a connected graph in $ER(18, 6, 1)$, see Figure 3.4. Note that $(K_3)^3 \in ER(27, 6, 1)$. Let H be C_{10} plus a 1-factor which creates no 3-cycles and let $G = L(H)$. By Corollary 3.1, $G \in ER(15, 4, 1)$, so the Cartesian product of G with K_3 is in $ER(15 \cdot 3, 4 + 2, 1) = ER(45, 6, 1)$. 30=15+15, 42=15+27, and 46=15+31 are in $S_1(6)$

because $S_1(6)$ is closed under addition. That $16 \notin S_1(6)$ follows from the main result of [7]. $\square$



Triangles not appearing above: ax_3y_2, by_3z_2, cx_2z_3, $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$

Figure 3.4: A graph in $ER(18, 6, 1)$

3.1.4 $d \geq 8$

As previously noted, $S_1(d) = \{n \mid ER(n, d, 1) \neq \emptyset\}$ is non-empty for every even positive integer d, and is closed under addition. For d not divisible by 3, we can deduce quite a lot about $S_1(d)$ with little effort.

Proposition 3.3. *Suppose that $d > 0$ is even and 3^q is the largest integer power of 3 that divides d. Then $S_1(d)$ contains all sufficiently large integer multiples of 3^{2q+1}.*

Proof. It has already been shown that $S_1(2) = \{3, 6, 9, \ldots\}$ and that $S_1(4) = \{9, 12, 15, \ldots\}$. Let $d = 2t$, $t \geq 3$. Then $3^q || t$. Since $(K_3)^t \in ER(3^t, d, 1)$, we have $3^t \in S_1(d)$. Since

$(K_t)^3 \in RCA(t^3, 3(t-1), t)$, it follows by Theorem 2.1 that $CL((K_t)^3) \in RCA(3t^2, d, 3) = ER(3t^2, d, 1)$, so $3t^2 \in S_1(d)$. Clearly $3^{2q+1} || 3t^2$. Since $q \leq \frac{\ln t}{\ln 3}$, it follows (applying a bit of calculus) that $2q+1 \leq 2\frac{\ln t}{\ln 3} + 1 \leq t$ for all $t \geq 3$. Therefore the greatest common divisor of $3t^2$ and 3^t is 3^{2q+1}.

Let $a = \frac{3t^2}{3^{2q+1}}$ and $b = 3^{t-2q-1}$. Then a and b are relatively prime positive integers. By a well-known theorem of Frobenius (and Sylvester, and others), every integer from $(a-1)(b-1)$ on is expressible as a combination $ax+by$, $x, y \in \mathbb{N}$. Since $S_1(d)$ is closed under addition, and contains $3^{2q+1}a$ and $3^{2q+1}b$, it follows that $S_1(d)$ contains $3^{2q+1}c$ for every integer $c \geq (a-1)(b-1)$. $\square$

Corollary 3.4. *If $d > 6$ is even and not divisible by 3 then $S_1(d)$ contains all integer multiples of 3 from $(\frac{d^2}{4} - 1)(3^{\frac{d}{2}-1} - 1) \cdot 3$ onward.*

Proof. Since $3 \nmid d$, $q = 0$, and a, b in the proof of Proposition 3.3 are $\frac{d^2}{4}$, $3^{\frac{d}{2}-1}$, respectively. The conclusion follows, not from Proposition 3.3, but from a conclusion at the end of its proof. $\square$

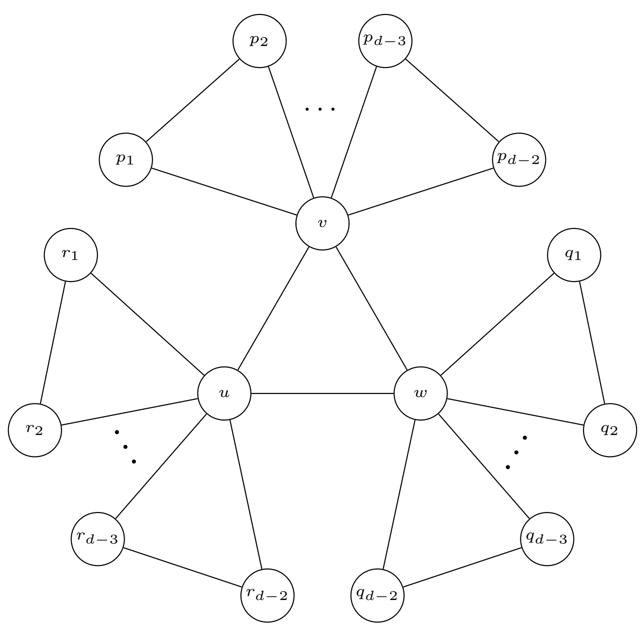


Figure 3.5: A spanning subgraph of $G[N[\{u, v, w\}]]$, $G \in ER(n, d, 1)$ for some n

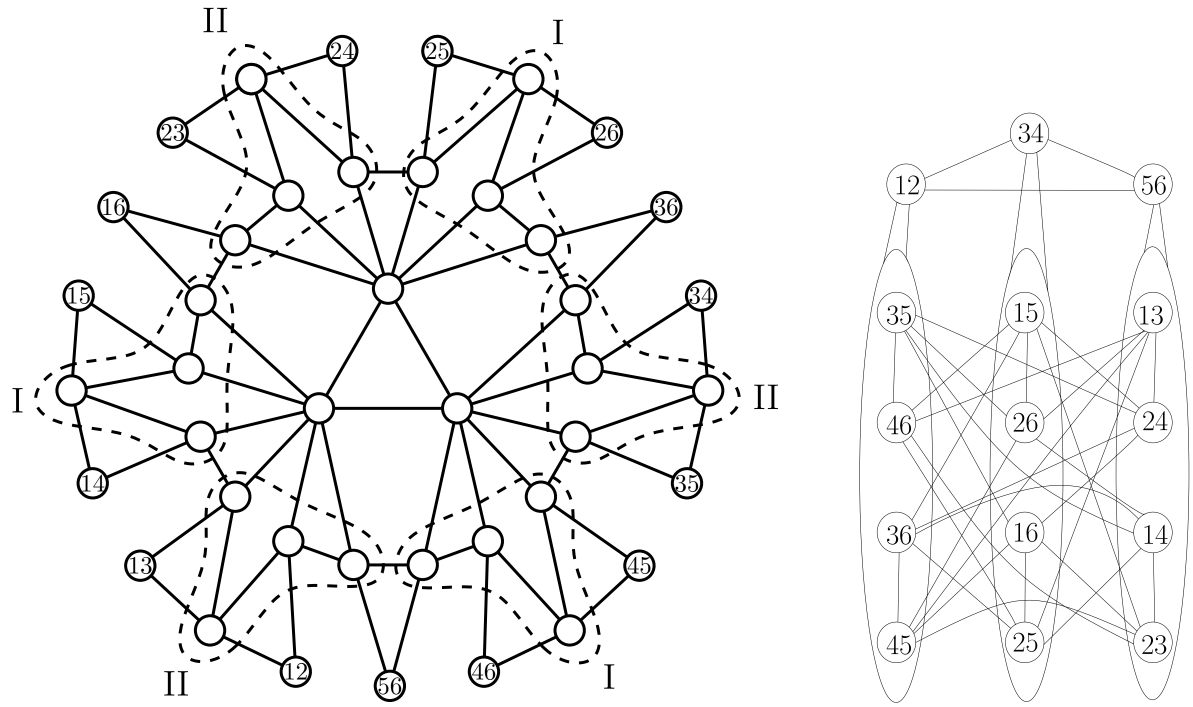


Figure 3.6: Left: a graph in $ER(42,8,1)$; right: $K(6,2)$, as used to complete the numbered vertices on the left

What about $S_1^c(d)$? For all $d > 2$, even, we have that $3^{\frac{d}{2}}$, $\frac{3d^2}{4} \in S_1^c(d)$ [because of $(K_3)^{\frac{d}{2}}$ and $CL\left((K_{\frac{d}{2}})^3\right)$]. Can we say more for $d > 6$?

Observe that the graph in Figure 3.5 has order $3(d-1)$, the famous lower bound for $n \in S_1(d)$. If there were a graph in $ER(3(d-1),d,1)$, as was the case when $d = 6$, then we could take the graph in Figure 3.5 as a preliminary scaffold and build larger scaffolds upon it, as we did in the case $d = 6$, with a view to "finishing" them to graphs in $ER(n,d,1)$ for some values of n.

Figure 3.6 shows a graph in $ER(42,8,1)$ which has been constructed by beginning with a scaffold as in Figure 3.5 and joining new vertices such that there remain 24 unfinished vertices of degree 4 and 15 unfinished vertices of degree 2. To finish the vertices of degree 4, we may cautiously add the edges of one of the graphs in $ER(12,4,1)$ to the vertices labelled I in Figure 3.6 and then do the same for the vertices labeled II. To finish the vertices of degree 2, we add the edges of the Kneser graph $K(6,2)$ as shown on the right of Figure 3.6.

This example was constructed much more carefully than the graphs in Section 3.1.3, and does not suggest an obvious generalization, so possible constructions for the case $d \geq 8$ remain of interest.

Proposition 3.4. *If $d > 2$ then $ER(3(d-1), d, 1) = SR(3(d-1), d, 1, \frac{d}{2})$.*

Proof. Clearly $ER(3(d-1), d, 1) \supseteq SR(3(d-1), d, 1, \mu)$ for any μ. If $G \in ER(3(d-1), d, 1)$ then d is even and for every triangle uvw in G, there is a spanning subgraph of G as depicted in Figure 3.5. The proof now proceeds by the argument in the proof of Theorem 3.1, about the case $d = 6$:

p_1 and p_2 can have no common neighbor but v, neither is adjacent to any p_i, $i > 2$, and neither can be adjacent to two adjacent vertices among the q_i, nor among the r_i; it follows that each has $\frac{d-2}{2}$ neighbors among the q_i and among the r_i. Therefore, p_1 and u have $1 + \frac{d-2}{2} = \frac{d}{2}$ common neighbors. Since p_1 and u could be any two vertices not adjacent in G, it follows that G is strongly regular with $\mu = \frac{d}{2}$. □

Corollary 3.5. *$ER(3(d-1), d, 1) \neq \emptyset$ if and only if $d \in \{2, 4, 6, 10\}$.*

Proof. The second-best-known necessary condition for $SR(n, d, \lambda, \mu) \neq \emptyset$, the *integrality condition* ([2], [4]), is that each of $\frac{1}{2}\left[(n-1) \pm \frac{(n-1)(\mu-\lambda)-2d}{\sqrt{(\mu-\lambda)^2+4(d-\mu)}}\right]$ is a non-negative integer. Plugging $n = 3(d-1)$, $\lambda = 1$, $\mu = \frac{d}{2}$, and simplifying, we find that $\frac{1}{2}(3d - 4 \pm (3d - 20 + \frac{48}{d+2}))$ must be non-negative integers.

Among even integers greater than 2, the possibilities for d are 4, 6, 10, and 22. Spence's website [18] shows a graph, and only one graph, in $SR(27, 10, 1, 5)$. That $SR(63, 22, 1, 11) = \emptyset$ can be shown using a less well-known necessary condition for the existence of a strongly regular graph, the *absolute bound*. See [14], Theorem 21.4. □

Corollary 3.6. $24 \in S_1^c(8)$.

Proof. Consider $G \in ER(27, 10, 1)$, and any one of the spanning "scaffolds" depicted in Figure 3.5, with $d = 10$. The edges of G not pictured in Figure 3.5 induce $H \in ER(24, 8, 1)$.

If H were not connected then one of its components would be edge-regular with $d = 8$, $\lambda = 1$, on no more than 12 vertices. Since $12 < 21 = 3(8-1)$, this is impossible. $\square$

Corollary 3.7. *$S_1^c(10)$ contains all sufficiently large multiples of 3.*

Proof. Starting with the scaffold in Figure 3.5, with $d = 10$, apply the scaffold-building analogs of Methods 1, 2, 3 that were used in the case $d = 6$ to build new scaffolds in which the number of unfinished vertices is a multiple of 24. Finish these off to make connected edge-regular graphs with $d = 10$ and $\lambda = 1$ by the method analogous to that used in the case $d = 6$, using the graph H referred to in the proof of Corollary 3.6 as $L(K_{4,4} - M)$ was used in the $d = 6$ constructions. There follows the verification that graphs in $ER(3q, 10, 1)$ can be so constructed for all sufficiently large integers q.

In the case $d = 10$, each instance of Method 1 increases the number of vertices by 8 and the number of unfinished vertices by 7; Method 2 increases the number of vertices by 13 and the number of unfinished vertices by 11; and Method 3 increases the number of vertices by 18 and the number of unfinished vertices by 15. As in the case $d = 6$, it is obvious that the distance requirements to be satisfied in applying these methods and in finishing scaffolds to edge-regular graphs with $d = 10$, $\lambda = 1$, are not a problem: distances between unfinished vertices in the scaffold remain the same or increase as the scaffolds are built. Therefore, since we are starting with a scaffold with 27 vertices, 24 of them unfinished, it suffices to show that $T = \{8a + 13b + 18c \mid a, b, c \in \mathbb{N} \text{ and } 7a + 11b + 15c \equiv 0 \mod 24\}$ contains all sufficiently large multiples of 3.

Clearly T is closed under addition; therefore, T is closed under taking non-negative integer combinations. We have that $8\cdot1+13\cdot1+18\cdot2 = 57 \in T$ and $8\cdot6+13\cdot0+18\cdot2 = 84 \in T$. Therefore, for all $d, e \in \mathbb{N}$, $57d+84e = 3(19d+28e) \in T$. By the famous theorem of Frobenius mentioned earlier, T contains $3t$ for all $t \geq (19-1)(28-1)$. $\square$

3.2 $\lambda > 1$

We begin with an observation which will prove useful in constructing certain graphs later.

Theorem 3.2. *Suppose* $G \in ER(n_1, d_1, \lambda)$ *and* $H \in ER(n_2, d_2, \lambda)$. *Then* $G \square H \in ER(n_1 n_2, d_1 + d_2, \lambda)$.

Proof. By definition, $V(G\square H) = V(G) \times V(H)$, so $|V(G\square H)| = |V(G)|\,|V(H)| = n_1 n_2$. Suppose $(g,h) \in V(G\square H)$. Then $d_{G\square H}(g,h) = |N_{G\square H}(g,h)| = |\{(g',h') \in V(G\square H) \mid gg' \in E(G) \text{or} hh' \in E(H)\} = d_G(g) + d_H(h) = d_1 + d_2$. If (g,h) and (g',h') are adjacent in $G\square H$, then either they are contained in the same copy of G or the same copy of H. Then their common neighbors will either be the λ common neighbors of g and g' in $V(G)$ or the λ common neighbors of h and h' in $V(H)$. Thus $G\square H \in ER(n_1n_2, d_1 + d_2, \lambda)$. $\square$

Taking $\lambda = 1$, $H = K_3 \in ER(3,2,1)$, we can see that $S_1^c(d)$ is infinite for all d even, $d > 2$: this claim holds for $d = 4$ by Corollary 2.1, for d even, $d > 4$, taking the Cartesian product of $(K_3)^{\frac{d-4}{2}}$ with each connected graph in $\bigcup_n ER(n,4,1)$ shows that $S_1^c(d)$ is infinite.

Proposition 3.5. *The cartesian product of strongly regular graphs is not strongly regular.*

Proof. Let G and H be strongly regular graphs. By definition G and H are neither empty nor complete. There exist $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$ such that $g_1g_2 \notin E(G)$ and $h_1h_2 \notin E(H)$. Then (g_1,h_1) and (g_2,h_2) are vertices in $V(G\square H)$ that are not adjacent and have no common neighbors. There also exist $g_1', g_2' \in V(G)$ and $h_1', h_2' \in V(H)$ such that $g_1'g_2' \in E(G)$ and $h_1'h_2' \in E(H)$. Then (g_1', h_1') and (g_2', h_2') are not adjacent in $G\square H$, and they have (g_1,h_2) and (g_2,h_1) as their common neighbors. Thus $G\square H$ is not strongly regular. $\square$

Given $G \in SR(n_1, d_1, \lambda_1, \mu_1)$ and $H \in SR(n_2, d_2, \lambda_2, \mu_2)$, $G\square H$ cannot be strongly regular, but if $\lambda_1 = \lambda_2 = \lambda$, $G\square H \in ER(n_1n_2, d_1 + d_2, \lambda)$ by Theorem 3.2. So the rather extensive collection of known strongly regular graphs is a powerful resource for generating new edge-regular graphs, as demonstrated by Proposition 3.7 at the end of this section.

We have seen that $ER(n,d,1) = RCA(n,d,3)$, and we will now show that for some values of n and d, $ER(n,d,2)\backslash RCA(n,d,4) \neq \emptyset$. If $G \in ER(n,d,2)$ and $u, v \in V(G)$ are adjacent, then the subgraph induced by $N[u] \cap N[v]$ will be as in one of the following:

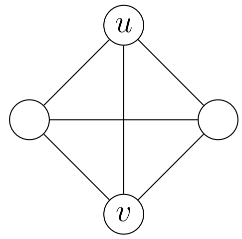


Figure 3.7: $G\,[N[u] \cap N[v]] = K_4$ if u and v are adjacent in $G \in ER(n,d,2)$ and G is clique-friendly

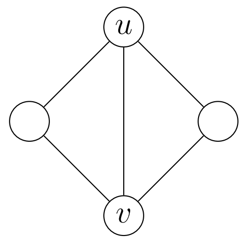


Figure 3.8: If $G \in ER(n,d,2)$ is not clique-friendly, $G[N[u] \cap N[v]] = K_2 \vee 2K_1$ for some pair of adjacent vertices u and v.

The octahedral and icosahedral graphs are examples of edge-regular graphs which are not clique friendly and which have $N[u] \cap N[v] = K_2 \vee 2K_1$ for all pairs of adjacent vertices u and v.

Thus far we have only described graphs in $ER(n,d,2)$, for some n and d, for which $G[N[u] \cap N[v]]$ is the same for any choice of adjacent vertices u and v. For a graph $G \in ER(n,d,2)$, could $u \in V(G)$ have distinct neighbors v and w such that $G[N[u] \cap N[v]] =$

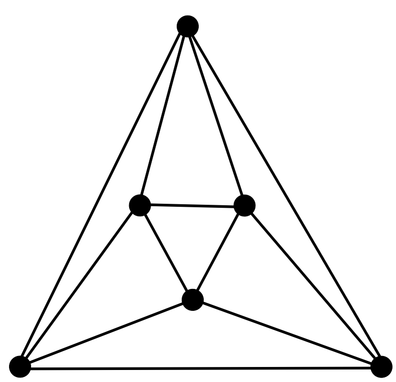


Figure 3.9: The octahedral graph is in $ER(6,4,2)$.

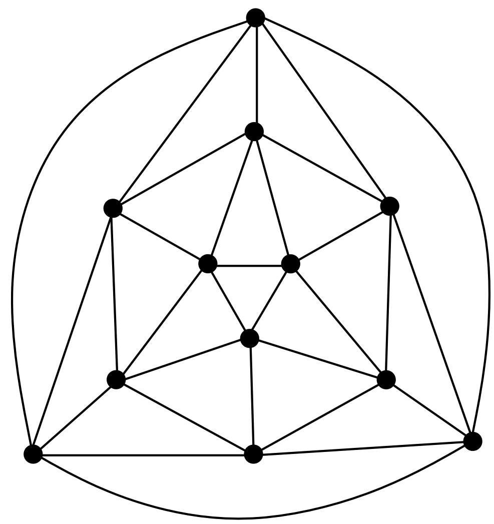


Figure 3.10: The icosahedral graph is in $ER(12, 5, 2)$.

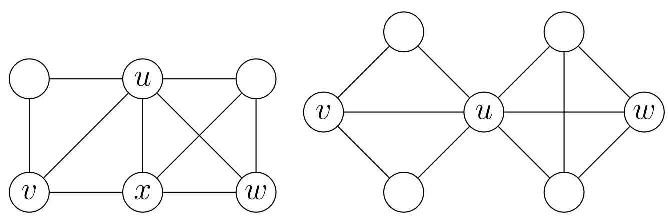


Figure 3.11: Possibilities for $G[N[u] \cap (N[v] \cup N[w])]$

$K_2 \vee 2K_1$ and $G[N[u] \cap N[w]] = K_4$? If so, the subgraph induced by $N[u] \cap (N[v] \cup N[w])$ must be one of the graphs in Figure 3.11.

Upon closer examination we see that the graph on the left in Figure 3.11 is not permissible, because $|N(u) \cap N(x)| = 3$, so in a graph with the local neighborhood structure described above, $G[N[u] \cap (N[v] \cup N[w])]$ would be $K_1 \vee (P_3 + K_3)$, as depicted on the right of Figure 3.11.

Let G be the octahedral graph shown in Figure 3.9. Note that $G \in ER(6, 4, 2)$ and $K_4 \in ER(4, 3, 2)$. By Theorem 3.2, $G \square K_4 \in ER(24, 7, 2)$. If $v \in V(G \square K_4)$, then the subgraph generated by the closed neighborhood of v is $K_1 \vee (C_4 + K_3)$ and contains $K_1 \vee (P_3 + K_3)$.

By Corollary 2.2, $G \in RCA(n, 6, 4)$ if and only if G is the line graph of some triangle-free 4-regular graph. Such a line graph would be in $ER(n, 6, 2)$ for some n; however, not every graph in $\bigcup_n ER(n,6,2)$ is the line graph of some triangle-free 4-regular graph. For instance, the graph in $ER(25,6,2)$ depicted in Figure 3.12 is not the line graph of anything because it contains induced subgraphs of the form $K_{1,3}$, known as *claws*, and it is known that any line graph must be claw-free [7].

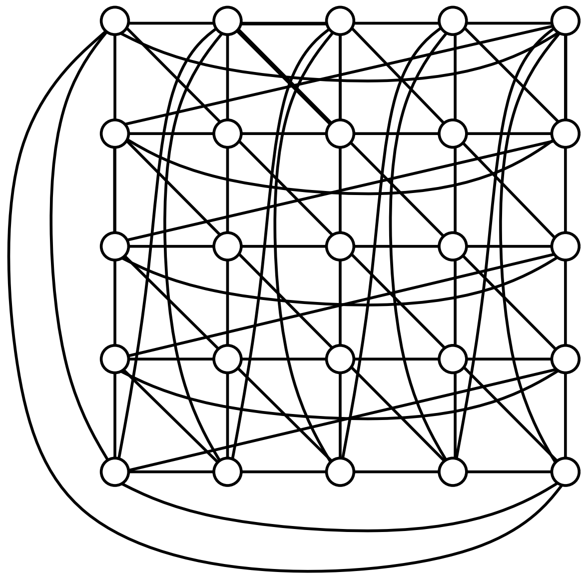


Figure 3.12: A graph in $ER(25,6,2)$

Proposition 3.6. *For all $d \geq 3$ there exists n such that $ER(n,d,2) \neq \emptyset$.*

Proof. As previously noted, $K_4 \in ER(4,3,2)$, the octahedral graph is in $ER(6,4,2)$, and the icosahedral graph is in $ER(12,5,2)$. Let $G_{\mathcal{O}}$ denote the octahedral graph. If $d \geq 6$, then $d = 3a+4b$ for some non-negative integers a and b and $(K_4)^a \,\square\, (G_{\mathcal{O}})^b \in ER(4^a 6^b, 3a+4b, 2)$. $\square$

Proposition 3.7. *For all even $d \geq 4$, there exists n such that $ER(n,d,3) \neq \emptyset$.*

Proof. Clearly $K_5 \in ER(5,4,3)$. The triangular graph T_m is the line graph of the complete graph K_m [3] and is known to be strongly regular with parameters $\left(\frac{m(m-1)}{2}, 2(m-2), m-2, 4\right)$. In particular, $T_5 \in SR(10,6,3,4) \subseteq ER(10,6,3)$. If $d \geq 4$ is even, then $d = 4a+6b$ for non-negative integers a and b and $(K_5)^a \,\square\, (T_5)^b \in ER(5^a 10^b, 4a+6b, 3)$. $\square$

Here we end our consideration of edge-regular graphs. Several open questions for $\lambda = 1$, $d > 6$ and for $\lambda > 1$ are given in Chapter 5. In the chapter to follow, we revisit the definition of regular clique assemblies and consider the implications of relaxing certain conditions of that definition.

Chapter 4

Relaxations of the definition of RCAs

Recall from Chapter 2:

G is a *regular clique assembly* if G is regular, $\omega(G) \geq 2$, and

(1) every maximal clique of G is maximum;

(2) each edge of G is in exactly one maximum clique of G.

In constructing this definition, it was conjectured privately that condition (1) above may not be necessary. The idea was quickly dispelled by the counterexample seen in Figure 4.1, but led to the consideration of what the graphs with (1) removed would look like. The three classes of graphs that follow are the results of relaxing, one at a time, (1), (2), and the requirement of regularity.

4.1 RCA*

Define $G \in RCA^*(n, d, k)$ if G is regular on n vertices, $k = \omega(G) \geq 2$, and each edge of G is in exactly one maximum clique of G.

Proposition 4.1. *$RCA^*(n, d, k) \supseteq RCA(n, d, k)$, with equality when $k \in \{2, 3\}$.*

Proof. Clearly $RCA^*(n, d, k) \supseteq RCA(n, d, k)$. Suppose $G \in RCA^*(n, d, 2)$ for some n and d. A maximal clique in G which is not maximum would be an isolated vertex. Since $\omega(G) = 2$, G contains at least one edge. G cannot be regular of degree $d > 0$ and contain isolates. So every maximum clique of G is also maximum. Thus $RCA^*(n, d, 2) = RCA(n, d, 2)$. Now suppose $G \in RCA^*(n, d, 3)$ for some n and d. A maximal clique in G which is not maximum is either an isolate or an edge which is contained in no K_3. G cannot contain isolates for the same reason mentioned above, and every edge must be contained in a K_3 by definition. Thus every maximal clique in G is maximum, and $RCA^*(n, d, 3) = RCA(n, d, 3)$. $\square$

We have ruled out maximal cliques of order 1 or 2, but for $k \geq 4$, could $G \in RCA^*(n, d, k)$ contain a maximal clique of order less than k? An example is given below.

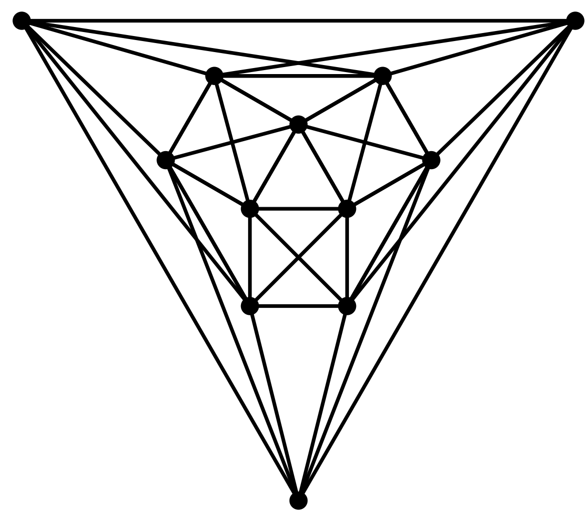


Figure 4.1: $G \in RCA^*(12, 6, 4) \backslash RCA(12, 6, 4)$

It is shown in Ch. 3, and is easy to see, that the line graph of a triangle-free k-regular graph is in $RCA(n, 2(k-1), k)$ for some n. Similarly, if $k > 2$, the line graph of any k-regular graph is in $RCA^*(n, 2(k-1), k)$. (When $k = 2$, this conclusion fails for any graph with K_3 among its components.)

It is possible for a graph $G \in RCA^*(n, d, k)$, $k \geq 5$ to contain maximal cliques of more than one order. A configuration is given below which corresponds to an element of $RCA^*(25, 12, 5)$ containing some maximal cliques of size 3 and some of size 4.

The following result may help the verification that the graph depicted in Figure 4.2 is in $RCA^*(25, 12, 5)$.

Proposition 4.2. *Suppose that G is d-regular on n vertices and $\omega(G) = k > 1$. Then $G \in RCA^*(n, d, k)$ if and only if:*

(i) any two maximum cliques of G have at most one vertex in common, and

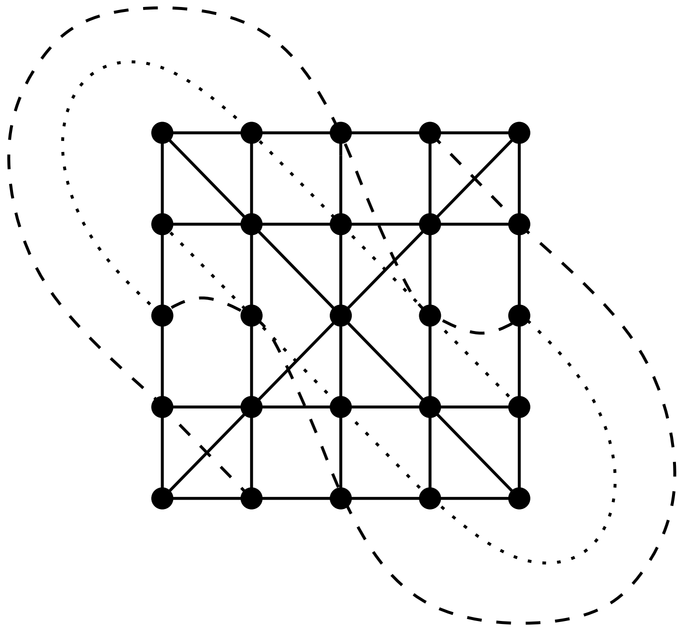


Figure 4.2: A configuration corresponding to $G \in RCA^*(25, 12, 5) \backslash RCA(25, 12, 5)$

(ii) each vertex of G *is in exactly* $\frac{d}{k-1}$ *different maximum cliques.*

Proof. Suppose that $G \in RCA^*(n, d, k)$. Then (i) holds because every edge of G is in exactly one maximum clique of G. For the same reason, the edges incident to any $v \in V(G)$ are partitioned into groups of $k-1$ each, one for each maximum clique containing v. Thus (ii) holds.

Now suppose that (i) and (ii) hold. Suppose that $e = uv \in E(G)$. Since v is of degree d and is in exactly $\frac{d}{k-1}$ different cliques of order $k = \omega(G)$, by (ii), and since these cliques are edge-disjoint, by (i), the edges of G incident to v must be partitioned into the sets of $k-1$ edges incident to v in cliques. Thus e belongs to exactly one maximum clique. Since $e \in E(G)$ was arbitrary, $G \in RCA^*(n, d, k)$. □

4.2 RCA**

Define $G \in RCA^{**}(n, d, k)$ if G is regular on n vertices, $k = \omega(G) \geq 2$, and every maximal clique of G is maximum. So each edge in G is in *at least* one maximum clique.

Proposition 4.3. *$RCA^{**}(n, d, k) \supseteq RCA(n, d, k)$, with equality when $k = 2$.*

Proof. Clearly $RCA^{**}(n, d, k) \supseteq RCA(n, d, k)$. Suppose $G \in RCA^{**}(n, d, 2)$ for some n and d. Since $\omega(G) = 2$, every edge is a maximum clique, so certainly every edge is in exactly one maximum clique. Thus $RCA^{**}(n, d, 2) = RCA(n, d, 2)$. $\square$

For $k \geq 3$ could $G \in RCA^{**}(n, d, k)$ contain an edge which is in more than one maximum clique? An example in which every edge is in two maximum cliques is given below.

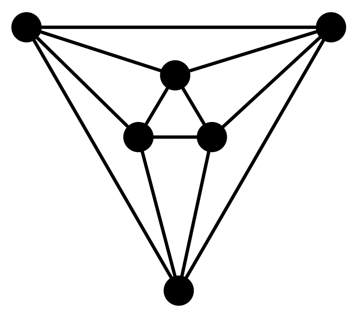


Figure 4.3: $G \in RCA^{**}(6, 4, 3) \backslash RCA(6, 4, 3)$

Note that while $G \notin ER(6, 4, 1) = RCA(6, 4, 3)$, G is, in fact, in $ER(6, 4, 2)$. If $G \in RCA^{**}(n, d, k)$, must all edges be in the same number of maximum cliques? Certainly not. In the graph below edges are in either one or three maximum cliques.

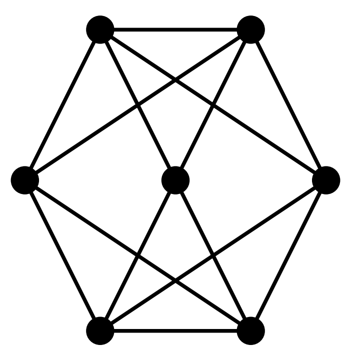


Figure 4.4: $2K_2 \vee 3K_1 \in RCA^{**}(7, 4, 3)$

Other graphs of this form are

$$mK_p \vee \left(\frac{p(m-1)}{q}+1\right) K_q \in RCA^{**}\left(p(2m-1)+q, mp+q-1, p+q\right).$$

If $p, q > 1$, different edges in one of these graphs could be in 1, m, or $\frac{p(m-1)}{q}+1$ maximal cliques.

4.3 Clique Assemblies

G is a *clique assembly* if $\omega(G) \geq 2$, and

(1) every maximal clique is maximum;

(2) every edge of G is in exactly one maximum clique of G.

If G is a clique assembly on n vertices with $k = \omega(G)$ we say $G \in CA(n, k)$. Clique assemblies are very easily constructed by connecting K_ks such that any two K_ks have at most one vertex in common and if any three K_ks are mutually adjacent, their intersection is a single vertex. Not only are all such graphs clique assemblies: it is easy to see that all clique assemblies are constructible in this way.

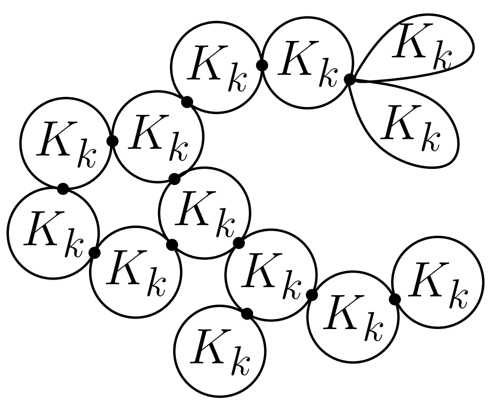


Figure 4.5: A clique assembly

Recall that the clique graph, $CL(G)$, is formed by replacing each maximal clique of G with a single vertex, and allowing two vertices in $CL(G)$ to be adjacent if and only if the

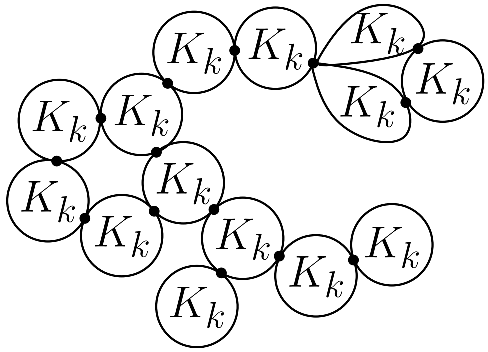


Figure 4.6: Not a clique assembly

corresponding cliques in G have at least one vertex in common. If $G \in CA(n,k)$ for some n and k, a clique of order m in $CL(G)$ necessarily corresponds to a set in G of m k-cliques which all have exactly one common vertex.

Among the clique assemblies:

1. Every triangle-free graph is in $CA(n,2)$ for some n.

2. The line graph of any triangle-free graph in which every vertex has degree either k or 1 and every component has at least one vertex of degree k is in $CA(n,k)$ for some n.

3. From a tree, T, on n vertices, we can construct $G \in CA(n(k-1)+1,k)$ by replacing each vertex in T with a clique of size k and for any adjacent vertices in T let the corresponding cliques in G share one vertex. (It does not matter which vertex is chosen, but different choices may give different resulting graphs G.)

4. From any graph, H, we can similarly construct $G \in CA(n,k)$ for some n, $k\,|V(H)| - |E(H)| \le n \le k\,|V(H)|$ by replacing each vertex in H with a clique of size k. For adjacent vertices in H, let the corresponding cliques in G share one vertex, and take care with any clique in H to let the corresponding set of cliques in G all have one vertex in common and be otherwise pairwise disjoint. Denote the clique in G induced by $v \in H$ by $C(v)$. Note that $uv \in E(H)$ implies that $C(u)$ and $C(v)$ have exactly one common vertex, but the converse is not true. In the example below, all cliques in

G must have a single vertex in common to satisfy the definition above, even though many of the corresponding vertices in H are not adjacent.

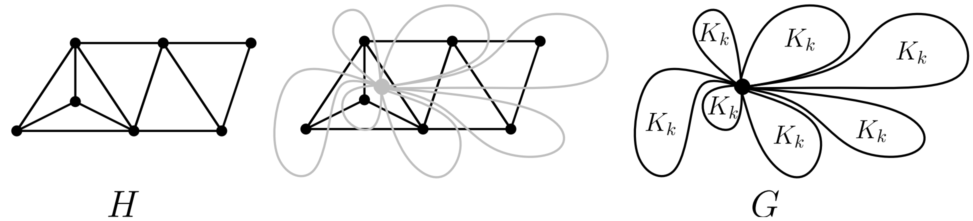


Figure 4.7: Due to the structure of H, there is a single vertex in G which is the intersection of every pair of maximal cliques in G.

Chapter 5

Conclusions and future work

Some open problems from each of the chapters as well as some concluding remarks are given.

5.1 Chapter 2

The discovery of the correspondence between regular clique assemblies and configurations has laid to rest many of the questions we might have posed, but it seems entirely likely that viewing these structures from a graph theoretic perspective could lend insight to questions in the future.

5.2 Chapter 3

1. Are there methods of constructing graphs in $ER(n, d, 1)$ for $d > 6$, $d \neq 10$ using a similar technique to the scaffold-building operations in 3.1.3 for $d = 6$ and in 3.1.4 for $d = 10$?

2. It is shown in Chapter 2 that if $G \in RCA(n, d, k)$ for some n, d, and $k \geq 2$, then $G \simeq CL(CL(G))$. Does this equation hold for any $G \in ER(n, d, \lambda) \backslash RCA(n, d, \lambda - 2)$, for some n, d, λ?

3. What else can we say about $ER(n, d, \lambda) \backslash RCA(n, d, \lambda - 2)$, for some n, d with $\lambda > 1$?

5.3 Chapter 4

1. For $X \in \{RCA^*, RCA^{**}\}$, for which (n, d, k) is $X(n, d, k)$ non-empty?

2. For $k \geq 5$ and $RCA^*(n, d, k) \neq \emptyset$, what orders of maximal cliques can occur in graphs in $RCA^*(n, d, k)$?

3. Is every graph H the clique graph of a clique assembly with clique number k, for k sufficiently large? If so, can the smallest k for which this happens be quickly determined by looking at H?

Bibliography

[1] K. Bragan, D. Hoffman, P. Johnson, Inferring regularity by a neglected theorem of Ryser, BICA 50 (2007), 47-51.

[2] A. E. Brouwer, Strongly Regular Graphs, Chapter VI.5 in *The CRC Handbook of Combinatorial Designs*, C. J. Colbourn, J. H. Dinitz, editors, CRC Press, New York, 1996, 667-685.

[3] R. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991, 152.

[4] Peter J. Cameron, Strongly Regular Graphs, Chapter 12 in *Selected Topics in Graph Theory*, Lowell W. Beineke and Robin J. Wilson, editors, Academic Press, 1978, 337-360.

[5] ——, Strongly regular graphs, Chapter 8 in *Topics in Algebraic Graph Theory*, Lowell W. Beineke and Robin J. Wilson, editors, Cambridge University Press, 2004, 203-221.

[6] K. Coolsaet, P. D. Johnson Jr., K. J. Roblee, and T. D. Smotzer, Some extremal problems for edge-regular graphs, *Ars Combinatoria* 105 (2012), 411-418.

[7] Ralph Faudree, Evelyne Flandrin, Zdeněk Ryjáček, Claw-free graphs — A survey, *Discrete Mathematics* 164 (13) (1997), 87147.

[8] Harald Gropp, Configurations, Chapter VI.7 in *The CRC Handbook of Combinatorial Designs*, 2nd edition, C. J. Colbourn, J. H. Dinitz, editors, CRC Press, New York, 2006, 352-354.

[9] Peter Johnson, Wendy Myrvold, and Kenneth Roblee, More extremal problems for edge-regular graphs, *Utilitas Mathematica* 73 (2007), 159-168.

[10] P. D. Johnson Jr. and K. J. Roblee, More extremal graphs for a maximum-joint-neighborhood, average-triangles-per-edge inequality, *Congressus Numerantium* 140 (1999), 87-95.

[11] ——, Non-existence of a nearly extremal family of edge-regular graphs, *Congressus Numerantium* 203 (2010), 161-166.

[12] ——, On an extremal subfamily of an extremal family of nearly strongly regular graphs, *Australasian Journal of Combinatorics* 25 (2002), 279-284.

[13] P. D. Johnson Jr., K. J. Roblee, and T. D. Smotzer, Another extremal family of edge-regular graphs, *Congressus Numerantium* 185 (2007), 61-64.

[14] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, 2nd edition, Cambridge University Press, 2001.

[15] Stephen C. Locke and Feng Lou, Finding independent set in K_4-free 4-regular connected graphs, *Journal of Combinatorial Theory, Series B* 71 (1997), 85-110.

[16] Michael W. Raney, On geometric trilateral-free (n_3) configurations, Ars Mathematica Contemporanea 6 (2013) 253259.

[17] K. J. Roblee and T. D. Smotzer, Some extremal families of edge-regular graphs, *European Journal of Combinatorics* 25 (2004), 927-933.

[18] Edward Spence, Strongly Regular Graphs on at most 64 vertices, http://www.maths.gla.ac.uk/∼es/