On the Countable Dense Homogeneity of Euclidean Spaces

by

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Abstract

A countable dense homogeneous space, in a general sense, is a topological space in which any two countable dense subsets of the space are “dispersed” the same way. In this thesis, we will show that some very well-known topological spaces, such as $n$-dimensional Euclidean space $\mathbb{R}^n$ and the $n$-sphere $S^n$ for all natural numbers $n$ is countable dense homogeneous.
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Table of Contents

Abstract ................................................................. ii
Acknowledgments ......................................................... iii
1  Preliminary Definition and Result ................................. 1
2  Countable Dense Homogeneity ...................................... 3
   2.1  \( \mathbb{R} \) is Countable Dense Homogeneous .................. 3
   2.2  \( \mathbb{Q} \) is not Countable Dense Homogeneous ................. 7
   2.3  \( \mathbb{R}^2 \) is Countable Dense Homogeneous ................. 8
   2.4  \( \mathbb{R}^n \) is Countable Dense Homogeneous .............. 11
   2.5  The \( n \)-sphere \( S^n \) is Countable Dense Homogeneous .... 14
Bibliography .............................................................. 16
Chapter 1
Preliminary Definition and Result

First, a definition and a lemma that we will use later.

**Definition 1.1.** Let \( f : A \to B \) and \( g : X \to Y \) be functions. The cartesian product of functions \( f \times g : A \times X \to B \times Y \) is defined by \((f \times g)(a, x) = (f(a), g(x))\). For a collection \( \{f_\alpha\}_{\alpha \in \Lambda} \) of functions, let \( \prod_{\alpha \in \Lambda} f_\alpha \) denote their cartesian product.

**Lemma 1.2.** Let \( \Lambda \) be an indexing set and \( f_\alpha : X_\alpha \to Y_\alpha \) be a homeomorphism for all \( \alpha \in \Lambda \). Then \( f = \prod_{\alpha \in \Lambda} f_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \to \prod_{\alpha \in \Lambda} Y_\alpha \) is a homeomorphism.

**Proof.** Let \( U \subset \prod_{\alpha \in \Lambda} Y_\alpha \) be an open set and \( \bar{x} \in f^{-1}(U) = (\prod_{\alpha \in \Lambda} f_\alpha)^{-1}(U) = (\prod_{\alpha \in \Lambda} f_\alpha^{-1})(U) \). Then there is a basic open set \( \prod_{\alpha \in \Lambda} U_\alpha \) in \( \prod_{\alpha \in \Lambda} Y_\alpha \) lying in \( U \) and containing \( f(\bar{x}) \). Thus \( \bar{x} \in f^{-1}(\prod_{\alpha \in \Lambda} U_\alpha) = \prod_{\alpha \in \Lambda} f_\alpha^{-1}(U_\alpha) \subset f^{-1}(U) \), where \( f_\alpha^{-1}(U_\alpha) \) is open for all \( \alpha \in \Lambda \) since \( f_\alpha \) is continuous for all \( \alpha \in \Lambda \). Therefore, \( f^{-1}(U) \) is open. Hence \( f \) is continuous. By a similar argument, \( f^{-1} \) is continuous.

We show injectivity. Let \( \bar{x}, \bar{y} \in \prod_{\alpha \in \Lambda} X_\alpha \) such that \( f(\bar{x}) = f(\bar{y}) \). Then \( f((x_\alpha)_{\alpha \in \Lambda}) = f((y_\alpha)_{\alpha \in \Lambda}) \), which implies \( (f_\alpha(x_\alpha))_{\alpha \in \Lambda} = (f_\alpha(y_\alpha))_{\alpha \in \Lambda} \). Thus \( f_\alpha(x_\alpha) = f_\alpha(y_\alpha) \) for all \( \alpha \in \Lambda \). Then \( f_\alpha \) being injective for all \( \alpha \in \Lambda \) implies \( x_\alpha = y_\alpha \) for all \( \alpha \in \Lambda \). Thus \( (x_\alpha)_{\alpha \in \Lambda} = (y_\alpha)_{\alpha \in \Lambda} \), or \( \bar{x} = \bar{y} \). This shows that \( f \) is injective.
Now, let $\vec{y} = (y_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} Y_\alpha$. Then $y_\alpha \in Y_\alpha$ for all $\alpha \in \Lambda$, and since $f_\alpha$ is surjective for all $\alpha \in \Lambda$, there exists an $x_\alpha \in X_\alpha$ such that $f_\alpha(x_\alpha) = y_\alpha$. It follows that there exists an $\vec{x} = (x_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} X_\alpha$ such that $f(\vec{x}) = f((x_\alpha)_{\alpha \in \Lambda}) = (f_\alpha(x_\alpha))_{\alpha \in \Lambda} = (y_\alpha)_{\alpha \in \Lambda} = \vec{y}$. \qed
Chapter 2
Countable Dense Homogeneity

Definition 2.1. A separable space $X$ is said to be countable dense homogeneous if, given any two countable dense sets $C, D \subset X$, there exists an autohomeomorphism $f$ of $X$ such that $f(C) = D$. For brevity, we will occasionally abbreviate “countable dense homogeneous” with CDH.

2.1 $\mathbb{R}$ is Countable Dense Homogeneous

First, an important lemma.

Lemma 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing surjection. Then $f$ is continuous.

Proof. Suppose $f$ is discontinuous. Then there is an $a \in \mathbb{R}$ at which $f$ is discontinuous. We have three cases for a discontinuity at $a$:

(i) Suppose $L = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, but $f(a) \neq L$. Assume $f(a) < L$. Since $\lim_{x \to a^-} f(x) = L > f(a)$, there is a neighborhood of $L$ that contains an $f(x)$ for $x < a$ for which $f(x) > f(a)$. This is a contradiction to $f$ being strictly increasing.

The proof when assuming $f(a) > L$ is similar.

(ii) Suppose $L = \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) = R$. Since $f$ is surjective, for every $y \in [L, R]$ there exists an $x_y \in \mathbb{R}$ such that $y = f(x_y)$. It follows that there exists some (and in fact, infinitely many) $y \in [L, R]$ such that $x_y < a$ or $a < x_y$. Suppose $x_y < a$. Since $f$ is strictly increasing, for every neighborhood $N$ of $a$ and every $x \in N$ such that $x < a$, $f(x) < L$. However, $x_y < a$, but $f(x_y) = y \geq L$, a contradiction.
The proof when assuming $a < x$ is similar.

(iii) Suppose one of the limits, say $L = \lim_{x \to a^-} f(x)$, does not exist or is infinite. If it is not infinite, then the set $\{f(x) \mid x < a\}$ is bounded above. However, since $f$ is strictly increasing, there must be a limit. This is a contradiction. Therefore, we suppose $L$ is infinite. If $L = \infty$, then for any $x > a$, there will exist an $x' < a$ such that $f(x') > f(x)$, contradicting $f$ being strictly increasing. Clearly, $L \neq -\infty$, for then that would say that $f$ were decreasing.

The proof when assuming $R = \lim_{x \to a^+} f(x)$ does not exist or is infinite is similar.

Thus the set of discontinuities of $f$ is empty. Therefore, $f$ is continuous.

Theorem 2.3. $\mathbb{R}$ is countable dense homogeneous.

Proof. Let $C = \{c_i\}_{i \in \mathbb{N}}$ and $D = \{d_i\}_{i \in \mathbb{N}}$ be countable dense subsets of $\mathbb{R}$. We will re-order the elements of $C$ and $D$ in the following manner:

Set $c'_0 = c_0$, and set $\Gamma_0 = \mathbb{R} \setminus \{c'_0\}$. For $i \in \mathbb{N}$, we call the connected components of $\Gamma_i = \mathbb{R} \setminus \{c'_j\}_{j=0}^i$ “cells” for brevity. Then for $i \geq 0$, we recursively define $c'_{i+1}, c'_{i+2}, \ldots, c'_{2(i+1)}$ for each of the cells $C_{i+1}, C_{i+2}, \ldots, C_{2(i+1)}$, respectively, of $\Gamma_i = \mathbb{R} \setminus \{c'_j\}_{j=0}^i$.

For $i = 0$, we have $\Gamma_0 = \mathbb{R} \setminus \{c'_0\} = (-\infty, c'_0) \cup (c'_0, \infty)$, so we can say, for example, that $C_1 = (-\infty, c'_0)$ and $C_2 = (c'_0, \infty)$ (Note that we could have interchanged them.). Then $c_1$ belongs to either $C_1$ or $C_2$. If the former occurs, then we set $c'_1 = c_1$. Otherwise, we set $c'_1 = c_m$, where $m = \min\{k \in \mathbb{N} \mid c_k \in C_1\}$, and $c'_2 = c_1$, etc.

For $i \geq 0$, suppose $\Gamma_i$ has been defined. Let $C_n, i + 1 \leq n \leq 2(i + 1)$, be the cell of $\Gamma_i$ in which we intend to place $c'_n$. Let $* : C \to C$ be defined by $*(c'_n) = c_m \Leftrightarrow c'_n = c_m$, where $m = \min\{k \in \mathbb{N} \mid c_k \in C_n\}$.

We claim that $*$ is a bijection.
Suppose $c'_m \neq c'_n$. Then either $m < n$ or $m > n$, so assume $m < n$. It’s obvious that $c'_n \in \Gamma_m$, so that $*(c'_n) \neq *(c'_m)$. Thus $*$ is injective.

Now, let $c_m \in C$. Clearly, for some $N < m$, $m = \min\{k \in \mathbb{N} \mid c_k \in C, \ C \text{ is some cell of } \Gamma_N\}$, in which case there exists an $n \in \mathbb{N}$ such that $*(c'_n) = c_m$. Thus $*$ is surjective, and this proves the claim.

We do the same for $D$ and impart on it the same ordering scheme used to order the $c'_i$s. That is, for $i, j \in \mathbb{N}$, if $C_i$ and $C_j$ are cells of $\Gamma_n$ for some $n \in \mathbb{N}$ with $c'_i \in C_i$ and $c'_j \in C_j$ such that $C_i < C_j$, then the cells $D_i$ and $D_j$ in which we intend to place $d'_i$ and $d'_j$, respectively, of $\Delta_n = \mathbb{R}\{d'_k\}_{k=0}^n$ also satisfy $D_i < D_j$.

Therefore, we conclude that if

\[
\begin{align*}
    c'_0 \\
    c'_1 &< c'_0 < c'_2 \\
    c'_3 &< c'_1 < c'_4 &< c'_0 < c'_5 < c'_2 < c'_6 \\
    \vdots
\end{align*}
\]

was the ordering scheme used to order the $c'_i$s, then we should (and do) also obtain

\[
\begin{align*}
    d'_0 \\
    d'_1 &< d'_0 < d'_2 \\
    d'_3 &< d'_1 < d'_4 < d'_0 < d'_5 < d'_2 < d'_6. \\
    \vdots
\end{align*}
\]

Then it is immediate that $c'_i < c'_j \Leftrightarrow d'_i < d'_j$.

Now, due to readability, we shall revert to the original notation used for the elements of $C$ and $D$, i.e. $c'_i \rightarrow c_i$ and $d'_i \rightarrow d_i$ for all $i$. 

5
There is some element $c_m$ of $C$ that is greater than $x$ and so bounds the set \{ $c_i \mid c_i < x$ \}. Then by the construction $d_m$ is an upper bound of the set \{ $d_i \mid c_i < x$ \} and so it has a supremum. Therefore, we define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sup\{d_i \mid c_i < x\}.$$ 

We want to show that $f$ is a homeomorphism.

Claim: $f$ is strictly increasing.

Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then $f(x) = \sup\{d_i \mid c_i < x\}$ and $f(y) = \sup\{d_i \mid c_i < y\}$. Since $x < y$, \{ $d_i \mid c_i < x$ \} $\subsetneq$ \{ $d_i \mid c_i < y$ \}. Furthermore, since $D$ is dense, there is a $d \in D \cap (x, y)$, and thus it follows that $d \in \{d_i \mid c_i < y\}$ but $d \notin \{d_i \mid c_i < x\}$. Therefore, $d$ is an upper bound for \{ $d_i \mid c_i < x$ \} but not an upper bound for \{ $d_i \mid c_i < y$ \}. Thus $f(x) = \sup\{d_i \mid c_i < x\} < \sup\{d_i \mid c_i < y\} = f(y)$.

This proves the claim, and it also implies $f$ is injective.

Claim: $f$ is surjective.

Let $y \in \mathbb{R}$. Then $y$ is an upper bound for the set $D' = \{d_i \mid d_i < y\}$. Let $y^* \in \mathbb{R}$ such that $d_i \leq y^*$ for all $d_i \in D'$. That is, let $y^*$ be an upper bound for $D'$. Suppose $y^* < y$. Then $(y^*, y)$ is open and thus contains an element $d$ of $D$ due to the denseness of $D$. Moreover, $d < y$, implying $d \in D'$ and showing that $y^* < d \in D'$, a contradiction. Therefore, $y^* \geq y$, implying $y = \sup D'$. Set $x = \sup\{c_i \mid d_i < y\}$. It follows that $y = \sup D' = \sup\{d_i \mid c_i < x\} = f(x)$. This proves the claim and furthermore shows that $f$ is a bijection.
By the previous lemma, since \( f \) is a strictly increasing surjection from \( \mathbb{R} \) to \( \mathbb{R} \), \( f \) is continuous.

Moreover, since \( f \) is a bijection, it has an inverse. We claim that

\[
g(y) = \sup\{c_j \mid d_j < y\}
\]

is this inverse.

Let \( y \in \mathbb{R} \). Let \( D^* = \{d_i \mid c_i < \sup\{c_j \mid d_j < y\}\} \). Then \( y \) is an upper bound of \( D^* \). Let \( y^* \in \mathbb{R} \) be an upper bound of \( D^* \). Suppose \( y^* < y \). Then \((y^*, y)\) is open and thus contains an element \( d_k \) of \( D \) due to the denseness of \( D \). Moreover, since \( d_k < y \), \( c_k \in \{c_j \mid d_j < y\} \), implying \( c_k < \sup\{c_j \mid d_j < y\} \) by considering the open interval \((d_k, y)\) and another element \( d_n \) of \( D \) such that \( d_k < d_n < y \), i.e. \( c_k < c_n \in \{c_j \mid d_j < y\} \). Thus \( d_k \in D^* \). However, this says that \( y^* < d_k \in D^* \), which is a contradiction to \( y^* \) being an upper bound of \( D^* \).

Therefore, \( y^* \geq y \), implying \( y = \sup D^* \). Thus

\[
f(g(y)) = f(\sup\{c_j \mid d_j < y\}) = \sup\{d_i \mid c_i < \sup\{c_j \mid d_j < y\}\} = \sup D^* = y,
\]

which shows that \( g = f^{-1} \). Clearly \( g \) is continuous by the same reasoning that \( f \) is continuous.

Furthermore, it’s readily seen that for any \( c_j \in C \), \( f(c_j) = \sup\{d_i \mid c_i < c_j\} = d_j \). Similarly, for any \( d_i \in D \), \( f^{-1}(d_i) = c_i \). Therefore, we have \( f(C) \subseteq D \) and \( C \supseteq f^{-1}(D) \), respectively, so we have \( f(C) \subseteq D \) and \( f(C) \supseteq f(f^{-1}(D)) = D \). Hence \( f(C) = D \).

Therefore, \( f \) is a homeomorphism such that \( f(C) = D \). We conclude that \( \mathbb{R} \) is countable dense homogeneous.

\[\square\]

2.2 \( \mathbb{Q} \) is not Countable Dense Homogeneous

We will show that \( \mathbb{Q} \) is not countable dense homogeneous, but first, we will show the more general result that countable dense spaces are not countable dense homogeneous.
Theorem 2.4. Countable dense spaces are not countable dense homogeneous.

Proof. Let $X$ be a countable dense space, and let $x \in X$. Suppose $X$ is CDH. Then since $X$ and $X \setminus \{x\}$ are countable dense subsets of $X$, there exists a homeomorphism $f : X \to X$ such that $f(X \setminus \{x\}) = X$. Thus there exists an $x' \in X \setminus \{x\}$ such that $f(x) = f(x')$, a contradiction to $f$ being injective. Thus $X$ is not CDH.

Corollary 2.5. $\mathbb{Q}$ is not CDH.

2.3 $\mathbb{R}^2$ is Countable Dense Homogeneous

Theorem 2.6. $\mathbb{R}^2$ is countable dense homogeneous.

Proof. Let $C, D \subset \mathbb{R}^2$ be countable dense. One can check that for $a, b \in \mathbb{R}^2$, the set $R^i(a, b) = \{\rho : \rho$ is a rotation of $\mathbb{R}^n$ such that $\pi_i(\rho(a)) = \pi_i(\rho(b))\}$ is countable. Thus $\bigcup_{x,y \in C} R^i(x, y)$ and $\bigcup_{w,z \in D} R^i(w, z)$ are countable for $i = 1, 2$ since $C$ and $D$ are countable. Then $\left(\bigcup_{i=1}^{2} \bigcup_{x,y \in C} R^i(x, y)\right) \cup \left(\bigcup_{i=1}^{2} \bigcup_{w,z \in D} R^i(w, z)\right)$ is also countable being a countable union of countable sets. Since there are uncountably many rotations of $\mathbb{R}^2$, there exists a rotation $R$ of $\mathbb{R}^2$ that is different from any of these rotations so that $\pi_i(R(x)) \neq \pi_i(R(y))$, $\pi_i(R(w)) \neq \pi_i(R(z))$ for any $1 \leq i \leq 2$ and for all $x, y \in C$ and $w, z \in D$. It follows that no vertical or horizontal line intersects $R(C)$ or $R(D)$ at more than one point. Moreover, since $R$ is a homeomorphism, $R(C)$ and $R(D)$ are countable dense. We set $R(C) = \{(a_i, b_i)\}_{i=0}^{\infty}$ and $R(D) = \{(r_i, s_i)\}_{i=0}^{\infty}$.

Now, for $(x_1, x_2) \in \mathbb{R}^2$, let $V_{x_1}$ and $H_{x_2}$ be the vertical and horizontal lines in the plane through $(x_1, x_2)$, respectively. Set $(a_0', b_0') = (a_0, b_0)$ and $(a_1', b_1') = (a_1, b_1)$. For $n \in \mathbb{N}$, $\mathbb{R}^2 \setminus \{V_{a_0'} \cup H_{b_0'}\} = C^1_n \cup C^2_n \cup C^3_n \cup C^4_n$, where $C^1_n = \{(x,y) \mid x > a_0', y > b_0'\}$, $C^2_n = \{(x,y) \mid x < a_0', y > b_0'\}$, $C^3_n = \{(x,y) \mid x < a_1', y < b_1'\}$, and $C^4_n = \{(x,y) \mid x > a_1', y < b_1'\}$. Set $(r_0', s_0') = (r_0, s_0)$. For $n \in \mathbb{N}$, $\mathbb{R}^2 \setminus \{V_{r_0'} \cup H_{s_0'}\} = D^1_n \cup D^2_n \cup D^3_n \cup D^4_n$, where the $D^i_n$ are defined similar to the $C^i_n$. 

8
Then \((a'_1, b'_1) \in C_0^{n_0}\) for some \(1 \leq n_0 \leq 4\), and consequently, we set \((r'_1, s'_1) = (r_{d_1}, s_{d_1})\), where \(d_1 = \min\{m \in \mathbb{N} \mid (r_m, s_m) \in D_0^{n_0}\}\). Set \((r'_2, s'_2) = (r_{d_2}, s_{d_2})\), where \(d_2 = \min\{m \in \mathbb{N} \mid (r_m, s_m) \neq (r'_j, s'_j)\text{ for any } j < 2\}\). Then there exist \(n_0, n_1 \in \{1, 2, 3, 4\}\) such that \((r'_2, s'_2) \in D_0^{n_0} \cap D_1^{n_1}\). Set \((a'_2, b'_2) = (a_{c_2}, b_{c_2})\), where \(c_2 = \min\{m \in \mathbb{N} \mid (a_m, b_m) \in C_0^{n_0} \cap C_1^{n_1}\}\).

After this, set \((a'_3, b'_3) = (a_{c_3}, b_{c_3})\), where \(c_3 = \min\{m \in \mathbb{N} \mid (a_m, b_m) \neq (a'_j, b'_j)\text{ for any } j < 3\}\). Observe by this construction that \((a'_i, b'_i) \in \bigcap_{k=0}^{i-1} C_k^{n_k}\) if and only if \((r'_i, s'_i) \in \bigcap_{k=0}^{i-1} D_k^{n_k}\). By the construction, we see that for any \(i, j \in \{0, 1, 2\}\), \(a'_i < a'_j \Leftrightarrow r'_i < r'_j\) and \(b'_i < b'_j \Leftrightarrow s'_i < s'_j\).
We continue the process above and assume that for some odd \( i > 0 \), \((a_0', b_0'), (a_1', b_1'), \ldots, (a_i', b_i') \) and \((r_0', s_0'), \ldots, (r_{i-1}', s_{i-1}')\) have been defined as above. Then there exist \( n_0, n_1, \ldots, n_{i-1} \in \{1, 2, 3, 4\} \) such that \((a_i', b_i') \in \bigcap_{j=0}^{i-1} C_j^{n_j} \). Set \((r_i', s_i') = (r_{d_i}, s_{d_i})\), where \(d_i = \min\{m \in \mathbb{N} | (r_m, s_m) \in \bigcap_{j=0}^{i-1} D_j^{n_j} \}\). After this, set \((r_{i+1}', s_{i+1}') = (r_{d_{i+1}}, s_{d_{i+1}})\), where \(d_{i+1} = \min\{m \in \mathbb{N} | (r_m, s_m) \neq (r_j', s_j') \text{ for any } j < i + 1 \}\). Then there exist \( n_0, n_1, \ldots, n_i \in \{1, 2, 3, 4\} \) such that \((r_{i+1}', s_{i+1}') \in \bigcap_{j=0}^i D_j^{n_j} \). Then set \((a_{i+1}', b_{i+1}') = (a_{c_{i+1}}, b_{c_{i+1}})\), where \(c_{i+1} = \min\{m \in \mathbb{N} | (a_m, b_m) \in \bigcap_{j=0}^i C_j^{n_j} \}\). After this, set \((a_{i+2}', b_{i+2}') = (a_{c_{i+2}}, b_{c_{i+2}})\), where \(c_{i+2} = \min\{m \in \mathbb{N} | (a_m, b_m) \neq (a_j', b_j') \text{ for any } j < i + 2 \}\). Thus \((a_0', b_0'), (a_1', b_1'), \ldots, (a_i', b_i'), (a_{i+1}', b_{i+1}'), (a_{i+2}', b_{i+2}')\) and \((r_0', s_0'), \ldots, (r_{i-1}', s_{i-1}'), (r_i', s_i'), (r_{i+1}', s_{i+1}')\) have been defined, so we continue repeating this process and proceed by induction to define \((a_j', b_j')\) and \((r_j', s_j')\) for all \( j \in \mathbb{N} \).

Clearly the above assignment is a bijection on \( R(C) \) and \( R(D) \). Moreover, we claim that, for \( i, j \in \mathbb{N}, a'_i < a'_j \iff r'_i < r'_j \) and \( b'_i < b'_j \iff s'_i < s'_j \). To that end, we assume \( i > j \) and suppose that for \((a'_i, b'_i), (a'_j, b'_j) \in R(C) \) and \((r'_i, s'_i), (r'_j, s'_j) \in R(D) \), \( a'_i < a'_j \) and \( b'_i < b'_j \). Then \((a'_i, b'_i) \in C_j^3 \). Thus \((r'_i, s'_i) \in D_j^3 \) by the construction above, showing \( r'_i < r'_j \) and \( s'_i < s'_j \).

The rest of the proof is done similarly. Hence the claim is true.

Due to readability, we shall revert to the original notation used for the elements of \( R(C) \) and \( R(D) \). I.e., \((a_i', b_i') \rightarrow (a_i, b_i)\) and \((r_i', s_i') \rightarrow (r_i, s_i)\) for all \( i \).
Set $C_n := \pi_n R(C)$ and $D_n := \pi_n R(D)$ for $1 \leq n \leq 2$. Then $C_n$ and $D_n$ are countable dense subsets of $\mathbb{R}$ since $\pi_n$ is a continuous surjection for $1 \leq n \leq 2$. For $i, j \in \mathbb{N}$, $a_i < a_j \Leftrightarrow r_i < r_j$. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) = \sup \{r_m \mid a_m < x\}$. Then $f_1$ is a homeomorphism such that $f_1(C_1) = D_1$. More importantly, $f_1(a_i) = r_i$ for all $i \in \mathbb{N}$. Similarly, $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_2(y) = \sup \{s_m \mid b_m < y\}$ is a homeomorphism such that $f_2(C_2) = D_2$ with $f_2(b_i) = s_i$ for all $i \in \mathbb{N}$.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (f_1(x), f_2(y))$. Then $f$ is a homeomorphism by a previous theorem. We claim that $f(R(C)) = R(D)$. Clearly $f(R(C)) \subseteq R(D)$. Now, for $(r_i, s_i) \in R(D)$, since $r_i = f_1(a_i)$ and $s_i = f_2(b_i)$, $(r_i, s_i) = (f_1(a_i), f_2(b_i)) = f(a_i, b_i)$. Thus $f(R(C)) \supseteq R(D)$, proving the claim. Therefore, $R^{-1} f R(C) = R^{-1} R(D) = D$. Since $R^{-1} f R$ is a homeomorphism, $\mathbb{R}^2$ is countable dense homogeneous.

2.4 $\mathbb{R}^n$ is Countable Dense Homogeneous

For the following proof, it is necessary that we define some terminology.

**Definition 2.7.** By a hyperplane of $\mathbb{R}^n$ we mean a set of the form $S + x$, where $S$ is a subspace of $\mathbb{R}^n$ of dimension $n - 1$ and $x \in \mathbb{R}^n$.

**Definition 2.8.** By the $i^{th}$ hyperplane axis $A$ of $\mathbb{R}^n$ we mean the set $A = \{(x_1, x_2, \ldots, x_n) \mid x_i = 0\}$.

**Definition 2.9.** Let $H$ be a hyperplane parallel to a hyperplane axis in $\mathbb{R}^n$. By a half-space of $H$ we mean one of the two open sets whose union is $\mathbb{R}^n \setminus H$. 

11
Definition 2.10. By an $n$-orthant of the point $\vec{x} \in \mathbb{R}^n$ we mean the nonempty intersection of a collection of $n$ half-spaces that do not contain $\vec{x}$. An orthant of $\vec{x} \in \mathbb{R}^n$ is thus necessarily open being a finite intersection of open sets. Let $O_x^i$ denote the $i^{th}$ orthant of $\vec{x}$. It follows that there are $2^n$ orthants for each $\vec{x} \in \mathbb{R}^n$.

Theorem 2.11. $\mathbb{R}^n$ is countable dense homogeneous.

Proof. We’ve already shown that $\mathbb{R}$ and $\mathbb{R}^2$ are CDH, so the cases $n = 1$ and $n = 2$ are done. We now prove that $\mathbb{R}^n$ for $n > 2$ is CDH by using a generalization of our argument for $\mathbb{R}^2$. To that end, assume $n > 2$. Let $C, D \subset \mathbb{R}^n$ be countable dense. Let $\vec{x}$ denote the point $(x_j)_{j=1}^n \in \mathbb{R}^n$. For $1 \leq i \leq n$, define $p_i : \mathbb{R}^n \to \mathbb{R}^n$ by $p_i(\vec{x}) = (q_i(x_j))_{j=1}^n$, where $q_i(x_j) = x_j$ for $i \neq j$ and $q_i(x_j) = 0$ for $i = j$. That is, $p_i (1 \leq i \leq n)$ projects $\mathbb{R}^n$ onto the $(n-1)$-dimensional subspace $\{\vec{y} \in \mathbb{R}^n \mid \pi_i(\vec{y}) = 0\}$ of $\mathbb{R}^n$, which is homeomorphic to $\mathbb{R}^{n-1}$.

We now repeat the procedure in the previous proof. For each pair $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $1 \leq i \leq n$, the set $R_i(\vec{x}, \vec{y}) = \{\rho : \rho \text{ is a rotation of } \mathbb{R}^n \text{ such that } p_i(\rho(\vec{x})) = p_i(\rho(\vec{y}))\}$ is countable. Therefore $\bigcup_{i=1}^n \bigcup_{\vec{y} \in C} R_i(\vec{x}, \vec{y})$ and $\bigcup_{i=1}^n \bigcup_{\vec{z} \in D} R_i(\vec{w}, \vec{z})$ are countable since $C$ and $D$ are countable. Thus $\left(\bigcup_{i=1}^n \bigcup_{\vec{y} \in C} P_i(\vec{x}, \vec{y})\right) \cup \left(\bigcup_{i=1}^n \bigcup_{\vec{z} \in D} P_i(\vec{w}, \vec{z})\right)$ is also countable. Since there are uncountably many rotations of $\mathbb{R}^n$, there exists a rotation $R$ of $\mathbb{R}^n$ that is different from any of these rotations so that $p_i(R(\vec{x})) \neq p_i(R(\vec{y}))$ and $p_i(R(\vec{w})) \neq p_i(R(\vec{z}))$ for all $1 \leq i \leq n$, $\vec{x}, \vec{y} \in C$ and $\vec{w}, \vec{z} \in D$. It follows that no hyperplane parallel to any hyperplane axis intersects $R(C)$ or $R(D)$ at more than one point. Moreover, since $R$ is a homeomorphism, $R(C)$ and $R(D)$ are countable dense. We set $R(C) = \{\vec{a}_m\}_{m=0}^\infty$ and $R(D) = \{\vec{b}_m\}_{m=0}^\infty$.

Now, for $\vec{x} \in \mathbb{R}^n$, let $\mathcal{H}_x$ be the union of all hyperplanes containing $\vec{x}$ that are parallel to a hyperplane axis. Then $\mathbb{R}^n \setminus \mathcal{H}_x = \bigcup_{k=1}^{2^n} \mathcal{O}_x^k$, where $\mathcal{O}_x^k$ is the $k^{th}$ $n$-orthant of $\vec{x}$. Set $\vec{a}_0 = \vec{a}_0', \vec{a}_1 = \vec{a}_1$, and $\vec{b}_0 = \vec{b}_0$. Then $\vec{a}_1' = \mathcal{O}^n_{\vec{a}_0'}$ for some $1 \leq n_0 \leq 2^n$, and consequently, we set $\vec{b}_1' = \mathcal{O}^n_{\vec{b}_0}$, where $d_1 = \min\{m \in \mathbb{N} \mid \vec{b}_m \in \mathcal{O}^n_{\vec{b}_0}\}$. Set $\vec{b}_2' = \mathcal{O}^n_{\vec{b}_1'}$, where $d_2 = \min\{m \in \mathbb{N} \mid \vec{b}_m \neq \vec{b}_\ell' \text{ for any } \ell < 2\}$. Then there exist $n_0, n_1 \in \{1, \ldots, 2^n\}$ such that $\vec{b}_2' \in \mathcal{O}^n_{\vec{b}_0} \cap \mathcal{O}^n_{\vec{a}_1}$. Set $\vec{a}_2 = \vec{a}_c_2$, where $c_2 = \min\{m \in \mathbb{N} \mid \vec{a}_m \in \mathcal{O}^n_{\vec{a}_0} \cap \mathcal{O}^n_{\vec{a}_1}\}$. After this, set $\vec{a}_3 = \vec{a}_c_3$, where $c_3 = \min\{m \in \mathbb{N} \mid \vec{a}_m \neq \vec{a}_\ell' \text{ for any } \ell < 3\}$. By this construction, observe that for any
$1 \leq i \leq n$, $0 \leq j \leq 2$, and $1 \leq k \leq 2^{n-1}$, $p_i(\vec{a}_j) \in O_{\ell}(x_{i,j})$ if and only if $p_i(\vec{b}_j) \in O_{\ell}(x_{i,j})$ for some $\ell$, where $O_{\ell}(x_{i,j})$ and $O_{\ell}(x_{i,j})$ are the $k$th orthants of the points $p_i(\vec{a}_j)$ and $p_i(\vec{b}_j)$ in $\mathbb{R}^{n-1}$.

Also by the construction, we see that $\vec{a}_i > \vec{a}_n$, and so we continue repeating this process and proceed by induction to define $\vec{a}_n$ used for the elements of $R$ since $\vec{a}_n$ is defined by a previous proof, so $\vec{b}_m \in O_{\ell}(x_{i,j})$ if and only if $\vec{b}_m \in O_{\ell}(x_{i,j})$.

As in the previous proof, we can continue the process above and assume that for some odd $i > 0$, $\vec{a}_0', \vec{a}_1', \ldots, \vec{a}_m'$ and $\vec{b}_0', \ldots, \vec{b}_m'$ have been defined as above. By a similar proof to the previous proof at this stage, we see that $\vec{a}_0', \vec{a}_1', \ldots, \vec{a}_m', \vec{a}_m'+1, \vec{a}_m'+2$ and $\vec{b}_0', \ldots, \vec{b}_m'+1$ can be defined, so we continue repeating this process and proceed by induction to define $\vec{a}_m'$ and $\vec{b}_m'$ for all $m \in \mathbb{N}$. By similar proofs to the ones for $\mathbb{R}$ and $\mathbb{R}^2$ being CDH, this assignment is a bijection on $R(C)$ and $R(D)$. Due to readability, we shall revert to the original notation used for the elements of $R(C)$ and $R(D)$. I.e., $\vec{a}_m' \rightarrow \vec{a}_m$ and $\vec{b}_m' \rightarrow \vec{b}_m$ for all $m$.

Set $C_i := p_i R(C)$ and $D_i := p_i R(D)$ $(1 \leq i \leq n)$. Then for $1 \leq i \leq n$, $C_i$ and $D_i$ are countable dense subsets of $\mathbb{R}^{n-1}$ since $p_i$ is a continuous surjection onto $\mathbb{R}^{n-1}$. By induction, $\mathbb{R}^{n-1}$ is CDH, so for each $i$ $(1 \leq i \leq n)$ there exists a homeomorphism $f^i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ such that $f^i(C_i) = D_i$. Pick $f^1$ to be one of these. It is defined inductively by $f^1((x_j^{n-1})_{j=1}^{n-1}) = (f_j(x_j))_{j=1}^{n-1}$, where $f_j : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f_j(x_j) = \sup\{\pi_j(\vec{b}_m) \mid \pi_j(\vec{a}_m) < x_j\}$ for $1 \leq j \leq n - 1$ is a homeomorphism. Define $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $f(x) = (f^1((x_j^{n-1})_{j=1}^{n-1}), f_n(x_n)) = ((f_j(x_j))_{j=1}^{n-1}, f_n(x_n))$, where $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f_n(x_n) = \sup\{\pi_n(\vec{b}_m) \mid \pi_n(\vec{a}_m) < x_n\}$.

Then $f_n$ is a homeomorphism by a previous proof, so $f$ is a homeomorphism by a previous proof. Moreover, $f_n(\pi_n(\vec{a}_i)) = \pi_n(\vec{b}_i)$ for all $i \in \mathbb{N}$.

We claim that $f(R(C)) = R(D)$. Let $\vec{a}_i \in R(C)$. Then $f(\vec{a}_i) = ((f_j(\pi_j(\vec{a}_i)))_{j=1}^{n-1}, f_n(\pi_n(\vec{a}_i))) = ((\pi_j(\vec{b}_i))_{j=1}^{n-1}, f_n(\pi_n(\vec{a}_i))) = \vec{b}_i \in R(D)$. Thus $f(R(C)) \subseteq R(D)$. Also, for $\vec{b}_i \in R(D)$, since $\pi_j(\vec{b}_i) = f_j(\pi_j(\vec{a}_i))$ for all $j$,

$$\vec{b}_i = (\pi_j(\vec{b}_i))_{j=1}^{n-1} = (f_j(\pi_j(\vec{a}_i)))_{j=1}^{n-1} = ((f_j(\pi_j(\vec{a}_i)))_{j=1}^{n-1}, f_n(\pi_n(\vec{a}_i)))$$

$$= (f^1((\pi_j(\vec{a}_i))_{j=1}^{n-1}), f_n(\pi_n(\vec{a}_i))) = f(\vec{a}_i).$$
Thus $f(R(C)) \supset R(D)$, proving the claim. Therefore, $R^{-1}fR(C) = D$. Since $R^{-1}fR$ is a homeomorphism, $\mathbb{R}^n$ is countable dense homogeneous.

2.5 The $n$-sphere $S^n$ is Countable Dense Homogeneous

**Lemma 2.12.** Let $X$ be CDH. Then any space $Y$ homeomorphic to $X$ is also CDH. (I.e., CDH is a topological property.)

**Proof.** Let $Y$ be a space homeomorphic to $X$. Then there is a homeomorphism $f : Y \to X$. Let $C$ and $D$ be countable dense subsets of $Y$. Then $f(C)$ and $f(D)$ are countable dense subsets of $X$. Since $X$ is CDH, there is a homeomorphism $g : X \to X$ such that $g(f(C)) = f(D)$. Then $f^{-1}gf \in \mathcal{H}(Y)$, and

$$(f^{-1}gf)(C) = f^{-1}(g(f(C))) = f^{-1}(f(D)) = D,$$

showing $Y$ is CDH.

**Theorem 2.13.** $S^n$ is CDH.

**Proof.** Since $\mathbb{R}^n$ is separable and $S^n$ is homeomorphic to the one-point compactification of $\mathbb{R}^n$, $S^n$ is separable. Denote by $\infty$ this extra point that we are adjoining to $\mathbb{R}^n$.

Let $C = \{ \vec{c}_i \}_{i \in \mathbb{N}}$ and $D = \{ \vec{d}_i \}_{i \in \mathbb{N}}$ be countable dense subsets of $S^n$. If $C, D \subset S^n \setminus \{ \infty \}$, then we’re done by the previous proof since $S^n \setminus \{ \infty \} \cong \mathbb{R}^n$. If both $C$ and $D$ contain $\infty$, then set $\vec{c}_0 = \infty = \vec{d}_0$, and for $i > 0$, set $\vec{c}_i = \vec{c}_{i-1}$ and $\vec{d}_i = \vec{d}_{i-1}$. Then, let $f : S^n \to S^n$ be defined by

$$f(\vec{x}) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n)), \text{ and } f(\infty) = \infty \quad (*)$$

where $f_i(x_i) = \sup \{ \pi_i(\vec{d}_n) \mid \pi_i(\vec{c}_n) < x_i \}$ for all $1 \leq i \leq n$.

By a previous proof, $f$ is an autohomeomorphism of $\mathbb{R}^n$ with $f(C \setminus \{ \infty \}) = D \setminus \{ \infty \}$. It’s clear that $f$ is also an autohomeomorphism of $S^n$ with $f(C) = D$. 14
Now, suppose $\infty \in C$ but $\infty \notin D$. Set $\vec{c}_0 = \infty$ and $\vec{c}_i = \vec{c}_{i-1}$ for all $i$. Let $r : S^n \to S^n$ be a rotation of $S^n$ such that $r(\vec{d}_0) = \infty$. Then set $\vec{d}_0 = r(\vec{d}_0) = \infty$, and for $\vec{d}_i \in D \setminus \{\infty\}$, set $\vec{d}_i = r(\vec{d}_i)$. Let $f : S^n \to S^n$ be defined by (*) above.

Then $r(D)$ is countable dense since $r$ is a homeomorphism, so by the previous argument, $f$ is a homeomorphism such that $f(C) = r(D)$. Thus $r^{-1}f$ is a homeomorphism, and

$$(r^{-1}f)(C) = r^{-1}(f(C)) = r^{-1}(r(D)) = D.$$

Hence $S^n$ is CDH. \hfill \Box
Bibliography

