On the Countable Dense Homogeneity of Euclidean Spaces

by

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Abstract

A countable dense homogeneous space, in a general sense, is a topological space in which any two countable dense subsets of the space are "dispersed" the same way. In this thesis, we will show that some very well-known topological spaces, such as n-dimensional Euclidean space $\mathbb{R}^n$ and the n-sphere S^n for all natural numbers n is countable dense homogeneous.

Acknowledgments

I would like to thank God, for without divine intervention I would have never discovered the beauty that is mathematics. I would like to thank my entire family, for without their tireless efforts in raising me, I would have ended up in a terrible situation and gone nowhere fast. I would like to thank all of my friends, for without them my life and success would have no meaning. Finally, I would like to thank Charlotte, for she is the love of my life.

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Chapter 1

Preliminary Definition and Result

First, a definition and a lemma that we will use later.

Definition 1.1. *Let $f : A \to B$ and $g : X \to Y$ be functions. The cartesian product of functions $f \times g : A \times X \to B \times Y$ is defined by $(f \times g)(a, x) = (f(a), g(x))$. For a collection $\{f_\alpha\}_{\alpha \in \Lambda}$ of functions, let $\prod_{\alpha \in \Lambda} f_\alpha$ denote their cartesian product.*

Lemma 1.2. *Let Λ be an indexing set and $f_\alpha : X_\alpha \to Y_\alpha$ be a homeomorphism for all $\alpha \in \Lambda$. Then $f = \prod_{\alpha \in \Lambda} f_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \to \prod_{\alpha \in \Lambda} Y_\alpha$ is a homeomorphism.*

Proof. Let $\mathcal{U} \subset \prod_{\alpha \in \Lambda} Y_\alpha$ be an open set and $\vec{x} \in f^{-1}(\mathcal{U}) = (\prod_{\alpha \in \Lambda} f_\alpha)^{-1}(\mathcal{U}) = (\prod_{\alpha \in \Lambda} f_\alpha^{-1})(\mathcal{U})$. Then there is a basic open set $\prod_{\alpha \in \Lambda} U_\alpha$ in $\prod_{\alpha \in \Lambda} Y_\alpha$ lying in $\mathcal{U}$ and containing $f(\vec{x})$. Thus $\vec{x} \in f^{-1}(\prod_{\alpha \in \Lambda} U_\alpha) = \prod_{\alpha \in \Lambda} f_\alpha^{-1}(U_\alpha) \subset f^{-1}(\mathcal{U})$, where $f_\alpha^{-1}(U_\alpha)$ is open for all $\alpha \in \Lambda$ since f_α is continuous for all $\alpha \in \Lambda$. Therefore, $f^{-1}(\mathcal{U})$ is open. Hence f is continuous. By a similar argument, f^{-1} is continuous.

We show injectivity. Let $\vec{x}, \vec{y} \in \prod_{\alpha \in \Lambda} X_\alpha$ such that $f(\vec{x}) = f(\vec{y})$. Then $f((x_\alpha)_{\alpha \in \Lambda}) = f((y_\alpha)_{\alpha \in \Lambda})$, which implies $(f_\alpha(x_\alpha))_{\alpha \in \Lambda} = (f_\alpha(y_\alpha))_{\alpha \in \Lambda}$. Thus $f_\alpha(x_\alpha) = f_\alpha(y_\alpha)$ for all $\alpha \in \Lambda$. Then f_α being injective for all $\alpha \in \Lambda$ implies $x_\alpha = y_\alpha$ for all $\alpha \in \Lambda$. Thus $(x_\alpha)_{\alpha \in \Lambda} = (y_\alpha)_{\alpha \in \Lambda}$, or $\vec{x} = \vec{y}$. This shows that f is injective.

Now, let $\vec{y} = (y_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} Y_\alpha$. Then $y_\alpha \in Y_\alpha$ for all $\alpha \in \Lambda$, and since f_α is surjective for all $\alpha \in \Lambda$, there exists an $x_\alpha \in X_\alpha$ such that $f_\alpha(x_\alpha) = y_\alpha$. It follows that there exists an $\vec{x} = (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} X_\alpha$ such that $f(\vec{x}) = f((x_\alpha)_{\alpha\in\Lambda}) = (f_\alpha(x_\alpha))_{\alpha\in\Lambda} = (y_\alpha)_{\alpha\in\Lambda} = \vec{y}$. $\square$

Chapter 2

Countable Dense Homogeneity

Definition 2.1. *A separable space X is said to be countable dense homogeneous if, given any two countable dense sets $C, D \subset X$, there exists an autohomeomorphism f of X such that $f(C) = D$. For brevity, we will occasionally abbreviate "countable dense homogeneous" with CDH.*

2.1 $\mathbb{R}$ is Countable Dense Homogeneous

First, an important lemma.

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing surjection. Then f is continuous.*

Proof. Suppose f is discontinuous. Then there is an $a \in \mathbb{R}$ at which f is discontinuous. We have three cases for a discontinuity at a:

(i) Suppose $L = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, but $f(a) \neq L$. Assume $f(a) < L$. Since $\lim_{x \to a^-} f(x) = L > f(a)$, there is a neighborhood of L that contains an $f(x)$ for $x < a$ for which $f(x) > f(a)$. This is a contradiction to f being strictly increasing.

The proof when assuming $f(a) > L$ is similar.

(ii) Suppose $L = \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) = R$. Since f is surjective, for every $y \in [L, R]$ there exists an $x_y \in \mathbb{R}$ such that $y = f(x_y)$. It follows that there exists some (and in fact, infinitely many) $y \in [L, R]$ such that $x_y < a$ or $a < x_y$. Suppose $x_y < a$. Since f is strictly increasing, for every neighborhood N of a and every $x \in N$ such that $x < a$, $f(x) < L$. However, $x_y < a$, but $f(x_y) = y \geq L$, a contradiction.

The proof when assuming $a < x_y$ is similar.

(iii) Suppose one of the limits, say $L = \lim_{x\to a^-} f(x)$, does not exist or is infinite. If it is not infinite, then the set $\{f(x) \mid x < a\}$ is bounded above. However, since f is strictly increasing, there must be a limit. This is a contradiction. Therefore, we suppose L is infinite. If $L = \infty$, then for any $x > a$, there will exist an $x' < a$ such that $f(x') > f(x)$, contradicting f being strictly increasing. Clearly, $L \neq -\infty$, for then that would say that f were decreasing.

The proof when assuming $R = \lim_{x\to a^+} f(x)$ does not exist or is infinite is similar.

Thus the set of discontinuities of f is empty. Therefore, f is continuous. □

Theorem 2.3. $\mathbb{R}$ *is countable dense homogeneous.*

Proof. Let $C = \{c_i\}_{i\in\mathbb{N}}$ and $D = \{d_i\}_{i\in\mathbb{N}}$ be countable dense subsets of $\mathbb{R}$. We will re-order the elements of C and D in the following manner:

Set $c'_0 = c_0$, and set $\Gamma_0 = \mathbb{R}\backslash\{c'_0\}$. For $i \in \mathbb{N}$, we call the connected components of $\Gamma_i = \mathbb{R}\backslash\{c'_j\}_{j=0}^{i}$ "cells" for brevity. Then for $i \geq 0$, we recursively define $c'_{i+1}, c'_{i+2}, \ldots, c'_{2(i+1)}$ for each of the cells $\mathcal{C}_{i+1}, \mathcal{C}_{i+2}, \ldots, \mathcal{C}_{2(i+1)}$, respectively, of $\Gamma_i = \mathbb{R}\backslash\{c'_j\}_{j=0}^{i}$:

For $i = 0$, we have $\Gamma_0 = \mathbb{R}\backslash\{c'_0\} = (-\infty, c'_0) \cup (c'_0, \infty)$, so we can say, for example, that $\mathcal{C}_1 = (-\infty, c'_0)$ and $\mathcal{C}_2 = (c'_0, \infty)$ (Note that we could have interchanged them.). Then c_1 belongs to either $\mathcal{C}_1$ or $\mathcal{C}_2$. If the former occurs, then we set $c'_1 = c_1$. Otherwise, we set $c'_1 = c_m$, where $m = \min\{k \in \mathbb{N} \mid c_k \in \mathcal{C}_1\}$, and $c'_2 = c_1$, etc.

For $i \geq 0$, suppose Γ_i has been defined. Let $\mathcal{C}_n$, $i + 1 \leq n \leq 2(i + 1)$, be the cell of Γ_i in which we intend to place c'_n. Let $* : C \to C$ be defined by $*(c'_n) = c_m \Leftrightarrow c'_n = c_m$, where $m = \min\{k \in \mathbb{N} \mid c_k \in \mathcal{C}_n\}$.

We claim that $*$ is a bijection.

Suppose $c'_m \neq c'_n$. Then either $m < n$ or $m > n$, so assume $m < n$. It's obvious that $c'_n \in \Gamma_m$, so that $*(c'_n) \neq *(c'_m)$. Thus $*$ is injective.

Now, let $c_m \in C$. Clearly, for some $N < m$, $m = \min\{k \in \mathbb{N} \mid c_k \in \mathcal{C},\ \mathcal{C} \text{ is some cell of } \Gamma_N\}$, in which case there exists an $n \in \mathbb{N}$ such that $*(c'_n) = c_m$. Thus $*$ is surjective, and this proves the claim.

We do the same for D and impart on it the same ordering scheme used to order the c'_is. That is, for $i, j \in \mathbb{N}$, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are cells of Γ_n for some $n \in \mathbb{N}$ with $c'_i \in \mathcal{C}_i$ and $c'_j \in \mathcal{C}_j$ such that $\mathcal{C}_i < \mathcal{C}_j$, then the cells $\mathcal{D}_i$ and $\mathcal{D}_j$ in which we intend to place d'_i and d'_j, respectively, of $\Delta_n = \mathbb{R}\backslash\{d'_k\}_{k=0}^n$ also satisfy $\mathcal{D}_i < \mathcal{D}_j$.

Therefore, we conclude that if

$$
\begin{array}{ccc}
 & c'_0 & \\
c'_1 & < c'_0 < & c'_2 \\
c'_3 < c'_1 < c'_4 & < c'_0 < & c'_5 < c'_2 < c'_6 \\
 & \vdots &
\end{array}
$$

was the ordering scheme used to order the c'_is, then we should (and do) also obtain

$$
\begin{array}{ccc}
 & d'_0 & \\
d'_1 & < d'_0 < & d'_2 \\
d'_3 < d'_1 < d'_4 & < d'_0 < & d'_5 < d'_2 < d'_6 \\
 & \vdots &
\end{array}
.
$$

Then it is immediate that $c'_i < c'_j \Leftrightarrow d'_i < d'_j$.

Now, due to readability, we shall revert to the original notation used for the elements of C and D, i.e. $c'_i \to c_i$ and $d'_i \to d_i$ for all i.

There is some element c_m of C that is greater than x and so bounds the set $\{c_i \mid c_i < x\}$. Then by the construction d_m is an upper bound of the set $\{d_i \mid c_i < x\}$ and so it has a supremum. Therefore, we define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sup\{d_i \mid c_i < x\}.$$

We want to show that f is a homeomorphism.

Claim: f is strictly increasing.

Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then $f(x) = \sup\{d_i \mid c_i < x\}$ and $f(y) = \sup\{d_i \mid c_i < y\}$. Since $x < y$, $\{d_i \mid c_i < x\} \subsetneq \{d_i \mid c_i < y\}$. Furthermore, since D is dense, there is a $d \in D \cap (x, y)$, and thus it follows that $d \in \{d_i \mid c_i < y\}$ but $d \notin \{d_i \mid c_i < x\}$. Therefore, d is an upper bound for $\{d_i \mid c_i < x\}$ but not an upper bound for $\{d_i \mid c_i < y\}$. Thus $f(x) = \sup\{d_i \mid c_i < x\} < \sup\{d_i \mid c_i < y\} = f(y)$.

This proves the claim, and it also implies f is injective.

Claim: f is surjective.

Let $y \in \mathbb{R}$. Then y is an upper bound for the set $D' = \{d_i \mid d_i < y\}$. Let $y^* \in \mathbb{R}$ such that $d_i \leq y^*$ for all $d_i \in D'$. That is, let y^* be an upper bound for D'. Suppose $y^* < y$. Then (y^*, y) is open and thus contains an element d of D due to the denseness of D. Moreover, $d < y$, implying $d \in D'$ and showing that $y^* < d \in D'$, a contradiction. Therefore, $y^* \geq y$, implying $y = \sup D'$. Set $x = \sup\{c_i \mid d_i < y\}$. It follows that $y = \sup D' = \sup\{d_i \mid c_i < x\} = f(x)$. This proves the claim and furthermore shows that f is a bijection.

By the previous lemma, since f is a strictly increasing surjection from $\mathbb{R}$ to $\mathbb{R}$, f is continuous.

Moreover, since f is a bijection, it has an inverse. We claim that

$$g(y) = \sup\{c_j \mid d_j < y\}$$

is this inverse.

Let $y \in \mathbb{R}$. Let $D^* = \{d_i \mid c_i < \sup\{c_j \mid d_j < y\}\}$. Then y is an upper bound of D^*. Let $y^* \in \mathbb{R}$ be an upper bound of D^*. Suppose $y^* < y$. Then (y^*, y) is open and thus contains an element d_k of D due to the denseness of D. Moreover, since $d_k < y$, $c_k \in \{c_j \mid d_j < y\}$, implying $c_k < \sup\{c_j \mid d_j < y\}$ by considering the open interval (d_k, y) and another element d_n of D such that $d_k < d_n < y$, i.e. $c_k < c_n \in \{c_j \mid d_j < y\}$. Thus $d_k \in D^*$. However, this says that $y^* < d_k \in D^*$, which is a contradiction to y^* being an upper bound of D^*. Therefore, $y^* \geq y$, implying $y = \sup D^*$. Thus

$$f(g(y)) = f(\sup\{c_j \mid d_j < y\}) = \sup\{d_i \mid c_i < \sup\{c_j \mid d_j < y\}\} = \sup D^* = y,$$

which shows that $g = f^{-1}$. Clearly g is continuous by the same reasoning that f is continuous.

Furthermore, it's readily seen that for any $c_j \in C$, $f(c_j) = \sup\{d_i \mid c_i < c_j\} = d_j$. Similarly, for any $d_i \in D$, $f^{-1}(d_i) = c_i$. Therefore, we have $f(C) \subset D$ and $C \supset f^{-1}(D)$, respectively, so we have $f(C) \subset D$ and $f(C) \supset f(f^{-1}(D)) = D$. Hence $f(C) = D$.

Therefore, f is a homeomorphism such that $f(C) = D$. We conclude that $\mathbb{R}$ is countable dense homogeneous. $\square$

2.2 $\mathbb{Q}$ is not Countable Dense Homogeneous

We will show that $\mathbb{Q}$ is not countable dense homogeneous, but first, we will show the more general result that countable dense spaces are not countable dense homogeneous.

Theorem 2.4. *Countable dense spaces are not countable dense homogeneous.*

Proof. Let X be a countable dense space, and let $x \in X$. Suppose X is CDH. Then since X and $X \backslash \{x\}$ are countable dense subsets of X, there exists a homeomorphism $f : X \to X$ such that $f(X \backslash \{x\}) = X$. Thus there exists an $x' \in X \backslash \{x\}$ such that $f(x) = f(x')$, a contradiction to f being injective. Thus X is not CDH. $\square$

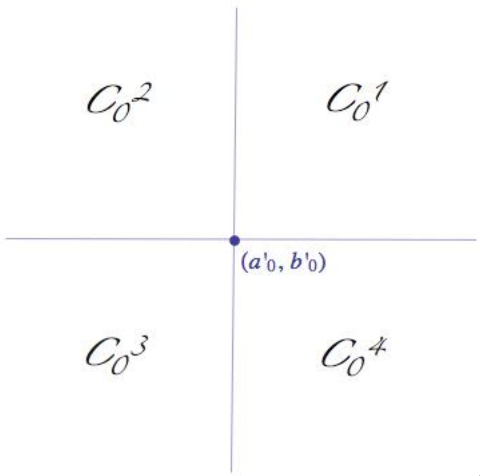
Corollary 2.5. $\mathbb{Q}$ *is not CDH.*

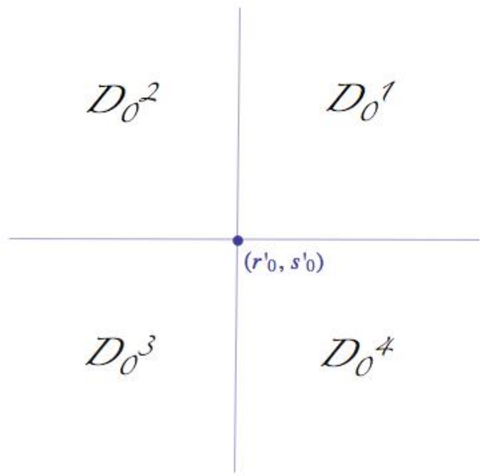
2.3 $\mathbb{R}^2$ is Countable Dense Homogeneous

Theorem 2.6. $\mathbb{R}^2$ *is countable dense homogeneous.*

Proof. Let $C, D \subset \mathbb{R}^2$ be countable dense. One can check that for $a, b \in \mathbb{R}^2$, the set $R^i(a, b) = \{\rho : \rho \text{ is a rotation of } \mathbb{R}^n \text{ such that } \pi_i(\rho(a)) = \pi_i(\rho(b))\}$ is countable. Thus $\bigcup_{x,y \in C} R^i(x, y)$ and $\bigcup_{w,z \in D} R^i(w, z)$ are countable for $i = 1, 2$ since C and D are countable. Then $\left(\bigcup_{i=1}^2 \bigcup_{x,y \in C} R^i(x, y)\right) \cup \left(\bigcup_{i=1}^2 \bigcup_{w,z \in D} R^i(w, z)\right)$ is also countable being a countable union of countable sets. Since there are uncountably many rotations of $\mathbb{R}^2$, there exists a rotation R of $\mathbb{R}^2$ that is different from any of these rotations so that $\pi_i(R(x)) \neq \pi_i(R(y))$, $\pi_i(R(w)) \neq \pi_i(R(z))$ for any $1 \leq i \leq 2$ and for all $x, y \in C$ and $w, z \in D$. It follows that no vertical or horizontal line intersects $R(C)$ or $R(D)$ at more than one point. Moreover, since R is a homeomorphism, $R(C)$ and $R(D)$ are countable dense. We set $R(C) = \{(a_i, b_i)\}_{i=0}^{\infty}$ and $R(D) = \{(r_i, s_i)\}_{i=0}^{\infty}$.

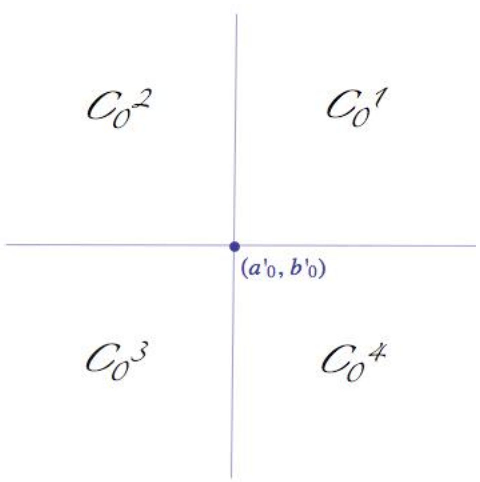
Now, for $(x_1, x_2) \in \mathbb{R}^2$, let V_{x_1} and H_{x_2} be the vertical and horizontal lines in the plane through (x_1, x_2), respectively. Set $(a'_0, b'_0) = (a_0, b_0)$ and $(a'_1, b'_1) = (a_1, b_1)$. For $n \in \mathbb{N}$, $\mathbb{R}^2 \backslash \{V_{a'_n} \cup H_{b'_n}\} = \mathcal{C}_n^1 \cup \mathcal{C}_n^2 \cup \mathcal{C}_n^3 \cup \mathcal{C}_n^4$, where $\mathcal{C}_n^1 = \{(x, y) \mid x > a'_n, y > b'_n\}$, $\mathcal{C}_n^2 = \{(x, y) \mid x < a'_n, y > b'_n\}$, $\mathcal{C}_n^3 = \{(x, y) \mid x < a'_n, y < b'_n\}$, and $\mathcal{C}_n^4 = \{(x, y) \mid x > a'_n, y < b'_n\}$. Set $(r'_0, s'_0) = (r_0, s_0)$. For $n \in \mathbb{N}$, $\mathbb{R}^2 \backslash \{V_{r'_n} \cup H_{s'_n}\} = \mathcal{D}_n^1 \cup \mathcal{D}_n^2 \cup \mathcal{D}_n^3 \cup \mathcal{D}_n^4$, where the $\mathcal{D}_n^j$ are defined similar to the $\mathcal{C}_n^j$.





,

Then $(a'_1, b'_1) \in \mathcal{C}_0^{n_0}$ for some $1 \leq n_0 \leq 4$, and consequently, we set $(r'_1, s'_1) = (r_{d_1}, s_{d_1})$, where $d_1 = \min\{m \in \mathbb{N} \mid (r_m, s_m) \in \mathcal{D}_0^{n_0}\}$. Set $(r'_2, s'_2) = (r_{d_2}, s_{d_2})$, where $d_2 = \min\{m \in \mathbb{N} \mid (r_m, s_m) \neq (r'_j, s'_j)$ for any $j < 2\}$. Then there exist $n_0, n_1 \in \{1, 2, 3, 4\}$ such that $(r'_2, s'_2) \in \mathcal{D}_0^{n_0} \cap \mathcal{D}_1^{n_1}$. Set $(a'_2, b'_2) = (a_{c_2}, b_{c_2})$, where $c_2 = \min\{m \in \mathbb{N} \mid (a_m, b_m) \in \mathcal{C}_0^{n_0} \cap \mathcal{C}_1^{n_1}\}$. After this, set $(a'_3, b'_3) = (a_{c_3}, b_{c_3})$, where $c_3 = \min\{m \in \mathbb{N} \mid (a_m, b_m) \neq (a'_j, b'_j)$ for any $j < 3\}$. Observe by this construction that $(a'_i, b'_i) \in \bigcap_{k=0}^{i-1} \mathcal{C}_k^{n_k}$ if and only if $(r'_i, s'_i) \in \bigcap_{k=0}^{i-1} \mathcal{D}_k^{n_k}$. By the construction, we see that for any $i, j \in \{0, 1, 2\}$, $a'_i < a'_j \Leftrightarrow r'_i < r'_j$ and $b'_i < b'_j \Leftrightarrow s'_i < s'_j$.



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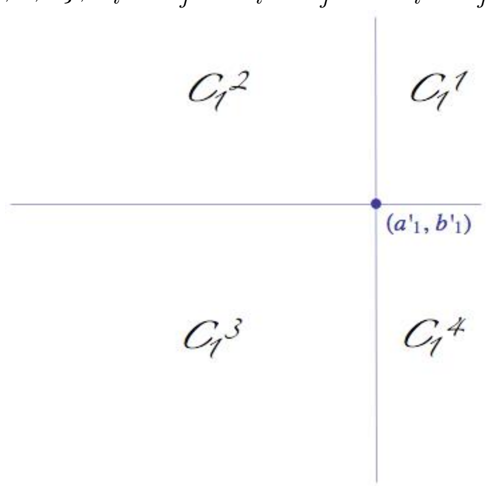

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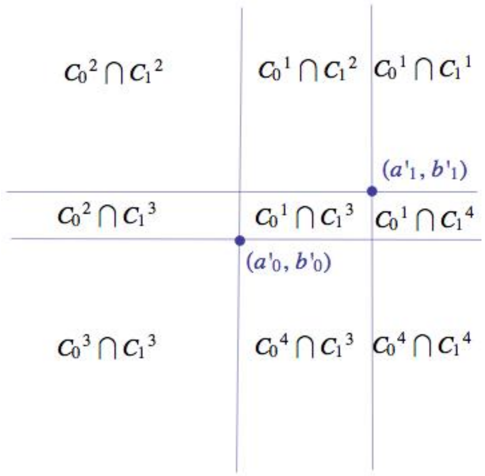

We continue the process above and assume that for some odd $i > 0$, $(a_0', b_0'), (a_1', b_1'), \ldots, (a_i', b_i')$ and $(r_0', s_0'), \ldots, (r_{i-1}', s_{i-1}')$ have been defined as above. Then there exist $n_0, n_1, \ldots, n_{i-1} \in \{1, 2, 3, 4\}$ such that $(a_i', b_i') \in \bigcap_{j=0}^{i-1} \mathcal{C}_j^{n_j}$. Set $(r_i', s_i') = (r_{d_i}, s_{d_i})$, where $d_i = \min\{m \in \mathbb{N} \mid (r_m, s_m) \in \bigcap_{j=0}^{i-1} \mathcal{D}_j^{n_j}\}$. After this, set $(r_{i+1}', s_{i+1}') = (r_{d_{i+1}}, s_{d_{i+1}})$, where $d_{i+1} = \min\{m \in \mathbb{N} \mid (r_m, s_m) \neq (r_j', s_j')$ for any $j < i+1\}$. Then there exist $n_0, n_1, \ldots, n_i \in \{1, 2, 3, 4\}$ such that $(r_{i+1}', s_{i+1}') \in \bigcap_{j=0}^{i} \mathcal{D}_j^{n_j}$. Then set $(a_{i+1}', b_{i+1}') = (a_{c_{i+1}}, b_{c_{i+1}})$, where $c_{i+1} = \min\{m \in \mathbb{N} \mid (a_m, b_m) \in \bigcap_{j=0}^{i} \mathcal{C}_j^{n_j}\}$. After this, set $(a_{i+2}', b_{i+2}') = (a_{c_{i+2}}, b_{c_{i+2}})$, where $c_{i+2} = \min\{m \in \mathbb{N} \mid (a_m, b_m) \neq (a_j', b_j')$ for any $j < i+2\}$. Thus $(a_0', b_0'), (a_1', b_1'), \ldots, (a_i', b_i'), (a_{i+1}', b_{i+1}'), (a_{i+2}', b_{i+2}')$ and $(r_0', s_0'), \ldots, (r_{i-1}', s_{i-1}'), (r_i', s_i'), (r_{i+1}', s_{i+1}')$ have been defined, so we continue repeating this process and proceed by induction to define (a_j', b_j') and (r_j', s_j') for all $j \in \mathbb{N}$.

Clearly the above assignment is a bijection on $R(C)$ and $R(D)$. Moreover, we claim that, for $i, j \in \mathbb{N}$, $a_i' < a_j' \Leftrightarrow r_i' < r_j'$ and $b_i' < b_j' \Leftrightarrow s_i' < s_j'$. To that end, we assume $i > j$ and suppose that for $(a_i', b_i'), (a_j', b_j') \in R(C)$ and $(r_i', s_i'), (r_j', s_j') \in R(D)$, $a_i' < a_j'$ and $b_i' < b_j'$. Then $(a_i', b_i') \in \mathcal{C}_j^3$. Thus $(r_i', s_i') \in \mathcal{D}_j^3$ by the construction above, showing $r_i' < r_j'$ and $s_i' < s_j'$. The rest of the proof is done similarly. Hence the claim is true.

Due to readability, we shall revert to the original notation used for the elements of $R(C)$ and $R(D)$. I.e., $(a_i', b_i') \to (a_i, b_i)$ and $(r_i', s_i') \to (r_i, s_i)$ for all i.

Set $C_n := \pi_n R(C)$ and $D_n := \pi_n R(D)$ for $1 \leq n \leq 2$. Then C_n and D_n are countable dense subsets of $\mathbb{R}$ since π_n is a continuous surjection for $1 \leq n \leq 2$. For $i, j \in \mathbb{N}$, $a_i < a_j \Leftrightarrow r_i < r_j$. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) = \sup\{r_m \mid a_m < x\}$. Then f_1 is a homeomorphism such that $f_1(C_1) = D_1$. More importantly, $f_1(a_i) = r_i$ for all $i \in \mathbb{N}$. Similarly, $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_2(y) = \sup\{s_m \mid b_m < y\}$ is a homeomorphism such that $f_2(C_2) = D_2$ with $f_2(b_i) = s_i$ for all $i \in \mathbb{N}$.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (f_1(x), f_2(y))$. Then f is a homeomorphism by a previous theorem. We claim that $f(R(C)) = R(D)$. Clearly $f(R(C)) \subset R(D)$. Now, for $(r_i, s_i) \in R(D)$, since $r_i = f_1(a_i)$ and $s_i = f_2(b_i)$, $(r_i, s_i) = (f_1(a_i), f_2(b_i)) = f(a_i, b_i)$. Thus $f(R(C)) \supset R(D)$, proving the claim. Therefore, $R^{-1}fR(C) = R^{-1}R(D) = D$. Since $R^{-1}fR$ is a homeomorphism, $\mathbb{R}^2$ is countable dense homogeneous. □

2.4 $\mathbb{R}^n$ is Countable Dense Homogeneous

For the following proof, it is necessary that we define some terminology.

Definition 2.7. *By a hyperplane of $\mathbb{R}^n$ we mean a set of the form $S + x$, where S is a subspace of $\mathbb{R}^n$ of dimension $n - 1$ and $x \in \mathbb{R}^n$.*

Definition 2.8. *By the i^{th} hyperplane axis A of $\mathbb{R}^n$ we mean the set $A = \{(x_1, x_2, \ldots, x_n) \mid x_i = 0\}$.*

Definition 2.9. *Let H be a hyperplane parallel to a hyperplane axis in $\mathbb{R}^n$. By a half-space of H we mean one of the two open sets whose union is $\mathbb{R}^n \backslash H$.*

Definition 2.10. *By an n-orthant of the point $\vec{x} \in \mathbb{R}^n$ we mean the nonempty intersection of a collection of n half-spaces that do not contain $\vec{x}$. An orthant of $\vec{x} \in \mathbb{R}^n$ is thus necessarily open being a finite intersection of open sets. Let $\mathcal{O}^i_{\vec{x}}$ denote the i^{th} orthant of $\vec{x}$. It follows that there are 2^n orthants for each $\vec{x} \in \mathbb{R}^n$.*

Theorem 2.11. $\mathbb{R}^n$ *is countable dense homogeneous.*

Proof. We've already shown that $\mathbb{R}$ and $\mathbb{R}^2$ are CDH, so the cases $n = 1$ and $n = 2$ are done. We now prove that $\mathbb{R}^n$ for $n > 2$ is CDH by using a generalization of our argument for $\mathbb{R}^2$. To that end, assume $n > 2$. Let $C, D \subset \mathbb{R}^n$ be countable dense. Let $\vec{x}$ denote the point $(x_j)_{j=1}^n \in \mathbb{R}^n$. For $1 \leq i \leq n$, define $p_i : \mathbb{R}^n \to \mathbb{R}^n$ by $p_i(\vec{x}) = (q_i(x_j))_{j=1}^n$, where $q_i(x_j) = x_j$ for $i \neq j$ and $q_i(x_j) = 0$ for $i = j$. That is, p_i $(1 \leq i \leq n)$ projects $\mathbb{R}^n$ onto the $(n-1)$-dimensional subspace $\{\vec{y} \in \mathbb{R}^n \mid \pi_i(\vec{y}) = 0\}$ of $\mathbb{R}^n$, which is homeomorphic to $\mathbb{R}^{n-1}$.

We now repeat the procedure in the previous proof. For each pair $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $1 \leq i \leq n$, the set $R^i(\vec{x}, \vec{y}) = \{\rho : \rho \text{ is a rotation of } \mathbb{R}^n \text{ such that } p_i(\rho(\vec{x})) = p_i(\rho(\vec{y}))\}$ is countable. Therefore $\bigcup_{i=1}^n \bigcup_{\vec{x},\vec{y} \in C} R^i(\vec{x}, \vec{y})$ and $\bigcup_{i=1}^n \bigcup_{\vec{w},\vec{z} \in D} R^i(\vec{w}, \vec{z})$ are countable since C and D are countable. Thus $\left(\bigcup_{i=1}^n \bigcup_{\vec{x},\vec{y} \in C} P^i(\vec{x}, \vec{y})\right) \cup \left(\bigcup_{i=1}^n \bigcup_{\vec{w},\vec{z} \in D} P^i(\vec{w}, \vec{z})\right)$ is also countable. Since there are uncountably many rotations of $\mathbb{R}^n$, there exists a rotation R of $\mathbb{R}^n$ that is different from any of these rotations so that $p_i(R(\vec{x})) \neq p_i(R(\vec{y}))$ and $p_i(R(\vec{w})) \neq p_i(R(\vec{z}))$ for all $1 \leq i \leq n$, $\vec{x}, \vec{y} \in C$ and $\vec{w}, \vec{z} \in D$. It follows that no hyperplane parallel to any hyperplane axis intersects $R(C)$ or $R(D)$ at more than one point. Moreover, since R is a homeomorphism, $R(C)$ and $R(D)$ are countable dense. We set $R(C) = \{\vec{a}_m\}_{m=0}^\infty$ and $R(D) = \{\vec{b}_m\}_{m=0}^\infty$.

Now, for $\vec{x} \in \mathbb{R}^n$, let $\mathfrak{H}_{\vec{x}}$ be the union of all hyperplanes containing $\vec{x}$ that are parallel to a hyperplane axis. Then $\mathbb{R}^n \backslash \mathfrak{H}_{\vec{x}} = \bigcup_{k=1}^{2^n} \mathcal{O}^k_{\vec{x}}$, where $\mathcal{O}^k_{\vec{x}}$ is the k^{th} n-orthant of $\vec{x}$. Set $\vec{a}'_0 = \vec{a}_0, \vec{a}'_1 = \vec{a}_1$, and $\vec{b}'_0 = \vec{b}_0$. Then $\vec{a}'_1 \in \mathcal{O}^{n_0}_{\vec{a}'_0}$ for some $1 \leq n_0 \leq 2^n$, and consequently, we set $\vec{b}'_1 = \vec{b}_{d_1}$, where $d_1 = \min\{m \in \mathbb{N} \mid \vec{b}_m \in \mathcal{O}^{n_0}_{\vec{b}'_0}\}$. Set $\vec{b}'_2 = \vec{b}_{d_2}$, where $d_2 = \min\{m \in \mathbb{N} \mid \vec{b}_m \neq \vec{b}'_\ell \text{ for any } \ell < 2\}$. Then there exist $n_0, n_1 \in \{1, \ldots, 2^n\}$ such that $\vec{b}'_2 \in \mathcal{O}^{n_0}_{\vec{b}'_0} \cap \mathcal{O}^{n_1}_{\vec{b}'_1}$. Set $\vec{a}'_2 = \vec{a}_{c_2}$, where $c_2 = \min\{m \in \mathbb{N} \mid \vec{a}_m \in \mathcal{O}^{n_0}_{\vec{a}'_0} \cap \mathcal{O}^{n_1}_{\vec{a}'_1}\}$. After this, set $\vec{a}'_3 = \vec{a}_{c_3}$, where $c_3 = \min\{m \in \mathbb{N} \mid \vec{a}_m \neq \vec{a}'_\ell \text{ for any } \ell < 3\}$. By this construction, observe that for any

$1 \le i \le n$, $0 \le j \le 2$, and $1 \le k \le 2^{n-1}$, $p_i(\vec{a}'_j) \in \mathcal{O}^k_{p_i(\vec{a}'_\ell)}$ if and only if $p_i(\vec{b}'_j) \in \mathcal{O}^k_{p_i(\vec{b}'_\ell)}$ for some ℓ, where $\mathcal{O}^k_{p_i(\vec{a}'_\ell)}$ and $\mathcal{O}^k_{p_i(\vec{b}'_\ell)}$ are the k^{th} orthants of the points $p_i(\vec{a}'_\ell)$ and $p_i(\vec{b}'_\ell)$ in $\mathbb{R}^{n-1}$. Also by the construction, we see that $\vec{a}'_m \in \bigcap_{i \in I \subset \mathbb{N}} \mathcal{O}^{k_i}_{\vec{a}'_i}$ if and only if $\vec{b}'_m \in \bigcap_{i \in I \subset \mathbb{N}} \mathcal{O}^{k_i}_{\vec{b}'_i}$.

As in the previous proof, we can continue the process above and assume that for some odd $i > 0$, $\vec{a}'_0, \vec{a}'_1, \ldots, \vec{a}'_i$ and $\vec{b}'_0, \ldots, \vec{b}'_{i-1}$ have been defined as above. By a similar proof to the previous proof at this stage, we see that $\vec{a}'_0, \vec{a}'_1, \ldots, \vec{a}'_i, \vec{a}'_{i+1}, \vec{a}'_{i+2}$ and $\vec{b}'_0, \ldots, \vec{b}'_{i-1}, \vec{b}'_i, \vec{b}'_{i+1}$ can be defined, so we continue repeating this process and proceed by induction to define $\vec{a}'_m$ and $\vec{b}'_m$ for all $m \in \mathbb{N}$. By similar proofs to the ones for $\mathbb{R}$ and $\mathbb{R}^2$ being CDH, this assignment is a bijection on $R(C)$ and $R(D)$. Due to readability, we shall revert to the original notation used for the elements of $R(C)$ and $R(D)$. I.e., $\vec{a}'_m \to \vec{a}_m$ and $\vec{b}'_m \to \vec{b}_m$ for all m.

Set $C_i := p_i R(C)$ and $D_i := p_i R(D)$ $(1 \le i \le n)$. Then for $1 \le i \le n$, C_i and D_i are countable dense subsets of $\mathbb{R}^{n-1}$ since p_i is a continuous surjection onto $\mathbb{R}^{n-1}$. By induction, $\mathbb{R}^{n-1}$ is CDH, so for each i $(1 \le i \le n)$ there exists a homeomorphism $f^i : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $f^i(C_i) = D_i$. Pick f^1 to be one of these. It is defined inductively by $f^1((x_j)_{j=1}^{n-1}) = (f_j(x_j))_{j=1}^{n-1}$, where $f_j : \mathbb{R} \to \mathbb{R}$ is defined by $f_j(x_j) = \sup\{\pi_j(\vec{b}_m) \mid \pi_j(\vec{a}_m) < x_j\}$ for $1 \le j \le n-1$ is a homeomorphism. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(\vec{x}) = (f^1((x_j)_{j=1}^{n-1}), f_n(x_n)) = ((f_j(x_j))_{j=1}^{n-1}, f_n(x_n))$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x_n) = \sup\{\pi_n(\vec{b}_m) \mid \pi_n(\vec{a}_m) < x_n\}$. Then f_n is a homeomorphism by a previous proof, so f is a homeomorphism by a previous proof. Moreover, $f_n(\pi_n(\vec{a}_i)) = \pi_n(\vec{b}_i)$ for all $i \in \mathbb{N}$.

We claim that $f(R(C)) = R(D)$. Let $\vec{a}_i \in R(C)$. Then $f(\vec{a}_i) = ((f_j(\pi_j(\vec{a}_i)))_{j=1}^{n-1}, f_n(\pi_n(\vec{a}_i)))$ $= ((\pi_j(\vec{b}_i))_{j=1}^{n-1}, \pi_n(\vec{b}_i)) = \vec{b}_i \in R(D)$. Thus $f(R(C)) \subseteq R(D)$. Also, for $\vec{b}_i \in R(D)$, since $\pi_j(\vec{b}_i) = f_j(\pi_j(\vec{a}_i))$ for all j,

$$\begin{aligned}
\vec{b}_i &= (\pi_j(\vec{b}_i))_{j=1}^{n} = (f_j(\pi_j(\vec{a}_i)))_{j=1}^{n} = ((f_j(\pi_j(\vec{a}_i)))_{j=1}^{n-1}, f_n(\pi_n(\vec{a}_i))) \\
&= (f^1((\pi_j(\vec{a}_i))_{j=1}^{n-1}), f_n(\pi_n(\vec{a}_i))) = f(\vec{a}_i).
\end{aligned}$$

Thus $f(R(C)) \supset R(D)$, proving the claim. Therefore, $R^{-1}fR(C) = D$. Since $R^{-1}fR$ is a homeomorphism, $\mathbb{R}^n$ is countable dense homogeneous. $\square$

2.5 The n-sphere S^n is Countable Dense Homogeneous

Lemma 2.12. *Let X be CDH. Then any space Y homeomorphic to X is also CDH. (I.e., CDH is a topological property.)*

Proof. Let Y be a space homeomorphic to X. Then there is a homeomorphism $f : Y \to X$. Let C and D be countable dense subsets of Y. Then $f(C)$ and $f(D)$ are countable dense subsets of X. Since X is CDH, there is a homeomorphism $g : X \to X$ such that $g(f(C)) = f(D)$. Then $f^{-1}gf \in \mathcal{H}(Y)$, and

$$(f^{-1}gf)(C) = f^{-1}(g(f(C))) = f^{-1}(f(D)) = D,$$

showing Y is CDH. $\square$

Theorem 2.13. *S^n is CDH.*

Proof. Since $\mathbb{R}^n$ is separable and S^n is homeomorphic to the one-point compactification of $\mathbb{R}^n$, S^n is separable. Denote by ∞ this extra point that we are adjoining to $\mathbb{R}^n$.

Let $C = \{\vec{c_i}\}_{i\in\mathbb{N}}$ and $D = \{\vec{d_i}\}_{i\in\mathbb{N}}$ be countable dense subsets of S^n. If $C, D \subset S^n \backslash \{\infty\}$, then we're done by the previous proof since $S^n\backslash\{\infty\} \cong \mathbb{R}^n$. If both C and D contain ∞, then set $\vec{c_0'} = \infty = \vec{d_0'}$, and for $i > 0$, set $\vec{c_i'} = \vec{c}_{i-1}$ and $\vec{d_i'} = \vec{d}_{i-1}$. Then, let $f : S^n \to S^n$ be defined by

$$f(\vec{x}) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n)), \text{ and } f(\infty) = \infty \quad (*)$$

where $f_i(x_i) = \sup\{\pi_i(\vec{d_n'}) \mid \pi_i(\vec{c_n'}) < x_i\}$ for all $1 \leq i \leq n$.

By a previous proof, f is an autohomeomorphism of $\mathbb{R}^n$ with $f(C\backslash\{\infty\}) = D\backslash\{\infty\}$. It's clear that f is also an autohomeomorphism of S^n with $f(C) = D$.

Now, suppose $\infty \in C$ but $\infty \notin D$. Set $\vec{c_0'} = \infty$ and $\vec{c_i'} = \vec{c}_{i-1}$ for all i. Let $r : S^n \to S^n$ be a rotation of S^n such that $r(\vec{d_0}) = \infty$. Then set $\vec{d_0'} = r(\vec{d_0}) = \infty$, and for $\vec{d_i} \in D\backslash\{\infty\}$, set $\vec{d_i'} = r(\vec{d_i})$. Let $f : S^n \to S^n$ be defined by $(*)$ above.

Then $r(D)$ is countable dense since r is a homeomorphism, so by the previous argument, f is a homeomorphism such that $f(C) = r(D)$. Thus $r^{-1}f$ is a homeomorphism, and

$$(r^{-1}f)(C) = r^{-1}(f(C)) = r^{-1}(r(D)) = D.$$

Hence S^n is CDH. $\square$

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