Transition Fronts in Time Heterogeneous Bistable Equations with Nonlocal Dispersal: Existence, Regularity, Stability and Uniqueness

by

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Abstract

This dissertation is devoted to the existence, regularity, stability and uniqueness of transition fronts in nonlocal bistable equations in time heterogeneous media.

Instead of existence, we start our study with qualitative properties. We first show that any transition front must be regular in space and propagates to the right with a continuously differentiable interface location function; this is also true in space-time heterogeneous media.

We then turn to the study of space nonincreasing transition fronts and prove various important properties such as uniform steepness, stability, uniform stability, exponential decaying estimates and so on.

Moreover, we show that any transition front, after certain space shift, coincides with a space nonincreasing transition front, verifying the uniqueness, up to space shifts, of transition fronts, and hence, all transition fronts satisfy just mentioned qualitative properties.

Also, we show that a transition front must be a periodic traveling wave in time periodic media and the asymptotic speeds of transition fronts exist in time uniquely ergodic media.

Finally, we prove the existence of space nonincreasing transition fronts by constructing appropriate approximating front-like solutions with regularities; this is done under certain additional assumptions.

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Chapter 1

Introduction

This dissertation is devoted to the study of front propagation phenomenon in nonlocal bistable equations in time heterogeneous media of the form

$$u_t = J * u - u + f(t, u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where t is the time variable, x is the space variable, $u(t, x)$ is the unknown function, J is the dispersal kernel with $[J * v](x) = \int_{\mathbb{R}} J(x - y)v(y)dy = \int_{\mathbb{R}} J(y)v(x - y)dy$, and the nonlinear term $f(t, u)$ is of bistable type.

In equation (1.1), the dispersal operator defined by $v \mapsto J * v - v$ on proper function spaces is a nonlocal analog of the classical random dispersal operator, i.e., the Laplacian ∂_{xx}, and hence, equation (1.1) is the nonlocal analog of the classical reaction-diffusion equation

$$u_t = u_{xx} + f(t, u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1.2}$$

In fact, if we consider the rescaled dispersal kernel $J^\epsilon(x) = \frac{1}{\epsilon}J(\frac{x}{\epsilon})$, then using Taylor expansion, it is not hard to see that the operator $v \mapsto J^\epsilon * v - v$ is quite close to ∂_{xx} up to some scale multiplication for all small $\epsilon > 0$. It is known that the Laplacian ∂_{xx} is very successful in modeling Brownian motion based continuous diffusive processes. But, in many applications, diffusion may not be continuous (see e.g. [22, 17, 25, 26, 32, 46, 62]), and hence, it is no longer suitable to use the Laplacian to model diffusive processes with jumps; it is where the nonlocal dispersal operator comes into play.

A time-independent function $f(u)$ being of bistable type means that there exist three zeros $u_- < u_* < u_+$ of $f(u)$ such that $f(u) < 0$ for $u \in (u_-, u_*)$ and $f(u) > 0$ for $u \in (u_*, u_+)$.

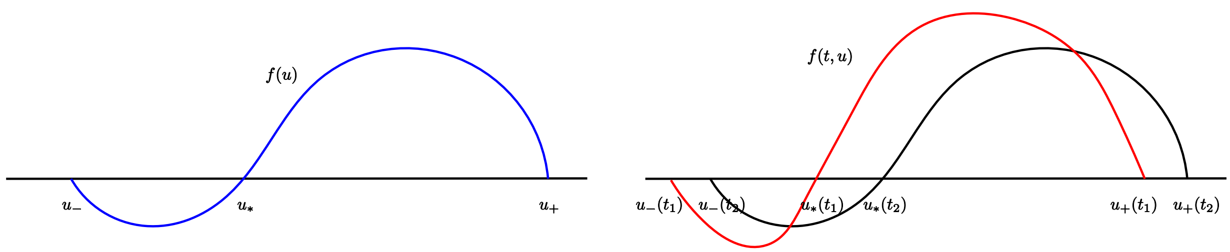

Figure 1.1: Bistable Nonlinearities

In applications, the values of $f(u)$ for $u \notin [u_-, u_+]$ are irrelevant. A time-dependent function $f(t,u)$ being of bistable type roughly means that for each t, $f(t,u)$ is of bistable type as a function of u. See Figure 1.1. Bistable nonlinearities arise in many scientific areas. In population dynamics, they correspond to the well-known phenomenon called Allee effect, after Warder C. Allee (see e.g. [2]), saying that the population grows logistically, while the reverse holds when the density is low. They also appear in the Fitzhugh-Nagumo model (more generally, the Hodgkin-Huxley model) describing the propagation of signals (see e.g. [29, 30, 34, 55]), as well as in phase transition models such as Allen-Cahn equation (see e.g. [3, 4]).

Motivated by application problems, the central problem concerning equations (1.1) and (1.2) is to understand the asymptotic dynamics of solutions with front-like or compactly supported initial data. This problem is then usually reduced to finding special solutions and studying their stability. Special solutions of particular interest are traveling waves (i.e., solutions of the form $u(t,x) = \phi(x - ct)$) in homogeneous media, i.e., $f(t,u) = f(u)$, and transition fronts (proper generalizations of traveling waves) in heterogeneous media. Results concerning equation (1.2) are quite complete in the literature. In the homogeneous media, traveling waves as well as their stability and uniqeness have been established (see e.g. [5, 6, 27]). In the time periodic media, periodic traveling waves as well as their stability and uniqueness have been established in [1]. In the time heterogeneous media, transition fronts as well as their stability and uniqueness have been established (see [63, 64, 65, 66]). There are also sharp transition phenomena in equation (1.2) (see [24, 59, 83]).

Due to the resemblance between (1.1) and (1.2), it is very natural to ask whether we can establish all the corresponding results for (1.1). However, this is not an easy job and results concerning (1.1) are far from enough. In fact, all we know about (1.1) in the literature is the existence, stability and uniqueness of traveling waves in the homogenous media (see [7, 15]). The main difficulty in treating equation (1.1) is caused by the lack of space regularity, since the semigroup generated by the nonlocal dispersal operator does not have regularizing effect. The lack of local comparison principle in nonlocal equations also causes lots of technical difficulties.

This dissertation is devoted to the existence, regularity, stability and uniqueness of transition fronts connecting 0 and 1 in nonlocal bistable equations in time heterogeneous media. By a transtion front connecting 0 and 1, we mean a global-in-time continuous solution $u(t,x)$ of (1.1) such that $u(t,x) \in (0,1)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$ and there exists an interface location function $X : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to-\infty} u(t,x+X(t)) = 1 \quad \text{and} \lim_{x\to\infty} u(t,x+X(t)) = 0 \quad \text{uniformly in} \quad t \in \mathbb{R}.$$

Instead of existence, we start our study with qualitative properties. We first prove the space regularity of transition front, that is,

- space regularity: any transition front $u(t,x)$ must be continuously differentiable in x and propagates to the right with a continuously differentiable interface location function $X(t)$, i.e., $\inf_{t\in\mathbb{R}} \dot{X}(t) > 0$.

We then turn to the study of space nonincreasing transition fronts and prove various important properties:

- uniform steepness: space derivative of the profile function $u(t, x + X(t))$ is negative uniformly in the time variable and locally uniformly in the space variable, that is, for any $M > 0$, $\sup_{|x|\le M} u_x(t, x + X(t)) < 0$;

- stability: the transition front exponentially attracts solutions with front-like initial data, that is, if the intial data $u_0 : \mathbb{R} \to [0, 1]$ is such that $u(x) \to 1$ as $x \to -\infty$ and $u(x) \to 0$ as $x \to \infty$, then there exists some $t_0 \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $\sup_{x\in\mathbb{R}} |u(t, x; t_0, u_0) - u(t, x + \xi)| \to 0$ exponentially as $t \to \infty$;

- exponential decaying estimates: the profile function $u(t, x + X(t))$ approaches to 1 exponentially as $x \to -\infty$ and to 0 exponentially as $x \to \infty$, uniformly in $t \in \mathbb{R}$.

Moreover, we show the uniqness of transition fronts, that is,

- uniqueness: any transition front, after certain space shift, coincides with a space nonincreasing transition front (if exists), verifying the uniqueness, up to space shifts, of transition fronts, and hence, all transition fronts satisfy just mentioned qualitative properties.

Also, we show that

- a transition front must be a periodic traveling wave in time periodic media and the asymptotic speeds of transition fronts exist in time uniquely ergodic media.

Finally, we prove

- the existence of space nonincreasing transition fronts by constructing appropriate approximating front-like solutions with regularities; this is done under certain additional assumptions.

Besides the study of transition fronts in time heterogeneous media of bistable type, it is also important to study transition fronts in space heterogeneous media of bistable type, that is, the equations

$$u_t = u_{xx} + f(x,u) \quad \text{and} \quad u_t = J * u - u + f(x,u),$$

where $f(x,u)$ is bistable in u for each $x \in \mathbb{R}$. However, much less is known in this case (see [12, 31, 56, 23]) because of the wave-blocking phenomenon (see [42]).

Also, there are lots of study of the equations

$$u_t = u_{xx} + f(t,x,u) \tag{1.3}$$

$$u_t = J * u - u + f(t,x,u) \tag{1.4}$$

in the monostable case and ignition case. Here, we just collect some existing results in the literature.

For equation (1.3), we refer to [5, 6, 27, 28, 35, 36, 37, 38, 39, 40, 78, 82] and references therein for works in the homogeneous media, i.e., $f(t,x,u) = f(u)$. We refer to [9, 23, 47, 48, 52, 56, 57, 77, 79, 84] and references therein for works in the space heterogeneous media, i.e., $f(t,x,u) = f(x,u)$, and to [1, 53, 63, 64, 66, 67, 69, 70] and references therein for works in the time heterogeneous media, i.e., $f(t,x,u) = f(t,x)$. There are also some works in the space-time heterogeneous media (see e.g. [41, 43, 44, 50, 51, 65, 68, 79]), but it remains widely open.

For equation (1.4), we refer to [7, 13, 15, 18, 19, 20, 61] and references therein for works in homogeneous media. The study of (1.4) in the heterogeneous media is rather recent and results concerning front propagation are very limited. In [21, 73, 74, 75], the authors investigated (1.4) in the space periodic monostable media and proved the existence of spreading speeds and periodic traveling waves. In [60], Rawal, Shen and Zhang studied the existence of spreading speeds and traveling waves of (1.4) in the space-time periodic monostable media. For (1.4) in the space heterogeneous monostable media, Berestycki, Coville and Vo studied in [8] the principal eigenvalue, positive solution and long-time behavior of solutions, while Lim and Zlatoš proved in [45] the existence of transition fronts. For (1.4) in the time heterogeneous media of ignition type, the authors of the present paper proved in [71, 72] the existence, regularity and stability of transition fronts.

The rest of the dissertation is organized as follows. In Chapter 2, we collect the main results obtained in this dissertation. In Chapter 3, we study the space regularity and propagation of transition fronts of, in particular, equation (1.1). In Chapter 4, we focus our study on space nonincreasing transition fronts of equation (1.1). We show that any space nonincreasing transition front enjoys uniform steepness, stability, uniform stability and exponential decaying estimates. In Chapter 5, we show that any transition front of equation (1.1), after certain space shift, coincides with a space nonincreasing transition front (if exists). In particular, any transtion front enjoys uniform steepness, stability, uniform stability and exponential decaying estimates. In Chapter 6, under the additional time periodic assumption on the nonlinearity, we show that any transition front must be a periodic traveling wave. In Chapter 7, under the assumption that the nonlinearity $f(t,u)$ is compact and uniquely ergodic, in the sense that the dynamical systems defined by shift operators on the hull of $f(t,u)$, is compact and uniquely ergodic, we show that the asymptotic speeds of transition fronts exist. In Chapter 8, we prove the existence of transition fronts. In Chapter 9, we make some remarks about the results obtained in the dissertation and mention some open problems along the line. We end the dissertation with two appendices: one on bistable traveling waves, and the other on comparison principles.

Chapter 2

Main results

For the moment, let us consider the following equation in space-time heterogeneous media

$$u_t = J * u - u + f(t, x, u), \quad (t, x) \in \mathbb{R} \times \mathbb{R} \tag{2.1}$$

with $f(t, x, 0) = 0 = f(t, x, 1)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. In this dissertation, we focus on the transition fronts of (2.1) connecting 0 and 1. Recall

Definition 2.1. *A global-in-time continuous solution $u(t, x)$ of (2.1) is called a transition front (connecting 0 and 1) in the sense of Berestycki-Hamel (see [10, 11], also see [63, 64]) if $u(t, x) \in (0, 1)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ and there exists a function $X : \mathbb{R} \to \mathbb{R}$, called interface location function, such that*

$$\lim_{x \to -\infty} u(t, x + X(t)) = 1 \text{ and } \lim_{x \to \infty} u(t, x + X(t)) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

We see that Definition 2.1 is equivalent to: a global-in-time continuous solution $u(t, x)$ of (2.1) is called a transition front if $u(t, -\infty) = 1$ and $u(t, \infty) = 0$ for any $t \in \mathbb{R}$, and for any $\epsilon \in (0, 1)$ there holds

$$\sup_{t \in \mathbb{R}} \operatorname{diam}\{x \in \mathbb{R} | \epsilon \le u(t, x) \le 1 - \epsilon\} < \infty.$$

This equivalent definition specifies the bounded interface width. We remark that neither the definition of transition front nor the equation (2.1) itself guarantees any space regularity of transition fronts beyond continuity. This lack of space regularity causes lots of troubles in studying transition fronts in nonlocal equations for that

- approximating solutions may not have a limit even bounded interface width can be verified;
- qualitative properties of transition fronts can hardly be reached.

Now, let us focus on equation (2.1) in the space-time heterogeneous media of bistable type, i.e.,

$$u_t = J * u - u + f(t,x,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2.2}$$

where the convolution kernel J and the nonlinearity f are assumed to satisfy Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.3, where

Hypothesis 2.1. *$J : \mathbb{R} \to \mathbb{R}$ satisfies $J \not\equiv 0$, $J \in C^1$, $J(x) = J(-x) \geq 0$ for all $x \in \mathbb{R}$, $\int_{\mathbb{R}} J(x)dx = 1$, and*

$$\int_{\mathbb{R}} J(x)e^{\gamma x}dx < \infty, \quad \int_{\mathbb{R}} |J'(x)|e^{\gamma x}dx < \infty, \quad \forall \gamma \in \mathbb{R}.$$

Hypothesis 2.2. *There exist $C^2(\mathbb{R})$ functions $f_B : [0,1] \to \mathbb{R}$ and $f_{\tilde{B}} : [0,1] \to \mathbb{R}$ such that*

$$f_B(u) \leq f(t,x,u) \leq f_{\tilde{B}}(u), \quad (t,x,u) \in \mathbb{R} \times \mathbb{R} \times [0,1].$$

Moreover, the following conditions hold:

- *$f : \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ is continuously differentiable in x and u, and satisfies*

$$\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_x(t,x,u)| < \infty \quad \text{and} \quad \sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_u(t,x,u)| < \infty;$$

- *f_B is of standard bistable type, that is, $f_B(0) = f_B(\theta) = f_B(1) = 0$ for some $\theta \in (0,1)$, $f_B(u) < 0$ for $u \in (0,\theta)$, $f_B(u) > 0$ for $u \in (\theta,1)$ and $\int_0^1 f_B(u)du > 0$;*
- *$f_{\tilde{B}}$ is also of standard bistable type, that is, $f_{\tilde{B}}(0) = f_{\tilde{B}}(\tilde{\theta}) = f_{\tilde{B}}(1) = 0$ for some $\tilde{\theta} \in (0,1)$, $f_{\tilde{B}}(u) < 0$ for $u \in (0,\tilde{\theta})$ and $f_{\tilde{B}}(u) > 0$ for $u \in (\tilde{\theta},1)$.*

Hypothesis 2.3. *There exist* $0 < \theta_0 < \tilde{\theta} \leq \theta < \theta_1 < 1$ *such that*

$$f_u(t,x,u) \leq 0, \quad u \in [0,\theta_0] \quad \textit{and} \quad f_u(t,x,u) \leq 0, \quad u \in [\theta_1, 1].$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Hypothesis 2.1 is a basic assumption on the dispersal kernel J. Hypothesis 2.2 is about the uniform bistability. In particular, we require $\int_0^1 f_B(u)du > 0$. Notice $\tilde{\theta} \leq \theta$. But, since it could happen $\tilde{\theta} < \theta$, $f(t,x,u)$ may not be of standard type for fixed $(t,x) \in \mathbb{R}$. Hypothesis 2.3 is about the uniform stability of constant solutions 0 and 1. We will take Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.3 as standard assumptions in the present dissertation. Note that we allow degeneracy at 0, 1 and other possible zeros. For different purposes, we will need enhanced verions of Hypothesis 2.2 and Hypothesis 2.3.

Instead of trying to prove the existence of transition fronts for equation (2.2), we start with various important qualitative properties of transition fronts, since this can be done in a more general setting.

Our first result concerning the space regularity and propagation of transition fronts of equation (2.2) is stated in the following

Theorem 2.2. *Suppose Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.3. Let* $u(t,x)$ *be an arbitrary transition front of equation* (2.2) *and* $X(t)$ *be its interface location function. Then,*

(i) $u(t,x)$ *is regular in space, that is, for any* $t \in \mathbb{R}$, $u(t,x)$ *is continuously differentiable in* x *and satisfies* $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)| < \infty$*;*

(ii) *there exists a continuously differentiable function* $\tilde{X} : \mathbb{R} \to \mathbb{R}$ *satisfying* $c_{\min} \leq \dot{\tilde{X}}(t) \leq c_{\max}$ *for some* $0 < c_{\min} \leq c_{\max} < \infty$ *such that* $\sup_{t\in\mathbb{R}} |X(t) - \tilde{X}(t)| < \infty$.

As it is known, space regularity of transition fronts is not a problem at all in the reaction-diffusion equation, but it is a big problem in the nonlocal equation (2.2), since the semigroup generated by the dispersal kernel J lacks of regularizing effects. But, in order to further study various imporant properties, such as stability and uniqueness, we need the space regularity to ensure the applicability of various techiniques for reaction-diffusion equations to nonlocal equations.

The proof of (i) in Theorem 2.2 relies on two things: rightward propagation estimates, and transition fronts being global-in-time. The rightward propagation estimates can be established due to the uniform bistability, i.e., $\int_0^1 f_B(u)du > 0$ in Hypothesis 2.2, which forces all front-like solutions to move to the right with their speeds controled from below by bistable traveling waves. Due to the rightward propagation estimates, we find that for any fixed x, only in certain period, the term $\left|\frac{u(t,x+\eta)-u(t,x)}{\eta}\right|$ may grow, which ensures the boundedness in the long run. But, there's another problem: lack of a priori information on transition fronts, since we are treating transition fronts. This lack of a priori information directly causes the boundedness problems of $\left|\frac{u(t_0,x+\eta)-u(t_0,x)}{\eta}\right|$ as $\eta \to 0$ at the initial time t_0. This is where the fact that transition fronts are global-in-time comes into play and helps out. The proof of (ii) in Theorem 2.2 is based on the rightward propagation estimates and a modification process. The fact that $\dot{\tilde{X}}(t)$ does not change its sign plays an important role in stability analysis.

We remark that the results in (i) in Theorem 2.2 are also true for equation (2.2) with ignition nonlinearities and some monostable nonlinearities; see Corollary 3.7 for more details.

With the understanding that all transition fronts are regular in space, we then turn to the study of various important qualitative properties, such as stability and uniquenss, of transition fronts of equation (2.2) in time heterogeneous media, i.e.,

$$u_t = J * u - u + f(t,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{2.3}$$

To do so, we need an enhanced version of Hypothesis 2.3 in the case $f(t,x,u) = f(t,u)$, that is,

Hypothesis 2.4. *There exist* $0 < \theta_0 < \tilde{\theta} \leq \theta < \theta_1 < 1$ *and* $\beta_0 > 0$, $\beta_1 > 0$ *such that*

$$f_u(t,u) \leq -\beta_0, \quad u \in (-\infty, \theta_0] \quad \textit{and} \quad f_u(t,u) \leq -\beta_1, \quad u \in [\theta_1, \infty)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

We remark that the values of $f(t,u)$ for $u \notin [0,1]$ plays no role in studying transtion fronts between 0 and 1. We prove

Theorem 2.3. *Suppose Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.4. Suppose, in addition, equation* (2.3) *admits a space decreasing transition front. Let* $u(t,x)$ *be an arbitrary transition front of equation* (2.3) *and* $X(t)$ *be its interface location function. Then, the following statements hold:*

(i) *monotonicity: for any* $t \in \mathbb{R}$, $u(t,x)$ *is decreasing in* x*;*

(ii) *uniform steepness: for any* $M > 0$, *there holds* $\sup_{t\in\mathbb{R}} \sup_{|x-X(t)|\leq M} u_x(t,x) < 0$*;*

(iii) *stability: let* $u_0 : \mathbb{R} \to [0,1]$ *be uniform continuous and satisfy*

$$\liminf_{x\to-\infty} u_0(x) > \theta_1 \quad \textit{and} \quad \limsup_{x\to\infty} u_0(x) < \theta_0,$$

and, $u(t,x;t_0,u_0)$ *be the solution of* (2.3) *with initial data* $u(t_0,\cdot;t_0,u_0) = u_0$. *Then, there exist* $t_0 = t_0(u_0) \in \mathbb{R}$, $\xi = \xi(u_0) \in \mathbb{R}$, $C = C(u_0) > 0$ *and* $\omega_* > 0$ *(independent of* u_0*) such that*

$$\sup_{x\in\mathbb{R}} |u(t,x;t_0,u_0) - u(t,x-\xi)| \leq Ce^{-\omega_*(t-t_0)}$$

for all $t \geq t_0$*;*

(iv) *uniform stability: let* $\{u_{t_0}\}_{t_0\in\mathbb{R}}$ *be a family of initial data satisfying*

$$u(t_0, x-\xi_0^-) - \mu_0 \leq u_{t_0}(x) \leq u(t_0, x-\xi_0^+) + \mu_0, \quad x \in \mathbb{R},\ t_0 \in \mathbb{R}$$

for $\xi_0^{\pm} \in \mathbb{R}$ *and* $\mu_0 \in (0, \min\{\theta_0, 1-\theta_1\})$ *being independent of* $t_0 \in \mathbb{R}$. *Then, there exist* t_0*-independent constants* $C > 0$ *and* $\omega_* > 0$, *and a family of shifts* $\{\xi_{t_0}\}_{t_0 \in \mathbb{R}} \subset \mathbb{R}$ *satisfying* $\sup_{t_0 \in \mathbb{R}} |\xi_{t_0}| < \infty$ *such that*

$$\sup_{x \in \mathbb{R}} |u(t, x; t_0, u_{t_0}) - u(t, x - \xi_{t_0})| \leq C e^{-\omega_*(t - t_0)}$$

for all $t \geq t_0$ *and* $t_0 \in \mathbb{R}$;

(v) *exponential decaying estimates: there exist two exponents* $c_{\pm} > 0$ *and two shifts* $h_{\pm} > 0$ *such that*

$$u(t, x + X(t) + h_+) \leq e^{-c_+ x} \quad \text{and} \quad u(t, x + X(t) - h_-) \geq 1 - e^{c_- x}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}$;

(vi) *uniqueness: if* $v(t, x)$ *is another transition front of* (2.3), *then there exists a shift* $\xi \in \mathbb{R}$ *such that* $v(t, x) = u(t, x + \xi)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$;

(vii) *periodicity: if, in addition,* $f(t, u)$ *is periodic in* t, *then* $u(t, x)$ *is a periodic traveling wave;*

(viii) *asymptotic speeds: if, in addition,* $f(t, u)$ *is compact and uniquely ergodic, then* $\lim_{t \to \pm\infty} \frac{X(t)}{t}$ *exist.*

Let us explain the strategies for proving Theorem 2.3. The proof of Theorem 2.3 starts with the verification of stability as in (iii). But, to prove the stability, we need the uniform steepness as in (ii), which however can not be checked for an arbitrary transition front. Hence, instead of trying to study an arbitrary transition front, we focus on an arbitrary space nonincreasing transition front for the moment. We then show that an arbitrary space nonincreasing transition front is uniformly steep as in (ii), which, together with the results in Theorem 2.2, allow us to establish the stability, the uniform stability and the exponential decaying estimates as in (iii), (iv) and (v), respectively, for this space nonincreasing transition front. Then, using the uniform stability of space nonincreasing transition fronts, we prove that an arbitrary transition front coincides with an space nonincreasing transition front, up to a space shift, and hence, any transition front satisfies (i)-(vi) in Theorem 2.3. Finally, we show (vii) and (viii) in Theorem 2.3. It is referred to Section 7 for the definition of $f(t,u)$ being compact and uniquely ergodic.

Finally, we investigate the existence of transition fronts of (2.3). To do so, we replace Hypothesis 2.2 by

Hypothesis 2.5. *There exist $C^2(\mathbb{R})$ functions $f_B:[0,1]\to\mathbb{R}$ and $f_{\tilde{B}}:[0,1]\to\mathbb{R}$ such that*

$$f_B(u)\leq f(t,u)\leq f_{\tilde{B}}(u),\quad (t,u)\in\mathbb{R}\times[0,1].$$

Moreover, the following conditions hold:

- $f:\mathbb{R}\times[0,1]\to\mathbb{R}$ *is continuously differentiable and satisfies*

$$\sup_{(t,u)\in\mathbb{R}\times[0,1]}|f_t(t,u)|<\infty\quad \text{and}\quad \sup_{(t,u)\in\mathbb{R}\times[0,1]}|f_u(t,u)|<\infty;$$

- f_B *satisfies* $f_B(0)=f_B(\theta)=f_B(1)=0$ *for some* $\theta\in(0,1)$, $f_B(u)<0$ *for* $u\in(0,\theta)$, $f_B(u)>0$ *for* $u\in(\theta,1)$, $f_B'(1)<0$ *and* $\int_0^1 f_B(u)du>0$;

- $f_{\tilde{B}}$ *satisfies* $f_{\tilde{B}}(0)=f_{\tilde{B}}(\theta)=f_{\tilde{B}}(1)=0$, $f_{\tilde{B}}(u)<0$ *for* $u\in(0,\theta)$ *and* $f_{\tilde{B}}(u)>0$ *for* $u\in(\theta,1)$.

The requirement $f_B(\theta)=0=f_{\tilde{B}}(\theta)$ in Hypothesis 2.5 is a little restrictive; see (i) in Section 9 for a possible variation. We prove

Theorem 2.4. *Suppose Hypothesis 2.1, Hypothesis 2.5 and Hypothesis 2.3. Then, equation* (2.3) *admits a transition front.*

Clearly, the transition front obtained in Theorem 2.4 must satisfy all properties in Theorem 2.2 and Theorem 2.3. The proof of Theorem 2.4 is constructive. We first construct approximating front-like solutions. However, due to the lack of space regularity in nonlocal equations, it is not clear that the approximating solutions will converge to some solution of (2.2). Also even if they do converge to some solution, it is difficult to see that the limiting solution is a transition front. We then first establish the uniform boundedness of the interface width and uniform decaying estimates of approximating solutions, which assure the limiting solution (if exists) is a transition front. Then, we show the uniform Lipschitz continuity in space of the approximating solutions, which of course implies the convergence of the approximating solutions thanks to Arzelà-Ascoli theorem. We have used this strategy to construct transition fronts in nonlocal equation in time heterogeneous media of ignition type (see [71]).

We point out that the transition front obtained in Theorem 2.4 through the approximating process is only uniformly Lipschitz continuous in space. Of course, we can then apply Theorem 2.2 to conclude the space regularity. Here, we want to mention another approach to space regularity, that is, putting more conditions on the nonlinearity f to ensure that the approximating solutions have more regularity in space. This approach has been used in [72] to the nonlocal equation in time heterogeneous media of ignition type.

Chapter 3

Properties of transition fronts in space-time heterogeneous media

In this chapter, we study some general properties, especially the space regularity, of arbitrary transition fronts of equation (2.2). Throughout this section, we assume Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.3.

In what follows in this section, $u(t,x)$ will be an arbitrary transition front with the interface location function $X(t)$ (see Definition 2.1).

In this section, we first establish in Section 3.1 the rightward propagation estimate of the transition front $u(t,x)$, that is, it travels from the left infinity to the right infinity with certain speed in the average sense. We then use this propagation estimate to prove the space regularity, i.e., (i) in Theorem 2.2, in Section 3.2. We remark that our arguments for the space regularity can be applied to ignition nonlinearities and some monostable nonlinearities (see Remark 3.6 for more details). Finally, in Section 3.3, we use the rightward propagation estimate to find a modification of $X(t)$, which is of great technical importance.

3.1 Rightward propagation estimates

In this section, we study the rightward propagation of $u(t,x)$. For $\lambda \in (0,1)$, let $X_\lambda^-(t)$ and $X_\lambda^+(t)$ be the leftmost and rightmost interface locations at λ, that is,

$$X_\lambda^-(t) = \min\{x \in \mathbb{R} | u(t,x) = \lambda\} \quad \text{and} \quad X_\lambda^+(t) = \max\{x \in \mathbb{R} | u(t,x) = \lambda\}. \tag{3.1}$$

Trivially, $X_\lambda^-(t) \leq X_\lambda^+(t)$ and $X^\pm(t)$ are decreasing in λ. By the continuity of $u(t,x)$ in x, there holds $u(t, X_\lambda^-(t)) = \lambda = u(t, X_\lambda^+(t))$. But, due to the nonlocality, it is not sure whether $X^\pm(t)$ are continuous in t.

From the definition of transition fronts, we have the following simple lemma.

Lemma 3.1. *The following statements hold:*

(i) *for any* $0 < \lambda_1 \leq \lambda_2 < 1$, *there holds* $\sup_{t\in\mathbb{R}}[X^+_{\lambda_1}(t) - X^-_{\lambda_2}(t)] < \infty$*;*

(ii) *for any* $\lambda \in (0,1)$, *there hold* $\sup_{t\in\mathbb{R}} |X(t) - X^{\pm}_{\lambda}(t)| < \infty$.

Proof. (i) By the uniform-in-t limits $\lim_{x\to-\infty} u(t, x+X(t)) = 1$ and $\lim_{x\to\infty} u(t, x+X(t)) = 0$, there exist x_1 and x_2 such that $u(t, x+X(t)) > \lambda_2$ for all $x \leq x_2$ and $u(t, x+X(t)) < \lambda_1$ for all $x \geq x_1$. It then follows from the definition of $X^-_{\lambda_2}(t)$ and $X^+_{\lambda_1}(t)$ that $x_2 + X(t) \leq X^-_{\lambda_2}(t)$ and $x_1 + X(t) \geq X^+_{\lambda_1}(t)$. The result then follows.

(ii) Let $\lambda_1 = \lambda = \lambda_2$ in the proof of (i), we have $x_2 + X(t) \leq X^-_{\lambda}(t)$ and $x_1 + X(t) \geq X^+_{\lambda}(t)$. In particular,

$$x_2 + X(t) \leq X^-_{\lambda}(t) \leq X^+_{\lambda}(t) \leq x_1 + X(t).$$

This completes the proof. □

The next result gives the rightward propagation of $u(t, x)$ in terms of $X(t)$.

Theorem 3.2. *There exist* $c^{\rm avg}_{\rm max} \geq c^{\rm avg}_{\rm min} > 0$ *and* $T^{\rm avg}_{\rm min}, T^{\rm avg}_{\rm max} > 0$ *such that*

$$c^{\rm avg}_{\rm min}(t - t_0 - T^{\rm avg}_{\rm min}) \leq X(t) - X(t_0) \leq c^{\rm avg}_{\rm max}(t - t_0 + T^{\rm avg}_{\rm max}), \quad t \geq t_0.$$

Proof. We here only prove the first inequality; the second one can be proven along the same line due to the bistability. Fix some $\lambda \in (\theta, 1)$. We write $X^-(t) = X^-_{\lambda}(t)$. Since $\sup_{t\in\mathbb{R}} |X(t) - X^-(t)| < \infty$ by Lemma 3.1, it suffices to show

$$X^-(t) - X^-(t_0) \geq c(t - t_0 - T), \quad t \geq t_0 \tag{3.2}$$

for some $c > 0$ and $T > 0$.

Let (c_B, ϕ_B) with $c_B > 0$ be the unique solution of

$$\begin{cases} J * \phi - \phi + c\phi_x + f_B(\phi) = 0, \\ \phi_x < 0, \ \phi(0) = \theta, \ \phi(-\infty) = 1 \text{ and } \quad \phi(\infty) = 0. \end{cases}$$

See Appendix A for more properties about ϕ_B.

Let $u_0 : \mathbb{R} \to [0, 1]$ be a uniformly continuous and nonincreasing function satisfying $u_0(x) = \lambda$ for $x \le x_0$ and $u_0(x) = 0$ for $x \ge 0$, where $x_0 < 0$ is fixed. Since $X^-(t)$ is the leftmost interface location at λ, we see that for any $t_0 \in \mathbb{R}$, there holds $u(t_0, x + X^-(t_0)) \ge u_0(x)$ for all $x \in \mathbb{R}$, and then, by $f(t, x, u) \ge f_B(u)$ and the comparison principle, we find

$$u(t, x + X^-(t_0)) \ge u_B(t - t_0, x; u_0), \quad x \in \mathbb{R}, \ t \ge t_0,$$

where $u_B(t, x; u_0)$ is the unique solution to $u_t = J * u - u + f_B(u)$ with $u_B(0, \cdot; u_0) = u_0$. By the choice of u_0 and the stability of bistable traveling waves (see Lemma A.2), there are constants $x_B = x_B(\lambda) \in \mathbb{R}$, $q_B = q_B(\lambda) > 0$ and $\omega_B > 0$ such that

$$u_B(t - t_0, x; u_0) \ge \phi_B(x - x_B - c_B(t - t_0)) - q_B e^{-\omega_B (t - t_0)}, \quad x \in \mathbb{R}, \ t \ge t_0.$$

Hence,

$$u(t, x + X^-(t_0)) \ge \phi_B(x - x_B - c_B(t - t_0)) - q_B e^{-\omega_B (t - t_0)}, \quad x \in \mathbb{R}, \ t \ge t_0.$$

Let $T_0 = T_0(\lambda) > 0$ be such that $q_B e^{-\omega_B T_0} = \frac{1-\lambda}{2}$ and denote by $\xi_B(\frac{1+\lambda}{2})$ the unique point such that $\phi_B(\xi_B(\frac{1+\lambda}{2})) = \frac{1+\lambda}{2}$. Setting $x_* = x_B + c_B(t - t_0) + \xi_B(\frac{1+\lambda}{2})$, the monotonicity of ϕ_B implies that for all $t \ge t_0 + T_0$ and $x < x^*$

$$u(t, x + X^-(t_0)) > \phi_B(x_* - x_B - c_B(t - t_0)) - q_B e^{-\omega_B T_0} = \phi_B(\xi_B(\frac{1 + \lambda}{2})) - q_B e^{-\omega_B T_0} = \lambda.$$

This says $x_* + X^-(t_0) \le X^-(t)$ for all $t \ge t_0 + T_0$, that is,

$$X^-(t) - X^-(t_0) \ge x_B + c_B(t - t_0) + \xi_B(\frac{1+\lambda}{2}), \quad t \ge t_0 + T_0. \tag{3.3}$$

We now estimate $X^-(t) - X^-(t_0)$ for $t \in [t_0, t_0 + T_0]$. We claim that there exists $z = z(T_0) < 0$ such that

$$X^-(t) - X^-(t_0) \ge z, \quad t \in [t_0, t_0 + T_0]. \tag{3.4}$$

Let $u_B(t, x; u_0)$ and $u_B(t; \lambda) := u_B(t, x; \lambda)$ be solutions of $u_t = J * u - u + f_B(u)$ with $u_B(0, x; u_0) = u_0(x)$ and $u_B(0; \lambda) = u_B(0, x; \lambda) \equiv \lambda$, respectively. By the comparison principle, we have $u_B(t, x; u_0) < u_B(t; \lambda)$ for all $x \in \mathbb{R}$ and $t > 0$, and $u_B(t, x; u_0)$ is strictly decreasing in x for $t > 0$.

We see that for any $t > 0$, $u_B(t, -\infty; u_0) = u_B(t; \lambda)$. This is because that $\frac{d}{dt} u_B(t, -\infty; u_0) = f_B(u_B(t, -\infty; u_0))$ for $t > 0$ and $u_B(0, -\infty; u_0) = \lambda$. Since $\lambda \in (\theta, 1)$, as a solution of the ODE $u_t = f_B(u)$, $u_B(t; \lambda)$ is strictly increasing in t, which implies that $u_B(t, -\infty; u_0) = u_B(t; \lambda) > \lambda$ for $t > 0$. As a result, for any $t > 0$ there exists a unique $\xi_B(t) \in \mathbb{R}$ such that $u_B(t, \xi_B(t); u_0) = \lambda$. Moreover, $\xi_B(t)$ is continuous in t.

Since $f(t, x, u) \ge f_B(u)$ and $u(t_0, \cdot + X^-(t_0)) \ge u_0$, the comparison principle implies that

$$u(t, x + X^-(t_0)) > u_B(t - t_0, x; u_0), \quad x \in \mathbb{R},\ t > t_0.$$

Setting $x_{**} = \xi_B(t - t_0)$, we find $u(t, x + X^-(t_0)) > \lambda$ for all $x < x_{**}$ by the monotonicity of $u_B(t, x; u_0)$ in x, which implies that $X^-(t) \ge x_{**} + X^-(t_0) = \xi_B(t - t_0) + X^-(t_0)$ for $t > t_0$. Thus, (3.4) follows if $\inf_{t \in (t_0, t_0 + T_0]} \xi_B(t - t_0) > -\infty$, that is,

$$\inf_{t \in (0, T_0]} \xi_B(t) > -\infty. \tag{3.5}$$

We now show (3.5). Since $u_0(x) = \lambda$ for $x \le x_0$, continuity with respect to the initial data (in sup norm) implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$u_B(t;\lambda) - \lambda \le \epsilon \quad \text{and} \quad \sup_{x \le x_0}[u_B(t;\lambda) - u_B(t,x;u_0)] = u_B(t;\lambda) - u_B(t,x_0;u_0) \le \epsilon$$

for all $t \in [0,\delta]$, where the equality is due to monotonicity. By Hypothesis 2.1, J concentrates near 0 and decays very fast as $x \to \pm\infty$. Thus, we can choose $x_1 = x_1(\epsilon) << x_0$ such that $\int_{-\infty}^{x_0} J(x-y)dy \ge 1-\epsilon$ for all $x \le x_1$. Now, for any $x \le x_1$ and $t \in (0,\delta]$, we have

$$\begin{aligned}
\frac{d}{dt}u_B(t,x;u_0) &= \int_{\mathbb{R}} J(x-y)u_B(t,y;u_0)dy - u_B(t,x;u_0) + f_B(u_B(t,x;u_0)) \\
&\ge \int_{-\infty}^{x_0} J(x-y)u_B(t,y;u_0)dy - u_B(t,x;u_0) + f_B(u_B(t,x;u_0)) \\
&\ge (1-\epsilon)\inf_{x \le x_0} u_B(t,x;u_0) - u_B(t;\lambda) + f_B(u_B(t,x;u_0)) \\
&= -(1-\epsilon)\sup_{x \le x_0}[u_B(t;\lambda) - u_B(t,x;u_0)] - \epsilon u_B(t;\lambda) + f_B(u_B(t,x;u_0)) \\
&\ge -\epsilon(1-\epsilon) - \epsilon(\lambda+\epsilon) + f_B(u_B(t,x;u_0)) > 0
\end{aligned}$$

if we choose $\epsilon > 0$ sufficiently small, since then $f_B(u_B(t,x;u_0))$ is close to $f_B(\lambda)$, which is positive. This simply means that $u_B(t,x;u_0) > \lambda$ for all $x \le x_1$ and $t \in (0,\delta]$, which implies that $\xi_B(t) > x_1$ for $t \in (0,\delta]$. The continuity of ξ_B then leads to (3.5). This proves (3.4). (3.2) then follows from (3.3) and (3.4). This completes the proof. □

As a simple consequence of Theorem 3.2, we have

Corollary 3.3. *There holds $X(t) \to \pm\infty$ as $t \to \pm\infty$. In particular, $u(t,x) \to 1$ as $t \to \infty$ and $u(t,x) \to 0$ as $t \to -\infty$ locally uniformly in x.*

Proof. We have from Lemma 3.2 that

$$c_{\min}^{\text{avg}}(t - t_0 - T_{\min}^{\text{avg}}) \le X(t) - X(t_0).$$

Setting $t \to \infty$ in the above estimate, we find $X(t) \to \infty$ as $t \to \infty$. Setting $t_0 \to -\infty$, we find $X(t_0) \to -\infty$ as $t_0 \to -\infty$. $\square$

This corollary shows that any transition front travels from the left infinity to the right infinity. Thus, steady-state-like transition fronts, blocking the propagations of solutions, do not exist.

3.2 Space regularity

In this section, study the space regularity of the transition front $u(t,x)$, that is, we are going to prove (i) in Theorem 2.2. For convenience, we restate (i) in Theorem 2.2 as

Theorem 3.4. *For any $t \in \mathbb{R}$, $u(t,x)$ is continuously differentiable in x. Moreover, there holds $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)| < \infty$.*

Proof. For $(t,x) \in \mathbb{R}\times\mathbb{R}$ and $\eta \in \mathbb{R}$ with $|\eta| \le \delta_0 \ll 1$, we set

$$v^\eta(t,x) = \frac{u(t,x+\eta) - u(t,x)}{\eta}.$$

It's easy to see that $v^\eta(t,x)$ satisfies

$$v_t^\eta(t,x) = \int_{\mathbb{R}} J(x-y)v^\eta(t,y)dy - v^\eta(t,x) + a^\eta(t,x)v^\eta(t,x) + \tilde{a}^\eta(t,x), \tag{3.6}$$

where

$$a^\eta(t,x) = \frac{f(t,x,u(t,x+\eta)) - f(t,x,u(t,x))}{u(t,x+\eta) - u(t,x)},$$
$$\tilde{a}^\eta(t,x) = \frac{f(t,x+\eta,u(t,x+\eta)) - f(t,x,u(t,x+\eta))}{\eta}.$$

Hence, for any fixed x, treating (3.6) as an ODE in the variable t, we find for any $t \geq t_0$

$$\begin{aligned} v^\eta(t,x) = v^\eta(t_0,x)e^{-\int_{t_0}^t (1-a^\eta(s,x))ds} + \int_{t_0}^t b^\eta(\tau,x)e^{-\int_\tau^t (1-a^\eta(s,x))ds}d\tau \\ + \int_{t_0}^t \tilde{a}^\eta(\tau,x)e^{-\int_\tau^t (1-a^\eta(s,x))ds}d\tau \end{aligned} \tag{3.7}$$

where

$$b^\eta(t,x) = \int_{\mathbb{R}} J(x-y)v^\eta(t,y)dy = \int_{\mathbb{R}} \frac{J(x-y+\eta) - J(x-y)}{\eta} u(t,y)dy.$$

Note that for any $(t,x) \in \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} a^\eta(t,x) &\to f_u(t,x,u(t,x)), \\ \tilde{a}^\eta(t,x) &\to f_x(t,x,u(t,x)), \\ b^\eta(t,x) &\to \int_{\mathbb{R}} J'(x-y)u(t,y)dy. \end{aligned} \tag{3.8}$$

as $\eta \to 0$.

To show the existence of the limit $\lim_{\eta\to 0} v^\eta(t,x)$, we first do some preparations. We set

$$L_0 = \delta_0 + \sup_{t\in\mathbb{R}} |X(t) - X^+_{\theta_0}(t)| \quad \text{and} \quad L_1 = \delta_0 + \sup_{t\in\mathbb{R}} |X(t) - X^-_{\theta_1}(t)|,$$

where θ_0 and θ_1 are as in Hypothesis 2.3. By Lemma 3.1, $L_0 < \infty$ and $L_1 < \infty$. We also set

$$\begin{aligned} I_l(t) &= (-\infty, X(t) - L_1), \\ I_m(t) &= [X(t) - L_1, X(t) + L_0], \\ I_r(t) &= (X(t) + L_0, \infty). \end{aligned}$$

Trivially, for any $t \in \mathbb{R}$, $I_l(t)$, $I_m(t)$ and$I_r(t)$ are pairwise disjoint and $I_l(t) \cup I_m(t) \cup I_r(t) = \mathbb{R}$. However, since $X(t)$ may not be continuous, so are $I_l(t)$, $I_m(t)$ and $I_r(t)$.

Since $X(t) \to \pm\infty$ as $t \to \pm\infty$ by Corollary 3.3, for any fixed $x \in \mathbb{R}$, there hold $x \in I_r(t)$ for all $t \ll -1$ and $x \in I_l(t)$ for all $t \gg 1$. Thus, for any fixed $x \in \mathbb{R}$, we may define

$$t_{\text{first}}(x) = \sup\{\tilde{t} \in \mathbb{R} | x \in I_r(t) \text{ for all } t \leq \tilde{t}\},$$
$$t_{\text{last}}(x) = \inf\{\tilde{t} \in \mathbb{R} | x \in I_l(t) \text{ for all } t \geq \tilde{t}\}.$$

Clearly, the fact that $X(t) \to \pm\infty$ as $t \to \pm\infty$ ensures $-\infty < t_{\text{first}}(x) \leq t_{\text{last}}(x) < \infty$ (notice, $t_{\text{first}}(x) = t_{\text{last}}(x)$ may happen since $X(t)$ may jump). By the definition of $t_{\text{first}}(x)$ and $t_{\text{last}}(x)$, we see

$$x \in \begin{cases} I_r(t), & t < t_{\text{first}}(x), \\ I_l(t), & t > t_{\text{last}}(x). \end{cases} \tag{3.9}$$

Moreover, there holds

$$T := \sup_{x \in \mathbb{R}} [t_{\text{last}}(x) - t_{\text{first}}(x)] < \infty. \tag{3.10}$$

To see this, we suppose $t_{\text{first}}(x) < t_{\text{last}}(x)$ and for technical reasons, we consider two cases:

Case 1. $x \notin I_r(t_{\text{first}}(x))$ In this case, we have $x \in I_l(t_{\text{first}}(x)) \cup I_m(t_{\text{first}}(x))$, that is, $x \leq X(t_{\text{first}}(x)) + L_0$. Thus, for all $t \geq t_{\text{first}}(x) + T^{\text{avg}}_{\min} + \frac{L_0+L_1+1}{c^{\text{avg}}_{\min}}$, we see from Lemma 3.2 that

$$x \leq X(t_{\text{first}}(x)) + L_0 \leq X(t) - c^{\text{avg}}_{\min}(t - t_{\text{first}}(x) - T^{\text{avg}}_{\min}) + L_0 \leq X(t) - L_1 - 1.$$

This, implies that $x \in I_l(t)$ for all $t \geq t_{\text{first}}(x) + T^{\text{avg}}_{\min} + \frac{L_0+L_1+1}{c^{\text{avg}}_{\min}}$, and hence, by definition

$$t_{\text{last}}(x) \leq t_{\text{first}}(x) + T^{\text{avg}}_{\min} + \frac{L_0 + L_1 + 1}{c^{\text{avg}}_{\min}}. \tag{3.11}$$

Case 2. $x \in I_r(t_{\text{first}}(x))$ In this case, we can find a sequence $\{t_n\}$ satisfying $t_n > t_{\text{first}}(x)$, $t_n \to t_{\text{first}}(x)$ as $n \to \infty$ and $x \notin I_r(t_n(x))$. Then, similar arguments as in the case $x \notin I_r(t_{\text{first}}(x))$ lead to

$$t_{\text{last}}(x) \leq t_n + T^{\text{avg}}_{\min} + \frac{L_0 + L_1 + 1}{c^{\text{avg}}_{\min}}.$$

Passing to the limit $n \to \infty$, we find (3.11) again. Hence, we have shown (3.10).

Also, by Hypothesis 2.3 and the choices of L_0 and L_1, we find that for any $(t,x) \in \mathbb{R} \times \mathbb{R}$ and $0 < |\eta| \leq \delta_0$

$$a^\eta(t,x) \leq \begin{cases} 0, & x \in I_r(t), \\ 0, & x \in I_l(t). \end{cases}$$

In particular, we have from (3.9) that for any $(t,x) \in \mathbb{R} \times \mathbb{R}$ and $0 < |\eta| \leq \delta_0$, there holds

$$a^\eta(t,x) \leq \begin{cases} 0, & t < t_{\text{first}}(x), \\ 0, & t > t_{\text{last}}(x). \end{cases} \tag{3.12}$$

By (3.8), (3.12) holds with $a^\eta(t,x)$ replaced by $f_u(t,x,u(t,x))$, that is,

$$f_u(t,x,u(t,x)) \leq \begin{cases} 0, & t < t_{\text{first}}(x), \\ 0, & t > t_{\text{last}}(x). \end{cases} \tag{3.13}$$

Now, we are ready to prove the existence of $\lim_{\eta\to 0} v^\eta(t,x)$. To do so, we fix any $x \in \mathbb{R}$. We are going to take $t_0 \to -\infty$ along some subsequence, and so $t_0 \ll t_{\text{first}}(x)$. For t, there are three cases: (i) $t \leq t_{\text{first}}(x)$; (ii) $t \in [t_{\text{first}}(x), t_{\text{last}}(x)]$; (iii) $t \geq t_{\text{last}}(x)$. Here, we only consider the case $t \geq t_{\text{last}}(x)$; other two cases can be treated similarly and are simpler.

Hence, we assume $t_0 \ll t_{\text{first}}(x)$ and $t \geq t_{\text{last}}(x)$ in the rest of the proof. We treat three terms on the right hand side of (3.7) separately.

For the second term on the right hand side of (3.7), we claim

$$\int_{t_0}^{t} b^\eta(\tau,x) e^{-\int_\tau^t (1-a^\eta(s,x))ds} d\tau \to \int_{t_0}^{t} \left(\int_{\mathbb{R}} J'(x-y) u(\tau,y) dy \right) e^{-\int_\tau^t (1-f_u(s,x,u(s,x)))ds} d\tau$$
$$\text{as } \eta \to 0 \ \text{ uniformly in } t_0 \ll t_{\text{first}}(x). \tag{3.14}$$

To see this, we notice

$$\begin{aligned}
&\bigg|\int_{t_0}^t b^\eta(\tau,x)e^{-\int_\tau^t(1-a^\eta(s,x))ds}d\tau-\int_{t_0}^t\bigg(\int_{\mathbb{R}}J'(x-y)u(\tau,y)dy\bigg)e^{-\int_\tau^t(1-f_u(s,x,u(s,x)))ds}d\tau\bigg|\\
&\qquad\le\int_{-\infty}^t\bigg|b^\eta(\tau,x)e^{-\int_\tau^t(1-a^\eta(s,x))ds}-\bigg(\int_{\mathbb{R}}J'(x-y)u(\tau,y)dy\bigg)e^{-\int_\tau^t(1-f_u(s,x,u(s,x)))ds}\bigg|d\tau.
\end{aligned}$$

By (3.8), the integrand converges to 0 as $\eta\to 0$ pointwise. Thus, by dominated convergence theorem, we only need to make sure that the integrand is controlled by some integrable function. Writing $b^0(\tau,x)=\int_{\mathbb{R}}J'(x-y)u(\tau,y)dy$ and $a^0(\tau,x)=f_u(\tau,x,u(\tau,x))$, we only need to make sure that the function

$$\tau\mapsto\sup_{0\le|\eta|\le\delta_0}\bigg|b^\eta(\tau,x)e^{-\int_\tau^t(1-a^\eta(s,x))ds}\bigg|$$

is integrable over $(-\infty,t]$. To see this, we first note $M:=\sup_{0\le|\eta|\le\delta_0}|b^\eta(\tau,x)|<\infty$ and the following uniform-in-η estimates hold:

$$\begin{aligned}
e^{-\int_r^{t_{\rm first}(x)}(1-a^\eta(s,x))ds}&\le e^{-(t_{\rm first}(x)-r)},\quad r\le t_{\rm first}(x),\\
e^{-\int_r^{t_{\rm last}(x)}(1-a^\eta(s,x))ds}&\le e^{C_aT},\quad r\in[t_{\rm first}(x),t_{\rm last}(x)],\\
e^{-\int_r^t(1-a^\eta(s,x))ds}&\le e^{-(t-r)},\quad r\in[t_{\rm last}(x),t],
\end{aligned}\tag{3.15}$$

where

$$C_a:=\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}\sup_{0<|\eta|\le\delta_0}|1-a^\eta(t,x)|<\infty$$

by Hypothesis 2.2. They are simple consequences of (3.10), (3.12) and (3.13). It then follows that

$$\sup_{0\le|\eta|\le\delta_0}\bigg|b^\eta(\tau,x)e^{-\int_\tau^t(1-a^\eta(s,x))ds}\bigg|\le M\sup_{0\le|\eta|\le\delta_0}e^{-\int_\tau^t(1-a^\eta(s,x))ds}.$$

To bound the last integral uniformly in $0\le|\eta|\le\delta_0$, according to (3.15), we consider three cases:

Case (i). $\tau < t_{\text{first}}(x)$ In this case,

$$\sup_{0\le|\eta|\le\delta_0} e^{-\int_\tau^t(1-a^\eta(s,x))ds} = \sup_{0\le|\eta|\le\delta_0} e^{-[\int_\tau^{t_{\text{first}}(x)} + \int_{t_{\text{first}}(x)}^{t_{\text{last}}(x)} + \int_{t_{\text{last}}(x)}^t](1-a^\eta(s,x))ds}$$
$$\le e^{-(t_{\text{first}}(x)-\tau)} e^{C_a T} e^{-(t-t_{\text{last}}(x))};$$

Case (ii). $\tau \in [t_{\text{first}}(x), t_{\text{last}}(x)]$ In this case,

$$\sup_{0\le|\eta|\le\delta_0} e^{-\int_\tau^t(1-a^\eta(s,x))ds} = \sup_{0\le|\eta|\le\delta_0} e^{-[\int_\tau^{t_{\text{last}}(x)} + \int_{t_{\text{last}}(x)}^t](1-a^\eta(s,x))ds} \le e^{C_a T} e^{-(t-t_{\text{last}}(x))};$$

Case (iii). $\tau \in (t_{\text{last}}(x), t]$ In this case,

$$\sup_{0\le|\eta|\le\delta_0} e^{-\int_\tau^t(1-a^\eta(s,x))ds} \le e^{-(t-\tau)}.$$

Thus, setting

$$h(\tau) = \begin{cases} e^{-(t_{\text{first}}(x)-\tau)} e^{C_a T} e^{-(t-t_{\text{last}}(x))}, & \tau < t_{\text{first}}(x) \\ e^{C_a T} e^{-(t-t_{\text{last}}(x))}, & \tau \in [t_{\text{first}}(x), t_{\text{last}}(x)] \\ e^{-(t-\tau)}, & \tau \in (t_{\text{last}}(x), t], \end{cases}$$

we find for any $\tau \in (-\infty, t]$

$$\sup_{0<|\eta|\le\delta_0} \left| b^\eta(\tau,x) e^{-\int_\tau^t(1-a^\eta(s,x))ds} - \left(\int_{\mathbb{R}} J'(x-y)u(\tau,y)dy\right) e^{-\int_\tau^t(1-f_u(s,x,u(s,x)))ds} \right| \le 2h(\tau).$$

To show (3.14), it remains to show $\int_{-\infty}^{t} h(\tau)d\tau < \infty$. But, we readily compute

$$
\begin{aligned}
\int_{-\infty}^{t} h(\tau)d\tau &= \int_{-\infty}^{t_{\text{first}}(x)} h(\tau)d\tau + \int_{t_{\text{first}}(x)}^{t_{\text{last}}(x)} h(\tau)d\tau + \int_{t_{\text{last}}(x)}^{t} h(\tau)d\tau \\
&\le \int_{-\infty}^{t_{\text{first}}(x)} e^{-(t_{\text{first}}(x)-\tau)} e^{C_a T} e^{-(t-t_{\text{last}}(x))} d\tau \\
&\quad + \int_{t_{\text{first}}(x)}^{t_{\text{last}}(x)} e^{C_a T} e^{-(t-t_{\text{last}}(x))} d\tau + \int_{t_{\text{last}}(x)}^{t} e^{-(t-\tau)} d\tau \\
&\le e^{C_a T} + T e^{C_a T} + 1.
\end{aligned}
\tag{3.16}
$$

Thus, we have shown (3.14). Note that the last bound is uniform in $(t, x) \in \mathbb{R} \times \mathbb{R}$.

For the third term on the right hand side of (3.7), we claim

$$
\begin{aligned}
&\int_{t_0}^{t} \tilde{a}^{\eta}(\tau, x) e^{-\int_{\tau}^{t}(1-a^{\eta}(s,x))ds} d\tau \to \int_{t_0}^{t} f_x(t, x, u(t, x)) e^{-\int_{\tau}^{t}(1-f_u(s,x,u(s,x)))ds} d\tau \\
&\qquad\qquad\qquad \text{as } \eta \to 0 \ \text{ uniformly in } t_0 \ll t_{\text{first}}(x).
\end{aligned}
\tag{3.17}
$$

The proof of (3.17) is similar to that of (3.14). So, we omit it here. Notice

$$
\begin{aligned}
&\int_{-\infty}^{t} \left| f_x(t, x, u(t, x)) e^{-\int_{\tau}^{t}(1-f_u(s,x,u(s,x)))ds} \right| d\tau \\
&\quad \le \left[\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_x(t, x, u)| \right] \int_{-\infty}^{t} h(\tau)d\tau \\
&\quad \le \left[\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_x(t, x, u)| \right] \left(e^{C_a T} + T e^{C_a T} + 1 \right).
\end{aligned}
\tag{3.18}
$$

For the first term on the right hand side of (3.7), we choose t_0 to be such that $t_{\text{first}}(x) - t_0 = \frac{1}{|\eta|}$ and claim that

$$
v^{\eta}(t_0, x) e^{-\int_{t_0}^{t}(1-a^{\eta}(s,x))ds} \to 0 \quad \text{as} \quad \eta \to 0. \tag{3.19}
$$

In fact, from $|v^\eta(t_0,x)| \le \frac{1}{|\eta|}$ and (3.15), we see

$$\begin{aligned}\left|v^\eta(t_0,x)e^{-\int_{t_0}^t(1-a^\eta(s,x))ds}\right| &\le \frac{1}{|\eta|}e^{-[\int_{t_0}^{t_{\text{first}}(x)}+\int_{t_{\text{first}}(x)}^{t_{\text{last}}(x)}+\int_{t_{\text{last}}(x)}^{t}](1-a^\eta(s,x))ds}\\ &\le \frac{1}{|\eta|}e^{-(t_{\text{first}}(x)-t_0)}e^{C_aT}e^{-(t-t_{\text{last}}(x))}\\ &\le \frac{1}{|\eta|}e^{-\frac{1}{|\eta|}}e^{C_aT}\to 0 \quad \text{as} \quad \eta\to 0.\end{aligned}$$

This proves (3.19).

Hence, choosing t_0 to be such that $t_{\text{first}}(x)-t_0=\frac{1}{|\eta|}$ and passing to the limit $\eta\to 0$ in (3.7), we conclude from (3.14), (3.17) and (3.19) that

$$\begin{aligned}u_x(t,x) &= \lim_{\eta\to 0}v^\eta(t,x)\\ &= \int_{-\infty}^t\left[\int_{\mathbb{R}}J'(x-y)u(\tau,y)dy+f_x(\tau,x,u(\tau,x))\right]e^{-\int_\tau^t(1-f_u(s,x,u(s,x)))ds}d\tau.\end{aligned} \tag{3.20}$$

From which, we see that $u_x(t,x)$ is continuous in $(t,x)\in\mathbb{R}\times\mathbb{R}$. Moreover, by (3.16) and (3.18), we have $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)|<\infty$. □

Remark 3.5. *From* (3.20), (3.16) *and* (3.18), *we see*

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)| \le \left[\|J'\|_{L^1(\mathbb{R})}+\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]}|f_x(t,x,u)|\right]\left(e^{C_aT}+Te^{C_aT}+1\right),$$

where C_a depends only on $\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]}|f_u(t,x,u)|$ and T is controled by (3.11), *and hence, T depends only on f_B and the shape of $u(t,x)$.*

Remark 3.6. *The proof of Theorem 3.4 relies on Theorem 3.2. But, notice we only used the first estimate in Theorem 3.2, i.e.,*

$$c_{\min}^{\text{avg}}(t-t_0-T_{\min}^{\text{avg}}) \le X(t)-X(t_0),$$

whose proof only needs $f(t,x,u) \geq f_B(u)$. This observation allows us to show that the regularity result of transition fronts in Theorem 3.4 (or (i) *in Theorem 2.2) is true if we replace $f_{\tilde{B}}$ by some monostable nonlinearity, and replace $f_u(t,x,u) \leq 0$ for $u \in [0,\theta_0]$ by $f_u(t,x,u) \leq 1-\kappa_0$ for $u \in [0,\theta_0]$ for some $\kappa_0 \in (0,1)$. More precisely, we have*

Corollary 3.7. *If J satisfies Hypothesis 2.1 and $f(t,x,u)$ satisfies*

(i) *there exist $C^2(\mathbb{R})$ function $f_B : [0,1] \to \mathbb{R}$ and $f_M : [0,1] \to \mathbb{R}$ such that*

$$f_B(u) \leq f(t,x,u) \leq f_M(u), \quad (t,x,u) \in \mathbb{R} \times \mathbb{R} \times [0,1];$$

moreover, the following conditions hold:

- *$f : \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ is continuously differentiable in x and u, and satisfies*

$$\sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_x(t,x,u)| < \infty \quad \text{and} \quad \sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}\times[0,1]} |f_u(t,x,u)| < \infty;$$

- *f_B is of standard bistable type, that is, $f_B(0) = f_B(\theta) = f_B(1) = 0$ for some $\theta \in (0,1)$, $f_B(u) < 0$ for $u \in (0,\theta)$, $f_B(u) > 0$ for $u \in (\theta,1)$ and $\int_0^1 f_B(u)du > 0$;*
- *f_M is of standard monostable type, that is, $f_M(0) = f_M(1) = 0$ and $f_M(u) > 0$ for $u \in (0,1)$,*

and

(ii) *there exist $0 < \theta_0 < \theta_1 < 1$ and $\kappa_0 > 0$ such that*

$$f_u(t,x,u) \leq 1-\kappa_0, \quad u \in [0,\theta_0] \quad \text{and} \quad f_u(t,x,u) \leq 0, \quad u \in [\theta_1,1]$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$,

then any transition front $u(t,x)$ of the equation (2.2) *is continuously differentiable in x and satisfies $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)| < \infty$.*

Clearly, the nonlinearity $f(t,x,u)$, as in (i) *and* (ii)*, covers ignition nonlinearities and some monostable nonlinearities.*

3.3 Modified interface locations

For the convenience of later analysis, we modify $X(t)$ in this section. We restate (ii) in Theorem 2.2 as

Theorem 3.8. *There are constants $\tilde{c}_{\min} > 0$, $\tilde{c}_{\max} > 0$ and $\tilde{d}_{\max} > 0$, and a continuously differentiable function $\tilde{X}(t)$ satisfying $\tilde{c}_{\min} \le \dot{\tilde{X}}(t) \le \tilde{c}_{\max}$ for all $t \in \mathbb{R}$ such that*

$$0 \le \tilde{X}(t) - X(t) \le \tilde{d}_{\max}, \quad t \in \mathbb{R}.$$

In particular, for any $\lambda \in (0,1)$, there hold $\sup_{t\in\mathbb{R}} |\tilde{X}(t) - X^{\pm}_{\lambda}(t)| < \infty$.

Proof. We prove the theorem within two steps. The first step gives a continuous modification. The second step gives the continuously differentiable modification as in the statement of the theorem.

Step 1. We show there is a continuous function $\tilde{X} : \mathbb{R} \to \mathbb{R}$ such that $\sup_{t\in\mathbb{R}} |\tilde{X}(t) - X(t)| < \infty$. Fix some $T > 0$. At $t = 0$, let

$$Z^{+}(t;0) = X(0) + c^{\rm avg}_{\max}(T + T^{\rm avg}_{\max}) + \frac{c^{\rm avg}_{\min}}{2}t, \quad t \ge 0$$

By Lemma 3.2, $X(t) < Z^{+}(t;0)$ for all $[0,T)$. By Lemma 3.2, we have $X(t) > Z^{+}(t;0)$ for all large t. Define $T^{+}_1 = \inf\{t \ge 0 | X(t) \ge Z^{+}(t;0)\}$. By Lemma 3.2, it is easy to see that $T^{+}_1 \in [T, \frac{c^{\rm avg}_{\max}(T+T^{\rm avg}_{\max})+c^{\rm avg}_{\min}T^{\rm avg}_{\min}}{c^{\rm avg}_{\min}/2}]$. At the moment T^{+}_1, $X(t)$ may jump, and, due to Lemma 3.2, the jump is at most $c^{\rm avg}_{\max}T^{\rm avg}_{\max}$. Thus, we obtain

$$X(t) < Z^{+}(t;0) \text{ for } t \in [0, T^{+}_1) \quad \text{and} \quad X(T^{+}_1) \in [Z^{+}(t;0), Z^{+}(t;0) + c^{\rm avg}_{\max}T^{\rm avg}_{\max}).$$

Next, at $t = T_1^+$, let

$$Z^+(t; T_1^+) = X(T_1^+) + c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) + \frac{c_{\min}^{\text{avg}}}{2}(t - T_1^+), \quad t \geq T_1^+.$$

Then, $T_2^+ = \inf\{t \geq T_1^+ | X(t) \geq Z^+(t; T_1^+)\}$ is well-defined, and $T_2^+ - T_1^+ \in [T, \frac{c_{\max}^{\text{avg}}(T+T_{\max}^{\text{avg}})+c_{\min}^{\text{avg}}T_{\min}^{\text{avg}}}{c_{\min}^{\text{avg}}/2}]$. Moreover, there hold

$$X(t) < Z^+(t; T_1^+) \text{ for } t \in [T_1^+, T_2^+) \quad \text{and} \quad X(T_2^+) \in [Z^+(T_2^+; T_1^+), Z^+(T_2^+; T_1^+) + c_{\max}^{\text{avg}} T_{\max}^{\text{avg}}).$$

Repeating the above arguments, we obtain the following, there is a sequence of times $\{T_{n-1}^+\}_{n\geq 1}$ satisfying $T_0^+ = 0$, $T_n^+ - T_{n-1}^+ \in [T, \frac{c_{\max}^{\text{avg}}(T+T_{\max}^{\text{avg}})+c_{\min}^{\text{avg}}T_{\min}^{\text{avg}}}{c_{\min}^{\text{avg}}/2}]$ and

$$X(t) < Z^+(t; T_{n-1}^+) \text{ for } t \in [T_{n-1}^+, T_n^+) \quad \text{and} \quad X(T_n^+) \in [Z^+(T_n^+; T_{n-1}^+), Z^+(T_n^+; T_{n-1}^+) + c_{\max}^{\text{avg}} T_{\max}^{\text{avg}}), \tag{3.21}$$

for all $n \geq 1$, where $Z^+(t; T_{n-1}^+) = X(T_{n-1}^+) + c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) + \frac{c_{\min}^{\text{avg}}}{2}(t - T_{n-1}^+)$.

We define $Z^+ : [0, \infty) \to \mathbb{R}$ by setting

$$Z^+(t) = Z^+(t; T_{n-1}^+), \quad t \in [T_{n-1}^+, T_n^+), \quad n \geq 1$$

Since $\sup_{n\geq 1}[T_{n-1}^+, T_n^+) = [0, \infty)$, $Z^+(t)$ is well-defined. It follows from (3.21) that $X(t) < Z^+(t)$ for all $t \geq 0$. Moreover, for $t \in [T_{n-1}^+, T_n^+)$,

$$\begin{aligned} Z^+(t) - X(t) &\leq X(T_{n-1}^+) + c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) + \frac{c_{\min}^{\text{avg}}}{2}(t - T_{n-1}^+) - [X(T_{n-1}^+) + c_{\min}^{\text{avg}}(t - T_{n-1}^+ - T_{\min}^{\text{avg}})] \\ &\leq c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) - \frac{c_{\min}^{\text{avg}}}{2}(t - T_{n-1}^+) + c_{\min}^{\text{avg}} T_{\max}^{\text{avg}} \leq c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) + c_{\min}^{\text{avg}} T_{\min}^{\text{avg}}. \end{aligned}$$

Hence, $0 < Z^+(t) - X(t) \leq c_{\max}^{\text{avg}}(T + T_{\max}^{\text{avg}}) + c_{\min}^{\text{avg}} T_{\min}^{\text{avg}}$ for all $t \in [0, \infty)$. Modifying $Z^+(t)$ near T_{n-1}^+ for $n \geq 1$, we find a continuous function $\tilde{Z}^+ : [0, \infty) \to \mathbb{R}$ such that $\sup_{t\in[0,\infty)} |\tilde{Z}^+(t) - X(t)| < \infty$.

Clearly, we can mimic the above arguments to find a continuous function $\tilde{Z}^- : (0, \infty] \to \mathbb{R}$ such that $\sup_{t\in(-\infty,0]} |\tilde{Z}^-(t) - X(t)| < \infty$. Combining $\tilde{Z}^{\pm}(t)$ and modifying near 0, we find a continuous function $\tilde{X} : \mathbb{R} \to \mathbb{R}$ such that $\sup_{t\in\mathbb{R}} |\tilde{X}(t) - X(t)| < \infty$.

Step 2. By Step 1, we assume, without loss of generality, that $X(t)$ is continuous. Fix any $t_0 \in \mathbb{R}$ and consider it as an initial moment. At the initial moment t_0, we define

$$Z(t; t_0) = X(t_0) + C_0 + \frac{1}{2} c_{\min}^{\text{avg}} (t - t_0), \quad t \geq t_0,$$

where $c_{\min}^{\text{avg}}$ is as in Lemma 3.2 and $C_0 > 0$ is so large that $C_0 > c_{\max}^{\text{avg}} T_{\max}^{\text{avg}}$. Clearly, $X(t_0) < Z(t_0; t_0)$. By Lemma 3.2, $X(t)$ will hit $Z(t; t_0)$ sometime after t_0. Let $T_1(t_0)$ be the first time that $X(t)$ hits $Z(t; t_0)$, that is, $T_1(t_0) = \min \left\{ t \geq t_0 \middle| X(t) = Z(t; t_0) \right\}$. It follows that

$$X(t) < Z(t; t_0) \text{ for } t \in [t_0, T_1(t_0)) \quad \text{and} \quad X(T_1(t_0)) = Z(T_1(t_0); t_0).$$

Moreover, $T_1(t_0) - t_0 \in \left[\frac{C_0 - C_{\max}^{\text{avg}} T_{\max}^{\text{avg}}}{C_{\max}^{\text{avg}} - C_{\min}^{\text{avg}}/2}, \frac{C_0 + C_{\min}^{\text{avg}} T_{\min}^{\text{avg}}}{C_{\min}^{\text{avg}}/2} \right]$, which is a simple result of Lemma 3.2 and the assumption on $X(t)$ as in the statement.

Now, at the moment $T_1(t_0)$, we define

$$Z(t; T_1(t_0)) = X(T_1(t_0)) + C_0 + \frac{1}{2} c_{\min}^{\text{avg}} (t - T_1(t_0)), \quad t \geq T_1(t_0).$$

Similarly, $X(T_1(t_0)) < Z(T_1(t_0); T_1(t_0))$ and $X(t)$ will hit $Z(t; T_1(t_0))$ sometime after $T_1(t_0)$. Denote by $T_2(t_0)$ the first time that $X(t)$ hits $Z(t; T_1(t_0))$. Then,

$$X(t) < Z(t; T_1(t_0)) \text{ for } t \in [T_1(t_0), T_2(t_0)) \quad \text{and} \quad X(T_2(t_0)) = Z(T_2(t_0); T_1(t_0)),$$

and $T_2(t_0) - T_1(t_0) \in \left[\frac{C_0 - C_{\max}^{\text{avg}} T_{\max}^{\text{avg}}}{C_{\max}^{\text{avg}} - C_{\min}^{\text{avg}}/2}, \frac{C_0 + C_{\min}^{\text{avg}} T_{\min}^{\text{avg}}}{C_{\min}^{\text{avg}}/2} \right]$.

Repeating the above arguments, we obtain the following: there is a sequence of times $\{T_{n-1}(t_0)\}_{n\in\mathbb{N}}$ satisfying $T_0(t_0)=t_0$ and

$$T_n(t_0)-T_{n-1}(t_0)\in\left[\frac{C_0-C_{\max}^{\text{avg}}T_{\max}^{\text{avg}}}{C_{\max}^{\text{avg}}-C_{\min}^{\text{avg}}/2},\frac{C_0+C_{\min}^{\text{avg}}T_{\min}^{\text{avg}}}{C_{\min}^{\text{avg}}/2}\right],\quad\forall\, n\in\mathbb{N}, \tag{3.22}$$

and for any $n\in\mathbb{N}$

$$X(t)<Z(t;T_{n-1}(t_0))\text{ for }t\in[T_{n-1}(t_0),T_n(t_0))\quad\text{and}\quad X(T_n(t_0))=Z(T_n(t_0);T_{n-1}(t_0)),$$

where

$$Z(t;T_{n-1}(t_0))=X(T_{n-1}(t_0))+C_0+\frac{1}{2}c_{\min}^{\text{avg}}(t-T_{n-1}(t_0)).$$

Moreover, for any $n\in\mathbb{N}$ and $t\in[T_{n-1}(t_0),T_n(t_0))$, we conclude from Lemma 3.2 that

$$\begin{aligned}
&Z(t;T_{n-1}(t_0))-X(t)\\
&\quad\le X(T_{n-1}(t_0))+C_0+\frac{1}{2}c_{\min}^{\text{avg}}(t-T_{n-1}(t_0))-\left[X(T_{n-1}(t_0))+c_{\min}^{\text{avg}}(t-T_{n-1}(t_0)-T_{\min}^{\text{avg}})\right]\\
&\quad=C_0+c_{\min}^{\text{avg}}T_{\min}^{\text{avg}}-\frac{1}{2}c_{\min}^{\text{avg}}(t-T_{n-1}(t_0))\le C_0+c_{\min}^{\text{avg}}T_{\min}^{\text{avg}}.
\end{aligned}$$

Next, define $\tilde{Z}(\cdot;t_0):[t_0,\infty)\to\mathbb{R}$ by setting

$$\tilde{Z}(t;t_0)=Z(t;T_{n-1}(t_0))\quad\text{for }t\in[T_{n-1}(t_0),T_n(t_0)),\ n\in\mathbb{N}. \tag{3.23}$$

Since $[t_0,\infty)=\cup_{n\in\mathbb{N}}[T_{n-1}(t_0),T_n(t_0))$ by (3.22), $\tilde{Z}(t;t_0)$ is well-defined for all $t\ge t_0$. Notice $\tilde{Z}(t;t_0)$ is strictly increasing and is linear on $[T_{n-1}(t_0),T_n(t_0))$ with slope $\frac{1}{2}c_{\min}^{\text{avg}}$ for each $n\in\mathbb{N}$, and satisfies

$$0\le\tilde{Z}(t;t_0)-X(t)\le C_0+c_{\min}^{\text{avg}}T_{\min}^{\text{avg}},\quad t\ge t_0.$$

Due to (3.22), we can modify $\tilde{Z}(t;t_0)$ near each $T_n(t_0)$ for $n\in\mathbb{N}$ as follows. Fix some $\delta_*\in\left(0,\frac{1}{2}\frac{C_0-C_{\max}^{\text{avg}}T_{\max}^{\text{avg}}}{C_{\max}^{\text{avg}}-C_{\min}^{\text{avg}}/2}\right)$. We modify $\tilde{Z}(t;s)$ by redefining it on the intervals $(T_n(t_0)-\delta_*,T_n(t_0))$,

$n \in \mathbb{N}$ as follows: define

$$X(t;t_0) = \begin{cases} \tilde{Z}(t;t_0), \quad t \in [t_0,\infty) \backslash \cup_{n\in\mathbb{N}} (T_n(t_0) - \delta_*, T_n(t_0)), \\ X(T_n(t_0)) + \delta(t - T_n(t_0)), \quad t \in (T_n(t_0) - \delta_*, T_n(t_0)),\ n \in \mathbb{N}, \end{cases}$$

where $\delta : [-\delta_*, 0] \to [-\frac{1}{2}c_{\min}^{\text{avg}}\delta_*, C_0]$ is twice continuously differentiable and satisfies

$$\begin{aligned} &\delta(-\delta_*) = -\frac{1}{2}c_{\min}^{\text{avg}}\delta_*, \quad \delta(0) = C_0, \\ &\dot{\delta}(-\delta_*) = \frac{1}{2}c_{\min}^{\text{avg}} = \dot{\delta}(0), \quad \dot{\delta}(t) \geq \frac{1}{2}c_{\min}^{\text{avg}} \text{ for } t \in (-\delta_*, 0) \quad \text{and} \\ &\ddot{\delta}(-\delta_*) = 0 = \ddot{\delta}(0). \end{aligned}$$

The existence of such a function $\delta(t)$ is clear. Moreover, there exist $c_{\max} = c_{\max}(\delta_*) > 0$ and $\tilde{c}_{\max} = \tilde{c}_{\max}(\delta_*) > 0$ such that $\dot{\delta}(t) \leq c_{\max}$ and $|\ddot{\delta}(t)| \leq \tilde{c}_{\max}$ for $t \in (-\delta_*, 0)$. Notice the above modification is independent of $n \in \mathbb{N}$ and t_0. Hence, $X(t;t_0)$ satisfies the following uniform in t_0 properties:

- $0 \leq X(t;t_0) - X(t) \leq d_{\max}$ for some $d_{\max} > 0$,
- $\frac{1}{2}c_{\min}^{\text{avg}} \leq \dot{X}(t;t_0) \leq c_{\max}$,
- $|\ddot{X}(t;t_0)| \leq \tilde{c}_{\max}$.

Since $X(t)$ locally bounded, we may apply Arzelà-Ascoli theorem to conclude the existence of some function continuously differentiable function $X : \mathbb{R} \to \mathbb{R}$ such that $X(t;t_0) \to \tilde{X}(t)$ and $\dot{X}(t;t_0) \to \dot{X}(t)$ locally uniformly in t as $t_0 \to -\infty$ along some subsequence. It's easy to see that $\tilde{X}(t)$ satisfies all the properties as in the statement of the theorem. □

Remark 3.9. *In what follows, replacing $X(t)$ by $\tilde{X}(t)$, we may assume, without loss of generality, that $X(t)$ is continuously differentiable and there exist $c_{\min} > 0$ and $c_{\max} > 0$ such that $c_{\min} \leq \dot{X}(t) \leq c_{\max}$ for all $t \in \mathbb{R}$.*

Chapter 4

Properties of space nonincreasing transition fronts in time heterogeneous media

In this chapter, we study some qualitative properties, such as uniform steepness, stability and exponential decaying estimates, of space nonincreasing transition fronts in time heterogeneous media. That is, we study the equation (2.3), i.e.,

$$u_t = J * u - u + f(t, u).$$

Throughout this section, we assume Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.4. Thus, all results obtained in Chapter 3 apply here.

In what follows in this chapter, $u(t, x)$ will be an arbitrary transition front of equation (2.3) that is *space nonincreasing*, i.e.,

$$u_x(t, x) \le 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$$

By comparison principle, $u(t, x)$ is decreasing in x for any $t \in \mathbb{R}$. As a result, for any $\lambda \in (0, 1)$, the leftmost and rightmost interface locations coincide, i.e., $X_\lambda^+(t) = X_\lambda^-(t)$, which will be denoted by $X_\lambda(t)$. Again, let $X(t)$ be the interface location function corresponding to $u(t, x)$. By Remark 3.9, we assume $X(t)$ is continuously differentiable and satisfies $c_{\min} \le \dot{X}(t) \le c_{\max}$ for all $t \in \mathbb{R}$

In this chapter, we first study the uniform steepness of $u(t, x)$ in Section 4.1. Then, we turn to the stability of $u(t, x)$ in Section 4.2. Uniform stability, in the sense of attracting a special family of initial data, is also obtained there. Finally, in Section 4.3, exponential decaying estimates of $u(t, x)$ are obtained.

In the next chapter, Chapter 5, we will show that an arbitrary transition front of equation (2.3) coincides with a space nonincreasing transition front of equation (2.3) up to a space shift. Hence, all transition fronts of equation (2.3) enjoy all properties obtained in this chapter.

4.1 Uniform steepness

In this section, we study the uniform steepness of $u(t,x)$, that is, the uniform-in-t negativity of $u(t,x+X(t))$ on bounded intervals. The main result in this section is given in

Theorem 4.1. *For any $M>0$, there holds*

$$\sup_{t\in\mathbb{R}}\sup_{|x-X(t)|\leq M}u_x(t,x)<0.$$

A direct consequence of Theorem 4.1 is the boundedness of the oscillation of $u(t,x)$.

Corollary 4.2. *For any $\lambda\in(0,1)$, $X_\lambda(t)$ is continuously differentiable and satisfies*

$$\sup_{t\in\mathbb{R}}|\dot{X}_\lambda(t)|<\infty.$$

Proof. Let $\lambda\in(0,1)$. By Theorem 4.1 and the fact that $\sup_{t\in\mathbb{R}}|X_\lambda(t)-X(t)|<\infty$, there exists some $\alpha_\lambda>0$ such that

$$\sup_{t\in\mathbb{R}}u_x(t,X_\lambda(t))\leq-\alpha_\lambda. \tag{4.1}$$

Then, since $u(t,X_\lambda(t))=\lambda$, implicit function theorem says that $X_\lambda(t)$ is continuously differentiable. Differentiating the equation $u(t,X_\lambda(t))=\lambda$ with respect to t, we find

$$\dot{X}_\lambda(t)=-\frac{u_t(t,X_\lambda(t))}{u_x(t,X_\lambda(t))}.$$

The result then follows from (4.1) and the fact $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_t(t,x)|<\infty$. □

To prove Theorem 4.1, we need the following

Lemma 4.3. *For any $t > t_0$, $h > 0$ and $z \in \mathbb{R}$, there holds*

$$u(t, x+\epsilon) - u(t,x) \leq C \int_{z-h}^{z+h} [u(t_0, y+\epsilon) - u(t_0, y)]dy, \quad \forall x \in \mathbb{R}, \quad \epsilon > 0,$$

where $C = C(t - t_0, |x - z|, h) > 0$ satisfies

(i) *$C \to 0$ polynomially as $t - t_0 \to 0$ and $C \to 0$ exponentially as $t - t_0 \to \infty$;*

(ii) *$C : (0, \infty) \times [0, \infty) \times (0, \infty) \to (0, \infty)$ is locally uniformly positive in the sense that for any $0 < t_1 < t_2 < \infty$, $M_1 > 0$ and $h_1 > 0$, there holds*

$$\inf_{t \in [t_1, t_2], M \in [0, M_1], h \in (0, h_1]} C(t, M, h) > 0.$$

Proof. Let $\epsilon > 0$ and $t > t_0$. Let $v_1(t,x) = u(t, x+\epsilon)$ and $v_2(t,x) = u(t,x)$. We see that $v(t,x) := v_1(t,x) - v_2(t,x) < 0$ by monotonicity and satisfies

$$v_t = J * v - v + f(t, v_1) - f(t, v_2).$$

By (H2), we can find $K > 0$ such that $f(t, v_1) - f(t, v_2) \leq -K(v_1 - v_2)$, which implies that

$$v_t \leq J * v - v - Kv.$$

Setting $\tilde{v}(t,x) = e^{(1+K)(t-t_0)} v(t,x)$, we see

$$\tilde{v}_t \leq J * \tilde{v}. \tag{4.2}$$

Since $v < 0$, we have $\tilde{v} < 0$, which implies $J * \tilde{v} < 0$ by the nonnegativity of J by (H1), and therefore, $\tilde{v}_t < 0$ by (4.2). In particular, $\tilde{v}(t,x) < \tilde{v}(t_0, x)$. It then follows from the nonnegativity of J and (4.2) that

$$\tilde{v}_t(t,x) \le [J * \tilde{v}(t,\cdot)](x) \le [J * \tilde{v}(t_0,\cdot)](x). \tag{4.3}$$

For each $x \in \mathbb{R}$, (4.3) is an ordinary differential inequality. Integrating (4.2) over $[t_0, t]$ with respect to the time variable, we find from $\tilde{v}(t_0, x) < 0$ that

$$\tilde{v}(t,x) \le (t-t_0)[J * \tilde{v}(t_0,\cdot)](x) + \tilde{v}(t_0,x) < (t-t_0)[J * \tilde{v}(t_0,\cdot)](x).$$

In particular, for any $T > 0$, we have

$$\tilde{v}(t_0+T,x) < T[J * \tilde{v}(t_0,\cdot)](x). \tag{4.4}$$

Then, considering (4.2) with initial time at $t_0 + T$ and repeating the above arguments, we find

$$\tilde{v}(t_0+T+T,x) < T[J * \tilde{v}(t_0+T,\cdot)](x) < T^2[J * J * \tilde{v}(t_0,\cdot)](x),$$

where we used (4.4) in the second inequality. Repeating this, we conclude that for any $T > 0$ and any $N = 1, 2, 3, \ldots$, there holds

$$\tilde{v}(t_0+NT,x) < T^N[J^N * \tilde{v}(t_0,\cdot)](x), \tag{4.5}$$

where $J^N = \underbrace{J * J * \cdots * J}_{N \text{ times}}$. Note that J^N is nonnegative, and if J is compactly supported, then J^N is not everywhere positive no matter how large N is. But, since J is nonnegative and positive on some open interval, J^N can be positive on any fixed bounded interval if N is large. Moreover, since J is symmetric, so is J^N.

Now, let $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $h > 0$, and let $N := N(|x-z|, h)$ be large enough so that

$$\tilde{C} = \tilde{C}(|x-z|, h) := \inf_{y \in [x-z-h, x-z+h]} J^N(y) > 0.$$

Note that the dependence of N on $x-z$ through $|x-z|$ is due to the symmetry of J^N. Moreover, the positivity of $\tilde{C}:[0,\infty)\times(0,\infty)\to(0,\infty)$ is uniform on compacts sets, which is because N can be chosen to be nondecreasing in $|x-z|$ and in h.

Then, for $t>t_0$, we see from (4.5) with $T=\frac{t-t_0}{N}$ that

$$\begin{aligned}\tilde{v}(t,x) &< \left(\frac{t-t_0}{N}\right)^N \int_{\mathbb{R}} J^N(x-y)\tilde{v}(t_0,y)dy \\ &\leq \left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h} J^N(x-y)\tilde{v}(t_0,y)dy \leq \tilde{C}\left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h} \tilde{v}(t_0,y)dy,\end{aligned}$$

since $x-y\in[x-z-h,x-z+h]$ when $y\in[z-h,z+h]$. Going back to $u(t,x)$, we find

$$u(t,x+\epsilon)-u(t,x)\leq \tilde{C}e^{-(1+K)(t-t_0)}\left(\frac{t-t_0}{N}\right)^N \int_{z-h}^{z+h}[u(t_0,y+\epsilon)-u(t_0,y)]dy$$

The result the follows with $C=\tilde{C}e^{-(1+K)(t-t_0)}\left(\frac{t-t_0}{N}\right)^N$. □

Now, we prove Theorem 4.1.

Proof of Theorem 4.1. Since $u_x(t,x)<0$ for $(t,x)\in\mathbb{R}\times\mathbb{R}$ by comparison principle, we have $X_\lambda^-(t)=X_\lambda^+(t)$ for all $\lambda\in(0,1)$ and $t\in\mathbb{R}$, where $X_\lambda^\pm(t)$ are leftmost and rightmost interface locations defined in (3.1). Thus, we write $X_\lambda(t):=X_\lambda^\pm(t)$ for $\lambda\in(0,1)$ and $t\in\mathbb{R}$. It then follows from Lemma 3.1 that for any $\lambda\in(0,1)$, $\sup_{t\in\mathbb{R}}|X(t)-X_\lambda(t)|<\infty$.

Fix any $\lambda_0\in(0,1)$ and set

$$h_{\lambda_0}:=\max\left\{\sup_{t\in\mathbb{R}}|X(t)-X_{\frac{\lambda_0}{2}}(t)|,\sup_{t\in\mathbb{R}}|X(t)-X_{\frac{1+\lambda_0}{2}}(t)|\right\}.$$

Then, $h_{\lambda_0}<\infty$ and

$$X(t)+h_{\lambda_0}\geq X_{\frac{\lambda_0}{2}}(t),\quad X(t)-h_{\lambda_0}\leq X_{\frac{1+\lambda_0}{2}}(t) \tag{4.6}$$

for all $t \in \mathbb{R}$. Now, fix $\tau > 0$. For $t \in \mathbb{R}$, we apply Lemma 4.3 with $z = X(t)$ and $h = h_{\lambda_0}$ to see that if $|x - X(t)| \leq M$, then

$$
\begin{aligned}
u_x(\tau + t, x) &\leq C(\tau, M, h_{\lambda_0}) \int_{X(t)-h_{\lambda_0}}^{X(t)+h_{\lambda_0}} u_x(t, y) dy \\
&= C(\tau, M, h_{\lambda_0})[u(t, X(t) + h_{\lambda_0}) - u(t, X(t) - h_{\lambda_0})] \\
&\leq C(\tau, M, h_{\lambda_0})[u(t, X_{\frac{\lambda_0}{2}}(t)) - u(t, X_{\frac{1+\lambda_0}{2}}(t))] = -\frac{C(\tau, M, h_{\lambda_0})}{2},
\end{aligned} \tag{4.7}
$$

where we used (4.6) and the monotonicity in the second inequality. To apply (4.7), we see that if $|x - X(t+1)| \leq M$, then $|x - X(t)| \leq |x - X(t+1)| + |X(t+1) - X(t)| \leq M + c_{\max}$, where we used Remark 3.9. We then apply (4.7) with M replaced by $M + c_{\max}$ and τ replaced by 1 to conclude that $u_x(t+1, x) \leq -\frac{C(1, M+c_{\max}, h_{\lambda_0})}{2}$. Since $t \in \mathbb{R}$ is arbitrary, we arrive at the result. □

4.2 Stability

In this section, we study the stability of $u(t, x)$. We prove

Theorem 4.4. *Let $u_0 : \mathbb{R} \to [0, 1]$ be uniform continuous and satisfies*

$$
\liminf_{x \to -\infty} u_0(x) > \theta_1 \quad \text{and} \quad \limsup_{x \to \infty} u_0(x) < \theta_0,
$$

where θ_0 and θ_1 are as in Hypothesis 2.3. Then, there exist $t_0 = t_0(u_0) \in \mathbb{R}$, $\xi = \xi(u_0) \in \mathbb{R}$, $C = C(u_0) > 0$ and $\omega_* > 0$ (independent of u_0) such that

$$
\sup_{x \in \mathbb{R}} |u(t, x; t_0, u_0) - u(t, x - \xi)| \leq C e^{-\omega_*(t - t_0)}
$$

for all $t \geq t_0$.

To prove Theorem 4.4, we first show

Lemma 4.5. *Let u_0 be as in Theorem 4.4. Then, there exist $t_0 = t_0(u_0) \in \mathbb{R}$, $\xi_0^\pm = \xi_0^\pm(u_0) \in \mathbb{R}$, $\mu = \mu(u_0) > 0$ and $\omega = \min\{\beta_0, \beta_1\} > 0$ (independent of u_0) such that*

$$u(t, x - \xi^-(t)) - \mu e^{-\omega(t-t_0)} \leq u(t, x; t_0, u_0) \leq u(t, x - \xi^+(t)) + \mu e^{-\omega(t-t_0)}, \quad x \in \mathbb{R} \tag{4.8}$$

for $t \geq t_0$, where β_0 and β_1 are as in Hypothesis 2.4, and

$$\xi^\pm(t) = \xi_0^\pm \pm \frac{A\mu}{\omega}(1 - e^{-\omega(t-t_0)}), \quad t \geq t_0$$

for some universal constant $A > 0$.

In particular, there holds

$$u(t, x - \xi^-) - \mu e^{-\omega(t-t_0)} \leq u(t, x; t_0, u_0) \leq u(t, x - \xi^+) + \mu e^{-\omega(t-t_0)}, \quad x \in \mathbb{R}$$

for $t \geq t_0$, where $\xi^\pm = \xi_0^\pm \pm \frac{A\mu}{\omega}$.

Proof. Let u_0 be as in the statement of Theorem 4.4. Let $\mu_0^\pm = \mu_0^\pm(u_0)$ be such that

$$\theta_1 < 1 - \mu_0^- < \liminf_{x \to -\infty} u_0(x) \quad \text{and} \quad \limsup_{x \to \infty} u_0(x) < \mu_0^+ < \theta_0.$$

Then, we can find $t_0 = t_0(u_0)$ and $\xi_0^\pm = \xi_0^\pm(u_0)$ such that

$$u(t_0, x - \xi_0^-) - \mu_0^- \leq u_0(x) \leq u(t_0, x - \xi_0^+) + \mu_0^+, \quad x \in \mathbb{R}. \tag{4.9}$$

To show the lemma, we then construct appropriate sub- and super-solutions and apply comparison principle. We here only prove the first inequality in (4.10); the second one can be proven along the same line. To do so, we fix $\omega > 0$, $A > 0$ (to be chosen) and set

$$u^-(t, x) = u(t, x - \xi(t)) - \mu_0^- e^{-\omega(t-t_0)},$$

where $\xi(t) = \xi_0^- - \frac{A\mu_0^-}{\omega}(1 - e^{-\omega(t-t_0)})$. We then compute

$$
\begin{aligned}
&u_t^- - [J * u^- - u^-] - f(t, u^-) \\
&\qquad = f(t, u(t, x - \xi(t))) - f(t, u^-(t, x)) + A\mu_0^- e^{-\omega(t-t_0)} u_x(t, x - \xi(t)) + \omega\mu_0^- e^{-\omega(t-t_0)}.
\end{aligned}
$$

Now, we let $M > 0$ be so large that

$$
\forall t \in \mathbb{R}, \quad \begin{cases} u(t, x) \leq \theta_0 & \text{if} \quad x - X(t) \geq M, \\ u(t, x) \geq \theta_1 + \mu_0^- & \text{if} \quad x - X(t) \leq -M. \end{cases}
$$

Notice such an M exists due to Lemma 3.1. Then, we see

- if $x - \xi(t) - X(t) \geq M$, then $u^-(t, x) \leq u(t, x - \xi(t)) \leq \theta_0$, and then by (H4),

  $$
  f(t, u(t, x - \xi(t))) - f(t, u^-(t, x)) \leq -\beta_0[u(t, x - \xi(t)) - u^-(t, x)] = -\beta_0\mu_0^- e^{-\omega(t-t_0)}.
  $$

  Since $A\mu_0^- e^{-\omega(t-t_0)} u_x(t, x - \xi(t)) \leq 0$, we find

  $$
  u_t^- - [J * u^- - u^-] - f(t, u^-) \leq -\beta_0\mu_0^- e^{-\omega(t-t_0)} + \omega\mu_0^- e^{-\omega(t-t_0)} \leq 0
  $$

  if $\omega \leq \beta_0$;

- if $x - \xi(t) - X(t) \leq -M$, then

  $$
  u(t, x - \xi(t)) \geq u^-(t, x) = u(t, x - \xi(t)) - \mu_0^- e^{-\omega(t-t_0)} \geq \theta_1 + \mu_0^- - \mu_0^1 = \theta_1,
  $$

  and then by (H4), $f(t, u(t, x - \xi(t))) - f(t, u^-(t, x)) \leq -\beta_1\mu_0^- e^{-\omega(t-t_0)}$. Hence, $u_t^- - [J * u^- - u^-] - f(t, u^-) \leq 0$ if $\omega \leq \beta_1$;

- if $|x-\xi(t)-X(t)|\leq M$, then by Theorem 4.1,

$$C_M=\sup_{t\in\mathbb{R}}\sup_{|x-\xi(t)-X(t)|\leq M}u_x(t,x-\xi(t))=\sup_{t\in\mathbb{R}}\sup_{|x-X(t)|\leq M}u_x(t,x)<0.$$

Since $|f(t,u(t,x-\xi(t)))-f(t,u^-(t,x))|\leq C_*\mu_0^- e^{-\omega(t-t_0)}$ for some $C_*>0$, we find

$$u_t^- -[J*u^- -u^-]-f(t,u^-)\leq \big(C_*\mu_0^- + A\mu_0^- C_M+\omega\mu_0^-\big)e^{-\omega(t-t_0)}\leq 0$$

if $A\geq\frac{C_*+\omega}{-C_M}$.

Hence, if we choose $\omega=\min\{\beta_0,\beta_1\}$ and $A=\frac{2C_*}{-C_M}$ (note $\omega=\min\{\beta_0,\beta_1\}\leq C_*$), we find

$$u_t^-\leq J*u^- -u^- +f(t,u^-),\quad x\in\mathbb{R},\quad t>t_0,$$

that is, $u^-(t,x)$ is a sub-solution on (t_0,∞). Since $u^-(t_0,x)=u(t_0,x-\xi_0^-)-\mu_0^-\leq u_0(x)$ due to (4.9), we conclude from comparison principle that

$$u(t,x-\xi(t))-\mu_0^- e^{-\omega(t-t_0)}=u^-(t,x)\leq u(t,x;t_0,u_0),\quad x\in\mathbb{R},\quad t\geq t_0.$$

Setting $\mu=\max\{\mu_0^-,\mu_0^+\}$, we completes the proof. □

The proof of Lemma 4.5 gives the following

Corollary 4.6. *Suppose that $\tilde{u}_0:\mathbb{R}\to[0,1]$ is uniformly continuous and satisfies*

$$u(t_0,x-\tilde{\xi}_0^-)-\tilde{\mu}_0^-\leq\tilde{u}_0(x)\leq u(t_0,x-\tilde{\xi}_0^+)+\tilde{\mu}_0^+,\quad x\in\mathbb{R}$$

for $t_0\in\mathbb{R}$, $\tilde{\xi}_0^\pm\in\mathbb{R}$ and $\tilde{\mu}_0^\pm>0$ satisfying $\theta_1<1-\tilde{\mu}_0^-$ and $\tilde{\mu}_0^+<\theta_0$, where θ_0 and θ_1 are as in Hypothesis 2.3. Then, there exist $\tilde{\mu}=\max\{\tilde{\mu}_0^-,\tilde{\mu}_0^+\}>0$ and $\omega=\min\{\beta_0,\beta_1\}>0$ such *that*

$$u(t,x-\tilde{\xi}^{-}(t))-\tilde{\mu}e^{-\omega(t-t_0)}\le u(t,x;t_0,\tilde{u}_0)\le u(t,x-\tilde{\xi}^{+}(t))+\tilde{\mu}e^{-\omega(t-t_0)},\quad x\in\mathbb{R}\quad (4.10)$$

for $t\ge t_0$, *where*

$$\tilde{\xi}^{\pm}(t)=\tilde{\xi}_0^{\pm}\pm\frac{A\tilde{\mu}}{\omega}(1-e^{-\omega(t-t_0)}),\quad t\ge t_0$$

for some universal constant $A>0$.

In particular, we have

$$u(t,x-\tilde{\xi}^{-})-\tilde{\mu}e^{-\omega(t-t_0)}\le u(t,x;t_0,\tilde{u}_0)\le u(t,x-\tilde{\xi}^{+})+\tilde{\mu}e^{-\omega(t-t_0)},\quad x\in\mathbb{R}$$

for $t\ge t_0$, *where* $\tilde{\xi}^{\pm}=\tilde{\xi}_0^{\pm}\pm\frac{A\tilde{\mu}}{\omega}$.

The next lemma is the key to the proof of Theorem 4.4. We will let $u(t,x;t_0)$, $t\ge t_0$ be a solution with initial data at time $t_0\in\mathbb{R}$.

Lemma 4.7. *There exists* $\epsilon_*\in(0,1)$ *such that if there holds*

$$u(\tau,x-\hat{\xi})-\hat{\delta}\le u(\tau,x;t_0)\le u(\tau,x-\hat{\xi}-\hat{h})+\hat{\delta},\quad x\in\mathbb{R}\qquad (4.11)$$

for some $\tau\ge t_0$, $\hat{\xi}\in\mathbb{R}$, $\hat{h}>0$ *and* $\hat{\delta}\in(0,\min\{\theta_0,1-\theta_1\})$, *then there exist* $\hat{\xi}(t)$, $\hat{h}(t)$ *and* $\hat{\delta}(t)$ *satisfying*

$$\begin{aligned}\hat{\xi}(t)\in[\hat{\xi}-\frac{2A\hat{\delta}}{\omega},\hat{\xi}+\epsilon^*\min\{1,\hat{h}\}]\\ 0\le&\hat{h}(t)\le\hat{h}-\epsilon^*\min\{1,\hat{h}\}+\frac{4A\hat{\delta}}{\omega}\\ 0\le&\hat{\delta}(t)\le[\hat{\delta}e^{-\omega}+C^*\epsilon^*\min\{1,\hat{h}\}]e^{-\omega(t-\tau-1)}\end{aligned}$$

such that

$$u(t,x-\hat{\xi}(t))-\hat{\delta}(t)\le u(t,x;t_0)\le u(t,x-\hat{\xi}(t)-\hat{h}(t))+\hat{\delta}(t),\quad x\in\mathbb{R}$$

for $t \geq \tau + 1$.

Proof. Applying Corollary 4.6 to (4.11), we find

$$u(t, x - \hat{\xi}^-(t)) - \hat{\delta} e^{-\omega(t-\tau)} \leq u(t, x; t_0) \leq u(t, x - \hat{\xi}^+(t) - \hat{h}) + \hat{\delta} e^{-\omega(t-\tau)}, \quad x \in \mathbb{R} \quad (4.12)$$

for $t \geq \tau$, where $\omega = \min\{\beta_0, \beta_1\}$ and $\hat{\xi}^\pm(t) = \hat{\xi} \pm \frac{A\hat{\delta}}{\omega}(1 - e^{-\omega(t-\tau)})$.

We now modify (4.12) at $t = \tau + 1$ to get a new estimate for $u(\tau + 1, x; t_0)$, and then apply Corollary 4.6 to this new estimate to conclude the result. To this end, we set

$$h = \min\{\hat{h}, 1\} \quad \text{and} \quad C_{\text{steep}} = \frac{1}{2} \sup_{t \in \mathbb{R}} \sup_{|x - X(t)| \leq 2} u_x(t, x).$$

By Theorem 4.1, $C_{\text{steep}} < 0$. Taylor expansion then yields $\int_{X(t)-\frac{1}{2}}^{X(t)+\frac{1}{2}} [u(t, x - h) - u(t, x)]dx \geq -2C_{\text{steep}} h$ for all $t \in \mathbb{R}$. In particular, at $t = \tau$, either

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(t, x - h) - u(t, x + \hat{\xi}; t_0)]dx \geq -C_{\text{steep}} h \quad (4.13)$$

or

$$\int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(t, x + \hat{\xi}; t_0) - u(t, x)]dx \geq -C_{\text{steep}} h \quad (4.14)$$

must be the case.

Suppose first that (4.14) holds. We estimate $u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h)$ from below, where $\epsilon^* > 0$ is to be chosen. To do so, let $M > 0$ and consider two cases: (i) $|x - \hat{\xi} - X(\tau)| \leq M$; (ii) $|x - \hat{\xi} - X(\tau)| \geq M$.

(i) $|x - \hat{\xi} - X(\tau)| \leq M$ In this case, we write

$$\begin{aligned} &u(\tau + 1, x; t_0) - u(\tau + 1, x - \hat{\xi}^-(\tau + 1) - \epsilon^* h) \\ &\quad = [u(\tau + 1, x; t_0) - u(\tau + 1, x - \hat{\xi}^-(\tau + 1))] \\ &\qquad + [u(\tau + 1, x - \hat{\xi}^-(\tau + 1)) - u(\tau + 1, x - \hat{\xi}^-(\tau + 1) - \epsilon^* h)] =: \text{(I)} + \text{(II)}. \end{aligned}$$

For (I), we argue

$$
\begin{aligned}
(\mathrm{I}) + \hat{\delta}e^{-\omega} &= u(\tau+1, x; t_0) - [u(\tau+1, x - \hat{\xi} + \frac{A\hat{\delta}}{\omega}(1-e^{-\omega})) - \hat{\delta}e^{-\omega}] \\
&= u(\tau+1, y+\hat{\xi}; t_0) - [u(\tau+1, y + \frac{A\hat{\delta}}{\omega}(1-e^{-\omega})) - \hat{\delta}e^{-\omega}] \\
&\quad (\text{by } y = x - \hat{\xi} \in X(\tau) + [-M, M]) \\
&= u(\tau+1, y+\hat{\xi}; t_0) - \hat{u}(\tau+1, y) \\
&\quad (\text{where } \hat{u}(t,y) = u(t, y + \frac{A}{\omega}(1-e^{-\omega(t-\tau)})) - \hat{\delta}e^{-\omega(t-\tau)}) \\
&\geq C(M) \int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y+\hat{\xi}; t_0) - \hat{u}(\tau, y)] dy \\
&\geq C(M) \int_{X(\tau)-\frac{1}{2}}^{X(\tau)+\frac{1}{2}} [u(\tau, y+\hat{\xi}; t_0) - u(\tau, y)] dy \geq -C(M) C_{\mathrm{steep}} h,
\end{aligned}
$$

where the first inequality follows as in the proof of Lemma 4.3. In fact, we know $u(t, y+\hat{\xi}; t_0)$ is a solution of $v_t = J * v - v + f(t, v)$, while $\hat{u}(t,y)$ is a subsolution by the proof of Theorem 4.4. Moreover, $u(t, y+\hat{\xi}; t_0) \geq \hat{u}(t,y)$ by (4.12). Based on these information, we can repeat the arguments as in the proof of Lemma 4.3 to conclude the inequality. Hence,

$$
(\mathrm{I}) \geq -\hat{\delta}e^{-\omega} - C(M) C_{\mathrm{steep}} h.
$$

For (II), Taylor expansion yields for some $x_* \in (0, \epsilon^* h)$

$$
(\mathrm{II}) = u_x(\tau+1, x - \hat{\xi}^-(\tau+1) - x_*)\epsilon^* h \geq -\epsilon^* h \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)| \geq C(M) C_{\mathrm{steep}} h
$$

if we choose $\epsilon^* = \min\left\{1, \frac{-C(M)C_{\mathrm{steep}}}{\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x(t,x)|}\right\}$. It then follows that

$$
u(\tau+1, x; t_0) - u(\tau+1, x - \hat{\xi}^-(\tau+1) - \epsilon^* h) \geq -\hat{\delta}e^{-\omega}. \tag{4.15}
$$

(ii) $|x-\hat{\xi}-X(\tau)|\geq M$ In this case,

$$
\begin{aligned}
&u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^*h)\\
&\quad=[u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^-(\tau+1))]\\
&\qquad+[u(\tau+1,x-\hat{\xi}^-(\tau+1))-u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^*h)]\\
&\quad\geq-\hat{\delta}e^{-\omega}-\epsilon^*h\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)|,
\end{aligned}
\tag{4.16}
$$

where we used the first inequality in (4.12) and Taylor expansion.

Hence, by (4.15), (4.16) and the second inequality in (4.12), we find

$$
\begin{aligned}
&u(\tau+1,x-\hat{\xi}^-(\tau+1)-\epsilon^*h)-\hat{\delta}e^{-\omega}-C^*\epsilon^*h\\
&\quad\leq u(\tau+1,x;t_0)\leq u(\tau+1,x-\hat{\xi}^+(\tau+1)-\hat{h})+\hat{\delta}e^{-\omega},
\end{aligned}
\tag{4.17}
$$

where $C^*=\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)|$. Taking ϵ_* smaller, if necessary, so that $\hat{\delta}e^{-\omega}+C^*\epsilon^*h<1-\theta_1$, and applying Corollary 4.6 to (4.17), we conclude

$$
u(t,x-\tilde{\xi}^-(t))-\tilde{\delta}e^{-\omega(t-\tau-1)}\leq u(t,x;t_0)\leq u(t,x-\tilde{\xi}^+(t))+\tilde{\delta}e^{-\omega(t-\tau-1)}\tag{4.18}
$$

for $t\geq\tau+1$, where $\omega=\min\{\beta_0,\beta_1\}$, $\tilde{\delta}=\max\{\hat{\delta}e^{-\omega}+C^*\epsilon^*h,\hat{\delta}e^{-\omega}\}=\hat{\delta}e^{-\omega}+C^*\epsilon^*h$ and

$$
\begin{aligned}
\tilde{\xi}^-(t)&=\hat{\xi}^-(\tau+1)+\epsilon^*h-\frac{A\hat{\delta}}{\omega}(1-e^{-\omega(t-\tau-1)})=\hat{\xi}-\frac{2A\hat{\delta}}{\omega}+\epsilon^*h+\frac{A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}],\\
\tilde{\xi}^+(t)&=\hat{\xi}^+(\tau+1)+\hat{h}+\frac{A\hat{\delta}}{\omega}(1-e^{-\omega(t-\tau-1)})=\hat{\xi}+\frac{2A\hat{\delta}}{\omega}+\hat{h}-\frac{A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}].
\end{aligned}
$$

Setting

$$
\begin{aligned}
\hat{\xi}(t)&=\tilde{\xi}^-(t)=\hat{\xi}-\frac{2A\hat{\delta}}{\omega}+\epsilon^*h+\frac{A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}],\\
\hat{h}(t)&=\tilde{\xi}^+(t)-\tilde{\xi}^-(t)=\hat{h}-\epsilon^*h+\frac{4A\hat{\delta}}{\omega}-\frac{2A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}],\\
\hat{\delta}(t)&=\tilde{\delta}e^{-\omega(t-\tau-1)}=[\hat{\delta}e^{-\omega}+C^*\epsilon^*h]e^{-\omega(t-\tau-1)},
\end{aligned}
$$

the estimate (4.18) can be written as

$$u(t,x-\hat{\xi}(t))-\hat{\delta}(t)\leq u(t,x;t_0)\leq u(t,x-\hat{\xi}(t)-\hat{h}(t))+\hat{\delta}(t),\quad x\in\mathbb{R},\ t\geq\tau+1. \tag{4.19}$$

Note that (4.19) is obtained under the assumption (4.14).

Now, we assume (4.13) and estimate $u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^+(\tau+1)-\hat{h}+\epsilon^*h)$ from above. Arguing as before and replacing $\hat{h}$ by h at appropriate steps lead to

$$u(\tau+1,x;t_0)-u(\tau+1,x-\hat{\xi}^+(\tau+1)-\hat{h}+\epsilon^*h)\leq\hat{\delta}e^{-\omega}+C^*\epsilon^*h,$$

where $C^*=\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)|$. This, together with the first inequality in (4.12), yields

$$\begin{aligned}&u(\tau+1,x-\hat{\xi}^-(\tau+1))-\hat{\delta}e^{-\omega}\\&\qquad\leq u(\tau+1,x;t_0)\leq u(\tau+1,x-\hat{\xi}^+(\tau+1)-\hat{h}+\epsilon^*h)+\hat{\delta}e^{-\omega}+C^*\epsilon^*h.\end{aligned} \tag{4.20}$$

Then, applying Corollary 4.6 to (4.20), we find (4.18) again with

$$\begin{aligned}\hat{\xi}(t)&=\hat{\xi}-\frac{2A\hat{\delta}}{\omega}+\frac{A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}],\\\hat{h}(t)&=\hat{h}-\epsilon^*h+\frac{4A\hat{\delta}}{\omega}-\frac{2A\hat{\delta}}{\omega}[e^{-\omega}+e^{-\omega(t-\tau-1)}],\\\hat{\delta}(t)&=[\hat{\delta}e^{-\omega}+C^*\epsilon^*h]e^{-\omega(t-\tau-1)}.\end{aligned}$$

This completes the proof. □

Now, we use the "squeezing technique" (see e.g. [15, 14, 16, 49, 48, 63, 70, 72, 76]) to prove Theorem 4.4.

Proof of Theorem 4.4. Let u_0 be the initial data as in the statement of the theorem. Lemma 4.5 ensures the existence of $t_0=t_0(u_0)\in\mathbb{R}$, $\xi^\pm=\xi^\pm(u_0)\in\mathbb{R}$ and $\mu=\mu(u_0)$ such that

$$u(t,x-\xi^-)-\mu e^{-\omega(t-t_0)}\leq u(t,x;t_0,u_0)\leq u(t,x-\xi^+)+\mu e^{-\omega(t-t_0)}$$

for $t \geq t_0$, where $\omega = \min\{\beta_0, \beta_1\}$. Choosing $T_0 = T_0(u_0) > 0$ be such that

$$\delta_0 := \mu e^{-\omega T_0} < \delta_* := \min\left\{\theta_0, 1-\theta_1, \frac{\epsilon^*\omega}{8A}\right\} < 1,$$

we find

$$u(t_0+T_0, x-\xi_0) - \delta_0 \leq u(t_0+T_0, x; t_0, u_0) \leq u(t_0+T_0, x-\xi_0-h_0) + \delta_0, \tag{4.21}$$

where $\xi_0 = \xi^-$ and $h_0 = \xi^+ - \xi^-$. Notice, we may assume, without loss of generality, that $\xi^+ > \xi^-$, so $h_0 > 0$. But, h_0 depends on u_0, so we may assume, without loss of generality, that $h_0 > 1$. Let $T > 1$ be such that

$$[e^{-\omega} + C^*\epsilon^*]e^{-\omega(T-1)} \leq \delta_* := \min\left\{\theta_0, 1-\theta_1, \frac{\epsilon^*\omega}{8A}\right\}.$$

We are going to reduce h_0.

Applying Lemma 4.7 to (4.21), we find

$$u(t_0+T_0+T, x-\xi_1) - \delta_1 \leq u(t_0+T_0+T, x; t_0, u_0) \leq u(t_0+T_0+T, x-\xi_1-h_1) + \delta_1, \tag{4.22}$$

where

$$\begin{aligned}
\xi_1 \in [\xi_0 - \frac{2A\delta_0}{\omega}, \xi_0 + \epsilon^* \min\{1, h_0\}] = [\xi_0 - \frac{2A\delta_0}{\omega}, \xi_0 + \epsilon^*] \subset [\xi_0 - \frac{\epsilon^*}{4}, \xi_0 + \epsilon^*], \\
0 \leq & h_1 \leq h_0 - \epsilon^* \min\{1, h_0\} + \frac{4A\delta_0}{\omega} = h_0 - \epsilon^* + \frac{4A\delta_0}{\omega} \leq h_0 - \frac{\epsilon^*}{2}, \\
0 \leq & \delta_1 \leq [\delta_0 e^{-\omega} + C^*\epsilon^* \min\{1, h_0\}]e^{-\omega(T-1)} = [\delta_0 e^{-\omega} + C^*\epsilon^*]e^{-\omega(T-1)} \leq \delta_*.
\end{aligned}$$

If $h_1 \leq 1$, we stop. Otherwise, we apply Lemma 4.7 to (4.22) to find

$$u(t_0+T_0+2T, x-\xi_2) - \delta_2 \leq u(t_0+T_0+2T, x; t_0, u_0) \leq u(t_0+T_0+2T, x-\xi_2-h_2) + \delta_2, \tag{4.23}$$

where

$$\begin{aligned}\xi_2 &\in [\xi_1 - \frac{2A\delta_1}{\omega}, \xi_1 + \epsilon^* \min\{1, h_1\}] = [\xi_1 - \frac{2A\delta_1}{\omega}, \xi_1 + \epsilon^*] \subset [\xi_1 - \frac{\epsilon^*}{4}, \xi_1 + \epsilon^*],\\
0 \leq &h_2 \leq h_1 - \epsilon^* \min\{1, h_1\} + \frac{4A\delta_1}{\omega} = h_1 - \epsilon^* + \frac{4A\delta_1}{\omega} \leq h_0 - 2\big(\frac{\epsilon^*}{2}\big),\\
0 \leq &\delta_2 \leq [\delta_1 e^{-\omega} + C^* \epsilon^* \min\{1, h_1\}] e^{-\omega(T-1)} = [\delta_1 e^{-\omega} + C^* \epsilon^*] e^{-\omega(T-1)} \leq \delta_*.\end{aligned}$$

If $h_2 \leq 1$, we stop. Otherwise, we apply Lemma 4.7 to (4.23), and repeat this. Suppose $h_i > 1$ for all $i = 0, 1, 2, \dots n-1$, we then have

$$u(t_0+T_0+nT, x-\xi_n)-\delta_n \leq u(t_0+T_0+nT, x; t_0, u_0) \leq u(t_0+T_0+nT, x-\xi_n-h_n)+\delta_n, \quad (4.24)$$

where

$$\begin{aligned}\xi_n &\in [\xi_{n-1} - \frac{2A\delta_{n-1}}{\omega}, \xi_{n-1} + \epsilon^* \min\{1, h_{n-1}\}] \subset [\xi_{n-1} - \frac{\epsilon^*}{4}, \xi_{n-1} + \epsilon^*],\\
0 \leq &h_n \leq h_{n-1} - \epsilon^* \min\{1, h_{n-1}\} + \frac{4A\delta_{n-1}}{\omega} = h_{n-1} - \epsilon^* + \frac{4A\delta_{n-1}}{\omega} \leq h_0 - n\big(\frac{\epsilon^*}{2}\big),\\
0 \leq &\delta_n \leq [\delta_{n-1} e^{-\omega} + C^* \epsilon^* \min\{1, h_{n-1}\}] e^{-\omega(T-1)} = [\delta_{n-1} e^{-\omega} + C^* \epsilon^*] e^{-\omega(T-1)} \leq \delta_*.\end{aligned}$$

Note that since $h_0 > 1$ and $\frac{\epsilon^*}{2} \in (0, 1)$, we must exist some $N = N(u_0) > 0$ such that $h_i > 1$ for $i = 0, 1, 2, \dots, N-1$ and $0 < h_0 - N(\frac{\epsilon^*}{2}) \leq 1$. In particular, $h_N \leq 1$. Then, we stop and obtain from (4.23) that

$$u(\tilde{t}_0, x - \tilde{\xi}_0) - \tilde{\delta}_0 \leq u(\tilde{t}_0, x; t_0, u_0) \leq u(\tilde{t}_0, x - \tilde{\xi}_0 - \tilde{h}_0) + \tilde{\delta}_0, \quad (4.25)$$

where $\tilde{t}_0 = t_0 + T_0 + NT$, $\tilde{\xi}_0 = \xi_N$, $\tilde{\delta}_0 = \delta_N \leq \delta_*$ and $\tilde{h}_0 = h_N \leq 1$.

Now, we treat (4.25) as the new initial estimate and run the iteration argument again. Let $\tilde{T} > 1$ be such that

$$[e^{-\omega} + C^* \epsilon^*] e^{-\omega(\tilde{T}-1)} \leq \min\Big\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A} \frac{\epsilon_*}{2} \big(1 - \frac{\epsilon^*}{2}\big)\Big\}.$$

Applying Lemma 4.7 to (4.25), we find

$$u(\tilde{t}_0 + \tilde{T}, x - \tilde{\xi}_1) - \tilde{\delta}_1 \le u(\tilde{t}_0 + \tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + \tilde{T}, x - \tilde{\xi}_1 - \tilde{h}_1) + \tilde{\delta}_1, \tag{4.26}$$

where

$$\begin{aligned} \tilde{\xi}_1 &\in [\tilde{\xi}_0 - \frac{2A\tilde{\delta}_0}{\omega}, \tilde{\xi}_0 + \epsilon^* \tilde{h}_0], \\ 0 \le &\tilde{h}_1 \le \tilde{h}_0 - \epsilon^* \tilde{h}_0 + \frac{4A\tilde{\delta}_0}{\omega} \le 1 - \frac{\epsilon^*}{2}, \\ 0 \le &\tilde{\delta}_1 \le [\tilde{\delta}_0 e^{-\omega} + C^* \epsilon^* \tilde{h}_0] e^{-\omega(T-1)} \le \min\Big\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon_*}{2}\big(1 - \frac{\epsilon^*}{2}\big)\Big\}. \end{aligned}$$

Applying Lemma 4.7 to (4.26), we find

$$u(\tilde{t}_0 + 2\tilde{T}, x - \tilde{\xi}_2) - \tilde{\delta}_2 \le u(\tilde{t}_0 + 2\tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + 2\tilde{T}, x - \tilde{\xi}_2 - \tilde{h}_2) + \tilde{\delta}_2,$$

where

$$\begin{aligned} \tilde{\xi}_2 &\in [\tilde{\xi}_1 - \frac{2A\tilde{\delta}_1}{\omega}, \tilde{\xi}_1 + \epsilon^* \tilde{h}_1], \\ 0 \le &\tilde{h}_2 \le \tilde{h}_1 - \epsilon^* \tilde{h}_1 + \frac{4A\tilde{\delta}_1}{\omega} \le (1 - \frac{\epsilon^*}{2})(1 - \epsilon^*) + \frac{\epsilon_*}{2}\big(1 - \frac{\epsilon^*}{2}\big) = (1 - \frac{\epsilon^*}{2})^2, \\ 0 \le &\tilde{\delta}_2 \le [\tilde{\delta}_1 e^{-\omega} + C^* \epsilon^* \tilde{h}_1] e^{-\omega(T-1)} \le (1 - \frac{\epsilon^*}{2}) \times \min\Big\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon_*}{2}\big(1 - \frac{\epsilon^*}{2}\big)\Big\}. \end{aligned}$$

Applying Lemma 4.7 repeatedly, we find for $n \ge 3$

$$u(\tilde{t}_0 + n\tilde{T}, x - \tilde{\xi}_n) - \tilde{\delta}_n \le u(\tilde{t}_0 + n\tilde{T}, x; t_0, u_0) \le u(\tilde{t}_0 + n\tilde{T}, x - \tilde{\xi}_n - \tilde{h}_n) + \tilde{\delta}_n,$$

where

$$\begin{aligned}
&\tilde{\xi}_n \in [\tilde{\xi}_{n-1} - \frac{2A\tilde{\delta}_{n-1}}{\omega}, \tilde{\xi}_{n-1} + \epsilon^* \tilde{h}_{n-1}],\\
0 \leq &\tilde{h}_n \leq \tilde{h}_{n-1} - \epsilon^* \tilde{h}_{n-1} + \frac{4A\tilde{\delta}_{n-1}}{\omega} \leq (1 - \frac{\epsilon^*}{2})^{n-1}(1-\epsilon^*) + \frac{\epsilon_*}{2}\big(1 - \frac{\epsilon^*}{2}\big)^{n-1} = (1 - \frac{\epsilon^*}{2})^n,\\
0 \leq &\tilde{\delta}_n \leq [\tilde{\delta}_{n-1} e^{-\omega} + C^* \epsilon^* \tilde{h}_{n-1}] e^{-\omega(T-1)} \leq (1 - \frac{\epsilon^*}{2})^{n-1} \times \min\Big\{\delta_*, 1 - \frac{\epsilon^*}{2}, \frac{\omega}{4A}\frac{\epsilon_*}{2}\big(1 - \frac{\epsilon^*}{2}\big)\Big\}.
\end{aligned}$$

The result then follows readily. We remark that the dependence of C on u_0 in the statement of the theorem is because T_0 is u_0 dependent. □

Checking the dependence of C and ξ on u_0 in the statement of Theorem 4.4, we have the following uniform stability for a family of initial data satisfying certain uniform conditions.

Corollary 4.8. *Let $\{u_{t_0}\}_{t_0 \in \mathbb{R}}$ be a family of initial data satisfying*

$$u(t_0, x - \xi_0^-) - \mu_0 \leq u_{t_0}(x) \leq u(t_0, x - \xi_0^+) + \mu_0, \quad x \in \mathbb{R},\ t_0 \in \mathbb{R}$$

for $\xi_0^\pm \in \mathbb{R}$ and $\mu_0 \in (0, \min\{\theta_0, 1-\theta_1\})$ being independent of $t_0 \in \mathbb{R}$. Then,

(i) *there holds*

$$u(t, x - \xi^-) - \mu_0 e^{-\omega(t-t_0)} \leq u(t, x; t_0, u_{t_0}) \leq u(t, x - \xi^+) + \mu_0 e^{-\omega(t-t_0)}, \quad x \in \mathbb{R}$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$, where $\omega = \min\{\beta_0, \beta_1\}$ and $\xi^\pm = \xi^\pm \pm \frac{A\mu_0}{\omega}$.

(ii) *there exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and a family of shifts $\{\xi_{t_0}\}_{t_0 \in \mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0 \in \mathbb{R}} |\xi_{t_0}| < \infty$ such that*

$$\sup_{x \in \mathbb{R}} |u(t, x; t_0, u_{t_0}) - u(t, x - \xi_{t_0})| \leq C e^{-\omega_*(t-t_0)}$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$.

We will use this corollary in the next section, Section 4.3, to show the exponential decaying estimates of space nonincreasing transition fronts, and in the next chapter, Chapter 5, to show the uniquenss of transiton fronts.

4.3 Exponential decaying estimates

In this section, we prove exponential decaying estimates of $u(t,x)$.

Theorem 4.9. *There exist $c^{\pm}>0$ and $h^{\pm}>0$ such that*

$$u(t,x)\le e^{-c^{+}(x-X(t)-h^{+})}\quad \textit{and}\quad 1-u(t,x)\le e^{c^{-}(x-X(t)+h^{-})}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}$.

In particular, for any $\lambda\in(0,1)$, there exist $h^{\pm}_{\lambda}>0$ such that

$$u(t,x)\le e^{-c^{+}(x-X_{\lambda}(t)-h^{+}_{\lambda})}\quad \textit{and}\quad 1-u(t,x)\le e^{c^{-}(x-X_{\lambda}(t)+h^{-}_{\lambda})}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}$.

To prove Theorem 4.9, we first prove several lemmas. Let $\theta_2\in(0,\min\{\frac{1}{4},\theta_0,1-\theta_1\})$ be small and $h>0$, and define $u_0^{\pm}:\mathbb{R}\to[0,1]$ to be smooth and nonincreasing functions satisfying

$$u_0^{+}(x)=\begin{cases}1-\theta_2, & x\le -h,\\ 0, & x\ge 0,\end{cases}\quad\text{and}\quad u_0^{-}(x)=\begin{cases}1, & x\le 0,\\ \theta_2, & x\ge h.\end{cases}\tag{4.27}$$

Moreover, we can make $u_0^{\pm}$ so that u_0^{+} is decreasing on $(-h,0)$ and u_0^{-} is decreasing on $(0,h)$. For $t_0\in\mathbb{R}$, we define

$$u^{+}(t,x;t_0):=u(t,x;t_0,u_0^{+}(\cdot-X_{1-\theta_2}(t_0))),$$
$$u^{-}(t,x;t_0):=u(t,x;t_0,u_0^{-}(\cdot-X_{\theta_2}(t_0)))$$

for $t \geq t_0$. Note that by the choice of θ_2 and the asymptotically stability of the constant solutions 0 and 1, we have

$$\lim_{x\to-\infty} u^+(t,x;t_0) > 1-\theta_2, \quad \lim_{x\to\infty} u^+(t,x;t_0) = 0,$$
$$\lim_{x\to-\infty} u^-(t,x;t_0) = 1 \quad \text{and} \quad \lim_{x\to\infty} u^-(t,x;t_0) < \theta_2$$

for all $t > t_0$. Moreover, since $u_0^\pm$ are nonincreasing, $u^\pm(t,x;t_0)$ are decreasing in x for all $t > t_0$. Hence, for any $\lambda \in (\theta_2, 1-\theta_2)$, the interface locations $X_\lambda^\pm(t;t_0) \in \mathbb{R}$ such that $u^\pm(t, X_\lambda^\pm(t;t_0);t_0) = \lambda$ are well-defined for all $t \geq t_0$.

The first lemma gives the uniform boundedness of the gap between the interface locations of $u^\pm(t,x;t_0)$ and $u(t,x)$.

Lemma 4.10. *For any $\lambda \in (\theta_2, 1-\theta_2)$, there hold*

$$\sup_{t_0\in\mathbb{R}} \sup_{t\geq t_0} |X_\lambda^\pm(t;t_0) - X(t)| < \infty.$$

Proof. Let $\lambda \in (\theta_2, 1-\theta_2)$. By the definition of u_0^+, we see that $u_0^+(x - X_{1-\theta_2}(t_0)) \leq u(t_0,x)$ for $x \in \mathbb{R}$. Comparison principle then yields $u^+(t,x;t_0) \leq u(t,x)$ for $x \in \mathbb{R}$ and $t \geq t_0$. In particular, $X_\lambda^+(t;t_0) \leq X_\lambda(t)$ for all $t \geq t_0$.

Moreover, we readily check that $u_0^+(x - X_{\theta_2}(t_0) - h) + \theta_2 \geq u(t_0,x)$, which is equivalent to

$$u(t_0, x + X_{\theta_2}(t_0) + h - X_{1-\theta_2}(t_0)) - \theta_2 \leq u_0^+(x - X_{1-\theta_2}(t_0)) = u^+(t_0,x;t_0).$$

Setting $L = \sup_{t_0\in\mathbb{R}} |X_{\theta_2}(t_0) + h - X_{1-\theta_2}(t_0)| < \infty$, we see from the monotonicity of $u(t,x)$ in x that

$$u(t_0, x-(-L)) - \theta_2 \leq u^+(t_0,x;t_0).$$

Since L and θ_2 are t_0-independent, we apply Corollary 4.8 to conclude that

$$u(t, x-(-L-\frac{A\theta_2}{\omega})) - \theta_2 \leq u(t, x-(-L-\frac{A\theta_2}{\omega})) - \theta_2 e^{-\omega(t-t_0)} \leq u^+(t,x;t_0), \quad x \in \mathbb{R}$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$. Setting $x = -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t)$, we find $\lambda \leq u^+(t, -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t); t_0)$, which implies by monotonicity that $X_\lambda^+(t;t_0) \geq -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t)$ for all $t \geq t_0$.

Hence, we have shown that

$$X_\lambda^+(t;t_0) \leq X_\lambda(t) \quad \text{and} \quad X_\lambda^+(t;t_0) \geq -L - \frac{A\theta_2}{\omega} + X_{\lambda+\theta_2}(t)$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$. Since $\sup_{t\in\mathbb{R}} |X_\lambda(t) - X_{\lambda+\theta_2}(t)| < \infty$, we arrive at the result $\sup_{t_0\in\mathbb{R}} \sup_{t\geq t_0} |X_\lambda^+(t;t_0) - X(t)| < \infty$. The another result $\sup_{t_0\in\mathbb{R}} \sup_{t\geq t_0} |X_\lambda^-(t;t_0) - X(t)| < \infty$ follows along the same line. □

Next, we prove the uniform exponential decaying estimates of $u^\pm(t,x;t_0)$.

Lemma 4.11. *There exist $c^\pm > 0$ and $h^\pm > 0$ such that*

$$u^+(t,x;t_0) \leq e^{-c^+(x-X(t)-h^+)} \quad \textit{and} \quad u^-(t,x;t_0) \geq 1 - e^{c^-(x-X(t)+h^-)}$$

for all $x \in \mathbb{R}$, $t \geq t_0$ and $t_0 \in \mathbb{R}$.

Proof. We prove the first estimate; the second one can be proven in a similar way. Note first that $f(t,u) \leq -\beta_0 u$ for $u \in [0,\theta_0]$. Let $h := \sup_{t\geq t_0} |X_{\theta_0}^+(t;t_0) - X(t)| < \infty$ by Lemma 4.10, since $\theta_0 \in (\theta_2, 1-\theta_2)$. We consider

$$N[u] = u_t - [J * u - u] + \beta_0 u.$$

Since $u^+(t,x;t_0) \leq \theta_0$ for $x \geq X_{\theta_0}^+(t;t_0)$, we find $N[u^+] = \beta_0 u + f(t,u) \leq 0$ for $x \geq X_{\theta_0}^+(t;t_0)$. In particular, $N[u^+] \leq 0$ for $x \geq X(t) + h$.

Now, let $c > 0$. We see

$$N[e^{-c(x-X(t)-h)}] = \left[c\dot{X}(t) - \int_{\mathbb{R}} J(y)e^{cy}dy + 1 + \beta_0\right] e^{-c(x-X(t)-h)}.$$

Since $\dot{X}(t) \geq c_{\min} > 0$ by Remark 3.9 and $\int_{\mathbb{R}} J(y)e^{cy}dy \to 1$ as $c \to 0$, we can find some $c_* > 0$ such that $N[e^{-c_*(x-X(t)-h)}] \geq 0$. Thus, we have

- $N[u^+(t,x;t_0)] \leq 0 \leq N[e^{-c_*(x-X(t)-h)}]$ for $x \geq X(t)+h$ and $t > t_0$,
- $u^+(t,x;t_0) < 1 \leq e^{-c_*(x-X(t)-h)}$ for $x \leq X(t)+h$ and $t > t_0$,
- $u^+(t_0,x;t_0) = u_0^+(x - X_{1-\theta_2}(t_0)) \leq e^{-c_*(x-X(t_0)-h)}$ for $x \in \mathbb{R}$.

We then conclude from the comparison principle (see e.g. Lemma B.1) that $u^+(t,x;t_0) \leq e^{-c_*(x-X(t)-h)}$ for all $x \in \mathbb{R}$, $t \geq t_0$ and $t_0 \in \mathbb{R}$. This completes the proof. $\square$

We also need the uniform-in-t_0 exponential convergence of $u^\pm(t,x;t_0)$ to $u(t,x)$.

Lemma 4.12. *There exist t_0-independent constants $C > 0$ and $\omega_* > 0$, and two families of shifts $\{\xi_{t_0}^\pm\}_{t_0 \in \mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}^\pm| < \infty$ such that*

$$\sup_{x\in\mathbb{R}} |u^\pm(t,x;t_0) - u(t,x-\xi_{t_0}^\pm)| \leq Ce^{-\omega_*(t-t_0)}$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$.

Proof. Let $C_2 = \sup_{t\in\mathbb{R}} |X_{\theta_2}(t) - X_{1-\theta_2}(t)| < \infty$. Then, it's easy to see that for any $t_0 \in \mathbb{R}$

$$u(t_0, x+C_2+h) - \theta_2 \leq u_0^+(x - X_{1-\theta_2}(t_0)) \leq u(t_0,x) + \epsilon_0, \quad x \in \mathbb{R}$$
$$u(t_0,x) - \epsilon_0 \leq u_0^-(x - X_{\theta_2}(t_0)) \leq u(t_0, x-C_2-h) + \theta_2, \quad x \in \mathbb{R}$$

for arbitrary fixed $\epsilon_0 \in (0, \min\{\frac{1}{4}, \theta_0, 1-\theta_1\})$, that is,

$$u(t_0, x+C_2+h) - \mu_0 \leq u^+(t_0,x;t_0) \leq u(t_0,x) + \mu_0, \quad x \in \mathbb{R}$$
$$u(t_0,x) - \mu_0 \leq u_0^-(t_0,x;t_0) \leq u(t_0, x-C_2-h) + \mu_0, \quad x \in \mathbb{R},$$

where $\mu_0 = \max\{\theta_2, \epsilon_0\}$. Since C_2, h and μ_0 are independent of $t_0 \in \mathbb{R}$, we apply Corollary 4.8 to conclude the result. $\square$

Finally, we prove Theorem 4.9.

Proof of Theorem 4.9. By Lemma 4.11 and Lemma 4.12, we have

$$u(t,x-\xi_{t_0}^+) \leq u^+(t,x;t_0) + Ce^{-\omega_*(t-t_0)} \leq e^{-c^+(x-X(t)-h^+)} + Ce^{-\omega_*(t-t_0)}$$

for all $x \in \mathbb{R}$ and $t \geq t_0$. Since $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}^+| < \infty$, there exists $\xi^+ \in \mathbb{R}$ such that $\xi_{t_0}^+ \to \xi^+$ as $t_0 \to -\infty$ along some subsequence. Thus, for any $(t,x) \in \mathbb{R}\times\mathbb{R}$, setting $t_0 \to -\infty$ along this subsequence, we find $u(t,x-\xi^+) \leq e^{-c^+(x-X(t)-h^+)}$. The lower bound for $u(t,x)$ follows similarly. The "in particular" part then is a simple consequence of the fact that $\sup_{t\in\mathbb{R}} |X_\lambda(t) - X(t)| < \infty$ for any $\lambda \in (0,1)$. □

Chapter 5

Uniquenss of transition fronts in time heterogeneous media

In this chapter, we study the uniqueness of transition fronts in time heterogenous media. Therefore, we consider equation (2.3), i.e.,

$$u_t = J * u - u + f(t, u),$$

under the assumptions Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.4 as in Chapter 4.

Let $v(t, x)$ be an arbitrary transition front (not necessarily nonicreasing in space), and $u(t, x)$ be an arbitrary space nonincreasing transition front of (2.3). Hence, all results obtained in Chapter 3 apply to $v(t, x)$, and all results obtained in Chapter 3 and Chapter 4 apply to $u(t, x)$.

Let $Y(t)$, $Y_\lambda^\pm(t)$ be the interface locations of $v(t, x)$, and $X(t)$, $X_\lambda(t) = X_\lambda^\pm(t)$ be the interface locations of $u(t, x)$. By Remark 3.9, both $X(t)$ and $Y(t)$ are continuously differentiable. By Corollary 4.2, $X_\lambda(t)$ is continuously differentiable. But, $Y_\lambda^\pm(t)$ may jump.

We prove

Theorem 5.1. *There exists some* $\xi \in \mathbb{R}$ *such that* $v(t, x) = u(t, x + \xi)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Combining all results obtained in Section 4 and Theorem 5.1, we have

Corollary 5.2. *Statements* (i)-(vi) *in Theorem 2.3 hold.*

To show Theorem 5.1, we first prove a lemma.

Lemma 5.3. *There holds* $\sup_{t\in\mathbb{R}} |X(t) - Y(t)| < \infty$.

Proof. Note that since $\sup_{t\in\mathbb{R}} |X_{\frac{1}{2}}(t) - X(t)| < \infty$, it suffices to show: (i) $\sup_{t\geq 0} |Y(t) - X_{\frac{1}{2}}(t)| < \infty$; (ii) $\sup_{t\leq 0} |Y(t) - X_{\frac{1}{2}}(t)| < \infty$.

(i) Let $\mu \in (0, \min\{\frac{1}{4}, \theta_0, 1-\theta_1\})$ be small. We first see that

$$u(0, x - Y^-_{1-\mu}(0) + X_\mu(0)) - \mu \le v(0,x) \le u(0, x - Y^+_\mu(0) + X_{1-\mu}(0)) + \mu, \quad x \in \mathbb{R}. \quad (5.1)$$

In fact, if $x \ge Y^-_{1-\mu}(0)$, then by the monotonicity of $u(t,x)$ in x, we have

$$u(0, x - Y^-_{1-\mu}(0) + X_\mu(0)) - \mu \le u(0, X_\mu(0)) - \mu = 0 < v(0,x).$$

If $x \le Y^-_{1-\mu}(0)$, then $v(0,x) \ge 1 - \mu \ge u(0, x - Y^-_{1-\mu}(0) + X_\mu(0)) - \mu$. This proves the first inequality. The second one is checked similarly.

Setting $\xi^-_0 = Y^-_{1-\mu}(0) - X_\mu(0)$ and $\xi^+_0 = Y^+_\mu(0) - X_{1-\mu}(0)$ in (5.1), and then, applying Corollary 4.6 to (5.1), we find

$$u(t, x - \xi^-) - \mu \le v(t,x) \le u(t, x - \xi^+) + \mu, \quad x \in \mathbb{R} \quad (5.2)$$

for all $t \ge 0$, where $\xi^\pm = \xi^\pm_0 \pm \frac{A\mu}{\omega}$. It then follows from the first inequality in (5.2) and the monotonicity of $u(t,x)$ in x that

$$\frac{1}{2} - \mu = u(t, X_{\frac{1}{2}}(t)) - \mu < u(t, x - \xi^-) - \mu \le v(t,x) \quad \text{for all} \quad x < \xi^- + X_{\frac{1}{2}}(t),$$

which implies that $\xi^- + X_{\frac{1}{2}}(t) \le Y^-_{\frac{1}{2}-\mu}(t)$ for $t \ge 0$. Similarly, the second inequality in (5.2) and the monotonicity of $u(t,x)$ in x implies that

$$v(t,x) \le u(t, x - \xi^+) + \mu < u(t, X_{\frac{1}{2}}(t)) + \mu = \frac{1}{2} + \mu \quad \text{for all} \quad x > \xi^+ + X_{\frac{1}{2}}(t),$$

which leads to $Y^+_{\frac{1}{2}+\mu}(t) \le \xi^+ + X_{\frac{1}{2}}(t)$ for $t \ge 0$. Since $\sup_{t\in\mathbb{R}} |Y^-_{\frac{1}{2}-\mu}(t) - Y(t)| < \infty$ and $\sup_{t\in\mathbb{R}} |Y(t) - Y^+_{\frac{1}{2}+\mu}(t)| < \infty$ by Lemma 3.1, we conclude that $\sup_{t\ge 0} |X_{\frac{1}{2}}(t) - Y(t)| < \infty$.

(ii) Suppose on the contrary that $\sup_{t\leq 0}|Y(t)-X_{\frac{1}{2}}(t)| = \infty$. Since both $Y(t)$ and $X_{\frac{1}{2}}(t)$ are continuous, there exists a sequence $t_n \to -\infty$ as $n\to\infty$ such that either $Y(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ or $Y(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n \to \infty$.

Suppose first that $Y(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ as $n\to\infty$. Since $\sup_{t\in\mathbb{R}}|Y(t) - Y^-_{\frac{1}{2}}(t)| < \infty$, we in particular have $Y^-_{\frac{1}{2}}(t_n) - X_{\frac{1}{2}}(t_n) \to \infty$ as $n\to\infty$. Then, for any $\mu > 0$ and $\xi_0 \in \mathbb{R}$, we can find an $N = N(\mu, \xi_0) > 0$ such that $t_N < 0$ and $u(t_N, x-\xi_0) - \mu \leq v(t_N, x)$ for $x\in\mathbb{R}$. We then apply Corollary 4.6 to conclude that

$$u(t, x-\xi_0 + \frac{A\mu}{\omega}) - \mu \leq v(t,x), \quad x\in\mathbb{R}, \quad t\geq t_N.$$

Then, setting $t=0$ in the above estimate, we find from the monotonicity of $u(t,x)$ in x that

$$\frac{1}{2} - \mu = u(0, X_{\frac{1}{2}}(0)) - \mu < u(0, x-\xi_0+\frac{A\mu}{\omega}) - \mu \leq v(0,x), \quad \forall x < \xi_0 - \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0),$$

which implies that $\xi_0 - \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0) \leq Y^-_{\frac{1}{2}-\mu}(0)$. Setting $\xi_0 \to \infty$, we arrive at a contradiction.

Now, suppose $Y(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n\to\infty$. Then, we have in particular $Y^+_{\frac{1}{2}}(t_n) - X_{\frac{1}{2}}(t_n) \to -\infty$ as $n \to \infty$. Then, for any $\mu > 0$ and $\xi_0 \in \mathbb{R}$, we can find some $N = N(\mu,\xi_0) > 0$ such that $t_N < 0$ and $v(t_n, x) \leq u(t_N, x-\xi_0) + \mu$ for $x \in \mathbb{R}$. Applying Corollary 4.6, we find

$$v(t,x) \leq u(t, x-\xi_0 - \frac{A\mu}{\omega}) + \mu, \quad x\in\mathbb{R}, \quad t\geq t_N.$$

Setting $t=0$ in the above estimate, we find

$$v(0,x) \leq u(0, x-\xi_0 - \frac{A\mu}{\omega}) + \mu < u(0, X_{\frac{1}{2}}(0)) + \mu = \frac{1}{2} + \mu, \quad \forall x > \xi_0 + \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0),$$

which implies that $Y^+_{\frac{1}{2}+\mu}(0) \leq \xi_0 + \frac{A\mu}{\omega} + X_{\frac{1}{2}}(0)$. This leads to a contradiction if we set $\xi_0 \to -\infty$. Hence, we have $\sup_{t\leq 0}|Y(t) - X_{\frac{1}{2}}(t)| < \infty$. This completes the proof. $\square$

Now, we prove Theorem 5.1.

Proof of Theorem 5.1. Let $\theta_3 \in (0, \min\{\theta_0, 1-\theta_1\})$. For $t_0 \in \mathbb{R}$, we define

$$u^-(t_0,x) = u(t_0, x - Y^-_{1-\theta_3}(t_0) + X_{\theta_3}(t_0)) - \theta_3,$$
$$u^+(t_0,x) = u(t_0, x - Y^+_{\theta_3}(t_0) + X_{1-\theta_3}(t_0)) + \theta_3.$$

We claim

$$u^-(t_0,x) \le v(t_0,x) \le u^+(t_0,x), \quad x \in \mathbb{R}.$$

In fact, if $x \ge Y^-_{1-\theta_3}(t_0)$, then by monotonicity, $u^-(t_0,x) \le u(t_0, X_{\theta_3}(t_0)) - \theta_3 = 0 < v(t_0,x)$. If $x \le Y^-_{1-\theta_3}(t_0)$, then by the definition of $Y^-_{1-\theta_3}(t_0)$, $v(t_0,x) \ge 1-\theta_3 > u^-(t_0,x)$. Hence, $u^-(t_0,x) \le v(t_0,x)$. The inequality $v(t_0,x) \le u^+(t_0,x)$ is checked similarly.

By Lemma 3.1 and Lemma 5.3, we have

$$L := \max\left\{ \sup_{t_0\in\mathbb{R}} |Y^-_{1-\theta_3}(t_0) - X_{\theta_3}(t_0)|, \sup_{t_0\in\mathbb{R}} |Y^+_{\theta_3}(t_0) - X_{1-\theta_3}(t_0)| \right\} < \infty.$$

Then, shifting $u^-(t_0,x)$ to the left and $u^+(t_0,x)$ to the right, we conclude from the monotonicity of $u(t,x)$ in x that for all $t_0 \in \mathbb{R}$, there holds

$$u(t_0, x+L) - \theta_3 \le u^-(t_0,x) \le v(t_0,x) \le u^+(t_0,x) \le u(t_0, x-L) + \theta_3. \tag{5.3}$$

That is, we are in the position to apply Corollary 4.8. So, we apply Corollary 4.8 to (5.3) to conclude that there exist t_0-independent constants $C > 0$ and $\omega > 0$, and a family of shifts $\{\xi_{t_0}\}_{t_0\in\mathbb{R}} \subset \mathbb{R}$ satisfying $\sup_{t_0\in\mathbb{R}} |\xi_{t_0}| < \infty$ such that

$$\sup_{x\in\mathbb{R}} |v(t,x) - u(t, x-\xi_{t_0})| \le Ce^{-\omega_*(t-t_0)}$$

for all $t \ge t_0$. We now pass to the limit $t_0 \to -\infty$ along some subsequence to conclude $\xi_{t_0} \to \xi$ for some $\xi \in \mathbb{R}$, and then conclude that $v(t,x) = u(t, x-\xi)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. This completes the proof. □

Chapter 6

Periodic traveling waves in time periodic media

In this chapter, we consider equation (2.3), i.e.,

$$u_t = J * u - u + f(t, u),$$

under the assumptions Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.4 as in Chapter 4, and the additional time periodic assumption, that is, there exists $T > 0$ such that $f(t + T, u) = f(t, u)$ for all $t \in \mathbb{R}$ and $u \in [0, 1]$. We also suppose equation (2.3) admits a space decreasing transition front so that Theorem 5.1 is applied here.

Let $u(t, x)$ be an arbitrary transition front of (2.3). Note $u(t, x)$ is decreasing in x by Theorem 5.1. We restate (vii) in Theorem 2.3 as

Theorem 6.1. *$u(t, x)$ must be a T-periodic traveling wave, that is, there are a constant $c > 0$ and a function $\psi : \mathbb{R} \times \mathbb{R} \to (0, 1)$ satisfying*

$$\begin{cases} \psi_t = J * \psi - \psi + c\psi_x + f(t, \psi), \\ \lim_{x\to-\infty} \psi(t, x) = 1, \ \lim_{x\to\infty} \psi(t, x) = 0 \text{ uniformly in } t \in \mathbb{R}, \\ \psi(t, \cdot) = \psi(t + T, \cdot) \text{ for all } t \in \mathbb{R} \end{cases} \tag{6.1}$$

such that $u(t, x) = \psi(t, x - ct)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Proof. By periodicity, $u(t + T, x)$ is also a transition front of (2.3). Theorem 5.1 then yields the existence of some $\xi \in \mathbb{R}$ such that

$$u(t + T, x) = u(t, x + \xi), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{6.2}$$

Fix some $\theta_* \in (0,1)$. Setting $t = 0$ and $x = X_{\theta_*}(T)$ in (6.2), we find $\theta_* = u(T, X_{\theta_*}(T)) = u(0, X_{\theta_*}(T) + \xi)$, which leads to $X_{\theta_*}(0) = X_{\theta_*}(T) + \xi$ by monotonicity. It then follows from (6.2) that

$$u(t+T, x) = u(t, x + X_{\theta_*}(0) - X_{\theta_*}(T)), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{6.3}$$

Setting $c = \frac{X_{\theta_*}(T) - X_{\theta_*}(0)}{T}$ and

$$\psi(t,x) = u(t, x + ct), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R},$$

we readily verify that (c, ψ) satisfies (6.1). The fact that $c > 0$ follows from the fact that $u(t,x)$ moves to the right. □

Chapter 7

Asymptotic speeds of transition fronts in time uniquely ergodic media

In this chapter, we consider equation (2.3), i.e.,

$$u_t = J * u - u + f(t,u),$$

under the assumptions Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.4 as in Chapter 4, and the following additional assumption on f: the dynamical system $\{\sigma_t\}_{t\in\mathbb{R}}$ defined by

$$\sigma_t : H(f) \to H(f), \quad f \mapsto f(\cdot + t, \cdot) \tag{7.1}$$

is compact and uniquely ergodic, where $H(f) = \overline{\{f(\cdot + t, \cdot) : t \in \mathbb{R}\}}$ with the closure taken under the open-compact topology (which is equivalent to locally uniform convergence in our case). We also suppose equation (2.3) admits a space decreasing transition front so that Theorem 5.1 is applied here

Let $u(t,x)$ be an arbitrary transition front of (2.3) and let $X(t)$ be the corresponding interface locations and $X_{\frac{1}{2}}(t)$ be the interface locations at $\frac{1}{2}$.

We restate (viii) in Theorem 2.3 as

Theorem 7.1. *The asymptotic speeds* $\lim_{t\to\pm\infty} \frac{X(t)}{t}$ *exist.*

To prove Theorem 7.1, let us first do some preparation. Note that any $g \in H(f)$ satisfies Hypothesis 2.2 and Hypothesis 2.4. Let $u^g(t,x)$ be the unique transition front of

$$u_t = J * u - u + g(t,x) \tag{7.2}$$

satisfying the normalization $X^g_{\frac{1}{2}}(0)=0$, where $X^g_{\frac{1}{2}}(t)$ is the interface locations of $u^g(t,x)$ at $\frac{1}{2}$, i.e., $u^g(t,X^g_{\frac{1}{2}}(t))=\frac{1}{2}$.

Let

$$\psi^g(t,x)=u^g(t,x+X^g_{\frac{1}{2}}(t)),\quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}$$

be the profiles of $u^g(t,x)$. Then, $\psi^g(t,0)=\frac{1}{2}$ for all $t\in\mathbb{R}$.

We prove

Lemma 7.2. *There hold the following statements:*

(i) *for any $g\in H(f)$, there holds*

$$\psi^g(t+\tau,x)=\psi^{g\cdot\tau}(t,x),\quad \forall (t,\tau,x)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},$$

where $g\cdot\tau=g(\cdot+\tau,\cdot)$;

(ii) *there holds $\sup_{(t,\tau)\in\mathbb{R}\times\mathbb{R}}|\dot{X}^{f\cdot\tau}_{\frac{1}{2}}(t)|<\infty$;*

(iii) *the limits*

$$\lim_{x\to-\infty}\psi^g(t,x)=1\quad \textit{and}\quad \lim_{x\to\infty}\psi^g(t,x)=0$$

are unfiormly in $t\in\mathbb{R}$ and $g\in H(f)$;

(iv) *there holds $\sup_{g\in H(f)}\sup_{t\in\mathbb{R}}|\dot{X}^g_{\frac{1}{2}}(t)|<\infty$.*

We remark that (ii) is a special case of (iv), but it plays an important role in proving the lemma, so we state it explicitly.

Proof of Lemma 7.2. For notational simplicity, we write $X^g(t)=X^g_{\frac{1}{2}}(t)$. Therefore, $u^g(t,X^g(t))=\frac{1}{2}$ and $X^g(0)=0$.

(i) Fix any $\tau\in\mathbb{R}$. We see that both

$$u_1(t,x)=\psi^{g\cdot\tau}(t,x-X^{g\cdot\tau}(t))\quad\text{and}\quad u_2(t,x)=\psi^g(t+\tau,x-X^g(t+\tau))$$

are transition fronts of $u_t = J * u - u + g(t+\tau, x)$. Then, by uniqueness, i.e., Theorem 5.1, there exists $\xi \in \mathbb{R}$ such that $u_1(t,x) = u_2(t, x+\xi)$. Moreover, since

$$u_1(t, X^{g\cdot\tau}(t)) = \psi^{g\cdot\tau}(t,0) = \frac{1}{2} \quad \text{and} \quad u_2(t, X^g(t+\tau)) = \psi^g(t+\tau, 0) = \frac{1}{2},$$

we find

$$u_1(t, X^g(t+\tau) - \xi) = u_2(t, X^g(t+\tau)) = \frac{1}{2},$$

and hence, $X^{g\cdot\tau}(t) = X^g(t+\tau) - \xi$ by monotonicity. It then follows that

$$\begin{aligned}\psi^{g\cdot\tau}(t,x) &= u_1(t, x + X^{g\cdot\tau}(t)) \\ &= u_2(t, x + X^{g\cdot\tau}(t) + \xi) \\ &= u_2(t, x + X^g(t+\tau)) \\ &= \psi^g(t+\tau, x).\end{aligned}$$

(ii) By (i), we in particular have

$$\psi^{f\cdot\tau}(t,x) = \psi^f(t+\tau, x), \quad \forall (t,\tau,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}. \tag{7.3}$$

Since the limits $\psi^f(t,x) \to 1$ as $x \to -\infty$ and $\psi^f(t,x) \to 0$ as $x \to \infty$ are uniformly in $t \in \mathbb{R}$, we find

$$\lim_{x\to-\infty} \psi^{f\cdot\tau}(t,x) = 1 \quad \textit{and} \quad \lim_{x\to\infty} \psi^{f\cdot\tau}(t,x) = 0 \quad \text{uniformly in} \quad (t,\tau) \in \mathbb{R}\times\mathbb{R}. \tag{7.4}$$

From (7.3), we also have

$$u^{f\cdot\tau}(t, x + X^{f\cdot\tau}(t)) = u^f(t+\tau, x + X^f(t+\tau)), \quad \forall (t,\tau,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}. \tag{7.5}$$

Setting $x = 0$ and differentiating the resulting equality with respect to t, we find

$$\begin{aligned}\dot{X}^{f\cdot\tau}(t) &= \frac{\frac{d}{dt}[u^f(t+\tau, X^f(t+\tau))] - u_t^{f\cdot\tau}(t, X^{f\cdot\tau}(t))}{u_x^{f\cdot\tau}(t, X^{f\cdot\tau}(t))} \\ &= \frac{\frac{d}{dt}[u^f(t+\tau, X^f(t+\tau))] - u_t^{f\cdot\tau}(t, X^{f\cdot\tau}(t))}{u_x^{f}(t+\tau, X^{f}(t+\tau))},\end{aligned}$$

where we used $u_x^{f\cdot\tau}(t, X^{f\cdot\tau}(t)) = u_x^f(t+\tau, X^f(t+\tau))$, which comes from (7.5). We see that both $\frac{d}{dt}[u^f(t+\tau, X^f(t+\tau))]$ and $u_t^{f\cdot\tau}(t, X^{f\cdot\tau}(t))$ are bounded uniformly in $(t,\tau) \in \mathbb{R}\times\mathbb{R}$. Moreover, $u_x^f(t+\tau, X^f(t+\tau))$ is bounded uniformly in $(t,\tau) \in \mathbb{R}\times\mathbb{R}$ due to the uniform steepness, i.e., Lemma 4.1. It then follows that $\sup_{(t,\tau)\in\mathbb{R}\times\mathbb{R}} |\dot{X}^{f\cdot\tau}(t)| < \infty$.

(iii) For any $g \in H(f)$, there is a sequence $\{t_n\}$ such that $g_n := f\cdot t_n \to g$ in $H(f)$. Trivially, $\sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_t^{g_n}(t,x)| < \infty$, and by (i), $\sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x^{g_n}(t,x)| < \infty$. It then follows from (ii) that

$$\begin{aligned}\sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |\psi_t^{g_n}(t,x)| &= \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_t^{g_n}(t, x+X^{g_n}(t)) + \dot{X}^{g_n}(t) u_x^{g_n}(t, x+X^{g_n}(t))| \\ &\leq \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_t^{g_n}(t, x+X^{g_n}(t))| \\ &\quad + \sup_{(t,\tau)\in\mathbb{R}\times\mathbb{R}} |\dot{X}^{f\cdot\tau}(t)| \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x^{g_n}(t, x+X^{g_n}(t))| < \infty, \\ \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |\psi_x^{g_n}(t,x)| &= \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |u_x^{g_n}(t, x+X^{g_n}(t))| < \infty.\end{aligned}$$

In particular, by Arzelà-Ascoli theorem, there exists a continuous function $\psi(\cdot,\cdot;g) : \mathbb{R}\times\mathbb{R} \to [0,1]$ such that $\lim_{n\to\infty} \psi^{g_n}(t,x) = \psi(t,x;g)$ locally uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}$. We then conclude from (7.4) that

$$\lim_{x\to-\infty} \psi(t,x;g) = 1 \quad \text{and} \quad \lim_{x\to\infty} \psi(t,x;g) = 0 \quad \text{uniformly in } t \in \mathbb{R} \text{ and } g \in H(f). \tag{7.6}$$

It remains to show $\psi^g(t,x)=\psi(t,x;g)$. Fix any $g\in H(f)$. By (ii), there exists a continuous function $X(\cdot;g):\mathbb{R}\to\mathbb{R}$ such that, up to a subsequence,

$$X^{g_n}(t)\to X(t;g) \quad \text{and} \quad \psi^{g_n}(t,x-X^{g_n}(t))\to\psi(t,x-X(t;g);g) \tag{7.7}$$

as $n\to\infty$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$. Since, trivially,

$$\begin{aligned}\sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}\left|\frac{d}{dt}\psi^{g_n}(t,x-X^{g_n}(t))\right| &= \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_t^{g_n}(t,x)|<\infty,\\ \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}\left|\frac{d^2}{dt^2}\psi^{g_n}(t,x-X^{g_n}(t))\right| &= \sup_n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_{tt}^{g_n}(t,x)|<\infty,\end{aligned}$$

we will also have

$$\frac{d}{dt}\psi^{g_n}(t,x-X^{g_n}(t))\to\frac{d}{dt}\psi(t,x-X(t;g);g) \tag{7.8}$$

as $n\to\infty$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$. Thus, $\psi(t,x-X(t;g);g)$ is a global-in-time solution of (7.2), and hence, it is a transition front due to (7.6). Uniqueness of transition fronts and the normalization $X^{g_n}(0)=0$ then imply that $\psi^g(t,x)=\psi(t,x;g)$.

(iv) It's a simple consequence of (ii) and the proof of (iii). $\square$

Now, we prove Theorem 7.1.

Proof of Theorem 7.1. Again, write $X^g(t)=X^g_{\frac{1}{2}}(t)$. Since $\sup_{t\in\mathbb{R}}|X^f(t)-X(t)|<\infty$, it suffices to show the existence of the limits $\lim_{t\to\pm\infty}\frac{X^f(t)}{t}$. Since

$$\lim_{t\to\pm\infty}\frac{X^f(t)}{t}=\lim_{t\to\pm\infty}\frac{X^f(t)-X^f(0)}{t}=\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\dot{X}^f(s)ds,$$

we only need to show the dynamical system (i.e., the shift operators) generated by $\dot{X}^f(t)$ is compact and uniquely ergodic.

To this end, we first derive a formula for $\dot{X}^g(t)$. We claim

$$\dot{X}^g(t)=-\frac{\int_{\mathbb{R}}J(y)\psi^g(t,-y)dy-\frac{1}{2}+g(t,\frac{1}{2})}{\psi^g_x(t,0)},\quad \forall t\in\mathbb{R}. \tag{7.9}$$

In fact, differentiating $u^g(t, X^g(t)) = \frac{1}{2}$, we find

$$\dot{X}^g(t) = -\frac{u_t^g(t, X^g(t))}{u_x^g(t, X^g(t))} = -\frac{[J * u^g(t,\cdot)](X^g(t)) - u^g(t, X^g(t)) + f(t, u^g(t, X^g(t)))}{u_x^g(t, X^g(t))}.$$

The equality (7.9) then follows from $u^g(t, x + X^g(t)) = \psi^g(t, x)$ and $u^g(t, X^g(t)) = \frac{1}{2}$. Note that due to (i) in Lemma 7.2 and (7.9), there holds $\dot{X}^{g\cdot\tau}(t) = \dot{X}^g(t+\tau)$ for all $t, \tau \in \mathbb{R}$.

Next, we define

- the phase space $\tilde{H} = \{(\psi^g, \dot{X}^g) | g \in H(f)\}$;
- the shift operators $\{\tilde{\sigma}\}_{t\in\mathbb{R}}$, i.e., the dynamical system on $\tilde{H}$,

$$\tilde{\sigma}_t : \tilde{H} \to \tilde{H}, \quad (\psi^g, \dot{X}^g) \mapsto (\psi^{g\cdot t}, \dot{X}^{g\cdot t}) = (\psi^g(\cdot + t, \cdot), \dot{X}^g(\cdot + t));$$

- an operator $\Omega : H(f) \to \tilde{H}$, $g \mapsto (\psi^g, \dot{X}^g)$.

Clearly,

$$\tilde{\sigma}_t \circ \Omega = \Omega \circ \sigma_t, \quad \forall t \in \mathbb{R}, \tag{7.10}$$

where $\{\sigma_t\}_{t\in\mathbb{R}}$ is given in (7.1).

We show that Ω is a homeomorphism. We first claim that Ω is continuous. By (7.9), the continuity of Ω is the case if we can show that if $g_n \to g_*$ in $H(f)$ as $n \to \infty$, then

$$\psi^{g_n}(t, x) \to \psi^{g_*}(t, x) \text{ locally uniform in } t \in \mathbb{R} \text{ and uniformly in } x \in \mathbb{R} \tag{7.11}$$

as $n \to \infty$. To see this, let $g_n \to g_*$ in $H(f)$ as $n \to \infty$, then as in the proof of (iii) in Lemma 7.2, there exist continuous functions $X^* : \mathbb{R} \to \mathbb{R}$ and $\psi^* : \mathbb{R} \times \mathbb{R} \to [0, 1]$ such that

$$X^{g_n}(t) \to X^*(t) \text{ and } \psi^{g_n}(t, x - X^{g_n}(t)) \to \psi^*(t, x - X^*(t)) \text{ locally uniformly in } (t, x) \in \mathbb{R}\times\mathbb{R}$$

as $n \to \infty$. As (7.8), we also have

$$\frac{d}{dt}\psi^{g_n}(t, x - X^{g_n}(t)) \to \frac{d}{dt}\psi^*(t, x - X^*(t)) \text{ locally uniformly in } (t, x) \in \mathbb{R} \times \mathbb{R}$$

as $n \to \infty$. In particular, $\psi^*(t, x - X^*(t))$ is global-in-time solution of (7.2) with g replaced by g^*. Moreover, (iii) in Lemma 7.2 forces $\psi^*(t, x - X^*(t))$ to be a transition front, and hence, $\psi^*(t, x) = \psi^{g_*}(t, x)$ by uniqueness and normalization. It then follows that $\psi^{g_n}(t, x) \to \psi^{g_*}(t, x)$ locally uniform in $(t, x) \in \mathbb{R} \times \mathbb{R}$ as $n \to \infty$. But, this actually leads to (7.11) due to the uniform limits as $x \to \pm\infty$ as in (iii) in Lemma 7.2. Hence, Ω is continuous.

Clearly, from the continuity of Ω and the compactness of $H(f)$, $\tilde{H} = \Omega(H(f))$ is compact, and hence, $\tilde{H} = \overline{\{(\psi^{f\cdot t}, \dot{X}^{f\cdot t}) | t \in \mathbb{R}\}}$. Thus, if we can show that Ω is one-to-one, then its inverse Ω^{-1} exists and must be continuous, and hence, Ω is a homeomorphism.

We show Ω is one-to-one. For contradiction, suppose there are $g_1, g_2 \in H(f)$ with $g_1 \neq g_2$, but $\Omega g_1 = \Omega g_2$, i.e., $(\psi^{g_1}, \dot{X}^{g_1}) = (\psi^{g_2}, \dot{X}^{g_2})$. In particular, $\dot{X}^{g_1} = \dot{X}^{g_2}$, which together with the normalization $X^{g_1}(0) = 0 = X^{g_2}(0)$ gives $X^{g_1} = X^{g_2}$. It then follows from (**??**) that

$$u^{g_1}(t, x) = u^{g_2}(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

which then leads to $g_1(t, u(t, x)) = g_2(t, u(t, x))$, where $u = u^{g_1} = u^{g_2}$. Since $u(t, x)$ is continuous and connects 0 and 1 for any $t \in \mathbb{R}$, we conclude that $g_1 = g_2$ on $\mathbb{R} \times [0, 1]$. It is a contradiction. Hence, Ω is one-to-one, and therefore, Ω is a homeomorphism.

Since Ω is a homeomorphism, invariant measures on $H(f)$ and $\tilde{H}$ are related by Ω. We then conclude from (7.10) and the fact $\{\sigma_t\}_{t\in\mathbb{R}}$ is compact and uniquely ergodic that $\{\tilde{\sigma}_t\}_{t\in\mathbb{R}}$ is compact and uniquely ergodic. The result follows. □

Chapter 8

Construction of transition fronts in time-heterogeneous media

In this chapter, we construct transition fronts for the equation (2.3), i.e.,

$$u_t = J * u - u + f(t,u),$$

Throughout this chapter, we assume Hypothesis 2.1, Hypothesis 2.5 and Hypothesis 2.3.

We restate Theorem 2.4 in some details as

Theorem 8.1. *Equation* (2.3) *admits a transition front* $u(t,x)$ *that is strictly decreasing in space and uniformly Lipschitz continuous in space, that is,*

$$\sup_{x\neq y, t\in\mathbb{R}} \left| \frac{u(t,y)-u(t,x)}{y-x} \right| < \infty.$$

Moreover, there exists a continuous differentiable function $X:\mathbb{R}\to\mathbb{R}$ *such that the following hold:*

(i) *there exist* $c_{\min}>0$ *and* $c_{\max}>0$ *such that* $c_{\min}\leq \dot{X}(t)\leq c_{\max}$ *for all* $t\in\mathbb{R}$*;*

(ii) *there exist two exponents* $c_{\pm}>0$ *and two shifts* $h_{\pm}>0$ *such that*

$$u(t,x+X(t)+h_+)\leq e^{-c_+x} \quad \textit{and} \quad u(t,x+X(t)-h_-)\geq 1-e^{c_-x}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}$.

The proof of Theorem 8.1 is constructive. In Section 8.1, we first construct appropriate approximating front-like solutions. We next show that the approximating solutions enjoys bounded interface width and exponential decaying estimates in Section 8.2 and Section 8.3, respectively. This ensures the limit function of the approximating solutions (if exists) must be a transition front. We then show that the approximating solutions are actually uniformly Lipschitz continuous in space in Section 8.4, and hence, the limit function exists. This finishes the construction. We also study the space regularity of this constructed transition front in Section 8.5 via improving the reglarity of approximating solutions.

8.1 Approximating front-like solutions

In this section, we construct approximating front-like solutions of (2.3), which will be shown to converge to a transition solution of (2.3).

Note that, by general semigroup theory (see e.g. [58]), for any $u_0 \in C^b_{\rm unif}(\mathbb{R},\mathbb{R})$ and $s \in \mathbb{R}$, (2.2) has a unique (local) solution $u(t,\cdot;s,u_0) \in C^b_{\rm unif}(\mathbb{R},\mathbb{R})$ with $u(s,x;s,u_0) = u_0(x)$, where

$$C^b_{\rm unif}(\mathbb{R},\mathbb{R}) = \{u \in C(\mathbb{R},\mathbb{R}) \,|\, u \text{ is uniformly continuous on } \mathbb{R} \text{ and } \sup_{x\in\mathbb{R}} |u(x)| < \infty\}$$

equipped with the norm $\|u\| = \sup_{x\in\mathbb{R}} |u(x)|$. Moreover, $u(t,\cdot;s,u_0)$ is continuous in $s \in \mathbb{R}$ and $u_0 \in C^b_{\rm unif}(\mathbb{R},\mathbb{R})$. By the comparison principle, if $u_0(x) \geq 0$ for $x \in \mathbb{R}$, then $u(t,\cdot;s,u_0)$ exists for all $t \geq s$ and $u(t,x;s,u_0) \geq 0$ for $t \geq s$ and $x \in \mathbb{R}$.

Recall that ϕ_B is the profile of bistable traveling waves given in (A.2). For $s < 0$ and $y \in \mathbb{R}$, denote by $u(t,x;s,\phi_B(\cdot - y))$ the classical solution of (2.3) with initial data $u(s,x;s,\phi_B(\cdot - y)) = \phi_B(x - y)$. The next lemma gives the approximating solutions.

Lemma 8.2. *For any $s < 0$, there exists a unique $y_s \in \mathbb{R}$ such that $u(0,0;s,\phi_B(\cdot - y_s)) = \theta$. Moreover, $y_s \to -\infty$ as $s \to -\infty$.*

Proof. Let $s < 0$. We first see that comparison principle gives

$$u(t,x;s,\phi_B(\cdot - y)) \geq \phi_B(x - y - c_B(t - s)), \quad x \in \mathbb{R},\ t \geq s.$$

In particular, $u(0,0;s,\phi_B(\cdot-y))\geq\phi_B(c_Bs-y)$. Thus, the monotonicity and the normalization of ϕ_B ensure that $u(0,0;s,\phi_B(\cdot-y))\geq\theta$ if $y\geq c_Bs$.

To bound $u(t,x;s,\phi_{\min}(\cdot-y))$ from above, we see that by Lemma A.1, $\phi_B(\cdot-y)\leq e^{-c_B^+(\cdot-y-x_B^+)}$ for all $y\in\mathbb{R}$. Setting $v^y(t,\cdot;s)=e^{-c_B^+(\cdot-y-x_B^+-c(t-s))}$, where $c>0$ is to be chosen, we compute

$$v_t^y-[J*v^y-v^y]=\left[c_B^+c-\int_{\mathbb{R}}J(z)e^{c_B^+z}dz+1\right]v^y\geq f(t,v^y)$$

provided $c>0$ is so large that $c_B^+c-\int_{\mathbb{R}}J(z)e^{c_B^+z}dz+1\geq\sup_{u\in(0,1)}\frac{f_{\tilde{B}}(u)}{u}$, where $f_{\tilde{B}}$ is as in Hypothesis 2.5. Comparison principle then leads to $u(t,x;s,\phi_B(\cdot-y))\leq e^{-c_B^+(x-y-x_B^+-c(t-s))}$ for all $x\in\mathbb{R}$ and $t\geq s$. In particular, $u(0,0;s,\phi_B(\cdot-y))\leq\theta$ if $y\leq\frac{\ln\theta-x_B^++c_B^+cs}{c_B^+}$.

Continuity of $u(0,0;s,\phi_B(\cdot-y))$ in y then yields the existence of some $y_s\in\mathbb{R}$ such that $u(0,0;s,\phi_B(\cdot-y_s))=\theta$. The uniqueness of such an y_s is a simple consequence of the comparison principle. Moreover, the above analysis implies that

$$\frac{\ln\theta-x_B^++c_B^+cs}{c_B^+}\leq y_s\leq c_Bs, \tag{8.1}$$

and hence, $y_s\to-\infty$ as $s\to-\infty$. □

For notational simplicity, in what follows, we put

$$u(t,x;s)=u(t,x;s,\phi_B(\cdot-y_s)). \tag{8.2}$$

Thus, $u(s,\cdot;s)=\phi_B(\cdot-y_s)$. The next lemma provides some fundamental properties of $u(t,x;s)$.

Lemma 8.3. *For any $s<0$ and $t\geq s$ there hold*

(i) *the limits $u(t,-\infty;s)=1$ and $u(t,\infty;s)=0$ are locally uniformly in s and t;*

(ii) $u(t,x;s)$ *is strictly decreasing in* x. *In particular,* $u(t,x;s)$ *is almost everywhere differentiable in* x.

Proof. (i) It follows from the fact $u(t,x;s)\in(0,1)$ by the comparison principle, the estimate (8.1) and the following estimate

$$\phi(x-y_s-c_B(t-s))\leq u(t,x;s)\leq e^{-c_B^+(x-y_s-x_B^+-c(t-s))} \tag{8.3}$$

for some sufficiently large $c>0$, which is derived in Lemma 8.2.

(ii) For the monotonicity, we first see that $u(s,x;s)=\phi_B(x-y_s)$ is strictly decreasing in x. For any $y>0$, we apply comparison principle to $u(t,x-y;s)-u(t,x;s)$ to conclude that that $u(t,x-y;s)>u(t,x;s)$ for $t>s$. The result then follows. □

8.2 Bounded interface width

For $s<0$, $t\geq s$ and $\lambda\in(0,1)$, let $X_\lambda(t;s)$ be such that $u(t,X_\lambda(t;s);s)=\lambda$. By Lemma 8.3, it is well-defined and continuous in t (but not sure whether it is differentiable in t right now). Moreover, $X_{\lambda_1}(t;s)>X_{\lambda_2}(t;s)$ if $\lambda_1<\lambda_2$ by monotonicity of $u(t,x;s)$ in x.

The main result in this section is stated in the following

Theorem 8.4. *There exists* $\lambda_*\in(\theta,1)$ *such that for any* $0<\lambda_1<\lambda_2\leq\lambda_*$, *there holds*

$$\sup_{s<0,t\geq s}\left[X_{\lambda_1}(t;s)-X_{\lambda_2}(t;s)\right]<\infty.$$

Theorem 8.4 shows the uniform boundedness of the width between interfaces below λ_*. Later in Corollary 8.11, it is extended to any $0<\lambda_1<\lambda_2<1$.

The proof Theorem 8.4 is a little long and technical. To do so, we first study the rightward propagation of $X_\lambda(t;s)$. We have

Lemma 8.5. *Let $\lambda \in (\theta, 1)$. For any $\epsilon > 0$ there exists $t_{\epsilon,\lambda} > 0$ such that*

$$X_\lambda(t; s) - X_\lambda(t_0; s) \geq (c_B - \epsilon)(t - t_0 - t_{\epsilon,\lambda})$$

for all $s < 0$, $t \geq t_0 \geq s$.

Proof. Fix some $\lambda \in (\theta, 1)$. Let $u_0 : \mathbb{R} \to [0, 1]$ be a uniformly continuous and nonincreasing function satisfying $u_0(x) = \lambda$ for $x \leq x_0$ and $u_0(x) = 0$ for $x \geq 0$, where $x_0 < 0$ is fixed. Clearly, space monotonicity of $u(t, x; s)$ implies that

$$u(t_0, x + X_\lambda(t_0; s); s) \geq u_0(x), \quad x \in \mathbb{R},\ t_0 \geq s,$$

and then, by $f(t, u) \geq f_{\min}(u) \geq f_B(u)$ and the comparison principle, we find

$$u(t, x + X_\lambda(t_0; s); s) \geq u_B(t - t_0, x; u_0), \quad x \in \mathbb{R},\ t \geq t_0 \geq s.$$

By Lemma A.2, there are constants $\xi_B^- = \xi_B^-(\lambda) \in \mathbb{R}$, $q_B = q_B(\lambda) > 0$ and $\omega_B > 0$ such that

$$u_B(t - t_0, x; u_0) \geq \phi_B(x - \xi_B^- - c_B(t - t_0)) - q_B e^{-\omega_B(t-t_0)}, \quad x \in \mathbb{R},\ t \geq t_0 \geq s.$$

Hence,

$$u(t, x + X_\lambda(t_0; s); s) \geq \phi_B(x - \xi_B^- - c_B(t - t_0)) - q_B e^{-\omega_B(t-t_0)}, \quad x \in \mathbb{R},\ t \geq t_0 \geq s.$$

Let $T_0 = T_0(\lambda)$ be such that $q_B e^{-\omega_B T_0} = \frac{1-\lambda}{2}$ and denote by $\xi_B(\frac{1+\lambda}{2})$ the unique point such that $\phi_B(\xi_B(\frac{1+\lambda}{2})) = \frac{1+\lambda}{2}$. Setting $x = \xi_B^- + c_B(t - t_0) + \xi_B(\frac{1+\lambda}{2})$, we find for $t \geq t_0 + T_0$

$$u(t, \xi_B^- + c_B(t - t_0) + \xi_B(\frac{1+\lambda}{2}) + X_\lambda(t_0; s); s) \geq \phi_B(\xi_B(\frac{1+\lambda}{2})) - q_B e^{-\omega_B T_0} = \lambda,$$

which together with monotonicity implies that

$$X_\lambda(t;s) - X_\lambda(t_0;s) \geq \xi_B^- + c_B(t-t_0) + \xi_B(\frac{1+\lambda}{2}), \quad t \geq t_0 + T_0. \tag{8.4}$$

We now estimate $X_\lambda(t;s) - X_\lambda(t_0;s)$ for $t \in [t_0, t_0+T_0]$. We claim that there exists $z = z(T_0) < 0$ such that

$$X_\lambda(t;s) - X_\lambda(t_0;s) \geq z \quad \text{for all } s < 0,\ s \leq t_0 \leq t \leq t_0 + T_0. \tag{8.5}$$

Let $u_B(t,x;u_0)$ and $u_B(t;\lambda) := u_B(t,x;\lambda)$ be solutions of (A.1) with $u_B(0,x;u_0) = u_0(x)$ and $u_B(0;\lambda) = u_B(0,x;\lambda) \equiv \lambda$, respectively. By the comparison principle, we have $u_B(t,x;u_0) < u_B(t;\lambda)$ for all $x \in \mathbb{R}$ and $t > 0$, and $u_B(t,x;u_0)$ is strictly decreasing in x for $t > 0$.

We see that for any $t > 0$, $\lim_{x\to-\infty} u_B(t,x;u_0) = u_B(t;\lambda)$. This is because that $\frac{d}{dt}u_B(t,-\infty;u_0) = f_B(u_B(t,-\infty;u_0))$ for $t > 0$ and $u_B(0,-\infty;u_0) = \lambda$. Since $\lambda \in (\theta, 1)$, as a solution of the ODE $u_t = f_B(u)$, $u_B(t;\lambda)$ is strictly increasing in t, which implies that $u_B(t,-\infty;u_0) = u_B(t;\lambda) > \lambda$ for $t > 0$. As a result, for any $t > 0$ there exists a unique $\xi_B(t) \in \mathbb{R}$ such that $u_B(t,\xi_B(t);u_0) = \lambda$. Moreover, $\xi_B(t)$ is continuous in t.

Since $f(t,u) \geq f_B(u)$ and $u(t_0, \cdot + X_\lambda(t_0;s);s) \geq u_0$, the comparison principle implies that

$$u(t, x + X_\lambda(t_0;s);s) \geq u_B(t-t_0, x; u_0), \quad x \in \mathbb{R},\ t \geq t_0 \geq s.$$

Setting $x = \xi_B(t-t_0)$, we find $u(t, \xi_B(t-t_0) + X_\lambda(t_0;s);s) \geq \lambda$, which together with the monotonicity implies that $X_\lambda(t;s) \geq \xi_B(t-t_0) + X_\lambda(t_0;s)$ for $t \geq t_0 \geq s$. Thus, (8.5) follows if $\inf_{t\in(t_0,t_0+T_0]} \xi_B(t-t_0) > -\infty$, that is,

$$\inf_{t\in(0,T_0]} \xi_B(t) > -\infty. \tag{8.6}$$

We now show (8.6). Since $u_0(x) = \lambda$ for $x \le x_0$, continuity with respect to the initial data (in sup norm) implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$u_B(t;\lambda) - \lambda \le \epsilon \quad \text{and} \quad \sup_{x \le x_0}[u_B(t;\lambda) - u_B(t,x;u_0)] = u_B(t;\lambda) - u_B(t,x_0;u_0) \le \epsilon$$

for all $t \in [0,\delta]$, where the equality is due to monotonicity. By Hypothesis 2.1, J concentrates near 0 and decays very fast as $x \to \pm\infty$. Thus, we can choose $x_1 = x_1(\epsilon) << x_0$ such that $\int_{-\infty}^{x_0} J(x-y)dy \ge 1-\epsilon$ for all $x \le x_1$. Now, for any $x \le x_1$ and $t \in (0,\delta]$, we have

$$\begin{aligned}
\frac{d}{dt}u_B(t,x;u_0) &= \int_{\mathbb{R}} J(x-y)u_B(t,y;u_0)dy - u_B(t,x;u_0) + f_B(u_B(t,x;u_0)) \\
&\ge \int_{-\infty}^{x_0} J(x-y)u_B(t,y;u_0)dy - u_B(t,x;u_0) + f_B(u_B(t,x;u_0)) \\
&\ge (1-\epsilon)\inf_{x \le x_0} u_B(t,x;u_0) - u_B(t;\lambda) + f_B(u_B(t,x;u_0)) \\
&= -(1-\epsilon)\sup_{x \le x_0}[u_B(t;\lambda) - u_B(t,x;u_0)] - \epsilon u_B(t;\lambda) + f_B(u_B(t,x;u_0)) \\
&\ge -\epsilon(1-\epsilon) - \epsilon(\lambda+\epsilon) + f_B(u_B(t,x;u_0)) \\
&> 0
\end{aligned}$$

if we choose $\epsilon > 0$ sufficiently small, since then $f_B(u_B(t,x;u_0))$ is close to $f_B(\lambda)$, which is positive. This simply means that $u_B(t,x;u_0) > \lambda$ for all $x \le x_1$ and $t \in (0,\delta]$, which implies that $\xi_B(t) > x_1$ for $t \in (0,\delta]$. The continuity of ξ_B then leads to (8.6). This proves (8.5). The result of the proposition then follows from (8.4) and (8.5). □

We remark that the estimate for $X_\lambda(t;s)$ of the form

$$X_\lambda(t;s) - X_\lambda(t_0;s) \le (c_{\tilde{B}} + \epsilon)(t - t_0 + t_{\epsilon,\lambda})$$

can also be established due to the bistability, but we are not going to use this.

Next, we define a new interface location and study the boundedness between this new interface location and the original ones. For $\kappa > 0$, set

$$c_\kappa := \inf_{\lambda>0} \frac{1}{\lambda}\Big(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1 + \kappa\Big) > 0.$$

It is not hard to see that there exists a unique $\lambda_\kappa > 0$ such that

$$c_\kappa = \frac{1}{\lambda_\kappa}\Big(\int_{\mathbb{R}} J(y)e^{\lambda_\kappa y}dy - 1 + \kappa\Big). \tag{8.7}$$

We remark that c_κ corresponds to the minimal speed of traveling waves in the KPP case for nonlocal equations. As in the classical random dispersal case, we have

Lemma 8.6. *$c_\kappa \to 0$ and $\lambda_\kappa \to 0$ as $\kappa \to 0$.*

Proof. We see

$$c_\kappa \le \frac{1}{\sqrt{\kappa}}\Big(\int_{\mathbb{R}} J(y)e^{\sqrt{\kappa}y}dy - 1 + \kappa\Big) \to 0 \quad \text{as} \quad \kappa \to 0,$$

since $\frac{1}{\sqrt{\kappa}}\big(\int_{\mathbb{R}} J(y)e^{\sqrt{\kappa}y}dy - 1\big) \to 0$ as $\kappa \to 0$.

We show $\lambda_\kappa \to 0$ as $\kappa \to 0$. It is understood that λ_κ is the unique positive point such that $\frac{d}{d\lambda}\big[\frac{1}{\lambda}\big(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1 + \kappa\big)\big] = 0$, that is,

$$\lambda\int_{\mathbb{R}} J(y)ye^{\lambda y}dy - \int_{\mathbb{R}} J(y)e^{\lambda y}dy + 1 = \kappa.$$

Setting $g(\lambda) := \lambda\int_{\mathbb{R}} J(y)ye^{\lambda y}dy - \int_{\mathbb{R}} J(y)e^{\lambda y}dy + 1$, we see $g(0) = 0$ and

$$g'(\lambda) = \lambda\int_{\mathbb{R}} J(y)y^2e^{\lambda y}dy > 0 \quad \text{for } \lambda > 0.$$

This simply means that the unique solution of $g(\lambda) = \kappa$ goes to 0 as $\kappa \to 0$. This completes the proof. □

For $\kappa > 0$, $s < 0$ and $t \geq s$, define

$$Y_\kappa(t;s) = \inf\left\{y \in \mathbb{R}\middle| u(t,x;s) \leq e^{-\lambda_\kappa(x-y)}, \quad x \in \mathbb{R}\right\}. \tag{8.8}$$

From the proof of Lemma 8.2, we see that $Y_\kappa(t;s)$ is well-defined if $\lambda_\kappa \leq c_B^+$, which is the case if κ is sufficiently small due to Lemma 8.6. Notice the definition does not guarantee any continuity or monotonicity of $Y_\kappa(t;s)$ since $u(t,x;s)$ is not monotone in time. The following result controls the propagation of $Y_\kappa(t;s)$.

Lemma 8.7. *Let* $\kappa_0 = \sup_{u\in(0,1)} \frac{f_{\tilde{B}}(u)}{u}$. *For* $\kappa > 0$, *set* $\tilde{c}_\kappa := \frac{1}{\lambda_\kappa}\big(\int_{\mathbb{R}} J(y)e^{\lambda_\kappa y}dy - 1 + \kappa_0\big)$, *where* λ_κ *is given in* (8.7). *Then, for any small* $\kappa > 0$, *we have* $\tilde{c}_\kappa > 0$ *and*

$$Y_\kappa(t;s) - Y_\kappa(t_0;s) \leq \tilde{c}_\kappa(t - t_0)$$

for all $s < 0$, $t \geq t_0 \geq s$.

Proof. For small $\kappa > 0$, we have $\tilde{c}_\kappa \geq c_\kappa > 0$. For $s < 0$, $t \geq t_0 \geq s$, define

$$v(t,x;t_0) = e^{-\lambda_\kappa(x - Y_\kappa(t_0;s) - \tilde{c}_\kappa(t-t_0))}, \quad x \in \mathbb{R}.$$

By the definition of $\tilde{c}_\kappa$, we readily check that $v_t = J * v - v + \kappa_0 v$. By the definition of κ_0, we have $\kappa_0 v \geq f_{\tilde{B}}(v)$ for all $v \geq 0$. It then follows from $v(t_0,x;t_0) = e^{-\lambda_\kappa(x - Y_\kappa(t_0;s))} \geq u(t_0,x;s)$ by (8.8) and the comparison principle that $v(t,x;t_0) \geq u(t,x;s)$ for $t \geq t_0$, which leads to the result. □

We now prove the main result leading to Theorem 8.4. In what follows, we fix some small $\kappa > 0$ such that $c_\kappa < c_B$, and for this fixed κ, we write $Y_\kappa(t;s)$ as $Y(t;s)$, and set

$$\lambda_* := \min\left\{u > 0\middle| f_{\tilde{B}}(u) = \kappa u\right\} \in (\theta, 1). \tag{8.9}$$

We remark that it is important to have $\lambda_* > \theta$, and this is the reason why we need $f_B(\theta) = 0 = f_{\tilde{B}}(\theta)$ in Hypothesis 2.5.

Proposition 8.8. *For any $\lambda \in (\theta, \lambda_*]$, there is $C(\lambda) > 0$ such that*

$$|X_\lambda(t;s) - Y(t;s)| \leq C(\lambda)$$

for all $s < 0$, $t \geq s$.

Proof. From the definition of λ_*, we readily see that

$$f(t,u) \leq f_{\tilde{B}}(u) \leq \kappa u, \quad u \in [0, \lambda_*]. \tag{8.10}$$

Fix an $\lambda \in (\theta, \lambda_*]$. Set $C_0 = \max\{Y(s;s) - X_\lambda(s;s), 1\}$. We see that C_0 is independent of $s < 0$. This is because that $u(s, \cdot; s) = \phi_B(\cdot - y_s)$, and hence, space translations do not change $Y(s;s) - X_\lambda(s;s)$. Clearly, we have the estimate $\sup_{s<0, t\geq s}[X_\lambda(t;s) - Y(t;s)] \leq C$ for some large $C > 0$.

Set $\epsilon = \frac{c_B - c_\kappa}{2}$ and $C_1 = C_0 + c_B t_{\epsilon,\lambda}$, where $t_{\epsilon,\lambda}$ is as in Lemma 8.5. To finish the proof, we only need to show $\sup_{s<0,t\geq s}[Y(t;s) - X_\lambda(t;s)] \leq C_1$. Suppose this is not the case, then we can find some $t_1 \geq s_1$ such $Y(t_1;s_1) - X_\lambda(t_1;s_1) > C_1$. Since $Y(s_1;s_1) - X_\lambda(s_1;s_1) \leq C_0 < C_1$, there holds $t_1 > s_1$. Let

$$t_0 = \sup\big\{t \in [s_1, t_1] \big| Y(t;s_1) - X_\lambda(t;s_1) \leq C_0\big\}.$$

We claim $Y(t_0;s_1) - X_\lambda(t_0;s_1) \leq C_0$. It is trivial if there are only finitely many $t \in [s_1, t_1]$ such that $Y(t;s_1) - X_\lambda(t;s_1) \leq C_0$. So we assume there are infinitely many such t and the claim is false. Then, there exists a sequence $\{\tilde{t}_n\}_{n\in\mathbb{N}} \subset [s_1, t_0)$ such that $Y(\tilde{t}_n;s_1) - X_\lambda(\tilde{t}_n;s_1) \leq C_0$ for $n \in \mathbb{N}$ and $\tilde{t}_n \to t_0$ as $n \to \infty$. Moreover, $Y(t_0;s_1) - X_\lambda(t_0;s_1) = \tilde{C}_1 > C_0$.

It then follows that for all $n \in \mathbb{N}$

$$Y(\tilde{t}_n; s_1) - X_\lambda(\tilde{t}_n; s_1) \le C_0 = C_0 - \tilde{C}_1 + Y(t_0; s_1) - X_\lambda(t_0; s_1),$$

that is,

$$\tilde{C}_1 - C_0 + X_\lambda(t_0; s_1) - X_\lambda(\tilde{t}_n; s_1) \le Y(t_0; s_1) - Y(\tilde{t}_n; s_1) \le \tilde{c}_\kappa(t_0 - \tilde{t}_n),$$

where the second inequality is due to Lemma 8.7. Passing $n \to \infty$, we easily deduce a contradiction from the continuity of $X_\lambda(t; s_1)$. Hence, the claim is true, that is, $Y(t_0; s_1) - X_\lambda(t_0; s_1) \le C_0$. It follows that $t_0 < t_1$.

We show

$$Y(t_0; s_1) - X_\lambda(t_0; s_1) = C_0. \tag{8.11}$$

Suppose (8.11) is not true, then we can find some $\delta_0 > 0$ such that $Y(t_0; s_1) - X_\lambda(t_0; s_1) = C_0 - \delta_0$. Since $Y(t; s_1) - X_\lambda(t; s_1) > C_0$ for $t \in (t_0, t_1]$ by the definition of t_0, we deduce from Lemma 8.7 that for $t \in (t_0, t_1]$

$$\begin{aligned} C_0 < Y(t; s_1) - X_\lambda(t; s_1) &\le Y(t_0; s_1) + \tilde{c}_\kappa(t - t_0) - X_\lambda(t_0; s_1) + X_\lambda(t_0; s_1) - X_\lambda(t; s_1) \\ &= C_0 - \delta_0 + \tilde{c}_\kappa(t - t_0) + X_\lambda(t_0; s_1) - X_\lambda(t; s_1). \end{aligned}$$

This leads to a contradiction when t approaches t_0 due to the continuity of $X_\lambda(t; s_1)$ in t. Hence, (8.11) holds.

Next, we look at the time interval $[t_0, t_1]$ and set $\tilde{Y}(t; s_1) = Y(t_0; s_1) + c_\kappa(t - t_0)$ for $t \in [t_0, t_1]$. Note both $X_\lambda(t; s_1)$ and $\tilde{Y}(t; s_1)$ are continuous, and $X_\lambda(t_0; s_1) < \tilde{Y}(t_0; s_1)$ by (8.11). We claim that $X_\lambda(t; s_1) < \tilde{Y}(t; s_1)$ for all $t \in [t_0, t_1]$. Suppose this is not the case and let

$$t_2 = \min\left\{t \in [t_0, t_1] \middle| X_\lambda(t; s_1) = \tilde{Y}(t; s_1)\right\}.$$

Clearly, $t_2 \in (t_0, t_1]$. We define

$$v(t,x;t_0) = e^{-\lambda_\kappa(x-\tilde{Y}(t;s_1))}, \quad x \in \mathbb{R},\ t \in [t_0, t_2].$$

We easily check $v_t = J * v - v + \kappa v$. Moreover, we see

- at the initial moment t_0, we have $u(t_0,x;s_1) \le e^{-\lambda_\kappa(x-Y(t_0;s_1))} = v(t_0,x;t_0)$ for $x \in \mathbb{R}$,
- for $x \le \tilde{Y}(t;s_1)$ and $t \in (t_0,t_2)$, we have $u(t,x;s_1) < 1 \le v(t,x;t_0)$,
- for $x > \tilde{Y}(t;s_1)$ and $t \in (t_0,t_2)$, we have $x > X_\lambda(t;s_1)$, and hence $u(t,x;s_1) \le \lambda$. As a result, we have $u_t = J * u - u + f(t,u) \le J * u - u + \kappa u$ by (8.10).

Note, by Lemma 8.3 and the definition of $v(t,x;t_0)$, the limit $v(t,x;t_0) - u(t,x;s_1) \to 0$ as $x \to \infty$ is uniformly in $t \in [t_0,t_2]$. Then, applying the comparison principle (see Proposition B.1) to $v(t,x;t_0) - u(t,x;s_1)$, we conclude

$$u(t,x;s_1) \le v(t,x;t_0) = e^{-\lambda_\kappa(x-\tilde{Y}(t;s_1))}, \quad x \in \mathbb{R},\ t \in [t_0,t_2].$$

It follows that $Y(t;s_1) \le \tilde{Y}(t;s_1)$ for $t \in [t_0,t_2]$ by definition in (8.8). In particular, $Y(t_2;s_1) \le \tilde{Y}(t_2;s_1) = X_\lambda(t_2;s_1)$. Since $t_2 \in (t_0,t_1]$, we have $Y(t_2;s_1) - X_\lambda(t_2;s_1) > C_0$ by the definition of t_0. It is a contradiction. Thus, the claim follows, that is, $X_\lambda(t;s_1) < \tilde{Y}(t;s_1)$ for all $t \in [t_0,t_1]$, and repeating the above arguments, we see

$$Y(t;s_1) \le \tilde{Y}(t;s_1) = Y(t_0;s_1) + c_\kappa(t-t_0), \quad t \in [t_0,t_1]. \tag{8.12}$$

It follows from (8.12) and Lemma 8.5 that for any $t \in [t_0,t_1]$

$$\begin{aligned} Y(t;s_1) - X_\lambda(t;s_1) &\le Y(t_0;s_1) + c_\kappa(t-t_0) - [X_\lambda(t_0;s_1) + (c_B - \epsilon)(t - t_0 - t_{\epsilon,\lambda})] \\ &= C_0 + (c_B-\epsilon)t_{\epsilon,\lambda} - (c_B - c_\kappa - \epsilon)(t-t_0) \\ &\le C_0 + c_B t_{\epsilon,\lambda} = C_1. \end{aligned}$$

This is a contradiction. Consequently, $Y(t;s) - X_\lambda(t;s) \le C_1$ for all $s < 0$, $t \ge s$. This completes the proof. □

We remark that the proof of Proposition 8.8 is based on the rightward propagation estimate as in Lemma 8.5 and an idea of Zlatoš (see [85, Lemma 2.5]).

Finally, we prove Theorem 8.4.

Theorem 8.4. Let λ_* be as in (8.9) and fix any $0 < \lambda_1 < \lambda_2 \le \lambda_*$. Consider the function

$$g(t,x;s) = e^{-\lambda_\kappa (x - Y(t;s))}, \quad x \in \mathbb{R}.$$

Since $g(t, Y(t;s); s) = 1$, there exists a unique $x_1 > 0$ (independent of $s < 0$, $t \ge s$) such that $g(t, Y(t;s) + x_1; s) = \lambda_1$. Since $g(t,x;s) \ge e^{-\lambda_\kappa(x - Y(t;s))} \ge u(t,x;s)$ for $x \ge Y(t;s)$, we have $Y(t;s) + x_1 \ge X_{\lambda_1}(t;s)$. It then follows that $X_{\lambda_1}(t;s) - X_{\lambda_2}(t;s) \le Y(t;s) - X_{\lambda_2}(t;s) + x_1$. The result then follows from Proposition 8.8. □

8.3 Modified interface locations and exponential decaying estimates

In the study of the propagation of the solution $u(t,x;s)$, the propagation of the interface location $X_\lambda(t;s)$, more precisely, how fast it moves, plays a crucial role. In the classical random dispersal case, this problem is transferred into the study of uniform steepness, that is, whether $u_x(t, X_\lambda(t;s); s)$ is uniformly negative, since there holds the formula

$$\dot{X}_\lambda(t;s) = -\frac{u_t(t, X_\lambda(t;s); s)}{u_x(t, X_\lambda(t;s); s)}.$$

Clearly, this approach does not work here since we are lack of space regularity of $u(t,x;s)$. Moreover, we do not know if $X_\lambda(t;s)$ is differentiable in t and it moves back and forth in general. To circumvent these difficulties, we look at the problem from a different viewpoint. Instead of studying $X_\lambda(t;s)$ directly, we modify it to get a new interface location of expected properties, such as moving in one direction with certain speed and staying within a certain distance from $X_\lambda(t;s)$, which captures the propagation nature of $u(t,x;s)$. This is the main purpose of this subsection. As an application of the new interface location, we derive uniform exponential decaying estimates of $u(t,x;s)$.

We first modify $X_\lambda(t;s)$ properly by proving the following

Theorem 8.9. *Let λ_* be as in (8.9). There are $c_{\min}>0$, $c_{\max}>0$, $\tilde{c}_{\max}>0$ and $d_{\max}>0$ such that for any $s<0$, there exists a continuously differentiable function $X(t;s):[s,\infty)\to\mathbb{R}$ satisfying*

$$c_{\min}\le \dot{X}(t;s)\le c_{\max},\quad t\ge s,$$
$$|\ddot{X}(t;s)|\le \tilde{c}_{\max},\quad t\ge s$$

such that

$$0\le X(t;s)-X_{\lambda_*}(t;s)\le d_{\max},\quad t\ge s.$$

Moreover, $\{\dot{X}(\cdot,s)\}_{s<0}$ and $\{\ddot{X}(\cdot,s)\}_{s<0}$ are uniformly bounded and uniformly Lipschitz continuous.

In particular, for any $\lambda\in(0,\lambda_*]$, there exists $d_{\max}(\lambda)>0$ such that

$$|X(t;s)-X_\lambda(t;s)|\le d_{\max}(\lambda)\quad \text{if } \lambda\in(0,\lambda_*]$$

for all $s<0$, $t\ge s$.

Proof. By Lemma 8.5, there exists $t_B>0$ such that

$$X_{\lambda_*}(t;s)-X_{\lambda_*}(t_0;s)\ge \frac{3}{4}c_B(t-t_0-t_B),\quad s<0,\ t\ge t_0\ge s. \tag{8.13}$$

Recall $Y(t;s)$ is $Y_\kappa(t;s)$ for some fixed small $\kappa>0$ and we have

$$C_0:=\sup_{s<0,t\ge s}|X_{\lambda_*}(t;s)-Y(t;s)|<\infty \tag{8.14}$$

by Proposition 8.8, and

$$Y(t;s) - Y(t_0;s) \le c_0(t-t_0), \quad s<0,\ t\ge t_0 \ge s \tag{8.15}$$

by Lemma 8.7, where $c_0 = \tilde{c}_\kappa$ for the fixed $\kappa > 0$. We interpret that (8.13), (8.14) and (8.15) imply that $X_{\lambda_*}(t;s)$ moves with a uniformly positive and uniformly bounded average speed. This observation is crucial in the following modification.

We modify $X_{\lambda_*}(t;s)$ as follows. At the initial moment s, we define

$$Z(t;s) = X_{\lambda_*}(s;s) + 2C_0 + 1 + \frac{c_B}{2}(t-s), \quad t \ge s.$$

Clearly, $X_{\lambda_*}(s;s) < Z(s;s)$. By (8.13), $X_{\lambda_*}(t;s)$ will hit $Z(t;s)$ sometime after s. Let $T_1(s)$ be the first time that $X_{\lambda_*}(t;s)$ hits $Z(t;s)$, that is, $T_1(s) = \min\big\{t \ge s \big| X_{\lambda_*}(t;s) = Z(t;s)\big\}$. It follows that

$$X_{\lambda_*}(t;s) < Z(t;s) \text{ for } t \in [s, T_1(s)) \quad \text{and} \quad X_{\lambda_*}(T_1(s);s) = Z(T_1(s);s).$$

Moreover, $T_1(s) - s \in \big[\frac{2}{2c_0 - c_B}, \frac{4(2C_0+1)}{c_B} + 3t_B\big]$, which is a simple result of (8.13), (8.14) and (8.15).

Now, at the moment $T_1(s)$, we define

$$Z(t;T_1(s)) = X_{\lambda_*}(T_1(s);s) + 2C_0 + 1 + \frac{c_B}{2}(t - T_1(s)), \quad t \ge T_1(s).$$

Similarly, $X_{\lambda_*}(T_1(s);s) < Z(T_1(s);T_1(s))$ and $X_{\lambda_*}(t;s)$ will hit $Z(t;T_1(s))$ sometime after $T_1(s)$. Denote by $T_2(s)$ the first time that $X_{\lambda_*}(t;s)$ hits $Z(t;T_1(s))$. Then,

$$X_{\lambda_*}(t;s) < Z(t;T_1(s)) \text{ for } t \in [T_1(s), T_2(s)) \quad \text{and} \quad X_{\lambda_*}(T_2(s);s) = Z(T_2(s);T_1(s)),$$

and $T_2(s) - T_1(s) \in \big[\frac{2}{2c_0 - c_B}, \frac{4(2C_0+1)}{c_B} + 3t_B\big]$ by (8.13), (8.14) and (8.15).

Repeating the above arguments, we obtain the following: there is a sequence of times $\{T_{n-1}(s)\}_{n\in\mathbb{N}}$ satisfying $T_0(s)=s$ and

$$T_n(s)-T_{n-1}(s)\in\left[\frac{2}{2c_0-c_B},\frac{4(2C_0+1)}{c_B}+3t_B\right]\quad\text{for all } n\in\mathbb{N},$$

and for any $n\in\mathbb{N}$

$$X_{\lambda_*}(t;s)<Z(t;T_{n-1}(s))\text{ for } t\in[T_{n-1}(s),T_n(s))\quad\text{and}\quad X_{\lambda_*}(T_n(s);s)=Z(T_n(s);T_{n-1}(s)),$$

where

$$Z(t;T_{n-1}(s))=X_{\lambda_*}(T_{n-1}(s);s)+2C_0+1+\frac{c_B}{2}(t-T_{n-1}(s)).$$

Moreover, for any $n\in\mathbb{N}$ and $t\in[T_{n-1}(s),T_n(s))$

$$\begin{aligned}
&Z(t;T_{n-1}(s))-X_{\lambda_*}(t;s)\\
&\quad\le X_{\lambda_*}(T_{n-1}(s);s)+2C_0+1+\frac{c_B}{2}(t-T_{n-1}(t_0))\\
&\qquad-\left[X_{\lambda_*}(T_{n-1}(s);s)+\frac{3}{4}c_B(t-T_{n-1}(s)-t_B)\right]\\
&\quad=2C_0+1+\frac{3}{4}c_Bt_B-\frac{1}{4}c_B(t-T_{n-1}(s))\le 2C_0+1+\frac{3}{4}c_Bt_B.
\end{aligned}$$

Now, define $\tilde{Z}(t;s):[s,\infty)\to\mathbb{R}$ by setting

$$\tilde{Z}(t;s)=Z(t;T_{n-1}(s))\quad\text{for } t\in[T_{n-1}(s),T_n(s)),\ n\in\mathbb{N}. \tag{8.16}$$

Since $[s,\infty)=\cup_{n\in\mathbb{N}}[T_{n-1}(s),T_n(s))$, $\tilde{Z}(t;s)$ is well-defined for all $t\ge s$. Notice $\tilde{Z}(t;s)$ is strictly increasing and is linear on $[T_{n-1}(s),T_n(s))$ with slope $\frac{c_B}{2}$ for each $n\in\mathbb{N}$, and satisfies

$$0\le\tilde{Z}(t;s)-X_{\lambda_*}(t;s)\le 2C_0+1+\frac{3}{4}c_Bt_B,\quad t\ge s.$$

Finally, we can modify $\tilde{Z}(t;s)$ near each $T_n(s)$ for $n\in\mathbb{N}$ to get the expected modification. In fact, fix some $\delta_*\in(0,\frac{1}{2}\frac{2}{2c_0-c_B})$. We modify $\tilde{Z}(t;s)$ by redefining it on the intervals $(T_n(s)-\delta_*,T_n(s))$, $n\in\mathbb{N}$ as follows: define

$$X(t;s)=\begin{cases}\tilde{Z}(t;s), & t\in[s,\infty)\backslash\cup_{n\in\mathbb{N}}(T_n(s)-\delta_*,T_n(s)),\\ X_{\lambda_*}(T_n(s))+\delta(t-T_n(s)), & t\in(T_n(s)-\delta_*,T_n(s)),\ n\in\mathbb{N},\end{cases}$$

where $\delta:[-\delta_*,0]\to[-\frac{c_B}{2}\delta_*,1]$ is twice continuously differentiable and satisfies

$$\begin{aligned}&\delta(-\delta_*)=-\frac{c_B}{2}\delta_*,\quad \delta(0)=1,\\ &\dot{\delta}(-\delta_*)=\frac{c_B}{2}=\dot{\delta}(0),\quad \dot{\delta}(t)\geq\frac{c_B}{2}\text{ for }t\in(-\delta_*,0)\quad\text{and}\\ &\ddot{\delta}(-\delta_*)=0=\ddot{\delta}(0).\end{aligned}$$

The existence of such a function $\delta(t)$ is clear. Moreover, there exist $c_{\max}=c_{\max}(\delta_*)>0$ and $\tilde{c}_{\max}=\tilde{c}_{\max}(\delta_*)>0$ such that $\dot{\delta}(t)\leq c_{\max}$ and $|\ddot{\delta}(t)|\leq\tilde{c}_{\max}$ for $t\in(-\delta_*,0)$. Notice the above modification is independent of $n\in\mathbb{N}$ and $s<0$. As a result, we readily check that $X(t;s)$ satisfies all expected properties. This completes the proof. □

We now apply Theorem 8.9 to study uniform exponential decaying estimates of $u(t,x;s)$ behind and ahead of interfaces. Let λ_* be as in (8.9) and $X(t;s)$ be as in Theorem 8.9. Since $f_B'(1)<0$, we see that there exist $\theta_*\in(\theta,\lambda_*]$ and $\beta>0$ such that

$$f_B(u)\geq\beta(1-u),\quad u\in[\theta_*,1].\tag{8.17}$$

Set

$$\hat{X}(t;s)=X(t;s)-d_{\max}-\hat{C},\tag{8.18}$$

where $\hat{C}>0$ is some constant (to be chosen) introduced only for certain flexibility. Theorem 8.4 and Theorem 8.9 then imply that $\hat{X}(t;s)\leq X_{\theta_*}(t;s)$, and hence, $u(t,x+\hat{X}(t;s);s)\geq\theta_*$

for all $x \le 0$. We also set

$$\tilde{X}(t;s) = X(t;s) + \sup_{s<0, t\ge s} \big| X_\theta(t;s) - X_{\lambda_*}(t;s) \big|. \tag{8.19}$$

Since $X(t;s) \ge X_{\lambda_*}(t;s)$ by Theorem 8.9, we have $\tilde{X}(t;s) \ge X_\theta(t;s)$, and hence, $u(t, x + \tilde{X}(t;s); s) \le \theta$ for $x \ge 0$.

We now prove the main result in this subsection.

Theorem 8.10. *There exist $c_\pm > 0$ such that*

$$u(t,x;s) \ge 1 - e^{c_-(x-\hat{X}(t;s))}, \quad x \le \hat{X}(t;s),$$
$$u(t,x;s) \le e^{-c_+(x-\tilde{X}(t;s))}, \quad x \ge \tilde{X}(t;s)$$

for all $s < 0$, $t \ge s$.

Proof. Define

$$N_-[v] = v_t - [J * v - v] - \beta(1-v).$$

For $x \le \hat{X}(t;s)$, we have $u(t,x;s) \ge \theta_*$, which together with (8.17) implies that

$$f(t, u(t,x;s)) \ge f_B(u(t,x;s)) \ge \beta(1 - u(t,x;s)), \quad x \le \hat{X}(t;s)$$

It then follows that for $x \le \hat{X}(t;s)$

$$N_-[u] = u_t - [J * u - u] - \beta(1-u) = f(t,u) - \beta(1-u) \ge 0.$$

For $c > 0$, we compute

$$\begin{aligned} N_-[1 - e^{c(x-\hat{X}(t;s))}] &= \left[c\dot{\hat{X}}(t;s) + \int_{\mathbb{R}} J(y) e^{-cy} dy - 1 - \beta \right] e^{c(x-\hat{X}(t;s))} \\ &\le \left[c c_{\max} + \int_{\mathbb{R}} J(y) e^{-cy} dy - 1 - \beta \right] e^{c(x-\hat{X}(t;s))}, \end{aligned}$$

where we used the definition of $\hat{X}(t;s)$ and Theorem 8.9. Since $\int_{\mathbb{R}} J(y)e^{-cy}dy \to 1$ as $c \to 0$, we can choose $c > 0$ so small that $cc_{\max} + \int_{\mathbb{R}} J(y)e^{-cy}dy - 1 - \beta \leq 0$, and then, $N_-[1 - e^{c(x-\hat{X}(t;s))}] \leq 0$. Hence, we have shown

$$N_-[u] \geq 0 \geq N_-[1 - e^{c_-(x-\hat{X}(t;s))}], \quad x \leq \hat{X}(t;s).$$

for some small $c_- > 0$. Trivially, we have $u(t,x;s) > 0 \geq 1 - e^{c_-(x-\hat{X}(t;s))}$ for $x \geq \hat{X}(t;s)$. At the initial moment s, we obtain from Lemma A.1 that $u(s,x;s) = \phi_B(x-y_s) \geq 1-e^{c_-(x-\hat{X}(s;s))}$ if we choose c_- smaller and $\hat{C}$ sufficiently large. Then, we conclude from the comparison principle (see (ii) in Proposition B.1) that $u(t,x;s) \geq 1 - e^{c_-(x-\hat{X}(t;s))}$ for $x \leq \hat{X}(t;s)$. This proves half of the theorem.

We now prove the other half. To do so, we define

$$N_+[v] = v_t - [J * v - v].$$

Since $\tilde{X}(t;s) \geq X_\theta(t;s)$ by construction, we have $u(t,x;s) \leq \theta$ for $x \geq \tilde{X}(t;s)$, and hence, $f(t,u(t,x;s)) \leq 0$ for $x \geq \tilde{X}(t;s)$. From which, we deduce

$$N_+[u] = u_t - [J * u - u] = f(t,u) \leq 0, \quad x \geq \tilde{X}(t;s).$$

Let $c > 0$. We compute

$$\begin{aligned} N_+[e^{-c(x-\tilde{X}(t;s))}] &= \left[c\dot{\tilde{X}}(t;s) - \int_{\mathbb{R}} J(y)e^{cy}dy + 1\right]e^{-c(x-\tilde{X}(t;s))} \\ &\geq \left[cc_{\min} - \int_{\mathbb{R}} J(y)e^{cy}dy + 1\right]e^{-c(x-\tilde{X}(t;s))}, \end{aligned}$$

where we used Theorem 8.9. Set $g(c) = cc_{\min} - \int_{\mathbb{R}} J(y)e^{cy}dy + 1$. Clearly, $g(0) = 0$ and $g'(c) = c_{\min} - \int_{\mathbb{R}} J(y)ye^{cy}dy$. Due to the symmetry of J, $\int_{\mathbb{R}} J(y)ye^{cy}dy \to 0$ as $c \to 0$. As a result, $g'(c) > 0$ for all small $c > 0$. Hence, we can find some $c_+ > 0$ such that $g(c^+) > 0$, and therefore, $N_+[e^{-c_+(x-\tilde{X}(t;s))}] \geq g(c_+)e^{-c_+(x-\tilde{X}(t;s))} \geq 0$. Hence, we have shown

$$N_+[u] \leq 0 \leq N_+[e^{-c_+(x-\tilde{X}(t;s))}], \quad x \geq \tilde{X}(t;s).$$

for some small $c_+ > 0$. Since we have trivially $\tilde{u} < 1 \leq e^{-c_+(x-\tilde{X}(t;s))}$ for $x \leq \tilde{X}(t;s)$ and, by Lemma A.1 and $\tilde{X}(s;s) \geq X_\theta(s;s) = y_s$, $u(s,x;s) = \phi_B(x-y_s) \leq e^{-c_+(x-\tilde{X}(t;s))}$ if we choose c_+ smaller, we conclude from the comparison principle (see (i) in Proposition B.1) that $u(t,x;s) \leq e^{-c_+(x-\tilde{X}(t;s))}$ for $x \geq \tilde{X}(t;s)$. This completes the proof. □

As a simple consequence of Theorem 8.10, we have

Corollary 8.11. *For any* $0 < \lambda_1 < \lambda_2 < 1$, *there holds*

$$\sup_{s<0,t\geq s} \left[X_{\lambda_1}(t;s) - X_{\lambda_2}(t;s)\right] < \infty.$$

In particular, for any $\lambda \in (0,1)$, *there holds*

$$\sup_{s<0,t\geq s} \left|X_\lambda(t;s) - X(t;s)\right| < \infty.$$

Proof. By Theorem 8.10, we have

$$\max\left\{0, 1-e^{c_-(x-\hat{X}(t;s))}\right\} \leq u(t,x;s) \leq \min\left\{1, e^{-c_+(x-\tilde{X}(t;s))}\right\}.$$

The result then follows from the fact that $\tilde{X}(t;s) - \hat{X}(t;s) \equiv \text{const}$. □

8.4 Construction of transition fronts

In this section, we prove Theorem 8.1. To do so, we prove uniform Lipschitz continuity of $u(t,x;s)$ in the space variable x.

Lemma 8.12. *There holds*

$$\sup_{\substack{x\neq y\\ s<0,t\geq s}} \left|\frac{u(t,y;s)-u(t,x;s)}{y-x}\right| < \infty.$$

Proof. Since $u(t,x;s)\in(0,1)$, there holds trivially

$$\forall\delta>0,\qquad \sup_{\substack{|y-x|\geq\delta\\ s<0,t\geq s}} \left|\frac{u(t,y;s)-u(t,x;s)}{y-x}\right| < \infty.$$

Thus, to finish the proof of the lemma, it suffices to show the local uniform Lipschitz continuity, that is,

$$\forall\delta>0,\qquad \sup_{\substack{0<|y-x|\leq\delta\\ s<0,t\geq s}} \left|\frac{u(t,y;s)-u(t,x;s)}{y-x}\right| < \infty. \tag{8.20}$$

To this end, we fix $\delta>0$. Let $X(t;s)$ be as in Theorem 8.9 and define

$$L_1=\delta+\sup_{s<0,t\geq s}\left|X_{\theta_0}(t;s)-X(t;s)\right| \quad\text{and}\quad L_2=\delta+\sup_{s<0,t\geq s}\left|X_{\theta_1}(t;s)-X(t;s)\right|,$$

where θ_0 and θ_1 are as in Hypothesis 2.3. Notice $L_1<\infty$ and $L_2<\infty$ by Corollary 8.11. Then, for any $0<|y-x|\leq\delta$ we have

- if $x\geq X(t;s)+L_1$, then $y\geq x-\delta\geq X_\theta(t;s)$, which implies that $u(t,y;s)\leq\theta_0$, $u(t,x;s)\leq\theta_0$, and hence by Hypothesis 2.3

$$\frac{f(t,u(t,y;s))-f(t,u(t,x;s))}{u(t,y;s)-u(t,x;s)}\leq 0; \tag{8.21}$$

- if $x\leq X(t;s)-L_2$, then $y\leq x+\delta\leq X_{\tilde\theta}(t;s)$, which implies that $u(t,y;s)\geq\tilde\theta$ and $u(t,x;s)\geq\tilde\theta$, and hence by Hypothesis 2.3

$$\frac{f(t,u(t,y;s))-f(t,u(t,x;s))}{u(t,y;s)-u(t,x;s)}\leq 0. \tag{8.22}$$

According to (8.21) and (8.22), we consider time-dependent disjoint decompositions of $\mathbb{R}$ into

$$\mathbb{R} = R_l(t;s) \cup R_m(t;s) \cup R_r(t;s),$$

where

$$\begin{aligned} R_l(t;s) &= (-\infty, X(t;s) - L_2), \\ R_m(t;s) &= [X(t;s) - L_2, X(t;s) + L_1] \quad \text{and} \\ R_r(t;s) &= (X(t;s) + L_1, \infty). \end{aligned}$$

Since $X(t;s)$ is continuous in t, this three regions change continuously in t. As $X(t;s)$ moves to the right by Theorem 8.9, any fixed point will eventually enter into $R_l(t;s)$ and stay there forever.

For $s < 0$ and $x_0 \in \mathbb{R}$, let $t_{\text{first}}(x_0;s)$ be the first time that x_0 is in $R_m(t;s)$, that is,

$$t_{\text{first}}(x_0;s) = \min\left\{t \geq s \middle| x_0 \in R_m(t;s)\right\},$$

and $t_{\text{last}}(x_0;s)$ be the last time that x_0 is in $R_m(t;s)$, that is,

$$t_{\text{last}}(x_0;s) = \max\left\{t_0 \in \mathbb{R} \middle| x_0 \in R_m(t_0;s) \text{ and } x_0 \notin R_m(t,s) \text{ for } t > t_0\right\}.$$

Since $X(t;s)$ moves to the right, if $x_0 \in R_l(s;s)$, then $x_0 \in R_l(t;s)$ for all $t > s$. In this case, $t_{\text{first}}(x_0;s)$ and $t_{\text{last}}(x_0;s)$ are not well-defined, but it will not cause any trouble. Clearly, $x_0 \in R_l(t;s)$ for all $t > t_{\text{last}}(x_0;s)$.

We see that either both $t_{\text{first}}(x_0;s)$ and $t_{\text{last}}(x_0;s)$ are well-defined, or both of them are not well-defined. In fact, $t_{\text{first}}(x_0;s)$ and $t_{\text{last}}(x_0;s)$ are well-defined only if $x_0 \notin R_l(s;s)$. As a simple consequence of Lemma 8.5 and the fact that the length of $R_m(t;s)$ is $L_1 + L_2$, we have

$$T = T(\delta) := \sup_{s<0, x_0 \notin R_l(s;s)} \left[t_{\text{last}}(x_0;s) - t_{\text{first}}(x_0;s)\right] < \infty. \tag{8.23}$$

Now, we are ready to prove the lemma. Fix $x_0 \in \mathbb{R}$, $s < 0$ and $0 < |\eta| \le \delta$. Set

$$v^y(t,x;s) = \frac{u(t,x+\eta;s) - u(t,x;s)}{\eta}.$$

Clearly, $v^\eta(t,x_0;s)$ satisfies

$$v_t^\eta(t,x_0;s) = \int_{\mathbb{R}} J(x_0-y)v^\eta(t,y;s)dy - v^\eta(t,x_0;s) + a^\eta(t,x_0;s)v^\eta(t,x_0;s),$$

where

$$a^\eta(t,x_0;s) = \frac{f(t,u(t,x_0+\eta;s)) - f(t,u(t,x_0;s))}{u(t,x_0+\eta;s) - u(t,x_0;s)}$$

is uniformly bounded. Notice $\int_{\mathbb{R}} J(x_0-y)v^\eta(t,y;s)dy$ is bounded uniformly in x_0 and η. In fact, the change of variable gives

$$\int_{\mathbb{R}} J(x_0-y)v^\eta(t,y;s)dy = \int_{\mathbb{R}} \frac{J(x_0-y+\eta) - J(x_0-y)}{\eta} u(t,y;s)dy.$$

The uniform boundedness then follows from the fact $u(t,x;s) \in (0,1)$ and the assumption $J' \in L^1(\mathbb{R})$ by Hypothesis 2.1.

Setting $M = M(\delta) := \sup_{x_0\in\mathbb{R},0<|\eta|\le\delta} |\int_{\mathbb{R}} J(x_0-y)v^\eta(t,y;s)dy|$, we see that $v^\eta(t,x_0;s)$ satisfies

$$\begin{aligned} &-M - v^\eta(t,x_0;s) + a^\eta(t,x_0;s)v^\eta(t,x_0;s) \\ &\quad \le v_t^\eta(t,x_0;s) \le M - v^\eta(t,x_0;s) + a^\eta(t,x_0;s)v^\eta(t,x_0;s), \end{aligned} \tag{8.24}$$

which essentially are ordinary differential inequalities. The solution of (8.24) satisfies

$$\begin{aligned} &v^\eta(t_0,x_0;s)e^{-\int_{t_0}^t(1-a^\eta(\tau,x_0;s))d\tau} - M\int_{t_0}^t e^{-\int_r^t(1-a^\eta(\tau,x_0;s))d\tau}dr \\ &\quad \le v^\eta(t,x_0;s) \le v^\eta(t_0,x_0;s)e^{-\int_{t_0}^t(1-a^\eta(\tau,x_0;s))d\tau} + M\int_{t_0}^t e^{-\int_r^t(1-a^\eta(\tau,x_0;s))d\tau}dr \end{aligned} \tag{8.25}$$

for $s \leq t_0 \leq t$. Notice $1 - a^\eta(t, x_0; s)$ controls the behavior of $v^\eta(t, x_0; s)$. We see, initially,

$$v^\eta(s, x_0; s) = \frac{u(s, x_0 + \eta; s) - u(s, x_0; s)}{\eta} = \frac{\phi_B(x_0 + \eta - y_s) - \phi_B(x_0 - y_s)}{\eta},$$

which is uniformly bounded (uniform in $s < 0$, $x_0 \in \mathbb{R}$ and $0 < |\eta| \leq \delta$, and even in $\delta > 0$) since $\phi_B \in C^1(\mathbb{R})$. Now,

(i) if $x_0 \in R_l(s; s)$, then $x_0 \in R_l(t; s)$ for all $t \geq s$, which implies that $a^\eta(t, x_0; s) \leq 0$ for all $t \geq s$ by (8.22). We then conclude from (8.25) that $\sup_{t \geq s} |v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})})$;

(ii) if $x_0 \in R_m(s; s)$, then $t_{\text{first}}(x_0; s) = s$. For $t \in [t_{\text{first}}(x_0; s), t_{\text{last}}(x_0; s)]$, we conclude from (8.25) and the fact that $a^\eta(t, x_0; s)$ is uniformly bounded that $v^\eta(t, x_0; s)$ at most grows exponentially with growth rate not larger than some universal constant, and this expnential growth can only last for a period not longer than T, which is given in (8.23). As a result, $|v^\eta(t_{\text{last}}(x_0; s), x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})}, T)$. As mentioned before, for $t > t_{\text{last}}(x_0; s)$, we have $x_0 \in R_l(t; s)$, which together with (8.22) implies that $a^\eta(t, x_0; s) \leq 0$. Then, as in (i), we conclude from (8.25) that $|v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})}, T)$ for $t > t_{\text{last}}(x_0; s)$. Hence, $\sup_{t \geq s} |v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})}, T)$;

(iii) if $x_0 \in R_r(s; s)$, then $t_{\text{first}}(x_0; s) > s$. For $t \in [s, t_{\text{first}}(x_0; s))$, we have $x_0 \in R_r(t; s)$, which implies $a^\eta(t, x_0; s) \leq 0$ by (8.21). Notice the interval $[s, t_{\text{first}}(x_0; s))$ may not have uniformly bounded length, but (8.25) says that as long as $t \in [s, t_{\text{first}}(x_0; s))$, we have $|v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})})$, which implies, $v^\eta(t_{\text{first}}(x_0; s), x_0; s) \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})})$. Then, we can follow the arguments in (ii) to conclude that $\sup_{t \geq s} |v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})}, T)$.

Consequently, $\sup_{t \geq s} |v^\eta(t, x_0; s)| \leq C(M, \|\phi_B\|_{C^1(\mathbb{R})}, T)$. Since $x_0 \in \mathbb{R}$ and $s < 0$ are arbitrary, and T and M depend only on δ, we find (8.20), and hence, finish the proof of the lemma. □

Now, we prove Theorem 8.1.

Proof of Theorem 8.1. Since $u = u(t,x;s)$ satisfies $u_t = J * u - u + f(t,u)$, we conclude from (H2) and the fact that $u(t,x;s) \in (0,1)$ that

$$\sup_{s<0,t>s} |u_t(t,x;s)| < \infty. \tag{8.26}$$

Then, since $v(t,x;s) := u_t(t,x;s)$ satisfies

$$v_t = J * v - v + f_t(t,u(t,x;s)) + f_u(t,u(t,x;s))v,$$

we conclude from (8.26) and Hypothesis 2.5 that

$$\sup_{s<0,t>s} |u_{tt}(t,x;s)| = \sup_{s<0,t>s} |v_t(t,x;s)| < \infty. \tag{8.27}$$

Now, by Lemma 8.12, (8.26), (8.27), Arzelà-Ascoli theorem and the diagonal argument, we are able to find some continuous function $u(t,x) : \mathbb{R}\times\mathbb{R} \to [0,1]$ that is differentiable in t and nonincreasing in x such that $u(t,x;s) \to u(t,x)$ and $u_t(t,x;s) \to u_t(t,x)$ locally uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}$ as $s \to -\infty$ along some subsequence. In particular, $u(t,x)$ is an entire classical solution of (2.3). Moreover, as an entire solution, $u(t,x) \in (0,1)$ and it is strictly decreasing in x. The uniform Lipschitz continuity in space of $u(t,x)$ also follows.

We now show the decaying properties of $u(t,x)$. Recall $\hat{X}(t;s)$ and $\tilde{X}(t;s)$ are given in (8.18) and (8.19), respectively. By Theorem 8.9, $\{X(t;s)\}_{s<0}$, $\{\hat{X}(t;s)\}_{s<0}$ and $\{\tilde{X}(t;s)\}_{s<0}$ converge locally uniformly to a continuously differentiable functions $X(t)$, $\hat{X}(t)$ and $\tilde{X}(t)$, respectively. Clearly, $\frac{c_B}{2} \leq \dot{X}(t) \leq c_{\max}$, $X(t) - \hat{X}(t) = h_-$ and $\tilde{X} - X(t) = h_+$ for all $t \in \mathbb{R}$, where $h_\pm > 0$ are constants. In particular,

$$u(t,x+\hat{X}(t;s);s) \to u(t,x+\hat{X}(t)) \quad \text{and} \quad u(t,x+\tilde{X}(t;s);s) \to u(t,x+\tilde{X}(t))$$

locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}$ as $s\to-\infty$ along some subsequence, which together with Theorem 8.10 implies that

$$u(t,x+\hat{X}(t))\geq 1-e^{c-x}\quad\text{and}\quad u(t,x+\tilde{X}(t))\leq e^{-c+x}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}$. This completes the proof. □

8.5 Conditional space regularity

Let $u(t,x)$ be the transition front constructed in Theorem 8.1. Then, it is continuously differentiable in x according to Theorem 2.2. In this subsection, we investigate the space regularity of $u(t,x)$ from a different viewpoint: we improve the space regularity of approximating solutions $u(t,x;s)$ to ensure the space regularity of $u(t,x)$. To do so, besides assumptions Hypothesis 2.1, Hypothesis 2.5 and Hypothesis 2.3, we assume in addition

Hypothesis 8.1. *$f(t,u)$ is twice continuously differentiable in u and satisfies*

$$\sup_{(t,u)\in\mathbb{R}\times[0,1]}|f_{uu}(t,u)|<\infty.$$

Then, we prove

Theorem 8.13. *Let $u(t,x)$ be the transition front in Theorem 8.1. Then, for any $t\in\mathbb{R}$, $u(t,x)$ is continuously differentiable in x. Moreover, $u_x(t,x)$ is uniformly bounded and uniformly Lipschitz continuous in x, that is,*

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}|u_x(t,x)|<\infty\quad\textit{and}\quad\sup_{\substack{x\neq y\\ t\in\mathbb{R}}}\left|\frac{u_x(t,x)-u_x(t,y)}{x-y}\right|<\infty,\tag{8.28}$$

respectively.

To prove Theorem 8.13, we first investigate the space regularity of $u(t,x;s)$. We have

Lemma 8.14. *For any $s < 0$ and $t \geq s$, $u(t,x;s)$ is continuously differentiable in x. Moreover,*

(i) *$u_x(t,x;s)$ is uniformly bounded, that is,*

$$\sup_{\substack{x\neq y\\ s<0,t\geq s}} |u_x(t,x;s)| < \infty;$$

(ii) *$u_x(t,x;s)$ is uniformly Lipschitz continuous in space, that is,*

$$\sup_{\substack{x\neq y\\ s<0,t\geq s}} \left|\frac{u_x(t,x;s)-u_x(t,y;s)}{x-y}\right| < \infty.$$

Assuming Lemma 8.14, let us prove Theorem 8.13.

Proof of Theorem 8.13. It follows from Lemma 8.14, Arzelà-Ascoli theorem and the diagonal argument. More precisely, besides $u(t,x;s) \to u(t,x)$ and $u_t(t,x;s) \to u_t(t,x)$ locally uniformly, we also have

$$u_x(t,x;s) \to u_x(t,x) \quad \text{locally uniformly in } (t,x) \in \mathbb{R}\times\mathbb{R} \tag{8.29}$$

as $s \to -\infty$ along some subsequence. The properties of $u(t,x)$ then inherit from that of $u(t,x;s)$. □

In the rest of this subsection, we prove Lemma 8.14.

Proof of Lemma 8.14. (i) Setting

$$v^\eta(t,x;s) := \frac{u(t,x+\eta;s)-u(t,x;s)}{\eta}.$$

By Lemma 8.12, $\sup_{\substack{x\in\mathbb{R},\eta\neq 0\\ s<0,t\geq s}} |v^\eta(t,x;s)| < \infty$. Clearly, $v^\eta(t,x;s)$ satisfies

$$v_t^\eta(t,x;s) = \int_{\mathbb{R}} J(x-y)v^\eta(t,y;s)dy - v^\eta(t,x;s) + a^\eta(t,x;s)v^\eta(t,x;s), \tag{8.30}$$

where

$$a^{\eta}(t,x;s) = \frac{f(t,u(t,x+\eta;s)) - f(t,u(t,x;s))}{u(t,x+\eta;s) - u(t,x;s)}$$

is uniformly bounded by Hypothesis 2.5. Setting

$$b^{\eta}(t,x;s) := \int_{\mathbb{R}} J(x-y)v^{\eta}(t,y;s)dy = \int_{\mathbb{R}} \frac{J(x-y+\eta) - J(x-y)}{\eta} u(t,y;s)dy,$$

we see that $\sup_{\substack{x\in\mathbb{R},\eta\neq 0\\ s<0,t\geq s}} |b^{\eta}(t,x;s)| < \infty$, since $J' \in L^1(\mathbb{R})$ and $u(t,x;s) \in (0,1)$.

The solution of (8.30) is given by

$$v^{\eta}(t,x;s) = v^{\eta}(s,x;s)e^{-\int_s^t (1-a^{\eta}(\tau,x;s))d\tau} + \int_s^t b^{\eta}(r,x;s)e^{-\int_r^t (1-a^{\eta}(\tau,x;s))d\tau} dr. \tag{8.31}$$

Notice as $\eta \to 0$, the following pointwise limits hold:

$$\begin{aligned} v^{\eta}(s,x;s) &= \frac{\phi_B(x+\eta-y_s) - \phi_B(x-y_s)}{\eta} \to \phi_B'(x-y_s), \\ a^{\eta}(t,x;s) &\to f_u(t,u(t,x;s)) \quad \text{and} \\ b^{\eta}(t,x;s) &\to \int_{\mathbb{R}} J'(x-y)u(t,y;s)dy. \end{aligned}$$

Then, setting $\eta \to 0$ in (8.31), we conclude from the dominated convergence theorem that for any $s < 0$, $t \geq s$ and $x \in \mathbb{R}$, the limit $u_x(t,x;s) = \lim_{\eta\to 0} v^{\eta}(t,x;s)$ exists and

$$u_x(t,x;s) = \phi_B'(x-y_s)e^{-\int_s^t (1-f_u(\tau,u(\tau,x;s)))d\tau} + \int_s^t b(r,x;s)e^{-\int_r^t (1-f_u(\tau,u(\tau,x;s)))d\tau} dr, \tag{8.32}$$

where $b(t,x;s) = \int_{\mathbb{R}} J'(x-y)u(t,y;s)dy = \int_{\mathbb{R}} J'(y)u(t,x-y;s)dy$. In particular, for any $s < 0$ and $t \geq s$, $u(t,x;s)$ is continuously differentiable in x. The uniform boundedness of $u_x(t,x;s)$, i.e., $\sup_{\substack{x\neq y\\ s<0,t\geq s}} |u_x(t,x;s)| < \infty$, then follows from Lemma 8.12.

(ii) Since $u_x(t,x;s)$ is uniformly bounded by (i), we trivially have

$$\forall\delta>0,\qquad \sup_{\substack{|x-y|\geq\delta\\ s<0,t\geq s}}\left|\frac{u_x(t,x;s)-u_x(t,y;s)}{x-y}\right|<\infty.$$

Thus, to show the uniform Lipschitz continuity of $u_x(t,x;s)$, it suffices to show the local uniform Lipschitz continuity, i.e.,

$$\forall\delta>0,\qquad \sup_{\substack{|x-y|\leq\delta\\ s<0,t\geq s}}\left|\frac{u_x(t,x;s)-u_x(t,y;s)}{x-y}\right|<\infty. \tag{8.33}$$

To this end, we fix $\delta>0$. Let $X(t;s)$ and $X_\lambda(t;s)$ for $\lambda\in(0,1)$ be as in Theorem 8.9 and define

$$L_1=\delta+\sup_{s<0,t\geq s}\big|X_{\theta_0}(t;s)-X(t;s)\big|\quad\text{and}\quad L_2=\delta+\sup_{s<0,t\geq s}\big|X_{\theta_1}(t;s)-X(t;s)\big|.$$

Notice $L_1<\infty$ and $L_2<\infty$. Then, for any $x\in\mathbb{R}$ and $|\eta|\leq\delta$ we have

- if $x\geq X(t;s)+L_1$, then $x+\eta\geq x-\delta\geq X_{\theta_0}(t;s)$, which implies that $u(t,x+\eta;s)\leq\theta_0$ by monotonicity, and hence

$$f_u(t,u(t,x+\eta;s))\leq 0; \tag{8.34}$$

- if $x\leq X(t;s)-L_2$, then $x+\eta\leq x+\delta\leq X_{\theta_1}(t;s)$, which implies that $u(t,x+\eta;s)\geq\theta_1$ by monotonicity, and hence

$$f_u(t,u(t,x+\eta;s))\leq 0. \tag{8.35}$$

According to (8.34) and (8.35), we consider time-dependent disjoint decompositions of $\mathbb{R}$ into

$$\mathbb{R}=R_l(t;s)\cup R_m(t;s)\cup R_r(t;s),$$

where

$$\begin{aligned} R_l(t;s) &= (-\infty, X(t;s) - L_2), \\ R_m(t;s) &= [X(t;s) - L_2, X(t;s) + L_1] \quad \text{and} \\ R_r(t;s) &= (X(t;s) + L_1, \infty). \end{aligned} \tag{8.36}$$

For $s < 0$ and $x_0 \in \mathbb{R}$, let $t_{\text{first}}(x_0; s)$ be the first time that x_0 is in $R_m(t;s)$, that is,

$$t_{\text{first}}(x_0; s) = \min \big\{ t \geq s \big| x_0 \in R_m(t;s) \big\},$$

and $t_{\text{last}}(x_0; s)$ be the last time that x_0 is in $R_m(t;s)$, that is,

$$t_{\text{last}}(x_0; s) = \max \big\{ t_0 \in \mathbb{R} \big| x_0 \in R_m(t_0; s) \text{ and } x_0 \notin R_m(t,s) \text{ for } t > t_0 \big\}.$$

Since $\dot{X}(t;s) \geq c_{\min} > 0$ by Theorem 8.9, if $x_0 \in R_l(s;s)$, then $x_0 \in R_l(t;s)$ for all $t > s$. In this case, $t_{\text{first}}(x_0;s)$ and $t_{\text{last}}(x_0;s)$ are not well-defined, but it will not cause any trouble. We see that $t_{\text{first}}(x_0;s)$ and $t_{\text{last}}(x_0;s)$ are well-defined only if $x_0 \notin R_l(s;s)$. As a simple consequence of $\dot{X}(t;s) \in [c_{\min}, c_{\max}]$ in Theorem 8.9 and the fact that the length of $R_m(t;s)$ is $L_1 + L_2$, we have

$$T = T(\delta) := \sup_{s<0, x_0 \notin R_l(s;s)} \big[t_{\text{last}}(x_0;s) - t_{\text{first}}(x_0;s) \big] < \infty. \tag{8.37}$$

Moreover, we see that for any $|\eta| \leq \delta$,

$$\begin{aligned} f_u(t, u(t, x_0 + \eta; s)) &\leq 0 \quad \text{if} \quad t \in [s, t_{\text{first}}(x_0;s)], \\ f_u(t, u(t, x_0 + \eta; s)) &\leq 0 \quad \text{if} \quad t \geq t_{\text{last}}(x_0;s). \end{aligned} \tag{8.38}$$

We now show that

$$\sup_{\substack{x_0\in\mathbb{R},0<|\eta|\le\delta\\ s<0,t\ge s}}\left|\frac{u_x(t,x_0+\eta;s)-u_x(t,x_0;s)}{\eta}\right|<\infty. \tag{8.39}$$

Using (8.32), we have

$$\begin{aligned}
&\frac{u_x(t,x_0+\eta;s)-u_x(t,x_0;s)}{\eta}\\
&\quad=\underbrace{\frac{\phi_B'(x_0+\eta-y_s)-\phi_B'(x_0-y_s)}{\eta}e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau}}_{\text{(I)}}\\
&\quad\quad+\underbrace{\phi_B'(x_0-y_s)\frac{e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau}-e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0;s)))d\tau}}{\eta}}_{\text{(II)}}\\
&\quad\quad+\underbrace{\int_s^t\frac{b(r,x_0+\eta;s)-b(r,x_0;s)}{\eta}e^{-\int_r^t(1-f_u(\tau,u(\tau,x_0;s)))d\tau}dr}_{\text{(III)}}\\
&\quad\quad+\underbrace{\int_s^t b(r,x_0;s)\frac{e^{-\int_r^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau}-e^{-\int_r^t(1-f_u(\tau,u(\tau,x_0;s)))d\tau}}{\eta}dr}_{\text{(IV)}}.
\end{aligned}$$

Hence, it suffice to bound terms (I)-(IV). To do so, we need to consider three cases: $x_0\in R_l(s;s)$, $x_0\in R_m(s;s)$ and $x_0\in R_r(s;s)$. We here focus on the last case, i.e., $x_0\in R_r(s;s)$, which is the most involved one. The other two cases are simpler and can be treated similarly. Also, for fixed $s<0$ and $x_0\in R_r(s;s)$, we will focus on $t\ge t_{\text{last}}(x_0;s)$; the case with $t\in[t_{\text{first}}(x_0;s),t_{\text{last}}(x_0;s)]$ or $t\le t_{\text{first}}(x_0;s)$ will be clear. Thus, we assume $x_0\in R_r(s;s)$ and $t\ge t_{\text{last}}(x_0;s)$.

We will frequently use the following estimates: for any $|\tilde{\eta}| \leq \delta$ there hold

$$\begin{aligned} e^{-\int_r^{t_{\text{first}}(x_0;s)}(1-f_u(\tau,u(\tau,x_0+\tilde{\eta};s)))d\tau} &= e^{-(t_{\text{first}}(x_0;s)-r)}, \quad r \in [s, t_{\text{first}}(x_0;s)] \\ e^{-\int_r^{t_{\text{last}}(x_0;s)}(1-f_u(\tau,u(\tau,x_0+\tilde{\eta};s)))d\tau} &\leq e^{T \sup_{(t,u)\in\mathbb{R}\times[0,1]} |1-f_u(t,u)|}, \quad r \in [t_{\text{first}}(x_0;s), t_{\text{last}}(x_0;s)] \\ e^{-\int_r^{t}(1-f_u(\tau,u(\tau,x_0+\tilde{\eta};s)))d\tau} &\leq e^{-(t-r)}, \quad r \in [t_{\text{last}}(x_0;s), t]. \end{aligned} \tag{8.40}$$

They are simple consequences of (8.37) and (8.38). Set

$$\begin{aligned} &C_0 := \sup_{(t,u)\in\mathbb{R}\times[0,1]} |1-f_u(t,u)|, \quad C_1 := \sup_{x\neq y}\left|\frac{\phi_B'(x)-\phi_B'(y)}{x-y}\right|, \quad C_2 := \sup_{x\in\mathbb{R}}|\phi_B'(x)| \\ &C_3 := \sup_{(t,u)\in\mathbb{R}\times[0,1]} |f_{uu}(t,u)| \times \sup_{\substack{x\in\mathbb{R}\\ s<0,t\geq s}} |u_x(t,x;s)|, \quad C_4 = \sup_{\substack{x\neq y\\ s<0,t\geq s}}\left|\frac{u(t,x;s)-u(t,y;s)}{x-y}\right|. \end{aligned}$$

Note that all these constants are finite. In fact, $C_0 < \infty$ by Hypothesis 2.5, $C_1 < \infty$ by (A.3), $C_3 < \infty$ by Hypothesis 8.1 and (i) in Theorem 8.14, and $C_4 < \infty$ by Lemma 8.12.

We are ready to bound (I)-(IV). For the term (I), using (A.3) and (8.40), we see that

$$\begin{aligned} |(\mathrm{I})| &\leq C_1 e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau} \\ &= C_1 e^{-\left[\int_s^{t_{\text{first}}(x_0;s)}+\int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)}+\int_{t_{\text{last}}(x_0;s)}^{t}\right](1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau} \\ &\leq C_1 e^{-(t_{\text{first}}(x_0;s)-s)} e^{C_0T} e^{-(t-t_{\text{last}}(x_0;s))} \leq C_1 e^{C_0T}. \end{aligned} \tag{8.41}$$

For the term (II), we have from Taylor expansion of the function $\eta \mapsto e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau}$ at $\eta = 0$ that

$$\begin{aligned} |(\mathrm{II})| &\leq C_2 \left|\frac{e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta;s)))d\tau} - e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0;s)))d\tau}}{\eta}\right| \\ &\leq C_2 e^{-\int_s^t(1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} \int_s^t \Big|f_{uu}(\tau,u(\tau,x_0+\eta_*;s))u_x(\tau,x_0+\eta_*;s)\Big| d\tau, \end{aligned}$$

where η_* is between 0 and η, and hence, $|\eta_*| \le \delta$. We see

$$\int_s^t \Big| f_{uu}(\tau, u(\tau, x_0+\eta_*; s)) u_x(\tau, x_0+\eta_*; s) \Big| d\tau \le C_3(t-s).$$

It then follows from (8.40) that

$$\begin{aligned} |(\mathrm{II})| &\le C_2 C_3 e^{-(t_{\mathrm{first}}(x_0;s)-s)} e^{-(t-t_{\mathrm{last}}(x_0;s))}(t-s) \\ &= C_2 C_3 e^{-(t_{\mathrm{first}}(x_0;s)-s)} e^{-(t-t_{\mathrm{last}}(x_0;s))} \\ &\quad \times \big[(t-t_{\mathrm{last}}(x_0;s)) + (t_{\mathrm{last}}(x_0;s) - t_{\mathrm{first}}(x_0;s)) + (t_{\mathrm{first}}(x_0;s)-s)\big] \\ &\le C_2 C_3 \big[e^{-(t-t_{\mathrm{last}}(x_0;s))}(t-t_{\mathrm{last}}(x_0;s)) + T + e^{-(t_{\mathrm{first}}(x_0;s)-s)}(t_{\mathrm{first}}(x_0;s)-s)\big] \\ &\le C_2 C_3 \left(\frac{2}{e} + T\right). \end{aligned} \tag{8.42}$$

For the term (III), we first see that

$$\left|\frac{b(r, x_0+\eta; s) - b(r, x_0; s)}{\eta}\right| = \left|\int_{\mathbb{R}} J'(y) \frac{u(r, x_0+\eta-y; s) - u(r, x_0-y; s)}{\eta} dy\right| \le C_4 \|J'\|_{L^1(\mathbb{R})}.$$

Thus,

$$\begin{aligned} |(\mathrm{III})| &\le C_4 \|J'\|_{L^1(\mathbb{R})} \int_s^t e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr \\ &= C_4 \|J'\|_{L^1(\mathbb{R})} \Big[\underbrace{\int_s^{t_{\mathrm{first}}(x_0;s)} e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\mathrm{III}\text{-}1)} + \underbrace{\int_{t_{\mathrm{first}}(x_0;s)}^{t_{\mathrm{last}}(x_0;s)} e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\mathrm{III}\text{-}2)} \\ &\qquad + \underbrace{\int_{t_{\mathrm{last}}(x_0;s)}^t e^{-\int_r^t (1-f_u(\tau, u(\tau, x_0; s)))d\tau} dr}_{(\mathrm{III}\text{-}3)} \Big]. \end{aligned}$$

We estimate (III-1), (III-2) and (III-3). For (III-1), we obtain from (8.40) that

$$
\begin{aligned}
\text{(III-1)} &= \int_s^{t_{\text{first}}(x_0;s)} e^{-\left[\int_r^{t_{\text{first}}(x_0;s)} + \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} + \int_{t_{\text{last}}(x_0;s)}^{t}\right](1-f_u(\tau,u(\tau,x_0;s)))d\tau} dr \\
&\leq e^{C_0 T} \int_s^{t_{\text{first}}(x_0;s)} e^{-(t_{\text{first}}(x_0;s)-r)} e^{-(t-t_{\text{last}}(x_0;s))} dr \\
&= e^{C_0 T} e^{-(t-t_{\text{last}}(x_0;s))}(1-e^{-(t_{\text{first}}(x_0;s)-s)}) \leq e^{C_0 T}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\text{(III-2)} &= \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} e^{-\left[\int_r^{t_{\text{last}}(x_0;s)} + \int_{t_{\text{last}}(x_0;s)}^{t}\right](1-f_u(\tau,u(\tau,x_0;s)))d\tau} dr \\
&\leq e^{C_0 T} \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} e^{-(t-t_{\text{last}}(x_0;s))} dr \leq e^{C_0 T} T e^{-(t-t_{\text{last}}(x_0;s))} \leq T e^{C_0 T}
\end{aligned}
$$

and $\text{(III-3)} \leq \int_{t_{\text{last}}(x_0;s)}^{t} e^{-(t-r)} dr = 1 - e^{-(t-t_{\text{last}}(x_0;s))} \leq 1$. Hence,

$$
\text{(III)} \leq C_4 \|J'\|_{L^1(\mathbb{R})} (e^{C_0 T} + T e^{C_0 T} + 1). \tag{8.43}
$$

For the term (IV), using $|b(r, x_0; s)| \leq \|J'\|_{L^1(\mathbb{R})}$ and Taylor expansion as in the treatment of the term (II), we have

$$
\begin{aligned}
|\text{(IV)}| &\leq \|J'\|_{L^1(\mathbb{R})} \int_s^t e^{-\int_r^t (1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} \left(\int_r^t \Big| f_{uu}(\tau, u(\tau, x_0+\eta_*; s)) u_x(\tau, x_0+\eta_*; s) \Big| d\tau \right) dr \\
&\leq C_3 \|J'\|_{L^1(\mathbb{R})} \int_s^t (t-r) e^{-\int_r^t (1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr \\
&= C_3 \|J'\|_{L^1(\mathbb{R})} \Bigg[\underbrace{\int_s^{t_{\text{first}}(x_0;s)} (t-r) e^{-\int_r^t (1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr}_{\text{(IV-1)}} \\
&\qquad + \underbrace{\int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} (t-r) e^{-\int_r^t (1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr}_{\text{(IV-2)}} \\
&\qquad + \underbrace{\int_{t_{\text{last}}(x_0;s)}^{t} (t-r) e^{-\int_r^t (1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr}_{\text{(IV-3)}} \Bigg],
\end{aligned}
$$

where $|\eta_*| \le |\eta| \le \delta$. Similar to (III-1), (III-2) and (III-3), we have

$$
\begin{aligned}
\text{(IV-1)} &= \int_s^{t_{\text{first}}(x_0;s)} (t-r) e^{-\left[\int_r^{t_{\text{first}}(x_0;s)} + \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} + \int_{t_{\text{last}}(x_0;s)}^{t}\right](1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr \\
&\le e^{C_0T} \int_s^{t_{\text{first}}(x_0;s)} \left[(t-t_{\text{last}}(x_0;s)) + T + (t_{\text{first}}(x_0;s)-r)\right] e^{-(t_{\text{first}}(x_0;s)-r)} e^{-(t-t_{\text{last}}(x_0;s))} dr \\
&\le e^{C_0T}\Bigg[(t-t_{\text{last}}(x_0;s)) e^{-(t-t_{\text{last}}(x_0;s))} \int_s^{t_{\text{first}}(x_0;s)} e^{-(t_{\text{first}}(x_0;s)-r)} dr \\
&\qquad + T\int_s^{t_{\text{first}}(x_0;s)} e^{-(t_{\text{first}}(x_0;s)-r)} dr + \int_s^{t_{\text{first}}(x_0;s)} (t_{\text{first}}(x_0;s)-r)) e^{-(t_{\text{first}}(x_0;s)-r)} dr\Bigg] \\
&\le e^{C_0T}\left[\frac{1-e^{-(t_{\text{first}}(x_0;s)-s)}}{e} + T(1-e^{-(t_{\text{first}}(x_0;s)-s)}) + \left(1-(1+t_{\text{first}}(x_0;s)-s)e^{-(t_{\text{first}}(x_0;s)-s)}\right)\right] \\
&\le e^{C_0T}\left(\frac{1}{e}+T+1\right),
\end{aligned}
$$

$$
\begin{aligned}
\text{(IV-2)} &= \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} (t-r) e^{-\left[\int_r^{t_{\text{last}}(x_0;s)} + \int_{t_{\text{last}}(x_0;s)}^{t}\right](1-f_u(\tau,u(\tau,x_0+\eta_*;s)))d\tau} dr \\
&\le e^{C_0T} \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} [(t-t_{\text{last}}(x_0;s)) + (t_{\text{last}}(x_0;s)-r)] e^{-(t-t_{\text{last}}(x_0;s))} dr \\
&\le e^{C_0T}\left[T(t-t_{\text{last}}(x_0;s)) e^{-(t-t_{\text{last}}(x_0;s))} + \int_{t_{\text{first}}(x_0;s)}^{t_{\text{last}}(x_0;s)} (t_{\text{last}}(x_0;s)-r) dr\right] \\
&\le e^{C_0T}\left(\frac{T}{e}+\frac{T^2}{2}\right)
\end{aligned}
$$

and

$$
\text{(IV-3)} \le \int_{t_{\text{last}}(x_0;s)}^{t} (t-r)e^{-(t-r)} dr = 1-(1+t-t_{\text{last}}(x_0;s))e^{-(t-t_{\text{last}}(x_0;s))} \le 1.
$$

Hence,

$$
|\text{(IV)}| \le C_3 \|J'\|_{L^1(\mathbb{R})}\left[e^{C_0T}\left(\frac{1}{e}+T+1\right) + e^{C_0T}\left(\frac{T}{e}+\frac{T^2}{2}\right)+1\right]. \tag{8.44}
$$

Consequently, (8.39) follows from (8.41), (8.42), (8.43) and (8.44). □

Chapter 9

Concluding remarks and future plans

In this chapter, we make some remarks about the results obtained in the present dissertation and mention some open problems.

(i) Let f_B and $f_{\tilde{B}}$ be two standard homogeneous bistable nonlinearies on $[0,1]$ with $f_B \leq f_{\tilde{B}}$ and $\int_0^1 f_B(u)du > 0$, as in Hypothesis 2.2. Thus, if $\theta \in (0,1)$ and $\tilde{\theta} \in (0,1)$ are the unstable (or middle) zeros of f_B and $f_{\tilde{B}}$, respectively, we have $\tilde{\theta} \leq \theta$. In proving the existence of transition fronts in Section 8, we require $\tilde{\theta} = \theta$, which is kind of restrictive, but allows degeneracy at θ, that is, $f'_B(\theta) = 0 = f'_{\tilde{B}}(\theta)$.

Here's a possible variation: suppose that $\tilde{\theta} < \theta$, $f(t,u)$ is of standard bistable in u for any $t \in \mathbb{R}$. Let $\theta(t) \in (0,1)$ be such that $f(t,\theta(t)) = 0$, and assume $\theta(t)$ is an exponentially unstable solution to $u_t = f(t,u)$. Then, the techniques in [64] can be adapted to prove the existence of transition fronts.

(ii) The condition $\tilde{\theta} = \theta$ is not needed in studying the qualitative properties of transition fronts in Chapter 3-Chapter 7.

(iii) The regularity result in Section 3.2 only covers some monostable nonlinearites in space-time heterogeneous (see Corollary 3.7). It would be interesting to know if the regularity can be established for all monostable nonlinearities.

(iv) The condition $\int_0^1 f_B(u)du > 0$ forces all front-like solutions to move to the right, and due to this, we can prove rightward propagation estimates of approximating front-like solutions in Section 8, which plays an important role in our regularity arguments leading to the construction of transition fronts.

Thus, it would be interesting and important to know if transition fronts can be constructed if we assume $\int_0^1 f_B(u)du < 0 < \int_0^1 f_{\tilde{B}}(u)du$. Under this assumption, we will lose rightward propagation estimates, and hence, our regularity arguments fail.

9.1 Future plans

Here are some problems I am interested in.

(i) There's few work (see [12]) concerning the equation $u_t = J * u - u + f(x,u)$ in the bistable case. I would like to know whether transition fronts, or periodic traveling waves in the periodic case, can be found.

(ii) Consider the equation $u_t = J*u-u+f(t,u)$ in the monostable case. Periodic traveling waves in the periodic case have been established (see [60]), but no result exist in the literature concerning transition fronts in the general case.

(iii) Consider the equation $u_t = u_{xx}+f(t,x,u)$ in the monostable, bistable or ignition case. Almost all results concerning this equation were obtained when $f(t,x,u) = f(t,u)$ or $f(t,x,u) = f(x,u)$ or $f(t,x,u)$ is periodic in t or x. Thus, it is very interesting and important to establish transition front in the general case. However, this seem unavailable so far.

(iv) Lots of effort and work has been carried out to the understanding of the single equation $u_t = u_{xx}+f(t,x,u)$ or $u_t = J*u-u+f(t,x,u)$. As opposed to this, results concerning systems, say

$$\begin{cases} u_t = u_{xx} + u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v) \\ v_t = u_{xx} + v(a_2(t,x) - c_2(t,x)v - b_2(t,x)u) \end{cases}$$

or

$$\begin{cases} u_t = J * u - u + u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v) \\ v_t = J * v - v + v(a_2(t,x) - c_2(t,x)v - b_2(t,x)u) \end{cases}$$

are much less known. But, in applications, systems are much more meaningful than single equations. Therefore, it is very important to have a better undertanding of the dynamics of systems. This is a challenging problem.

Appendix A

Bistable traveling waves

We collect some results about bistable traveling waves for convenience (see [7]). Consider the following homogeneous equation

$$u_t = J * u - u + f_B(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{A.1}$$

where f_B, given in Hypothesis 2.2, is of standard bistable type. There are a unique $c_B > 0$ and a unique continuously differentiable function $\phi = \phi_B : \mathbb{R} \to (0, 1)$ satisfying

$$\begin{cases} J * \phi - \phi + c_B \phi_x + f_B(\phi) = 0, \\ \phi_x < 0,\ \phi(0) = \theta,\ \phi(-\infty) = 1 \text{ and } \phi(\infty) = 0. \end{cases} \tag{A.2}$$

That is, ϕ_B is the normalized wave profile and $\phi_B(x - c_B t)$ is the traveling wave of (A.1). It is not hard to see that ϕ'_B is uniformly Lipschitz continuous, that is,

$$\sup_{x \neq y} \left| \frac{\phi'_B(x) - \phi'_B(y)}{x - y} \right| < \infty. \tag{A.3}$$

Moreover, ϕ_B enjoys exponential decaying estimates as in

Lemma A.1. *There exist $c_B^{\pm} > 0$ and $x_B^{\pm} > 0$ such that*

$$\phi_B(x) \leq e^{-c_B^+(x - x_B^+)} \quad \textit{and} \quad 1 - \phi_B(x) \leq e^{c_B^-(x + x_B^-)}$$

for all $x \in \mathbb{R}$.

Also, the following stability result for $\phi_B(x - c_B t)$ holds:

Lemma A.2. *Let $u_0 : \mathbb{R} \to [0,1]$ be uniformly continuous and satisfy*

$$\limsup_{x\to\infty} u_0(x) < \theta < \liminf_{x\to-\infty} u_0(x).$$

Then, there exists $\xi_B^{\pm} = \xi_B^{\pm}(u_0) \in \mathbb{R}$, $q_B = q_B(u_0) > 0$ and $\omega_B > 0$ (independent of u_0) such that

$$\phi_B(x - \xi_B^- - c_B t) - q_B e^{-\omega_B t} \leq u_B(t, x; u_0) \leq \phi_B(x - \xi_B^+ - c_B t) + q_B e^{-\omega_B t}, \quad x \in \mathbb{R}$$

for all $t \geq 0$, where $u_B(t, x; u_0)$ is the solution of (A.1) *with initial data $u_B(0, \cdot; u_0) = u_0$.*

Appendix B

Comparison principles

We prove comparison principles used in the previous chapters.

Proposition B.1. *Let* $K : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *be continuous and satisfy* $\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} K(x, y)dy < \infty$. *Let* $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be continuous and uniformly bounded.*

(i) *Suppose that* $X : [0, \infty) \to \mathbb{R}$ *is continuous and that* $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following:* $u, u_t : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous, the limit* $\lim_{x\to\infty} u(t, x) = 0$ *is locally uniformly in* t*, and*

$$\begin{cases} u_t(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), \quad x > X(t),\ t > 0, \\ u(t,x) \geq 0, \quad x \leq X(t),\ t > 0, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}. \end{cases}$$

Then $u(t, x) \geq 0$ *for* $(t, x) \in (0, \infty) \times \mathbb{R}$.

(ii) *Suppose that* $X : [0, \infty) \to \mathbb{R}$ *is continuous and that* $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following:* $u, u_t : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous, the limit* $\lim_{x\to-\infty} u(t, x) = 0$ *is locally uniformly in* t*, and*

$$\begin{cases} u_t(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), \quad x < X(t),\ t > 0, \\ u(t,x) \geq 0, \quad x \geq X(t),\ t > 0, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}. \end{cases}$$

Then $u(t, x) \geq 0$ *for* $(t, x) \in (0, \infty) \times \mathbb{R}$.

(iii) *Suppose that* $u : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ *satisfies the following:* $u, u_t : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ *is continuous,* $\inf_{t\geq 0, x\in\mathbb{R}} u(t,x) > -\infty$*, and*

$$\begin{cases} u_t(t,x) \geq \int_{\mathbb{R}} K(x,y)u(t,y)dy + a(t,x)u(t,x), \quad x\in\mathbb{R},\ t>0, \\ u(0,x) = u_0(x) \geq 0, \quad x\in\mathbb{R}. \end{cases}$$

Then $u(t,x)\geq 0$ *for* $(t,x)\in(0,\infty)\times\mathbb{R}$*. Moreover, if* $u_0(x)\not\equiv 0$*, then* $u(t,x)>0$ *for* $(t,x)\in(0,\infty)\times\mathbb{R}$*.*

Proof. (i) We follow [33, Proposition 2.4]. Note first that replacing $u(t,x)$ by $e^{rt}u(t,x)$ for sufficiently large $r>0$, we many assume, without loss of generality, that $a(t,x)>0$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$.

Set $K_0 := \sup_{x\in\mathbb{R}}\int_{\mathbb{R}} K(x,y)dy < \infty$, $a_0 := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} a(t,x)$ and $\tau := \frac{1}{2(K_0+a_0)}$. Suppose there exists $(t_0,x_0)\in[0,\tau]\times\mathbb{R}$ such that $u(t_0,x_0)<0$. Then, by the assumption, there exists $(t_1,x_1)\in\Omega_{0,\tau} := \{(t,x)\in\mathbb{R}\times\mathbb{R} | x > X(t),\ t\in(0,\tau]\}$ such that

$$u(t_1,x_1) = \min_{(t,x)\in\Omega_{0,\tau}} u(t,x) < 0.$$

Define

$$s_1 := \max\{t\in[0,t_1] | u(t,x_1)\geq 0\}.$$

By continuity of $u(t,x)$, $s_1<t_1$ and $u(s_1,x_1)\geq 0$. Moreover, by the definition of s_1, we see that $x_1 > X(t)$ for $t\in(s_1,t_1]$. In particular, we have

$$u_t(t,x_1) \geq \int_{\mathbb{R}} K(x_1,y)u(t,y)dy + a(t,x_1)u(t,x_1), \quad t\in(s_1,t_1].$$

Integrating the above inequality with respect to t from s_1 to t_1, we conclude from the facts that $u(t,x) \geq 0$ for $x \leq X(t)$ and $u(t_1,x_1) < 0$ that

$$
\begin{aligned}
u(t_1,x_1) - u(s_1,x_1) &\geq \int_{s_1}^{t_1} \int_{\mathbb{R}} K(x_1,y)u(t,y)dydt + \int_{s_1}^{t_1} a(t,x_1)u(t,x_1)dt \\
&\geq \int_{s_1}^{t_1} \int_{X(t)}^{\infty} K(x_1,y)u(t,y)dydt + \int_{s_1}^{t_1} a(t,x_1)u(t,x_1)dt \\
&\geq u(t_1,x_1)\left[\int_{s_1}^{t_1} \int_{X(t)}^{\infty} K(x_1,y)dydt + \int_{s_1}^{t_1} a(t,x_1)dt\right] \\
&\geq u(t_1,x_1)(K_0 + a_0)(t_1 - s_1) \\
&\geq u(t_1,x_1)(K_0 + a_0)\tau,
\end{aligned}
$$

which implies that $[1-(K_0+a_0)\tau]u(t_1,x_1) \geq u(s_1,x_1) \geq 0$. It then follows from the choice of τ that $u(t_1,x_1) \geq 0$. It's a contradiction. Thus, $u(t,x) \geq 0$ for $(t,x) \in [0,\tau] \times \mathbb{R}$. Repeating the above arguments with initial times $\tau, 2\tau, 3\tau, \dots,$ we find the result.

(ii) It can be proved by the similar arguments as in (i).

(iii) It follows from the arguments in [73, Propositions 2.1 and 2.2]. □

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