Asymptotic Dynamics of Competition Systems with Immigration and/or Time Periodic Dependence

by

Peng Zhao

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Approved by

Wenxian Shen, Chair, Professor of Mathematics and Statistics
Georg Hetzer, Professor of Mathematics and Statistics
Xiaoying Han, Associate Professor of Mathematics and Statistics
Thomas H. Pate, Professor of Mathematics and Statistics

Abstract

This dissertation is devoted to the study of the asymptotic dynamics, in particular, coexistence states and convergence of nonnegative solutions to the competition systems with immigration and time periodic dependence. This includes two species competition systems of ordinary differential equations with positive sources and periodic dependence, two species competition systems of nonlocal evolution equations with inhomogeneous boundary conditions and cancer models with radiation treatment. The main results of the dissertation consist of the following parts.

Firstly, we look at the Volterra-Lotka competition systems of ordinary differential equations with positive sources. We show that (i) if the competition is weak between two species, a unique stationary solution can be obtained in the time independent case. (ii) As long as the positive sources are large enough, a unique positive stationary solution exists no matter the competition is weak or not in the time independent case. (iii) If the system is time periodic, uniqueness can also be achieved under weak competition.

Secondly, we obtain the existence and uniqueness of continuous coexistence states of competition systems with nonlocal dispersal. It is shown that inhomogeneous Neumann condition or/and Dirichlet condition guarantee not only the persistence of the two species, but also the continuous coexistence when the competition is weak between two species. Once again, it can also be shown that some large enough inhomogeneous boundary conditions allow the continuous coexistence even if the competition between two species is strong. A sufficient condition is also obtained for such continuous coexistence to be unique. In particular, this condition always holds true when the coefficients that account for the competition between the species are both small.

Thirdly, we investigate the cancer model with periodic radiation treatment. Normal cell, tumor cell, radiated normal cell and radiated tumor cell are being considered in the model. We have found that (i) in the absence of cancer cells, if the trivial solution is a stable solution, then it is globally stable. If the trivial solution is an unstable solution, then a unique periodic positive solution exists and it is globally asymptotically stable. (ii) Any solution of the four-species cancer model system converges to a time periodic nonnegative solution. Moreover, if the competition coefficients between unaffected normal and cancer cells, as well as the recombining rates for radiated normal and tumor cells are sufficiently small, the uniqueness of such periodic solution can be obtained.

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Chapter 1

Introduction

Competition exists between many species in biology. There exist various types of competition systems to model the dynamics of competing species. This dissertation is devoted to the study of the asymptotic dynamics of the following three types of competition systems,

$$\begin{cases} u_t = u(a_1(t) - b_1(t)u - c_1(t)v) + d_1(t) \\ v_t = v(a_2(t) - b_2(t)u - c_2(t)v) + d_2(t), \end{cases} \tag{1.1}$$

where a_i, b_i, c_i, and d_i are periodic functions with period T;

$$\begin{cases} u_t = \int_\Omega J(y-x)u(t,y)dy - u(t,x) + u(a_1(x) - b_1(x)u - c_1(x)v) + h_1(x), & x \in \bar{\Omega} \\ v_t = \int_\Omega J(y-x)v(t,y)dy - v(t,x) + v(a_2(x) - b_2(x)u - c_2(x)v) + h_2(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $J(\cdot)$ is C^1, nonnegative, $J(-y) = J(y)$, and $\int_{\mathbb{R}^N} J(y)dy = 1$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain; and

$$\begin{cases} \dot{u} = uf(t,u) - \epsilon D(t)u + p(t)v - a(t)ux \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \\ \dot{x} = xg(t,x) - D(t)x + q(t)y - b(t)ux \\ \dot{y} = D(t)x - q(t)y - \delta(t)y \end{cases} \tag{1.3}$$

where $f(t,u) < 0$ and $g(t,u) < 0$ for $u \gg 1$, and $f_u(t,u) < 0$ and $g_u(t,u) < 0$ for $u \geq 0$. All the functions in (1.3) are periodic in t with period T.

1.1 Competition systems of ordinary differential equations

System (1.1) arises in modeling the dynamics of two competing species in which the internal interaction between organisms can be neglected. In (1.1), $u(t)$ and $v(t)$ denote the population densities of the two species at time t. The functions a_i, b_i, c_i, and d_i ($i = 1, 2$) are assumed to be smooth and and nonnegative. In the species population context, the functions a_1, a_2 represent the respective growth rates of the two species, b_1, c_2 account for self-regulation of the respective species, and c_1, b_2 account for competition between the two species. The positive d_i, $i = 1, 2$ terms can be understood as positive supplies to the system. The periodicity of a_i, b_i, c_i, and d_i reflects the seasonal variation of the underlying environment.

From the biological meaning perspective, we are only interested in nonnegative solutions of (1.1). The central issues in (1.1) include the existence and uniqueness of coexistence states, extinction of one of the two species, and asymptotic behavior of nonnegative solutions. A *coexistence state* of (1.1) is a positive stationary solution in the time independent case and a positive periodic solution in the time periodic case.

When $d_i(\cdot) \equiv 0$, the results of autonomous case and also the periodic case are complete. Under the assumption that a_i, b_i, c_i are positive constants, (see, for example [25], pp 46-50), it has been shown that the conditions

$$a_1 > \frac{c_1 a_2}{c_2}, \quad a_2 > \frac{b_2 a_1}{b_1} \tag{1.4}$$

are necessary and sufficient for the existence of the stable positive equilibrium (u^{**}, v^{**}),

$$u^{**} = \frac{a_1 c_2 - c_1 a_2}{b_1 c_2 - c_1 b_2}, \quad v^{**} = \frac{b_1 a_2 - b_2 a_1}{b_1 c_2 - c_1 b_2}.$$

Moreover, (u^{**}, v^{**}) attracts all solutions with initial values in the open first quadrant of the (u, v)- plane. Note that (1.1) has two semi-trivial equilibria, $(u^*, 0) = (\frac{a_1}{b_1}, 0)$ and $(0, v^*) =$

$(0, \frac{a_2}{c_2})$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, if

$$a_1 > \frac{c_1 a_2}{c_2}, \quad a_2 \le \frac{b_2 a_1}{b_1},$$

then any solution $(u(t), v(t))$ with $(u(0), v(0)) \in (0, \infty) \times (0, \infty)$ converges to $(u^*, 0)$ and hence the species v eventually extincts. If

$$a_1 \le \frac{c_1 a_2}{c_2}, \quad a_2 > \frac{b_2 a_1}{b_1},$$

then any solution $(u(t), v(t))$ with $(u(0), v(0)) \in (0, \infty) \times (0, \infty)$ converges to $(0, v^*)$ and hence the species u eventually extincts. In general, it is known that any solution $(u(t), v(t))$ with $(u(0), v(0)) \in \mathbb{R}^+ \times \mathbb{R}^+$ converges to an equilibrium solution. The reader is referred to [18, 19, 20, 32, 33] and references therein for the study of more general competition systems of autonomous ordinary differential equations.

Alvarez and Lazer considered the system (1.1) with $d_i = 0$ under the assumptions that a_i, b_i, c_i were positive and T-periodic ($T > 0$). They have shown that if

$$a_{1L} > \frac{c_{1M} a_{2M}}{c_{2L}}, \quad a_{2L} > \frac{b_{2M} a_{1M}}{b_{1L}} \tag{1.5}$$

hold (see [1]), then there exists a unique positive T-periodic solution $(u^{**}(t), v^{**}(t))$ which is globally stable in the open first quadrant. Alvarez and Lazer also established upper and lower bounds for the components of the unique T-periodic solution $(u^{**}(t), v^{**}(t))$. Note that (1.1) has two semi-trivial periodic solutions $(u^*(t), 0)$ and $(0, v^*(t))$ in $\mathbb{R}^+ \times \mathbb{R}^+$. It is proved in [26] (see also [34]) that any bounded solution of (1.1) in $\mathbb{R}^+ \times \mathbb{R}^+$ converges to a periodic solution.

There is also a huge amount of research being done on the extension of existing results for time periodic two species competition systems with $d_i = 0$ to time almost periodic and general nonautonomous two species competition systems, see, for example, [2, 12, 13, 14, 31], etc.

In this dissertation, we study the asymptotic dynamics of (1.1) in the case that $d_1, d_2 > 0$. In such case, persistence always occurs, that is, there is some $\delta_0 > 0$ such that any solution $(u(t), v(t))$ in $\mathbb{R}^+ \times \mathbb{R}^+$ satisfies

$$u(t) \geq \delta_0, \quad v(t) \geq \delta_0$$

when t is sufficiently large. Uniqueness of positive equilibrium solutions in the time independent case and uniqueness of positive time periodic solutions are among interesting issues. We prove

- (Weak competition)

  •• *Suppose that a_i, b_i, c_i, d_i are positive constants, $i = 1, 2$. If*

$$\frac{b_1}{c_1} \geq \frac{b_2}{c_2}, \tag{1.6}$$

  then (1.1) *has a unique positive stationary solution* $(u^{**}, v^{**}) \in (0, \infty) \times (0, \infty)$ *(see Theorem 2.2.1).*

  •• *Suppose that $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, and $d_i(\cdot)$ are positive periodic functions with period T. Let $a_{iL} = \min_{t\in[0,T]} a_i(t)$ and $a_{iM} = \max_{t\in[0,T]} a_i(t)$ for $i = 1, 2$. b_{iL}, b_{iM}, c_{iL}, and c_{iM} are defined similarly. Assume $\sup_{t\in\mathbb{R}} d_i(t) > 0$ and*

$$\frac{b_{1L}}{b_{2M}} > \frac{c_{1M}}{c_{2L}}. \tag{1.7}$$

  Then (1.1) *has a unique time periodic positive solution (see Theorem 2.3.1)*

- (Large sources) *Suppose that* a_i, b_i, c_i, d_i *are positive constants,* $i = 1, 2$. *If*

$$d_1 + d_2 \gg 1,$$

then (1.1) *has a unique positive stationary solution* $(u^{**}, v^{**}) \in (0,\infty) \times (0,\infty)$ *(see Theorems 2.2.2 and 2.2.3).*

We would like to make the following three remarks about the results we obtained for (1.1).

First, we remark that condition (1.4) implies (1.6) and that the condition (1.5) implies (1.7). (1.6) and (1.7) are referred to as the weak competition in the time independent case and time periodic case respectively. The results of the dissertation then show that, under the weak competition assumption, the system has a unique coexistence state, which is of great biologic interest.

Next, we remark that the second result stated in the above shows that, when a_i, b_i, c_i, and d_i are time independent and the sources d_1 and d_2 are large enough, (1.1) has a unique coexistence state, which is of great biological interest too. In the periodic case, it remains open whether (1.1) can have only one coexistence state or not if the sources are large enough.

Finally, we remark that the results obtained for (1.1) will play an important role in the study of coexistence states of (1.2).

1.2 Competition systems with nonlocal dispersal

As it is mentioned above, system (1.1) is used to model the dynamics of two competing species when the internal interaction or movement of organisms can be neglected. In many cases, the internal movement of organisms is not negligible and omitting it in the model would make the systems unable to characterize real-world phenomena. Needless to say that many biological systems exhibit genuine long-range internal interactions. System (1.2) is widely used to model such scenarios. The reader is referred to [4], [6], [7], [8], [22], [23],

[15], [21], [35], [36], etc. for various mathematical models for biological systems involving nonlocal internal dispersal.

In (1.2), $u(t,x)$ and $v(t,x)$ denote the population densities of the two species at time t and location x. The functions a_i, b_i, c_i, and d_i ($i = 1, 2$) are assumed to be smooth and nonnegative. As in (1.1), the functions a_1, a_2 represent the respective growth rates of the two species, b_1, c_2 account for self-regulation of the respective species, and c_1, b_2 account for competition between the two species. $\int_\Omega J(y-x)u(t,y)dy - u(t,x)$ and $\int_\Omega J(y-x)v(t,y) - v(t,x)$ represent the internal dispersal of the species u and v, respectively, and $h_i(\cdot)$ can be understood as proper inhomogeneous boundary conditions.

More precisely, we consider the following nonlocal dispersal evolution systems modeling the dynamics of two competing species with Dirichlet and Neumann type boundary conditions,

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\mathbb{R}^{\mathbf{N}}} J(y-x)(u(t,y)-u(t,x))dy \\ \qquad\qquad +u(t,x)(a_1(x)-b_1(x)u-c_1(x)v), & x \in \bar{\Omega} \\ v_t(t,x) = \displaystyle\int_{\mathbb{R}^{\mathbf{N}}} J(y-x)(v(t,y)-v(t,x))dy \\ \qquad\qquad +v(t,x)(a_2(x)-b_2(x)u-c_2(x)v), & x \in \bar{\Omega} \\ u(t,x) = g_1(x), & x \in \mathbb{R}^{\mathbf{N}} \backslash \bar{\Omega} \\ v(t,x) = g_2(x), & x \in \mathbb{R}^{\mathbf{N}} \backslash \bar{\Omega}, \end{cases} \tag{1.8}$$

and

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(y-x)(u(t,y)-u(t,x))dy \\ \qquad\qquad +u(t,x)(a_1(x)-b_1(x)u-c_1(x)v), \qquad x \in \bar{\Omega} \\ \\ v_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(y-x)(v(t,y)-v(t,x))dy \\ \qquad\qquad +v(t,x)(a_2(x)-b_2(x)u-c_2(x)v), \qquad x \in \bar{\Omega} \\ \\ \int_{\mathbb{R}^N\setminus\Omega} J(y-x)(u(t,y)-u(t,x))dy = g_1(x), \quad x \in \bar{\Omega} \\ \\ \int_{\mathbb{R}^N\setminus\Omega} J(y-x)(v(t,y)-v(t,x))dy = g_2(x), \quad x \in \bar{\Omega}, \end{cases} \tag{1.9}$$

where $u(t,x)$ and $v(x,t)$ represent the population density of two species at time t and space position x.

In (1.8), the integral $\displaystyle\int_{\mathbb{R}^N} J(x-y)(u(t,y)-u(t,x))dy$ takes into account the individuals arriving or leaving position $x \in \bar{\Omega}$ from or to other places while we are prescribing the values of u outside the domain Ω by imposing $u = g_i$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}$, $i = 1, 2$, which is so called Dirichlet type boundary condition. When $g_i = 0$, $i = 1, 2$ we get that any individuals that leave $\bar{\Omega}$, die, this is the case when Ω is surrounded by a hostile environment. Note that (1.8) can be written as (1.2) with

$$h_1(x) = \int_{\mathbb{R}^N\setminus\Omega} J(y-x)g_1(y)dy, \quad h_2(x) = \int_{\mathbb{R}^N\setminus\Omega} J(y-x)g_2(y)dy.$$

Similarly, in (1.9), the integral $\displaystyle\int_{\mathbb{R}^N} J(x-y)(u(t,y)-u(t,x))dy$ takes into account the individuals arriving or leaving position $x \in \bar{\Omega}$ from or to other places while we are prescribing the values of $\int_{\mathbb{R}^N\setminus\Omega} J(y-x)(u(t,y)-u(t,x))dy$ and $\int_{\mathbb{R}^N\setminus\Omega} J(y-x)(v(t,y)-v(t,x))dy$, which is so called the Neumann type boundary condition. Note that (1.9) can be written as (1.2) with

$$h_1(x) = g_1(x), \quad h_2(x) = g_2(x)$$

and with $a_i(x)$ being replaced by $a_i(x) + 1 - \int_{\mathbb{R}^N \setminus \Omega} J(y-x)dy$ $(i = 1, 2)$.

Similar to (1.1), the central issues in (1.2) include existence and uniqueness of continuous coexistence states and extinction of one of the two species. A continuous coexistence state of (1.2) is a continuous function $(u^{**}(x), v^{**}(x))$ satisfying

$$\begin{cases} \int_\Omega J(y-x)u^{**}(y)dy - u^{**}(x) \\ \qquad\qquad +u^{**}(x)(a_1(x) - b_1(x)u^{**}(x) - c_1(x)v^{**}(x)) + h_1(x) = 0, \quad x \in \bar{\Omega} \\ \int_\Omega J(y-x)v^{**}(y)dy - v^{**}(x) \\ \qquad\qquad +v^{**}(x)(a_2(x) - b_2(x)u^{**}(x) - c_2(x)v^{**}(x)) + h_2(x) = 0, \quad x \in \Omega. \end{cases}$$

The asymptotic dynamics of (1.8) with $g_1 \equiv g_2 \equiv 0$ has been studied in [15], [21], etc.. In particular, sufficient conditions for persistence and coexistence are obtained in [15].

Set

$$a_{iL(M)} = \inf_{x\in\bar{\Omega}} (\sup_{x\in\bar{\Omega}})(a_i(x)),$$

$$b_{iL(M)} = \inf_{x\in\bar{\Omega}} (\sup_{x\in\bar{\Omega}})(a_i(x)),$$

$$c_{iL(M)} = \inf_{x\in\bar{\Omega}} (\sup_{x\in\bar{\Omega}})(a_i(x)).$$

Let λ_0 be the principal eigenvalue of the following spectral problem in $C(\bar{\Omega})$

$$\int_\Omega J(y-x)u(y)dy - u(x) = \lambda u(x), \quad x \in \bar{\Omega}$$

(see [35] for the definition of principal eigenvalues of nonlocal dispersal operators). Hetzer, Nguyen and Shen have obtained the following.

Assume $a_{iL} > -\lambda_0$, for $i = 1, 2$.

* If $a_{1L} > -\lambda_0 + \frac{c_{1M}a_{2M}}{c_{2L}}$ and $a_{2L} > -\lambda_0 + \frac{b_{2M}a_{1M}}{b_{1L}}$, then (1.8) has at least one coexistence state. This result is actually stated in a more general setting with different dispersal rates and conditions are depended on the dispersal rate.

* If $a_1(x) = a_2(x)$, $b_1(x) > b_2(x)$ and $c_1(x) < c_2(x)$ for $x \in \bar{\Omega}$, then (1.8) has at least one coexistence state.

* If $a_1(x) = a_2(x)$ for $x \in \bar{\Omega}$ and b_i, c_i $(i = 1, 2)$ are constant functions with $b_1 > b_2$ and $c_1 < c_2$, then (1.8) has a unique globally stable coexistence state.

The last two statements are in fact true as long as the dispersal rates are equal for two species.

In this dissertation, we investigate the existence and uniqueness of continuous coexistence states of (1.2) with $h_1(x) \not\equiv 0$ and $h_2(x) \not\equiv 0$. Note that the inhomogeneity of the boundary conditions enables both species to persist (see Proposition 3.2.3). Among others, we prove

- (Coexistence: weak competition) *Assume* $\dfrac{b_1(x)}{c_1(x)} \geq \dfrac{b_2(x)}{c_2(x)}$ *for* $x \in \bar{\Omega}$. *Then* (1.2) *has a positive continuous stationary solution* $(u^{**}(\cdot), v^{**}(\cdot)) \in C(\bar{\Omega}, (0, \infty)) \times C(\bar{\Omega}, (0, \infty))$ *(see Theorem 3.3.1)*

- (Coexistence: large inhomogeneous condition) *Assume that there is* $\delta_0 > 0$ *such that* $h_1(x) \gg 1$ *or* $h_2(x) \gg 1$ *for* $x \in \bar{\Omega}$ *with* $d(x, \partial\Omega) \leq \delta_0$, *then* (1.2) *has a positive continuous stationary solution* $(u^{**}(\cdot), v^{**}(\cdot)) \in C(\bar{\Omega}, (0, \infty)) \times C(\bar{\Omega}, (0, \infty))$ *(see Theorem 3.3.2).*

- (Uniqueness)(1.2) *has a unique positive stationary solution provided that* c_2 *and* b_1 *are sufficiently small (see Theorem 3.4.2.)*

The above results show that if the competition between two species is weak in the sense that $\dfrac{b_1(x)}{c_1(x)} \geq \dfrac{b_2(x)}{c_2(x)}$ for $x \in \bar{\Omega}$, any inhomogeneity of the boundary conditions enables the system to have a continuous coexistence state. Moreover, the continuous coexistence state is unique in (1.2) provided that competition coefficients c_1 and b_2 are sufficiently small.

1.3 Cancer model

System (1.3) is known as the Cancer Model with radiation treatment. Mathematical modelling of cancer growth and its treatments is a burgeoning field as scientists are still exploring unknown causes and treatments of this deadly illness. Many articles were devoted to cancer modelling in the past decade. The main types of cancer treatments involve surgery, chemotherapy, radiotherapy and immunotherapy, either in isolation or in combination of two or more of these. See [17], [16], [9], [10], [11], [37], [28], [30], [29].

In (1.3), u and x denote the population density at time t of the host and cancer cell respectively, with v and y being the population density of radiated host and tumor cell populations. $a > 0$ and $b > 0$ represent the competition coefficients between unaffected normal and cancer cells. $q > 0$ and $p > 0$ are recombining rates for radiated host and tumor cells. We assume the same washout rate δ for both radiated host and tumor cells. The radiotherapy treatment is given by $D \geq 0$. The radiotherapy is designed so that the full radiation concentration affects the cancer cells. However, only a small proportion of the radiation ϵD affects the normal cells.

Radiotherapy is a treatment procedure that uses radiation to kill malignant tumor cells. This treatment targets rapidly growing and dividing cells such as those in cancer. Radiation destroys cells by causing one or more chromosomes to break. When this happens, the cells cannot reproduce and eventually die off. Hence the question of persistence or extinction of a community of cells exposed to radiation is of great interest. Sometimes it is also possible for the broken chromosomes to recombine. The radiation protocol may be constant dosage or periodic dosage. In this thesis, we focus on the coexistence/extinction of (1.3) under period dosage.

It should be pointed out that the coexistence/extinction of (1.3) under the constant dosage has been studied. For example, Freedman and Pinho have studied (1.3) with $f(t,u) = r(1 - u/K)$ and $g(t,x) = s(1 - x/L)$. This is a special case of (1.3) with f and g being logistic functions (see [10]). In the absence of radiation ($D = 0$) and competition

($a = b = 0$), normal and cancer cells grow with specific birth rates $r > 0$ and $s > 0$ respectively to their respective carrying capacities, $K > 0$ and $L > 0$. They have explicitly given three of the four possible equilibria for the system, namely $(0,0,0,0)$, cure state $(\frac{K[r(p+\delta)-\epsilon D\delta]}{r(p+\delta)}, \frac{K\epsilon D[r(p+\delta)-\epsilon D\delta]}{r(p+\delta)^2}, 0, 0)$ which exists if $0 < \epsilon D < \frac{r(p+\delta)}{\delta}$, cancer state $(0, 0, \frac{L[s(q+\delta)-D\delta]}{s(q+\delta)}, \frac{LD[s(q+\delta)-D\delta]}{s(q+\delta)^2})$ which exists if $0 < D < \frac{s(q+\delta)}{\delta}$. They have also attempted to get the criteria for the stability of the origin, the cure state and the tumor state.

In another paper, Freedman and Pinho investigated the system (1.3) in absence of cancer cells and radiated cancer cells

$$\begin{cases} \dot{u} = uf(u) - \epsilon D(t) + p(t)v \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \end{cases} \tag{1.10}$$

with $f(t,u) = r(1-u/K)$ under three scenarios(see [11]. That is, the model being considered was

$$\begin{cases} \dot{u} = ru(1-u/K) - \epsilon D(t)u + p(t)v \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \end{cases} \tag{1.11}$$

Three scenarios are constant dosage, decaying dosage and periodic dosage. That is, $D(t) = D_0$, D_0 is a positive constant, $D(t) = D_0e^{-\alpha t}$ and $D(t) = \Delta + \epsilon D_1(t), D_1(t+T) = D_1(t)$, perturbed periodic.

Freedman and Pinho obtained some precise criteria for persistence under constant scenario and periodically perturbed scenario.

In a more recent paper ([9]), Freedman and Belostotski have discussed (1.3) in absence of v and y using perturbation techniques. More precisely, they considered

$$\begin{cases} \dot{u} = ru(1-\frac{u}{K_1}) - \beta_1 uv - \epsilon\eta_1(t,u,v) \\ \dot{v} = rv(1-\frac{v}{K_2}) - \beta_2 uv - \eta_2(t,u,v) \end{cases} \tag{1.12}$$

under the following scenarios:

i. $\eta_1(t,u,v) = \gamma_1, \eta_2(t,u,v) = \gamma_2$,

ii. $\eta_1(t,u,v) = \gamma_1 v, \eta_2(t,u,v) = \gamma_2 v$,

iii. $\eta_1(t,u,v) = \gamma_1 \frac{v}{u}, \eta_2(t,u,v) = \gamma_2 \frac{v}{u}$,

iv. $\eta_1(t,u,v) = \gamma_1$,

$$\eta_2(t,u,v) = \begin{cases} \gamma, nT \leq t < nT + L \\ 0, nT + L \leq t < (n+1)T, n = 0, 1, 2, ... \end{cases} \tag{1.13}$$

Equilibriums and the local stability were discussed under each case. Numerical results were presented as well. In particular, the periodic solution and its stability is preserved under small perturbation. This did not take critical cases into account.

In this dissertation, we investigate the asymptotic behavior of system (1.3) in a generalized setting, namely, with $f(t,u) < 0$ and $g(t,u) < 0$ for $u \gg 1$, and $f_u(t,u) < 0$ and $g_u(t,u) < 0$ for $u \geq 0$. We first study the asymptotic dynamics of (1.3) in the absence of cancer cells, that is, the following system,

$$\begin{cases} \dot{u} = uf(t,u) - \epsilon D(t)u + p(t)v \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v. \end{cases} \tag{1.14}$$

- (In the absence of cancer cells).

  •• *If* $(0,0)$ *is a stable solution of* (1.14), *then it is globally stable (see Theorem 4.2.1).*

  •• *If* $(0,0)$ *is an unstable solution of* (1.14), *then there exists a unique periodic positive solution* $(u^*, v^*) \in Z^{++}$ *which is globally asymptotically stable on* $\mathbb{R}^2 \backslash (0,0)$ *(see Theorem 4.2.2).*

  Among others, we prove

- (In the general case)

  - •• *Any solution* $(u(t), v(t), x(t), y(t))$ *of* (1.3) *in* $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *converges to a time periodic positive solution* $(u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t))$ *(see Theorem 4.2.3)*

  - •• *There is a unique positive periodic solution of* (1.3) *provided that* $p(t)$*,* $q(t)$*,* $a(t)$*, and* $b(t)$ *are sufficiently small (see Theorem 4.2.4).*

The above results extend many existing results in literature for time independent cancer models to time periodic ones.

1.4 Outline

The rest of this thesis is organized as following. In Chapter 2, we investigate the dynamics of two species competition systems of ODEs with sources. We explore the dynamics of competition systems with nonlocal dispersal and inhomogeneous boundary conditions in Chapter 3. In Chapter 4, we study the dynamics of time periodic competition systems modeling the competition of normal and cancer cells. The dissertation is ended with conclusions and open problems in Chapter 5.

Chapter 2

Systems of Two Species Competition Systems of ODEs with Immigration or Sources

In this chapter, we study the asymptotic dynamics, in particular, the uniqueness of coexistence states of two species competition systems of ODEs in the form

$$\begin{cases} u_t = u(a_1(t) - b_1(t)u - c_1(t)v) + d_1(t) \\ v_t = v(a_2(t) - b_2(t)u - c_2(t)v) + d_2(t), \end{cases} \tag{2.1}$$

where $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, and $d_i(\cdot)$ $(i = 1, 2)$ are continuous periodic functions with period T. We first present in section 2.1 some basic properties of (2.1). In section 2.2, we investigate the uniqueness of positive coexistence states of (2.1) in the case that the coefficients are time independent. We study the uniqueness of coexistence states of (2.1) in the time periodic case in section 2.3.

Throughout this chapter, we let

$$X = \mathbb{R} \times \mathbb{R},$$

$$X^+ = \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$X^{++} = \mathrm{Int}(X^+).$$

For (u_1, v_1), $(u_2, v_2) \in X$, we define

$$(u_1, u_2) \leq_1 (\ll_1)(u_2, v_2) \quad \text{if} \quad (u_2 - u_1, v_2 - v_1) \in X^+ (\in X^{++})$$

and

$$(u_1, v_1) \leq_2 (\ll_2)(u_2, v_2) \quad \text{if} \quad (u_2 - u_1, v_1 - v_2) \in X^+ (\in X^{++}).$$

We assume

(HA) *$a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, and $d_i(\cdot)$ $(i = 1, 2)$ are continuous periodic functions with period T and* $\inf_{t\in\mathbb{R}} b_i(t) > 0$, $\inf_{t\in\mathbb{R}} c_i(t) > 0$, $\inf_{t\in\mathbb{R}} d_i(t) > 0$ *$(i = 1, 2)$.*

2.1 Basic properties

In this section, we present some basic properties of (2.1).

For given $(u_0, v_0) \in X$, let $(u(t; u_0, v_0), v(t; u_0, v_0))$ be the solution of (2.1) with $(u(0; u_0, v_0), v(0; u_0, v_0)) = (u_0, v_0)$. A given differentiable function $(u(t), v(t))$ on an interval I is called a supersolution of (2.1) on I if

$$\begin{cases} u_t \geq u(a_1(t) - b_1(t)u - c_1(t)v) + d_1(t) \\ v_t \leq v(a_2(t) - b_2(t)u - c_2(t)v) + d_2(t) \end{cases}$$

for any $t \in I$, and is called a subsolutuion of (2.1) on I if

$$\begin{cases} u_t \leq u(a_1(t) - b_1(t)u - c_1(t)v) + d_1(t) \\ v_t \geq v(a_2(t) - b_2(t)u - c_2(t)v) + d_2(t) \end{cases}$$

for any $t \in I$.

Proposition 2.1. *(1) Suppose that $(u_1(t), v_1(t))$ is a subsolution of (2.1) on $[0, \infty)$ and $(u_2(t), v_2(t))$ is a supersolution of (2.1) on $[0, \infty)$. If $(u_i(t), v_i(t)) \in X^+$ $(i = 1, 2)$ and $(u_1(0), v_1(0)) \leq_2 (u_2(0), v_2(0))$, then $(u_1(t), v_1(t)) \leq_2 (u_2(t), v_2(t))$ for all $t \geq 0$.*

(2) For given $(u_0, v_0) \in X^+$, $(u(t; u_0, v_0), v(t; u_0, v_0))$ exists for all $t \geq 0$ and $(u(t; u_0, v_0), v(t; u_0, v_0)) \in X^+$ for all $t \geq 0$.

Proof. It follows from standard comparison principle for two species competition systems of ordinary differential equations (see [34]). □

Proposition 2.2. *Assume (HA).*

(1) There is a unique stable time periodic positive solution $u^*(t)$ of

$$u_t = u(a_1(t) - b_1(t)u) + d_1(t). \tag{2.2}$$

(2) There is a unique stable time periodic positive solution $v^*(t)$ of

$$v_t = v(a_2(t) - c_2(t)v) + d_2(t). \tag{2.3}$$

(3) $(0, v^*(t))$ is a subsolution of (2.1) and $(u^*(t), 0)$ is a supersolution of (2.1).

Proof. (1) Let $u(t; u_0)$ be the solution of (2.2) with $u(0; u_0) = u_0$. Note that for any $M \geq \frac{a_{1M}}{b_{1L}} + \left(\frac{d_{1M}}{b_{1L}}\right)^{1/2}$, then $u^+ = M$ is a supersolution and $u^- = 0$ is a subsolution of (2.2), respectively.

Notice that $u(T; 0) > 0$. This implies that

$$u(nT; 0) < u((n+1)T, 0).$$

Let $u^*_- = \lim_{n\to\infty} u(nT; 0)$, then $u^-(t) := u(t; u^*_-)$ is a periodic solution. Moreover $lim_{t\to\infty} u(t; u_0) = u^*_-(t)$ for $u_0 \leq u^-(0)$.

Similarly, we can get $u^*_+ = \lim_{n\to\infty} u(nT; M)$ and $u^+(t) := u(t; u^*_+)$ is a periodic solution. Moreover, $\lim_{t\to} u(t; u_0) = u^+(t)$ for $u_0 \geq u^*_+$.

It then suffices to prove that $u^+(t) \equiv u^-(t)$. Let

$$a^-(t) = a_1(t) - b_1(t)u^-(t)$$

and

$$a^+(t) = a_1(t) - b_1(t)u^+(t).$$

Then

$$a^-(t) \geq a^+(t).$$

Note that $u = u^-(t)$ is a periodic solution of

$$\dot{u} = a^-(t)u + d_1(t)$$

and $u = u^+(t)$ is a periodic solution of

$$\dot{u} = a^+(t)u + d_1(t).$$

Then by Floquet theory for periodic ordinary differential equations, we must have

$$\int_0^T a^-(t)dt < 0 \quad \text{and} \quad \int_0^T a^+(s)ds < 0.$$

Note that $u = u^+(t) - u^-(t)$ is a periodic solution of

$$\dot{u} = a^+(t)u + (a^+(t) - a^-(t))u^-(t).$$

We then have

$$0 \leq u^+(t) - u^-(t) = e^{\int_{t_0}^t a^+(s)ds}(u^+(t_0) - u^-(t_0)) + \int_{t_0}^t e^{\int_s^t a^+(\tau)d\tau}(a^+(s) - a^-(s))u^-(s)ds.$$

Let $t_0 \to -\infty$, we have

$$0 \leq \int_{-\infty}^t e^{\int_s^t a^+(\tau)d\tau}(a^+(s) - a^-(s))u^-(s)ds \leq 0$$

for any $t \in \mathbb{R}$. Therefore, we must have $u^+(t) \equiv u^-(t)$ and (2.2) has a unique positive periodic solution $u^*(\cdot)$.

(2). By the similar arguments as in (1), we can prove that (2.3) has a unique positive periodic solution $(0, v^*(\cdot))$.

(3). It is easy to see. In fact, since $u(a_1(t) - b_1(t)u) + d_1(t) = c_1 uv$, $(u^*, 0)$ is a supersolution of (2.1). Similarly, $(0, v^*)$ is a subsolution of (2.1). □

Proposition 2.3. *(1) There is $\delta^* > 0$ such that for any $(u_0, v_0) \in X^+$, there is $T(u_0, v_0) > 0$ such that*

$$(\delta^*, \delta^*) \leq u(t; u_0, v_0), v(t; u_0, v_0)) \leq (u^*(t), v^*(t))$$

for $t \geq T(u_0, v_0) > 0$.

(2) There is a time periodic positive solution $(u^{**}(t), v^{**}(t))$ of (2.1).

Proof. (1) This can be proved similar to Proposition 3.2.3 in the next chapter.

(2) It is a special case of Lemma 2.3.1. □

2.2 Uniqueness of coexistence in the time independent case

In this section, we investigate the uniqueness of coexistence states of (2.1) in the case that the coefficients are time independent, that is,

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v) + d_1 \\ v_t = v(a_2 - b_2 u - c_2 v) + d_2 \end{cases} \tag{2.4}$$

where a_i, b_i, c_i, d_i are constants, $i = 1, 2$. By Proposition 2.3, (2.4) has a positive coexistence state $(u^{**}, v^{**}) (\in X^{++})$. The main results of this section are stated in the following three theorems, which show that (2.4) has a unique positive coexistence state provided that the competition is weak or the sources are large. In the rest of this section, we assume that a_i, b_i, c_i, d_i are positive constants, unless otherwise specified.

Theorem 2.2.1 (Weak competition)**.** *If* $\frac{b_1}{c_1} \geq \frac{b_2}{c_2}$, (2.4) *always has a unique positive stationary solution* $(u^{**}, v^{**}) \in (0, \infty) \times (0, \infty)$.

Theorem 2.2.2 (Strong competition)**.** *If* $\frac{b_1}{c_1} < \frac{b_2}{c_2}$, *then* (2.4) *has a unique positive stationary solution* $(u^{**}, v^{**}) \in (0, \infty) \times (0, \infty)$ *provided that*

$$\sqrt{d_2}\left(\frac{c_2}{b_2}\sqrt{\frac{1}{Ab_2}} - \sqrt{\frac{A}{b_2}}\right) + \sqrt{d_1}\sqrt{\frac{c_2}{b_1 b_2 A}} > \frac{a_2}{b_2}, \tag{2.5}$$

where $A = \frac{c_1}{b_1} - \frac{c_2}{b_2}$, *or*

$$\sqrt{d_1}\left(\frac{b_1}{c_1}\sqrt{\frac{1}{c_1 B}} - \sqrt{\frac{B}{c_1}}\right) + \sqrt{d_2}\sqrt{\frac{b_1}{c_1 c_2 B}} > \frac{a_1}{c_1} \tag{2.6}$$

where $B = \frac{b_2}{c_2} - \frac{b_1}{c_1}$.

Remark 2.2.1. *(1) For fixed* a_i, b_i, c_i *and* d_2, *i=1, 2,* (2.5) *holds if* d_1 *is large enough; for fixed* a_i, b_i, c_i *and* d_1, *i=1, 2,* (2.6) *holds if* d_2 *is large enough.*

(2) If $\frac{b_2}{2c_2} < \frac{b_1}{c_1}$, *then* $\left(\frac{c_2}{b_2}\sqrt{\frac{1}{Ab_2}} - \sqrt{\frac{A}{b_2}}\right) > 0$ *and* $\left(\frac{b_1}{c_1}\sqrt{\frac{1}{c_1 B}} - \sqrt{\frac{B}{c_1}}\right) > 0$. *Therefore, by Theorem 2.2.2, as long as* $\frac{b_2}{2c_2} < \frac{b_1}{c_1}$, *sufficiently large* d_1 *or* d_2 *can guarantee the uniqueness.*

Theorem 2.2.3 (Large immigration)**.** *There is* $d^* > 0$ *such that, if* $d_1 \geq d^*$ *or* $d_2 \geq d^*$, *then* (2.4) *has a unique positive stationary solution* $(u^{**}, v^{**}) \in (0, \infty) \times (0, \infty)$.

Remark 2.2.2. *(1) If* $\frac{b_2}{2c_2} < \frac{b_1}{c_1}$, *Theorem 2.2.2 implies Theorem 2.2.3.*

(2) By standard theory for two species competition systems of ODEs, for any $(u_0, v_0) \in X^+$, $(u(t; u_0, v_0), v(t; u_0, v_0))$ *converges to an equilibrium solution (see [34]). By Proposition 3.2.3, under the conditions of Theorem 2.2.1 or Theorem 2.2.2 or Theorem 2.2.3, for any* $(u_0, v_0) \in X^+$,

$$(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^{**}, v^{**}) \to 0$$

as $t \to \infty$.

Proof of Theorem 2.2.1. Let (u^{**}, v^{**}) be a positive stationary solution of (2.4), i.e. it satisfies

$$\begin{cases} u^{**}(a_1 - b_1 u^{**} - c_1 v^{**}) + d_1 = 0 \\ v^{**}(a_2 - b_2 u^{**} - c_2 v^{**}) + d_2 = 0 \end{cases} \tag{2.7}$$

We view $u(a_1 - b_1 u - c_1 v) + d_1 = 0$ and $v(a_2 - b_2 u - c_2 v) + d_2 = 0$ as two different functions,

$$v = \frac{a_1 - b_1 u}{c_1} + \frac{d_1}{c_1 u} \quad \text{for} \quad u > 0 \tag{2.8}$$

and

$$u = \frac{a_2 - c_2 v}{b_2} + \frac{d_2}{b_2 v} \quad \text{for} \quad v > 0. \tag{2.9}$$

We are going to prove that their graphs have only one intersection in the first quadrant, i.e., (u^{**}, v^{**}) is unique. Since the existence of the intersection follows from the coexistence. We only need to rule out the possibility of having more than one intersection by restricting the number of intersections of the corresponding first derivatives to zero.

Notice that the function in (2.8) gives

$$\frac{dv_1}{du_1} = -\frac{b_1}{c_1} - \frac{d_1}{c_1 u_1^2}$$

the function in (2.9) gives

$$\frac{du_2}{dv_2} = -\frac{c_2}{b_2} - \frac{d_2}{b_2 v_2^2}.$$

It is easy to see both $\frac{dv_1}{du_1}$and $\frac{dv_2}{du_2}$ are increasing functions. Note that $\frac{dv_1}{du_1}$ is bounded above by $-\frac{b_1}{c_1}$, $\frac{dv_2}{du_2}$ is bounded below by $-\frac{b_2}{c_2}$ and bounded above by 0. If $\frac{b_1}{c_1} \geq \frac{b_2}{c_2}$, then the graphs of the first derivatives can not have intersections. Therefore, (2.7) can have at most one solution. $\square$

Proof of Theorem 2.2.2. We let u_{01} be the positive solution of

$$\frac{dv_1}{du_1} = -\frac{b_1}{c_1} - \frac{d_1}{c_1 u_1^2} = -\frac{b_2}{c_2}$$

and v_{01} be the corresponding v value when $u = u_{01}$. We have

$$u_{01} = \sqrt{\frac{d_1}{c_1 B}}, \quad v_{01} = \frac{a_1 - b_1 u_{01}}{c_1} + \frac{d_1}{c_1 u_{01}}.$$

Let v_{02} be the positive solution of

$$\frac{du_2}{dv_2} = -\frac{c_2}{b_2} - \frac{d_2}{b_2 v_2^2} = -\frac{c_1}{b_1}$$

and u_{02} be the corresponding u value when $v = v_{02}$. We have

$$v_{02} = \sqrt{\frac{d_2}{b_2 A}}, \quad u_{02} = \frac{a_2 - c_2 v_{02}}{b_2} + \frac{d_2}{b_2 v_{02}}.$$

Observe that, if $u_{01} > u_{02}$, the graph of $\frac{du_2}{dv_2}$ does not intersect with the graph of $\frac{du_1}{dv_1}$. Observe also that $u_{01} > u_{02}$ is equivalent to

$$\begin{aligned}\sqrt{\frac{d_1}{c_1 B}} = \sqrt{\frac{d_1 c_2}{b_1 b_2 A}} &> \frac{a_2 - c_2 v_{02}}{b_2} + \frac{d_2}{b_2 v_{02}} \\ &= \frac{a_2 - c_2 \sqrt{\frac{d_2}{b_2 A}}}{b_2} + \frac{d_2}{b_2}\sqrt{\frac{b_2 A}{d_2}},\end{aligned}$$

which is equivalent to

$$\frac{a_2}{b_2} < \sqrt{d_1}\sqrt{\frac{c_2}{b_1 b_2 A}} + \sqrt{d_2}\Big(\frac{c_2}{b_2}\sqrt{\frac{1}{b_2 A}} - \sqrt{\frac{A}{b_2}}\Big).$$

We have that, if (2.5) holds, (2.7) has only one positive solution.

Similarly, if $v_{01} < v_{02}$, the graph of $\dfrac{du_2}{dv_2}$ does not intersect with the graph of $\dfrac{du_1}{dv_1}$. Observe that $v_{01} < v_{02}$ is equivalent to

$$\sqrt{\frac{d_2}{b_2 A}} = \sqrt{\frac{b_1 d_2}{c_1 c_2 B}} > \frac{a_1 - b_1 u_{01}}{c_1} + \frac{d_1}{c_1 u_{01}}$$
$$= \frac{a_1 - b_1\sqrt{\frac{d_1}{c_1 B}}}{c_1} + \frac{d_1}{c_1}\sqrt{\frac{c_1 B}{d_1}},$$

which is equivalent to

$$\frac{a_1}{c_1} < \sqrt{d_1}\Big(\frac{b_1}{c_1}\sqrt{\frac{1}{c_1 B}} - \sqrt{\frac{B}{c_1}}\Big) + \sqrt{d_2}\sqrt{\frac{b_1}{c_1 c_2 B}}.$$

We have that, if (2.6) holds, then (2.7) has only one positive solution. □

Proof of Theorem 2.2.3. First of all, by Theorem 2.2.1, if $\frac{b_1}{c_1} \geq \frac{b_2}{c_2}$, then for any $d_1 > 0$ and $d_2 > 0$, (2.7) has only one positive solution. Hence we only need to prove the case $\frac{b_1}{c_1} < \frac{b_2}{c_2}$

Throughout the rest of the proof, we assume that $\frac{b_1}{c_1} < \frac{b_2}{c_2}$. Suppose that $(u^{**}, v^{**}) \in X^{++}$ is a solution of (2.7).

We first prove that there is $d^* > 0$ such that for any $d_1 \geq d^*$ and $d_2 > 0$, (2.1) has a unique positive solution (u^{**}, v^{**}).

To do so, solving $v > 0$ from $v(a_2 - b_2 u - c_2 v) + d_2 = 0$, we have

$$v = \frac{(a_2 - b_2 u) + \sqrt{(a_2 - b_2 u)^2 + 4c_2 d_2}}{2c_2}.$$

Plugging this v into $u(a_1 - b_1 u - c_1 v) + d_1 = 0$, we get

$$u\Big(a_1 - b_1 u - c_1\frac{(a_2 - b_2 u) + \sqrt{(a_2 - b_2 u)^2 + 4c_2 d_2}}{2c_2}\Big) + d_1 = 0. \tag{2.10}$$

Note that u^{**} must be a solution of (2.10). To prove (2.7) has a unique solution (u^{**}, v^{**}) in $(0,\infty) \times (0,\infty)$, it suffices to prove that (2.10) has a unique solution $u^{**} \in (0,\infty)$.

Rewrite (2.10) as

$$a_1u - b_1u^2 - \frac{c_1}{2c_2}(a_2 - b_2u)u + d_1 = \frac{c_1\sqrt{(a_2-b_2u)^2+4c_2d_2}}{2c_2}u. \tag{2.11}$$

This implies that

$$\big(a_1u - b_1u^2 - \frac{c_1}{2c_2}(a_2 - b_2u)u + d_1\big)^2 = \frac{c_1^2\big((a_2-b_2u)^2+4c_2d_2\big)}{4c_2^2}u^2. \tag{2.12}$$

Let

$$\alpha = \frac{c_1^2b_2^2}{4c_2^2} - \big(b_1 - \frac{c_1b_2}{2c_2}\big)^2,$$

$$\beta = 2a_1b_1 - \frac{a_1b_2c_1 + a_2c_1b_1}{c_2},$$

and

$$\gamma(d_1,d_2) = \frac{a_2^2c_1^2 + 4c_1^2c_2d_2}{4c_2^2} - \big(a_1 - \frac{a_2c_1}{2c_2}\big)^2 + 2d_1\big(b_1 - \frac{c_1b_2}{2c_2}\big).$$

Then (2.12) can be written as

$$\alpha u^4 + \beta u^3 + \gamma(d_1,d_2)u^2 = (2a_1 - \frac{a_2c_1}{c_2})d_1u + d_1^2. \tag{2.13}$$

It then suffices to prove that (2.13) has only one solution $u^{**} \in (0,\infty)$ when $d_1 \gg 1$.

By $\frac{b_1}{c_1} < \frac{b_2}{c_2}$, we have $\alpha > 0$. Let

$$f(u) = \alpha u^4 + \beta u^3 + \gamma(d_1,d_2)u^2$$

and

$$g(u) = (2a_1 - \frac{a_2c_1}{c_2})d_1u + d_1^2.$$

It suffices to prove that the graph of f and g only intersects at one point in the half plane $u > 0$ provided that $d_1 \gg 1$.

We first prove a claim.

Claim. *Suppose that the graph of* $f(u^{**}) = g(u^{**})$ *for some* $u^{**} > |\frac{\beta}{2\alpha}|$. *Then*

$$f^{'}(u^{**}) > 2g^{'}(u^{**}).$$

Proof of the claim. Observe that

$$\begin{aligned} g^{'}(u^{**}) &= (2a_1 - \frac{a_2c_1}{c_2})d_1 \\ &= \frac{g(u^{**})}{u^{**}} - \frac{d_1^2}{u^{**}} \end{aligned}$$

and

$$\begin{aligned} f^{'}(u^{**}) &= 4\alpha(u^{**})^3 + 3\beta(u^{**})^2 + 2\gamma(d_1, d_2)u^{**} \\ &= 2\alpha(u^{**})^3 + \beta(u^{**})^2 + \frac{2}{u^{**}}\big(\alpha(u^{**})^4 + \beta(u^{**})^3 + \gamma(d_1, d_2)(u^{**})^2\big) \\ &= 2\alpha(u^{**})^2 + \beta(u^{**})^2 + \frac{2g(u^{**})}{u^{**}} \\ &= 2\alpha(u^{**})^3 + \beta(u^{**})^2 + 2g^{'}(u^{**}) + \frac{d_1^2}{u^{**}}. \end{aligned}$$

When $u^{**} > |\frac{\beta}{2\alpha}|$, we have

$$2\alpha(u^{**})^3 + \beta(u^{**})^2 = (u^{**})^2(2\alpha u^{**} + \beta) > 0.$$

It then follows that

$$f^{'}(u^{**}) > 2g^{'}(u^{**}).$$

This proves the claim.

We now divide the proof into two cases.

Case I. $\gamma(d_1, d_2) \geq 0$.

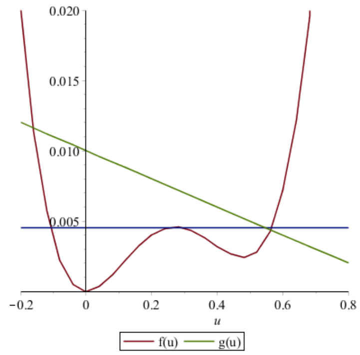

Figure 2.1: Case I ($\gamma(d_1, d_2) > 0$, $\beta < 0$)

Case II. $\gamma(d_1, d_2) < 0$.

Assume that **case I** occurs and $\beta < 0$. For example, $f(u) = u^2(u^2 - u + 0.26)$ (see Figure 2.1.)

Observe that

$$f^{'}(u) = 4\alpha u^3 + 3\beta u^2 + 2\gamma(d_1, d_2)u$$

and

$$f^{''}(u) = 12\alpha u^2 + 6\beta u + 2\gamma.$$

Hence

$$f^{'}(u) > 0, \; f^{''}(u) > 0 \quad \forall \; u > -\frac{3\beta}{4\alpha}$$

and the local maximum of f appears before $-\frac{3\beta}{4\alpha}$.

Observe also that

$$f(u) = u(\alpha u^3 + \beta u^2 + \gamma(d_1, d_2)u) = u(f^{'}(u) - 3\alpha u^3 - 2\beta u^2 - \gamma(d_1, d_2)u).$$

Hence at $\tilde{u} > 0$ with $f^{'}(\tilde{u}) = 0$,

$$f(\tilde{u}) = \tilde{u}(-3\alpha\tilde{u}^3 - 2\beta\tilde{u}^2 - \gamma(d_1, d_2)\tilde{u}) < \tilde{u}(-3\alpha\tilde{u}^3 - 2\beta\tilde{u}^2).$$

Let

$$M = \max_{u\in[0,\frac{3|\beta|}{4\alpha}]} \{u^3(-3\alpha u - 2\beta)\}.$$

We have

$$f(u) \leq M \quad \text{for} \quad 0 \leq u \leq -\frac{3\beta}{4\alpha}.$$

If $\frac{2a_1}{a_2} > \frac{c_1}{c_2}$ then $g^{'}(u) > 0$. Let $d_1 > \sqrt{M}$, then

$$g(u) > M \quad \text{for} \quad 0 \leq u \leq -\frac{3\beta}{4\alpha}$$

and hence the graphs of $f(u)$ and $g(u)$ do not intersect for $0 \leq u \leq -\frac{3\beta}{4\alpha}$. Since $g^{'}(u)$ is a constant and $f^{'}(u)$ increases in u for $u > -\frac{3\beta}{4\alpha}$, the graphs of $f(u)$ and $g(u)$ in the half plane $u > 0$ intersect at exactly one point.

If $\frac{2a_1}{a_2} \leq \frac{c_1}{c_2}$, $g^{'}(u) \leq 0$. Let $d_1 > \frac{1}{2}A + \sqrt{A^2/4 + M}$, where $A = \frac{3\beta}{4\alpha}(2a_1 - \frac{a_2c_1}{c_2})$. Then $g(-\frac{3\beta}{4\alpha}) > M$ and hence

$$g(u) > M \quad \text{for} \quad 0 \leq u \leq -\frac{3\beta}{4\alpha}.$$

This implies that the graphs of $f(u)$ and $g(u)$ do not intersect for $0 \leq u \leq -\frac{3\beta}{4\alpha}$. Recall that $f^{'}(u) > 0$ for $u > -\frac{3\beta}{4\alpha}$. Hence the graphs of $f(u)$ and $g(u)$ intersect at exactly one point again in the half plane $u > 0$.

Let $d^* = \max\{\sqrt{M}, \frac{1}{2}A + \sqrt{A^2/4 + M}\}$. Then for $d_1 > d^*$ and $d_2 > 0$, (2.1) has a unique positive equilibrium (u^{**}, v^{**}).

Assume that **Case I** occurs and $\beta \geq 0$. The uniqueness can be easily obtained as $f^{'}(u)$ is strictly increasing when $u > 0$ (see Figure 2.2) and $g^{'}(u)$ is a constant. Hence for any $d^* > 0$, any $d_1 \geq d^*$ and $d_2 > 0$, (2.1) has a unique positive equilibrium (u^{**}, v^{**}).

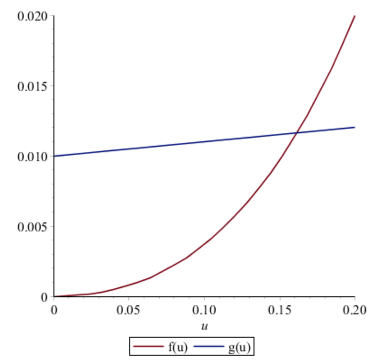

Figure 2.2: Case II ($\gamma(d_1, d_2) > 0$, $\beta \geq 0$)

Assume that **case II** occurs. Figure 2.3 demonstrates the graph of $f(u)$. Observe that

$$f(u) = \alpha u^4 + \beta u^3 + \gamma(d_1, d_2)u^2 < \alpha u^4 + \beta u^3 \quad \forall\ u > 0.$$

Let

$$M = \max\{\alpha u^4 + \beta u^3 \,|\, 0 \leq u \leq |\frac{\beta}{2\alpha}|\}.$$

Let d^* be such that

$$d_1^2 - |2a_1 - \frac{a_2 c_1}{c_2}|d_1 u > M \quad \text{for} \quad 0 \leq u \leq |\frac{\beta}{2\alpha}|$$

for any $d_1 \geq d^*$. Then for $d_1 \geq d^*$ and $d_2 > 0$,

$$f(u) < M \quad \text{and} \quad g(u) > M \quad \text{for} \quad 0 < u \leq |\frac{\beta}{2\alpha}|$$

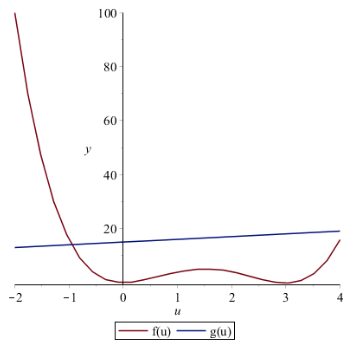

Figure 2.3: Case III ($\gamma(d_1, d_2) < 0$)

and hence the graph of f and the graph of g do not intersect for $0 < u \leq |\frac{\beta}{2\alpha}|$. By the claim, the graph of f and the graph of g intersects at most at one point for $u > |\frac{\beta}{2\alpha}|$. Therefore, for any $d_1 \geq d^*$ and $d_2 > 0$, (2.1) has a unique positive equilibrium (u^{**}, v^{**}).

Similarly, we can prove that there is $d_2^* > 0$ such that if $d_2 \geq d_2^*$, then for any $d_1 > 0$, (2.7) has a unique solution $(u^{**}, v^{**}) \in X^{++}$. The theorem is thus proved. $\square$

Remark 2.2.3. *(a) One application of theorems 2.2.1-2.2.3 is when* (2.4) *is replaced with*

$$\begin{cases} u_t = u(a_1 - 1 - b_1 u - c_1 v) + d_1 \\ v_t = v(a_2 - 1 - b_2 u - c_2 v) + d_2 \end{cases} \tag{2.14}$$

where we still assume a_i, b_i, c_i, d_i (i=1, 2) are positive constants. This result will be used when we prove the continuity of positive solutions of nonlocal dispersal system.

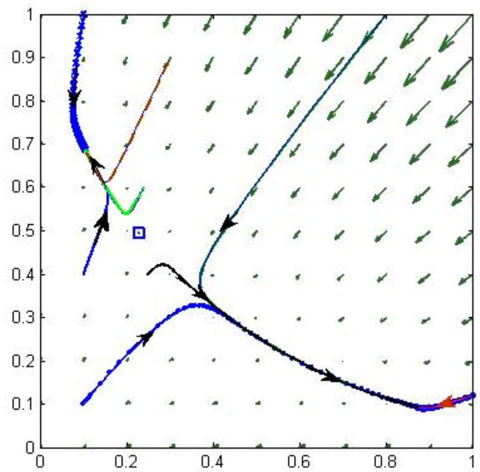

Figure 2.4: Three steady states

(b) If d_1 or d_2 is not big enough, Theorem 2.2.3 may not hold true under assumption $\frac{b_1}{c_1} < \frac{b_2}{c_2}$. For example,

$$\begin{cases} u_t = u(1 - u - 2v) + 0.05 \\ v_t = v(0.75 - 2u - v) + 0.1 \end{cases}$$

(See Figure 2.4). Three positive stationary solutions are

$u \approx 0.104806 + 2.99787 \times 10^{-17}i,\ v \approx 0.686132 + 3.45398 \times 10^{-17}$

$u \approx 0.228172 - 2.02215 \times 10^{-17}i,\ v \approx 0.495481 + 7.65668 \times 10^{-17}$

$u \approx 0.873579 + 1.18535 \times 10^{-18}i,\ v \approx 0.0918285 + 1.78183 \times 10^{-17}$

2.3 Uniqueness of coexistence in the time periodic case

In this section, we explore the uniqueness of coexistence states of (2.1) in the case that $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, and $d_i(\cdot)$ are periodic in t with period T. Let $a_{iL} = \min_{t\in[0,T]} a_i(t)$ and $a_{iM} = \max_{t\in[0,T]} a_i(t)$ for $i = 1, 2$. b_{iL}, b_{iM}, c_{iL}, and c_{iM} are defined similarly.

The main result of this section is stated in the following theorem.

Theorem 2.3.1. *Assume* $\sup_{t\in\mathbb{R}} d_i(t) > 0$ *and*

$$\frac{b_{1L}}{b_{2M}} > \frac{c_{1M}}{c_{2L}}.$$

Then (2.1) *has a unique time periodic positive solution.*

To prove the above theorem, we first prove a lemma.

Lemma 2.3.1. *There are positive periodic solutions* $(u^+(\cdot), v^+(\cdot))$ *and* $(u^-(\cdot), v^-(\cdot))$ *of* (2.1) *such that*

$$u^-(t) \leq u^+(t),\ \ v^-(t) \geq v^+(t) \quad \forall\ t \in \mathbb{R}.$$

Moreover, for any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *with* $(u^+(0), v^+(0) \leq_2 (u_0, v_0)$,

$$(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^+(t), v^+(t)) \to 0$$

as $t \to \infty$, *and for any* $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *with* $(u_0, v_0) \leq_2 (u^-, v^-)$,

$$(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^-(t), v^-(t)) \to 0$$

as $t \to \infty$.

Proof. First note that

$$(u(T; u^*(0), 0), v(T; u^*(0), 0)) \ll_2 (u^*(0), 0).$$

This implies that

$$(u(nT; u^*(0), 0), v(nT; u^*(0), 0)) \ll_2 (u((n+1)T; u^*(0), 0), v((n+1)T; u^*(0), 0)).$$

Let $(u_0^+, v_0^+) = \lim_{n\to\infty}(u(nT; u^*(0), 0), v(nT; u^*(0), 0))$. Then

$$(u^+(t), v^+(t)) := (u^+(t; u^*(0), 0), v^+(t; u^*(0), 0))$$

is a periodic solution of (2.1). For any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(u_0, v_0) \geq_2 (u^+(0), v^+(0))$,

$$(u^+(t), v^+(t)) \leq_2 (u(t; u_0, v_0), v(t; u_0, v_0))$$

for all $t \geq 0$. Note that $u(t; u_0, v_0)$ satisfies

$$\dot{u} = u(a_1(t) - b_1(t)u - c_1(t)v(t; u_0, v_0)) + d_1(t) < u(a_1(t) - b_1(t)u) + d_1(t).$$

Then there is $N^* \geq 1$ such that

$$u(NT; u_0, v_0) \leq u^*(0)$$

for $n \geq N^*$. This implies that

$$(u(NT; u_0, v_0), v(NT; u_0, v_0)) \leq_2 (u^*(0), 0)$$

for $N \geq N^*$. It then follows that

$$(u^+(NT+t), v^+(NT+t)) \leq_2 (u(NT+t; u_0, v_0), v(NT+t; u_0, v_0)) \leq_2 (u(t; u^*(0), 0), v(t; u^*(0), 0))$$

for any $N \geq N^*$ and then

$$\lim_{t\to\infty}[(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^+(t), v^+(t))] = 0.$$

Similarly, we can prove the existence of the limit

$$(u_0^-, v_0^-) = \lim_{n\to\infty}(u(nT; 0, v^*(0)), v(nT; 0, v^*(0))$$

and that $(u^-(t), v^-(t)) := (u(t; u_0^-, v_0^-), v(t; u_0^-, v_0^-))$ is a periodic solution of (2.1) satisfying that

$$\lim_{t\to\infty}[(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^-(t), v^-(t))] = 0$$

for any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(u_0, v_0) \leq_2 (u^-(0), v^-(0))$. □

Proof of Theorem 2.3.1. It suffices to prove that

$$(u^-(t), v^-(t)) \equiv (u^+(t), v^+(t)).$$

Assume that

$$(u^-(t), v^-(t)) \not\equiv (u^+(t), v^+(t)).$$

Then we have

$$u^-(t) < u^+(t), \;\; v^-(t) > v^+(t) \quad \forall\, t \in \mathbb{R}.$$

Observe that

$$\frac{d}{dt}\ln u^\pm(t) = a_1(t) - b_1(t)u^\pm(t) - c_1(t)v^\pm(t) + \frac{d_1(t)}{u^\pm(t)}$$

and

$$\frac{d}{dt}\ln v^\pm(t) = a_2(t) - b_2(t)u^\pm(t) - c_2(t)v^\pm(t) + \frac{d_2(t)}{v^\pm(t)}.$$

Hence

$$\frac{d}{dt}\ln\frac{u^-(t)}{u^+(t)} = b_1(t)[u^+(t) - u^-(t)] + c_1(t)[v^+(t) - v^-(t)] + d_1(t)\Big[\frac{1}{u^-(t)} - \frac{1}{u^+(t)}\Big]$$

and

$$\frac{d}{dt}\ln\frac{v^-(t)}{v^+(t)} = b_2(t)[u^+(t)-u^-(t)] + c_2(t)[v^+(t)-v^-(t)] + d_2(t)\Big[\frac{1}{v^-(t)} - \frac{1}{v^+(t)}\Big].$$

It then follows that

$$0 = \int_0^T \frac{d}{dt}\ln\frac{u^-(t)}{u^+(t)}dt > \int_0^T \Big[b_1(t)[u^+(t)-u^-(t)] + c_1(t)[v^+(t)-v^-(t)]\Big]dt$$

and

$$0 = \int_0^T \frac{d}{dt}\ln\frac{v^-(t)}{v^+(t)}dt < \int_0^T \Big[b_2(t)[u^+(t)-u^-(t)] + c_2(t)[v^+(t)-v^-(t)]\Big]dt.$$

This implies that

$$b_{1L}\int_0^T [u^+(t)-u^-(t)]dt < c_{1M}\int_0^T [v^-(t)-v^+(t)]dt$$

and

$$b_{2M}\int_0^T [u^+(t)-u^-(t)]dt > c_{1L}\int_0^T [v^-(t)-v^+(t)]dt.$$

Hence

$$\frac{b_{1L}}{c_{1M}} < \frac{\int_0^T [v^-(t)-v^+(t)]dt}{\int_0^T [u^+(t)-u^-(t)]dt} < \frac{b_{2M}}{c_{2L}}.$$

This is a contradiction. The theorem is thus proved. □

Chapter 3

Competition Systems with Nonlocal Dispersal and Inhomogeneous Boundary Conditions

In this chapter, we study the asymptotic dynamics of the following two species competition systems with nonlocal dispersal and inhomogeneous boundary condition,

$$\begin{cases} u_t(t,x) = \int_{\bar{\Omega}} J(x-y)u(t,y)dy - u(t,x) + h_1(x) \\ \qquad\qquad +u(t,x)(a_1(x) - b_1(x)u - c_1(x)v), \qquad x \in \bar{\Omega} \\ v_t(t,x) = \int_{\bar{\Omega}} J(x-y)v(t,y)dy - v(t,x) + h_2(x) \\ \qquad\qquad +v(t,x)(a_2(x) - b_2(x)u - c_2(x)v), \qquad x \in \bar{\Omega}. \end{cases} \tag{3.1}$$

Throughout this chapter, we let

$$X = C(\bar{\Omega}) \times C(\bar{\Omega}),$$

$$X^+ = \{(u,x) \in X \,|\, u(x) \geq 0,\ v(x) \geq 0 \text{ for } x \in \bar{\Omega}\}$$

and

$$X^{++} = \{(u,v) \in X^+ \,|\, u(x) > 0,\ v(x) > 0 \text{ for } x \in \bar{\Omega}\}.$$

For given (u_1,v_1), $(u_2,v_2) \in X$, we define

$$(u_1,v_1) \leq_1 \ (\ll_1)(u_2,v_2) \quad \text{if} \quad (u_2-u_1, v_2-v_1) \in X^+\,(X^{++}),$$

and

$$(u_1,v_1) \leq_2 \ (\ll_2)(u_2,v_2) \quad \text{if} \quad (u_2-u_1, v_1-v_2) \in X^+\,(X^{++}).$$

We assume

(HB) *$a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, $h_i(\cdot)$ are continuous functions in $x \in \bar{\Omega}$, and* $\inf_{x\in\bar{\Omega}} b_i(x) > 0$, $\inf_{x\in\bar{\Omega}}(c_i(x) > 0$, *$h_1(x) \geq 0$, $h_2(x) \geq 0$ and $h_1(x) \not\equiv 0$, $h_2(x) \not\equiv 0$.*

We first in section 3.1 study the basic properties and asymptotic dynamics of one species nonlocal evolution equations with inhomogeneous boundary. Next, we present some basic properties of (3.1) in section 3.2. We then investigate the existence of continuous coexistence states of (3.1) in section 3.3. In section 3.4, we explore the uniqueness of continuous coexistence states of (3.1).

3.1 One species nonlocal evolution equation

In this section, we study basic properties and asymptotic dynamics of the following one species nonlocal equation,

$$\begin{aligned} u_t = \int_{\bar{\Omega}} & J(x-y)u(t,y)dy - u(t,x) \\ & +u(t,x)f(x,u(t,x)) + h(x) \qquad x \in \bar{\Omega}, \end{aligned} \tag{3.2}$$

where $h(x)$ is continuous in $x \in \bar{\Omega}$, $h(x) \geq 0$, $h(\cdot) \not\equiv 0$, and $f(x,u)$ is continuous in x and C^1 in u, $f(x,u) < 0$ for $u \gg 1$, and $f_u(x,u) < 0$ for $u \geq 0$.

Note that for any $u_0 \in C(\bar{\Omega})$, there is a unique (local) solution $u(t,\cdot;u_0)$ of (3.2) with $u(,\cdot;u_0) = u_0(\cdot)$. We denote $[0, t_{\max}(u_0))$ the maximal existence interval of $u(t,\cdot;u_0)$ for nonnegative time. A function $u(t,x)$ is called a sub-solution (super-solution) of (3.2) on an interval I if

$$\begin{aligned} u_t(t,x) \leq (\geq) \int_{\bar{\Omega}} & J(x-y)u(t,y)dy - u(t,x) \\ & +u(t,x)f(x,u(t,x)) + h(x) \qquad\qquad x \in \bar{\Omega} \end{aligned}$$

for $t \in I$. We define

$$C^+(\bar{\Omega}) = \{u \in C(\bar{\Omega}) \,|\, u(x) \geq 0, \quad x \in \bar{\Omega}\}$$

and

$$C^{++}(\bar{\Omega}) = \{u \in C(\bar{\Omega}) \,|\, u(x) > 0, \quad x \in \bar{\Omega}\}.$$

For $u \in C(\bar{\Omega})$,

$$u \geq 0 \quad \text{if} \quad u \in C^{+}(\bar{\Omega})$$

and

$$u \gg 0 \quad \text{if} \quad u \in C^{++}(\bar{\Omega}).$$

Proposition 3.1.1 (Comparison principle)**.** *(1) Let $u(t,x)$ (resp. $v(t,x)$) be a supersolution (resp. subsolution) of* (3.2) *on $[0,T)$. If $v(0,\cdot) \leq u(0,\cdot)$, then $v(t,\cdot) \leq u(t,\cdot)$ for $t \in [0,T)$.*

(2) Let $u(t,x;u_1), u(t,x;u_2)$ be solutions of (3.2) *with $u(0,x;u_1) = u_1(x)$, $u(0,x;u_2) = u_2(x)$. If $u_1 \leq u_2$, then $u(t,\cdot;u_1) \leq u(t,\cdot;u_2)$ for $t \in [0,t_{\max}(u_1)) \cap [0,t_{\max}(u_2))$. If $u_1 \geq 0$, then $u(t,\cdot;u_1) \geq 0$, for $t \in [0,t_{\max}(u_1))$.*

(3) Let $u(t,x;u_1), u(t,x;u_2)$ be solutions of (3.2) *with $u(0,x;u_1) = u_1(x)$, $u(0,x;u_2) = u_2(x)$. If $u_1 \leq u_2$ and $u_1 \not\equiv u_2$, then $u(t,\cdot;u_1) \ll u(t,\cdot;u_2)$ for $t \in [0,t_{\max}(u_1)) \cap [0,t_{\max}(u_2))$. If $u_1 \geq 0$, then $u(t,\cdot;u_1) \gg 0$, for $t \in [0,t_{\max}(u_1))$.*

Proof. (1) First note that we just need to consider $t \in [0,T_0]$ for any $T_0 \in [0,T)$.

Let $w(t,x) = u(t,x) - v(t,x)$. Then $w(0,\cdot) = u(0,\cdot) - v(0,\cdot) \geq 0$. To prove $w(t,\cdot) \geq 0$ for $t \in [0,T_0]$, we argue by contradiction.

Given $t \in [0,T_0]$. Let $\widetilde{w}(t,x) = e^{-Mt}w(t,x)$, where

$$M = \sup_{\substack{x \in \bar{\Omega} \\ t \in [0,T_0]}} \{\|u(t,x)f(x,u(t,x)) - v(t,x)f(x,v(t,x))\|\} + 1.$$

Note that

$$\widetilde{w}_t = -Me^{-Mt}w(t,x) + e^{-Mt}w_t(t,x) = e^{-Mt}w_t(t,x) - M\widetilde{w}(t,x).$$

Suppose w is negative in some points, then $\widetilde{w}$ is negative at some points. Let $\widetilde{w}(t^*, x^*) = \min_{\substack{x\in\bar{\Omega} \\ t\in[0,T_0]}} \widetilde{w}(t, x)$, then $w_t(t^*, x^*) \leq 0$ and $\widetilde{w}_t(t^*, x^*) \leq 0$. Further

$$\begin{aligned}
&\widetilde{w}_t(t^*, x^*) \\
&= e^{-Mt^*}(u_t(t^*, x^*) - v_t(t^*, x^*)) - M\widetilde{w}(t^*, x^*) \\
&\geq \int_\Omega J(x^* - y)(\widetilde{w}(t^*, y) - \widetilde{w}(t^*, x^*))dy \\
&+e^{-Mt^*}[u(t^*, x^*)f(x^*, u) - v(t^*, x^*)f(x^*, v)] - M\widetilde{w}(t^*, x^*)
\end{aligned}$$

By mean value theorem, there exist $u^* \in C(\bar{\Omega})$ and $u^*(t^*, x^*)$ is in the interior of $(v(t^*, x^*), u(t^*, x^*))$ satisfying $u(t^*, x^*)f(x^*, u) - v(t^*, x^*)f(x^*, v) = (u^*(t^*, x^*)f_u(x^*, u^*) + f(x^*, u^*))(u(t^*, x^*) - v(t^*, x^*))$. Therefore,

$$\begin{aligned}
&\widetilde{w}_t(t^*, x^*) \\
&\geq \int_\Omega J(x^* - y)(\widetilde{w}(t^*, y) - \widetilde{w}(t^*, x^*))dy \\
&+[u^*(t^*, x^*)f_u(x^*, u^*) + f(x^*, u^*) - M]\widetilde{w}(t^*, x^*) \\
&> 0
\end{aligned}$$

since $\widetilde{w}(t^*, x^*) < 0$, $u^*(t^*, x^*)f_u(x^*, u^*) + f(x^*, u^*) - M < 0$ and $\int_\Omega J(x^* - y)(\widetilde{w}(t^*, y) - \widetilde{w}(t^*, x^*))dy \geq 0$. This contradicts with $\widetilde{w}_t(t^*, x^*) \leq 0$. Hence, $v(t, x; v_0, h) \leq u(t, x; u_0, g)$, for $x \in \bar{\Omega}$, $t \in [0, T_0]$. Since T_0 is picked arbitrarily from $(0, T)$, therefore the statement (1) holds.

(2) The first part of (2) statement follows immediately from (1). Since $u \equiv 0$ is a subsolution, thus following statement (1), the second part of (2) also holds.

(3) We first prove the second part of (3). For given $u \in C(\bar{\Omega})$, define

$$(Ku)(x) = \int_{\bar{\Omega}} J(x - y)u(y)dy, x \in \bar{\Omega}.$$

For given $u_1 \in C(\bar{\Omega})$ and $T_0 \in (0, t_{\max}(u_1))$, define

$$l(t,x) = -1 + f(x, u(t,x;u_1)) \quad \text{for} \quad t \in [0, t_{\max}(u_1)),\ x \in \bar{\Omega}$$

and $m \in (0, \infty)$ such that

$$m > -\min_{t \in [0,T_0], x \in \bar{\Omega}} l(t,x).$$

Then $u(t, \cdot; u_1)$ satisfies

$$u_t = (K - mI)u + (m + l(t, \cdot))u + h(\cdot)$$

for $t \in (0, T_0]$. Let

$$T(t) = e^{(K-mI)t} \quad \text{for} \quad t \geq 0.$$

Then

$$u(t, \cdot; u_1) = T(t)u_1 + \int_0^t T(t-s)\big(m + l(s, \cdot)\big)u(t, \cdot; u_1)ds + \int_0^t T(t-s)h(\cdot)ds$$

for $t \in [0, T_0]$. Observe that for any $u_0 \in C(\bar{\Omega})$,

$$e^{(K-mI)t}u_0 = e^{-mt}e^{Kt}u_0$$

and

$$e^{Kt}u_0 = u_0 + tKu_0 + \frac{t^2K^2u_0}{2!} + \cdots + \frac{t^nK^nu_0}{n!} + \cdots .$$

Observe also that if $u_0 \geq 0$, then

$$e^{Kt}u_0 \geq 0 \quad \forall t \geq 0$$

and if $u_0 \geq 0$ and $u_0 \not\equiv 0$, then

$$e^{Kt}u_0 \gg 0 \quad \forall t > 0.$$

By (H2), $h(\cdot) \geq 0$ and $h(\cdot) \not\equiv 0$. By (2), $u(t,\cdot;u_1) \geq 0$ for $t \in [0, t_{\max}(u_1))$. We then have

$$T(t)u_1 \geq 0 \quad \text{for} \quad t \geq 0$$

and

$$T(t)h(\cdot) \gg 0 \quad \text{for} \quad t \geq 0.$$

It then follows that

$$u(t,\cdot;u_1) \gg 0 \quad \text{for} \quad t \in (0, T_0]$$

for any $T_0 \in (0, t_{\max}(u_1))$ and hence

$$u(t,\cdot;u_1) \gg 0 \quad \text{for} \quad t \in (0, t_{\max}(u_1)).$$

Next, we prove the first part of (3). Let

$$u(t,x) = u(t,x;u_2) - u(t,x;u_1) \quad \text{for} \quad t \in [0, t_{\max}(u_1)) \cap [0, t_{\max}(u_2)),\ x \in \bar{\Omega}.$$

Then for any $T_0 \in (0, t_{\max}(u_1)) \cap (0, t_{\max}(u_2))$, $u(t,x)$ satisfies

$$u_t = (K - mI)u + (m + l(t,\cdot))u, \quad t \in [0, T_0]$$

where

$$l(t,x) = -1 + \frac{u(t,x;u_2)f(x,u(t,x;u_2)) - u(t,x;u_1)f(x,u(t,x;u_1))}{u(t,x;u_2) - u(t,x;u_1)}$$

for $t \in [0, T_0]$ and $x \in \bar{\Omega}$, and $m > 0$ is such that

$$m > -\min_{t \in [0,T_0], x \in \bar{\Omega}} l(t,x).$$

Note that $u(0,\cdot) = u_2(\cdot) - u_1(\cdot) \geq 0$ and $u(0,\cdot) \not\equiv 0$. By (2), $u(t,\cdot) \geq 0$ for $t \in [0, t_{\max}(u_1)) \cap [0, t_{\max}(u_2))$. Then following the above arguments,

$$u(t,\cdot) \gg 0 \quad \text{for} \quad t \in (0, T_0]$$

for any $T_0 \in (0, t_{\max}(u_1)) \cap (0, t_{\max}(u_2))$ and hence

$$u(t,\cdot) \gg 0 \quad \text{for} \quad t \in (0, t_{\max}(u_1)) \cap (0, t_{\max}(u_2)).$$

This completes the proof of (3). □

Proposition 3.1.2 (Global existence)**.** *For any $u_0 \in C(\bar{\Omega})$, $u_0 \geq 0$, $u(t,x;u_0)$ exists for all $t \geq 0$.*

Proof. Observe that $u \equiv 0$ is a solution of (3.2) and $u \equiv M$ is a super-solution of (3.2) for any $M \gg 1$. For given $u_0 \in C^+(\bar{\Omega})$, let $M \gg 1$ be such that

$$0 \leq u_0(x) \leq M \quad \forall\, x \in \bar{\Omega}.$$

Then by Proposition 3.1.1,

$$0 \leq u(t,x;u_0) \leq M \quad \forall\, x \in \bar{\Omega},\ \forall t \in [0, t_{\max}(u_0)).$$

It then follows from fundamental theory for ordinary differential equations in Banach space, $t_{\max}(u_0) = \infty$. □

Proposition 3.1.3 (Positive stationary solution)**.** *There is a unique positive equilibrium solution $u^* \in C^{++}(\bar{\Omega})$ of (3.2) which is globally stable in the sense that for any $u_0 \in C^+(\bar{\Omega})$,*

$$u(t,x;u_0) \to u^*(x) \quad \text{as} \quad t \to \infty.$$

To prove 3.1.3, we first prove a lemma. For given $u, v \in C^{++}(\bar{\Omega})$, we define

$$\rho(u,v) = \inf\{\ln\alpha \mid \alpha \geq 1,\ \frac{1}{\alpha}u(\cdot) \leq v(\cdot) \leq \alpha u(\cdot)\}.$$

Lemma 3.1.1. *For any given $u_1, u_2 \in C^{++}(\bar{\Omega})$ with $u_1(\cdot) \not\equiv u_2(\cdot)$, $\rho(u(t,\cdot;u_1), u(t,\cdot;u_2))$ is strictly decreasing in $t > 0$.*

Proof. First, for any $\alpha > 1$ with $\frac{1}{\alpha}u_1(\cdot) \leq u_2(\cdot) \leq \alpha u_1(\cdot)$, by comparison principle,

$$u(t,\cdot;u_2) \leq u(t,\cdot;\alpha u_1)$$

for $t > 0$. Let $\tilde{u}(t,x) = \alpha u(t,x;u_1)$. Then $\tilde{u}(t,x)$ satisfies

$$\begin{aligned}\tilde{u}_t(t,x) &= \int_\Omega J(y-x)\tilde{u}(t,y)dy - \tilde{u}(t,y) + \alpha h(x) + \tilde{u}(t,x)f(u(t,x;u_1)) \\ &> \int_\Omega J(y-x)\tilde{u}(t,y)dy - \tilde{u}(t,y) + h(x) + \tilde{u}(t,x)f(u(t,x;\tilde{u}_1)).\end{aligned}$$

By comparison principle, we have

$$\tilde{u}(t,x) > u(t,x;\alpha u_1) \quad \forall\ x \in \bar{\Omega},\ t > 0.$$

Hence for any $t > 0$,

$$u(t,\cdot;u_2) \ll \alpha u(t,\cdot;u_1)$$

for any $t > 0$.

Similarly, we have

$$\frac{1}{\alpha}u(t,\cdot;u_1) \ll u(t,\cdot;u_2)$$

for any $t > 0$. It then follows that

$$\rho(u(t,\cdot;u_1), u(t,\cdot;u_2)) < \rho(u_1,u_2)$$

for $t > 0$ and then $\rho(u(t,\cdot;u_1),u(t,\cdot;u_2))$ is strictly decreasing in $t > 0$. □

Proof of Proposition 3.1.3. First we show the existence of a positive equilibrium. We argue by super- and sub-solutions methods.

Let $u^+ = M,\ M \gg 1$. Then

$$\int_{\bar{\Omega}} J(x-y)u^+(y)dy - u^+(x) + h(x) + u^+ f(x,u^+) < 0.$$

Hence $u = u^+$ is a supersolution of (3.2). This implies that $u(t,\cdot;u^+) < u^+$ for $0 < t \ll 1$ and $u(t_2 - t_1,\cdot;u^+) \le u^+$, for $t_2 > t_1 > 0$. It follows $u(t_2,\cdot;u^+) = u(t_1,\cdot;u(t_2-t_1;u^+)) \le u(t_1,\cdot;u^+) < u^+$ for any $t_2 > t_1 > 0$. Therefore, there is a bounded measurable function $u_+^* : \bar{\Omega} \to [0,\infty)$ such that $u(t,x;u_0^+) \to u_+^*(x)$ as $t \to \infty$. Note that

$$\begin{aligned} & u(t+s,x;u_0^+) - u(t,x;u_0^+) \\ = & \int_0^s [\int_{\bar{\Omega}} J(x-y)u(t+\tau,y,u_0^+)dy - u(t+\tau,x,u_0^+) + h(x) \\ + & u(t+\tau;u_0^+)f(x,u(t+\tau;u_0^+)]d\tau. \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, letting $t \to \infty$, we have

$$\int_{\bar{\Omega}} J(x-y)u_+^*(y)dy - u_+^*(x) + h(x) + u_+^*(x)f(x,u_+^*(x)) = 0,\ \forall x \in \bar{\Omega}.$$

Next, let $u^- \equiv 0$. By strong comparison principle, $u(t;\cdot u^-) \gg u^-$. Further, we have $u(t_2,\cdot;u^-) \ge u(t_1,\cdot;u^-)$ for any $t_2 > t_1 > 0$. Thus, there exists a bounded measurable function $u_-^* : \bar{\Omega} \to (0,\infty)$ such that $u(t,x;u_0^-) \to u_-^*(x)$ as $t \to \infty$. Similarly,

$$\int_{\bar{\Omega}} J(x-y)u_-^*(y)dy - u_-^*(x) + h(x) + u_-^*(x)f(x,u_-^*(x)) = 0,\ \forall x \in \bar{\Omega}.$$

Observe that there is $\delta_0 > 0$ such that $u_-^*(x) \ge \delta_0$ and $u_-^*(x) \le u_+^*(x)$ for all $x \in \bar{\Omega}$. Thus $u_+^*(x) \ge \delta_0$ for all $x \in \bar{\Omega}$. Further, we prove u_+^* and u_-^* are continuous. Without lost of generalization, we shall prove u_+^* is continuous. The continuity of u_-^* can be obtained in the same way. Let

$$F(x,\alpha) := \int_{\bar{\Omega}} J(x-y)u_+^*(y)dy + h(x) - \alpha + \alpha f(x,\alpha).$$

Then

$$F(x, u_+^*(x)) = 0, \quad \forall x \in \bar{\Omega}.$$

We use Implicit Function Theorem to prove that $u_+^*(x)$ is continuous in x. It is obvious that for any $x_0 \in \bar{\Omega}$, there is $\alpha_0 > 0$, s.t. $F(x_0, \alpha_0) = 0$. It is also obvious that $F(x,\alpha)$ is continuously differentiable. Note that $F_\alpha'(x_0,\alpha_0) \not\equiv 0$. Indeed, $F(x_0,\alpha_0) = \int_{\mathbb{R}^{\mathbf{N}}} J(x-y)u_+^*(y)dy + h(x) - a(x_0)\alpha_0 + \alpha_0 f(x_0,\alpha_0) = 0$ implies $-a(x_0) + f(x_0,\alpha_0) < 0$. Therefore,

$$F_\alpha'(x_0,\alpha_0) = -a(x_0) + \alpha f_\alpha'(x_0,\alpha_0) + f(x_0,\alpha_0) < 0.$$

This implies that $u_+^*(x) \in C(\bar{\Omega})$. Similarly, $u_-^* \in C(\bar{\Omega})$.

Then, we show $u_+^* = u_-^*$. Suppose not, since $u_-^* \le u_+^*$, we assume $u_-^* < u_+^*$. By Lemma 3.1.1, it follows $\rho(u(t,\cdot,u_-^*), u(t,\cdot,u_+^*)) < \rho(u_-^*, u_+^*)$ which contradicts $u_+^*,\ u_-^*$ are stationary. Therefore $u_+^* = u_-^*$.

Finally, by $u_-^* = u_+^*$, for any $u_0 \in C^+(\bar{\Omega})$, $u(t,\cdot;u_0) \to u^*$ as $t \to \infty$, where $u^* = u_+^*(= u_-^*)$. □

3.2 Basic properties of two species competition systems with nonlocal dispersal

In this section, we present some basic properties of (3.1).

Observe that, by general semigroup theory, for any $(u_0, v_0) \in X$, there is a unique (local) solution $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ of (3.1) with $(u(0,\cdot;u_0,v_0), v(0,\cdot;u_0,v_0)) = (u_0(\cdot), v_0(\cdot))$. We denote $[0, t_{\max}(u_0,v_0))$ the maximal existence interval of $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ for nonnegative time.

Definition 3.2.1. *$u(t,x), v(t,x)$ is called a supersolution (subsolution) of* (3.1) *if*

$$\begin{cases} u_t(x,t) \geq (\leq) \int_{\bar{\Omega}} J(x-y)u(y,t)dy - u(x,t) \\ \qquad +h_1(x) + u(x,t)(a_1(x) - b_1(x)u - c_1(x)v), \qquad x \in \bar{\Omega}. \\ v_t(x,t) \leq (\geq) \int_{\bar{\Omega}} J(x-y)v(y,t)dy - v(x,t) \\ \qquad +h_2(x) + v(x,t)(a_2(x) - b_2(x)u - c_2(x)v), \qquad x \in \bar{\Omega}. \end{cases} \tag{3.3}$$

Proposition 3.2.1 (Comparison principle)**.** *(1) If $(0,0) \leq_1 (u_0, v_0)$, then $(0,0) \leq_1 (u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ for $t \in [0, t_{\max}(u_0,v_0))$.*

(2) If $(0,0) \leq_1 (u_i, v_i)$, for $i = 1,2$, and $(u_1,v_1) \leq_2 (u_2,v_2)$, then

$$(u(t,\cdot;u_1,v_1), v(t,\cdot;u_1,v_1)) \leq_2 (u(t,\cdot;u_2,v_2), v(t,\cdot;u_2,v_2))$$

for $t \in [0, t_{\max}(u_1,v_1)) \cap [0, t_{\max}(u_2,v_2))$.

(3) For any $(u_0,v_0) \in X^+$, $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0)) \in Z^{++}$ for $t \in [0, t_{\max}(u_0,v_0))$.

Proof. (1) Note that $u(t,x;u_0,v_0)$ is the solution of

$$u_t(x,t) = \int_{\bar{\Omega}} J(x-y)u(y,t)dy - u(x,t) + h_1(x) + u(x,t)(a_1(x) - b_1(x)u - c_1(x)v(t,x;u_0,v_0)), \ x \in \bar{\Omega}.$$

Then by Proposition 3.1.1, $u(t,x;u_0,v_0) \geq 0$ for $t \in [0, t_{\max}(u_0,v_0))$ and $x \in \bar{\Omega}$. Similarly, $v(t,x;u_0,v_0) \geq 0$ for $t \in [0, t_{\max}(u_0,v_0))$ and $x \in \bar{\Omega}$. (1) thus follows.

(2) Let $t_{\max} = \min\{t_{\max}(u_1,v_1), t_{\max}(u_2,v_2)\}$. Pick any $T \in (0, t_{\max})$, we just need to consider $t \in [0,T]$. Let $M = \max_{\substack{x \in \bar{\Omega} \\ 0 \leq t \leq T}} ||u(t,x;u_2,v_2)||$ Let

$$(u(t,\cdot), v(t,\cdot)) = (u(t,u_2,v_2) - u(t;u_1,v_1) + \varepsilon e^{\alpha t}, v(t;u_2,v_2) - v(t,u_1,v_1) - \varepsilon e^{\alpha t})$$

and

$$\alpha = M[||b_1(\cdot)|| + ||c_1(\cdot)||] + ||a_1(\cdot)|| + 1.$$

Then $u(0,x)>0$ and $v(0,x)<0$ in $\bar{\Omega}$. We claim that $u(t,x)>0$ and $v(t,x)<0$ for $x\in\bar{\Omega}$, $t\in[0,T]$. To prove this, we argue by contradiction. If not, define $t_0=\inf\{t:u(t,x)\leq 0 \text{ or } v(t,x)\geq 0 \text{ for some } x\in\bar{\Omega}\}$. Then $t_0>0$, and $u(t,x)>0>v(t,x)$, for all $t<t_0$ and $x\in\bar{\Omega}$, and there exist some $x_0\in\bar{\Omega}$ such that either $u(t_0,x_0)=0$ or $v(t_0,x_0)=0$. Without generalization, assume $u(t_0,x_0)=0$ and $v(t_0,x_0)\leq 0$. Note that

$$
\begin{aligned}
u_t(t,x) &= u_t(t;u_2,v_2)-u_t(t;u_1,v_1)+\alpha\varepsilon e^{\alpha t}\\
&\geq \int_{\bar{\Omega}} J(x-y)u(t,y)dy-u(t,x)+u(t,x)[a_1(x)-b_1(x)u(t,x;u_2,v_2)-\\
&\quad c_1(x)v(t,x;u_2,v_2)]-u(\cdot;u_1,v_1)[b_1(x)u(t,x)+c_1(x)v(t,x)]+\\
&\quad \varepsilon e^{\alpha t}\{\alpha-a_1(x)+[b_1(x)-c_1(x)]u(;u_1,v_1)\\
&\quad +b_1(x)u(;u_2,v_2)+c_1(x)v(;u_2,v_2)\} \qquad (3.4)
\end{aligned}
$$

for $x\in\bar{\Omega}$. Since $u(t_0,x_0)=0$ and $u(t,x)\geq 0$, for $t\leq t_0$, $u_t(t_0,x_0)$ should be non-positive. However, the right hand side of (3.4) at (t_0,x_0) is positive. Hence, $u(t,x)>0$ and $v(t,x)<0$, for $t\in[0,T]$ and $x\in\bar{\Omega}$. By letting $\varepsilon\to 0$, we see that $u(t,x;u_2,v_2)\geq u(t,x;u_1,v_1)$ and $v(t,x;u_2,v_2)\leq v(t,x;u_1,v_1)$, for $t\in[0,T]$ and $x\in\bar{\Omega}$. Since t is arbitrarily picked from $(0,t_{\max})$, so we get the conclusion.

(3) By (1), $(u(t,\cdot;u_0,v_0),v(t,\cdot;u_0,v_0))\in X^+$. By (HB), $h_1(x)\not\equiv 0$ and $h_2(x)\not\equiv 0$, we have $u(t,x;u_0,v_0)\not\equiv 0$ and $v(t,x;u_0,v_0)\equiv 0$. Then by Proposition 3.1.1, $(u(t,\cdot;u_0,v_0),v(t,\cdot;u_0,v_0))\in X^{++}$. □

Proposition 3.2.2. *For any $(u_0,v_0)\in X^+$, $(u(t;u_0,v_0),v(t;u_0,v_0))$ exists for all $t>0$.*

Proof. By (HB), there is $M_0>0$ such that $(M,0)$ is a super-solution of (3.1) and $(0,M)$ is a sub-solution of (3.1) when $M\geq M_0$. For given $(u_0,v_0)\in X^+$, choose $M>0$ such that

$$(0,M)\leq_2 (u_0,v_0)\leq_2 (M,0).$$

Then by Proposition 3.2.1,

$$(0, M) \leq_2 (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \leq_2 (M, 0)$$

for $t \in [0, t_{\max}(u_0, v_0))$. It then follows that $t_{\max}(u_0, v_0) = \infty$. □

Proposition 3.2.3. *(Persistence) Assume (H3) and (H4). Then* $\exists\, \delta_0 > 0$*, such that for any* $(u_0, v_0) \in Z^+$*, the solution* $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ *of* (3.1) *satisfies that* $u(t, x; u_0, v_0) \geq \delta_0$ *and* $v(t, x; u_0, v_0) \geq \delta_0$*, for* $t \gg 1$ *and* $x \in \bar{\Omega}$*.*

Proof. First by Proposition 3.2.1 (2), (3), for $M_1, M_2 \gg 1$,

$$\begin{aligned} & (u(t_2, \cdot; M_1, 0), v(t_2, \cdot; M_1, 0)) \\ \leq_2 \;& (u(t_1, \cdot; M_1, 0), v(t_1, \cdot; M_1, 0)) \ll_2 (M_1, 0), \text{ for } t_2 > t_1 > 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (0, M_2) \;&\ll_2\; (u(t_1, \cdot; 0, M_2), v(t_1, \cdot; 0, M_2)) \\ &\leq_2\; (u(t_2, \cdot; 0, M_2), v(t_2, \cdot; 0, M_2)), \text{ for } t_2 > t_1 > 0. \end{aligned} \tag{3.6}$$

Choose any $t_0 > 0$, define $\delta_0 = \inf_{x \in \bar{\Omega}} \{v(t_0, x; M_1, 0), u(t_0, x, 0, M_2)\}$. (3.5) and (3.6) imply $\delta_0 > 0$. By comparison principle,

$$(u(t, \cdot; M_1, 0), v(t, \cdot; M_1, 0)) \geq_1 (\delta_0, \delta_0)$$

and

$$(u(t, \cdot; 0, M_2), v(t, \cdot; 0, M_2)) \geq_1 (\delta_0, \delta_0)$$

for $t \geq t_0$.

Next, for fixed $v \equiv 0$, we can get the solution $u(t, x; u_0)$ from

$$u_t = \int_{\bar{\Omega}} J(x-y)u(y)dy - u(x) + h_1(x) + u(a_1(x) - b_1(x)u).$$

with initial value $u(0, x; u_0) = u_0 > 0$. It follows that $(u(t, x; u_0), 0)$ is a supersolution of (3.1). Then for any fixed $(u_0, v_0) \in X^+$, we have

$$(u(t, x; u_0, v_0), v(t, x; u_0, v_0)) \leq_2 (u(t, x; u_0), 0).$$

By Proposition 3.1.3, $(u(t, x; u_0), 0) \to (u^*, 0)$ as $t \to \infty$. Therefore, there exist $T_1(u_0) > 0$, such that for any $t > T_1(u_0)$, $u(t, x; u_0, v_0) < M_1$.

Similarly, by fixing $u \equiv 0$ and introducing the solution $v(t, x; v_0)$ which satisfies

$$v_t = \int_{\bar{\Omega}} J(x-y)v(y)dy - v(x) + h_2(x) + v(a_2(x) - c_2(x)v).$$

with initial value $v(0, x; v_0) = v_0 > 0$, we have $(0, v(t, x; v_0))$ is a sub-solution of (3.1). Then for any fixed $(u_0, v_0) \in X^+$,

$$(0, v(t, x; v_0)) \leq_2 (u(t, x; u_0, v_0), v(t, x; u_0, v_0))$$

holds and Proposition 3.1.3 yields $(0, v(t, x; u_0) \to (0, v^*)$ as $t \to \infty$. Thus, there exist $T_2(v_0) > 0$, such that for any $t > T_2(v_0)$, $v(t, x; u_0, v_0) < M_2$. Let $T(u_0, v_0) = \max\{T_1, T_2\}$, for any $t > T(u_0, v_0)$,

$$(0, M_2) \leq_2 (u(t, x; u_0, v_0), v(t, x; u_0, v_0)) \leq_2 (M_1, 0).$$

By comparison principle,

$$\begin{aligned}(u(t_0,x;0,M_2),v(t_0,x;0,M_2)) &\le_2 (u(t+t_0,x;u_0,v_0),v(t+t_0,x;u_0,v_0))\\ &\le_2 (u(t_0,x;M_1,0),v(t_0,x;M_1,0).\end{aligned}$$

It follows $u(t,x;u_0,v_0)\ge\delta_0$ and $v(t,x;u_0,v_0)\ge\delta_0$, for $t\ge t_0+T(u_0,v_0)$ and $x\in\bar\Omega$. □

3.3 Existence of continuous coexistence states

In this section, we study the existence of continuous coexistence states of (3.1).

A function $(u^{**}(x),v^{**}(x))$ is called a coexistence state of (3.1) if it is measurable and

$$\begin{cases}\int_{\bar\Omega}J(x-y)u^{**}(y)dy-u^{**}(x)+h_1(x)+u^{**}(x)(a_1(x)-b_1(x)u^{**}(x)-c_1(x)v^{**}(x)), & x\in\bar\Omega\\ \int_{\bar\Omega}J(x-y)v^{**}(y)dy-v^{**}(x)+h_2(x)+v^{**}(x)(a_2(x)-b_2(x)u^{**}(x)-c_2(x)v^{**}(x)), & x\in\bar\Omega.\end{cases}$$

The main results of this section are stated in the following three theorems.

Theorem 3.3.1. *(Coexistence: weak Competition) Assume (HB). If* $\dfrac{b_1(x)}{c_1(x)}\ge\dfrac{b_2(x)}{c_2(x)}$ *for* $x\in\bar\Omega$*, then* (3.1) *has a positive continuous stationary solution* $(u^{**}(\cdot),v^{**}(\cdot))\in C(\bar\Omega,(0,\infty))\times C(\bar\Omega,(0,\infty))$.

Theorem 3.3.2. *(Coexistence: strong inhomogeneous boundary condition) Let* $d_0>0$ *be a given positive constant and* $\Omega_0=\{x\in\bar\Omega\,|\,d(x,\partial\Omega)\le d_0\}$*. If* $h_1(x)\gg1$ *or* $h_2(x)\gg1$ *for* $x\in\Omega_0$*, then any coexistence state is continuous.*

Proof of Theorem 3.3.1. Let M_1 be defined as in the proof of Proposition 3.2.3. In inequality (3.5), we proved the monotonicity of solution with initial value $(M_1,0)$. Since $(u(t,x;M_1,0),v(t,x;M_1,0))$ is also bounded and by Proposition 3.2.3, we know there exist measurable functions $u_+^{**},v_+^{**}:\bar\Omega\to[\delta_0,M_1]$ such that

$$(u(t,x;M_1,0),v(t,x;M_1,0))\to(u_+^{**}(x),v_+^{**}(x))\text{ as }t\to\infty.$$

Repeating the similar argument as we did in Proposition 3.1.3 yields that $(u_+^{**}(x), v_+^{**}(x))$ satisfies

$$\begin{cases} K_1(x) + u_+^{**}(x)\big(a_1(x) - 1 - b_1(x)u_+^{**}(x) - c_1(x)v_+^{**}(x)\big) = 0, \quad x \in \bar{\Omega} \\ K_2(x) + v_+^{**}(x)\big(a_2(x) - 1 - b_2(x)u_+^{**}(x) - c_2(x)v_+^{**}(x)\big) = 0, \quad x \in \bar{\Omega}, \end{cases} \tag{3.7}$$

where

$$K_1(x) = \int_\Omega J(y-x)u_+^{**}(y)dy + h_1(x), \quad K_2(x) = \int_\Omega J(y-x)v_+^{**}(y)dy + h_2(x).$$

It then suffices to prove that $(u_+^{**}(\cdot), v_+^{**}(\cdot)) \in X^{++}$. We prove this by contradiction.

Assume that there is $x_0 \in \bar{\Omega}$ such that $(u_+^{**}(x), v_+^{**}(x))$ is not continuous at x_0. Without loss of generality, we may assume that there is $\{x_n\} \subset \bar{\Omega}$ such that $x_n \to x_0$ and $\lim_{n\to\infty} u_+^{**}(x_n) = \alpha \neq u_+^{**}(x_0)$, $\lim_{n\to\infty} v_+^{**}(x_0) = \beta$. Observe that $K_1(x)$, $K_2(x)$ are continuous in x. It then follows from (3.7) that

$$\begin{cases} K_1(x_0) + \alpha\big(a_1(x_0) - 1 - b_1(x_0)\alpha - c_1(x_0)\beta\big) = 0 \\ K_2(x_0) + \beta\big(a_2(x_0) - 1 - b_2(x_0)\alpha - c_2(x_0)\beta\big) = 0. \end{cases} \tag{3.8}$$

It follows from Theorem 2.2.1, (3.8) has a unique solution. Therefore $u_+^{**}(x_0) = \alpha$ and $v_+^{**}(x_0) = \beta$, that is, $(u_+^{**}(\cdot), v_+^{**}(\cdot)) \in X^{++}$. □

Proof of Theorem 3.3.2. Suppose (u^{**}, v^{**}) is a positive solution of (3.1). For simplicity in notation, we put $(u(x), v(x)) = (u^{**}(x), v^{**}(x))$. Let

$$g_1(x) = \int_\Omega J(y-x)u(y)dy + h_1(x),$$

$$g_2(x) = \int_\Omega J(y-x)v(y)dy + h_2(x).$$

Suppose that $\mathrm{supp}(J(\cdot)) \subset B(0, r_0)$ for some $r_0 > 0$. Let $m_0 > 0$ and $k_0 > 0$ be such that

$$\Omega_1 = \{x \in \bar{\Omega} \setminus \Omega_0 \,|\, m\big(B(x, r_0) \cap \Omega_0\big) \geq m_0, \ \text{ and for any } \tilde{\Omega}_1 \subset B(x, r_0) \cap \Omega_0$$
$$\text{with } m(\tilde{\Omega}_1) \geq \frac{m_0}{2}, \ \int_{\tilde{\Omega}_1} J(y - x) dy \geq k_0\},$$

$$\Omega_2 = \{x \in \bar{\Omega} \setminus (\Omega_0 \cup \Omega_1) \,|\, m\big(B(x, r_0) \cap (\Omega_0 \cup \Omega_1)\big) \geq m_0, \ \text{ and for any}$$
$$\tilde{\Omega}_2 \subset B(x, r_0) \cap (\Omega_0 \cap \Omega_1) \ \text{ with } m(\tilde{\Omega}_2) \geq \frac{m_0}{2},$$
$$\int_{\tilde{\Omega}_2} J(y - x) dy \geq k_0\}$$

$$\cdots,$$

$$\Omega_k = \{x \in \bar{\Omega} \setminus \{\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{k-1}) \,|\, m\big(B(x, r_0) \cap (\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{k-1})\big) \geq m_0,$$
$$\text{for any } \tilde{\Omega}_k \subset B(x_0 \cap (\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{k-1})$$
$$\text{with } m(\tilde{\Omega}_k) \geq \frac{m_0}{2}, \ \int_{\tilde{\Omega}_k} J(y - x) dy \geq k_0\}$$

$(k = 1, 2, \cdots)$.

Observe that there is $k \geq 1$ such that

$$\Omega_0 \cup \Omega_1 \cup \cdots \Omega_k = \bar{\Omega}.$$

We prove the theorem for the case that $k = 2$. In general, it can be proved by induction.

First of all, by Theorem 2.2.3, there is $d_0 \gg 1$ such that for any $x \in \bar{\Omega}$,

$$\begin{cases} u(a_1(x) - 1 - b_1(x)u - c_1(x)v) + d_1 = 0 \\ v(a_2(x) - 1 - b_1(x)u - c_2(x)v) + d_2 = 0 \end{cases}$$

has a unique positive solution provided that $d_1 \geq d_0$ or $d_2 \geq d_0$ Let

$$n_1 = 9, \quad n_2 = 4, \quad n_3 = 1.$$

Let $d \geq d_0$ be such that

$$d^{0.1} k_0 \geq 1.$$

Assume that for any $x \in \Omega_0$, $h_1(x) \geq d^{n_1}$ or $h_2(x) \geq d^{n_1}$. Then by Theorem 2.2.3, $(u(x), v(x))$ is continuous in $x \in \Omega_0$.

Claim 1. For any $x \in \Omega_1$, $g_1(x) \geq d^{n_2}$ or $g_2(x) \geq d^{n_2}$ provided that $d \gg 1$.

Let

$$\tilde{n}_1 = 9.1, \quad \tilde{n}_2 = 4.1, \quad \tilde{n}_3 = 1.1,$$

To prove the claim, we first prove that for any $x \in \Omega_0$, $u(x) \geq d^{\tilde{n}_2}$ or $v(x) \geq d^{\tilde{n}_2}$ provided that $d \gg 1$.

For given $x \in \Omega_0$, assume $h_2(x) \geq d^{n_1}$. If $u(x) < d^{\tilde{n}_2}$, then by

$$v(x)(a_2 - 1 - b_2 u(x) - c_2 v(x)) + h_2(x) \leq 0,$$

we must have $v(x) \geq d^{\tilde{n}_2}$ provided that $d \gg 1$. If $v(x) < d^{\tilde{n}_2}$, by

$$v(x)(a_2 - 1 - b_2 u(x) - c_2 v(x)) + h_2(x) \leq 0,$$

we must have $u(x) \geq d^{\tilde{n}_2}$ provided that $d \gg 1$.

Similarly, for given $x \in \Omega_0$, if $h_1(x) \geq d^{n_1}$, we have either $u(x) \geq d^{\tilde{n}_2}$ or $v(x) \geq d^{\tilde{n}_2}$.

Next, note that for $x \in \Omega_1$,

$$g_1(x) \geq \int_{B(x,r_0) \cap \Omega_0} J(y - x) u(y) dy$$

and

$$g_2(x) \geq \int_{B(x,r_0)\cap\Omega_0} J(y-x)v(y)dy.$$

For given $x \in \Omega_1$, let

$$\Omega_1(x) = \{y \in B(x,r_0)\cap\Omega_0 \,|\, u(y) \geq d^{\tilde{n}_2}\}$$

and

$$\Omega_2(x) = \{y \in B(x,r_0)\cap\Omega_0 \,|\, v(y) \geq d^{\tilde{n}_2}\}.$$

Then

$$B(x,r_0)\cap\Omega_0 = \Omega_1(x)\cup\Omega_2(x).$$

We must have either $m\big(\Omega_1(x)\big) \geq \frac{m_0}{2}$ or $m\big(\Omega_2(x)\big) \geq \frac{m_0}{2}$. This implies that, either

$$g_1(x) \geq \int_{\Omega_1(x)} J(y-x)u(y)dy \geq d^{\tilde{n}_2}\int_{\Omega_1(x)} J(y-x)dy \geq d^{n_2}d^{0.1}k_0 \geq d^{n_2}$$

or

$$g_2(x) \geq \int_{\Omega_2(x)} J(y-x)v(y)dy \geq d^{\tilde{n}_2}\int_{\Omega_2(x)} J(y-x)dy \geq d^{n_2}d^{0.1}k_0 \geq d^{n_2}.$$

The claim 1 is thus proved.

By Claim 1 and Theorem 2.2.3, $(u(x), v(x))$ is continuous in $x \in \Omega_1$.

Claim 2. For any $x \in \Omega_2$, there holds $g_1(x) \geq d^{n_3}$ or $g_2(x) \geq d^{n_3}$ provided that $d \gg 1$.

Claim 2 can be proved by the similar arguments as in Claim 1. By Claim 2 and Theorems 2.2.2 and 2.2.3, $(u(x), v(x))$ is continuous in $x \in \Omega_2$. We then have that $(u(x), v(x))$ is continuous in $x \in \bar{\Omega}$. The theorem is thus proved. □

3.4 Uniqueness of coexistence states

Before we proof such coexistence state is unique, we are going to develop the bounds for such solutions first. For given $z \in C(\bar{\Omega})$, define

$$K(z)(x) = \int_{\Omega} J(y-x)z(y)dy, \quad x \in \bar{\Omega}$$

Set

$$a_{iL(M)} = \inf_{x\in\bar{\Omega}}(\sup_{x\in\bar{\Omega}})(a_i(x)),$$

$$b_{iL(M)} = \inf_{x\in\bar{\Omega}}(\sup_{x\in\bar{\Omega}})(a_i(x)),$$

$$c_{iL(M)} = \inf_{x\in\bar{\Omega}}(\sup_{x\in\bar{\Omega}})(a_i(x)),$$

and

$$h_{iL} = \inf_{x\in\bar{\Omega}}(\sup_{x\in\bar{\Omega}}) \int_{\mathbb{R}^{\mathbf{N}}\setminus\bar{\Omega}} J(y-x)g_i(y)dy$$

where $i = 1, 2$. If (u, v) is a coexistence state of (3.1), then it satisfies

$$\begin{cases} (K-I)u + u(a_1(x) - b_1(x)u - c_1(x)v) + h_1(x) = 0, \\ (K-I)v + v(a_2(x) - b_2(x)u - c_2(x)v) + h_2(x) = 0. \end{cases} \tag{3.9}$$

Consider

$$(K-I)u + u(a_1(x) - b_1(x)u) + h_1(x) = 0. \tag{3.10}$$

By Proposition 3.1.3, (3.10) has a unique positive solution. Let θ_{a_1,b_1} be such solution. Set

$$z^+ = \frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}.$$

It is easy to see $(K-I)z^+ + z^+(a_1 - b_1 z^+) + h_1 < 0$ on $\bar{\Omega}$. So z^+ is a supersoluion and therefore

$$\theta_{a_1,b_1} < \frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}.$$

Similarly, consider

$$(K-I)v + v(a_2(x) - c_2(x)v) + h_2(x) = 0. \tag{3.11}$$

The unique positive solution of (3.11) is denoted as θ_{a_2,c_2}. We have

$$\theta_{a_2,c_2} < \frac{a_{2M}}{c_{2L}} + \Big(\frac{h_{2M}}{c_{2L}}\Big)^{1/2}.$$

The main results of this section are stated in the following two theorems.

Theorem 3.4.1. *Assume that*

$$a_1 - c_1\Big[\frac{a_{2M}}{c_{2L}} + \Big(\frac{h_{2M}}{c_{2L}}\Big)^{1/2}\Big] > 0 \quad \text{and} \quad a_2 - b_2\Big[\frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}\Big] > 0.$$

Suppose that (u^{**}, v^{**}) *satisfies* (3.9). *Then*

$$\theta_{a_1 - c_1\left[\frac{a_{2M}}{c_{2L}} + \left(\frac{h_{2M}}{c_{2L}}\right)^{1/2}\right], b_1} \le u^{**} \le \theta_{a_1,b_1}$$

$$\theta_{a_2 - b_2\left[\frac{a_{1M}}{b_{1L}} + \left(\frac{h_{1M}}{b_{1L}}\right)^{1/2}\right], c_2} \le v^{**} \le \theta_{a_2,c_2}. \tag{3.12}$$

Remark 3.4.1. *(1) Note that* $\theta_{a_1,b_1} \gg 0$ *and* $\theta_{a_2,c_2} \gg 0$, *thus*

$$\sup_{x\in D} \frac{\theta_{a_1,b_1}}{\theta_{a_2,c_2}} :\equiv R((a_1,b_1),(a_2,c_2)) < \infty.$$

(2) Let

$$A_1 := a_1 - c_1\Big[\frac{a_{2M}}{c_{2L}} + \Big(\frac{h_{2M}}{c_{2L}}\Big)^{1/2}\Big]$$

and

$$A_2 := a_2 - b_2\Big[\frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}\Big]$$

Assume

$$A_1 > 0 \quad \text{and} \quad A_2 > 0. \tag{3.13}$$

Consider the following condition

$$4b_1c_2 > c_1^2 R((a_1, b_1), (A_2, c_2)) + 2c_1b_2 + b_2^2 R((a_2, c_2), (A_1, b_1)) \tag{3.14}$$

For fixed a_i, b_1, c_2 and h_i, $i = 1, 2$, (3.14) will be satisfied for b_2 and c_1 sufficiently small. In fact, θ_{A_2,b_1} (θ_{A_1,c_2}) increases as c_1 (b_2) decreases for $x \in \Omega$, $R((a_1, b_1), (A_2, c_2))$ ($R((a_2, c_2), (A_1, b_1))$) decreases as c_1 (b_2) decreases.

Theorem 3.4.2. *Assume that (3.13) and (3.14) are satisfied, then (3.9) has a unique solution (u^{**}, v^{**}) with $u^{**} > 0$ and $v^{**} > 0$ on $\bar{\Omega}$*

Proof of Theorem 3.4.1. Observe that

$$(K - I)u^{**} + u^{**}(a_1(x) - b_1(x)u^{**} - c_1(x)v^{**}) + h_1(x) = 0, \quad x \in \bar{\Omega}$$

and

$$(K - I)\theta_{a_1,b_1} + \theta_{a_1,b_1}(a_1(x) - b_1(x)\theta_{a_1,b_1}) + h_1(x) = 0, \quad x \in \bar{\Omega}.$$

By comparison principle, we have

$$u^{**} \le \theta_{a_1,b_1}.$$

Similarly, we have

$$v^{**} \le \theta_{a_2,c_2}.$$

Next, observe that

$$\theta_{a_1,b_1} \le \frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}$$

and

$$\theta_{a_2,c_2} \le \frac{a_{2M}}{c_{2L}} + \Big(\frac{h_{2M}}{c_{2L}}\Big)^{1/2}.$$

Hence

$$(K-I)u^{**} + u^{**}\Big[a_1(x) - b_1(x)u^{**} - c_1(x)\big(\frac{a_{2M}}{c_{2L}} + \Big(\frac{h_{2M}}{c_{2L}}\Big)^{1/2}\big)\Big] + h_1(x) \le 0$$

and

$$(K-I)v^{**} + v^{**}\Big[a_2(x) - b_2(x)\big(\frac{a_{1M}}{b_{1L}} + \Big(\frac{h_{1M}}{b_{1L}}\Big)^{1/2}\big) - c_2(x)v^{**}\Big] + h_2(x) \le 0.$$

It then follows that

$$\theta_{a_1 - c_1\left[\frac{a_{2M}}{c_{2L}} + \left(\frac{h_{2M}}{c_{2L}}\right)^{1/2}\right], b_1} \le u^{**}$$

and

$$\theta_{a_2 - b_2\left[\frac{a_{1M}}{b_{1L}} + \left(\frac{h_{1M}}{b_{1L}}\right)^{1/2}\right], c_2} \le v^{**}.$$

□

Proof of Theorem 3.4.2. Suppose that $(u_1, v_1), (u_2, v_2)$ are two solutions of (3.9) with $u_i > 0, v_i > 0$ on Ω for $i = 1, 2$. If $p = u_1 - u_2$, $q = v_1 - v_2$ then

$$\begin{cases} \int_\Omega k(y-x)p(y)dy - p + p(a_1(x) - b_1(x)u_1 - c_1(x)v_1) - b_1u_2p - c_1u_2q = 0, \\ \qquad , \qquad x \in \bar{\Omega} \\ \int_\Omega k(y-x)q(y)dy - q + q(a_2(x) - b_2(x)u_2 - c_2(x)v_2) - b_2v_1p - c_2v_1q = 0, \\ \qquad x \in \bar{\Omega}. \end{cases} \tag{3.15}$$

Let

$$L_1(z)(x) = (a_1(x) - b_1(x)u_1 - c_1(x)v_1)z, \quad x \in \bar{\Omega}$$

Let $\lambda(L_1)$ be the principal spectrum point of $K - I + L_1$ in $C(\bar{\Omega})$ (see [35] for the definition of principal spectrum point for nonlocal dispersal operators). Since u_1 is a solution of

$$(K - I + L_1)z + h_1(x) = 0.$$

This implies that

$$\lambda(K - I + L_1) \leq 0.$$

By the argument in [22], the principal spectrum point of $K - I + L_1$ in $C(\bar{\Omega})$ is also the principal spectrum point of $K - I + L_1$ in $L^2(\Omega)$. It follows from the variational property of $\lambda(L_1)$, for any $\phi \in L^2(\Omega)$ with $\int_\Omega \phi^2(x)dx \neq 0$

$$\begin{aligned} 0 \geq \lambda(L_1) \geq \frac{1}{\|\phi\|^2_{L^2(\Omega)}}[\int_\Omega \int_\Omega k(y-x)\phi(x)\phi(y)dydx - \int_\Omega \phi^2(x)dx \\ + \int_\Omega (a_1(x) - b_1(x)u_1 - c_1(x)v_1)\phi^2(x)dx] \end{aligned}$$

Multiplying the first equation of (3.15) by $-p$ and the second by $-q$ integrating over Ω and adding, we obtain, using the inequalities above,

$$\int_\Omega (b_2u_2p^2 + (c_1u_2 + b_2v_1)pq + c_2v_1q^2)dx \leq 0$$

If the quadratic form $Q_x(\xi, \eta) = b_1u_2(x)\xi^2 + (c_1u_2(x) + b_2v_1(x))\xi\eta + c_2v_1(x)\eta^2$ is positive definite for each $x \in \Omega$, then $p \equiv 0$ and $q \equiv 0$ proving uniqueness. For $x \in \Omega$, Q_x is positive definite if

$$4b_1c_2 > c_1^2(u_2(x)/v_1(x)) + 2c_1b_1 + b_2^2(v_1(x)/u_2(x))$$

Since the right hand of above inequality is no greater than the right hand of (3.14). So if (3.14) holds, Q_x is positive definite for each $x \in \Omega$ and this proves the theorem. $\square$

Chapter 4

Asymptotic Dynamics in a Cancer Model with Radiation Treatment

In this chapter, we investigate the asymptotic dynamics of the following cancer model,

$$\begin{cases} \dot{u} = uf(t,u) - \epsilon D(t)u + p(t)v - a(t)ux \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \\ \dot{x} = xg(t,x) - D(t)x + q(t)y - b(t)ux \\ \dot{y} = D(t)x - q(t)y - \delta(t)y. \end{cases} \tag{4.1}$$

Throughout this chapter, we assume

(HC) *$f(t,u)$ and $g(t,u)$ are C^1 in u and continuous and periodic in t with period T, $f(t,u) < 0$ and $g(t,u) < 0$ for $u \gg 1$, and $\inf_{t\in\mathbb{R},u\geq 0} f_u(t,u) < 0$ and $\inf_{t\in\mathbb{R},u\geq 0} g_u(t,u) < 0$; $D(t)$, $p(t)$, $a(t)$, $\delta(t)$, $q(t)$, and $b(t)$ are positive and continuous functions and are periodic in t with period T.*

We first present some basic properties of (4.1) in section 4.1. Next, we study in section 4.2 the asymptotic dynamics of (4.1) in the absence of cancer or normal cells as well as in presence of both cancer and normal cells.

Throughout this chapter,

$$X = \mathbb{R} \times \mathbb{R},$$

$$X^+ = \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$X^{++} = \mathrm{Int}(X^+).$$

For given (u_1, v_1), $(u_2, v_2) \in X$, we define

$$(u_1, v_1) \leq_1 (\ll_1)(u_2, v_2) \quad \text{if} \quad (u_2 - u_1, v_2 - v_1) \in X^+(X^{++}).$$

For given (u_1, v_1, x_1, y_1), $(u_2, v_2, x_2, y_2) \in X \times X$, we define

$$(u_1, v_1, x_1, y_1) \leq_1 (\ll_1)(u_2, v_2, x_2, y_2) \quad \text{if}$$

$$(u_2 - u_1, v_2 - v_1, x_2 - x_1, y_2 - y_1) \in X^+ \times X^+(X^{++} \times X^{++}),$$

and

$$(u_1, v_1, x_1, y_1) \leq_2 (\ll_2)(u_2, v_2, x_2, y_2) \quad \text{if}$$

$$(u_2 - u_1, v_2 - v_1, x_1 - x_2, y_1 - y_2) \in X^+ \times X^+(X^{++} \times X^{++}).$$

4.1 Basic properties

In this section, we present some basic properties of (4.1).

Note that, by the fundamental theory for ordinary differential equations, for any given (u_0, v_0, x_0, y_0), there is a unique (local) solution $(u(t; u_0, v_0, x_0, y_0), v(t; u_0, v_0, x_0, y_0), x(t; u_0, v_0, x_0, y_0), y(t; u_0, v_0, x_0, y_0))$ with $(u(0; u_0, v_0, x_0, y_0), v(0; u_0, v_0, x_0, y_0), x(0; u_0, v_0, x_0, y_0), y(0; u_0, v_0, x_0, y_0)) = (u_0, v_0, x_0, y_0)$.

Note also that when the cancer cells are absent, (4.1) becomes

$$\begin{cases} \dot{u} = uf(t, u) - \epsilon D(t)u + p(t)v \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v. \end{cases} \tag{4.2}$$

When the normal cells are absent, (4.1) becomes

$$\begin{cases} \dot{x} = xg(t,x) - D(t)x + q(t)y \\ \dot{y} = D(t)x - q(t)y - \delta(t)y. \end{cases} \tag{4.3}$$

For given $(u_0, v_0) \in X$ (resp. $(x_0, y_0) \in X$), we denote $(u(t; u_0, v_0), v(t; u_0, v_0))$ (resp. $(x(t; x_0, y_0), y(t; x_0, y_0)))$ the solution of (4.2) (resp. (4.3)) with $(u(0; u_0, v_0), v(0; u_0, v_0)) = (u_0, v_0)$ (resp. $(x(0; x_0, y_0), y(0; x_0, y_0)) = (x_0, y_0)$). Clearly,

$$(x(t; u_0, v_0, 0, 0), y(t; u_0, v_0, 0, 0)) = (0, 0),$$

$$(u(t; 0, 0, x_0, y_0), v(t; 0, 0, x_0, y_0)) = (0, 0),$$

and

$$(u(t; u_0, v_0), v(t; u_0, v_0)) = (u(t; u_0, v_0, 0, 0), v(t; u_0, v_0, 0, 0)),$$

$$(x(t; x_0, y_0), y(t; x_0, y_0)) = (x(t; 0, 0, x_0, y_0), y(t; 0, 0, x_0, y_0)).$$

We call $(u(t), v(t))$ a sub-solution (super-solution) of (4.2) on an interval $I \subset \mathbb{R}$ if

$$\begin{cases} \dot{u} \leq (\geq) uf(t,u) - \epsilon D(t)u + p(t)v \\ \dot{v} \leq (\geq) \epsilon D(t)u - p(t)v - \delta(t)v \end{cases}$$

for $t \in I$. $(x(t), y(t))$ is called a sub-solution (super-solution) of (4.3) on an interval $I \subset \mathbb{R}$ if

$$\begin{cases} \dot{x} \leq (\geq) xg(t,x) - D(t)x + q(t)y \\ \dot{y} \leq (\geq) D(t)x - q(t)y - \delta(t)y \end{cases}$$

We call $(u(t), v(t), x(t), y(t))$ is a sub-solution (super-solution) of (4.1) on an interval I if

$$\begin{cases} \dot{u} \le (\ge) uf(t,u) - \epsilon D(t)u + p(t)v - a(t)ux \\ \dot{v} \le (\ge) \epsilon D(t)u - p(t)v - \delta(t)v \\ \dot{x} \ge (\le) xg(t,x) - D(t)x + q(t)y - b(t)ux \\ \dot{y} \ge (\le) D(t)x - q(t)y - \delta(t)y \end{cases}$$

for $t \in I$.

Proposition 4.1.1 (Comparison principle)**.** *Consider* (4.2),

(1) Suppose that (u^-, v^-) *and* (u^+, v^+) *are sub- and super-solutions of* (4.2) *on* $[0, t_{max}(u^-(0), v^-(0)))$ *and* $[0, t_{max}(u^+(0), v^+(0)))$ *respectively with* $(u^-(0), v^-(0)) \le_1 (u^+(0), v^+(0))$*. Let* $u^+(t)$ *be* $u^+(t; (u^+(0), v^+(0))$*. We define notation* u^-*,* v^+*,* v^- *in the same way. Then* $(u^-(t), v^-(t)) \le_1 ((u^+(t), v^+(t))$ *on* $[0, t_{max}(u^-(0), v^-(0)) \cap [0, t_{max}(u^+(0), v^+(0)))$*.*

(2) If $(0,0) \le_1 (u_0, v_0)$*, then* $(0,0) \le_1 (u(t; u_0, v_0), v(t; u_0, v_0))$ *for* $t \in [0, t_{\max}(u_0, v_0))$*.*

(3) If $(0,0) \le_1 (u_i, v_i)$*, for* $i = 1, 2$*, and* $(u_1, v_1) \le_1 (u_2, v_2)$*, then*

$$(u(t, u_1, v_1), v(t; u_1, v_1)) \le_1 (u(t; u_2, v_2), v(t; u_2, v_2))$$

for $t \in [0, t_{\max}(u_1, v_1)) \cap [0, t_{\max}(u_2, v_2))$*.*

Proof. (1) By the positivity of $D(t)$ and $p(t)$, (4.2) is a cooperative system. (1) then follows from the order preserving property of general cooperative systems of ordinary differential equations (see [34]).

(2) Notice that $(u, v) \equiv (0,0)$ is a subsolution, so (2) follows immediately from (1).

(3) It follows from (1) and (2). □

Proposition 4.1.2 (Strong comparison principle)**.** *Consider* (4.2)*.*

(1) For any $(u_0, v_0) \in X^+\backslash(0,0)$,

$(u(t;u_0,v_0), v(t;u_0,v_0)) \in X^{++}$, *for* $t \in (0, t_{\max}(u_0,v_0))$.

(2) For any $u_i, v_i \in X^+\backslash(0,0)$, $i = 1,2$, *if* $(u_1,v_1) \leq_1 (u_2,v_2)$ *and* $(u_1,v_1) \not\equiv (u_2,v_2)$, *then* $(u(t;(u_1,v_1)), v(t;(u_1,v_1))) \ll_1 (u(t;(u_2,v_2)), v(t;(u_2,v_2)))$ *for* $t \in (0, t_{\max}(u_1,v_1)) \cap (0, t_{\max}(u_2,v_2))$.

Proof. (1) Fix any $T_0 \in (0, t_{\max}(u_0,v_0))$. Set $D_- = \min_{t\in\mathbb{R}} D(t)$, $D_+ = \max_{t\in\mathbb{R}} D(t)$, $p_- = \min_{t\in\mathbb{R}} p(t)$, $p_+ = \max_{t\in\mathbb{R}} p(t)$, $\delta_+ = \max_{t\in\mathbb{R}} \delta(t)$. Let $m = \max_{t\in[0,T_0]} |f(t,u(t;u_0,v_0))|$ and $b = \max\{m + \epsilon D_+, p_+ + \delta_+\}$. Define

$$A = \begin{pmatrix} -(m+\epsilon D_+) & p_- \\ \epsilon D_- & -(p_+ + \delta_+) \end{pmatrix} = -bI + \begin{pmatrix} b-(m+\epsilon D_+) & p_- \\ \epsilon D_- & b-(p_+ + \delta_+) \end{pmatrix}$$

and

$$C = \begin{pmatrix} b-(m+\epsilon D_+) & p_- \\ \epsilon D_- & b-(p_+ + \delta_+) \end{pmatrix}.$$

Then $A = -bI + C$ and C is a positive 2×2 matrix. By comparison principle for cooperative systems of ordinary differential equations, we have

$$e^{At}(u_0,v_0) \leq_1 (u(t;u_0,v_0), v(t;u_0,v_0))$$

for $t \in [0,T_0]$. Observe that for any $x_0 \in \mathbb{R}^2$

$$e^{At}x_0 = e^{-bt}e^{Ct}x_0$$

and

$$e^{Ct}x_0 = x_0 + tCx_0 + \frac{t^2C^2x_0}{2!} + \cdots + \frac{t^nC^nx_0}{n!} + \cdots$$

Notice also that if $x_0 \in X^+$, then

$$e^{Ct}x_0 \geq_1 0, \quad \forall t \geq 0$$

and if $x_0 \in X^+ \backslash (0,0)$, then

$$e^{Ct}x_0 \gg_1 0, \forall t > 0.$$

We then have $(u(t;u_0,v_0), v(t;u_0,v_0)) \in X^{++}$, for $t \in [0, t_{\max}(u_0,v_0))$.

(2) It can be proved by the similar arguments as in (1). □

Proposition 4.1.3. *Consider* (4.2),

(1) For any given $T_0 \in [0, t_{\max}(u_0,v_0))$,

$$u(t;u_0,v_0) + v(t;u_0,v_0) \leq \max\{u_0 + v_0, \frac{(\delta_{\max} + f_{\max})^2}{4M\delta_{\min}}\}, \tag{4.4}$$

where $\delta_{\min} = \min_{t\in\mathbb{R}} \delta(t)$*,* $\delta_{\max} = \max_{t\in\mathbb{R}} \delta(t)$*,* $f_{\max} = \max_{t\in\mathbb{R}} |f(t,0)|$*, and*

$$M = \min_{t\in\mathbb{R}, u\geq 0}\{|f_u(t,u)|\}. \tag{4.5}$$

(2) For any $(u_0,v_0) \in Z^+$*,* $(u(t;(u_0,v_0)), v(t;u_0,v_0))$ *exists and*

$$u(t;u_0,v_0) + v(t;u_0,v_0) \leq \max\{u_0 + v_0, \frac{(\delta_{\max} + f_{\max})^2}{4M\delta_{\min}}\}$$

for all $t \geq 0$*.*

Proof. (1) Let $U(t) = u(t; u_0, v_0) + v(t; u_0, v_0)$. Note that $U(t) \geq 0$ for $t \in [0, t_{\max}(u_0, v_0))$. Using mean value theorem, we get

$$\begin{aligned}
\dot{U} &= uf(t,u) - \delta(t)(u+v) + \delta(t)u \\
&\leq u(\delta(t) + f(t,0) - Mu) - \delta(t)U \\
&= -M(u - \frac{\delta(t) + f(t,0)}{2M})^2 - \delta U + M(\frac{\delta(t) + f(t,0)}{2M})^2 \\
&\leq \frac{(\delta_{\max} + f_{max})^2}{4M} - \delta_{\min} U
\end{aligned}$$

for $t \in [0, t_{\max}(u_0, v_0))$. Solving the corresponding equality, we get

$$\begin{aligned}
U(t) &\leq (U(0) - \frac{(\delta_{\max} + f_{\max})^2}{4M\delta_{\min}})e^{-\delta_{\min} t} + \frac{(\delta_{\max} + f_{\max})^2}{4M\delta_{\min}} \\
&\leq \max\{u_0 + v_0, \frac{(\delta_{\max} + f_{\max})^2}{4M\delta_{\min}}\}
\end{aligned}$$

for $t \in [0, t_{\max}(u_0, v_0))$. This proves (1).

(2) It follows from (1) and fundamental theory of ordinary differential equations. □

Remark 4.1.1. *Propositions similar to Propositions 4.1.1, 4.1.2, and 4.1.3 can be proved for* (4.3).

For given $(u_0, v_0, x_0, y_0) \in \mathbb{R}^4$, for simplicity in notation, let $\Pi_t(u_0, v_0, x_0, y_0)$ be the solution of (4.1) with initial condition $\Pi_0(u_0, v_0, x_0, y_0) = (u_0, v_0, x_0, y_0)$. Let $I(u_0, v_0, x_0, y_0)$ be the existence interval of $\Pi_t(u_0, v_0, x_0, y_0)$.

Proposition 4.1.4. *Consider* (4.1),

(1) If $(0,0,0,0) \leq_1 (u_0, v_0, x_0, y_0)$, *then*

$$(0,0,0,0) \leq_1 \Pi_t(u_0, v_0, x_0, y_0)$$

for $t \in I(u_0, v_0, x_0, y_0) \cap \mathbb{R}^+$.

(2) If $(0,0,0,0) \leq_1 (u_i, v_i, x_i, y_i)$ *for* $i = 1,2$ *and* $(u_1, v_1, x_1, y_1) \leq_2 (u_2, v_2, x_2, y_2)$*, then*

$$\Pi_t(u_1, v_1, x_1, y_1) \leq_2 \Pi_t(u_2, v_2, x_2, y_2)$$

for $t \in \mathbb{R}^+ \cap I(u_1, v_1, x_1, y_1) \cap I(u_2, v_2, x_2, y_2)$.

Proof. 1) It suffices to prove that $(u(t; u_0, v_0, x_0, y_0), v(t; u_0, v_0, x_0, y_0)) \in X^+$ and $(x(t; u_0, v_0, x_0, y_0), y(t; u_0, v_0, x_0, y_0)) \in X^+$ for $t \in I(u_0, v_0, x_0, y_0) \cap \mathbb{R}^+$. To this end, note that $(u(t; u_0, v_0, x_0, y_0), v(t; u_0, v_0, x_0, y_0))$ is the solution of

$$\begin{cases} \dot{u} = uf(t,u) - \epsilon D(t)u + p(t)v - a(t)x(t; u_0, v_0, x_0, y_0)u \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \end{cases} \tag{4.6}$$

with $(u(0; u_0, v_0, x_0, y_0), v(0; u_0, v_0, x_0, y_0)) = (u_0, v_0)$. Note also that (4.6) is a cooperative system and $(0,0)$ is an equilibrium solution of (4.6). It then follows from comparison principle for cooperative systems of ordinary differential equations, $(u(t; u_0, v_0, x_0, y_0), v(t; u_0, v_0, x_0, y_0)) \in X^+$ for $t \in I(u_0, v_0, x_0, y_0) \cap \mathbb{R}^+$. Similarly, we have $(x(t; u_0, v_0, x_0, y_0), y(t; u_0, v_0, x_0, y_0)) \in X^+$ for $t \in I(u_0, v_0, x_0, y_0) \cap \mathbb{R}^+$.

(2) By the continuity of $\Pi_t(u_0, v_0, x_0, y_0)$ with respect to initial conditions, it suffice to prove that, if $(u_1, v_1, x_1, y_1) \ll_2 (u_2, v_2, x_2, y_2)$, then

$$\Pi_t(u_1, v_1, x_1, y_1) \ll \Pi_t(u_2, v_2, x_2, y_2)$$

for $t \in \mathbb{R}^+ \cap I(u_1, v_1, x_1, y_1) \cap I(u_2, v_2, x_2, y_2)$. We then assume that $(u_1, v_1, x_1, y_1) \ll_2 (u_2, v_2, x_2, y_2)$. Fix any $T \in (0, \infty) \cap I(u_1, v_1, x_1, y_1) \cap I(u_2, v_2, x_2, y_2)$. Assume there is $\tilde{t}_0 \in (0, T]$ such that

$$\Pi_{\tilde{t}_0}(u_1, v_1, x_1, y_1) \not\ll_2 \Pi_{\tilde{t}_0}(u_2, v_2, x_2, y_2).$$

Then there is $t_0 \in (0, \tilde{t}_0]$ such that

$$\Pi_t(u_1, v_1, x_1, y_1) \ll_2 \Pi_t(u_2, v_2, x_2, y_2).$$

for $t \in (0, t_0)$, but

$$\Pi_{t_0}(u_1, v_1, x_1, y_1) \not\ll_2 \Pi_{t_0}(u_2, v_2, x_2, y_2).$$

Without loss of generality, we assume that

$$u(t_0; u_1, v_1, x_1, y_1) = u(t_0; u_2, v_2, x_2, y_2) \quad \text{or} \quad v(t_0; u_1, v_1, x_1, y_1) = v(t_0; u_2, v_2, x_2, y_2).$$

Observe that $(u(t; u_1, v_1, x_1, y_1), x(t; u_1, v_1, x_1, y_1))$ is a sub-solution of

$$\begin{cases} \dot{u} = uf(t, u) - \epsilon D(t)u + p(t)v - a(t)x(t; u_2, v_2, x_2, y_2)u \\ \dot{v} = \epsilon D(t)u - p(t)v - \delta(t)v \end{cases} \tag{4.7}$$

for $t \in [0, t_0]$ and $(u(t; u_2, v_2, x_2, y_2), v(t; u_2, v_2, x_2, y_2))$ is solution of (4.7) for $t \in [0, t_0]$. It then follows from strong comparison principle for cooperative systems of ordinary differential equations, we have

$$u(t_0; u_1, v_1, x_1, y_1) < u(t_0; u_2, v_2, x_2, y_2) \quad \text{and} \quad v(t_0; u_1, v_1, x_1, y_1) < v(t_0; u_2, v_2, x_2, y_2).$$

This is a contradiction. Hence

$$\Pi_t(u_1, v_1, x_1, y_1) \ll \Pi_t(u_2, v_2, x_2, y_2)$$

for $t \in \mathbb{R}^+ \cap I(u_1, v_1, x_1, y_1) \cap I(u_2, v_2, x_2, y_2)$. □

Proposition 4.1.5. *Consider* (4.1), *(1) For any given* $(u_0, v_0, x_0, y_0) \in X^+ \times X^+$ *and* $t \in \mathbb{R}^+ \cap I(u_0, v_0, x_0, y_0)$*,*

$$\begin{aligned} & u(t; u_0, v_0, x_0, y_0) + v(t; u_0, v_0, x_0, y_0) + x(t; u_0, v_0, x_0, y_0) + y(t; u_0, v_0, x_0, y_0) \\ & \leq \max\{u_0 + v_0 + x_0 + y_0, \frac{(\delta_{\max} + f_{\max})^2}{4M_1\delta_{\min}} + \frac{(\delta_{\max} + g_{\max})^2}{4M_2\delta_{\min}}\}\}, \end{aligned}$$

where $\delta_{min} = \min_{t\in\mathbb{R}} \delta(t)$*,* $\delta_{\max} = \max_{t\in\mathbb{R}} \delta(t)$*,*

$$f_{\max} = \max_{t\in\mathbb{R}} |f(t, 0)|, \quad g_{\max} = \max_{t\in\mathbb{R}} |g(t, 0)|,$$

and

$$M_1 = \min_{t\in\mathbb{R}, u\geq 0}\{|f_u(t, u)|\}, \quad M_2 = \min_{t\in\mathbb{R}, u\geq 0}\{|g_u(t, u)|\}.$$

(2) For any given $(u_0, v_0, x_0, y_0) \in X^+ \times X^+$*,* $\Pi_t(u_0, v_0, x_0, y_0)$ *exists for all* $t \geq 0$*.*

Proof. (1) First, for simplicity in notation, let $u(t) = u(t; u_0, v_0, x_0, y_0)$, $v(t) = v(t; u_0, v_0, x_0, y_0)$, $x(t) = x(t; u_0, v_0, x_0, y_0)$, and $y(t) = y(t; u_0, v_0, x_0, y_0)$. Let

$$W(t) = u(t) + v(t) + x(t) + y(t).$$

Using mean value theorem, we get

$$\begin{aligned} \dot{W}(t) &= \dot{u}(t) + \dot{v}(t) + \dot{x}(t) + \dot{y}(t) \\ &\leq -\delta(t)W + (\delta(t)u + uf(t, u)) + (xg(t, x) + \delta(t)x) \\ &\leq -\delta_{\min}W + (\delta_{\max} + f_{\max})^2/4M_1 + (\delta_{\max} + g_{\max})^2/4M_2. \end{aligned} \tag{4.8}$$

The rest of the argument is similar to Proposition 4.1.3.

(2) It follows from (1) and fundamental theory of ordinary differential equations. □

4.2 Asymptotic dynamics

In this section, we study the asymptotic dynamics of (4.1) as well as the asymptotic dynamics of (4.2) and (4.3). The main results of this section are stated in the following theorems.

Theorem 4.2.1. *(1) Consider* (4.2)*. If* $(0,0)$ *is a stable solution of* (4.2) *, then it is globally stable.*

(2) Consider (4.3)*. If* $(0,0)$ *is a stable solution of* (4.3)*, then it is globally stable.*

Theorem 4.2.2. *(1) Consider* (4.2)*. If* $(0,0)$ *is unstable, then there exists a solution* $(u^*, v^*) \in X^{++}$ *which is periodic with period* T *and is globally asymptotically stable with respect to perturbations in* X^+*.*

(2) Consider (4.3)*. If* $(0,0)$ *is unstable, then there exists a solution* $(x^*, y^*) \in X^{++}$ *which is globally asymptotically stable with respect to perturbations in* X^+*.*

Theorem 4.2.3. *Consider* (4.1)*. For any given* $(u_0, v_0, x_0, y_0) \in X^+ \times X^+$*, there is a periodic solution* $(u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t))$ *of* (4.1) *such that*

$$\lim_{t\to\infty} \left[\Pi_t(u_0, v_0, x_0, y_0) - (u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t))\right] = 0.$$

Observe that, if $(0,0)$ is an unstable solution of (4.2) (resp. (4.3)), then $(u^*, v^*, 0, 0)$ (resp. $(0, 0, x^*, y^*)$) is a periodic solution of (4.1).

Theorem 4.2.4. *Consider* (4.1)*. Suppose that* $f(t,u) = a_1(t) - b_1(t)u$ *and* $g(t,x) = a_2(t) - b_2(t)x$*, where* $a_i(t)$ *and* $b_i(t)$ *are positive, continuous and periodic functions with period* T*. Assume that* $a_i(\cdot)$*,* $b_i(\cdot)$*,* $D(\cdot)$*,* $\delta(\cdot)$*,* $p(\cdot)$*, and* $q(\cdot)$ *are fixed and* $(0,0)$ *is an unstable solution of* (4.2) *and* (4.3)*. Then* (4.1) *has a unique time periodic solution* $(u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t))(\in X^{++} \times X^{++})$ *provided that* $p(t)$*,* $q(t)$*,* $a(t)$ *and* $b(t)$ *are sufficiently small.*

To prove the above theorems, we first prove a lemma.

We introduce the so called part metric in X^{++} as follows.

Definition 4.2.1 (Part metric)**.** *For* (u_1, v_1), $(u_2, v_2) \in X^{++}$,

$$\rho((u_1, v_1), (u_2, v_2)) = \inf\{\ln \alpha | \alpha \geq 1, u_1/\alpha \leq u_2 \leq \alpha u_1, v_1/\alpha \leq v_2 \leq \alpha v_2\}.$$

Note that if $\alpha_n > 1$, $u_1/\alpha_n \leq u_2 \leq \alpha_n u_1$, $v_1/\alpha_n \leq v_2 \leq \alpha_n v_1$ and $\alpha_n \to \alpha$, then $u_1/\alpha \leq u_2 \leq \alpha u_1$ and $v_1/\alpha \leq v_2 \leq \alpha v_1$. Hence

$$\rho((u_1, v_1), (u_2, v_2)) = \min\{\ln \alpha | \alpha \geq 1, u_1/\alpha \leq u_2 \leq \alpha u_1, v_1/\alpha \leq v_2 \leq \alpha v_2\}.$$

Lemma 4.2.1. *(1) Consider* (4.2)*. For any* (u_1, v_1), $(u_2, v_2) \in X^{++}$, $\rho((u_1(t), v_1(t)), (u_2(t), v_2(t)))$ *decreases as* t *increases, where* $(u_i(t), v_i(t)) = ((u(t; u_i, v_i), v(t; u_i, v_i))$ *for* $i = 1, 2$*.*

(2) Consider (4.3)*. For any* (x_1, y_1), $(x_2, y_2) \in X^{++}$, $\rho((x_1(t), y_1(t)), (x_2(t), y_2(t)))$ *decreases as* t *increases, where* $(x_i(t), y_i(t)) = ((x(t; x_i, y_i), y(t; x_i, y_i))$ *for* $i = 1, 2$*.*

Proof. (1) For given (u_1, v_1), $(u_2, v_2) \in X^{++}$ and $\alpha > 1$, suppose that

$$\frac{1}{\alpha}(u_1, v_1) \leq_1 (u_2, v_2) \leq_1 \alpha(u_1, v_1).$$

Then

$$(u(t; u_2, v_2), v(t; u_2, v_2)) \leq_1 (u(t; \alpha u_1, \alpha v_1), v(t; \alpha u_1, \alpha v_1))$$

for $t \geq 0$. Next, we show that

$$(u(t; \alpha u_1, \alpha v_1), v(t; \alpha u_1, \alpha v_1)) \ll_1 \alpha(u(t; u_1, v_1), v(t; u_1, v_1))$$

for $t > 0$. Let $\tilde{u_1}(t) = \alpha u_1(t; u_1, v_1)$ and $\tilde{v_1}(t) = \alpha v_1(t; u_1, v_1)$. Then

$$\begin{aligned} \dot{\tilde{u_1}} &= \alpha u_1(t) f(t, u_1) - D(t)\alpha u_1(t) + p(t)\alpha v_1(t) \\ &> \tilde{u_1}(t) f(t, \tilde{u}_1(t)) - D(t)\tilde{u}_1(t) + p(t)\tilde{v}_1(t) \\ \dot{\tilde{v_1}}(t) &= D(t)\tilde{u}_1(t) - (p(t) + \delta(t))\tilde{v}_1(t) \end{aligned} \tag{4.9}$$

By strong comparison principle,

$$(u(t; \alpha u_1, \alpha v_1), v(t; \alpha u_1, \alpha v_1)) \ll_1 \alpha(u_1(t), v_1(t))$$

for $t > 0$. Therefore,

$$(u(t; u_2, v_2), v(t; u_2, v_2)) \ll_1 \alpha(u(t; u_1, v_1), v(t; u_1, v_1))$$

for $t > 0$.

Similarly, we have

$$\frac{1}{\alpha}(u(t; u_1, v_1), v(t; u_1, v_1)) \ll_1 u(t; u_2, v_2), v(t; u_2, v_2))$$

for $t > 0$. It follows that

$$\rho((u_1(t), v_1(t)), (u_2(t), v_2(t))) < \rho((u_1, v_1), (u_2, v_2))$$

for $t > 0$ and then $\rho((u_1(t), v_1(t)), (u_2(t), v_2(t)))$ is strictly decreasing as t increases.

(2) It can be proved by the similar arguments as in (1). □

Next, we recall some convergence results from [34]. Consider tridiagonal competitive or cooperative system of differential equations of the form,

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2) \\ \dot{y}_j = f(t; y_{j-1}, y_j, y_{j+1}), \quad 2 \leq j \leq n-1 \\ \dot{y}_n = f_n(t; y_{n-1}, y_n), \end{cases} \tag{4.10}$$

where f_i are continuous and periodic in t with period T and are C^1 in y_j.

Lemma 4.2.2. *Suppose that $\mathcal{O}$ is a nonempty open subset of $\mathbb{R}^n$ and there are $\delta_i \in \{+1, -1\}$, $1 \leq i \leq n-1$, such that*

$$\delta_i \frac{\partial f_i}{\partial y_{i+1}} > 0, \quad \delta_i \frac{\partial f_{i+1}}{\partial y_i} > 0$$

for $(t, y) \in \mathbb{R} \times \mathcal{O}$. Suppose also that the coordinate projections $\mathcal{O}_1 \subset \mathbb{R}^2$ of $\mathcal{O}$ onto the (y_1, y_2)- plane, $\mathcal{O}_n$ of $\mathcal{O}$ onto the (y_{n-1}, y_n)- plane, and $\mathcal{O}_j$ of $\mathcal{O}$ onto the (y_{j-1}, y_j, y_{j+1})-space, $2 \leq< j \leq n-1$, are nonempty convex subsets. Then for any bounded solution $(y_1(t), y_2(t), \cdots, y_n(t))$ of (4.10) in $\mathcal{O}$, there is a periodic solution $(y_1^{**}(t), y_2^{**}(t), \cdots, y_n^{**}(t))$ of (4.10) such that

$$\lim_{t \to \infty} [y_i(t) - y_i^{**}(t)] = 0, \quad i = 1, 2, \cdots, n.$$

Proof. See [34, Theorem 2.2]. □

Proof of Theorem 4.2.1. (1) Let $\mathcal{O} = X^{++}$. For any given $(u_0, v_0) \in X^+ \setminus \{(0,0)\}$, $(u(t; u_0, v_0), v(t; u_0, v_0)) \in \mathcal{O}$ and is bounded. By Lemma 4.2.2, there is a periodic solution $(u^*(t), v^*(t))$ of (4.2) such that

$$\lim_{t \to \infty} [(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^*(t), v^*(t))] = (0, 0).$$

Now for $(u_0, v_0) \in \mathcal{O}$ with $u_0 \ll 1$ and $v_0 \ll 1$,

$$\lim_{t\to\infty}(u(t; u_0, v_0), v(t; u_0, v_0)) = (0, 0).$$

Then by Lemma 4.2.1, we must have

$$(u^*(t), v^*(t)) = (0, 0).$$

(2) It can be proved by the similar arguments as in (1). □

Proof of Theorem 4.2.2. (1) As in the proof of Theorem 4.2.1, let $\mathcal{O} = X^{++}$. For any given $(u_0, v_0) \in X^+ \setminus \{(0, 0)\}$, $(u(t; u_0, v_0), v(t; u_0, v_0)) \in \mathcal{O}$ and is bounded. By Lemma 4.2.2, there is a periodic solution $(u^*(t), v^*(t))$ of (4.2) such that

$$\lim_{t\to\infty}[(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^*(t), v^*(t))] = (0, 0).$$

Since $(0, 0)$ is unstable, we must have

$$(u^*(t), v^*(t)) \in X^{++}$$

for $t \in \mathbb{R}$. It then remains to show that (4.2) has only one periodic solution in X^{++}.

Assume that $(u_1^*(t), v_1^*(t))$ and $(u_2^*(t), v_2^*(t))$ are two periodic solutions of (4.2) in X^{++}. By Lemma 4.2.2, we have

$$\rho((u_1^*(t), v_1^*(t)), (u_2^*(t), v_2^*(t))) = 1$$

for all $t \in \mathbb{R}$ and then

$$(u_1^*(t), v_1^*(t)) \equiv (u_2^*(t), v_2^*(t)).$$

(1) is thus proved.

(2) It can be proved by the similar arguments as in (1). □

Proof of Theorem 4.2.3. First, let $y_1 = v$, $y_2 = u$, $y_3 = x$, $y_4 = y$. Then (4.1) becomes

$$\begin{cases} \dot{y}_1 = \epsilon D(t) y_2 - p(t) y_1 - \delta(t) y_1 \\ \dot{y}_2 = y_2 f(t, y_2) - \epsilon D(t) y_2 + p(t) y_1 - a(t) y_2 y_3 \\ \dot{y}_3 = y_3 g(t, y_3) - D(t) y_3 + q(t) y_4 - b(t) y_2 y_3 \\ \dot{y}_4 = D(t) y_3 - q(t) y_4 - \delta(t) y_4. \end{cases} \tag{4.11}$$

Equation (4.11) is the same form as (4.10). The theorem then follows from Lemma 4.2.2. □

Proof of Theorem 4.2.4. First, by Theorems 4.2.1-4.2.3 and the assumptions of Theorem 4.2.4, (4.1) have periodic solutions $(u_-^{**}, v_-^{**}, x_-^{**}, y_-^{**})$ and $(u_+^{**}, v_+^{**}, x_+^{**}, y_+^{**})$ such that

$$(u_-^{**}(t), v_-^{**}(t), x_-^{**}(t), y_-^{**}(t)) \leq_2 (u_+^{**}(t), v_+^{**}(t), x_+^{**}(t), y_+^{**}(t))$$

for any $t \in \mathbb{R}$ and for any other periodic solution $(u^{**}, v^{**}, x^{**}, y^{**})$ of (4.1), there holds

$$(u_-^{**}(t), v_-^{**}(t), x_-^{**}(t), y_-^{**}(t) \leq_2 (u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t)) \leq_2 (u_+^{**}(t), v_+^{**}(t), x_+^{**}(t), y_+^{**}(t))$$

for any $t \in \mathbb{R}$. Moreover, it is not difficult to prove that there are $K^* > 0$, $k^* > 0$ and $\epsilon^* > 0$ such that

$$(k^*, k^*, k^*, k^*) \leq_1 (u^{**}(t), v^{**}(t), x^{**}(t), y^{**}(t)) \leq_1 (K^*, K^*, K^*, K^*)$$

for any $t \in \mathbb{R}$ provided that $p(t), q(t), a(t), b(t) \leq \epsilon^*$.

It the suffices to prove that

$$(u_-^{**}(t), v_-^{**}(t), x_-^{**}(t), y_-^{**}(t)) = (u_+^{**}(t), v_+^{**}(t), x_+^{**}(t), y_+^{**}(t))$$

for any $t \in \mathbb{R}$. Assume that

$$(u_-^{**}(t), v_-^{**}(t), x_-^{**}(t), y_-^{**}(t)) \not\equiv (u_+^{**}(t), v_+^{**}(t), x_+^{**}(t), y_+^{**}(t)).$$

Then by strong comparison principle,

$$(u_-^{**}(t), v_-^{**}(t), x_-^{**}(t), y_-^{**}(t)) \ll_2 (u_+^{**}(t), v_+^{**}(t), x_+^{**}(t), y_+^{**}(t))$$

for any $t \in \mathbb{R}$. Observe that

$$\frac{d}{dt} \ln \frac{u_-^{**}(t)}{u_+^{**}(t)} = b_1(t)\big(u_+^{**}(t) - u_-^{**}(t)\big) + p(t)\Big(\frac{v_-^{**}(t)}{u_-^{**}(t)} - \frac{v_+^{**}(t)}{u_+^{**}(t)}\Big) + a(t)\big(x_+^{**}(t) - x_-^{**}(t)\big)$$

and

$$\frac{d}{dt} \ln \frac{x_-^{**}(t)}{x_+^{**}(t)} = b_2(t)\big(x_+^{**}(t) - x_-^{**}(t)\big) + q(t)\Big(\frac{y_-^{**}(t)}{x_-^{**}(t)} - \frac{y_+^{**}(t)}{x_+^{**}(t)}\Big) + b(t)\big(x_+^{**}(t) - x_-^{**}(t)\big).$$

Hence

$$\int_0^T b_1(t)\big(u_+^{**}(t) - u_-^{**}(t)\big)dt + \int_0^T p(t)\Big(\frac{v_-^{**}(t)}{u_-^{**}(t)} - \frac{v_+^{**}(t)}{u_+^{**}(t)}\Big)dt + \int_0^T a(t)\big(x_+^{**}(t) - x_-^{**}(t)\big)dt = 0 \tag{4.12}$$

and

$$\int_0^T b_2(t)\big(x_+^{**}(t) - x_-^{**}(t)\big)dt + \int_0^T q(t)\Big(\frac{y_-^{**}(t)}{x_-^{**}(t)} - \frac{y_+^{**}(t)}{x_+^{**}(t)}\Big)dt + \int_0^T b(t)\big(u_+^{**}(t) - u_-^{**}(t)\big)dt = 0. \tag{4.13}$$

Observe that

$$\epsilon \int_0^T D(t)\big(u_+^{**}(t) - u_-^{**}(t)\big)dt = \int_0^T \big(p(t) + \delta(t)\big)\big(v_+^{**}(t) - v_-^{**}(t)\big)dt$$

and

$$\int_0^T D(t)\big(x_-^{**}(t) - x_+^{**}(t)\big)dt = \int_0^T \big(q(t) + \delta(t)\big)\big(y_-^{**}(t) - y_+^{**}(t)\big)dt.$$

We then have

$$(p_{\min} + \delta_{\min}) \int_0^T (v_+^{**}(t) - v_-^{**}(t))dt \leq \epsilon D_{\max} \int_0^T (u_+^{**}(t) - u_-^{**}(t))dt \tag{4.14}$$

and

$$(q_{\min} + \delta_{\min}) \int_0^T (y_-^{**}(t) - y_+^{**}(t))dt \leq D_{\max} \int_0^T (x_-^{**}(t) - x_+^{**}(t))dt. \tag{4.15}$$

Let $K^* = \max\{u^*(t), v^*(t), x^*(t), y^*(t) | t \in \mathbb{R}\}$. By (4.14), we have

$$\begin{aligned}
|\int_0^T p(t)\Big(\frac{v_-^{**}(t)}{u_-^{**}(t)} - \frac{v_+^{**}(t)}{u_+^{**}(t)}\Big)dt| &\leq p_{\max} \int_0^T \Big|\frac{v_-^{**}(t)u_+^{**}(t) - u_-^{**}(t)v_+^{**}(t)}{u_-^{**}(t)u_+^{**}(t)}\Big|dt \\
&\leq \frac{p_{max}}{(k^*)^2} \int_0^T [u_+^{**}(t)|v_-^{**}(t) - v_+^{**}(t)| + v_+^{**}(t)|u_+^{**}(t) - u_-^{**}(t)|]dt \\
&\leq \frac{p_{\max}K^*}{(k^*)^2} \int_0^T [[|v_-^{**}(t) - v_+^{**}(t)| + |u_+^{**}(t) - u_-^{**}(t)|]dt \\
&\leq \frac{p_{\max}K^*}{(k^*)^2}\Big(\frac{\epsilon D_{\max}}{p_{\min} + \delta_{min}} + 1\Big) \int_0^T (u_+^{**}(t) - u_-^{**}(t))dt.
\end{aligned}$$

By (4.15), we have

$$\begin{aligned}
\Big|\int_0^T q(t)\Big(\frac{y_-^{**}(t)}{x_-^{**}(t)} - \frac{y_+^{**}(t)}{y_-^{**}(t)}\Big)dt\Big| &\leq q_{\max} \int_0^T \Big|\frac{y_-^{**}(t)x_+^{**}(t) - x_-^{**}(t)y_+^{**}(t)}{x_-^{**}(t)x_+^{**}(t)}\Big|dt \\
&\leq \frac{q_{max}}{(k^*)^2} \int_0^T [x_+^{**}(t)|y_-^{**}(t) - y_+^{**}(t)| + y_+^{**}(t)|x_+^{**}(t) - x_-^{**}(t)|]dt \\
&\leq \frac{q_{\max}K^*}{(k^*)^2} \int_0^T [[|y_-^{**}(t) - y_+^{**}(t)| + |x_+^{**}(t) - x_-^{**}(t)|]dt \\
&\leq \frac{q_{\max}K^*}{(k^*)^2}\Big(\frac{D_{\max}}{q_{\min} + \delta_{min}} + 1\Big) \int_0^T (x_-^{**}(t) - x_+^{**}(t))dt.
\end{aligned}$$

Then by (4.12), we have

$$\Big[b_{1,\min} - \frac{p_{\max}K^*}{(k^*)^2}\big(\frac{\epsilon D_{max}}{p_{\min} + \delta_{\min}} + 1\big)\Big] \int_0^T (u_+^{**}(t) - u_-^{**}(t))dt \leq a_{\max} \int_0^T (x_-^{**}(t) - x_+^{**}(t))dt$$

By (4.13), we have

$$\Big[b_{2,\max} - \frac{q_{\max}K^*}{(k^*)^2}\big(\frac{D_{max}}{q_{\min}+\delta_{\min}}+1\big)\Big]\int_0^T (x_-^{**}(t)-x_+^{**}(t))dt \le b_{max}\int_0^T (u_+^{**}(t)-u_-^{**}(t))dt.$$

It then follows that

$$\frac{1}{a_{max}}\Big[b_{1,\min} - \frac{p_{\max}K^*}{(k^*)^2}\big(\frac{\epsilon D_{max}}{p_{\min}+\delta_{\min}}+1\big)\Big] \le \frac{b_{max}}{\Big[b_{2,\max} - \frac{q_{\max}K^*}{(k^*)^2}\big(\frac{D_{max}}{q_{\min}+\delta_{\min}}+1\big)\Big]},$$

which is impossible when $a(t)$, $b(t)$, $p(t)$, and $q(t)$ are sufficiently small. The theorem is thus proved. □

Chapter 5

Concluding Remarks and Open Problems

In this dissertation, we studied the coexistence states and convergence of nonnegative solutions to the competition systems with immigration and time periodic dependence. We first investigated the coexistence and uniqueness of the Voterra-Lotka competition systems of ordinary differential equations with positive sources. It is not only an important preparation for proving asymptotic behavior of nonlocal dispersal systems, but also of great biological interest in its own.

Applying the comparison principle to sub- and super-solution as well as the part metric technique, we obtained the existence, uniqueness and global stability of one species nonlocal case with inhomogeneous boundary condition. The continuous coexistence of nonlocal dispersal system can be understood as the strong positive sources near the bounded domain will allow both species to survive no matter how strong the competition is between the two species. Notice that such continuous coexistence in the homogeneous boundary case requires other conditions on coefficients as indicated in the first chapter. The uniqueness of global stable coexistence state can be achieved using the technique employed in [5] which is true if the competition is weak.

In the cancer model, we have extended the results of periodic perturbation by Freedman and Pinho to a more general setting. When the cancer model is in absence of tumor cells and radiated tumor cells, we give the proof of global existence using the comparison principle for cooperative systems. The convergence follows from a result discovered by Smith (in [34]). The uniqueness of periodic positive coexistence is further discussed in this dissertation.

Some related problems that remain open are

Open problem 1. We proved that in the time independent case, (1.1) has a unique coexistence state provided that d_1 or d_2 is sufficiently larsge. In the periodic case of (1.1), it remains open whether d_i for $i = 1, 2$ are large enough in some sense will guarantee the uniqueness of the coexistence state or not.

Open problem 2. It is proved in this dissertation that any coexistence state of (1.2) is continuous provided that the inhomogeneous boundary conditions $h_1(x)$ and $h_2(x)$ are sufficiently large in certain sense. It remains open whether (1.2) has a unique coexistence state provided that $h_1(x)$ and $h_2(x)$ are sufficiently large. It also remains open whether coexistence states of time periodic two species competition systems with nonlocal dispersal and inhomogeneous boundary conditions are constinuous provided that the inhomogeneous boundary conditions are sufficiently large in certain sense.

Open Problem 3. Recall in the periodic cancer model (4.1), we proved in the special case, that is, $f(t, u) = a_1(t) - b_1(t)u$ and $g(t, x) = a_2(t) - b_2(t)x$, the uniqueness of positive periodic solution provided that $(0, 0)$ is an unstable solution of (4.2) and (4.3) when $p(t), q(t), a(t)$ and $b(t)$ are sufficiently small. It remains open whether we can get some conditions on $f(t, u)$ ad $g(t, x)$ in the general case that will result in a unique positive stable periodic solution.

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