Fair Factorizations of the Complete Multipartite Graph and Related Edge-Colorings

by

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Abstract

In this dissertation, first the technique of vertex amalgamations is used to extend known results on graph decompositions, and in particular on decompositions of the complete multipartite graph $K(n, p)$ with p parts, each of which has n vertices. The decompositions focus on hamilton cycles and 1-factors that satisfy certain fairness notions, as well as frame versions of these results where each color class (as defined by the decompositions) spans all vertices except for those in one part. Second, some edge-coloring results are proved, extending theorems in the literature on edge-colorings with different fairness properties. Finally, a related new topic is introduced, focusing on equalizing the number of vertices in each color class of an edge-coloring.

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Chapter 1

Introduction

A hamiltonian decomposition of a graph G is a partition of the edges of G into sets, each of which induces a spanning cycle, called a hamiltonian cycle. An early result on hamiltonian decompositions appeared in 1892 when Walecki [28] proved the famous result that the complete graph K_n on n vertices has a hamiltonian decomposition if and only if n is odd. The corresponding result for the existence of hamiltonian decompositions of the complete p-partite graph $K(n, p)$ with n vertices in each of p parts was settled in 1976 by Laskar and Auerbach [23], showing that such a decomposition exists if and only if $n(p-1)$ is even. A new technique called vertex amalgamations, which proved to be very useful in finding hamiltonian decompositions, was introduced 30 years ago by Hilton and Rodger [17, 20]. They used this technique to find alternative proofs of the aforementioned two results, and demonstrated the power of the technique by obtaining embeddings of edge-colorings into hamiltonian decompositions. In this context, a hamiltonian decomposition of G is represented as an edge-coloring of G in which each color class induces a hamiltonian cycle. Buchanan [4] in 1997 used amalgamations to prove that for any odd n, and any 2-regular spanning subgraph U of K_n, $K_n - E(U)$ has a hamiltonian decomposition. By generalizing amalgamation results, Leach and Rodger [24, 26] solved the corresponding problem for complete bipartite graphs, for complete tripartite graphs, and for complete multipartite graphs with any number of parts in the case when U has no cycles of small length. More recently, a neat observation using difference methods solved this and natural generalizations [3, 29]. In Chapter 4 and 5, these results are extended in various ways with further motivation and background being provided in Chapter 3.

In Chapter 7 and Chapter 8, some coloring results are proved with detailed background being provided in Chapter 6, thereby extending theorems in the literature described below. First we introduce some terminology. A graph G is called *even* if all vertices of G have even degree. Given a k-edge-coloring of a graph G, for each color $i \in \mathbb{Z}_k = \{0, 1, ..., k-1\}$ let $G[i]$ denote the spanning subgraph of G in which the edge-set contains precisely the edges colored i, and let $G(i)$ be the (not necessarily spanning) subgraph induced by the edges colored i. Then a k-edge-coloring of G is called an *even k-edge-coloring* if for each color $i \in \mathbb{Z}_k$, $G[i]$ is an even graph. A k-edge-coloring of G is said to be *equitable* if for each vertex $v \in V(G)$ and for each pair of colors $i, j \in \mathbb{Z}_k$, $|deg_{G[i]}(v) - deg_{G[j]}(v)| \in \{0, 1\}$. Moreover, a k-edge-coloring of G is said to be *evenly-equitable* if

(i) for each color $i \in \mathbb{Z}_k$, $G[i]$ is an even graph, and

(ii) for each vertex $v \in V(G)$ and for any pair of colors $i, j \in \mathbb{Z}_k$, $|deg_{G[i]}(v) - deg_{G[j]}(v)| \in \{0, 2\}$.

For any pair of vertices $\{v, w\}$, let $m_G(\{v, w\})$ be the number of edges between v and w in G (we allow $v = w$, so $m_G(\{v, v\})$ is the number of loops incident with v). A k-edge-coloring of G is said to be *balanced* if for all pairs of colors i and j and all pairs of vertices v and w (possibly $v = w$), $|m_{G[i]}(\{v, w\}) - m_{G[j]}(\{v, w\})| \leq 1$. A k-edge-coloring of G is said to be *equalized* if $||E(G[i])| - |E(G[j])|| \leq 1$ for each pair of colors $i, j \in \mathbb{Z}_k$.

In 1970's de Werra studied these special types of edge-colorings for bipartite graphs. Due to his work in [9, 10, 11, 12] it is known that for each $k \in \mathbb{N}$ every bipartite graph has a k-edge-coloring that is balanced, equitable and equalized at the same time. Several other results exist for more general graphs. In particular, Hilton proved in [16] that each even graph has an evenly-equitable k-edge-coloring for each $k \in \mathbb{N}$, thereby completely settling this problem. The existence of equitable k-edge-colorings is much more problematic, and very unlikely to be completely solved. For example, settling the existence of equitable Δ-edge-colorings is equivalent to classifying the Class 1 graphs (see [21, 22] for example). One general result on this topic was found by Hilton and de Werra [19] who proved that if $k \geq 2$ and G is a simple graph such that no vertex in G has degree equal to a multiple of k, then G has an equitable k-edge-coloring. More recently, Zhang and Liu [33] extended this result by proving that for each $k \geq 2$, if the subgraph of G induced by the vertices with degree divisible by k is a forest, then G has an equitable k-edge-coloring, thereby verifying a conjecture made by Hilton in [18].

A related, new topic is introduced in Chapter 8, focusing on the vertices in each color class of an edge-coloring. As with edge-colorings, notions of fairly distributing vertices among color classes have been considered. For example it is known that if $k \geq \Delta(G)$ then there exists an equalized vertex-coloring of G with k colors [15], and that if $\Delta(G) \geq |V(G)|/2$ then G has an equalized vertex-coloring with $\Delta(G)$ colors [6]. In these results, vertex-colorings are determining subgraphs with similar number of vertices. In Chapter 8 the number of vertices of subgraphs induced by color classes of an edge-coloring are considered.

Chapter 2

Amalgamations

2.1 Amalgamations

A graph H is said to be an *amalgamation* of a graph G if there exists a function ψ from $V(G)$ onto $V(H)$ and a bijection $\psi' : E(G) \to E(H)$ such that $e = \{u, v\} \in E(G)$ if and only if $\psi'(e) = \{\psi(u), \psi(v)\} \in E(H)$. The function ψ is called an amalgamation function. We say that G is a *detachment* of H, where each vertex v of H splits into the vertices of $\psi^{-1}(\{v\})$. An η-*detachment* of H is a detachment in which each vertex v of H splits into $\eta(v)$ vertices. Amalgamating a finite graph G to form the corresponding amalgamated graph H can be thought of as partitioning the vertices of G and forming one supervertex for each part by squashing together the original vertices in the same part. An edge incident with a vertex in G is then incident with the corresponding new vertex in H; in particular an edge joining two vertices from the same part becomes a loop on the corresponding new vertex in H.

In what follows, $G(j)$ denotes the subgraph of G induced by the edges colored j, $d_G(u)$ denotes the degree of vertex u in G, and $m_G(u, v)$ denotes the number of edges between u and v in G, $\omega(G)$ denotes the number of components of G, and $l_G(u)$ denotes the number of loops at u in G. The following theorem was proved in more generality by Bahmanian and Rodger in [1], but this is sufficient for our purposes.

Theorem 2.1.1. *Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $w \in V(H)$, $\eta(w) = 1$ implies $l_H(w) = 0$. Then there exists a loopless η-detachment G of H with amalgamation function $\psi : V(G) \to V(H)$, η being the number function associated with ψ, such that G satisfies the following property:*

(i) $d_G(u) \in \{\lfloor d_H(w)/\eta(w)\rfloor, \lceil d_H(w)/\eta(w)\rceil\}$ *for each* $w \in V(H)$ *and each* $u \in \psi^{-1}(w)$*,*

(ii) $d_{G(j)}(u) \in \{\lfloor d_{H(j)}(w)/\eta(w)\rfloor, \lceil d_{H(j)}(w)/\eta(w)\rceil\}$ *for each* $w \in V(H)$ *and each* $u \in \psi^{-1}(w)$ *and each* $j \in \mathbb{Z}_k$*,*

(iii) $m_G(u,v) \in \{\lfloor m_H(w,z)/\eta(w)\eta(z)\rfloor, \lceil m_H(w,z)/\eta(w)\eta(z)\rceil\}$ *for every pair of distinct vertices* $w, z \in V(H)$*, each* $u \in \psi^{-1}(w)$ *and each* $v \in \psi^{-1}(z)$*,*

(iv) $m_G(u,u') \in \{\lfloor l_H(w)/(\eta(w)(\eta(w)-1)/2)\rfloor, \lceil l_H(w)/(\eta(w)(\eta(w)-1)/2)\rceil\}$ *for each* $w \in V(H)$ *with* $\eta(w) \geq 2$ *and every pair of distinct vertices* $u, u' \in \psi^{-1}(w)$*, and*

(v) if for some $j \in \mathbb{Z}_k$*,* $d_{H(j)}(w)/\eta(w)$ *is an even integer for each* $w \in V(H)$*, then* $\omega(G(j)) = \omega(H(j))$*.*

Three corollaries of Theorem 2.1.1 will be introduced in Chapter 3, and these corollaries will be used in Chapter 4 and Chapter 5.

Chapter 3

Introduction to Fair Factorizations of Complete Multipartite Graphs

3.1 Introduction to Fair Factorizations of Complete Multipartite Graphs

In 2002 Leach and Rodger [25] completely settled the existence problem for fair hamiltonian decompositions of $K(n,p)$ (the notion of fairness being defined below), showing that they exist if and only if $n(p-1)$ is even. The aim of the next two chapters is to extend these results on fair hamiltonian cycle decompositions to other natural decompositions of $K(n,p)$, also described below.

A *k-factor* of a graph G is a k-regular spanning subgraph of G. A *k-factorization* is a partition of $E(G)$ into k-factors. For each $v \in V(G)$, a k-factor of $G-v$ is said to be an *almost parallel class* (or *near k-factor*) of *G with deficiency v*. An *almost resolvable k-factorization* of G is a partition of $E(G)$ into almost parallel classes each of which is a k-factor of $G-v$ for some $v \in V(G)$. If $V_1, ..., V_p$ are the p parts of $V(K(n,p))$, then a *holey k-factor* of deficiency V_i of $K(n,p)$ is a k-factor of $K(n,p) - V_i$ for some i satisfying $1 \leq i \leq p$. Hence a *holey k-factorization* is a set of holey k-factors whose edges partition $E(K(n,p))$. When $k = 2$ and each holey k-factor is connected, then this is called a holey hamiltonian decomposition. It is useful to represent a holey k-factorization of $K(n,p)$ as an edge-coloring of $K(n,p)$: an edge-coloring of $K(n,p)$ is said to be a *holey edge-coloring* if each color class induces a holey k-factor of $K(n,p)$. In holey edge-colorings of $K(n,p)$, a color c is said to be *permitted* on an edge joining two vertices from parts V_i and V_j respectively if c is the color of a holey k-factor of deficiency V_x where $x \notin \{i,j\}$. (Similarly, a part V_i is said to be permitted for a color c if the holey k-factor induced by the edges colored c has deficiency V_x where $x \neq i$.) In edge-colorings of $K(n,p)$ induced by k-factors (unlike in holey edge-colorings of $K(n,p)$) all colors appearing on any k-factor are permitted on any edge. For any simple graph G, let λG denote the multigraph in which if two vertices are joined by $\epsilon \in \{0,1\}$ edges in G then they are joined by $\lambda\epsilon$ edges in λG. Then, an edge-coloring of $\lambda K(n,p)$ is said to be *fair* if for each pair of parts V_x and V_y and for each pair of permitted colors i and j on edges joining two vertices from V_x and V_y respectively, $||c_i(V_x, V_y)| - |c_j(V_x, V_y)|| \leq 1$, where $c_i(V_x, V_y)$ denotes the set of edges colored i between V_x and V_y. In Chapter 4 the existence of fair 1-factorizations of $K(n,p)$ is completely settled (see Theorem 4.2.1), as is the existence of fair holey 1-factorizations of $K(n,p)$ (see Theorem 4.2.2).

It is not hard to see that, from a design theoretic perspective, a holey 1-factorization of $K(n,p)$ is the same as a $(2,1,1)$-frame of type n^p. [A $(k,\alpha;\lambda)$-*frame of Type* n^p is an ordered triple $(V,\mathcal{G},B)$ where: B is a collection of subsets of V, each of size k; $\mathcal{G}$ is a partition of V into p sets, each of size n; and each 2-element subset S of V is in zero elements of B if $S \subseteq G$ for some $G \subseteq \mathcal{G}$ and is in λ elements of B otherwise, such that there exists a partition $\mathcal{P}$ of the subsets of B with the property that for each $P \in \mathcal{P}$: there exists a $G \in \mathcal{G}$ such that $p \cap G = \emptyset$ for all $p \in P$, and each element of $V \setminus G$ occurs in exactly α elements of P. For other information on these structures, see page 261 of [8].] Indeed, there is a clear one-to-one correspondence between holey 1-factorizations of $K(n,p)$ and symmetric quasigroups of order np with holes of size n: cells (i,j) and (j,i) are filled with symbol k if and only if the edge $\{i,j\}$ is colored k. Such quasigroups can often be constructed using direct products, but in so doing the edge coloring corresponding to the resulting quasigroup is as far as possible from being fair. The following two quasigroups of order 20 with holes of size 4 are constructed using the well-known direct product construction and the fair holey 1-factorization of $K(4,5)$ obtained from the construction described in the proof of Theorem 4.2.2, respectively. Comparing these two quasigroups, the effect of the fair property is clearly seen in the 4×4 "boxes" of the latter quasigroup, in which each permitted symbol appears once or twice in each such "box", as opposed to the direct product construction, which produces a quasigroup in which each permitted symbol appears 0 or 4 times in each 4×4 "box".

				8	9	a	b	c	d	e	f	g	h	i	j	4	5	6	7
				9	8	b	a	d	c	f	e	h	g	j	i	5	4	7	6
				a	b	8	9	e	f	c	d	i	j	g	h	6	7	4	5
				b	a	9	8	f	e	d	c	j	i	h	g	7	6	5	4
8	9	a	b					g	h	i	j	0	1	2	3	c	d	e	f
9	8	b	a					h	g	j	i	1	0	3	2	d	c	f	e
a	b	8	9					i	j	g	h	2	3	0	1	e	f	c	d
b	a	9	8					j	i	h	g	3	2	1	0	f	e	d	c
c	d	e	f	g	h	i	j					4	5	6	7	0	1	2	3
d	c	f	e	h	g	j	i					5	4	7	6	1	0	3	2
e	f	c	d	i	j	g	h					6	7	4	5	2	3	0	1
f	e	d	c	j	i	h	g					7	6	5	4	3	2	1	0
g	h	i	j	0	1	2	3	4	5	6	7					8	9	a	b
h	g	j	i	1	0	3	2	5	4	7	6					9	8	b	a
i	j	g	h	2	3	0	1	6	7	4	5					a	b	8	9
j	i	h	g	3	2	1	0	7	6	5	4					b	a	9	8
4	5	6	7	c	d	e	f	0	1	2	3	8	9	a	b				
5	4	7	6	d	c	f	e	1	0	3	2	9	8	b	a				
6	7	4	5	e	f	c	d	2	3	0	1	a	b	8	9				
7	6	5	4	f	e	d	c	3	2	1	0	b	a	9	8				

				8	9	a	b	g	h	i	j	4	5	6	7	e	d	f	c
				9	8	b	a	c	d	e	f	g	h	i	j	5	6	7	4
				c	d	e	f	h	g	j	i	5	4	7	6	b	8	a	9
				g	h	i	j	4	5	6	7	8	9	a	b	f	e	c	d
8	9	c	g					j	0	d	3	h	a	b	i	2	f	e	1
9	8	d	h					e	i	f	c	j	g	0	3	1	b	2	a
a	b	e	i					2	f	c	g	1	8	j	h	0	9	d	3
b	a	f	j					1	e	h	d	9	i	g	2	8	c	3	0
g	c	h	4	j	e	2	1					i	7	3	0	d	5	6	f
h	d	g	5	0	i	f	e					6	j	2	1	c	3	4	7
i	e	j	6	d	f	c	h					3	0	5	g	7	4	1	2
j	f	i	7	3	c	g	d					2	1	h	4	6	0	5	e
4	g	5	8	h	j	1	9	i	6	3	2					a	7	0	b
5	h	4	9	a	g	8	i	7	j	0	1					3	2	b	6
6	i	7	a	b	0	j	g	3	2	5	h					4	1	9	8
7	j	6	b	i	3	h	2	0	1	g	4					9	a	8	5
e	5	b	f	2	1	0	8	d	c	7	6	a	3	4	9				
d	6	8	e	f	b	9	c	5	3	4	0	7	2	1	a				
f	7	a	c	e	2	d	3	6	4	1	5	0	b	9	8				
c	4	9	d	1	a	3	0	f	7	2	e	b	6	8	5				

In design theory context, a holey hamiltonian decomposition of $K(n,p)$ is the same as an $(l,1;\lambda)$-cycle frame of type n^p where $l = n(p-1)$ is the length of each cycle and $\lambda = 1$ is the number of cycles containing each pair of vertices. Several results have recently been established on cycle frames of $K(n,p)$ with small cycle length, all being of fixed cycle length as the number of vertices grows. Cao, Niu and Tang [5] established the result for cycles of length l for each $l \in \{4,5,6\}$, Tiemeyer [32] independently proved the case where $l = 4$. Another paper, published by Niu and Cao [30], settled the case where cycles of both length 3 and length 5 are allowed in the frame. The only result in the literature addressing cycles of larger (yet still fixed) lengths appears to be that of Chitra, Vadivu and Muthusamy [7], who solved the problem for $l = 4t$ where t is a prime. In Chapter 5 we settle the existence of cycle frames of type n^p for the longest possible cycle length (see Theorem 5.2.1). In fact, our result settles the existence of fair holey hamiltonian decompositions of $K(n,p)$, thereby extending results in the literature on fair graph decompositions, while simultaneously settling the existence of the largest cycle frames.

We present some notation that will be used throughout the next two chapters. Usually the vertex set of λK_p will be $\{\infty\} \cup \mathbb{Z}_{p-1}$. Since difference methods will be usefully employed, define $\infty + t = \infty$ and $(c_0, c_1, ..., c_{n-2})_z + t = (c_0 + t, c_1 + t, ..., c_{n-2} + t)$, with sums reduced modulo z. Also, define $f((c_0, c_1, ..., c_{n-2})) = (f(c_0), ..., f(c_{n-2}))$ where $f : V(G) \to V(G)$. The edge $\{a, b\} \subseteq \mathbb{Z}_n$ with $a < b$ is said to have *difference* $D_n(a, b) = \min\{b - a, n - (b - a)\}$, and the edge $\{a, \infty\}$ is said to have *difference* ∞. It will be convenient for each $a \in \mathbb{Z}_n \setminus \{0\}$ to define $D_n(a) = min\{a, n - a\}$. For each $d \in \mathbb{Z}_n \setminus \{0\}$, let S_d be the 2-factor of $2K_n$ on the vertex set $\mathbb{Z}_n$ induced by the edges in the multiset $\{\{i, i + d\} \mid i \in \mathbb{Z}_n\}$, reducing the sums modulo n (so $S_d = S_{n-d}$).

The following is well-known and easy to prove (see page 220ff. of [27], for example).

Lemma 3.1.1. *$S_{d_1} \cup S_{d_2}$ is connected if and only if $gcd(d_1, d_2, n) = 1$.*

In the following three corollaries the technique of amalgamations (which is described in Chapter 2) will be used to get certain factorizations of $K(n,p)$ from some special edge-colorings of n^2K_p. The required special edge-colorings of n^2K_p will be constructed in Chapter 4 and Chapter 5.

Corollary 3.1.2. *If there exists a fair $n(p-1)$-edge-coloring of n^2K_p in which each color class is n-regular, then there exists a fair 1-factorization of $K(n,p)$.*

Proof. Using the notation in Theorem 2.1.1, let $H = n^2K_p$. Then by Theorem 2.1.1 there exists an n-detachment G of H such that

(i) the degree of each vertex in G is $n(p-1)$ (by *(i)* of Theorem 2.1.1),

(ii) each color class induces a spanning 1-regular subgraph (since we are given that $d_{H(j)}(w) = n$ for each $w \in V(H)$ and $j \in \mathbb{Z}_{n(p-1)}$) (by *(ii)* of Theorem 2.1.1), and

(iii) there is exactly one edge between each pair of vertices u and v of G for which $\psi(u) \neq \psi(v)$, and no edges otherwise (by *(iii)* of Theorem 2.1.1).

So, by (i) and (iii) it is clear that $G = K(n,p)$ with partition $\{\psi^{-1}(w) \mid w \in V(H)\}$ of the vertex set, and by (ii) each color class induces a 1-factor of $K(n,p)$. This yields a 1-factorization of $K(n,p)$. There is a one-to-one correspondence between the edges joining any pair of vertices w and z in H and the edges between the two corresponding parts of $G = K(n,p)$, one of which contains the vertices in $\psi^{-1}(w)$ and the other one contains the vertices in $\psi^{-1}(z)$. Hence a fair edge-coloring of n^2K_p (in which the parts all have size 1) results in a fair edge-coloring of $K(n,p)$. □

Corollary 3.1.3. *If there exists a fair holey n-factorization of n^2K_p (np-edge-coloring of n^2K_p in which each color class is an n-regular subgraph of $n^2K_p - v$ for some $v \in V(n^2K_p)$) then there exists a fair holey 1-factorization of $K(n,p)$.*

Proof. Using the notation in Theorem 2.1.1, let $H = n^2K_p$. Then by Theorem 2.1.1 there exists an n-detachment G of H such that

(i) the degree of each vertex in G is $n(p-1)$ (by *(i)* of Theorem 2.1.1),

(ii) each color class induces a spanning 1-regular subgraph of $G - \psi^{-1}(v)$ for some $v \in V(H)$ (since we are given that $d_{H(j)}(w) = n$ for each $w \in V(H) \setminus \{v\}$ for some vertex $v \in V(H)$) (by *(ii)* of Theorem 2.1.1), and

(iii) there is exactly one edge between each pair of vertices u and w of G for which $\psi(u) \neq \psi(w)$, and no edges otherwise (by *(iii)* of Theorem 2.1.1).

So, by (i) and (iii) it is clear that $G = K(n,p)$ with partition $\{\psi^{-1}(w) \mid w \in V(H)\}$ of the vertex set, and by (ii) each color class induces a holey 1-factor of $K(n,p)$. This yields a holey 1-factorization of $K(n,p)$. There is a one-to-one correspondence between the edges colored c joining any pair of vertices u and w in H and the edges colored c between the two corresponding parts $\psi^{-1}(u)$ and $\psi^{-1}(w)$ of $G = K(n,p)$, so this fair edge-coloring of n^2K_p results in a fair holey edge-coloring of $K(n,p)$ as required. $\square$

The following corollary of Theorem 2.1.1 will be useful in Chapter 5.

Corollary 3.1.4. *If there exists a fair holey connected $2n$-factorization of n^2K_p (that is, an $np/2$-edge-coloring of n^2K_p in which each color class is a connected $2n$-regular subgraph of $n^2K_p - v$ for some $v \in V(n^2K_p)$), then there exists a fair holey hamiltonian decomposition of $K(n,p)$.*

Proof. Using the notation in Theorem 2.1.1, let $H = n^2K_p$. Then by Theorem 2.1.1 there exists an n-detachment G of H such that

(i) the degree of each vertex in G is $n(p-1)$ (by *(i)* of Theorem 2.1.1),

(ii) each color class induces a 2-regular subgraph of $G - \psi^{-1}(v)$ for some $v \in V(H)$ (since we are given that $d_{H(j)}(w) = 2n$ for each $w \in V(H) \setminus \{v\}$, each $j \in \mathbb{Z}_k$, and some vertex $v \in V(H)$) (by *(ii)* of Theorem 2.1.1),

(iii) there is exactly one edge between each pair of vertices u and w of G for which $\psi(u) \neq \psi(w)$, and no edges otherwise (by *(iii)* of Theorem 2.1.1), and

(iv) each color class has one component (by *(v)* of Theorem 2.1.1).

So, by (i) and (iii) it is clear that $G = K(n,p)$ with partition $\{\psi^{-1}(w) \mid w \in V(H)\}$ of the vertex set, by (ii) each color class induces a holey 2-factor of $K(n,p)$, and by (iv) each color class is connected. This yields a holey hamiltonian decomposition of $K(n,p)$. There is a one-to-one correspondence between the edges colored c joining any pair of vertices u and w in H and the edges colored c between the two corresponding parts $\psi^{-1}(u)$ and $\psi^{-1}(w)$ of $G = K(n,p)$, so this fair edge-coloring of n^2K_p results in a fair holey hamiltonian decomposition of $K(n,p)$ as required.

$\square$

Chapter 4

Fair 1-Factorizations and Fair Holey 1-Factorizations of Complete Multipartite Graphs

4.1 Coloring Results

In this section the coloring results are obtained that allow Theorem 4.2.1 and Theorem 4.2.2 to be deduced from Corollary 3.1.2 and Corollary 3.1.3, respectively.

Proposition 4.1.1. *Suppose np is even. There exists an edge-coloring of n^2K_p with $n(p-1)$ colors such that*

(i) the edge-coloring is fair, and

(ii) each color class induces an n-regular subgraph.

Proof. First note that condition (i) is equivalent to requiring that between each pair of vertices each color appears on $\lfloor n^2/(n(p-1))\rfloor$ or $\lceil n^2/(n(p-1))\rceil$ edges.

Now suppose that p is even. Then clearly, K_p has a 1-factorization consisting of $p-1$ 1-factors $F_0, F_1, ..., F_{p-2}$. Let $F = (F_0, F_1, ..., F_{n^2(p-1)-1})$ be a sequence of $n^2(p-1)$ 1-factors of K_p where $F_i = F_j$ if $i \equiv j$ modulo $p-1$. For each $i \in \mathbb{Z}_{n(p-1)}$, let $G(i)$ be the subgraph induced by the edges in $\bigcup_{j=in}^{(i+1)n-1} F_j$, and color all edges in $G(i)$ with i. Then $\{E(G(i)) \mid i \in \mathbb{Z}_{n(p-1)}\}$ is a partition of the edge set of n^2K_p. Clearly this coloring of n^2K_p satisfies condition (ii). To see that it also satisfies condition (i), note that for each $i \in \mathbb{Z}_{n(p-1)-(p-1)}$, $\{F_{i+j} \mid j \in \mathbb{Z}_{p-1}\}$ is a 1-factorization of K_p. So for each $i \in \mathbb{Z}_{n(p-1)}$, the $n(p/2)$ edges colored i are shared as evenly as possible among the $p(p-1)/2$ pairs of vertices, so the number of edges colored i between each pair of vertices is $\lfloor (np/2)/(p(p-1)/2)\rfloor = \lfloor n/(p-1)\rfloor$ or $\lceil n/(p-1)\rceil$ as required.

Finally, suppose that p is odd. Then n is even since we are given that np is even. K_p has a 2-factorization consisting of $(p-1)/2$ 2-factors $F_0, F_1, ..., F_{(p-3)/2}$ (by Petersen's Theorem

[31]). Let $F = (F_0, F_1, ..., F_{n^2(p-1)/2-1})$ be a sequence of $n^2(p-1)/2$ 2-factors of K_p where $F_i = F_j$ if $i \equiv j$ modulo $(p-1)/2$. For each $i \in \mathbb{Z}_{n(p-1)}$, let $G(i)$ be the subgraph induced by the edges in $\bigcup_{j=in/2}^{(i+1)n/2-1} F_j$, and color all edges in $G(i)$ with i. Then $\{E(G(i)) \mid i \in \mathbb{Z}_{n(p-1)}\}$ is a partition of the edge set of n^2K_p. Clearly this coloring of n^2K_p satisfies condition (ii). To see that it also satisfies condition (i), note that for each $i \in \mathbb{Z}_{n(p-1)-(p-1)}$, $\{F_{i+j} \mid j \in \mathbb{Z}_{p-1}\}$ is a 2-factorization of K_p, so for each $i \in \mathbb{Z}_{n(p-1)}$, the $(n/2)p$ edges colored i are shared as evenly as possible among the $p(p-1)/2$ pairs of vertices. $\square$

Proposition 4.1.2. *Suppose $p > 2$ is even. There exists a set $\mathcal{F} = \{F_d \mid d \in \mathbb{Z}_{(p-2)/2}\}$ of $(p-2)/2$ almost resolvable 2-factorizations of $2K_p$ such that for each vertex $v \in V(2K_p)$,*

(i) the almost parallel classes in $\bigcup_{F\in\mathcal{F}} F$ with deficiency v form a 2-factorization of K_{p-1}.

Proof. Let $V(2K_p) = \mathbb{Z}_{p-1} \cup \{\infty\}$. Define the $(p-1)$-cycle $C = (c_0, c_1, ..., c_{p-2})$ by

$$c_i = \begin{cases} 1 & \text{if } i = 0 \\ (i+3)/2 & \text{if } i \text{ is odd} \\ (p-1) - (i/2) & \text{if } i \text{ is even and } i \notin \{0, p-2\} \\ \infty & \text{if } i = p-2. \end{cases}$$

Then C has deficiency 0 and contains two edges of each difference in $(\mathbb{Z}_{(p-2)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of difference 2. Therefore $F_0 = \{C + t \mid t \in \mathbb{Z}_{p-1}\} \cup \{S_2\}$ is an almost resolvable 2−factorization of $2K_p$, where $C + t$ has deficiency t and S_2 has deficiency ∞.

To construct the remaining almost resolvable 2-factorizations of $2K_p$, for $1 \leq d \leq (p-4)/2$ define the $(p-1)$-cycle $g_d(C) = (g_d(c_0), g_d(c_1), ..., g_d(c_{p-2}))$ where

$$g_d(c_i) = \begin{cases} c_i + d + 1 & \text{if } i \in \{2j \mid 1 \leq j \leq d\} \\ c_i + d & \text{otherwise} \end{cases}$$

with additions defined modulo $p-1$. Clearly, since C is a cycle, $g_d(C)$ is also a cycle on the same vertex set as C, namely $V(C) = (\mathbb{Z}_{p-1} \setminus \{0\}) \cup \{\infty\}$. Also define $g_0(C) = C$. Then for each $d \in \mathbb{Z}_{(p-2)/2}$, $g_d(C)$ has deficiency 0 and contains two edges of each difference in $\mathbb{Z}_{(p-2)/2} \cup \{\infty\}$, except for just one edge of difference $D_{p-1}(2d+2)$. Therefore, for each $d \in \mathbb{Z}_{(p-2)/2}$, $F_d = \{g_d(C) + t \mid t \in \mathbb{Z}_{p-1}\} \cup \{S_{2d+2}\}$ is an almost resolvable 2-factorization of $2K_p$.

Now we show that for each $v \in V(2K_p)$, the almost parallel classes in $\bigcup_{d \in \mathbb{Z}_{(p-2)/2}} F_d$ with deficiency v form a 2-factorization of K_{p-1}.

First suppose $v = \infty$. Since $p-1$ is odd and since $S_d = S_{p-1-d}$, $\{S_{2d+2} \mid d \in \mathbb{Z}_{(p-2)/2}\} = \{S_1, S_2, ..., S_{(p-2)/2}\}$. Hence these almost parallel classes with deficiency ∞ form a 2-factorization of $K_p \setminus \{\infty\}$.

Next suppose $v = 0$. For each $w \in \mathbb{Z}_{p-1} \setminus \{0\}$, let $N(w) = (1, 2, ..., w-1, \infty, w+1, ..., p-2) = (N_0, N_1, ..., N_{p-3})$. For each $w \in \mathbb{Z}_{p-1} \setminus \{0\}$ and each $d \in \mathbb{Z}_{(p-2)/2}$, for some $i \in \mathbb{Z}_{p-1}$ the neighbors of w in $g_0(C)$ are N_i and N_{i+1}, and so then with this value i in mind it is easy to check that the neighbors of w in $g_d(C)$ are N_{i+2d} and N_{i+2d+1}, reducing the sums in the subscript modulo $p-2$. So,

(†) $\{g_d(C) \mid d \in \mathbb{Z}_{(p-2)/2}\}$ is a 2-factorization of K_{p-1} on the vertex set $(\mathbb{Z}_{p-1} \setminus \{0\}) \cup \{\infty\}$.

Finally, for each $v \in \mathbb{Z}_{p-1}$, the almost parallel classes with deficiency v are $\{g_d(C) + v \mid d \in \mathbb{Z}_{(p-2)/2}\}$, which by (†) is clearly a 2-factorization of K_{p-1} on the vertex set $(\mathbb{Z}_{p-1} \setminus \{v\}) \cup \{\infty\}$. □

In the following lemma, condition (iii) will be of use in the case where n is even and condition (iv) will be needed when n is odd when proving Theorem 4.2.2.

Proposition 4.1.3. *Suppose $p > 1$ is odd. There exists a set $\mathcal{F} = \{F_d \mid d \in \mathbb{Z}_{p-2}\}$ of $p-2$ almost resolvable 2-factorizations of $2K_p$ such that for each vertex $v \in V(2K_p)$*

(i) the almost parallel classes in $\bigcup_{F\in\mathcal{F}} F$ with deficiency v form a 2-factorization of $2K_{p-1}$,

(ii) there exists $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| = (p-3)/2$ and the almost parallel classes with deficiency v in $\mathcal{F}'$ are edge-disjoint,

(iii) there exists $F \in \mathcal{F} \setminus \mathcal{F}'$ such that for each $v \in V$ each edge in K_{p-1} on the vertex set $V \setminus \{v\}$ occurs in a 2-factor in $\mathcal{F}' \cup \{F\}$, and

(iv) there exists an almost resolvable 1-factorization F^1 of K_p such that for each $v \in V(2K_p)$ the almost parallel class with deficiency v in F^1 together with the almost parallel classes in $\mathcal{F}'$ with deficiency v partition the edges of K_{p-1} on the vertex set $V \setminus \{v\}$.

Proof. We begin by defining $\mathcal{F}$ on the vertex set $V = \mathbb{Z}_{p-1} \cup \{\infty\}$. Define the $(p-1)$-cycle $C = (c_0, c_1, ..., c_{p-2})$ by

$$c_i = \begin{cases} 1 & \text{if } i = 0 \\ (i+3)/2 & \text{if } i \text{ is odd and } i \neq p-2 \\ (p-1) - (i/2) & \text{if } i \text{ is even and } i \neq 0 \\ \infty & \text{if } i = p-2, \end{cases}$$

with all arithmetic reduced to modulo $p-1$. Then C has deficiency 0 and contains two edges of each difference in $\mathbb{Z}_{(p-1)/2} \cup \{\infty\}$, except for just one edge of each of the differences 2 and $(p-1)/2$. Therefore $F_0 = \{C+v \mid v \in \mathbb{Z}_{p-1}\} \cup \{S_2\}$ is an almost resolvable $2-$factorization of $2K_p$, where $C+v$ has deficiency v and S_2 has deficiency ∞ (each edge of difference $(p-1)/2$

appears twice in the cycles in F_0 since the edge $e = \{x, x + (p-1)/2\}$ is the same as the edge $e + v$ when $v = (p-1)/2$.

To construct the remaining almost resolvable 2-factorizations of $2K_p$, for $1 \leq d \leq (p-3)/2$ define the $(p-1)$-cycle $g_d(C)$ where

$$g_d(c_i) = \begin{cases} c_i + d + 1 & \text{if } i \in \{2j \mid 1 \leq j \leq d\} \\ c_i + d & \text{otherwise,} \end{cases}$$

and for $(p-1)/2 \leq d \leq p-3$ define $g_d(C)$ as

$$g_d(c_i) = \begin{cases} c_i + d + 1 & \text{if } i \neq 0 \text{ is even or } i \in \{p - 2j \mid 2 \leq j \leq d - (p-5)/2\} \\ c_i + d & \text{otherwise,} \end{cases}$$

with additions defined modulo $p-1$. Also define $g_0(C) = C$. Then for each $d \in \mathbb{Z}_{p-2}$, $g_d(C)$ has deficiency 0. For each $d \in \mathbb{Z}_{p-2}$, define $m_p(d)$ as follows.

Suppose $p = 4k+1$ for some integer k. If $0 \leq d \leq 2k-2$ and $d \neq k-1$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of each of the two differences $m_p(d) = D_{p-1}(2d+2)$ and $(p-1)/2$. If $d = k-1$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for no edge of difference $m_p(d) = (p-1)/2$. If $2k-1 \leq d \leq 3k-2$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of each of the two differences $m_p(d) = 2d - 4k + 3$ and $(p-1)/2$. If $3k-1 \leq d \leq 4k-2$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of each of the two differences $m_p(d) = 8k - 2d - 3$ and $(p-1)/2$. Notice that: $\{m_p(d) \mid 0 \leq d \leq 2k-2, d \neq k-1\}$ is the multiset consisting of two copies of each even difference except for the difference $(p-1)/2$; $m_p(k-1) = (p-1)/2$; and each of $\{m_p(d) \mid 2k-1 \leq d \leq 3k-2\}$ and $\{m_p(d) \mid 3k-1 \leq d \leq 4k-2\}$ is the set consisting of one copy of each odd difference. Hence $\{S_m \mid m = m_p(d), d \in \mathbb{Z}_{p-2}\}$ is a 2-factorization of $2K_{p-1}$ on the vertex set $V \setminus \{0\}$.

Suppose $p = 4k + 3$ for some integer k. If $0 \le d \le 2k - 1$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of each of the two differences $m_p(d) = D_{p-1}(2d + 2)$ and $(p - 1)/2$. If $2k \le d \le 4k$ and $d \ne 3k$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for just one edge of each of the two differences $m_p(d) = 2k + 1 - 2|d - 3k|$ and $(p - 1)/2$. If $d = 3k$ then $g_d(C)$ contains two edges of each difference in $(\mathbb{Z}_{(p+1)/2} \setminus \{0\}) \cup \{\infty\}$, except for no edge of difference $m_p(d) = (p - 1)/2$. Again notice that: $\{m_p(d) \mid 0 \le d \le 2k - 1\}$ is the multiset consisting of two copies of each even difference; each of $\{m_p(d) \mid 2k \le d \le 3k - 1\}$ and $\{m_p(d) \mid 3k + 1 \le d \le 4k\}$ is the set consisting of one copy of each odd difference except for the difference $(p - 1)/2$; and $m_p(3k) = (p - 1)/2$. Hence $\{S_m \mid m = m_p(d), d \in \mathbb{Z}_{p-2}\}$ is a 2-factorization of $2K_{p-1}$ on the vertex set $V \setminus \{0\}$.

Therefore, for each $d \in \mathbb{Z}_{p-2}$, $F_d = \{g_d(C) + v \mid v \in \mathbb{Z}_{p-1}\} \cup \{S_{m_p(d)}\}$ is an almost resolvable 2-factorization of $2K_p$. (Recall that $S_{(p-1)/2}$ is the 2-factor of $2K_{p-1}$ in which each edge of difference $(p - 1)/2$ appears twice.)

For each $d \in \mathbb{Z}_{p-2}$, $g_d(C)$ is a cycle on the vertex set $V = \mathbb{Z}_{p-1} \cup \{\infty\}$ with deficiency 0. Let H_d be formed from $g_d(C)$ by removing the vertex 0 and renaming vertex i with $i - 1$ for $1 \le i \le p - 2$. There is a clear one-to-one correspondence between $E(g_d(C))$ and $E(H_d)$, so showing that $\{H_d \mid d \in \mathbb{Z}_{p-2}\}$ is a 2-factorization of $2K_{p-1}$ proves condition (i) for the case $v = 0$. Under this bijection, for each $d \in \mathbb{Z}_{p-2}$, the near 2-factor $g_d(C)$ corresponds to the 2-factor $h_d = (h_d(0), ..., h_d(p - 2))$ where

$$h_d(i) = \begin{cases} (-1)^{i+1}\lceil i/2 \rceil + d & \text{if } i \in \mathbb{Z}_{p-2} \\ \infty & \text{if } i = p - 2, \end{cases}$$

with all arithmetic reduced to modulo $p - 2$. So H_d is a hamiltonian cycle formed in a way similar to Walecki's construction, and so clearly $\{H_d \mid d \in \mathbb{Z}_{p-2}\}$ is a hamiltonian decomposition of $2K_{p-1}$. For each $v \in \mathbb{Z}_{p-1}$, since $g_d(C) + v \equiv g_d(C)$, the same argument used when $v = 0$ shows that $\{g_d(C) + v \mid d \in \mathbb{Z}_{p-2}\}$ provides the 2-factorization required by condition (i) for the case where the deficiency is $v \in \mathbb{Z}_{p-1}$. Finally, note that the near 2-factors with deficiency $v = \infty$, namely those in $\{S_m \mid m = m_p(d), d \in \mathbb{Z}_{p-2}\}$, form a 2-factorization of $2K_{p-1}$. So, condition (i) is satisfied for all $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$.

We now consider condition (ii). Clearly, the cycles $g_d(C)$ with deficiency 0 are edge-disjoint for any set of $(p-3)/2$ consecutive values of d. Similarly, the cycles $g_d(C) + v$ with deficiency v where $v \in \mathbb{Z}_{p-1}$ are edge-disjoint for any set of $(p-3)/2$ consecutive values of d. In order to guarantee that the near 2-factors with deficiency $v = \infty$ are also edge-disjoint we must avoid $S_{(p-1)/2}$ (since it contains two copies of each edge). Therefore, recalling that $p \in \{4k+1, 4k+3\}$, we can define $\mathcal{F}^{'} = \{F_k, F_{k+1}, ..., F_{k+(p-5)/2}\}$, which provides (ii). Notice that

(‡) the near 2-factors in $F_k, F_{k+1}, ..., F_{k+(p-5)/2}$ with deficiency $v \neq \infty$ each contain two edges of each difference, except for one edge of difference $2(k-1), 2(k-2), ..., 2, 1, 3, ..., 2k-1$ respectively if $p = 4k+1$, and except for one edge of difference $2k, 2(k-1), ..., 2, 1, 3, ..., 2k-1$ respectively if $p = 4k+3$.

To see that condition (iii) is satisfied, first note that if $v \in \mathbb{Z}_{p-1}$ then each edge in K_{p-1} on the vertex set $V \setminus \{v\}$ occurs in one of the cycles $g_d(C) + v$ for any $(p-1)/2$ consecutive values of d. So, defining $F = F_{k-1}$ and $F = F_{3k} = F_{k+(p-3)/2}$ if $p = 4k+1$ and if $p = 4k+3$ respectively provides condition (iii), since in both cases the near 2-factor in F with deficiency ∞ is $S_{(p-1)/2}$.

Finally to show that condition (iv) is satisfied, for each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ let $F(v)$ be the near 2-factor in F with deficiency v, and let $F^1(v)$ be the subgraph of $K_p - v$ formed by the edges occurring in no near 2-factors with deficiency v in $\mathcal{F}^{'}$; so by (ii) and (iii) $F^1(v)$ is a subgraph of $F(v)$. By (ii) and (iii), each vertex in $(\mathbb{Z}_{p-1} \cup \{\infty\}) \setminus \{v\}$ must have degree 1 in $F^1(v)$, so $F^1(v)$ is a near 1-factor of K_p with deficiency v. We now show that $F^1 = \{F^1(v) \mid v \in \mathbb{Z}_{p-1} \cup \{\infty\}\}$ is the required near 1-factorization of K_p. Since $F^1(\infty) = \{\{i, i+(p-1)/2\} \mid i \in \mathbb{Z}_{(p-1)/2}\}$ uses all the edges of difference $(p-1)/2$ once, and since clearly for each $v \in \mathbb{Z}_{p-1}$ $F^1(v+1) = F^1(v) + 1$ (reducing the sums modulo $v-1$), it remains to show that the set of differences of the edges in $F^1(0)$ is $\{1, 2, ..., (p-3)/2\} \cup \{\infty\}$. To see this, note that by (‡) there are $(p-3)/2$ near 2-factors in $\mathcal{F}'$ with deficiency 0 which together contain exactly $p-4$ edges of each of the differences in $D = \{1, 2, ..., (p-3)/2\}$ and exactly $p-3$ edges of difference ∞. Since in the subgraph $K_{p-1} - v$ of K_p there are exactly $p-3$ edges of difference $d \in D$ (two of the $p-1$ edges of difference d in K_p are incident with v) and exactly $p-2$ edges of difference ∞ (one of the $p-1$ edges of difference ∞ in K_p is incident with v), one edge of each difference in $D \cup \{\infty\}$ must appear in $F^1(0)$.

□

4.2 Main Results

Theorem 4.2.1. *There exists a fair 1-factorization of $K(n,p)$ if and only if np is even.*

Proof. If np is odd then $K(n,p)$ has no 1-factors, so clearly no 1-factorization exists. If np is even then the result follows immediately by Proposition 4.1.1 and Corollary 3.1.2. □

Theorem 4.2.2. *There exists a fair holey 1-factorization of $K(n,p)$ if and only if $n(p-1)$ is even and $p \neq 2$.*

Proof. It is clear that $K(n,p)$ has no holey 1-factors when $n(p-1)$ is odd. Also, no holey 1-factors exist in $K(n,p)$ when $p = 2$. So the necessity is clear.

The result is trivial if $p = 1$, so to prove the sufficiency we can assume that $p \geq 3$.

First suppose that p and n are both even. By Proposition 4.1.2, there exist $(p-2)/2$ almost resolvable 2-factorizations of $2K_p$, say $F_0, ..., F_{(p-4)/2}$, on the vertex set $\mathbb{Z}_p$ such that for each $v \in \mathbb{Z}_p$ the almost parallel classes with deficiency v form a 2-factorization of K_{p-1} on the vertex set $\mathbb{Z}_p \setminus \{v\}$. Extend this list by defining $F_i = F_j$ if and only if $i \equiv j$ (mod $(p-2)/2$). From this extended list, form a sequence $T = (T_0, ..., T_{(np/2)-1})$ of $np/2$ almost parallel classes of nK_p where for $0 \leq i \leq p-1$ and $0 \leq k \leq (n/2)-1$, T_{i+kp} is the almost parallel class in F_k with deficiency i. For each $i \in \mathbb{Z}_p$ let $G(i)$ be the subgraph of n^2K_p induced by the edges in $\bigcup_{k=0}^{(n/2)-1} T_{i+kp}$, and color all edges in $G(i)$ with i. To complete this coloring to an np-edge-coloring of n^2K_p, for each $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{np}$, let $G(j) = G(i)$ if $i \equiv j \pmod{p}$. Then color class i consists of the $n(p-1)/2$ edges of $n/2$ almost parallel classes which all have the same deficiency i for some $i \in V(K_p)$, and hence each color class is an n-regular subgraph of $n^2K_p - i$. Furthermore, by condition (i) of Proposition 4.1.2, the $n(p-1)/2$ edges in each color class are shared out evenly among the $(p-1)(p-2)/2$ pairs of vertices in $\mathbb{Z}_p \setminus \{i\}$, so between any such pair of vertices there are $\lceil (n(p-1)/2)/((p-1)(p-2)/2) \rceil = \lceil n/(p-2) \rceil$ or $\lfloor (n(p-1)/2)/((p-1)(p-2)/2) \rfloor = \lfloor n/(p-2) \rfloor$ edges colored i. Therefore this np-edge-coloring is fair. By Corollary 3.1.3 we conclude that there exists a fair holey 1-factorization of $K(n,p)$.

Next suppose p is odd and n is even. By Proposition 4.1.3, there exist $p-2$ almost resolvable 2-factorizations of $2K_p$ on the vertex set $\mathbb{Z}_p$ such that

(1) the almost parallel classes with deficiency v form a 2-factorization of $2K_{p-1}$,

and in which there exists a subset Σ of $(p-1)/2$ almost resolvable 2-factorizations satisfying the two additional properties

(2) for each $v \in \mathbb{Z}_p$, each edge of K_p occurs in one of the $(p-1)/2$ almost parallel classes with deficiency v in Σ, and

(3) Σ contains $(p-3)/2$ almost resolvable 2-factorizations in which for each $v \in \mathbb{Z}_p$ the almost parallel classes with deficiency v are edge-disjoint.

Label the almost resolvable 2-factorizations of $2K_p$ with $F_0, ..., F_{p-3}$ such that for each $v \in \mathbb{Z}_p$ the almost parallel classes with deficiency v in $F_0, ..., F_{(p-5)/2}$ are edge-disjoint, and each edge of K_{p-1} on the vertex set $V \setminus \{v\}$ is contained in at least one almost parallel class in $\{F_0, ..., F_{(p-3)/2}\}$. Extend this list by defining $F_i = F_j$ if $i \equiv j \pmod{p-2}$. From this extended list form a sequence $T = (T_0, ..., T_{(np/2)-1})$ of $np/2$ almost parallel classes of nK_p where for $0 \leq i \leq p-1$ and $0 \leq k \leq (n/2)-1$, T_{i+kp} is the almost parallel class in F_k with deficiency i. For each $i \in \mathbb{Z}_p$ let $G(i)$ be the subgraph of n^2K_p induced by the edges in $\bigcup_{k=0}^{(n/2)-1} T_{i+kp}$, and color all edges in $G(i)$ with i. To complete this coloring to an np-edge-coloring of n^2K_p, for each $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{np}$, let $G(j) = G(i)$ if $i \equiv j \pmod p$. Then color class i consists of the $n(p-1)/2$ edges of $n/2$ almost parallel classes which all have the same deficiency i for some $i \in V(K_p)$, and hence each color class is an n-regular subgraph of $n^2K_p - i$. Furthermore, by Proposition 4.1.3, the $n(p-1)/2$ edges in each color class are shared out evenly among the $(p-1)(p-2)/2$ pairs of vertices in $\mathbb{Z}_p \setminus \{i\}$, so between any such pair of vertices there are $\lceil (n(p-1)/2)/((p-1)(p-2)/2) \rceil = \lceil n/(p-2) \rceil$ or $\lfloor (n(p-1)/2)/((p-1)(p-2)/2) \rfloor = \lfloor n/(p-2) \rfloor$ edges colored i. Therefore this np-edge-coloring is fair. By Corollary 3.1.3 we conclude that there exists a fair holey 1-factorization of $K(n,p)$.

Finally suppose that p and n are both odd. By condition (iv) of Proposition 4.1.3 there exists an almost resolvable 1-factorization F^1 such that for each $v \in V(2K_p)$ the almost parallel class with deficiency v in F^1, together with the almost parallel classes in $\mathcal{F}'$ with deficiency v partition the edges of K_{p-1} on the vertex set $V \setminus \{v\}$. Relabel the $(p-1)/2$ almost resolvable factorizations in $\mathcal{F}' \cup \{F^1\}$ with $F_0, F_1, ..., F_{(p-3)/2}$ such that $F_0 = F^1$. Extend this renaming so that $F_{i+(p-1)/2} = F_{(p-3)/2-i}$ for each $i \in \mathbb{Z}_{(p-1)/2}$ and so that $F_{i+j(p-1)} = F_i$ for each $i \in \mathbb{Z}_{p-1}$ and each positive integer j. The plan is to choose the factorizations in this order. So for example the ordering of the first $p-1$ near factorizations is $F_0, F_1, ..., F_{(p-3)/2}, F_{(p-3)/2}, F_{(p-5)/2}, ..., F_0$. This appears to be an unusual ordering, but is essential for the reason that is explained below. Form a sequence $T = (T_0, ..., T_{p(n+2\lfloor n/(2p-4)\rfloor+1)/2-1})$ of almost parallel classes of K_p, where for $0 \le i \le p-1$ and $0 \le k \le (n + \lceil n/(p-2) \rceil)/2 - 1$, T_{i+kp} is the almost parallel class (a near 1-factor or a near 2-factor) in F_k with deficiency i (it may help to note that exactly $p(2\lfloor n/(2p-4) \rfloor + 1)$ of the almost parallel classes in T are near 1-factors, the rest being near 2-factors). In this case, it is important to note that the ordering in which the factorizations are used is chosen to ensure that each color class is regular of odd degree and balanced (a multigraph G is said to be *balanced* if the multiplicity between any two pairs of vertices differs by at most

1). The role of F_0 is critical in this regard. Since the factorizations are chosen in the order $F_0, F_1, ..., F_{(p-3)/2}, F_{(p-3)/2}, F_{(p-5)/2}, ..., F_0$, the critical observations are

(∗) if $(X_0, ..., X_{p-2}) = (F_0, F_1, ..., F_{(p-3)/2}, F_{(p-3)/2}, F_{(p-5)/2}, ..., F_0)$, then for all $v \in \mathbb{Z}_p$, $\bigcup_{j \in \mathbb{Z}_{p-1}} X_j(v) = 2K_p$, and

(∗∗) for any $i < p-1$ and all $v \in \mathbb{Z}_p$, $\bigcup_{j \in \mathbb{Z}_i} X_j(v)$ is balanced (since $\bigcup_{j \in \mathbb{Z}_{(p-1)/2}} X_j(v) = K_p$) and regular of odd degree (since F_0 is the only graph where vertices have odd degree).

For each $i \in \mathbb{Z}_p$ let $G(i)$ be the subgraph of n^2K_p induced by the edges in $\bigcup_{k=0}^{(n+2\lfloor n/(2p-4) \rfloor+1)/2-1} T_{i+kp}$. Color all edges in $G(i)$ with i. To complete this coloring to an np-coloring of n^2K_p, for each $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{np}$, let $G(j) = G(i)$ if $i \equiv j \pmod p$. Then color class i consists of edges in $2\lfloor n/(2p-4) \rfloor + 1$ near 1-factors and $(n - 2\lfloor n/(2p-4) \rfloor - 1)/2$ near 2-factors which all have the same deficiency i for some $i \in V(K_p)$, and hence each color class is an n-regular subgraph of $n^2K_p - i$. By (∗) and (∗∗) this np-coloring is fair. By Corollary 3.1.3 we conclude that there exists a fair holey 1-factorization of $K(n, p)$.

□

4.3 Final Remark

It is worth noting that there is another notion of fairness one could define from the perspective of each color class of $K(n, p)$, requiring that its edges are shared out as evenly as possible among the permitted pairs of parts of $K(n, p)$ (if the edges colored c induce a holey 1-factor with deficiency V_i, then the permitted pairs of parts are those which do not include V_i since vertices in V_i are incident with no edges colored i). Theorems 4.2.1 and 4.2.2 each guarantee that each color class does satisfy this additional fairness property. To see this, note that in each theorem the n^2 edges between each pair of parts are colored with k colors, where $k = n(p-1)$ in Theorem 4.2.1 and $k = n(p-2)$ in Theorem 4.2.2. So for each color c, the number of edges between vertices in a permitted pair of parts is $\lfloor n^2/k \rfloor$ or

$\lceil n^2/k \rceil$ as required. In fact it is easy to observe that for any partition of the vertices of a graph G and for any edge-coloring (or holey edge-coloring) of G, the first notion of fairness together with the extra condition that the number of edges between a pair of parts differs by at most 1 from the number of edges between each other pair of parts implies the new fairness condition. Note that the extra condition is necessary in order to establish this implication. For example, consider the graph G with a partition $\{P_1, P_2, P_3\}$ of the vertex set of G, where $P_1 = \{v_1\}$, $P_2 = \{v_2, v_3\}$, and $P_3 = \{v_4, v_5\}$. Let $E(G) = \{v_1v_2, v_2v_4, v_2v_5, v_3v_5\}$. Color the edges v_1v_2 and v_2v_4 with 0 and the edges v_2v_5 and v_3v_5 with 1. This edge-coloring of G clearly satisfies the first notion of fairness, but not the new one. Notice that there are three edges between the parts P_2 and P_3, and there is only one edge between the parts P_1 and P_2. Conversely, it turns out that for any partition P of the vertices of a graph G into p parts and for any edge-coloring (or holey edge-coloring) of G with the color set $\mathbb{Z}_k$, the new fairness notion together with the condition $||E(G(i))| - |E(G(j))|| \leq 1$ (where $i, j \in \mathbb{Z}_k$) implies that the first notion of fairness is satisfied. Note that the implication is not necessarily true without assuming the extra condition. For example, consider the graph G with a partition $\{P_1, P_2, P_3\}$ of the vertex set of G, where $P_1 = \{v_1\}$, $P_2 = \{v_2, v_3\}$, and $P_3 = \{v_4, v_5\}$. Let $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_3v_5\}$. Color the edges $v_1v_2, v_1v_4, v_2v_4, v_3v_5$ with 0 and the edges v_1v_3, v_1v_5 with 1. This edge-coloring of G clearly satisfies the new notion of fairness, but not the original notion of fairness. Notice that $|E(G(0))| = 4$ and $|E(G(1))| = 2$.

Chapter 5

Fair Holey Hamiltonian Decompositions of Complete Multipartite Graphs and Long Cycle Frames

5.1 Coloring Results

In this section some special edge-colorings of $2K_p$ are found which are used in conjunction with Corollary 3.1.4 in the proof of the main theorem, Theorem 5.2.1 in the next section.

Proposition 5.1.1. *Suppose $p > 2$ is even. There exists a set $\mathcal{F} = \{F_d \mid d \in \mathbb{Z}_{(p-2)/2}\}$ of $(p-2)/2$ almost resolvable 2-factorizations of $2K_p$ such that*

(i) for each vertex $v \in V(2K_p)$ the $(p-2)/2$ almost parallel classes in $\bigcup_{F \in \mathcal{F}} F$ with deficiency v form a 2-factorization of K_{p-1}, and

(ii) there exists an ordering of the elements in $\mathcal{F}$ such that for each $v \in V(2K_p)$ and for each pair of consecutive almost resolvable 2-factorizations in $\mathcal{F}$, the union of the two almost parallel classes with deficiency v is connected.

Proof. The constructive proof of (i) is described in Proposition 4.1.2. With the notation of Proposition 4.1.2, $F_d = \{g_d(C)+t \mid t \in \mathbb{Z}_{p-1}\} \cup \{S_{x(d)}\}$ where $x(d) = \min\{2d+2, p-2d-3\}$ is an almost resolvable 2-factorization of $2K_p$, where $g_d(C)+t$ has deficiency t and $S_{x(d)}$ has deficiency ∞. Define $F_d(v)$ to be the near 2-factor with deficiency v in F_d, where $d \in \mathbb{Z}_{(p-2)/2}$ and $v \in V(2K_p)$. For each $v \in V \setminus \{\infty\}$ and for each $d \in \mathbb{Z}_{(p-2)/2}$, $F_d(v)$ is a near hamiltonian cycle and hence the edges in $F_d(v)$ form a connected subgraph of $2K_{p-1}$. So (ii) holds for each $v \neq \infty$ for any ordering of the almost resolvable 2-factorizations. Now suppose $v = \infty$. If $p = 4k$ for some positive integer k, then $(F_0(\infty), ..., F_{2k-2}(\infty)) = (S_2, S_4, ..., S_{2k-2}, S_{2k-1}, S_{2k-3}, ..., S_1)$, and if $p = 4k+2$ for some positive integer k, then

$(F_0(\infty), ..., F_{2k-1}(\infty)) = (S_2, S_4, ..., S_{2k}, S_{2k-1}, S_{2k-3}, ..., S_1)$. So, the union of any two consecutive almost parallel classes in this list is either of the form $S_d \cup S_{d+2}$ or $S_d \cup S_{d+1}$, both of which are clearly connected by Lemma 3.1.1 since $p-1$ is odd. So (ii) holds for all v with the ordering $F_0, F_1, ..., F_{(p-4)/2}$ of the elements of $\mathcal{F}$.

$\square$

We illustrate the construction defined in the proof of Proposition 5.1.1 with the following example.

Example 5.1.2. Proposition 5.1.1 yields the 4 almost resolvable 2-factorizations of $2K_{10}$ below.

$F_0 = \{g_0(C) + t \mid t \in \mathbb{Z}_9\} \cup \{F_0(\infty)\} = \{(1, 2, 8, 3, 7, 4, 6, 5, \infty) + t \mid t \in \mathbb{Z}_9\} \cup \{S_2\}$,

$F_1 = \{g_1(C) + t \mid t \in \mathbb{Z}_9\} \cup \{F_1(\infty)\} = \{(2, 3, 1, 4, 8, 5, 7, 6, \infty) + t \mid t \in \mathbb{Z}_9\} \cup \{S_4\}$,

$F_2 = \{g_2(C) + t \mid t \in \mathbb{Z}_9\} \cup \{F_2(\infty)\} = \{(3, 4, 2, 5, 1, 6, 8, 7, \infty) + t \mid t \in \mathbb{Z}_9\} \cup \{S_3\}$,

$F_3 = \{g_3(C) + t \mid t \in \mathbb{Z}_9\} \cup \{F_3(\infty)\} = \{(4, 5, 3, 6, 2, 7, 1, 8, \infty) + t \mid t \in \mathbb{Z}_9\} \cup \{S_1\}$.

Notice that for each $t \in \mathbb{Z}_9$, $g_0(C) + t$, $g_1(C) + t$, $g_2(C) + t$ and $g_3(C) + t$ form a 2-factorization of K_9 on the vertex set $(\mathbb{Z}_9 \setminus \{t\}) \cup \{\infty\}$; and $F_0(\infty) = S_2$, $F_1(\infty) = S_4$, $F_2(\infty) = S_3$ and $F_3(\infty) = S_1$ form a 2-factorization of K_9 on the vertex set $\mathbb{Z}_9$. Hence condition (i) is satisfied. Also note that taking the ordering $(F'_0, F'_1, F'_2, F'_3) = (F_0, F_1, F_2, F_3)$ of $\mathcal{F}$ satisfies condition (ii).

A companion result to Proposition 5.1.1 holds for the case when p is odd.

Proposition 5.1.3. *Suppose $p > 1$ is odd. There exists a set $\mathcal{F} = \{F_d \mid d \in \mathbb{Z}_{p-2}\}$ of $p-2$ almost resolvable 2-factorizations of $2K_p$ such that*

(i) for each vertex $v \in V(2K_p)$ the $p-2$ almost parallel classes in $\bigcup_{F \in \mathcal{F}} F$ with deficiency v form a 2-factorization of $2K_{p-1}$,

(ii) there exists $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| = (p-3)/2$ and for each vertex $v \in V(2K_p)$ the almost parallel classes with deficiency v in $\mathcal{F}'$ are edge-disjoint,

(iii) *there exists $F \in \mathcal{F} \setminus \mathcal{F}^{'}$ such that for each $v \in V$ each edge in K_{p-1} on the vertex set $V \setminus \{v\}$ occurs in a 2-factor in $\mathcal{F}^{'} \cup \{F\}$, and*

(iv) *there exists an ordering $(F_0^{'}, F_1^{'}, ..., F_{p-3}^{'})$ of the elements of $\mathcal{F}$ such that $F_i^{'} \in \mathcal{F}'$ if $0 \leq i \leq (p-5)/2$ and $F_{(p-3)/2}^{'} = F$, with the additional property that for each $v \in V(2K_p)$ and for each pair of consecutive almost resolvable 2-factorizations in $\mathcal{F}$, the union of the two almost parallel classes with deficiency v is connected.*

Proof. A constructive proof of (i), (ii) and (iii) is described in Proposition 4.1.3. With the notation of Proposition 4.1.3, we see that $F_d = \{g_d(C) + t \mid t \in \mathbb{Z}_{p-1}\} \cup \{S_{x(d)}\}$ where

$$x(d) = \begin{cases} \min\{2d+2, p-3-2d\} & \text{if } 0 \leq d \leq (p-5)/2 \\ \min\{2d-p+4, 2(p-d)-5\} & \text{if } (p-3)/2 \leq d \leq p-3 \end{cases}$$

is an almost resolvable 2-factorization of $2K_p$, where $g_d(C)+t$ has deficiency t and $S_{x(d)}$ has deficiency ∞. Note that for each $v \in V \setminus \{\infty\}$ and for each $d \in \mathbb{Z}_{p-2}$, $F_d(v)$ is a near hamiltonian cycle. So clearly the additional property described in (iv) holds for all $v \in V \setminus \{\infty\}$ regardless of the ordering of the elements of $\mathcal{F}$. To see that conditions (ii-iv) hold, consider the ordering $(F_0^{'}, ..., F_{p-3}^{'})$ of the almost resolvable 2-factorizations, where $F_i^{'} = F_{(-1)^i \lceil i/2 \rceil + (p-3)/2}$ for $0 \leq i \leq p-3$. Let $\mathcal{F}^{'} = \{F_i^{'} \mid i \in \mathbb{Z}_{(p-3)/2}\}$ and $F = F_{(p-3)/2}^{'}$. Then since $(F_0(\infty), ..., F_{4k-2}(\infty)) = (S_2, S_4, ..., S_{2k}, S_{2k-2}, S_{2k-4}, ..., S_2, S_1, S_3, ..., S_{2k-1}, S_{2k-1}, S_{2k-3}, ..., S_1)$ when $p = 4k+1$ for some k, and since $(F_0(\infty), ..., F_{4k}(\infty)) = (S_2, S_4, ..., S_{2k}, S_{2k}, S_{2k-2}, S_{2k-4}, ..., S_2, S_1, S_3, ..., S_{2k+1}, S_{2k-1}, ..., S_1)$ when $p = 4k+3$, it follows that in all cases $(F_0^{'}(\infty), F_1^{'}(\infty), ..., F_{p-3}^{'}(\infty)) = (S_1, S_2, ..., S_{(p-3)/2}, S_{(p-1)/2}, S_{(p-3)/2}, ..., S_1)$. So using this ordering, clearly properties (ii) and (iii) are satisfied, and since the union of any two consecutive almost parallel classes with deficiency ∞ is of the form $S_d \cup S_{d+1}$, which is clearly connected by Lemma 3.1.1, property (iv) is also satisfied.

□

We illustrate the construction defined in the proof of Proposition 4.1.2 with the following example.

Example 5.1.4. Proposition 4.1.2 yields the 5 almost resolvable 2-factorizations of $2K_7$ below.

$F_0 = \{g_0(C) + t \mid t \in \mathbb{Z}_6\} \cup \{F_0(\infty)\} = \{(1, 2, 5, 3, 4, \infty) + t \mid t \in \mathbb{Z}_6\} \cup \{S_2\}$,

$F_1 = \{g_1(C) + t \mid t \in \mathbb{Z}_6\} \cup \{F_1(\infty)\} = \{(2, 3, 1, 4, 5, \infty) + t \mid t \in \mathbb{Z}_6\} \cup \{S_2\}$,

$F_2 = \{g_2(C) + t \mid t \in \mathbb{Z}_6\} \cup \{F_2(\infty)\} = \{(3, 4, 2, 5, 1, \infty) + t \mid t \in \mathbb{Z}_6\} \cup \{S_1\}$,

$F_3 = \{g_3(C) + t \mid t \in \mathbb{Z}_6\} \cup \{F_3(\infty)\} = \{(4, 5, 3, 1, 2, \infty) + t \mid t \in \mathbb{Z}_6\} \cup \{S_3\}$,

$F_4 = \{g_4(C) + t \mid t \in \mathbb{Z}_6\} \cup \{F_4(\infty)\} = \{(5, 1, 4, 2, 3, \infty) + t \mid t \in \mathbb{Z}_6\} \cup \{S_1\}$.

Notice that for each $t \in \mathbb{Z}_6$, $g_0(C)+t$, $g_1(C)+t$, $g_2(C)+t$, $g_3(C)+t$ and $g_4(C)+t$ form a 2-factorization of $2K_6$ on the vertex set $(\mathbb{Z}_6 \setminus \{t\}) \cup \{\infty\}$; and $F_0(\infty) = S_2$, $F_1(\infty) = S_2$, $F_2(\infty) = S_1$, $F_3(\infty) = S_3$ and $F_4(\infty) = S_1$ form a 2-factorization of $2K_6$ on the vertex set $\mathbb{Z}_6$. Hence condition (i) is satisfied. Also note that taking the ordering $(F_0', F_1', F_2', F_3', F_4') = (F_2, F_1, F_3, F_0, F_4)$ of $\mathcal{F}$ satisfies conditions (ii)-(iv) where $\mathcal{F}' = \{F_2, F_1\}$ and $F = F_3$.

5.2 Main Result

Theorem 5.2.1. *There exists a fair holey hamiltonian decomposition of $K(n, p)$ if and only if n is even and $p \neq 2$.*

Proof. Each vertex of $K(n, p)$ has degree $n(p - 1)$ and such a decomposition requires $np/2$ almost parallel classes. Together, these two conditions imply that n must be even. Also, no holey 2-factors exist in $K(n, p)$ when $p = 2$. So the necessity is clear.

The result is trivial if $p = 1$, so to prove the sufficiency we can assume that $p \geq 3$.

Suppose that n and p are both even. Let $F_0, ..., F_{(p-4)/2}$ be the almost resolvable 2-factorizations of $2K_p$ as given in Proposition 5.1.1. Define $T = \{T(i, k) \mid i \in \mathbb{Z}_p, k \in \mathbb{Z}_n\}$, where $T(i, k)$ is the almost parallel class in $F_{k'}$ with deficiency i and $k' \equiv k$ (modulo $(p - 2)/2$). Define $G(i) = \bigcup_{k=0}^{n-1} T(i', k)$ where $i' \equiv i$ (modulo p), and color all edges in $G(i)$

with i. To complete this coloring to an $np/2$-edge-coloring of n^2K_p, for each $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{np/2}$, let $G(j) = G(i)$ if $i \equiv j$ (modulo p). Then for each $i \in \mathbb{Z}_p$ color class i consists of the $n(p-1)$ edges of n almost parallel classes which all have the same deficiency, namely $i \in V(K_p)$, and hence color class i is a $2n$-regular subgraph of $n^2K_p - i$. By condition (i) of Proposition 5.1.1, between each pair of vertices in $\mathbb{Z}_p \setminus \{i\}$, the number of edges colored i is $\lceil n/((p-2)/2) \rceil = \lceil 2n/(p-2) \rceil$ or $\lfloor n/((p-2)/2) \rfloor = \lfloor 2n/(p-2) \rfloor$ (since each color class is formed by n almost parallel classes). Therefore this $np/2$-edge-coloring is fair. Furthermore, by condition (ii) of Proposition 5.1.1, each color class is connected. By Corollary 3.1.4 we conclude that there exists a fair holey hamiltonian decomposition of $K(n,p)$.

Next suppose that p is odd and n is even. Let $F_0, ..., F_{p-3}$ be the almost resolvable 2-factorizations of $2K_p$ as given in Proposition 5.1.3. Define $T = \{T(i,k) \mid i \in \mathbb{Z}_p, k \in \mathbb{Z}_n\}$, where $T(i,k)$ is the almost parallel class in $F_{k'}$ with deficiency i and $k' \equiv k$ (modulo $p-2$). Define $G(i) = \bigcup_{k=0}^{n-1} T(i',k)$ where $i' \equiv i$ (modulo p), and color all edges in $G(i)$ with i. To complete this coloring to an $np/2$-edge-coloring of n^2K_p, for each $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{np/2}$, let $G(j) = G(i)$ if $i \equiv j$ (modulo p). Then for each $i \in \mathbb{Z}_p$ color class i consists of the $n(p-1)$ edges of n almost parallel classes which all have the same deficiency, namely $i \in V(K_p)$, and hence color class i is a $2n$-regular subgraph of $n^2K_p - i$. By condition (i) of Proposition 5.1.3 for each i, each set of $p-2$ consecutive almost parallel classes with deficiency i forms a 2-factorization of $2K_{p-1}$. Therefore the union of the first $\lfloor n/(p-2) \rfloor$ almost parallel classes accounts for exactly $2\lfloor n/(p-2) \rfloor$ edges colored i between each pair of vertices. Let x be the number of the remaining almost parallel classes. Consider the union of these x almost parallel classes. By (ii) and (iii) of Proposition 5.1.3, in this union the number of edges colored i between each pair of vertices is in $\{0,1\}$ if $0 \le x \le (p-3)/2$, and is in $\{1,2\}$ if $(p-1)/2 \le x < p-2$. Therefore this $np/2$-edge-coloring is fair. Furthermore, by condition (iv) of Proposition 5.1.3, each color class is connected. By Corollary 3.1.4 we conclude that there exists a fair holey hamiltonian decomposition of $K(n,p)$.

$\square$

Chapter 6

Introduction to Edge-Colorings with Special Fairness Properties

6.1 Introduction to Edge-Colorings with Special Fairness Properties

When considering edge-colorings of graphs it is usually desired to have some fairness properties imposed on the number of edges colored by each color. Due to de Werra's work in [9, 10, 11, 12] it has been known since the 1970's that for each $k \in \mathbb{N}$ every bipartite graph has a k-edge-coloring that is balanced, equitable and equalized at the same time. One important result for more general graphs is by Hilton, who proved in [16] that each even graph has an evenly-equitable k-edge-coloring for each $k \in \mathbb{N}$, thereby completely settling this problem (see Theorem 7.3.2).

In Chapter 7, first we extend Hilton's result [16] by finding a characterization for graphs that have an evenly-equitable, balanced k-edge-coloring for each $k \in \mathbb{N}$ (see Theorem 7.1.1). We then use this result to find a different kind of characterization for even graphs to have an evenly-equitable, balanced 2-edge-coloring (see Theorem 7.1.2). Then we prove Theorem 7.2.1 and Theorem 7.2.2, the latter of which uses the aforementioned characterization. The proof of Theorem 7.2.2 provides an instance of how evenly-equitable, balanced edge-colorings can be used to ensure that a certain fairness property of factorizations of some regular graphs is satisfied. This particular notion of fairness is defined as follows. A k-factorization of a graph in which the vertices have been partitioned into parts is said to be *fair* if for each two parts (possibly they are the same), the number of edges between these two parts in each factor differs from the number in each other factor by at most one. Finally we address the existence of all other combinations of the three edge-coloring properties (namely: evenly-equitable, balanced and equalized), finding weakest subsets of conditions that will guarantee (if possible) that a graph G has a k-edge-coloring which has the following properties in turn: (P_1) evenly-equitable, balanced and equalized, (P_2) evenly-equitable and equalized, (P_3) balanced and equalized, (P_4) evenly-equitable, (P_5) balanced, and (P_6) equalized.

We give some further terminology that will be useful in Chapter 7. For each proper subset S of the vertex set of a graph G, define the *edge-cut* $E(S, \overline{S}) = \{e = \{x, y\} \mid e \in E(G), x \in S, y \in V(G) \setminus S\}$. Let $r_{G,k}(\{v, w\}) \in \mathbb{Z}_k$ be such that $r_{G,k}(\{v, w\}) \equiv m_G(\{v, w\})$ (modulo k). Let $G^{(k)}$ be the spanning subgraph of G in which for each pair of vertices v and w the number of edges between v and w is $r_{G,k}(\{v, w\})$. Then clearly $deg_G(v) \equiv deg_{G^{(k)}}(v)$ (modulo k) for all $v \in V(G)$. For the purposes of Chapter 7, a vertex $v \in V(G^{(k)})$ is said to be *odd* (*even*) if $(deg_G(v) - deg_{G^{(k)}}(v))/k$ is an odd (even) integer.

In Chapter 8 we consider a new fairness notion, requiring that the number of vertices in the subgraphs induced by the edges of each color are within one of each other. Given a k-edge-coloring of a graph G, for each color $i \in \mathbb{Z}_k$ let $G(i)$ denote the (not necessarily spanning) subgraph of G induced by the edges colored i. Let $\nu_i(G) = |V(G(i))|$. Formally, a k-edge-coloring of a graph G is said to be *vertex-equalized* if for each pair of colors $i, j \in \mathbb{Z}_k$, $|\nu_i(G) - \nu_j(G)| \leq 1$. In Chapter 8, a characterization is found for connected graphs that have vertex-equalized k-edge-colorings for each $k \in \{2, 3\}$ (see Theorem 8.1.2 and Theorem 8.2.1).

Chapter 7

On Evenly-Equitable, Balanced Edge-Colorings and Related Notions

7.1 Coloring Results

The following characterization can be used to find evenly-equitable, balanced k-edge-colorings. The proof has the flavor of Hilton's proof in [16] of the case where the additional property of being balanced was not required, but is modified to deal with extra complications that arise in this new setting.

Theorem 7.1.1. *For each positive integer k, a graph G (possibly with loops) has an evenly-equitable, balanced k-edge-coloring if and only if it has an even, balanced k-edge-coloring.*

Proof. Proving the "only if" result is trivial. To show the "if" result, we first prove the assertion for the case when G is connected and loopless. Let f be an even, balanced k-edge-coloring of G. Among all pairs of colors $i, j \in \mathbb{Z}_k$ and all vertices $v \in V(G)$ suppose that $|deg_{G[i]}(v) - deg_{G[j]}(v)| = 2x$ is as large as possible (where $x \in \mathbb{N}$). If $x \in \{0, 1\}$, then this edge-coloring is evenly-equitable, so assume $x \geq 2$. Let G' be the spanning subgraph of G induced by the edges colored i and j. From G' form a new graph G'' by adding an uncolored loop at each vertex v satisfying $deg_{G'}(v) \equiv 2 \pmod 4$. Then

$$deg_{G''}(v) \equiv 0 \,(\text{mod } 4) \text{ for each vertex } v \in V(G''). \qquad (1)$$

For each pair of vertices $\{v, w\}$ with $v, w \in \mathbb{Z}_n$ and for any color $h \in \mathbb{Z}_k$, let $m_{G[i,j]}(\{v, w\}) = min\{m_{G[i]}(\{v, w\}), m_{G[j]}(\{v, w\})\}$, and let $S_{i,j}(\{v, w\})$ be a set of size $2m_{G[i,j]}(\{v, w\})$ containing precisely $m_{G[i,j]}(\{v, w\})$ edges of each color i and j joining vertices v and w. So $|S_{i,j}(\{v, w\})|$ is even. Let $S_{i,j}(v) = \bigcup_{w \in V(G)\setminus\{v\}} S_{i,j}(\{v, w\})$ and $S_{i,j} = \bigcup_{0 \leq v < w < n} S_{i,j}(\{v, w\})$.

Define $G''' = G'' - S_{i,j}$. Since $|S_{i,j}(v)|$ is even for each $v \in V(G)$, and since the original edge-coloring is even, each component of G''' is an eulerian graph, and has no multiple edges since f is balanced (possibly it has an uncolored loop at some of the vertices). The following argument establishes property (3) below, namely that each component of G''' has an even number of edges. First note that by the assumption of this theorem, for all $h \in \mathbb{Z}_k$ each component of $G[h]$ is eulerian, so

$$\text{the size of each edge-cut in } G[h] \text{ is even (so is also even in } G''[h]\,). \qquad (2)$$

Let C be any component of G''' and let $H = G[S_{i,j}]$. Let $E_1 = E(H[V(C)])$; so $|E_1|$ is even (since there is an even number of edges in $S_{i,j}$ between each pair of vertices). Let E_2 be the edge-cut $H[V(C), V(H) \setminus V(C)]$, which by the definition of $S_{i,j}$ satisfies $H[V(C), V(G) \setminus V(C)] = G''[V(C), V(G) \setminus V(C)]$. So $|E_2 \cap E(H[i])| = |E_2 \cap E(H[j])|$. Furthermore, since for each color $h \in \{i, j\}$ $E_2 \cap E(H[i])$ and $E_2 \cap E(H[j])$ are edge-cuts in $H[i]$ and $H[j]$ respectively, by (2) $|E_2 \cap E(H[i])|$ and $|E_2 \cap E(H[j])|$ are even. Hence $|E_2| = |E_2 \cap E(H[i])| + |E_2 \cap E(H[j])| = 2|E_2 \cap E(H(i))| \equiv 0 \pmod 4$. Then,

$$\begin{aligned}
\sum_{v \in V(C)} deg_{G'''}(v) &= \sum_{v \in V(C)} deg_{G''}(v) - 2|E_1| - |E_2| \\
&\equiv \sum_{v \in V(C)} deg_{G''}(v) \pmod 4 \\
&\equiv 0 \pmod 4 \text{ by (1).}
\end{aligned}$$

So,

$$|E(C)| = (\sum_{v \in V(C)} deg_{G'''}(v))/2 \equiv 0 \,(\mathrm{mod}\ 2). \qquad (3)$$

Let f' be a new 2-edge-coloring of G' formed as follows. For each component C of G''', alternately color the edges of an eulerian circuit of C with i and j. This yields a balanced 2-edge-coloring of G''' (G''' is simple) where by (3) for each vertex $v \in V(G)$,

$$deg_{G'''[i]}(v) = deg_{G'''[j]}(v). \qquad (4)$$

Now add the edges in $S_{i,j}$ with their original colors to G''' and remove the uncolored loops that were added when forming G''. Then clearly the resulting graph is G' and this new 2-edge-coloring f' satisfies $|deg_{G'[i]}(v) - deg_{G'[j]}(v)| \in \{0, 2\}$ for each $v \in V(G')$. To show that f' is also even, consider the following cases (in which $deg_{G'''[i]}(v)$ refers to edge-coloring G''' with f').

Case 1: $deg_{G'}(v) \equiv 0 \pmod 4$. Note that in this case we are not adding a loop at v when forming G''. Now look at the following subcases.

Subcase 1.1: $\sum_{w \in V(G') \setminus \{v\}} m_{G[i,j]}(\{v, w\})$ is odd. So, an odd number of edges incident with v of each color i and j were removed when forming G''' from G''. So, $deg_{G'''}(v) \equiv 2 \pmod 4$ and hence by (4) $deg_{G'''[i]}(v) \equiv deg_{G'''[j]}(v) \equiv 1 \pmod 2$. Putting back the removed edges shows that v is incident with an even number of edges of each color in the edge-coloring f' of G'.

Subcase 1.2: $\sum_{w \in V(G') \setminus \{v\}} m_{G[i,j]}(\{v, w\})$ is even. So, an even number of edges incident with v of each color i and j were removed when forming G'''. So, $deg_{G'''}(v) \equiv 0 \pmod 4$ and hence $deg_{G'''[i]}(v) \equiv deg_{G'''[j]}(v) \equiv 0 \pmod 2$. Putting back the removed edges shows that v is incident with an even number of edges of each color in the edge-coloring f' of G'.

Case 2: $deg_{G'}(v) \equiv 2 \pmod 4$. Note that in this case an uncolored loop is added to v when forming G''. Now look at the following subcases.

Subcase 2.1: $\sum_{w \in V(G') \setminus \{v\}} m_{G[i,j]}(\{v, w\})$ is odd. So, after adding an uncolored loop at v, an odd number of edges incident with v of each color i and j were removed when forming G'''. Then $deg_{G'''}(v) \equiv 2 \pmod 4$, so by (4) in the new edge-coloring $deg_{G'''}(v) = deg_{G'''}(w) \equiv 1 \pmod 2$. So, for each $u \in \{v, w\}$ and each $l \in \{i, j\}$ $deg_{G'[l]}(u) = deg_{G'''[l]}(u) + m_{G[i,j]}(\{v, w\}) \equiv 0 \pmod 2$.

Subcase 2.2: $\sum_{w \in V(G') \setminus \{v\}} m_{G[i,j]}(\{v, w\})$ is even. So, after adding an uncolored loop at v, an even number of edges incident with v of each color i and j were removed when forming G'''. Then $deg_{G'''}(v) \equiv 0 \pmod 4$, so by (4) in the new edge-coloring $deg_{G'''}(v) =$

$deg_{G'''}(w) \equiv 0 \pmod 2$. So, for each $u \in \{v, w\}$ and each $l \in \{i, j\}$ $deg_{G'[l]}(u) = deg_{G'''[l]}(u) + m_{G[i,j]}(\{v, w\}) \equiv 0 \pmod 2$.

Repetition of this procedure yields an evenly-equitably, balanced k-edge-coloring of G.

For the case when G has loops and is possibly disconnected, simply remove all the loops from G and apply this procedure to each component of the resulting loopless graph to get an evenly-equitable, balanced k-edge-coloring of each component. Then put back the loops; it is easy to color them in a balanced way without destroying the evenly-equitable property at each vertex. □

Note that in the statement of Theorem 7.1.1 we cannot replace the condition on the existence of an even, balanced k-edge-coloring by a weaker set of conditions, as is illustrated by the next two examples. A cycle of length 3 with a cycle of length 2 intersecting in one of its vertices is an even graph and clearly has a balanced (and equalized) 2-edge-coloring, but no 2-edge-coloring that is evenly-equitable and balanced. The graph $2K_2$ (the graph with two vertices and two edges joining these two vertices) has an even (actually evenly-equitable) 2-edge-coloring, but no 2-edge-coloring that is evenly-equitable and balanced. While these two graphs are trivial, they can be generalized to more complicated examples.

Theorem 7.1.1 leads to the problem of finding conditions guaranteeing that a graph has an even, balanced k-edge-coloring. The following result addresses that problem. Recall that our unusual definition of even and odd vertices, and of $G^{(2)}$ are given at the end of Section 1.

Theorem 7.1.2. *G has an even, balanced 2-edge-coloring if and only if G is even and $G^{(2)}$ has no components with an odd number of odd vertices.*

Proof. To prove the necessity, suppose that an even, balanced 2-edge-coloring of G is given. Since the given 2-edge-coloring is balanced, for each pair of vertices v and w, the $m_G(\{v, w\}) - r_{G,2}(\{v, w\})$ edges between v and w that are to be deleted when forming $G^{(2)}$ from G can be chosen so that they are shared evenly among the two color classes. Let C be a component in

$G^{(2)}$. Now since the given 2-edge-coloring of G is even, for each color $i \in \mathbb{Z}_2$, an odd vertex in C contributes an odd number to the degree sum of the graph $G^{(2)}[i]$, and an even vertex in C contributes an even number to the degree sum of the graph $G^{(2)}[i]$. Hence the number of odd vertices in C must be even.

To show the sufficiency, color the edges in G as follows. To satisfy the balanced property, for each pair of vertices $\{v, w\} \subseteq V(G)$ color $(m_G(\{v, w\}) - r_{G,2}(\{v, w\}))/2$ (note that by definition of $r_{G,2}$ this is an integer) of the edges between v and w with each color $i \in \mathbb{Z}_2$. Let G^* be the graph induced by the edges that have been colored so far, and note that the graph induced by the uncolored edges is $G^{(2)}$. Also note that by the definition of odd and even vertices, for each $i \in \mathbb{Z}_2$,

$$deg_{G^*[i]}(v) \text{ is odd (even) if and only if } v \text{ is an odd (even) vertex.} \qquad (*)$$

Since G is an even graph and since $m_G(\{v, w\}) - r_{G,2}(\{v, w\})$ is even for each $\{v, w\} \subseteq V(G)$, $G^{(2)}$ is also an even graph. For each component C in $G^{(2)}$ color the edges of an eulerian tour of C as follows. Start by coloring the first edge in the eulerian tour with $i \in \mathbb{Z}_2$ and then switch to $i + 1$ (modulo 2) whenever the eulerian tour reaches an odd vertex for the first time. Note that if the first vertex in the eulerian tour is even, then the first and last edges in the eulerian tour will have the same color because an even number of color switches will occur (by assumption there is an even number of odd vertices). Similarly, if the first vertex, say v, is odd then the first and the last edges will have different colors if $deg_{G^{(2)}}(v) = 2$ (since no color switch is made at v) and they will have the same color if $deg_{G^{(2)}}(v) > 2$ (since then the eulerian tour will pass through v, so a color switch will occur at v). This coloring of the edges in $G^{(2)}$ has the property that for each $v \in V(G)$ and for each $i \in \mathbb{Z}_2$

(i) if v is odd, then $deg_{G^{(2)}[i]}(v)$ is odd, and

(ii) if v is even, then $deg_{G^{(2)}[i]}(v)$ is even.

So, for each $i \in \mathbb{Z}_2$ and each $v \in V(G)$, since $deg_{G[i]}(v) = deg_{G^{(2)}[i]}(v) + deg_{G^*[i]}(v)$, by (*), (i) and (ii) each vertex in $G(i)$ has even degree and hence the given 2-edge-coloring has the desired properties. □

It appears to us that a generalization of Theorem 7.1.2 for three or more colors may be difficult to obtain.

The following result characterizes graphs which have an evenly-equitable, balanced 2-edge-coloring.

Corollary 7.1.3. *Suppose that* G *is an even graph. Then* G *has an evenly-equitable, balanced 2-edge-coloring if and only if* $G^{(2)}$ *has no components with an odd number of odd vertices.*

Proof. This follows immediately by Theorem 7.1.1 and Theorem 7.1.2. □

7.2 An Application Using Amalgamations

In this section edge-colorings that satisfy another notion of equally distributing edges across color classes is considered, namely that of fairness. Not only are the edge-colorings equitable, but also for any given partition P of the vertices, for each two parts in P (possibly they are the same) the edges between vertices in the two parts are equally divided among the color classes. While the results here (Theorem 7.2.1 and Theorem 7.2.2) address general partitions, these types of questions naturally arise when edge-coloring the complete multipartite graph $K_{a_1,\ldots,a_p}$, in which the partition is chosen to be the parts of the graph. For example, it has been shown when there exist fair equitable edge-colorings of $K_{a_1,\ldots,a_p}$ in which each color class induces a hamilton cycle [25] or a 1-factor [14] (see also Chapter 4 of this dissertation).

The following theorem provides a necessary condition for the existence of fair 2-factorizations of $4k$-regular graphs ($k \geq 1$). For any graph G and any partition P of $V(G)$, let $P(G)$ be the ψ-amalgamation of G where ψ maps two vertices in G to the same vertex in $P(G)$ if and only if they are in the same element of P.

Theorem 7.2.1. *Let G be a $4k$-regular graph ($k \geq 1$). Let P be any partition of $V(G)$. Let $H = P(G)$. Suppose that G has a fair $2k$-factorization. Then*

(1) $H^{(2)}$ has no components with an odd number of odd vertices.

Proof. Suppose that G has a fair $2k$-factorization. Let F_1 and F_2 be the subgraphs of H induced by the edges corresponding to the $2k$-factors of G. Since at each vertex in H the number of edge-ends incident with a vertex is a multiple of 4 and since these edge-ends are shared evenly among F_1 and F_2, the number of edge-ends incident with each vertex in H in each of F_1 and F_2 is even. So, by the definition of odd and even vertices, in $H^{(2)}$ an odd vertex is incident with an odd number of edge-ends in each of F_1 and F_2, and an even vertex is incident with an even number of edge-ends in each of F_1 and F_2. Let C be a component of $H^{(2)}$. Clearly $\sum_{v \in V(C)} deg_C(v)$ is an even number and

$$\sum_{v \in V(C)} deg_C(v) = \sum_{v \in V(C) \text{ is odd}} deg_C(v) + \sum_{v \in V(C) \text{ is even}} deg_C(v)$$

where $\sum_{v \in V(C) \text{ is even}} deg_C(v)$ is an even number and each term in $\sum_{v \in V(C) \text{ is odd}} deg_C(v)$ is an odd number by the above observation. Hence the number of odd vertices in $V(C)$ must be even. □

To investigate whether the necessary condition given in Theorem 7.2.1 is also sufficient for a graph to have a fair $2k$-factorization, we introduce the notion of P-equivalence.

Let G_1 and G_2 be two graphs with $V(G_1) = V(G_2) = V$, and let P be a partition of V. Then G_1 is said to be *P-equivalent* to G_2 if for all $V_i, V_j \in P$ (possibly $i = j$) $e(G_1(V_i, V_j)) = e(G_2(V_i, V_j))$, where $e(G_k(V_i, V_j))$ denotes the number of edges in G_k (for $k = 1, 2$) between the parts V_i and V_j. So if G_1 and G_2 are P-equivalent, then $H = P(G_1) = P(G_2)$. If either G_1 or G_2 has a fair $2k$-factorization, then Theorem 7.2.1 shows that (1) must be satisfied. To investigate the strength of (1), in the following Theorem 7.2.2 shows that if G is a 4-regular graph for which $H^{(2)} = P(G)^{(2)}$ satisfies (1), then G is P-equivalent to some graph (which is simple if a certain necessary condition is met) with a fair 2-factorization. Conjecture 1 goes on to make a much stronger claim that if G_1 is P-equivalent to G_2, then G_1 has a fair $2k$-factorization if and only if G_2 does.

Theorem 7.2.2. *Let G_1 be a 4-regular graph. Let P be any partition of $V(G_1)$. Let $H = P(G_1)$. Suppose $H^{(2)}$ has no components with an odd number of odd vertices. Then there exists a graph G_2 such that*

(i) $V(G_1) = V(G_2)$,

(ii) G_2 *is P-equivalent to* G_1,

(iii) G_2 *has a fair 2-factorization (with respect to the given partition P), and*

(iv) G_2 *can be chosen to be simple if and only if for all* $V_i, V_j \in P$, $e(V_i, V_j) \leq |V_i||V_j|$ *if* $i \neq j$, *and* $e(V_i, V_j) \leq |V_i|(|V_i| - 1)/2$ *if* $i = j$.

Note that it is long known by Petersen's 2-factor Theorem (see [2] for example) that every $2k$-regular graph has a 2-factorization. The importance of Theorem 7.2.2 is that if the condition of the theorem is satisfied, then regardless of the partition P that is chosen, the resulting factorization of G_2 (formed with P in mind) is fair.

Proof. By the supposition $H^{(2)}$ has no components with an odd number of odd vertices. Clearly H is even since G_1 is even. So H satisfies the conditions of Corollary 7.1.3 and hence it has an evenly-equitable, balanced 2-edge-coloring. By the evenly-equitable property of this 2-edge-coloring, each color appears on exactly half of the edge-ends incident with each vertex of H (a loop contributes two edge-ends to the incident vertex). Notice that H is the ψ-amalgamation of G_1 where $\psi(v_1) = \psi(v_2)$ if and only if v_1 and v_2 are in the same element of P. For each $v \in V(H)$ define $\eta(v) = deg_H(v)/4 = |\psi^{-1}(v)|$. By (ii) of Theorem 2.1.1, there exists an η-detachment G_2 of H such that

(1) G_2 is P-equivalent to G_1, and

(2) for each vertex v of H the edges of each color incident with v are shared as evenly as possible among the vertices in $\psi^{-1}(v)$ (that is, the vertices in the corresponding part of G_2).

Note that by (iv) and (iii) of Theorem 2.1.1, G_2 will be simple if for all $V_i, V_j \in P$, $e(V_i, V_j) \leq |V_i||V_j|$ if $i \neq j$, and $e(V_i, V_j) \leq |V_i|(|V_i|-1)/2$ if $i = j$. Clearly these are necessary conditions if the η-detachment of H is to be simple.

By (2), in G_2 each color is on two edges incident with each vertex. So, in G_2 the subgraph induced by the edges of each color is a 2-factor, and hence this 2-edge-coloring is a 2-factorization of G_2. The fairness of this 2-factorization follows from the following observation: There is a one-to-one correspondence between the edges colored c joining any pair of vertices u and w in H and the edges colored c between the two corresponding parts $\psi^{-1}(u)$ and $\psi^{-1}(w)$ of G_2. So, the balanced property of this 2-edge-coloring implies the required fairness property of the 2-factorization. □

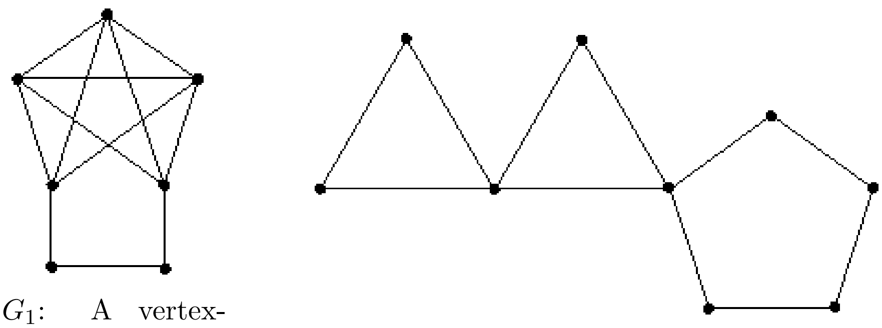
In the light of Theorem 7.2.1 and Theorem 7.2.2 we make the following conjecture.

Conjecture 1. *Let G be a $4k$-regular graph ($k \geq 1$). Let P be any partition of $V(G)$. Let $H = P(G)$. Suppose $H^{(2)}$ has no components with an odd number of odd vertices. Then G has a fair $2k$-factorization.*

7.3 Edge-Colorings with other Combinations of Fairness Requirements

As described in the introduction of this chapter we now consider other combinations of edge-coloring properties in turn. The results in this section are straight-forward to obtain, but are reported here for completeness.

(P_1) Evenly-equitable, balanced and equalized. As is discussed below, the examples in Figure 1 show that there are graphs which have an even, balanced, equalized 2-edge-coloring, but no 2-edge-coloring that is evenly-equitable and equalized. So, for each positive integer k, no matter which combination of the conditions on the existence of an even k-edge-coloring, balanced k-edge-coloring and equalized k-edge-coloring of a graph G is used, it is not possible to guarantee that G has a k-edge-coloring which is evenly-equitable, balanced and equalized.



(a) G_1: A vertex-minimum example

(b) G_2: An edge-minimum example

Figure 7.1: Examples of graphs that are not of color-type 1

A graph is said to be of *color-type* 1 if it is connected, simple and has an even, equalized 2-edge-coloring, but has no evenly-equitable, equalized 2-edge-coloring. Note that any edge-coloring of a color-type 1 graph is balanced because it is simple. In G_1 there are two 3-cycles that intersect in just the top vertex; color the six edges in these 3-cycles with color 0 and color the remaining edges with color 1 to produce an even, balanced, equalized 2-edge-coloring. G_1 does not have an evenly-equitable, equalized 2-edge-coloring, since in every evenly-equitable 2-edge-coloring one color class must be 2-regular and spanning, so has 7 edges. So, G_1 is of color-type 1. In fact, a basic search shows that there is no color-type 1 graph with fewer vertices nor one on 7 vertices with less than 12 edges.

In G_2 the six edges of the two 3-cycles can be colored with color 0 and the edges of the 5-cycle with color 1, thereby producing an even, balanced, equalized 2-edge-coloring. G_2 does not have an evenly-equitable, equalized 2-edge-coloring, since the only evenly-equitable 2-edge-coloring has one color class consisting of the three edges in the middle 3-cycle. So, G_2 is of color-type 1. In fact, another basic search shows that there is no color-type 1 graph with fewer edges nor one with 11 edges on less than 9 vertices.

Note that G_2 suggests a way to construct infinitely many color-type 1 graphs: Take any cycle of length a as the middle cycle, attach to it a cycle of length b on the left, and a cycle of length c on the right where $c \in \{a+b-1, a+b, a+b+1\}$, and $a, b, c \geq 3$.

Since we cannot guarantee the existence of an evenly-equitable, balanced and equalized k-edge-coloring of a graph G, even with the strong assumption that G has a k-edge-coloring which is even, balanced and equalized, we focus our attention on conditions implying the existence of k-edge-colorings that are (P_2) evenly-equitable and equalized, (P_3) balanced and equalized, (P_4) evenly-equitable, (P_5) balanced, and (P_6) equalized; evenly-equitable, balanced edge-colorings are the focus of Section 2.

(P_2) Evenly-equitable and equalized. The examples in Figure 1 show that even with the strong assumption that a graph G has an even, balanced, equalized k-edge-coloring, G does not necessarily have an evenly-equitable, equalized k-edge-coloring; characterizations of graphs with such edge-colorings would seem to be difficult to find.

(P_3) Balanced and equalized. Such edge-colorings are always easy to find as is stated in the following theorem.

Theorem 7.3.1. *For each positive integer k, each graph has a balanced, equalized k-edge-coloring.*

Proof. Let G be a graph with m edges (loops, being special types of edges, are also included in this count). Form an ordering $(e_1, e_2, ..., e_m)$ of the edges of G where loops incident with the same vertex appear consecutively in the list, as do the edges joining the same pair of vertices. For $1 \leq i \leq m$ color e_i with i (modulo k). This k-edge-coloring is clearly balanced and equalized. □

(P_4) Evenly-equitable. Hilton proved the following theorem in [16]:

Theorem 7.3.2. *For each $k \geq 1$, each even graph G has an evenly-equitable k-edge-coloring.*

Note that the condition that G is even is clearly necessary.

(P_5) Balanced. By Theorem 7.3.1 for each positive integer k, any graph G has a balanced k-edge-coloring.

(P_6) Equalized. By Theorem 7.3.1 for each positive integer k, any graph G has an equalized k-edge-coloring.

The discussion above leads to the following chart.

G has an even, balanced, equalized k-edge-coloring for each positive integer k	$\not\Rightarrow$ by (P_1)	G has an evenly-equitable, equalized k-edge-coloring for each positive integer k
G has an even, balanced k-edge-coloring for each positive integer k	$\Leftrightarrow$ by Theorem 7.1.1	G has an evenly-equitable, balanced k-edge-coloring for each positive integer k
G is any graph	$\Rightarrow$ by Theorem 7.3.1	G has a balanced, equalized k-edge-coloring for each positive integer k
G is even	$\Rightarrow$ by Theorem 7.3.2	G has an evenly-equitable k-edge-coloring for each positive integer k

Chapter 8

Vertex-Equalized Edge-Colorings

8.1 Vertex-Equalized 2-Edge-Colorings

In this chapter characterizations are provided for graphs to have vertex-equalized k-edge-colorings in the cases where $k = 2$ and $k = 3$.

If H is edge-colored with colors in $\mathbb{Z}_k$ then define $m(H)$ to be a color $c \in \mathbb{Z}_k$ for which $\nu_c(H) \leq \nu_{c'}(H)$ for all $c' \in \mathbb{Z}_k$. Throughout Chapter 8, S_i denotes a star with i edges.

The following lemma will be very useful in proving the main results of this chapter.

Lemma 8.1.1. *Each non-empty connected graph has a spanning subgraph that is a union of vertex-disjoint non-empty stars.*

Proof. Let G be a non-empty connected graph, and T be a spanning tree of G. Let H be formed from T by greedily removing the middle edge in any path of length 3 until no 3-path remains. Then clearly each component is a star and $\delta(H) \geq 1$ since removing a middle edge never creates a vertex of degree 0. □

Theorem 8.1.2. *Suppose G is a connected simple graph. Then G has a vertex-equalized 2-edge-coloring if and only if $G \neq K_2$.*

Proof. It is clear that K_2 has no vertex-equalized 2-edge-colorings. To prove sufficiency, assume that $G \neq K_2$. If G is empty, then the result is trivial; otherwise by Lemma 8.1.1, G has a spanning subgraph H consisting of vertex-disjoint non-empty stars. Form a non-decreasing ordering $(G_1, G_2, ..., G_s)$ of the components in H with respect to the number of edges in each component. Then form an ordering $(e'_1, e'_2, ..., e'_t)$ of the edges of H where if $e'_i \in G_k$, $e'_j \in G_l$ and $i < j$, then $k \leq l$. Alternately color these edges with 0 and 1. Suppose that in H the number of stars with exactly one edge is even. This procedure clearly yields a vertex-equalized 2-edge-coloring of H. If in H the number of stars with a single edge is odd, then $G_1 \cong K_2$, its edge e' is colored 0, and $\nu_0(H) \in \{\nu_1(H)+1, \nu_1(H)+2\}$. Also, since $G \neq K_2$, $s \geq 2$ (that is, G_2 exists). G is connected, so there must be an edge $e \neq e'$ incident with a vertex in G_1. Color e with 1. This gives a vertex-equalized 2-edge-coloring of $H + e$.

Let $H_0 = \begin{cases} H & \text{if the number of stars in } H \text{ with a single edge is even} \\ H + e & \text{if the number of stars in } H \text{ with a single edge is odd.} \end{cases}$

Now the vertex-equalized 2-edge-coloring of H_0 can be completed to a vertex-equalized 2-edge-coloring of G as follows. Let $E(G) \setminus E(H_0) = \bigcup_{i=1}^{p} e_i$ where $e_i = \{x_i, y_i\}$. For each k where $1 \leq k \leq p$, let $H_k = H_{k-1} + e_k$. Then for $1 \leq i \leq p$, if for some $c \in \{0, 1\}$ both x_i and y_i in H_{i-1} are incident with c then color e_i with c; otherwise color e_i with $m(H_{i-1})$. This gives a vertex-equalized 2-edge-coloring of G. □

8.2 Vertex-Equalized 3-Edge-Colorings

Theorem 8.2.1. *Suppose G is a connected simple graph. Then G has a vertex-equalized 3-edge-coloring if and only if $G \neq K_2, S_2$.*

Proof. It is clear that K_2 and S_2 have no vertex-equalized 3-edge-colorings. To prove sufficiency, assume that $G \neq K_2, S_2$. If G is empty, then the result is trivial; otherwise by Lemma 8.1.1, G has a spanning subgraph H consisting of vertex-disjoint non-empty stars. We begin by coloring the edges in H together with at most two edges in $G - E(H)$, considering five cases in turn. In H let $a \in \mathbb{N}$ be the number of S_1's, and $b \in \mathbb{N}$ be the number of S_2's. Let $m = min\{a, b\}$. Properly edge-color the $3m$ edges in m of the S_1's and m of the S_2's with m edges of each color. For each $i \in \mathbb{Z}_3$, color with i all edges in $\lfloor (a-m)/3 \rfloor$ of the uncolored S_1's and all edges in $\lfloor (b-m)/3 \rfloor$ of the uncolored S_2's. The components that are left uncolored in H are all S_i's with $i \geq 3$, along with exactly one of the following

(i) one K_2 and no S_2's,

(ii) two K_2's and no S_2's,

(iii) one S_2 and no K_2's

(iv) two S_2's and no K_2's, or

(v) no other components.

Let L be the subgraph of H consisting of the uncolored components. Form a non-decreasing ordering $(L_1, L_2, ..., L_s)$ of the components in L with respect to the number of edges in each component. Then form an ordering $(e'_1, e'_2, ..., e'_t)$ of the edges of L where if $e'_i \in L_k$ and $e'_j \in L_l$ with $i < j$ then $k \leq l$.

Suppose we are in case (i); so $L_1 \cong K_2$, its edge being e'_1. If $s \geq 2$ then $L_2 \cong S_i$ where $i \geq 3$, in which case a vertex-equalized 3-edge-coloring of L can be produced by coloring e'_1 with 0, e'_2 with 1, e'_3 with 1, e'_4 with 2, and for $5 \leq k \leq t$ coloring e'_k with k (modulo 3). So now we can assume $s = 1$; so in H there is no component isomorphic to S_i where $i \geq 3$. If in H there is a component isomorphic to S_2, then $m \geq 1$ and so H contains 3 components $L_1, H' \cong S_2$ and $H'' \cong K_2$, such that currently in H' one edge is colored 1 and the other edge is colored 2, and in H'' the only edge is colored 0. Produce a vertex-equalized 3-edge-coloring of L by coloring e'_1 with 0, recoloring the edge in H'' with 1, and recoloring both edges in H' with 2. Finally suppose that in H there is no component isomorphic to S_i where $i \geq 2$; so $s = 1$ and $m = 0$. Then since $G \neq K_2$, in H there exist four components L_1, H', H'', H''', each isomorphic to K_2, such that currently the edge in H' is colored 0, the edge in H'' is colored 1, and the edge in H''' is colored 2. Since G is connected, there are at least two edges $e, e' \neq e'_1$ in G incident with a vertex in $V(L_1 \cup H')$. Color e'_1 with 0, e with 1, and e' with 2. This 3-edge-coloring of $L + \{e, e'\}$ is vertex-equalized.

In case (ii) $L_1, L_2 \cong K_2$, and $E(L_1) = \{e'_1\}$, $E(L_2) = \{e'_2\}$. Color e'_1 with 0, and e'_2 with 1. Since G is connected, there must be an edge $e \notin E(L_1 \cup L_2)$ incident with at least one vertex in $L_1 \cup L_2$. Color e with 2. For $3 \leq k \leq t$, color e'_k with $k - 1$ (modulo 3). (In fact, thinking recursively, as $e_3', ..., e_t'$ are colored in turn, the resulting partial edge-coloring of G is vertex-equalized.)

In case (iii) $L_1 \cong S_2$, and $E(L_1) = \{e_1', e_2'\}$. Color e_1' with 0, and e_2' with 1. Since G is connected and $G \neq S_2$, there must be an edge $e \notin E(L_1)$ incident with at least one vertex in L_1. Color e with 2. For $3 \leq k \leq t$, color e_k' with $k-1$ (modulo 3) to produce a vertex-equalized 3-edge-coloring of $L + e$.

In case (iv) $L_1, L_2 \cong S_2$, and $E(L_1) = \{e_1', e_2'\}$, $E(L_2) = \{e_3', e_4'\}$. Color e_1' and e_2' with 0, e_3' with 1, and e_4' with 2. For $5 \leq k \leq t$, color e_k' with $k-1$ (modulo 3).

In case (v) for $1 \leq k \leq t$, color e_k' with $k-1$ (modulo 3).

It is important to note that in each of the above cases a vertex-equalized 3-edge-coloring of a spanning subgraph H_0 of G has been found. Now the vertex-equalized 3-edge-coloring of H_0 can be completed to a vertex-equalized 3-edge-coloring of G. Let $E(G) \setminus E(H_0) = \{e_i \mid 1 \leq i \leq p\}$ where $e_i = \{x_i, y_i\}$. For each i where $1 \leq i \leq p$, let $H_i = H_{i-1} + e_i$ and recursively (inductively) color the remaining uncolored edges to produce a vertex-equalized 3-edge-coloring of G as follows. For $1 \leq i \leq p$, assuming that H_{i-1} has a vertex-equalized 3-edge-coloring in which $\nu_0(H_{i-1}) \geq \nu_1(H_{i-1}) \geq \nu_2(H_{i-1})$ (rename colors if necessary), one of the following statements holds:

(i) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) = \nu_2(H_{i-1})$,

(ii) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) = \nu_2(H_{i-1}) + 1$,

(iii) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) + 1 = \nu_2(H_{i-1}) + 1$.

In case (i) color e_i with c where c is any color occurring on an edge in H_{i-1} incident with x_i. In case (ii) color e_i with 2. In case (iii): color e_i with 1 if there is an edge colored 1 in H_{i-1} incident with x_i or y_i; otherwise color e_i with 2 if there is an edge colored 2 in H_{i-1} incident with x_i or y_i; and if e_i is still uncolored then color it with 0 (note that in this case each of x_i and y_i must be incident with edges colored 0 in H_{i-1}).

$\square$

8.3 Further Remarks

Companion results for Theorem 8.1.2 and Theorem 8.2.1 follow easily for the case when G is connected, but not necessarily simple. In this section, it is assumed that edges join two distinct vertices; so loops are not described as edges.

Theorem 8.3.1. *Suppose G is a connected graph (possibly with loops and multiple edges) such that the underlying simple graph G_u has a vertex-equalized k-edge-coloring. Then G has a vertex-equalized k-edge-coloring.*

Proof. For each multiple edge $e = \{u, v\}$ in G, color e with $c \in \mathbb{Z}_k$ if $\{u, v\}$ in G_u is colored c. For each loop l at a vertex w, color l with $c \in \mathbb{Z}_k$ if c is the color of an edge in G_u that is incident with w. □

Corollary 8.3.2. *Suppose G is a connected graph (possibly with loops and multiple edges). Then G has a vertex-equalized 2-edge-coloring if and only if $G \neq K_2$.*

Proof. Clearly K_2 has no vertex-equalized 2-edge-coloring. To prove sufficiency let G be connected and $G \neq K_2$. Then in view of Theorems 8.1.2 and 8.3.1 we can assume that $G_u = K_2$. If G has any loops then color all loops with 0, and all edges with 1. If G has no loops, then color one edge with 0, and the remaining edges with 1. □

Corollary 8.3.3. *Suppose G is a connected graph (possibly with loops and multiple edges). Then G has a vertex-equalized 3-edge-coloring if and only if $G \notin \{S_2, K_2, 2K_2\}$.*

Proof. Clearly S_2, K_2 and $2K_2$ have no vertex-equalized 3-edge-coloring. To prove sufficiency let G be connected and $G \neq S_2, K_2, 2K_2$. Then in view of Theorems 8.2.1 and 8.3.1 we can assume that $G_u = K_2$ or $G_u = S_2$. Suppose $G_u = K_2$. Then there are at least 3 edges in G. Color one such edge with 0, one with 1, and color all the other edges and loops in G with 2. Suppose $G_u = S_2$. Let $\{x, y\}$ and $\{y, z\}$ be the edges in G_u. If G has a loop, then color all loops in G with 0, all edges that join x to y with 1, and all edges that join y to z with 2.

If G has no loops, then G has at least three edges. Color one edge with 0, one edge with 1, and the remaining edges with 2. $\square$

Note that a generalization of Corollary 8.3.2 and Corollary 8.3.3 for disconnected graphs does not seem to be easy to obtain. For example, to settle the case with two colors (see Corollary 8.3.2) such a result would require the classification of all graphs G for which all vertex-equalized 2-edge-colorings satisfy $\nu_0(G) = \nu_1(G)$, since the graph consisting of two components G and K_2 would have no vertex-equalized 2-edge-coloring.

Also note that extending Theorem 8.1.2 and Theorem 8.2.1 to edge-colorings with four or more colors would require a different approach. This is because the idea of taking a spanning subgraph of a graph G, finding a vertex-equalized k-edge-coloring of this subgraph and then completing this coloring to a vertex-equalized k-edge-coloring of G by coloring a single edge at a time rarely works if $k \geq 4$. On the other hand, for a graph G that has many edges it is not difficult to see that one can take a vertex-equalized 3-edge-coloring of G and then recolor some of the edges in G with a new color to get a vertex-equalized 4-edge-coloring of G. Another approach for dense simple graphs would be to somehow find k edge-disjoint spanning subgraphs (for example, use Dirac's Theorem [13] k times to find k hamiltonian cycles in a graph on n vertices with $\delta \geq 2(k-1) + n/2$, coloring the edges in the i^{th} such subgraph with color i and all the other edges with any color to obtain a vertex-equalized k-edge-coloring in which $\nu_i = n$ for $1 \leq i \leq k$). Nevertheless, new ideas will be needed to settle the problem in general.

Finally the authors would like to note that an interesting related problem is to find the spectrum of $\nu_c(G)$ among all vertex-equalized k-edge-colorings of a graph G; that is, find $N(G) = \{\nu_c(G) \mid c \in \mathbb{Z}_k,\, G \text{ has a vertex-equalized } k\text{-edge-coloring with colors in } \mathbb{Z}_k\}$.

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Appendices