Extensions of Monotonicity Results to Semisimple Lie Groups

by

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Abstract

We extend a monotonicity result of Wang and Gong on the product of powers of positive definite matrices. This result concerns the eigenvalues of such products which written as vectors have a log majorization relationship. Our extension is in the context of semisimple Lie groups which relies on the complete multiplicative Jordan decomposition and which we express with Kostant's preorder. An analogous result on singular values is also obtained.

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Chapter 1

Introduction

This dissertation concerns the extension of inequalities involving matrices to semisimple Lie groups. The particular inequalities studied were first investigated in mathematical physics. In [5], Lieb and Thirring showed that for any real number $\alpha \geq 1$,

$$\operatorname{tr}((AB)^{\alpha}) \leq \operatorname{tr}(A^{\alpha} B^{\alpha})$$

It was further studied by Araki in [10] and by Audenaert in [17] and [18]. This inequality was of interest outside of mathematical physics, in particular matrix theory. In 1993, Wang and Gong [13] proved an extension of the Lieb-Thirring inequality, which is also a generalization of results of Marcus [2], Le Couteur [9], and Bushell and Trustrum [11]. This dissertation further generalizes Wang and Gong's extension of the Lieb-Thirring inequality to a general semisimple Lie group, as well as a similar inequality involving singular values.

The theory of semisimple Lie groups is readily applicable to the study of matrix groups, and somewhat conversely matrix groups serve as important prototypes and examples for the study of Lie groups. The matrix groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ have a well known group structure, and various decompositions and results in these groups serve as natural prototypes for analogous results in the context of Lie groups. New results involving matrix groups are often extended to this more general context.

The aforementioned decompositions are important tools in the study of semisimple Lie groups. Extensive use will be made of the Cartan decomposition, which is the familiar polar decomposition in the context of $\mathrm{SL}_n(\mathbb{C})$. Use will also be made of the Iwasawa decomposition, analogous to the QR decomposition of matrices, as well as the KAK decomposition, analogous to the singular value decomposition. These Lie group decompositions are all approached similarly in that they are built from decompositions of Lie algebras, Lie algebras having in some respects a simpler algebraic structure which is approachable via the standard techniques of linear algebra and matrix theory.

Of particular importance is the Complete Multiplicative Jordan Decomposition, hereafter abbreviated CMJD. As the name would indicate, this decomposition, introduced by Kostant in [4], is an extension of the Jordan-Chevalley decomposition of matrices, itself a multiplicative version of the familiar Jordan canonical form. The Jordan canonical form presents a matrix or linear map as a sum of an essentially unique semisimple part and nilpotent part. The Jordan-Chevalley decomposition presents a nonsingular matrix, linear map, or Lie group element as the product of a semisimple part and a unipotent part, and Kostant's CMJD further decomposes the semisimple part into a hyperbolic part and an elliptic part. This decomposition of a Lie group element $g = ehu$, where e is elliptic, h is hyperbolic, and u is unipotent, has several desirable properties, among them the factors being unique and commuting.

The study of Lie algebras facilitates that of Lie groups because Lie algebras and Lie groups are closely intertwined. Not only are they diffeomorphic on neighborhoods of their respective identities, but they also share an important group action. There is a Weyl group formulated as a quotient of normal subgroups and also a Weyl group describing the symmetries of its Lie algebra. Although the definitions of these groups are quite different, they are in fact isomorphic.

A facet of the correspondence between a Lie group and its Lie algebra may also be described in the language of category theory. Let **FLGrp** be the category of finite dimensional Lie groups having smooth homomorphisms as morphisms, **FMan** be the category of finite dimensional smooth manifolds having smooth maps as morphisms, and **FLAlg** be the category of finite dimensional Lie algebras having Lie algebra homomorphisms as morphism. Let $F : \mathbf{FLGrp} \to \mathbf{FMan}$ be the forgetful functor. Every finite dimensional Lie algebra is also a vector space, and every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ is trivially a smooth manifold. Likewise, every linear map between finite dimensional vector spaces is also trivially a smooth map, so we may define a forgetful functor M : **FLAlg** $\to$ **FMan**. There is another functor natural to the study of Lie groups, and that is the functor L : **FLGrp** $\to$ **FLAlg**, defined as follows: for a Lie group G, $L(G) = \mathfrak{g}$, its associated Lie algebra, and for a smooth homomorphism, $\pi : \mathfrak{g} \to \mathfrak{h}$, $L(\pi) = d\pi$, the differential of π at the identity.

Although Lie theorists speak of the exponential map exp as though there's only one, there is actually an exponential map $\exp = \exp_G : G \to \mathfrak{g}$ for each pair of a Lie group and its associated Lie algebra. It is well known that $\pi \circ \exp = \exp \circ d\pi$ for a smooth homomorphism π, which is to say the diagram given in Figure 1.1 commutes. Moreover, since exp is smooth,

$$\begin{tikzcd} G \arrow[r, "\pi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\pi"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Figure 1.1: $\pi \circ \exp = \exp \circ d\pi$

we have the equivalent commuting diagram Figure 1.2 in **FMan**. Because this diagram

$$\begin{tikzcd} F(G) \arrow[r, "F(\pi)"] & F(H) \\ (M \circ L)(G) \arrow[u, "\exp_G"] \arrow[r, "(M \circ L)(\pi)"'] & (M \circ L)(H) \arrow[u, "\exp_H"'] \end{tikzcd}$$

Figure 1.2: Commutative diagram in **FMan**

commutes, exp is a natural transformation between the functors F and $M \circ L$.

This dissertation is organized as follows: in Chapter 2 we examine the definitions and basic results of Lie theory, having to detour briefly into a discussion of smooth manifolds to do so. In Chapter 3 we examine the various decompositions and tools of Lie groups in some details. Chapter 4 is an overview of the matrix inequalities generalized in this dissertation, as well as matrix theoretic results that will be useful. In Chapter 5, we state and prove generalizations of the Wang-Gong inequality. Throughout we assume that every vector space discussed is finite-dimensional.

Chapter 2

Lie Groups and Lie Algebras

This chapter contains the necessary background material needed to understand the subsequent chapters. We will, however, assume the reader has a grasp of the basic definitions and theorems of linear algebra, group theory, and topology.

A Lie group is both a group and a smooth manifold, and these two disparate mathematical structures are compatible in a certain specific way. This compatibility may be described both concisely and precisely by the following categorial definition: a Lie group is a group object in the category of smooth manifolds. Although this categorial definition is precise, it requires elaboration. We will expound upon it in some detail, starting with the basic theory of smooth manifolds. We follow the development of [1], [3], [6]and .

2.1 Smooth Manifolds

We first describe a *manifold structure* on a topological space M. Let M be a second countable Hausdorff space, and suppose that for every point m of M, there is an open neighborhood of m that is homeomorphic to $\mathbb{R}^n$. We call each open set - homeomorphism pair (U, φ_u) a *chart*, and we call a collection of charts covering all of M an *atlas* $\mathcal{A}$. We call such a topological space with an atlas of charts a *topological manifold* of dimension n.

If there are two charts (U, φ_U) and (V, φ_V) such that U and V intersect, then $\varphi_U \circ \varphi_V^{-1}$ and $\varphi_V \circ \varphi_U^{-1}$ are *transition maps* $\mathbb{R}^n \to \mathbb{R}^n$ and it is natural to consider the various properties that real maps may take. If we mandate that each coordinate function of each transition map has all partial derivatives up to a certain order k, we call the resulting structure a *differentiable manifold of order* k or a C^k manifold. If a map $f : \mathbb{R}^n \to \mathbb{R}^n$ has component functions with partial derivatives of all orders, it is called a C^∞ map. If all of the transition maps of a given atlas of charts of a differentiable manifold are in fact C^∞ maps, then the resulting structure is a *smooth manifold*.

Although not strictly necessary, it is often convenient to assume that the atlas of charts of a smooth manifold M is maximal in the certain following sense: the inclusion of any additional charts to the atlas would cause the manifold to no longer be smooth.

we now describe maps that preserve smooth structure. These are called *smooth maps*, and are defined as follows:

1. A smooth map $f : M \to N$ is first a continuous map between the underlying topological spaces of M and N and

2. for every chart (U, φ_U) of M and every chart (V, ψ_V) of N containing $f(U)$, $\psi_V \circ f \circ \varphi_U^{-1}$ is C^∞.

Smooth maps are the morphisms in the category of differentiable manifolds, and the isomorphisms in this category are referred to as *diffeomorphisms* and may be characterized as smooth and reversibly smooth bijections.

There is a feature of smooth manifolds that any student in a freshman calculus course would recognize. We may define intrinsically and in the abstract a tangent space to any point p on a smooth manifold M, including the more familiar tangent lines and tangent planes of undergraduate calculus. We first define a tangent vector.

Let $f : U \to \mathbb{R}$ be a function whose domain U is an open set containing p. Such a function is also called C^∞ if $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is C^∞ for all charts φ whose domain contains p. Let $\mathrm{C}^\infty(p)$ denote the set of all real-valued C^∞ functions whose domain is an open set containing p.

$\mathrm{C}^\infty(p)$ has a natural ring structure, where the ring operations are defined as follows for $f : U \to \mathbb{R}, g : V \to \mathbb{R} \in \mathrm{C}^\infty(p)$.

1. $f + g : U \cap V \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$

2. $fg : U \cap V \to \mathbb{R}$ by $(fg)(x) = f(x)g(x)$

This can be extended to an algebra structure by defining scalar multiplication for $\alpha \in \mathbb{R}$ by $f(\alpha x) = \alpha f(x)$, or by viewing scalars as constant functions $\alpha : M \to \mathbb{R}$ and using the ring multiplication above. We may also describe an algebra structure on $\mathrm{C}^\infty(M) = \{f : M \to \mathbb{R} | f \text{ is } C^\infty\}$, with addition, multiplication, and scalar multiplication defined exactly as above. In this case $U = V = M$ and hence $U \cap V = M$ also.

A *tangent vector* t *at* p is defined as a certain kind of operator on $\mathrm{C}^\infty(p)$. $t : \mathrm{C}^\infty(p) \to \mathbb{R}$ is a tangent vector if for $f, g \in \mathrm{C}^\infty(p)$ and $\alpha, \beta \in \mathbb{R}$

1. $t(\alpha f + \beta g) = \alpha t(f) + \beta t(g)$ *(t is linear)*

2. $t(fg) = t(f)g(p) + f(p)t(g)$ *(t satisfies the product rule)*

Such an operator is also known as a *derivation*, so we may more concisely state that tangent vectors at p are derivations on $\mathrm{C}^\infty(p)$.

The collection of tangent vectors to M at p is denoted M_p and in fact forms a vector space of the same dimension as M. M_p has a standard basis consisting of the operators $\frac{\partial}{\partial x^i}\big|_p$, $i = 1, \ldots, n$, defined by

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = (D_i(f \circ \varphi))(\varphi(p)).$$

The union

$$\bigcup_{p \in M} M_p = TM$$

of all the tangent spaces of M is known as the *tangent bundle* and is itself a smooth manifold. Note that while each of M_p is isomorphic to $\mathbb{R}^n$, this is a disjoint union, as for each p $v \in M_p$ has a different domain.

The manifold structure of TM becomes somewhat readily apparent once we define a topology on it. First we denote a tangent vector $v \in M_p$ by the pair (p, v). A subset of TM

is open if and only if it is of the form

$$U_{TM} := \{(p, v) : p \in U, v \in M_p\}$$

for a fixed open subset U of M. In this way the open sets of M induce the open sets of TM, and we can see that each open set of TM is homeomorphic to $U \times \mathbb{R}^n$. As U is itself homeomorphic via some chart φ_U to $\mathbb{R}^n$, we can conclude that each open set of TM is homeomorphic to $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$.

We may express this homeomorphism explicity by recalling that each $v \in M_p$ can be written as

$$\sum_{i=1}^{n} a_i \left.\frac{\partial}{\partial x^i}\right|_p .$$

Thus we may define a chart (U_{TM}, φ) by

$$\varphi(v) = ((\varphi_U(v))_1, \ldots, (\varphi_U(v))_n, a_1, \ldots, a_n),$$

where $(\varphi_U(v))_i$ denotes the ith coordinate of $\varphi_U(v) \in \mathbb{R}^n$.

If we have a smooth map $\pi : M \to N$ between smooth manifolds M and N and a fixed point $p \in M$, we can lift it to a linear map $d\pi_p : M_p \to N_{\pi(p)}$ between the tangent spaces M_p and $N_{\pi(p)}$. As a tangent vector is an operator, we may describe the operator $d\pi_p(v)$ for $v \in M_p$ by how it operates. Specifically, $d\pi_p$ is defined by

$$d\pi_p(v)(f) = v(f \circ \pi)$$

for $f \in \mathrm{C}^\infty(\pi(p))$. $d\pi_p$ is called the *differential of* π *at* p. To use the language of category theory, we have defined a functor F from the category of pointed manifolds to the category of vector spaces by $F((M, p)) = M_p$ and $F(\pi) = d\pi$. The collection $\{d\pi_p : p \in M\}$ gives rise to a map between tangent bundles $d\pi_* : TM \to TN$ by $d\pi_*(v) = d\pi_p(v)$ when $v \in M_p$.

In the case of smooth manifolds, we describe more than one kind of submanifold. Let $\mathcal{A}$ be an atlas of charts of M and N be an open subset of M. Define an atlas of charts $\mathcal{B} = \{(U \cap N, \varphi_U|_{U\cap N}) : (U, \varphi_U) \in \mathcal{A}, U \cap N \neq \emptyset\}$. If $\mathcal{B}$ is an atlas of charts for N such that N is itself a smooth manifold, then N is called an *open submanifold* of M.

Because $d\pi_p$ is a linear map, it has rank. If for all $p \in M$, $\operatorname{rank} d\pi_p = \dim M$, we call π an *immersion*. If a subset $N \subseteq M$ is itself a smooth manifold, with an atlas of charts not necessarily related to that of M, it is called an *immersed submanifold* so long as the inclusion map $\iota : N \to M$ is an immersion.

A map $X : M \to TM$ such that $X(p) \in M_p$ for all $p \in M$ is a *vector field*. We will sometimes refer to $X(p)$ as X_p for notational convenience. A vector field is called C^∞ or smooth in the case that X is itself a smooth map between smooth manifolds. We may be more specific in saying that as TM is a set of derivations, we may write each $X(p)$ as $\sum_{i=1}^n a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ by taking advantage of the standard bases of tangent spaces described above. We then call X smooth if each of the coordinate functions $a^i : M \to \mathbb{R}$ is smooth.

The set of vector fields $\mathcal{V}$ on X has the structure of a vector space over $\mathbb{R}$. Its vector space structure may be described, for $X, Y \in \mathcal{V}$, $p \in M$ and $\alpha \in \mathbb{R}$ by

1. $(X + Y)_p = X_p + Y_p$

2. $(\alpha X)_p = \alpha(X_p)$

A vector field X also acts on $\mathrm{C}^\infty(M)$ by

$$Xf_p = X_p(f).$$

Thus Xf is itself an element of $\mathrm{C}^\infty(M)$. This action on $\mathrm{C}^\infty(M)$ is compatible with its algebraic structure as seen in the following theorem.

Theorem 2.1 ([3, p. 83])**.** *For $X \in \mathcal{V}$ and $f, g \in \mathrm{C}^\infty(M)$,*

1. $X(f + g) = Xf + Xg$, and

2. $X(fg) = gXf + fXg$,

which is to say that every smooth vector field X is a derivation on $C^\infty(M)$.

There is one more operation that we may define on vector fields, known as the *Lie derivative* or *Lie bracket*. The Lie bracket is instrumental in the definition of a Lie algebra, so it is of value to have two different perspectives. First, we define the *flow* associated to a smooth vector field X.

A function $\gamma : \mathbb{R} \to M$ is called a *C^∞ curve* if for any chart (U, φ_U) of M, $\varphi_U \circ \gamma$ is C^∞ whenever the image of γ and U intersect. Suppose now that γ is a C^∞ curve, $p \in M$, and α is a real number such that $\gamma(\alpha) = p$. The *vector tangent to γ at p* is defined as the operator $\gamma_{*\alpha} : C^\infty(p) \to \mathbb{R}$ given by $f \mapsto (f \circ \gamma)'(\alpha)$ where $'$ denotes the ordinary derivative as a function $\mathbb{R} \to \mathbb{R}$.

For any vector field X and $p \in M$, there is a unique C^∞ curve γ^p satisfying the following on an open interval I containing 0:

1. $\gamma^p_{*\alpha} = X(\gamma^p(\alpha))$, and

2. $\gamma^p(0) = p$.

The flow associated with X is

$$\psi : \mathbb{R} \times M \to M$$

defined by

$$\psi(\alpha, p) = \gamma^p(\alpha).$$

It will be useful to fix a particular $\alpha \in \mathbb{R}$, and examine the map $\psi(\alpha, -) = \psi_\alpha : M \to M$, which in fact is a diffeomorphism.

We finally define, given a diffeomorphism $\theta : M \to N$, the vector field $\theta^* X$ on N induced by θ for $q \in N$ and $f \in C^\infty(q)$ as

$$(\theta^* X)_q f = X_{\theta^{-1}(q)}(f \circ \theta).$$

Definition 2.2. *Suppose X and Y are vector fields on the smooth manifold M and ψ is the flow associated with X. Then the Lie derivative of Y with respect to X is the vector field*

$$[L_X Y]_p = \lim_{h \to 0} \frac{Y_p - (\psi_h^* Y)_p}{h}$$

A much simpler operation to generate a new vector field from two old ones is given by the Lie bracket.

Definition 2.3. *Assume X, Y, and M as above. Then the Lie bracket of X and Y is the vector field given by*

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$$

or more concisely we write

$$[X, Y] = XY - YX$$

In fact

$$[L_X Y] = [X, Y].$$

The definition of the Lie derivative is clearly an analytic and geometric one, strikingly resembling the definition of a derivative of a function $f : \mathbb{R} \to \mathbb{R}$ and giving us a notion of the infinitesimal change of one vector field with respect to another. The Lie bracket, however, has a more algebraic flavor, and we can regard the Lie bracket as a sort of noncommutative, nonassociative multiplication. That the two are in fact the same is remarkable. Although the Lie bracket is favored by Lie theory texts due to its simplicity, a familiarity with the Lie derivative significantly reduces the mystique surrounding the relationship between a Lie group and its associated Lie algebra.

A large number of concepts from Euclidean geometry may be translated to smooth manifolds via the *Riemannian metric tensor*. The Riemannian metric tensor is a family of inner products

$$\{\langle \cdot, \cdot \rangle_p \text{ on } M_p : p \in M\}.$$

This family must satisfy the property that for all smooth vector fields X and Y the map $p \mapsto \langle X_p, Y_p \rangle_p$ is a smooth map $M \to \mathbb{R}$. A smooth manifold for which a Riemannian metric tensor exists is called a *Riemannian manifold.*

Example 2.4. We may define the length of a smooth curve $\gamma : [a, b] \to M$ via the Riemannian metric tensor as

$$L(\gamma) = \int_a^b \sqrt{\langle \gamma_{*\alpha}, \gamma_{*\alpha} \rangle_{\gamma(\alpha)}} d\alpha$$

2.2 Lie Groups and Lie Algebras

We begin our discussion by first defining a Lie algebra in the abstract, and only later asserting that Lie algebras may arise as certain vector spaces associated with Lie groups, equipped with the Lie derivative or equivalently the Lie bracket.

Definition 2.5. *A Lie algebra* $\mathfrak{g}$ *is a vector space over a field* $\mathbb{F}$*, typically* $\mathbb{R}$ *or* $\mathbb{C}$*, equipped with a bilinear map* $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, usually written* $\mu(X, Y) = [X, Y]$ *and called the Lie bracket, satisfying the following:*

1. $[X, X] = 0$ *for all* $X \in \mathfrak{g}$
2. $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$*, the Jacobi identity, holds for all* $X, Y,$ *and* $Z \in \mathfrak{g}$

It must first be observed that for all $X, Y \in \mathfrak{g}$,

$$\begin{aligned} [X, Y] + [Y, X] &= [X, Y] + [Y, X] - [X, X] - [Y, Y] \\ &= [X - Y, Y - X] \\ &= -[X - Y, X - Y] = 0 \end{aligned}$$

and hence $[X, Y] = -[Y, X]$. Thus if $[-, -]$ is commutative, in which case $\mathfrak{g}$ is said to be *abelian*, $[Y, X] = [X, Y] = -[Y, X]$ and hence $[X, Y] = 0$ for all X and Y. Conversely, if a Lie algebra with a trivial Lie bracket is abelian. Similarly, the typical Lie algebra is nonassociative, associativity being replaced by the Jacobi identity.

Example 2.6. Let V be a vector space. We define the Lie algebra $\mathfrak{gl}(V)$. The underlying vector space of $\mathfrak{gl}(V)$ is $\mathrm{End}(V)$, and the Lie bracket is defined as $[A, B] = AB - BA$ where juxtaposition denotes the ordinary composition of linear maps. This is known as the general linear algebra of the vector space V.

For the vector space $\mathbb{F}^n$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,) we fix a basis and view the elements of $\mathfrak{gl}(\mathbb{F}^n)$ as matrices. The resulting Lie algebra is written $\mathfrak{gl}_n(\mathbb{F})$.

The structure preserving maps between Lie algebras are *Lie algebra homomorphisms.* A Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ is a linear map between the underlying vector spaces of $\mathfrak{g}$ and $\mathfrak{h}$ that additionally satisfies

$$f([X, Y]) = [f(X), f(Y)]$$

Much like any other algebraic structure, the image and kernel of a Lie algebra homomorphism are both necessarily Lie algebras.

Similar to rings, Lie algebras have both subalgebras and ideals. A subalgebra of a Lie algebra $\mathfrak{g}$ is simply a subset $\mathfrak{h} \subseteq \mathfrak{g}$ that is itself a Lie algebra when equipped with the same bracket as $\mathfrak{g}$. An *ideal* $\mathfrak{i} \subseteq \mathfrak{g}$ is a subalgebra that satisfies the sponge property

$$[X, I] \in \mathfrak{i} \text{ whenever } I \in \mathfrak{i}.$$

The inclusion map from a subalgebra or ideal to its parent algebra is a Lie algebra homomorphism.

There are several important ideals in the study of Lie algebras.

Example 2.7.

1. The *center* of $\mathfrak{g}$, $Z_{\mathfrak{g}} = \{X : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$
2. The *commutator ideal*, $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{[X, Y] : X, Y \in \mathfrak{g}\}$
3. In more generality, if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then so is $[\mathfrak{a}, \mathfrak{b}] = \mathrm{span}\{[A, B] : A \in \mathfrak{a}, B \in \mathfrak{b}\}$

4. If $f : \mathfrak{a} \to \mathfrak{b}$ is a Lie algebra homomorphism, then $\ker f$ is an ideal of $\mathfrak{a}$.

Lie algebras are readily studied by the techniques of representation theory. A Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{gl}(V)$ into a general linear algebra is called a Lie algebra representation. A particularly important and useful representation is the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}(X)(Y) = [X, Y]$. Although ad is a useful representation for proving theorems in the field of Lie algebra, in light of the contents of this dissertation, it will be much more important as the Lie algebra associated with a particular Lie group representation.

We are now at the point where the definition of a Lie group is needed. Two will be given, one categorial in nature and one not. Let us first establish notation and basic definitions needed to discuss the categorial viewpoint. We follow [8, p. 52] and [16].

A category is a class C of *objects*, and for any two objects x and y there is a set $C(x, y)$ of *morphisms.* It is typical that objects are sets with some structure and morphisms are structure-preserving functions. Categories for which this is the case are called *concrete.*

For each pair of sets of morphisms $C(x, y)$ and $C(y, z)$ there is a function $\circ : C(y, z) \times C(x, y) \to C(x, z)$ known as *composition* and written with infix notation or juxtaposition, i.e. $f \circ g$ or fg rather than $\circ(f, g)$. For each object x there must also exist a certain special morphism 1_x called the *identity.* If the category C is concrete, then 1_x may be taken to be the identity map. Composition must satisfy the following properties for all morphisms $f \in C(x, y)$, $g \in C(y, z)$, and $h \in C(z, w)$ for which composition is defined:

1. $(hg)f = h(gf)$,

2. $1_y f = f$, and

3. $g1_y = g$.

Objects and morphisms are depicted visually via directed graphs called *diagrams.* In a diagram, an object $x \in C$ is a vertex, and a morphism $f \in C(x, y)$ is an edge from x to y. A diagram is said to *commute* if composing the morphisms along any path between two fixed objects yields the same result.

An object $t \in C$ is called *terminal* if for any object $x \in C$ there is a unique morphism in $C(x, t)$.

The product $x \times y$ of two objects may be defined. $x \times y$ is an object along with morphisms $\pi_x \in C(x \times y, x)$ and $\pi_y \in C(x \times y, y)$ so that for any object $z \in C$ and morphisms $f_x \in C(z, x)$ and $f_y \in C(z, y)$, there is a unique morphism $\varphi \in C(z, x \times y)$ so that the following diagram commutes.

$$\begin{tikzcd}
 & z \arrow[d, "\varphi"'] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_y"] & \\
 & x \times y \arrow[dl, "\pi_x"'] \arrow[dr, "\pi_y"] & \\
x & & y
\end{tikzcd}$$

Figure 2.1: Product diagram

We now define a *group object in a concrete category* C. C must have a terminal object t and binary products. A group object is an object $g \in C$ with morphisms

1. $\mu \in C(g \times g, g)$, thought of as multiplication

2. $\epsilon \in C(t, g)$, thought of as giving the identity of g

3. $\iota \in C(g, g)$, thought of as group inversion.

The elements of g must satisfy the usual group axioms, where the above maps determine multiplication, identity, and inversion. These axioms may be given as the statement that the following diagrams commute:

$$\begin{tikzcd}
g \times g \times g \arrow[r, "\mu \times \mathrm{id}\, g"] \arrow[d, "\mathrm{id}\, g \times \mu"'] & g \times g \arrow[d, "\mu"] \\
g \times g \arrow[r, "\mu"'] & g
\end{tikzcd}$$

Figure 2.2: Multiplication is associative

$$
\begin{tikzcd}
g \arrow[r, "(\mathrm{id}\, g, \epsilon)"] \arrow[d, "(\epsilon, \mathrm{id}\, g)"'] \arrow[dr, "\mathrm{id}\, g"] & g \times g \arrow[d, "\mu"] \\
g \times g \arrow[r, "\mu"'] & g
\end{tikzcd}
$$

Figure 2.3: Identity map gives identity element

$$
\begin{tikzcd}
g \arrow[rr, "(\mathrm{id}\, g, \iota)"] \arrow[dd, "(\iota, \mathrm{id}\, g)"'] \arrow[dr] & & g \times g \arrow[dd, "\mu"] \\
& t \arrow[dr, "\epsilon"] & \\
g \times g \arrow[rr, "\mu"'] & & g
\end{tikzcd}
$$

Figure 2.4: Inverse map gives inverses

If the category of C is the category of smooth manifolds, we call g a Lie group. That is, a *Lie group* is simultaneously a group and a smooth manifold where multiplication and inversion can be regarded as smooth maps.

Example 2.8. Matrix groups $G \subseteq \mathbb{R}^{n\times n}$ or $G \subseteq \mathbb{C}^{n\times n}$ are often Lie groups when given the submanifold structure inherited from their parent space $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ or $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$ The *general linear group*, $\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} : \det A \neq 0\}$, and the closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ are called *closed linear groups*. They are Lie groups of particular importance. Various closed linear groups are known as the *classical groups*, and several are listed below.

1. The real general linear group $\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$
2. The real or complex special linear group $\mathrm{SL}_n(\mathbb{F}) = \{A \in \mathbb{F}^{n\times n} : \det A = 1\}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$
3. The real orthogonal group $\mathrm{O}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^T A = I\}$
4. The real special orthogonal group $\mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n(\mathbb{R})$
5. The unitary group $\mathrm{U}_n = \{A \in \mathbb{C}^{n\times n} : A^* A = I\}$

6. The special unitary group $\mathrm{SU}_n = \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C})$

7. The complex symplectic group

$$\mathrm{Sp}_n(\mathbb{C}) = A \in \mathrm{SL}_{2n}(\mathbb{C}) : A^T J_n A = J_n\} \text{ where } J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

8. The real symplectic group $\mathrm{Sp}_n(\mathbb{R}) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{GL}_n(\mathbb{R})$

9. The compact symplectic group $\mathrm{Sp}(n) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{U}_n$

To every Lie group G, there is associated a Lie algebra $\mathfrak{g}$ which is unique up to isomorphism. The underlying set of this Lie algebra is a set of vector fields on G. Because multiplication is a smooth map $G \times G \to G$, if we fix a $g \in G$, we can define the left translation map $L_g : G \to G$ by $L_g(h) = gh$, which is a diffeomorphism from G to itself. A vector field X satisfying

$$dL_g(X_h) = X_{gh}$$

is called *left invariant*. The Lie algebra $\mathfrak{g}$ associated to G is the set of left invariant vector fields, which forms a vector space, where the bracket product is the Lie bracket of vector fields.

Equivalently, every left invariant vector field is of the form $dL_g(v)$ for a fixed $g \in G$ where $v \in M_e$ is a vector tangent to G at the group identity. The map $v \mapsto dL_g(v)$ is a vector space isomorphism, and after defining $[v, w]_{M_e} = [dL_g(v), dL_g(w)]$, it is a Lie algebra isomorphism.

Example 2.9. The matrix groups listed above, as Lie groups, have associated Lie algebras, which are isomorphic to the following matrix algebras, with $[X, Y] = XY - YX$.

1. The real general linear algebra $\mathfrak{gl}_n(\mathbb{R}) = \mathbb{R}^{n\times n}$ associated to $\mathrm{GL}_n(\mathbb{R})$.

2. The real or complex special linear algebra $\mathfrak{sl}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n\times n} : \operatorname{tr} X = 0\}$ associated to $\mathrm{SL}_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

3. The special orthogonal algebra $\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} : X^T = -X\}$ is the Lie algebra of both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, and hence Lie groups and their corresponding Lie algebras are not in one-to-one correspondence.

4. The unitary algebra $\mathfrak{u}(n) = \{X \in \mathbb{C}^{n\times n} : X^* = -X\}$ is the Lie algebra of $U(n)$

5. The complex symplectic algebra

$$\mathfrak{sp}_n(\mathbb{C}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{pmatrix} : X_i \in \mathbb{C}^{n\times n}, X_2 = X_2^T, \text{ and } X_3 = X_3^T \right\}$$

   associated to $\mathrm{Sp}_n(\mathbb{C})$

6. The real symplectic algebra $\mathfrak{sp}_n(\mathbb{R}) = \mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{gl}_n(\mathbb{R})$ associated to $\mathrm{Sp}_n(\mathbb{R})$

Because a smooth homomorphism $\pi : G \to H$ is a smooth map, $d\pi_g : M_g \to M_{\pi(g)}$ is a linear map of tangent spaces. Thus $d\pi_e : \mathfrak{g} \to \mathfrak{h}$ is a linear map between the underlying vector spaces of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. $d\pi_e$ is in fact a Lie algebra homomorphism, and for the remainder we will denote $d\pi_e$ as simply $d\pi$.

Lie groups are also commonly studied via the techniques of representation theory. A group representation is a smooth homomorphism $\pi : G \to \mathrm{GL}(V)$ for some vector space V. A representation is called *irreducible* if the only subspaces of V invariant under $\pi(G)$ are the trivial space 0 and V itself. Similar to the representation theory of finite groups, we may define the *character* of the representation π as $\chi_\pi = \mathrm{tr} \circ \pi$.

The real numbers $\mathbb{R}$ form a Lie group when equipped with addition, and its Lie algebra is again $\mathbb{R}$. A smooth homomorphism $\gamma : \mathbb{R} \to G$ is known as a *one-parameter group*. A one-parameter group is a C^∞ curve. Because $\mathfrak{g}$ is the tangent space to G at e, for each $X \in \mathfrak{g}$ there is a one-parameter group γ so that $d\gamma : \mathbb{R} \to \mathfrak{g}$ is a map satisfying $d\gamma(1) = X$. In fact such a γ is unique. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma(1)$. As mentioned in the introduction, the exponential map may be viewed in a categorial light as a natural transformation. We may more concisely state that for any smooth homomorphism $\pi : G \to H$, $\exp \circ d\pi = \pi \circ \exp$. exp is an extremely important tool in the study of Lie groups, in that it further tightens the correspondence between a Lie group and its Lie algebra.

A *Lie subgroup* of a Lie group is a submanifold that is additionally a subgroup. Suppose that G is a Lie group with Lie subgroup H with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Because the structure of G and $\mathfrak{g}$ are so closely related, it is only reasonable to suppose that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Indeed this is the case. Moreover, as given in a theorem in [6, p .112], each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one analytic Lie subgroup of G. That is to say, if H is connected as a topological space, then it is the unique connected subgroup of G with Lie algebra $\mathfrak{h}$. This correspondence between algebraic structures is a strong property reminiscent of the fundamental theorem of Galois theory [8, p. 245].

A Lie subgroup H that is topologically closed as a subset of G is called a *closed subgroup.* Closed subgroups, like connected Lie subgroups, have a remarkable uniqueness property, as demonstrated in the following theorem.

Theorem 2.10 ([6, p. 115])**.** *Let G and $\mathfrak{g}$ be as above. Suppose that, disregarding Lie structure, H is a subgroup of G. That is, H is a subgroup, but not necessarily a Lie subgroup. Suppose further that H is closed as a subset of G. Then there is a unique smooth manifold structure on H so that H is a Lie subgroup of G.*

That is to say, Lie groups are so tightly structured that there is exactly one way to embed a group into a Lie group as a closed set.

For $g \in G$ let $\Phi_g : G \to G$ by $\Phi_g(x) = gxg^{-1}$. Φ_g is a group automorphism. Because multiplication is smooth, Φ_g is in fact a smooth automorphism. Hence $d\Phi_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, which is often denoted $\mathrm{Ad}(g)$. The set $\mathrm{Ad}\, G = \{\mathrm{Ad}(g) : g \in G\}$ is a Lie group when $\mathrm{Ad}(g)\,\mathrm{Ad}(h)$ is defined to be $\mathrm{Ad}(gh)$. We may after this definition regard Ad itself as a smooth smooth homomorphism $G \to \mathrm{Ad}\, G$ where $g \mapsto \mathrm{Ad}(g)$. This is called the Ad *representation of G.*

Because Ad is a smooth homomorphism, we may discuss its differential $d\,\mathrm{Ad}$, known as $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$. Note that $\exp \circ \mathrm{ad} = \exp \circ d\,\mathrm{Ad} = \mathrm{Ad} \circ \exp$. ad is given by $\mathrm{ad}(X)(Y) = [X, Y]$ and is known as the ad *representation of* $\mathfrak{g}$. An automorphism of $\mathfrak{g}$ of the form $\mathrm{ad}(X)$ for some X is known as an *inner derivation.*

In the study of Lie algebras, we often want to study the action of $\mathrm{Ad}\,G$ with explicit reference to G itself. Thus we define $\mathrm{Int}\,\mathfrak{g}$ to be the analytic subgroup of $\mathrm{Aut}\,\mathfrak{g}$ having $\mathrm{ad}\,\mathfrak{g}$ as its Lie algebra. In fact $\mathrm{Int}\,\mathfrak{g} \subseteq \mathrm{Ad}\,G$, and the two are identical in the case that G is connected [6, p. 127]. Elements of $\mathrm{Int}\,\mathfrak{g}$ are called *inner automorphisms.*

Recall that the Lie algebra of a Lie group is a vector space, thus we may use familiar linear algebra techniques to study $\mathfrak{g}$ and hence G. $\mathfrak{g}$, however, is a real vector space whereas a variety of matrix theoretic tools require the base field to be algebraically closed. We therefore define here a complex vector space $\mathfrak{g}_{\mathbb{C}}$ closely related to $\mathfrak{g}$, known as the *complexification* of $\mathfrak{g}$. This idea of complexification runs quite deep, having ultimately a manifestation in the theory of modules over a ring [8, p. 216]. In our case, we follow the construction of [15, p. 33]. Because $\mathbb{C}$ is a field extension of $\mathbb{R}$ and hence can be viewed as a vector space over $\mathbb{R}$, we define

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

$\otimes_{\mathbb{R}}$ denotes a tensor product which is sesquilinear with respect to multiplication by a scalar in $\mathbb{R}$. We will assume this is always the case and use only the symbol $\otimes$ to follow. To give $\mathfrak{g}_{\mathbb{C}}$ a complex structure we define for a complex number z and $X \otimes a \in \mathfrak{g}_{\mathbb{C}}$, $z(X \otimes a) = X \otimes za$.

When beginning with a complex vector space we may restrict scalar multiplication to $\mathbb{R}$. The result of applying this to $\mathfrak{g}_{\mathbb{C}}$ is denoted $(\mathfrak{g}_{\mathbb{C}})^{\mathbb{R}}$ and is decomposed as $\mathfrak{g} \oplus i\mathfrak{g}$. If a complex vector space W and a real vector space V are such that $W^{\mathbb{R}} = V \oplus iV$, we say that V is a *real form* of W. This encapsulates the relationship between $\mathfrak{g}$ and its complexification. In short, the real form of $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{g}$ itself.

Having discussed the vector space structure of $\mathfrak{g}_{\mathbb{C}}$, we must now imbue it with the structure of a Lie algebra. For $X \otimes a$ and $Y \otimes b$ in $\mathfrak{g}_{\mathbb{C}}$, define

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes ab$$

We may now identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subseteq \mathfrak{g}_{\mathbb{C}}$ and work within the confines of a complex vector space should it prove useful.

There are several classes of Lie groups that are analogous to classifications in the theory of finite groups. We will now define what it means for a Lie group to be simple, solvable, or nilpotent. Unlike the theory of finite groups, however, these classes are defined in terms of Lie algebras and then lifted to Lie groups. We begin by defining the commutator series and lower central series.

We define recursively

$$\mathfrak{g}^0 = \mathfrak{g} \qquad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \qquad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$$

The sequence $\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$ is called the *commutator series* of $\mathfrak{g}$. Each of $\mathfrak{g}^i$ is by definition an ideal of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be *solvable* if the commutator series terminates. That is, there exists an integer k so that $\mathfrak{g}^k = 0$.

We again recursively define

$$\mathfrak{g}_0 = \mathfrak{g} \qquad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \qquad \mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$$

The sequence $\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$ is called the *lower central series* of $\mathfrak{g}$. Each of $\mathfrak{g}_i$ is again an ideal by definition, and $\mathfrak{g}$ is *nilpotent* if the lower central series terminates. Every nilpotent Lie algebra is solvable, however the converse is not necessarily true.

Finally, a Lie algebra is *simple* if it is nonabelian and has no nontrivial ideals. A Lie group is simple, solvable, or nilpotent if it is analytic and its Lie algebra is simple, solvable, or nilpotent respectively.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The sum of all solvable ideals of $\mathfrak{g}$ is solvable, and it is the unique maximal solvable ideal of $\mathfrak{g}$. This ideal is known as the *radical* of $\mathfrak{g}$ and is denoted $\operatorname{Rad}\mathfrak{g}$. $\mathfrak{g}$ is called *semisimple* if $\operatorname{Rad}\mathfrak{g} = 0$, that is, the only solvable ideal of $\mathfrak{g}$ is the trivial one. In particular, a semisimple Lie algebra is not itself solvable. In a certain sense, the radical of $\mathfrak{g}$ measures the degree to which $\mathfrak{g}$ is solvable, as is made precise by the following theorem.

Theorem 2.11 ([15, p. 33])**.** *If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ is semisimple.*

Semisimplicity also behaves well with regards to complexification, further justifying the use of complexification in studying semisimple Lie algebras. Specifically, $\mathfrak{g}^{\mathbb{C}}$ is semisimple if and only if $\mathfrak{g}$ is also.

We will frequently rely on a certain bilinear form on $\mathfrak{g}$ reflecting its Lie structure. The *Killing form* is defined as

$$B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$$

and satisfies the useful property that

$$B([X, Y], Z) = B(X, [Y, Z]).$$

The Killing form is in particular crucial to the study of semisimple Lie groups and semisimple Lie algebras, which are the focus in chapters to come.

The Killing form is a symmetric bilinear form in general. In addition, Cartan's criterion characterizes the Lie algebras for which the it is nondegenerate.

Theorem 2.12 (Cartan's Criterion, [15, p. 50])**.** *$B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$ is nondegenerate if and only if $\mathfrak{g}$ is semisimple.*

A consequence of cartan's Criterion is an equivalent characterization of semisimplicity that resembles the definition of semisimple for other algebraic objects. That is, $\mathfrak{g}$ is semisimple if and only if there exist simple ideals $\mathfrak{i}_1, \mathfrak{i}_2, \ldots, \mathfrak{i}_m$ such that

$$\mathfrak{g} = \mathfrak{i}_1 \oplus \mathfrak{i}_2 \oplus \cdots \oplus \mathfrak{i}_m$$

Chapter 3

Lie Group Decompositions

In this chapter we discuss the decompositions and structure of semisimple Lie groups which facilitates the results given later.

3.1 Cartan Decomposition

Our prototype for the Cartan decomposition is the polar decomposition of matrices. It is well known that an $n \times n$ nonsingular matrix A can be decomposed as $A = PU$ where P is positive semidefinite and U is unitary. P is $(AA^*)^{1/2}$ and if A is nonsingular, then U can easily be written as $U = P^{-1}A$. It is the aim of this section to give a similar decomposition for elements of Lie groups. We start with a decomposition on the Lie algebra level.

Every real semisimple Lie algebra $\mathfrak{g}$, which we will view as a matrix algebra, has a *Cartan involution* $\theta : \mathfrak{g} \to \mathfrak{g}$. A Cartan involution is an involution such that $B_\theta(X, Y) := -B(X, \theta Y)$ is positive definite. While there is not necessarily a unique Cartan involution, there is a canonical involution that may be easily defined for any semisimple matrix Lie algebra over $\mathbb{C}$, $\theta(X) = -X^*$, where X^* is the Hermitian adjoint of X.

Once we have a Cartan involution, we may decompose $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ and $\mathfrak{k}$ are the -1 and $+1$ eigenspaces of θ respectively.

Example 3.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\theta(X) = -X^*$. Then $\mathfrak{p}$ consists of Hermitian matrices and $\mathfrak{k}$ consists of skew-Hermitian matrices. We have the ordinary decomposition of a matrix into its Hermitian and skew-Hermitian parts.

We now extends this Lie algebra decomposition to a decomposition on the group level.

Theorem 3.2 ([15, p. 362]). *Let G be a semisimple Lie group with associated Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then $G = KP$ where K is the analytic subgroup of G with Lie algebra $\mathfrak{k}$ and $P = \{\exp X : X \in \mathfrak{p}\}$. So for any $g \in G$, $g = kp$ where $k \in K$ and $p \in P$, and $K \times P \to G$ via $(k, p) \mapsto kp$ is a diffeomorphism.*

We may further describe the Cartan decomposition in terms of the *global Cartan involution.* If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition with the Cartan involution θ as above, there must exist an automorphism Θ of G with $d\Theta = \theta$ and $\Theta^2 = 1$. Furthermore, the subgroup of G fixed by Θ is K. Θ is called the global Cartan involution corresponding to θ

3.2 Root Space Decomposition

The root space decomposition is a cornerstone in the complete classification of semisimple Lie algebras, and is a useful tool in other contexts as well. We begin by assuming that $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$. The analogous decomposition in the case of a real semisimple Lie algebra is called the restricted root space decomposition, and will be detailed later.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ such that the endomorphism ad H is semisimple for all $H \in \mathfrak{h}$. Recall that a linear map is semisimple if there exists a basis for which its matrix representation is diagonal. Such a subalgebra is called a *Cartan subalgebra.*

Theorem 3.3 ([15, p. 376]). *Every semisimple Lie algebra over $\mathbb{C}$ contains a Cartan subalgebra.*

Let $\alpha \in \mathfrak{h}^*$. That is, $\alpha : \mathfrak{h} \to \mathfrak{c}$ is a linear functional. Let $\mathfrak{g}^\alpha$ denote the subspace

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

of $\mathfrak{g}$. For almost every α, $\mathfrak{g}^\alpha$ will be the trivial subalgebra $\{0\}$. If, however, $\mathfrak{g}^\alpha \neq \{0\}$, then α is called a *root* of $\mathfrak{g}$ with respect to fh, and $\mathfrak{g}^\alpha$ is called a *root space*. Observe that

$$\mathfrak{g}^0 = \{X \in \mathfrak{g} : [H, X] = 0 \text{ for all } H \in \mathfrak{h}\},$$

so $\mathfrak{g}^0$ is an abelian subalgebra of $\mathfrak{g}$. In particular, $[H, H'] = 0$ for all $H, H' \in \mathfrak{h}$, and hence $\mathfrak{h} \subseteq \mathfrak{g}^0$. Since $\mathfrak{h}$ is a maximal abelian subalgebra, we have that $\mathfrak{h} = \mathfrak{g}^0$. Let Δ be the set of all nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Theorem 3.4 (Root Space Decomposition)**.**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

We reproduce here the proof given in [6, p. 166].

Proof. Suppose that the sum is not direct. In this case, there would exist some $H \in \mathfrak{h}$ and $X_i \in g^{\alpha_i}$ such that $H + \sum X_i = 0$ where each of the α_i are distinct and nonzero. We may also choose an H' such that $\alpha_i(H')$ are all distinct and nonzero. This means H and the X_i are in separate eigenspaces of $\operatorname{ad} H'$, and are therefore linearly independent. Thus we have a contradiction and the sum must be direct.

Now $\operatorname{ad} \mathfrak{h}$ is a collection of semisimple linear maps with $(\operatorname{ad} H)\mathfrak{h} = 0$ for all $H \in \mathfrak{h}$, and hence $(\operatorname{ad} H)(\operatorname{ad} H') = (\operatorname{ad} H')(\operatorname{ad} H)$, and $\operatorname{ad} \mathfrak{h}$ forms a commuting family. Thus they are simultaneously diagonalizable and there exists a collection of one-dimensional subspaces $\mathfrak{g}_i \subseteq \mathfrak{g}$ invariant under $\operatorname{ad} \mathfrak{h}$ so that $\mathfrak{g} = \sum \mathfrak{g}_i$. Hence each of $\mathfrak{g}_i \subseteq \mathfrak{g}^\alpha$ for a suitable α, and $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$. □

A useful property of the root space decomposition is that the Killing form is nondegenerate when restricted to $\mathfrak{h} \times \mathfrak{h}$. More specifically, for each root α, there is an $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$.

3.3 Restricted Root Space Decomposition

Suppose now that $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{R}$. We begin with fixing a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. $\mathfrak{a}$ will here play the role that $\mathfrak{h}$ does in the root space decomposition of a complex semisimple Lie algebra. We proceed analogously. The set $\{\operatorname{ad} H : H \in \mathfrak{a}\}$ is a commuting family of linear maps, giving us a decomposition of $\mathfrak{g}$ into a direct sum of one-dimensional eigenspaces. We may view the simultaneous eigenvalues of this collection as members of $\mathfrak{a}^*$, the dual space of $\mathfrak{a}$. Thus we may now define as before *restricted root spaces* for $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g} : (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Those $\lambda \in \mathfrak{a}^*$ such that $\mathfrak{g}^\lambda \neq \{0\}$ are called *restricted roots*, and Σ denotes the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

Theorem 3.5 (Restricted Root Space Decomposition)**.**

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}^\lambda$$

The proof is analogous to that of the root space decomposition.

For each restricted root λ we may define a hyperplane $P_\lambda = \{H \in \mathfrak{a} : \lambda(H) = 0\}$ having codimension 1 in $\mathfrak{a}$. The set

$$\mathfrak{a} \setminus \bigcup_{\lambda \in \Sigma} P_\lambda$$

consists then of a finite number of open connected components of $\mathfrak{a}$. We call each of these components a *Weyl chamber*.

3.4 Iwasawa Decomposition

We begin as above with a real semisimple Lie algebra $\mathfrak{g}$ and a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Suppose G is a Lie group having $\mathfrak{g}$ as its Lie algebra. In the case of $\mathrm{SL}_n(\mathbb{C})$, the Iwasawa decomposition is the Gram-Schmidt orthonormalization process, written as a matrix decomposition.

The Iwasawa decomposiion has three factors, the first of which being $\mathfrak{k}$ from the Cartan decomposition. For the second factor, let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ as in the restricted root space decomposition $\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}^\lambda$. We obtain the third by introducing a notion of positivity to the restricted root space decomposition as follows.

For a restricted root λ, our notion of positivity must satisfy λ:

- exactly one of λ and $-\lambda$ is positive, and
- any linear combination of positive restricted roots with positive scalar coefficients is positive.

Any notion of positivity meeting these two requirements is satisfactory for the Iwasawa decomposition, but we give the construction from [15, p. 155]. Fix a spanning set $\{A_1, \ldots, A_m\}$ of $\mathfrak{a}$. λ is called positive, written $\lambda > 0$, if there exists an index k, $1 \leq k \leq m$, satisfying $\lambda(A_i) = 0$ if $1 \leq i \leq k-1$ and $\lambda(A_k) > 0$. Let Σ^+ denote the set of positive roots once a notion of positivity has been established. The final factor of the Iwasawa decomposition is

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}^\lambda,$$

and we have

Theorem 3.6 (Iwasawa Decomposition, [15, p. 373]). *With,* $\mathfrak{g}, \mathfrak{g}, \mathfrak{a}$, *and* $\mathfrak{n}$ *as above,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Much like the Cartan decomposition, we now seek to lift the Iwasawa decomposition of $\mathfrak{g}$ to an Iwasawa decomposition of G. Unlike the global Cartan decomposition, the factors of the global Iwasawa decomposition are subgroups of G. Let K, A, and N be the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ respectively. We then have

Theorem 3.7 (Global Iwasawa Decomposition, [15, p.374])**.** *With $G, K, A,$ and N as above,*

$$G = KAN$$

3.5 Weyl Groups

The Weyl group is a group that reflects the symmetries of a Lie structure. It may be defined for both a Lie group and a Lie algebra, and there are Weyl group actions on each. The Weyl groups for a Lie group and its Lie algebra are isomorphic, serving as another close link between the two concepts. We provide both equivalent definitions of the Weyl group, starting with the more group-theoretic definition.

Fix an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Let

$$Z = \{k \in K : \mathrm{Ad}(k)X = X \text{ for each } X \in \mathfrak{a}\}$$

be the *centralizer of* $\mathfrak{a}$ *in* K and

$$N = \{k \in K : \mathrm{Ad}(k)\mathfrak{a} \subseteq \mathfrak{a}\}$$

be the *normalizer of* $\mathfrak{a}$ *in* K. The *(analytic) Weyl group* is the quotient

$$W = W(G, \mathfrak{a}) = N/Z.$$

Each of N and Z is compact, and they have isomorphic Lie algebras. For this reason, the Weyl group W is compact and 0-dimensional. It must therefore be a finite group. W acts on $\mathfrak{g}$ via Ad.

We may equivalently define the Weyl group without any reference to G whatsoever. Let α be a root, not necessarily restricted, of $\mathfrak{g}$. Define the reflection

$$s_\alpha(X) = X - 2\frac{B(X, X_\alpha)}{B(X_\alpha, X_\alpha)} X_\alpha = X - 2\frac{\alpha(X)}{\alpha(X_\alpha)} X_\alpha$$

where X_α is an element of $\mathfrak{a}$ such that $B(X, X_\alpha) = \alpha(X)$. The *Weyl group* is the group generated by these reflections. As previously mentioned, it is isomorphic to the analytic Weyl group. We will use the term Weyl group to refer to both, as the distiniction does not matter in practice.

The orbits of the action of the Weyl group will be of great importance to us, so we spend some time here describing them. Let $X \in \mathfrak{g}$ and $\mathfrak{w}(X) = \{Y \in \mathfrak{a} : (\exists g \in G)\,\mathrm{Ad}(g)X = Y\}$, the set of all elements in $\mathfrak{a}$ that are conjugate to X. Similarly, for $h \in G$, let $W(h) = \{a \in A : (\exists g \in G) ghg^{-1} = a\}$, the set of elements in A that are conjugate to h. X is said to be *real semisimple* if $\mathrm{ad}\, X$ is diagonalizable over $\mathbb{R}$. h is said to by *hyperbolic* if there exists a real semisimple Y such that $h = \exp Y$. From [4, p. 422] we know the following for real semisimple X and hyperbolic h.

Theorem 3.8. *Let $X \in \mathfrak{g}$. If X is real semisimple, then $\mathfrak{w}(X)$ is a single Weyl group orbit in $\mathfrak{a}$, and if h is hyperbolic, then $W(h)$ is a single Weyl group orbit in A. Moreover, for such an X, $\exp X$ is hyperbolic and $W(\exp X) = \exp(\mathfrak{w}(X))$.*

We may also lift the action of the Weyl group to act on Weyl chambers. Let C_1 and C_2 be Weyl chambers, and let $w \in W$. If there is a single $X \in C_1$ such that $w.X \in C_2$, then it is the case that $w.X \in C_2$ for all $X \in C_1$. So we may say $w.C_1 = C_2$. In fact this action on the level of Weyl chambers by the Weyl group is simply transitive.

We may also rephrase our notion of positivity in the language of Weyl chambers. The choice of positivity corresponds to a choice of Weyl chamber, which we will denote $\mathfrak{a}_+$ and call the *fundamental Weyl chamber*. Then a root α is positive if it is positive on $\mathfrak{a}_+$.

3.6 The KAK and KA_+K Decompositions

Let the notation be as in the previous sections. Let $G = PK$ be a global Cartan decomposition of G with the Cartan involution θ of $\mathfrak{g}$ and the global Cartan involution Θ of G. Let $g \in G$ and define g^* to be $\Theta(g^{-1})$. Then $(fg)^* = \Theta((fg)^{-1}) = \Theta(g^{-1}f^{-1}) = g^*f^*$.

If $g = pk$ is the Cartan decomposition of the element g, then $p^* = \Theta(p^{-1}) = (p^{-1})^{-1} = p$ and $k^* = \theta(k^{-1}) = k^{-1}$ so that $g^* = k^*p^* = k^{-1}p$. In fact, $p = (gg^*)^{1/2}$ because $(gg^*)^{1/2} = (pkk^{-1}p)^{1/2} = (p^2)^{1/2} = p$.

By [4, p. 434], any element $p \in P$ is conjugate via K to an element $a \in A$. That is

$$p = k_1 a k_1^{-1},$$

so that $g = pk = k_1ak_1^{-1}k = k_1a(K_1^{-1}k) = k_1ak_2$ and we have the $G = KAK$ decomposition.

Fix a Weyl chamber $\mathfrak{a}_+$ to be the fundamental Weyl chamber, and let $A_+ = \exp \mathfrak{a}_+$. Again from [4, p. 434], p is K-conjugate to a unique element in the closure of A_+ so that

$$p = k_1 a' k_1^{-1}.$$

Then we have that $g = pk = k_1ak_1^{-1}k$ and the $G = KA_+K$ decomposition.

Example 3.9. Let $G = \mathrm{SL}_n(\mathbb{R})$. Then $K = \mathrm{U}(n)$ and A is the set of diagonal matrices with positive elements. It is well know that a non-singular matrix $M \in \mathrm{SL}_n(\mathbb{C})$ has a non-unique singular value decomposition $M = V\Sigma W$ where V, W are unitary and Σ is diagonal with positive entries. V and W are non-unique, but Σ is unique up to permutation of the diagonal. Thus the singular value decomposition is the KAK decomposition in $\mathrm{SL}_n(\mathbb{R})$.

In this case the Weyl group W is the full symmetric group on n symbols, denoted $\mathcal{S}_n$, and its action on $GL_n(\mathbb{R})$ is by conjugation by a permutation matrix. In the case of a diagonal matrix, this is permutation of the diagonal. Thus a Weyl chamber $\mathfrak{a}_+$ (and hence A_+) may be selected by imposing a particular ordering on the diagonal of matrices in $\mathfrak{a}$ (and hence A.) Thus we arrive at the KA_+K decomposition, a special case of the singular value decomposition.

3.7 Complete Multiplicative Jordan Decomposition

Here we describe the complete multiplicative Jordan decomposition. For brevity we will henceforth refer to the complete multiplicative Jordan decomposition as CMJD. CMJD can be seen as an extension of the Jordan-Chevalley decomposition of a nonsingular linear map, so we begin by detailing that. We recall first some definitions.

The vector space automorphism α_s is said to be *semisimple* if there exists a basis $\mathcal{B}$ such that α_s written as a matrix with respect to $\mathcal{B}$ is diagonal. The vector space endomorphism α_n is said to be *nilpotent* if there exists a positive integer k such that $\alpha_n^k = 0$. Equivalently, α_n is nilpotent if and only if it has all 0 eigenvalues. A vector space automorphism α_u is *unipotent* if and only if its eigenvalues are all 1. We call an automorphism α_e *elliptic* if it is diagonalizable over $\mathbb{C}$ with modulus 1 eigenvalues, and we call an endomorphism *hyperbolic* if it is the exponential of a semisimple map.

Let id denote the identity map, which we may think of as the familiar identity matrix I.

Lemma 3.10 (Jordan Decomposition, [6])**.** *Let $\alpha : V \to V$ be a vector space automorphism. Then we may uniquely decompose $\alpha = \alpha_s\alpha_u = \alpha_u\alpha_s$ with α_s semisimple and α_u unipotent.*

Proof. Begin with the additive Jordan decomposition $\alpha = \alpha_s + \alpha_n$ where α_n is nilpotent, α_s is semisimple and $\alpha_s\alpha_n = \alpha_n\alpha_s$ [7]. This decomposition is the familiar Jordan canonical form of matrices, expressed as the sum of a semisimple and a nilpotent linear map which may, after appropriate choice of basis, be expressed as a diagonal matrix and a strictly upper triangular matrix. Let $\alpha_u = \operatorname{id} + \alpha_s^{-1}\alpha_n$.

The characteristic polynomial of α_u is

$$\begin{aligned} p_{\alpha_u}(t) &= \det(\alpha_u - t \cdot \operatorname{id}) \\ &= \det(\operatorname{id} + \alpha_s^{-1}\alpha_n - t \cdot \operatorname{id}) \\ &= \det(\alpha_s^{-1}\alpha_n + (1-t)\operatorname{id}) = p_{\alpha_s^{-1}n}(t-1) \end{aligned}$$

So the roots of $p_{\alpha_u}(t)$ are the roots of $p_{\alpha_s^{-1}\alpha_n}(t)$ plus 1. Note that because α_s and α_n commute, so do α_s^{-1} and α_n, and hence $(\alpha_s^{-1}\alpha_n)^k = 0$ for the same integers k that $\alpha_n^k = 0$. That is to say, $\alpha_s^{-1}\alpha_n$ is nilpotent, and hence α_u is unipotent.

Finally,

$$\alpha_u\alpha_s = (\operatorname{id} + \alpha_s^{-1}\alpha_n)\alpha_s = \alpha_s + \alpha_s^{-1}\alpha_n\alpha_s = \alpha_s + \alpha_s^{-1}\alpha_s\alpha_n = \alpha_s + \alpha_n = \alpha$$

and,

$$\alpha_s\alpha_n = \alpha_s(\operatorname{id} + \alpha_s^{-1}\alpha_n) = \alpha_s + \alpha_n = \alpha.$$

Uniqueness follows from the uniqueness of the additive Jordan decomposition. □

We now seek to develop fully CMJD in $\mathrm{GL}_n(\mathbb{C})$, which we will then extend to an arbitrary semisimple Lie group. Thus, we need to further decompose the semisimple element of the Jordan-Chevalley decomposition as follows.

Lemma 3.11. *$\alpha_s = \alpha_e\alpha_h = \alpha_h\alpha_e$, where α_e is elliptic and α_h is hyperbolic.*

Proof. Fix a basis and write α_s as a matrix. Let $d^{-1}\alpha_s d$ be diagonal, and write its diagonal entries in (complex) polar form

$$d^{-1}\alpha_s d = \begin{pmatrix} e^{i\theta_1}\lambda_1 & & & \\ & e^{i\theta_2}\lambda_2 & & \\ & & \ddots & \\ & & & e^{i\theta_n}\lambda_n \end{pmatrix}$$

Let $h = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $e = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^i\theta_n)$. Clearly $d^{-1}\alpha_s d = eh = he$. Let $\alpha_h = dhd^{-1}$ and $\alpha_e = ded^{-1}$. Then $\alpha_s = \alpha_h\alpha_e = \alpha_e\alpha_h$. Observe that $h = \exp(\operatorname{diag}(\ln\lambda_1, \ln\lambda_2, \ldots, \ln\lambda_n))$, where exp here is the ordinary matrix exponential functions. h and is therefore hyperbolic, and consequently so is α_h. Similarly α_e is elliptic. □

We have decomposed the vector space automorphism $\alpha = \alpha_e\alpha_h\alpha_u$. All three factors commute, and uniqueness follows from the uniqueness of the Jordan-Chevalley decomposition. With this prototype firmly in mind, we now move to constructing CMJD for an arbitrary semisimple Lie group. We follow the proof given in [4, p. 419].

As before, let G be a semisimple Lie group with associated Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Let $G_{\mathbb{C}}$ be the analytic subgroup of $\operatorname{Aut}\mathfrak{g}_{\mathbb{C}}$ having $\operatorname{ad}\mathfrak{g}_{\mathbb{C}}$ as its Lie algebra, that is $G_{\mathbb{C}} = \operatorname{Int}\mathfrak{g}_{\mathbb{C}}$. Note that Ad maps G into $G_{\mathbb{C}}$. Because $\operatorname{Ad} g$ for $g \in G$ is a Lie algebra homomorphism, it is also a linear map, and we may use the argument above to decompose it into $\operatorname{Ad} g = e'h'u'$ with e' elliptic, h' hyperbolic, and u' unipotent, all unique and commuting.

Each of e', h', and u' is a Lie algebra automorphism, and moreoever the exponential of a derivation. We now require the following theorem, as given in [15, p. 102].

Theorem 3.12. *Let $\mathfrak{g}$ be a semisimple Lie algebra, real or complex. Then every derivation of $\mathfrak{g}$ is $\operatorname{ad} X$ for some $X \in \mathfrak{g}$.*

Thus in particular $h' = \exp \operatorname{ad} X$ and $u' = \exp \operatorname{ad} Z$ for some unique $X, Z \in \mathfrak{g}_{\mathbb{C}}$, where X is semisimple and Z is nilpotent. Because h and u commute, it is the case that $h' \operatorname{ad} Z = (\operatorname{ad} Z)h'$, and similarly $\operatorname{ad} X \operatorname{ad} Z = \operatorname{ad} Z \operatorname{ad} X$. Hence $[X, Z] = 0$.

We now seek to show that e', h', and u' are genuinely in $\operatorname{Ad} G$ and not $G_{\mathbb{C}} \setminus \operatorname{Ad} G$. Let σ be the automorphism on $\mathfrak{g}_{\mathbb{C}}$ defined by $\sigma(A + iB) = A - iB$. If $a \in G_{\mathbb{C}}$, then it is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ and so the map $a \mapsto a^{\sigma} = \sigma a \sigma$ gives an automorphism of $G_{\mathbb{C}}$. We have that

$$\begin{aligned}
\sigma \operatorname{Ad} g \sigma(A + iB) &= \sigma \operatorname{Ad} g(A - iB) \\
&= \sigma(\operatorname{Ad} g A - i(\operatorname{Ad} g B)) \\
&= \operatorname{Ad} g A + i(\operatorname{Ad} g B) = \operatorname{Ad} g(A + iB)
\end{aligned}$$

because $\mathfrak{g}$ is invariant under $\operatorname{Ad} g$ as a subspace of $\mathfrak{g}_{\mathbb{C}}$. Thus $\operatorname{Ad} g$ is fixed under $(\cdot)^{\sigma}$. So

$$\operatorname{Ad} g = \sigma \operatorname{Ad} g \sigma = \sigma e' h' u' \sigma = \sigma e' \sigma \sigma h' \sigma \sigma u' \sigma$$

because $\sigma\sigma = \mathrm{id}$, and each of $\sigma e' \sigma$, $\sigma h' \sigma$, and $\sigma u' \sigma$ remains elliptic, hyperbolic, and unipotent respectively. Thus by the uniqueness of the decompositoin

$$\operatorname{Ad} g = e' h' u' = \sigma e' \sigma \sigma h' \sigma \sigma u' \sigma$$

we have that

$$e' = \sigma e' \sigma, \quad h' = \sigma h' \sigma, \quad u' = \sigma u' \sigma$$

and so each of e', h', and u' is invariant under $(\cdot)^{\sigma}$.

Lemma 3.13. *If $a \in G_{\mathbb{C}}$ is invariant under $(\cdot)^{\sigma}$ then $\mathfrak{g}$ is invariant under a.*

Proof. Suppose $a = \sigma a \sigma$ and $Y \in \mathfrak{g}$. Then $a(Y) = \sigma a \sigma(Y) = \sigma a(Y)$ and hence $a(Y) \in \mathfrak{g}$. $\square$

So we have that $\mathfrak{g}$ is invariant under each of e', h', and u'. In particular, $h'(Y) = (\exp \operatorname{ad} X)Y \in \mathfrak{g}$ and hence $(\operatorname{ad} X)Y \in \mathfrak{g}$. Similarly, $\mathfrak{g}$ is invariant under $\operatorname{ad} Z$, implying $X, Z \in \mathfrak{g}$. Let $h = \exp X$ and $u = \exp Z$, and then h and u are hyperbolic and unipotent, respectively, and moreover they commute. $\operatorname{Ad} g$ commutes with h' because $(\operatorname{Ad} g)h' = e'h'u'h' = h'e'h'u' = h'(\operatorname{Ad} g)$. Similarly $\operatorname{Ad} g$ commutes with u', and so g commutes with both h and u. Moreover, $\operatorname{Ad} h = h'$ and $\operatorname{Ad} u = u'$.

Let

$$e = gh^{-1}u^{-1}.$$

Then

$$\operatorname{Ad} e = \operatorname{Ad} g(\operatorname{Ad} h)^{-1}(\operatorname{Ad} u)^{-1} = \operatorname{Ad} gh'^{-1}u'^{-1} = e',$$

and so e is elliptic and $g = ehu$.

We now need only show uniqueness of this decomposition. Suppose $g = ehu = e_1h_1u_1$. Because the decomposition $\operatorname{Ad} g = e'h'u'$ is unique, it must be the case that $\operatorname{Ad} e_1 = e'$, $\operatorname{Ad} h_1 = h'$, and $\operatorname{Ad} u_1 = u'$. $h_1 = \exp(\log h_1)$ and so

$$\operatorname{Ad} h_1 = \operatorname{Ad} \exp \log h_1 = \exp \operatorname{ad} \log h_1 = \exp \operatorname{ad} X = h'$$

By the uniqueness of the decomposition of $\operatorname{Ad} g$, $\log h_1$ must be X. Similarly, $\log u_1$ must be Z, and thus $h_1 = h$, $u_1 = u$, and it must be the case that $e_1 = e$

3.8 Kostant's Preorder

Let the notation be as in the previous sections.

Recall that an element $h \in G$ is hyperbolic if there exists a unique real semisimple $X \in \mathfrak{g}$ such that $h = \exp X$, in which case it is said that $X = \log h$. Let $h(g)$ denote the unique hyperbolic part of g in CMJD.

Suppose $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$ with associated global Iwasawa decomposition $G = KAN$. Let $\mathfrak{w}(X)$ denote the set of all elements in $\mathfrak{a}$ conjugate to the real semisimple element $X \in \mathfrak{g}$. In this case, due to [4], $\mathfrak{w}(X)$ is a single Weyl-group orbit in $\mathfrak{a}$.

Define the set

$$A(g) = \exp(\operatorname{conv} \mathfrak{w}(\log h(g)))$$

where conv is the convex hull. Two elements $f, g \in G$ satisfy $f \prec_G g$ if and only if $A(f) \subseteq A(g)$. While $\prec_G$ is clearly a preorder on G, it is in fact not a partial order, as two distinct Lie group elements with the same hyperbolic part would satisfy both $f \prec_G g$ and $g \prec_G f$, but $f \neq g$. $\prec_G$ is called *Kostant's preorder.*

While the above definition of Kostant's preorder is good for geometric intuition, in practice it is unwieldy, and so there is the following equivalent formulation, also given in [4].

Theorem 3.14.

$$f \prec_G g$$

is and only if

$$\rho(\pi(f)) \leq \rho(\pi(g))$$

for all irreducible finite dimensional representations π *of* G*, where* ρ *is the ordinary spectral radius operator.*

Chapter 4

Matrices and Inequalities Involving Matrices

4.1 Positive Definite Matrices

We first recall some facts about positive definite matrices [12]. An $n \times n$ matrix A is *positive definite* if and only if A is Hermitian and $x^*Ax > 0$ for all $x \in \mathbb{C}^n$. Thus a positive definite matrix has all positive eigenvalues, and in fact its eigenvalues coincide with its singular values. We will denote the set of $n \times n$ positive definite matrices by $\mathbb{P}_n$.

We may define non-integer, and in fact arbitrary non-rational, exponents of positive definite matrices as follows. Let $A \in \mathbb{P}_n$, and let $A = V\Sigma W$ be a singular value decomposition of A with $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)^T$. Then we define $A^\alpha := V\Sigma^\alpha W$ where $\Sigma^\alpha = \mathrm{diag}(\sigma_1^\alpha, \sigma_2^\alpha, \ldots, \sigma_n^\alpha)^T$. Note that A^α is also positive definite and $A^{\alpha/2}A^{\alpha/2} = A^\alpha$.

Although $\mathbb{P}_n$ is a Riemannian manifold, it does not form a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$, as the product AB of $A, B \in \mathbb{P}_n$ is not necessarily in $\mathbb{P}_n$. Indeed, the set $\mathbb{P}_n^2 := \{AB : A, B \in \mathbb{P}_n\}$ can be characterized as the set of all matrices with positive eigenvalues. However, $A^{\alpha/2}B^\alpha A^{\alpha/2} \in \mathbb{P}_n$ for any real number α.

Example 4.1. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

which is not even Hermitian. However,

$$B^{1/2}AB^{1/2} = \begin{pmatrix} 19/5 & 8/5 \\ 8/5 & 6/5 \end{pmatrix}$$

which is positive definite.

4.2 Majorization, Weak Majorization, and Log Majorization

We use the standard definitions as found in, for example, [12]. Let $x, y \in \mathbb{R}^n$. Fix a basis for $\mathbb{R}^n$ so that we may treat x and y as n-tuples. Let $x^{\downarrow}$ be the vector having the same entries as x but in nonincreasing order, that is $x^{\downarrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})^T$ where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and each of $x_{[i]}$ is an entry of x. y is said to majorize x, or $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \text{ for all } k = 1, 2, \ldots, n-1 \text{ and} \tag{4.1}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{4.2}$$

There is an equivalent definition which appeals to a more geometric intuition. First, define the following action of $\mathcal{S}_n$, which permutes entries of a vector:

$$\sigma \cdot x = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)})^T.$$

Define the set $C(x) = \operatorname{conv} S_n \cdot x$, where $S_n \cdot x$ is the orbit of the above action and $\operatorname{conv} A$ is the convex hull of the set A. Now $x \prec y$ if and only if $C(x) \subseteq C(y)$. Note the similarity to Kostant's preorder, where $f \prec_G g$ if and only if $A(f) \subseteq A(g)$.

If we weaken the condition (4.2), then we have *weak majorization,*

$$x \prec_{\mathrm{w}} y \text{ if and only if } \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \tag{4.3}$$

for all integers k in $[1, n]$.

Now suppose that x and y are vectors with only positive entries. A slight mutation of (4.1) and (4.2) gives us *log majorization*: $x \prec_{\log} y$ if and only if

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]} \text{ for all } k = 1, 2, \ldots, n-1 \text{ and} \tag{4.4}$$

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i \tag{4.5}$$

There is no direct implication relationship between majorization and log majorization.

Example 4.2. Let $x = (2, 2)^T$ and $y = (3, 1)^T$. Then $x \prec y$, but x and y have no log majorization relationship, because $\prod x_i \neq \prod y_i$.

Similarly, let $x = (1, 1)^T$ and $y = \left(2, \frac{1}{2}\right)^T$. Then $x \prec_{\log} y$, but $x \not\prec y$ and $y \not\prec x$. Note, however, that $x \prec_{\mathrm{w}} y$.

We do have that log majorization is stronger than weak majorization:

Proposition 4.3 ([13]). *If $x \prec_{log} y$, then $x \prec_{w} y$.*

4.3 The Lieb-Thirring and Wang-Gong Inequalities

In [5], Lieb and Thirring proved the following:

Theorem 4.4. *Let $A, B \in \mathbb{P}_n$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 1$. Then*

$$\operatorname{tr}((AB)^{\alpha}) \leq \operatorname{tr}(A^{\alpha} B^{\alpha}). \tag{4.6}$$

Let $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))^T$, the vector of eigenvalues of A, in nonincreasing order if they are all real. For $\alpha \in \mathbb{R}$, let $\lambda^{\alpha}(A) = ((\lambda_1(A))^{\alpha}, (\lambda_2(A))^{\alpha}, \ldots, (\lambda_n(A))^{\alpha})$ Because $\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i(A)$, it makes sense to reinterpret the theorem as a statement about eigenvalues.

Wang and Gong extended (4.6) in [13] as a majorization relationship between vectors of eigenvalues. Specifically, they proved the following theorem.

Theorem 4.5. *If $A, B \in \mathbb{P}_n$ and $0 < \alpha \leq \beta$, then*

$$\lambda^{1/\alpha}(A^\alpha B^\alpha) \prec_{log} \lambda^{1/\beta}(A^\beta B^\beta). \tag{4.7}$$

The proof of Wang and Gong is matrix theoretic in nature, first requiring an assortment of lemmas regarding compound matrices and inequalities with rational exponents. We will establish these lemmas, and then these rational inequalities are extended to real ones via a continuity argument.

Recall that the kth compound matrix $C_k(A)$ of A is the $\binom{n}{k} \times \binom{n}{k}$ matrix of all $k \times k$ minors of A ordered lexicographically. To be more precise, let $Q_{k,n}$ be the set of strictly increasing length k tuples of the integers in $[1, n]$. Order $Q_{k,n}$ lexicographically, and for $\tau \in Q_{k,n}$ and $i = 1, 2, \ldots, n$, let $\tau(i)$ denote the ith entry of τ. Let $A[\tau, \upsilon]$ be the submatrix of A with rows given by $\tau \in Q_{k,n}$ and columns given by $\upsilon \in Q_{k,n}$. The τ, υ entry of $C_k(A)$ is $\det(A[\tau, \upsilon])$.

The following properties of $C_k(A)$ are known and relevant [12]:

1. $C_k(AB) = C_k(A)C_k(B)$
2. $C_k(A^T) = C_k(A)^T$ for matrices A over any field
3. $C_k(A^*) = C_k(A)^*$ for matrices A over $\mathbb{C}$

The final property above gives us that if A is Hermitian, so is $C_k(A)$ because

$$C_k(A)^* = C_k(A^*) = C_k(A).$$

Let $\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))^T$, the vector of singular values of A in nonincreasing order. We aim to show

Theorem 4.6. *The largest singular value of $C_k(A)$ is $\prod_{i=1}^{k} \sigma_i(A)$.*

This is a consequence of [13, Lemma 2]. Wang and Gong, however, offer no proof, so we do so here. We use the following lemmas.

Lemma 4.7. *If A and B are unitarily equivalent, then $\sigma(A) = \sigma(B)$.*

Proof. Suppose $A = U^*BU$. Then

$$AA^* = (U^*BU)(U^*BU)^* = U^*BUU^*BU = U^*BB^*U$$

so AA^* and BB^* have the same eigenvalues, and hence A and B have the same singular values. □

Lemma 4.8. *If U is unitary, then $C_k(U)$ is unitary.*

Proof.

$$C_k(I) = C_k(UU^*) = C_k(U)C_k(U)^*.$$

□

Lemma 4.9. *If T is triangular, then $C_k(T)$ is also triangular.*

Proof. Assume without loss of generality that T is upper triangular.

The subdiagonal elements of $C_k(T)$ are of the form $\det A[\tau, \upsilon]$ with $\upsilon < \tau$. Because $\upsilon < \tau$, there exists and integer i such that $\upsilon(i) < \tau(i)$, in which case there is a row of $T[\tau, \upsilon]$ with a zero on the diagonal. As $T[\tau, \upsilon]$ is upper triangular, we have that $\det T = \prod \operatorname{diag}(T) = 0$.

Hence, the subdiagonal elements of $C_k(T)$ are all 0, and $C_k(T)$ is upper triangular. □

A mutatis mutandis argument gives us that if L is lower triangular, so is $C_k(L)$. Thus it is also the case that if D is diagonal, then so is $C_k(D)$.

Now suppose that $A \in \mathbb{P}_n$. We unitarily triangularize $A = U^*TU$, assuming without loss of generality that the diagonal of T is in nonincreasing order, whence the diagonal of $C_k(T)$

is also in nonincreasing order. The $(1,1)$ entry of $C_k(T)$ is $\det T[\tau,\tau]$ where $\tau = (1,2,\dots,k)$, the minimum element of $Q_{k,n}$, and

$$\det T[\tau,\tau] = \prod_{i=1}^{k} \lambda_i(A) = \prod_{i=1}^{k} \sigma_i(A).$$

Hence we have proved (4.6). A similar argument gives us that the eigenvalues of $C_k(A)$ are all positive, and because $C_k(A)$ is Hermitian, it must be the case that $C_k(A)$ is itself positive definite.

We will now need several more lemmas to complete the proof of (4.7). Real numbers will be used whenever possible, however we will not be able to escape relying on the rationals.

Lemma 4.10. *If A is positive semi-definite, then for any real number α, $C_k(A^\alpha) = C_k(A)^\alpha$.*

Proof. Begin by unitarily diagonalizing $A = U^*DU$. Then $C_k(A^\alpha) = C_k(U^*D^\alpha U) = C_k(U)^*C_k(D^\alpha)C_k(U)$. Similarly, $C_k(A)^\alpha = C_k(U)^*C_k(D)^\alpha C_k(U)$. Thus it suffices to prove that $C_k(D^\alpha) = C_k(D)^\alpha$.

Let $\tau \in Q_{k,n}$, using the notation from the proof of (4.9). Then

$$(C_k(D^\alpha))_{\tau\tau} = \det D^\alpha[\tau,\tau] = \prod_{i=1}^{k} \lambda_{\tau(i)}^\alpha = \left(\prod_{i=1}^{k} \lambda_{\tau(i)}\right)^\alpha = \det(D[\tau,\tau])^\alpha = (C_k(D)^\alpha)_{\tau\tau}.$$

$\square$

Lemma 4.11 ([13])**.** *For $A, B \in \mathbb{P}_n$,*

$$\lambda_1(AB) \le \lambda_1(A)\lambda_1(B).$$

Moreover, for arbitrary $n \times n$ matrices X and Y,

$$|\lambda_1(XY)| \le \sigma_1(XY) \le \sigma_1(X)\sigma_1(Y).$$

Lemma 4.12. *Suppose A and B are positive semidefinite matrices and $m \in \mathbb{N}$. Then*

$$\lambda_1^{1/m}(A^m B^m) \leq \lambda_1^{1/m+1}(A^{m+1} B^{m+1}).$$

Proof. We prove by induction on m. In the case that $m = 1$, we have that

$$\lambda_1(AB) \leq \sigma_1(AB) = \lambda_1^{1/2}((AB)(AB)^*) = \lambda_1^{1/2}(ABBA) = \lambda_1^{1/2}(A^2 B^2).$$

Now with m arbitrary, define $X = A^{(m+1)/2} B^{(m+1)/2}$ and $Y = B^{(m-1)/2} A^{(m-1)/2}$.

$$|\lambda_1(A^m B^m)| = |\lambda_1(XY)| \leq \sigma_1(x)\sigma_1(Y) = \lambda_1^{1/2}(A^{m+1} B^{m+1})\lambda_1^{1/2}(A^{m-1} B^{m-1}).$$

Observe that

$$\lambda_1^{1/2}(A^{m-1} B^{m-1}) = \lambda_1^{(m-1)/2(m-1)}(A^{m-1} B^{m-1}) \leq \lambda_1^{(m-1)/2m}(A^m B^m)$$

by the inductive hypothesis. Thus

$$\lambda_1(A^m B^m) \leq \lambda_1^{1/2}(A^{m+1} B^{m+1})\lambda_1^{(m-1)/2m}(A^m B^m).$$

Dividing both sides of the inequality yields

$$\lambda_1^{(m+1)/2m}(A^m B^m) \leq \lambda_1^{1/2}(A^{m+1} B^{m+1}).$$

Finally we raise both sides of the inequality to the power of $2/(m+1)$ to obtain

$$\lambda_1^{1/m}(A^m B^m) \leq \lambda_1^{1/(m+1)}(A^{m+1} B^{m+1}).$$

□

We now give the rational version of the Wang-Gong inequality, which we we shortly extend to the real version given above.

Theorem 4.13.

$$\lambda^{1/m}(A^m B^m) \prec_{log} \lambda^{1/(m+1)}(A^{m+1} B^{m+1}). \tag{4.8}$$

Proof. Suppose $1 \leq k < n$. Observe that from (4.10) we have that

$$C_k(A^m B^m) = C_k(A^m) C_k(B^m) = C_k(A)^m C_k(B)^m$$

and from (4.6)

$$\lambda_1(C_k(A)^m C_k(B)^m) = \lambda_1(C_k(A^m B^m)) = \prod_{i=1}^{k} \lambda_i(A^m B^m).$$

We may now apply (4.12) to obtain

$$\begin{aligned}
\prod_{i=1}^{k} \lambda_1^{1/m}(A^m B^m) &= \lambda_1^{1/m}(C_k(A)^m C_k(B)^m) \\
&\leq \lambda_1^{1/(m+1)}(C_k(A)^{m+1} C_k(B)^{m+1}) \\
&= \prod_{i=1}^{k} \lambda_1^{1/(m+1)}(A^{m+1} B^{m+1})
\end{aligned}$$

Now in the case that $k = n$,

$$\prod_{i=1}^{n} \lambda_1^{1/m}(A^m B^m) = \det(A^m B^m)^{1/m} = \det A \det B = \prod_{i=1}^{n} \lambda_1^{1/(m+1)}(A^{m+1} B^{m+1}).$$

$\square$

We may now finally construct a proof of the Wang-Gong inequality. While our argument follows that of [13], we provide many details omitted therein.

Proof of (4.7). Suppose $0 < \alpha < \beta$, where α and β are rational numbers. So there exist $r, s, p \in \mathbb{N}$ such that $\alpha = r/p$ and $\beta = s/p$. We may conclude that $r < s$ and

$$\begin{aligned} \lambda^{1/\alpha}(A^\alpha B^\alpha) &= \lambda^{p/r}(A^{r/p} B^{r/p}) \\ &= [\lambda^{1/r}((A^{1/p})^r (B^{1/p})^r)]^p \\ &\prec_{\log} [\lambda^{1/s}((A^{1/p})^s (B^{1/p})^s)]^p \\ &= \lambda^{p/s}(A^{s/p} B^{s/p}) = \lambda^{1/\beta}(A^\beta B^\beta). \end{aligned}$$

Fixing A and B we may view $x \mapsto \lambda^{1/x}(A^x B^x)$ as a map $\mathbb{Q} \to \mathbb{R}^n$, which may then be uniquely extended to a continuous map $\mathbb{R} \to \mathbb{R}^n$. $\square$

Having proven the Wang-Gong inequality, we now seek to twist it to our purposes. Because $\lambda(AB) = \lambda(BA)$, we may equivalently write

$$\lambda(A^\alpha B^\alpha) = \lambda(B^{\alpha/2} B^{\alpha/2} A^\alpha) = \lambda(B^{\alpha/2} A^\alpha B^{\alpha/2})$$

and hence describe the inequality (4.7) as the statement that the map

$$\alpha \mapsto \lambda^{1/\alpha}(B^{\alpha/2} A^\alpha B^{\alpha/2}) \tag{4.9}$$

is monotonic increasing on $(0, \infty)$ with respect to log majorization. This has a geometric advantage relevant to a Lie theory perspective, in that this map describes a smooth, linearly ordered curve through the manifold $\mathbb{P}_n$.

In fact, because $\lambda^{1/\alpha}(B^{\alpha/2} A^\alpha B^{\alpha/2}) = \lambda^{-1/\alpha}((B^{\alpha/2} A^\alpha B^{\alpha/2})^{-1}) = \lambda^{-1/\alpha}(B^{-\alpha/2} A^{-\alpha} B^{-\alpha/2})$, we can also say that (4.7) is true for any $0 < |\alpha| \le |\beta|$.

4.4 A Singular Value Inequality

While Wang and Gong provided an inequality of Lieb-Thirring type for eigenvalues of positive definite matrices, it is also possible to arrive at such an inequality for singular values. We begin with the fact that

$$\alpha \mapsto \|A^\alpha B^\alpha\|^{1/\alpha} \tag{4.10}$$

is monotonic increasing on $(0, \infty)$ where $\|\cdot\|$ is the spectral norm [14]. Recall that the spectral norm of A is the largest singular value of A.

(4.10) gives us that the map

$$\alpha \to \|C_k(A)^\alpha C_k(B)^\alpha\|^{1/\alpha}$$

is monotonic increasing for all k, and hence

$$\prod_{i=1}^{k} \sigma_i^{1/\alpha}(A^\alpha B^\alpha) \leq \prod_{i=1}^{k} \sigma_i^{1/\beta}(A^\beta B^\beta)$$

when $0 < \alpha \leq \beta$ and $k = 1, 2, \ldots, n-1$. To establish equality in the $k = n$ case, note that

$$\begin{aligned}
\sigma_1^{1/\alpha}(C_n(A^\alpha B^\alpha)) &= \prod_{i=1}^{n} \sigma_i^{1/\alpha}(A^\alpha B^\alpha) \\
&= \prod_{i=1}^{n} \lambda_i^{1/\alpha}(A^\alpha B^\alpha) \\
&= (\det(A^\alpha B^\alpha))^{1/\alpha} \\
&= (\det(AB)^\alpha)^{1/\alpha} = \det AB.
\end{aligned}$$

Hence

$$\prod_{i=1}^{n} \sigma_i^{1/\alpha}(A^\alpha B^\alpha) = \det AB = \prod_{i=1}^{n} \sigma_{i=1}^{n} \sigma_i^{1/\beta}(A^\beta B^\beta)$$

and we have demonstrated the following inequality of Wang-Gong type for singular values. While the existence of this result is hinted at in [13], we state it explicitly here:

Proposition 4.14.

$$\sigma^{1/\alpha}(A^\alpha B^\alpha) \prec_{log} \sigma^{1/\beta}(A^\beta B^\beta)$$

when $0 < \alpha \leq \beta$.

Chapter 5

Extension to Semisimple Lie Groups

We now present novel results analogous to those of Wang and Gong in the setting of semisimple Lie groups.

5.1 Extension of the Wang-Gong Inequality

Theorem 5.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and Cartan decomposition $G = PK$. Suppose $p, q \in P$, and $\alpha, \beta \in \mathbb{R}$ with $0 < |\alpha| \leq |\beta|$. Then*

$$(p^{\alpha/2} q^{\alpha} p^{\alpha/2})^{1/\alpha} \prec_G (p^{\beta/2} q^{\beta} p^{\beta/2})^{1/\beta}.$$

To prove (5.1) we first need the following lemma.

Lemma 5.2. *Let $\pi : G \to \mathrm{GL}(V)$ be an irreducible finite-dimensional representation of $G = PK$. Let $p \in P$ and $\alpha \in \mathbb{R}$. Then*

$$\pi(p^{\alpha}) = \pi^{\alpha}(p)$$

Proof. Fix a basis of $\mathcal{B}$ of V such that $\pi(p)$, when written as a matrix with respect to $\mathcal{B}$, is positive definite for all $p \in P$ [4, p. 435]. Because every element of P is hyperbolic, we can say that $p = \exp X$ for some unique $X \in \mathfrak{g}$. Thus

$$\begin{aligned}
\pi(p^\alpha) &= \pi(\exp \alpha X) \\
&= \exp(d\pi(\alpha X)) \\
&= \exp(\alpha d\pi(X)) \\
&= (\pi(p))^\alpha = \pi^\alpha(p). \qquad \square
\end{aligned}$$

Proof of (5.1). Let $\pi : G \to \mathrm{GL}(V)$ be an irreducible finite-dimensional representation of $G = PK$. As above fix a basis of V such that $\pi(p)$ written as a matrix with respect to this basis is positive definite for all $p \in P$ [4, p. 435].

$$\begin{aligned}
\rho(\pi[(p^{\alpha/2}q^\alpha p^{\alpha/2})^{1/\alpha}]) &= \rho([\pi(p^{\alpha/2}q^\alpha p^{\alpha/2})]^{1/\alpha}) \\
&= (\rho[\pi(p^{\alpha/2}q^\alpha p^{\alpha/2})])^{1/\alpha} && \text{(because } \pi(p^{\alpha/2}q^\alpha p^{\alpha/2}) \text{ is positive definite)} \\
&= \rho^{1/\alpha}[(\pi(p))^{\alpha/2}(\pi(q))^\alpha(\pi(p))^{\alpha/2})] \\
&= \lambda_1^{1/\alpha}[(\pi(p))^{\alpha/2}(\pi(q))^\alpha(\pi(p))^{\alpha/2})] \\
&\le \lambda_1^{1/\beta}[(\pi(p))^{\beta/2}(\pi(q))^\beta(\pi(p))^{\beta/2})] && \text{(by (4.9))} \\
&= \rho^{1/\beta}[(\pi(p))^{\beta/2}(\pi(q))^\beta(\pi(p))^{\beta/2})] \\
&= \rho(\pi[(p^{\beta/2}q^\beta p^{\beta/2})^{1/\beta}]).
\end{aligned}$$

And thus by the equivalence (3.14) we have the desired result. $\square$

We have also the following equivalent inequality of Wang-Gong type involving only hyperbolic Lie group elements.

Theorem 5.3. *Let $G = PK$, p, q, α, and β as in* (5.1). *Then*

$$h^{1/\alpha}(p^\alpha q^\alpha) \prec_G h^{1/\beta}(p^\beta q^\beta).$$

Proof.

$$\begin{aligned}
\rho(\pi((p^{\alpha/2}q^{\alpha}p^{\alpha/2})))^{1/\alpha} &= \rho^{1/\alpha}(\pi^{\alpha/2}(p)\pi^{\alpha}(q)\pi^{\alpha/2}(p)) \\
&= \rho^{1/\alpha}(\pi^{\alpha}(p)\pi^{\alpha}(q)) \\
&= \rho^{1/\alpha}(\pi(p^{\alpha}q^{\alpha})) \\
&= \rho^{1/\alpha}(\pi(h(p^{\alpha}q^{\alpha}))) && \text{(by [4, Proposition 3.4])} \\
&= \rho(\pi(h^{1/\alpha}(p^{\alpha}q^{\alpha}))).
\end{aligned}$$

Thus,

$$(p^{\alpha/2}q^{\alpha}p^{\alpha/2})^{1/\alpha} \prec_G (p^{\beta/2}q^{\beta}p^{\beta/2})^{1/\beta}$$

if and only if

$$h^{1/\alpha}(p^{\alpha}q^{\alpha}) \prec_G h^{1/\beta}(p^{\beta}q^{\beta}).$$

$\square$

5.2 Extension of the Singular Value Inequality

We obtain another inequality of Wang-Gong type as an extension of the singular value inequality (4.14). First, decompose G as $G = KA_+K$, and for $g \in G$, let $a(g)$ be the unique element of A_+ such that $g = ka(g)k'$.

We need two lemmas.

Lemma 5.4. *Let $l \in L = P^2$ for $G = PK$. Then $a^2(l) = a(ll^*)$.*

Proof. Suppose $l = kak'$ so that $a(l) = a$. Then

$$ll^* = kak'(kak')^* = kak'(k')^{-1}a^*k^{-1} = kaa^*k^{-1}$$

and hence $a(ll^*) = aa^* = a(l)(a(l))^*$. Then because $(a(l))^* = a(l^*) = a(l)$, we have

$$a^2(l) = a(l)a(l) = a(l)(a(l))^* = a(ll^*).$$

$\square$

Lemma 5.5. *Let $f, g \in G$. If $f \prec_G g$ then $mfm^{-1} \prec_G ngn^{-1}$ for any $m, n \in G$. That is to say, conjugacy preserves Kostant's preorder.*

Proof. Let π be an irreducible finite-dimensional representation of G as above. Because for linear maps α and β we have that the eigenvalues of $\alpha\beta$ are also the eigenvalues of $\beta\alpha$, it must also be the case they have the same spectral radius. Thus

$$\begin{aligned}
\rho(\pi(mfm^{-1})) &= \rho(\pi(m)\pi(f)\pi(f)^{-1}) \\
&= \rho(\pi(f)\pi(m)^{-1}\pi(m)) \\
&= \rho(\pi(f)) \\
&\leq \rho(\pi(g)) = \rho(\pi(ngn^{-1}))
\end{aligned}$$

$\square$

Theorem 5.6. *Let notation be as above with $0 < |\alpha| \leq |\beta|$. Then*

$$a^{1/\alpha}(p^\alpha q^\alpha) \prec_G a^{1/\beta}(p^\beta q^\beta).$$

Proof. Without loss of generality, we examine $a^{2/\alpha}(p^\alpha q^\alpha)$. Note that

$$a^2(p^\alpha q^\alpha) = a((p^\alpha q^\alpha)(p^\alpha q^\alpha)^*) = a(p^\alpha q^{2\alpha} p^\alpha).$$

By [4], $a(p^\alpha q^{2\alpha} p^\alpha)$ is conjugate to $h(p^\alpha q^{2\alpha} p^\alpha)$ and in turn to $h(p^{2\alpha} q^{2\alpha})$. Conjugacy preserves Kostant's preorder, and by (5.3) we have that $h^{2/\alpha}(p^{2\alpha} q^{2\alpha}) \prec_G h^{2/\beta}(p^{2\beta} q^{2\beta})$. The desired result follows. $\square$

5.3 Character Inequalities

Due to the following theorem of Kostant, Huang, and Kim, we also obtain inequalities for the character χ_π of an irreducible finite dimensional representation π.

Theorem 5.7 ([4],[19]). *If $g, h \in G$ are hyperbolic, then*

$$g \prec_G h$$

if and only if

$$\chi_\pi(g) \leq \chi_\pi(h)$$

for all $\pi \in \hat{G}$.

We then have the corollary

Corollary 5.8. *If p, q in P, then*

1. $\chi_\pi((q^{\alpha/2}p^\alpha q^{\alpha/2})^{1/\alpha}) \leq \chi_\pi((q^{\beta/2}p^\beta q^{\beta/2})^{1/\beta})$
2. $\chi_\pi(h^{1/\alpha}(p^\alpha q^\alpha)) \leq \chi_\pi(h^{1/\beta}(p^\beta q^\beta))$
3. $\chi_\pi(a^{1/\alpha}(p^\alpha q^\alpha)) \leq \chi_\pi(a^{1/\beta}(p^\beta q^\beta))$

Bibliography

[1] Claude Chevalley. *Theory of Lie Groups. I.* Princeton Mathematical Series, vol. 8. Princeton University Press, Princeton, N. J., 1946, pp. ix+217.

[2] M. Marcus. “An eigenvalue inequality for the product of normal matrices”. In: *Amer. Math. Monthly* 63 (1956), pp. 173–174.

[3] Michael Spivak. *A comprehensive introduction to differential geometry. Vol. One.* Published by M. Spivak, Brandeis Univ., Waltham, Mass., 1970, iii+656 pp.

[4] Bertram Kostant. “On convexity, the Weyl group and the Iwasawa decomposition”. In: *Ann. Sci. École Norm. Sup. (4)* 6 (1973), 413–455 (1974).

[5] E. Lieb and W. Thirring. “Studies in Mathematical Physics”. In: ed. by B. Simon E. Lieb and A. Wightman. Princeton Press, 1976, pp. 301–302.

[6] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Vol. 80. Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, pp. xv+628.

[7] James E. Humphreys. *Introduction to Lie algebras and representation theory.* Vol. 9. Graduate Texts in Mathematics. Second printing, revised. Springer-Verlag, New York-Berlin, 1978, pp. xii+171.

[8] Thomas W. Hungerford. *Algebra.* Vol. 73. Graduate Texts in Mathematics. Reprint of the 1974 original. Springer-Verlag, New York-Berlin, 1980, pp. xxiii+502.

[9] K. J. Le Couteur. “Representation of the function $\mathrm{Tr}(\exp(A - \lambda B))$ as a Laplace transform with positive weight and some matrix inequalities”. In: *J. Phys. A* 13.10 (1980), pp. 3147–3159. ISSN: 0305-4470. URL: `http://stacks.iop.org/0305-4470/13/3147`.

[10] Huzihiro Araki. "On an inequality of Lieb and Thirring". In: *Lett. Math. Phys.* 19.2 (1990), pp. 167–170. ISSN: 0377-9017. DOI: 10.1007/BF01045887. URL: http://dx.doi.org/10.1007/BF01045887.

[11] P. J. Bushell and G. B. Trustrum. "Trace inequalities for positive definite matrix power products". In: *Linear Algebra Appl.* 132 (1990), pp. 173–178. ISSN: 0024-3795. DOI: 10.1016/0024-3795(90)90062-H. URL: http://dx.doi.org/10.1016/0024-3795(90)90062-H.

[12] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Corrected reprint of the 1985 original. Cambridge University Press, Cambridge, 1990, pp. xiv+561. ISBN: 0-521-38632-2.

[13] Bo-Ying Wang and Ming-Peng Gong. "Some eigenvalue inequalities for positive semidefinite matrix power products". In: *Linear Algebra Appl.* 184 (1993), pp. 249–260. ISSN: 0024-3795. DOI: 10.1016/0024-3795(93)90382-X. URL: http://dx.doi.org/10.1016/0024-3795(93)90382-X.

[14] Rajendra Bhatia. *Matrix analysis*. Vol. 169. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xii+347. ISBN: 0-387-94846-5. DOI: 10.1007/978-1-4612-0653-8. URL: http://dx.doi.org/10.1007/978-1-4612-0653-8.

[15] Anthony W. Knapp. *Lie groups beyond an introduction*. Second. Vol. 140. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002, pp. xviii+812. ISBN: 0-8176-4259-5.

[16] Serge Lang. *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+914. ISBN: 0-387-95385-X. DOI: 10.1007/978-1-4613-0041-0. URL: http://dx.doi.org/10.1007/978-1-4613-0041-0.

[17] Koenraad M. R. Audenaert. "On the Araki-Lieb-Thirring inequality". In: *Int. J. Inf. Syst. Sci.* 4.1 (2008), pp. 78–83. ISSN: 1708-296X.

[18] Koenraad M. R. Audenaert. “A Lieb-Thirring inequality for singular values”. In: *Linear Algebra Appl.* 430.11-12 (2009), pp. 3053–3057. ISSN: 0024-3795. DOI: 10.1016/j.laa.2009.01.021. URL: http://dx.doi.org/10.1016/j.laa.2009.01.021.

[19] Huajun Huang and Sangjib Kim. “On Kostant’s partial order on hyperbolic elements”. In: *Linear Multilinear Algebra* 58.5-6 (2010), pp. 783–788. ISSN: 0308-1087. DOI: 10.1080/03081080903062113. URL: http://dx.doi.org/10.1080/03081080903062113.