An Exploratory Study of the Impact of a Manipulatives-Intensive Fractions Unit during a Middle Grades Methods Course on Prospective Teachers' Relational Understanding of Fractions

by

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A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
December 10, 2016

Key words: manipulatives, fractions, middle grades methods course, prospective teachers, relational understanding, conceptual knowledge, procedural knowledge

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Abstract

“A critical pillar of a strong PreK–12 education is a well-qualified teacher in every classroom” (Conference Board of Mathematical Sciences, 2012, p. 1). In order for teachers to help students build procedural fluency from conceptual understanding and meet the standards set forth by governing agencies, prospective teachers will need deep conceptual understanding of the mathematics they teach and experiences learning mathematics in ways that they will be expected to teach (Kilpatrick, Swafford, & Findell, 2001). However, at least one study has suggested that there exists a preparation gap that may contribute to a lack of student achievement (Schmidt et al., 2007). Furthermore, prospective teachers who will likely be expected to use manipulatives in their future teaching practice may have limited experience using those materials to demonstrate their knowledge.

Data from the National Assessment of Educational Progress over the last decade indicates that student achievement in Grades 4 and 8 in the Number & Operations domain has only increased by seven points and two points, respectively (https://nces.ed.gov/). Since fraction proficiency is thought to be a predictor of later success in algebra (Booth, Newton, & Twiss-Garrity, 2014; National Mathematics Advisory Panel, 2008; Usiskin, 2007), further study on rational numbers may be beneficial.

This study, which was conducted at a mid-sized, four-year university in the southeastern United States, used a case-study design to examine the impact of a manipulatives-intensive middle grades mathematics methods course on prospective teachers' procedural and conceptual knowledge of fraction multiplication and division and the connections between the two types of knowledge, i.e. relational understanding (Skemp, 1987). Data for four participants were collected through tests of knowledge; observations; various assessments given by the instructor of the course; and one-on-one, task-based and semi-structured interviews.

Acknowledgements

I lift my eyes unto the hills. Where does my help come from? My help comes from the Lord, Maker of Heaven and Earth (Psalm 121:1-2). I am most thankful to my Lord and Savior for being gracious and merciful toward me, giving me the capability to accomplish this feat, and surrounding me with loving family and friends to support me.

Thank you to my husband, Glenn, and my children, Matthew, Caleb, Benjamin, and Jacob, for your love, support, and patience with me as I spent many long hours engrossed in completing this paper. I appreciate each act of kindness and helpfulness that you provided during this time. Words cannot express how much I love each one of you.

Thank you to my mother, Marie Burkhalter, and my maternal grandmother, Estin Harris, for exemplifying a life of determination that inspired me to persevere even when I was not sure that I could. Your examples set the bar very high. I love you both.

Thank you to two special friends, Esther Flitcroft and Bill Tinkler, who always believed in me and provided encouragement and wisdom for many years and through many hard times. I love you both and am very thankful for our enduring friendship.

Thank you to Dr. Debbie Gober for continually asking me, “What is your research question?” and “How does this answer your research question?” Each time I sought your advice, those two simple questions helped me to refocus every time you asked me. I appreciate your friendship.

Thank you to my colleagues, Dr. Deirdre Greer, Kenneth Jones, Hope Phillips, Nancy Mims, and Ruby Tucker for your advice, assistance, and encouragement. I appreciate each one of you and am thankful for our friendship.

Thank you to my friend, Dr. Anna Wan, for spending many hours talking to me, listening to me, encouraging me, and reminding me to listen more than I speak. You have been a very good friend to me. Your interpretation skills are priceless.

Last, but certainly not least, thank you to my committee for the many hours you mentored, advised, and counseled me. Thank you to Dr. Marilyn Strutchens for your patience with me and for sharing your wisdom. Thank you to Dr. Gary Martin for challenging me to continually improve. Thank you to "Dr. A," (Dr. Carey Andrzejewski) for always being available to answer my questions with questions that made me look within myself for the answers and for always reminding me that "it depends." Thank you to Dr. Dean Hoffman for encouraging me throughout my many years at Auburn University. Thank you to "Pete" (Dr. Pete Johnson) for sharing your insights as my University Reader.

To my friend Sharlyn Huber: You owe me a Frosty. It's your turn now.

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Glossary

Computer applets are programs that are executed from within other applications. An applet is a small computer program that has limited features, requires limited memory resources, and is downloaded from the Internet to run on a webpage. An applet cannot read or write data on the user's machine (http://dictionary.reference.com/browse/applet).

Conceptual knowledge is knowledge that is rich in relationships. It is how knowledge is networked with other existing knowledge, connected in such a way that the linking relationships are as prominent as the discrete pieces of information. “By definition, a piece of information is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information (Hiebert & LeFevre, 1986, p. 1-2). Hiebert and LeFevre (1986) suggested that conceptual knowledge is achieved when one recognizes the relationships between pieces of information.

Conceptual understanding is the comprehension of mathematical concepts, operations, and relations. (Kilpatrick, Swafford, & Findell, 2001)

Concrete manipulatives are “material objects designed to represent explicitly and concretely mathematical ideas that are abstract” (Moyer, 2001, p. 176).

Instrumental learning is learning that focuses on recall and procedural-skill development (Pesek & Kirshner, 2000).

Instrumental understanding is knowing "rules without reasons" and the ability to use the rules to compute (i.e., ability to perform conventional algorithms without an understanding of why the algorithms work) (Skemp, 1987, p. 153)

Mathematical proficiency consists of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, & Findell, 2001, p. 5).

Procedural fluency is "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (Kilpatrick, Swafford, & Findell, 2001, p. 121).

Procedural knowledge has two distinct parts: formal language (i.e., symbol representation system) and rules, algorithms, or procedures used for completing mathematical tasks (Hiebert & LeFevre, 1986). Hiebert and LeFevre (1986) distinguished between two kinds of procedures, noting that "some procedures manipulate written mathematical symbols whereas others operate on concrete objects, visual diagrams, or other entities" (p. 7).

Relational learning is learning that focuses on the meaning of mathematical concepts (Pesek & Kirshner, 2000).

Relational understanding is understanding both what procedures to perform and why to perform them (Skemp, 1987, p. 153). Relational understanding "consists of building up a conceptual structure (schema) from which its possessor can produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (Skemp, 1987, p. 163).

Virtual manipulatives are interactive, Web-based visual representations of dynamic objects that present opportunities for constructing mathematical knowledge (Moyer, Bolyard, & Spikell, 2002).

Chapter 1: Introduction

Consider the following scenario:

Teacher: Class, today we're going to learn how to divide fractions. Let's look at this example: $\frac{1}{2} \div \frac{1}{6}$. For dividing fractions, I'm going to teach you about KFC.

Student: KFC? Like Kentucky Fried Chicken?

Teacher: (laughing) Well…no. KFC means Keep, Flip, Change.

Student: Huh?

Teacher: It's an easy way to remember how to divide fractions. You keep the first fraction; flip the second fraction, and change the sign to multiply.

Student: Why do you do that?

Teacher: "Yours is not to ask why, just flip and multiply."

Student: I don't get it. Why does flipping and multiplying work? Why do you flip the second fraction and not the first?

Teacher: I've never really thought about why it works. This is the way I was taught, and it worked for me. So, that's the way I am going to teach it. Don't worry though; you'll get it.

Although this scenario may or may not have actually happened, its consequences are many. In order for students to use mathematics to solve problems, they need to understand the associated processes, why a particular process should be chosen, and what to expect as an outcome (Martin, 2009). However, if teachers only teach memorization of procedures without understanding, mathematics may not be seen by students as a useful tool to solve problems.

Since the recent wide-spread adoption of the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010), many states have begun to use the term *college and career ready*. The National Center on Education and the Economy [NCEE] sought to determine what it means for a student to be *college and career ready*. In order to do so, the NCEE (2013) posed the following question: How much and what kind of mathematics does a student have to know and be able to deploy to be successful in their initial credit-bearing community college courses? They provided three justifications behind posing this question: 1) a large fraction of high school graduates enter community colleges; 2) community colleges provide the bulk of vocational and technical education, including auto mechanics, nurses, emergency medical technicians, and police officers; and 3) a large portion of community college graduates continue their education at four-year institutions.

The NCEE (2013) found that most of the mathematics needed that will enable students to be successful is middle school mathematics, especially arithmetic, ratio, proportion, expressions and simple equations. However, they further noted that “a large proportion of our high school graduates do not have a sound command of this fundamental aspect of mathematics” (p. 2) as evidenced by the large failure rates in community colleges. The NCEE (2013) reasoned that students were taught procedures without “learning the concepts in any durable way” (p. 2) and that the instruction they received had “significant weaknesses in teaching the concepts on which these procedures are based” (p. 2). They suggested that the teaching of fundamental middle grades mathematics concepts needs to be addressed in prospective teacher education through the way prospective teachers are taught mathematics, as well as through mathematics education courses (NCEE, 2013). That is, instruction in these courses should focus on conceptual development of topics in concert with procedural development.

Although much research has been conducted regarding fractions, students’ NAEP data demonstrates there is room for growth (Walcott, Mohr, & Kloosterman, 2012). Suggestions made by the NCEE (2013) related to students’ lack of proficiency with ratio and proportion indicates the need for continuing the search for solutions to this problem. In addition, proficiency with fractions is thought to be a predictor of success in algebraic reasoning (Booth, Newton, & Twiss-Garrity, 2014; National Mathematics Advisory Panel [NMAP], 2008). If successful completion of algebra is a goal, then students need competence with fractions (Usiskin, 2007).

Data released by the National Assessment of Educational Progress (NAEP) showed the overall average score on the 2015 Mathematics NAEP as 282, a drop from 285 in 2013 (http://nces.ed.gov/nationsreportcard/). Considering the content NAEP aligns with a score above 300, there is still work to be done. For example, NAEP noted that a score above 300 would indicate that students could: add two fractions with unlike denominators, write an improper fraction as a decimal, determine the percent given the part and the whole, and identify fractional models. This comparison paints a bleak picture for student achievement in the United States.

If student achievement is indeed impacted by teacher knowledge, as has been indicated by the Conference Board of Mathematical Sciences [CBMS] (2012) and others (e.g., Kilpatrick, Swafford, & Findell, 2001; Darling-Hammond, 1997), then efforts need to be made to ensure that teachers entering the profession have a deep knowledge of the mathematics they will be required to teach in order to relate ideas to one another and address misconceptions (Darling-Hammond, 2008). In addition, teachers must also be able to use varying teaching strategies to reach diverse learners (Darling-Hammond, 2008). In order for prospective teachers to develop deep knowledge of mathematics and varying strategies, they need experiences themselves learning mathematics in engaging ways, similar to what they will need to provide for their students (Kilpatrick et al., 2001).

If prospective teachers do not have mastery of the content they are required to teach, then the issue becomes one of inequity. NCTM (2014) stated in *Principles to Actions* that "support for access and equity requires, but is not limited to, high expectations, access to high-quality mathematics curriculum and instruction, adequate time for students to learn, appropriate emphasis on differentiated processes that broaden students' productive engagement with mathematics, and human and material resources" (p. 60). Providing access to high-quality mathematics instruction requires having teachers with mastery of the content they are responsible for teaching. In order to achieve access and equity for all students, NCTM (2014) suggested teachers design instruction that attends to the following eight Mathematics Teaching Practices presented by NCTM (2014):

- Establish mathematics goals to focus learning;
- Implement tasks that promote reasoning and problem solving;
- Use and connect mathematical representations;
- Facilitate meaningful mathematical discourse;

- Pose purposeful questions;
- Build procedural fluency from conceptual understanding;
- Support productive struggle in learning mathematics; and
- Elicit and use evidence of student thinking (p. 10).

However, research cited by NCTM (2014) indicates that one obstacle to instruction that promotes access and equity for all students is the belief that learning is memorizing and practicing skills, which some parents and educators still view as "effective" (e.g., Sam & Ernest, 2000). NCTM (2014) suggested productive beliefs to counter unproductive beliefs related to teaching and learning mathematics. For example, the unproductive belief that "mathematics learning should focus on practicing procedures and memorizing basic number combinations" should be replaced by the productive belief that "mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse" (NCTM, 2014, p. 11). This productive belief is somewhat similar to statements by the NCEE (2013) regarding how students should learn mathematics, by developing understanding of concepts and procedures.

Statement of the Problem

Taking into consideration the adoption of the *Common Core State Standards for Mathematics* (*CCSSM*) (NGA Center & CCSSO, 2010), which emphasizes conceptual understanding to undergird students' procedural fluency, prospective teachers will need deep conceptual understanding of the mathematics they teach in order to meet these standards (Kilpatrick, Swafford, & Findell, 2001). Furthermore, the *CCSSM* (NGA Center & CCSSO, 2010) suggested the use of concrete models particularly with fractions;

therefore, prospective teachers will likely need learning opportunities with manipulatives to prepare them to meet the needs of diverse learners. Because these standards are new, prospective teachers currently in, or soon to be entering, colleges of education may have deficiencies in both their conceptual understanding of mathematics and their ability to use manipulatives.

Although much research has been done to examine prospective elementary teachers' knowledge of fractions (e.g., Ma, 1999; Ball, 1990a; Ball, 1990b; Lo & Lou, 2012), research has not examined middle school teachers' knowledge of fractions to the same extent (Li & Kulm, 2008; Schmidt et al., 2007). Because middle school teachers build upon concepts from the elementary grades and prepare students for algebraic reasoning, which requires an understanding of fractions, it is important for these teachers to have strong conceptual understanding of fractions (Kilpatrick et al., 2001) and to have preparation on how the mathematical ideas of the middle grades connect with ideas and topics of elementary school and high school (Suzuka, Sleep, Ball, Bass, Lewis, & Thames, 2007).

Since proficiency with fractions is thought to be a predictor of success in algebraic reasoning (Booth, Newton, & Twiss-Garrity, 2014; NMAP, 2008; Usiskin, 2007) and in being *college and career ready* (NCEE, 2013), further study on rational numbers may be beneficial. Based on the findings of their study, the National Center on Education and the Economy [NCEE] (2013) recommended additional time be spent mastering middle school mathematics rather than advancing quickly to advanced algebra. Their recommendation also came with an emphasis on learning mathematics conceptually, not just procedurally. On the following page, *Figure 1* shows the national average scaled scores on the NAEP in the Number & Operations domain for Grades 4 and 8 from 1990 to 2015 (http://nces.ed.gov/nationsreportcard/naepdata/dataset.aspx).

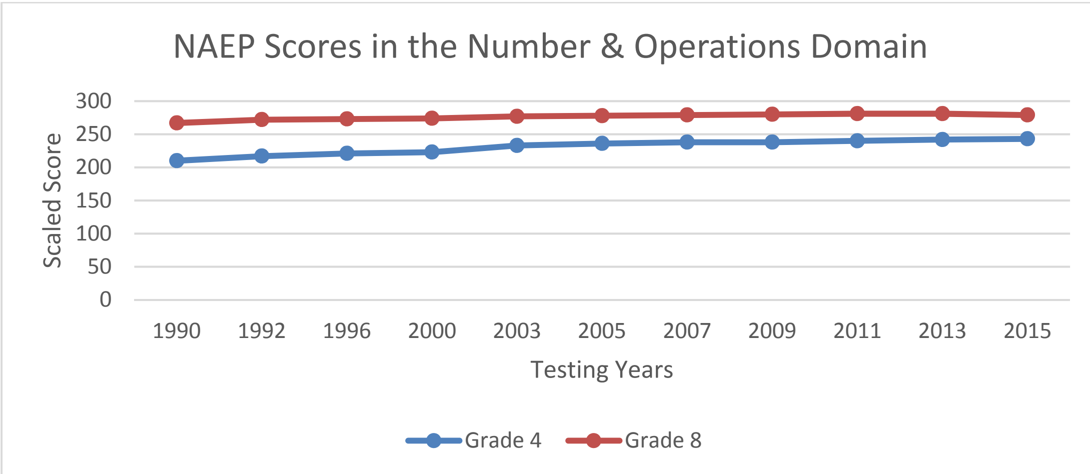


Figure 1. NAEP Average Scores in the Number & Operations Domain by Years Tested

Recommendations from the Conference Board of Mathematical Sciences [CBMS] (2012) focus not only on the practices that may improve students' learning, but also specifically the content in which their teachers need to be proficient. The CBMS (2012) noted that knowledge of unit rates is developed by connecting ratios to prior learning of fractions. In addition, the CBMS (2012) specifically address importance of knowing the connections between multiplication and division of fractions, as well as the two models of fraction division: sharing (partitive) and measurement.

The use of manipulatives is generally accepted as best practice in today's classroom to help students develop conceptual understanding (NCTM, 2000; NCTM, 2014). However, incorporating manipulatives into classroom instruction is not a simple matter (Burns, 1996; Puchner, Taylor, O'Donnell, & Fick, 2008; Moyer, 2001). For students to benefit from experiences with manipulatives, much planning and preparation must take place on the part of the teacher (Burns, 1996). Teachers must take into consideration the appropriateness of manipulative materials, as well as the limitations of the manipulative materials, in order for students to benefit from their use (Cramer & Wyberg, 2009; Moss & Case, 1999).

Teachers must recognize how to link the concrete representation to the symbolic representation in order to help students make those connections and develop conceptual understanding of mathematics (NCTM, 2000; Hiebert & Carpenter, 1992). Moreover, teachers must realize that how the manipulative is used in the classroom is of paramount importance (Ball, 1992; Moyer, 2001; Van de Walle, Karp, & Bay-Williams, 2013; Puchner, Taylor, O'Donnell, & Fick, 2008). That is, students who “mimic” a teacher’s actions with manipulatives may not benefit from the use of the manipulatives. Instead, students need opportunities to explore and develop their own mathematical reasoning through their experiences using the manipulatives (NCTM, 2014; Burns, 1996).

It is important that prospective teachers have significant learning opportunities to develop understanding of the mathematics they are required to teach (Schmidt et al., 2007). However, some prospective teachers’ prior learning experiences may have only focused on procedures (instrumental learning) instead of conceptual development (relational learning), and they may have never experienced learning with manipulatives. If prospective teachers engage in relational learning of fractions with manipulatives, they may experience interference due to their prior procedural learning similar to that experienced by grade school students studied by Pesek and Kirshner (2000). On the other hand, prospective teachers may be able to correct faulty understanding through the use of manipulatives similar to prospective elementary teachers studied by Green, Piel, and Flowers (2008).

Purpose of the Study

Since proficiency with fractions is thought to be a predictor of student success in algebraic reasoning (National Mathematics Advisory Panel, 2008), it is important to continue to explore methods of instruction that might be helpful to students as they learn fractions. Perhaps equally important in improving students' knowledge of fractions is ensuring teachers possess deep conceptual knowledge of fractions (Kilpatrick et al., 2001). Part of ensuring that teachers possess the content knowledge necessary to be effective teachers is providing them with instructional experiences during their teacher preparation that will help them develop deep conceptual understanding of mathematics content in ways similar to what they will be expected to carry out with their own future students (Kilpatrick et al., 2001; Schmidt et al., 2007). Expected methods of instruction may include the use of manipulatives.

Skemp (1987) suggested that having relational understanding is desirable due to the benefits of its adaptability to new tasks and being easier to remember. Relational understanding is having both procedural and conceptual knowledge and knowing how these are linked. Skemp (1987) noted that relational understanding is not only knowing what procedures to apply, but why to apply them.

Numerous studies have examined prospective elementary teachers' mathematics knowledge (e.g., Lo & Lou, 2012), specifically fraction knowledge, while fewer studies have examined prospective middle school teachers' mathematics knowledge (e.g., Li & Kulm, 2008). Furthermore, few studies have examined the use of manipulatives with prospective teachers (e.g., Green, Piel, & Flowers, 2008; Vinson, 2001). Therefore, the purpose of this study was to investigate the impact of a manipulatives-intensive fractions unit during a middle grades methods course on prospective teachers' relational understanding of fractions in the hope of informing teacher educators and others.

Prospective teachers in this study were enrolled in a middle grades methods course and were engaged in relational learning opportunities involving manipulatives and other best practices. Best practices carried out in the methods course included:

- Actively engaging learners as members of a learning community;
- Using worthwhile tasks;
- Facilitating student-student and student-teacher discourse;
- Inviting peer demonstrations;
- Reflecting on students' misconceptions;
- Using multiple representations;
- Taking learners' prior knowledge into consideration;
- Valuing alternative methods of problem solving;
- Attending to the development of conceptual and procedural knowledge;
- Facilitating connections with real-world phenomena and other content disciplines;
- Encouraging metacognition; and
- Encouraging abstraction when appropriate.

As I collected and analyzed observational, interview, and assessment data, I attempted to answer the following question:

> What is the impact of a manipulatives-intensive fractions unit in a middle grades methods course on prospective teachers' relational understanding of fractions?

Chapter 2: Literature Review

This chapter begins with a discussion of conceptual and procedural knowledge of mathematics and their importance with respect to mathematics education. Next, literature related to fractions, including learning trajectories, fraction literacy, and student achievement data is discussed. Then, a review of literature related to teacher knowledge is presented, including literature about prospective teachers. Finally, a review of the literature on manipulatives, including a historical look at manipulatives, is provided.

Knowledge and Understanding

In this section, Hiebert and Carpenter's (1992) conception of understanding is discussed. Conceptual and procedural knowledge from the perspectives of Hiebert and LeFevre (1986) and Hiebert and Carpenter (1992) are also mentioned and information about the importance of students and teachers having both types of knowledge is provided. The term *conceptual understanding* is discussed as it is commonly used in the literature, along with mathematical proficiency as it relates to conceptual understanding and procedural knowledge as defined by Kilpatrick, Swafford, and Findell (2001). Ball's (1988) and Ma's (1999) use of the term *conceptual understanding* is discussed and related to Hiebert and LeFevre's (1986) notion of conceptual knowledge, as well as Skemp's (1987) relational and instrumental understanding. Finally, instrumental and relational learning and Pesek and Kirshner's (2000) study about the possibility of interference of prior instrumental learning on subsequent relational learning are discussed.

Hiebert and Carpenter (1992) defined understanding in terms of the way information is represented and structured. That is, “mathematics is understood if its mental representation is part of a network of representations” (Hiebert & Carpenter, 1992, p. 67). Understanding also entails recognizing relationships between pieces of information (Hiebert & Carpenter, 1992). The number of connections that exists determines the depth of understanding (Hiebert & Carpenter, 1992).

Hiebert and Carpenter (1992) described the process of building mathematical understanding:

> Networks of mental representations are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information. Understanding grows as the networks become larger and more organized. Thus, understanding is not an all or none phenomenon. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. Connections that are weak and fragile may be useless in the face of conflicting or nonsupportive situations. Understanding increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring (p. 69).

For example, students who make the connection that lining up decimal places when adding decimals means like units are being combined (e.g. tenths with tenths) are building on their existing knowledge of place value of whole numbers (Hiebert & Carpenter, 1992).

Conceptual and procedural knowledge. In mathematics education, there are two types of knowledge that are discussed frequently: conceptual and procedural knowledge (Hiebert & LeFevre, 1986). Conceptual knowledge is “knowledge that is rich in relationships” (Hiebert & LeFevre, 1986, p. 3). It is how knowledge is networked with other existing knowledge, connected in such a way that “the linking relationships are as prominent as the discrete pieces of information” (Hiebert & LeFevre, 1986, p. 3). By definition, a piece of information “is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information” (Hiebert & LeFevre, 1986, p. 4). Eisenhart et al. (1993) defined conceptual knowledge as “knowledge of the underlying structure of mathematics-the relationships and interconnections of ideas that explain and give meaning to mathematical procedures” (p. 9).

Hiebert and LeFevre (1986) suggested that conceptual knowledge is achieved when one recognizes the relationships between pieces of information. These pieces of information can either be already stored in memory or new information linked with existing knowledge (Hiebert & LeFevre, 1986). Hiebert and LeFevre (1986) cited an example from Ginsburg (1977) of conceptual knowledge being formed between two existing pieces of information when a child understood multi-digit subtraction by recognizing the connection between the memorized algorithm and prior knowledge of the positional value of each digit. The authors further explained that conceptual knowledge of multi-digit subtraction could also be formed if the child recognized the connection between the algorithm and place value immediately upon being taught the algorithm (Hiebert & LeFevre, 1986).

In their explanation of this process, Hiebert and LeFevre (1986) stated, “Perhaps *understanding* is the term used most often to describe the state of knowledge when new mathematical information is connected appropriately to existing knowledge” (p. 4). In addition, Hiebert and LeFevre (1986) stated that the term *meaningful learning* conveys similar connotation. Hiebert and LeFevre (1986) further expressed that, “regardless of the term used, the heart of the process involves assimilating (Piaget, 1960) the new material into appropriate knowledge networks or structures. The result is that the new material becomes part of an existing network” (p. 4).

Hiebert and Carpenter (1992) stated that conceptual knowledge is either primary, where knowledge is tied to a specific context, or reflective, where knowledge becomes abstract when it is freed from specific contexts. As a result of learning procedures conceptually, ideas are linked to other networks and bring forth a greater potential to transfer the procedural use to additional applications (Hiebert & Carpenter, 1992). For example, when students learn about decimal numbers, they learn about place values of numbers and that it is necessary to line up the decimals when adding or subtracting decimals (Hiebert & LeFevre, 1986). However, this learning of lining up decimals can be thought of as a primary level. It is not until a student makes the connection that lining up decimals is a special case of the general idea that one must add like things that the learning becomes reflective (Hiebert & LeFevre, 1986). Similarly, when students recognize that getting a common denominator when adding fractions means that they are adding the same size unit pieces, the learning has proceeded to a reflective level (Hiebert & LeFevre, 1986).

Eisenhart et al. (1993) provided an example of conceptual knowledge of fractions. More specifically, conceptual knowledge of dividing fractions includes knowledge of fractions in general and the particular fractions to be divided, as well as the meaning of division. Eisenhart et al. (1993) suggested concrete referents could be used to illustrate fraction division, but also emphasized that discussing the links between and among mathematical ideas is necessary as well. For example, mathematical ideas associated with division of fractions include division of whole numbers, how multiplication is related to division, scaling and proportion, partitive and measurement division, and how story problems are related to number sentences, just to name a few (Eisenhart et al., 1993).

Procedural knowledge has two distinct parts: formal language (i.e., symbol representation system) and rules, algorithms, or procedures used for completing mathematical tasks (Hiebert & LeFevre, 1986). The formal language of mathematics includes "a familiarity with the symbols used to represent mathematical ideas and an awareness of the syntactic rules for writing symbols in an acceptable form" (Hiebert & LeFevre, 1986, p. 6). Procedures can be learned by rote and are generally followed in a step-by-step manner, done in a predetermined linear sequence (Hiebert & LeFevre, 1986). Hiebert and LeFevre (1986) distinguished between two kinds of procedures and noted, "Some procedures manipulate written mathematical symbols whereas others operate on concrete objects, visual diagrams, or other entities" (p. 7).

Procedural knowledge and conceptual knowledge differ in that the focus on procedural knowledge is on symbols and step-by-step procedures, while the focus of conceptual knowledge is how the knowledge is networked with other existing knowledge, as well as integrated with new knowledge (Hiebert & Carpenter, 1992). Both conceptual and procedural knowledge are necessary for mathematical understanding (Eisenhart et al., 1993). Hiebert and LeFevre (1986) indicated that conceptual knowledge must be learned meaningfully; however, procedural knowledge may or may not be learned with meaning (i.e., rote learning). However, Hiebert and LeFevre (1986) proposed that if procedures are learned with meaning, then they are linked to conceptual knowledge. Hiebert and LeFevre (1986) also suggested that the use of procedures provides a way for conceptual knowledge to be observable; therefore, conceptual and procedural knowledge are not necessarily distinct entities.

Hiebert and LeFevre (1986) discussed the relationship between conceptual and procedural knowledge:

> Mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge. Students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing (p. 9).

Hiebert and Carpenter (1992) suggested that how well students' conceptual and procedural knowledge are developed is related to their ability to apply skills and solve complex problems in new situations. That is, "procedural knowledge that is informed by conceptual knowledge results in symbols that have meaning and procedures that can be remembered better and used more effectively" (Hiebert & LeFevre, 1986, p. 16). For example, a student tasked with adding two decimal numbers like 3.5 and 1.76 would recognize that adding the 5 and 6 would not be appropriate because they are not of like size (e.g., tenths and hundredths).

A goal of education is to make knowledge useful to the learner, for the learner to become a competent user of his/her knowledge (Bransford, Brown, & Cocking, 2000). "A critical part of mathematical competence stipulates that entities in the symbol world must represent (for the learner) entities in the reference world" (Hiebert & LeFevre, 1986, p. 10). Building conceptual knowledge that is linked to procedural knowledge gives meaning to symbols, while building procedural knowledge that is linked to conceptual meaning contributes to memory of procedures for their effective use (Hiebert & LeFevre, 1986). In other words, procedures that are meaningful and make sense are more likely to be recalled and useful.

> If procedures are linked with conceptual knowledge, they become stored as part of a network of information, glued together with semantic relationships. Such a network is less likely to deteriorate than an isolated piece of information, because memory is especially good for relationships that are meaningful and highly organized (Hiebert & LeFevre, 1986, p. 11).

However, teachers typically focus on developing procedural skill rather than conceptual development due to their belief that conceptual development takes more time (Hiebert & Carpenter, 1992; Skemp, 1987). As a result of the focus on procedural skill, students' conceptual knowledge is often deficient (Moss & Case, 1999).

In fact, “what students learn through instruction at any moment is not just a function of the instruction; it is influenced by what they already know and by instruction in which they have already participated” (Thompson & Saldanha, 2003, p. 96). Once students have adopted procedural practices (even incorrect procedures) in solving mathematics problems, they are reluctant for those to be disturbed (Pesek & Kirshner, 2000). As knowledge is acquired and procedures are practiced repeatedly, individual parts of knowledge become a single unit (Anderson, 1983). In order to connect a procedure with its conceptual knowledge, one must separate the procedure into its individual steps so that reflection of its makeup is possible, which can be difficult to do (Hatano, 1988).

Conceptual understanding. Some of today’s mathematics education literature use the term *conceptual understanding* in ways that differ from others. Ball (1988) used the term *conceptual understanding* when she referred to what teachers need to know to be able to help students develop meaningful understanding of mathematics. She stated:

> In order to help students develop meaningful understanding of mathematics, teachers themselves need to have explicit and conceptual connected understandings of mathematical concepts and procedures. This includes being able to explain why a procedure works, to be able to connect one mathematical concept to another, or to make links between mathematics and other domains (p. 191).

Ball (1988) gave a specific example of conceptual understanding when she stated that “knowing that division by zero is undefined and knowing why go hand in hand to comprise conceptual understanding” (p. 40). In addition, Ball stated that a prospective teacher had conceptual understanding when the prospective teacher explained the regrouping algorithm using 64 as 50 plus 14 to clarify what is happening when the 6 changes to a 5 and the 4 changes to a 14 and indicated that the value of the number had not changed. Lastly, Ball (1988) stated that she asked prospective teachers to generate and explain representations of division with fractions, by zero, and in algebraic equations in order to get the candidates to "display explicit conceptual understanding" (p. 60).

When NCTM established the Learning Principle in *Principles and Standards for School Mathematics* (NCTM, 2000), they indicated that conceptual understanding plays an important role in proficiency and cited Bransford, Brown, and Cocking (1999) in saying that "conceptual understanding is an important component of proficiency, along with factual knowledge and procedural facility" (NCTM, 2000, p. 20). However, NCTM (2000) did not provide an explicit definition of conceptual understanding. In NCTM's *Principles to Actions* (2014), conceptual understanding is defined as "the comprehension of mathematical concepts, operations, and relations" (p. 7), which is the definition provided by Kilpatrick, Swafford, and Findell (2001) in the National Research Council's *Adding It Up: Helping Children Learn Mathematics*.

Finally, Ma (1999) used the term *conceptual understanding* when she discussed a teacher's understanding of regrouping when subtracting. The teacher suggested using coins as a manipulative to get students to understand what the teacher called "borrowing." The teacher indicated that a quarter could be changed to two dimes and a nickel so that one could "borrow a dime." Ma (1999) noted that this teacher's "manipulative would not convey any conceptual understanding of the mathematical topic he was supposed to teach" (p. 5) because it was not related to the base-ten structure of the number system.

Mathematical proficiency. In the National Research Council's *Adding It Up: Helping Children Learn Mathematics*, Kilpatrick, Swafford, and Findell (2001) discussed the concept of mathematical proficiency. According to Kilpatrick, Swafford, and Findell (2001), mathematical proficiency has five strands that are interwoven and interdependent. The five strands are:

- Conceptual understanding—comprehension of mathematical concepts, operations, and relations;
- Procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- Strategic competence—ability to formulate, represent, and solve mathematical problems;
- Adaptive reasoning—capacity for logical thought, reflection, explanation, and justification;
- Productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Kilpatrick et al., 2001, p. 5).

Kilpatrick et al. (2001) defined procedural fluency as "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121). In order to help students become mathematically proficient, teachers need to focus equally on conceptual understanding and procedural fluency (Kilpatrick et al., 2001; NCTM, 2000; NCTM, 2014). "It can be said that those who have conceptual understanding grasp the full meaning of knowledge, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations" (Panasuk, 2010, p. 237). *Principles and Standards for School Mathematics* (NCTM, 2000) suggested that if conceptual understanding is developed prior to procedural fluency, then less time will be spent addressing students' misconceptions and procedural difficulties. In addition, if students fail to develop a connected understanding of concepts and algorithms, they will not be able to use mathematics to its fullest potential (National Mathematics Advisory Panel, 2008).

Relational and instrumental understanding. Skemp (1987) stated that relational understanding is knowing both what procedures to perform and why to perform them, while instrumental understanding is "learning rules without reasons" (p. 9).

> What is inflicted on all too many children and older students is the manipulation of symbols with little or no meaning attached, according to a number of rote-memorized rules. This is not only boring (because meaningless); it is very much harder, because unconnected rules are much harder to remember than an integrated conceptual structure (Skemp, 1987, p. 18).

Skemp's (1987) statement regarding relational understanding as "an integrated conceptual structure" is similar to Hiebert and LeFevre's (1986) definition of conceptual knowledge, in that both definitions are grounded in the integration (linking) of mathematical concepts.

Skemp (1987) stated that instrumental understanding of mathematics does not involve an awareness of the relationship between successive stages of procedures. However, relational understanding "consists of building up a conceptual structure (schema) from which its possessor can produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (Skemp, 1987, p. 163). Skemp (1987) indicated several benefits of relational understanding: 1) it is more adaptable to new tasks; 2) it is easier to remember; 3) relational knowledge can be effective as a goal in itself (e.g., intrinsic motivation to learn); and 4) relational schemas are organic in quality (p. 158-159). With respect to the fourth benefit, Skemp (1987) noted, "If people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas" (p. 159).

Relational and instrumental learning. Relational learning is learning (instruction) that focuses on the meaning of mathematical concepts and the connections that exist among them, while instrumental learning is learning that focuses on recall and procedural-skill development (Pesek & Kirshner, 2000). Pesek and Kirshner (2000) studied the effectiveness of relational learning compared to instrumental learning that is followed by relational learning. In their study, Pesek and Kirshner (2000) examined the cognitive, attitudinal, and metacognitive interferences of prior instrumental learning on subsequent relational learning.

Their study focused on six intact regularly scheduled fifth-grade mathematics classes that were heterogeneously-grouped. Within the classes, students were separated into two treatment groups using random stratification by gender and achievement level. The two treatment groups were I-R (instrumental-relational) and R-O (relational only). The I-R group received five days of instrumental instruction focused on using and memorizing conventional formulas for calculating the area and perimeter of various geometric shapes. During instrumental instruction, formulas were not justified in terms of the characteristics of the geometric figures.

Relational instruction was then provided in three one-hour lessons over a three-day period to each intact class consisting of both I-R and R-O students. During relational instruction, all students were encouraged to use multiple modes of representation with no attention given to the use of conventional formulas (Pesek & Kirshner, 2000). Instruction focused on developing connections through the use of concrete materials, questioning, student communication, and problem solving. Instructional methods encouraged students to find efficient solution methods for calculating.

Pesek and Kirshner (2000) gathered data through the use of a pre-test, intermediate test, post-test, retention test, and interviews. The pre-test, post-test and retention test were nearly identical and consisted of 37 open-ended items worth one point each; three items were deleted in the analysis due to being poorly constructed. The intermediate test was administered to the I-R group only after the instrumental instruction was complete and was the same format used during the instrumental instruction. The post-test mean for the I-R group was 14.36, while the post-test mean for the R-O was 16.42. Trends for the retention test were similar. Although an analysis of covariance did not determine statistical significance between the means, the authors suggested that, given a larger sample size, there was a strong likelihood that the results would have been judged statistically significant.

Six students from each class were selected for three interviews each: before any instruction, following the intermediate test, and following the post-test. The interview data was gathered in order to better understand any interference effects. Analysis of interview data revealed evidence of interference of instrumental instruction on subsequent relational understanding (Pesek & Kirshner, 2000). The authors also identified cognitive, attitudinal, and metacognitive characteristics of the interference.

One example of cognitive differences between the I-R and R-O students' understanding was evident when discussing who needs to understand area and perimeter and why they need to understand it. The R-O interviewees gave examples of applications like carpet, painting, and wallpaper, while the I-R students gave examples of tests, later study, and college. Furthermore, a consistent misconception among I-R interviewees that was not present in R-O interviewees was the belief that one can only find the perimeter of walls, not the surface area, because walls "go around." The R-O students' responses are better aligned to the *CCSSM* (NGA Center & CCSSO, 2010) than the I-R students' responses because there is an emphasis in the *CCSSM* to "solve real-world problems."

With respect to attitudinal interference, all interviewees indicated that they enjoyed learning the relational unit, and they liked "playing" with the Geoboards and rubber bands. In addition, all interviewees indicated that their regular mathematics instruction was similar to the instrumental instruction because their teacher explained problems on the overhead instead of students using hands-on manipulatives. One interviewee from the I-R treatment group indicated that he preferred the relational instruction over the instrumental instruction because the formulas were confusing, and it was difficult to remember which formula went with which geometric figure.

Evidence of metacognitive interference was demonstrated when I-R students used operations incorrectly and used operations based on incorrect reasoning (Pesek & Kirshner, 2000). In addition, none of the I-R students could correctly explain the role of the “2” in the perimeter formula $P = 2(l + w)$. On the contrary, five of the six R-O interviewees were able to give a partial or a complete explanation of the perimeter formula for a rectangle despite the lack of instruction focused on formulas. Pesek and Kirshner (2000) suggested that maintaining applicative skills requires a higher level of cognitive demand and subsequent relational learning may cause students to be distracted.

Pesek and Kirshner (2000) concluded that students who first received instrumental instruction experienced metacognitive interference and subsequently did not benefit from the use of representations in the same way that the students who received only relational instruction did. The authors cautioned against teaching for rote skill development part of the time and for conceptual understanding part of the time. In fact, Pesek and Kirshner (2000) stated, “initial rote learning of a concept can create interference to later meaningful learning” (p. 537). It should be noted that Pesek and Kirshner (2000) recognized that students who were stronger, as determined by their California Achievement Test scores, “seemed able to overcome the negative effects of instrumental instruction more easily than weaker students” (p. 533).

Since the primary mode of relational instruction was through exploration using multiple representations, Pesek and Kirshner’s (2000) study may have implications on the effects of using multiple representations with prospective teachers in relational learning that has been preceded by instrumental learning. That is, if teacher educators focus on the conceptual development of middle grades mathematics, particularly through the use of manipulatives, prospective teachers in such courses may experience cognitive, attitudinal, and metacognitive interference from prior instrumental learning similar to that experienced by students in Pesek and Kirshner’s (2000) study.

Summary of knowledge and understanding. In this section, knowledge and understanding were examined from multiple perspectives, and mathematical proficiency as it relates to conceptual and procedural knowledge was discussed. In order for students to be considered mathematically proficient, they need both conceptual and procedural knowledge, as well as a connectedness between the two. How students learn depends on how they are taught (Pesek & Kirshner, 2000). If students are taught with a focus on procedures (instrumental learning) without the meaning associated with those procedures (relational understanding), it is likely that their mathematical proficiency will suffer. Therefore, it may be said that one is mathematically proficient if he/she has relational understanding, which entails having conceptual and procedural knowledge and an understanding of the connections between the two (Skemp, 1987).

Teacher Knowledge

This section begins with a discussion of the importance of qualified teachers in the middle school. Next, mathematical knowledge for teaching, specialized content knowledge, and pedagogical content knowledge are discussed. Finally, literature related to prospective teachers and how it relates to literature presented earlier in this paper is explicated.

Why teacher knowledge matters. “A critical pillar of a strong PreK–12 education is a well-qualified teacher in every classroom” (Conference Board of Mathematical Sciences [CBMS], 2012, p. 1). Teaching is a complex endeavor requiring mathematical knowledge, knowledge of teaching, knowledge of learning, knowledge of students, and much more (CBMS, 2012; NCTM, 2000).

However, in order to be well-qualified, teachers need more than a “student’s understanding” of the content they are required to teach, as well as an understanding of the curricular coherence of the content (CBMS, 2012). Teachers also need a balance of conceptual understanding and procedural knowledge of the mathematics they are required to teach, which is not always the case (Tchoshanov, 2011).

Middle school is one of the critical points in a child’s education, and teacher knowledge likely has some impact on student success (Schmidt, Blömeke, & Tatto, 2011). “Because the middle grades are ‘in the middle,’ it is critical that middle grades teachers be aware of the mathematics that students will study before and after the middle grades” (CBMS, 2012, p. 45). In addition to their responsibilities for middle grades content, middle grades teachers also need to have an elementary teacher’s perspective on mathematics content because they may need to provide additional instructional support for students who have not yet achieved mastery (CBMS, 2012). Moreover, prospective middle grades teachers need to be familiar with representations used in the earlier grades and how these representations support the extension of mathematical ideas into the middle grades and beyond (CBMS, 2012). Hence, prospective middle grades teachers need specialized preparation that addresses mathematics relevant for teaching grades 5-8, as well as methods courses which integrate the study of mathematics and pedagogy (CBMS, 2012).

Why prospective teacher education is important. In August 2013, the Council for the Accreditation of Educator Preparation (CAEP) Board approved new CAEP standards for accreditation. CAEP's first recommended standard is prospective teachers' development of content and pedagogical knowledge. The authors cited Ball, Thames, and Phelps (2008) as the source of their definition of content knowledge, which they articulated as "the depth of understanding of critical concepts, theories, skills, processes, principles, and structures that connect and organize ideas within a field" (p. 12). In their rationale for this standard, they cited a need for prospective teachers to develop "deep understanding of the major concepts and principles within the candidate's field, including college- and career-ready expectations" (p. 11). This expectation stems partly from the recent adoption of the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010).

"Effective programs of teacher preparation and professional development help teachers understand the mathematics they teach, how their students learn that mathematics, and how to facilitate that learning" (Kilpatrick et al., 2001, p. 10). Teacher preparation programs need to provide experiences for prospective teachers that help them to develop an integrated knowledge of mathematics, knowledge of the development of students' mathematical understanding, and pedagogical practices that take into account the mathematics being taught and the students who are learning it (Kilpatrick et al., 2001).

In order for teachers to develop deeper understanding of mathematics and how to effectively teach it, "teachers themselves need experiences in doing mathematics—in exploring, guessing, testing, estimating, arguing, and proving…they should learn mathematics in a manner that encourages active engagement with mathematical ideas" (Mathematical Sciences Education Board and National Research Council, 1989, p. 65). Prospective teachers need to be engaged in this type of learning as well as practicing teachers (Kilpatrick et al., 2001).

Rationale for studying prospective middle grades teachers. For many years there has been a focus on the content knowledge of elementary teachers, or lack thereof as the case may be (e.g. Ma, 1999; Ball, 1990a; Ball, 1990b; Lo & Lou, 2012). However, less attention has been paid to what knowledge middle school mathematics teachers possess (Li & Kulm, 2008; Schmidt et al., 2007). Because middle school teachers build upon concepts from the elementary grades and prepare students for algebraic reasoning, which requires an understanding of fractions, it is important for these teachers to have strong conceptual understanding of fractions (Kilpatrick et al., 2001) and to have preparation on how the mathematical ideas of the middle grades connect with ideas and topics of elementary school and high school (Suzuka et al., 2007).

In their report *The Preparation Gap: Teacher Education for Middle School Mathematics in Six Countries* (Schmidt et al., 2007), the authors stated that "the results clearly suggest that teacher education as defined by the learning opportunities provided likely has an impact on what future teachers know and believe as they leave their teacher preparation program" (p. 41). Schmidt et al. (2007) chose the term "preparation gap" to sum up their findings. They found that the top performing countries, as determined by the 8^{th}-grade results of the Third International Mathematics and Science Study (TIMSS), had future middle school teachers who had higher mathematical knowledge than those in the United States. Schmidt et al. (2007) attributed the higher achievement of Asian middle school students to the learning experiences of their teachers during the teachers' preparation. Similarly, Li and Kulm (2008) stated that a focus on the preparation of middle school mathematics teachers in different education systems suggests the importance of understanding not only the status of middle school teachers' knowledge preparation they received through different programs of study, but also possible connections with middle school students' mathematics performance (p. 836-837).

Opportunities to develop conceptual knowledge of mathematics are among these important learning opportunities for prospective teachers (Li & Kulm, 2008). It is likely that prospective middle school teachers enter college with a procedural knowledge of mathematics, but their conceptual knowledge of mathematics may be lacking (Graeber, 1999). Prospective teachers need to experience activities as learners, opportunities to examine their own misconceptions, and to design lessons that will facilitate students' conceptual understanding (Graeber, 1999). If the cause of the achievement gap is truly a "preparation gap," then mathematics teacher educators need to contemplate the learning opportunities they provide to future middle school mathematics teachers to help them develop their own conceptual understanding (Li & Kulm, 2008).

Mathematical knowledge for teaching. The need for well-prepared, knowledgeable teachers has been a concern of many (NCTM, 2000; CBMS, 2012; Kilpatrick et al., 2001). Hill, Schilling, and Ball (2004) developed and tested measures that would determine the level of teachers' mathematical knowledge for teaching. Hill, Rowan, and Ball (2005) defined mathematical knowledge for teaching as the mathematical knowledge used to carry out the work of teaching mathematics (p. 373).

Hill, Rowan, and Ball's (2005) study used their previously-developed test measures to determine whether teachers' mathematical knowledge for teaching was a predictor of student achievement. They found that "teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grades" (p. 399). With respect to student gains in the first grade, Hill, Schilling, and Ball (2004) suggested that teachers' content knowledge is important even in very elementary mathematics content.

Similarly, in their work to identify knowledge that is specific to teaching mathematics, Ball, Thames, and Phelps (2008) built upon Shulman's (1986) conception of pedagogical content knowledge. The authors suggested that Shulman's content knowledge could be subdivided into common content knowledge and specialized content knowledge. Ball et al. (2008) further suggested that Shulman's pedagogical content knowledge could be subdivided into knowledge of content and students and knowledge of content and teaching.

Ball et al. (2008) defined common content knowledge as the mathematical knowledge and skill used in settings other than teaching (p. 399). The authors proposed that this knowledge encompasses, for example, being able to do the work assigned to students and use terms and notation correctly. However, because this knowledge is not unique to teaching and is used in a wide variety of settings, Ball et al. (2008) chose to describe it as "common." In order to study CCK, the authors posed questions that would be answerable by persons other than mathematics teachers. For example, Ball et al. (2008) asked questions which required knowing that a square is a rectangle; that 0/7 is 0; and that the diagonals of a parallelogram are not necessarily perpendicular, as well as a question requiring knowledge of what numbers lie between 1.1 and 1.11 (p. 399). Ball et al. (2008) indicated that these questions do not require special knowledge and would be answerable by persons who know mathematics.

Next, Ball et al. (2008) defined specialized content knowledge (SCK) as mathematical knowledge and skills unique to teaching (p. 400). They stated that SCK is not typically needed for purposes other than teaching. Ball et al. (2008) listed 16 mathematical tasks for teaching, as shown in Table 1 below and continued on the following page.

Table 1

Mathematical Tasks of Teaching

Presenting mathematical ideas
Responding to students' "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use

Asking productive mathematical questions

Selecting representations for particular purposes

Inspecting equivalencies (p. 400)

Ball et al. (2008) provided an example problem, shown below in Table 2, that requires specialized content knowledge. The focus of the problem is mathematical concepts that are commonly confused, dividing by 2 (part a in Table 2) and dividing by $\frac{1}{2}$ (part c in Table 2). Part c of the problem also involves measurement division of fractions.

Table 2

Sample Problem to measure Specialized Content Knowledge (SCK)

Which of the following story problems can be used to represent $1\frac{1}{4}$ divided by $\frac{1}{2}$?

	Yes	No
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need? (p. 400)	1	2

The questions Ball et al. (2008) used to measure SCK are aligned with those examined by Ma (1999) to determine the extent of teachers' conceptual understanding of division of fractions.

The third type of knowledge Ball et al. (2008) defined was knowledge of content and students (KCS), which combines knowledge about students and knowledge about mathematics (p. 401). Within this domain, teachers must be able to anticipate students' thinking and what they are likely to find confusing, including common conceptions and misconceptions. Problems Ball et al. (2008) asked in developing their measures for KCS included problems about the likelihood that students may write 405 for 45, the shapes students are likely to identify as triangles, and problems where confusion between area and perimeter lead to wrong answers.

Finally, Ball et al. (2008) defined knowledge of content and teaching (KCT) as knowing about teaching and knowing about mathematics (p. 401). This knowledge domain includes knowing about the design of instruction. For example, sequencing content for instruction; choosing appropriate examples with specific goals in mind; evaluating advantages and disadvantages of particular representations for specific content; identifying advantages and disadvantages of specific instructional methods and strategies; responding to student contributions; and posing questions are all encompassed in KCT. Ball et al. (2008) gave the example of knowing which instructional models are appropriate for learning place value, knowing what each model can reveal about an algorithm, and how the differences matter for the development of a topic as being a part of KCT. In developing measures for KCT, Ball et al. (2008) asked questions like whether a tape measure would be a good tool for learning about place value, how to choose examples to demonstrate multiple strategies for simplifying radicals, and how to sequence subtraction problems with and without regrouping.

Ball et al. (2008) concluded, “Teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content. At the same time, just knowing a subject well may not be sufficient for teaching” (p. 404). This statement is similar to that made by Kilpatrick et al. (2001). Ball et al. (2008) suggested that further research needs to be done to possibly refine their concept map, as well as look at relationships among specialized content knowledge, student achievement, and “whether and how different approaches to teacher development have different effects on particular aspects of teachers’ pedagogical content knowledge” (p. 405). Finally, Ball et al. (2008) suggested that a clearer sense of the knowledge domains might inform teacher education and professional development, and result in a curriculum for the content preparation of teachers. Therefore, one should wonder what learning experiences during prospective teacher education contribute to the development of MKT.

Pedagogical content knowledge. “Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies” (NCTM, 2000, p.17). An, Kulm, and Wu (2004) suggested that the most important element in the domain of mathematics teachers’ knowledge is pedagogical content knowledge, the integration of pedagogy and content knowledge.

Pedagogical content knowledge encompasses both how to teach mathematics content and how to understand students’ thinking (An, Kulm, & Wu, 2004). Pedagogical content knowledge also refers to the teachers’ ability to transform content into pedagogically rich experiences for students while also taking into consideration their varying abilities and prior knowledge (Shulman, 1987).

An, Kulm, and Wu (2004) compared the pedagogical content knowledge of United States (U.S.) and Chinese middle school teachers, within a cultural context, by examining how the teachers developed students’ mathematical thinking. An, Kulm, and Wu (2004) expanded upon Shulman’s (1987) conception of pedagogical content knowledge by including knowledge of content, knowledge of curriculum, and knowledge of teaching. Knowledge of content includes a broad mathematics knowledge, as well as more specific content knowledge of the mathematics particular to a given grade level (An, Kulm, & Wu, 2004).

Knowledge of curriculum includes selecting curricular materials with specific goals in mind (NCTM, 2000). Knowledge of teaching includes selecting appropriate instructional strategies while taking students’ thinking into consideration (NCTM, 2000). With respect to knowledge of teaching, An, Kulm, and Wu (2004) listed the following as being encompassed in knowing students’ thinking: addressing students’ misconceptions; building on students’ math ideas; engaging students in math learning; and promoting students’ thinking in mathematics.

The subjects for An, Kulm, and Wu’s (2004) study were 28 U.S. mathematics teachers, ranging from fifth to eighth grade, from 12 schools in four school districts in a large metropolitan area in Texas and 33 Chinese fifth- and sixth-grade mathematics teachers from 22 schools in four school districts in a large city in Jiangsu province in eastern China. Data were collected using an author-constructed Mathematics Teaching Questionnaire, an author-constructed Teachers’ Beliefs about Mathematics Teaching and

Learning Questionnaire, and interviews and observations with selected teachers (An, Kulm, & Wu, 2004).

The Mathematics Teaching Questionnaire consisted of four problems that were designed to examine teachers' pedagogical content knowledge of fractions, ratios, and proportion. The problems focused on the following components of teachers' knowledge of teaching: building on students' mathematical ideas, identifying and correcting students' misconceptions, engaging students in learning, and promoting student thinking (An, Kulm, & Wu, 2004). "On the one hand, the results [of the Mathematics Teaching Questionnaire] show that both groups of teachers have extensive and broad pedagogical content knowledge and are able to apply various methods to help students learn mathematics" (An, Kulm, & Wu, 2004, p. 155). However, there were marked differences in each of the four components of teaching for understanding.

The Mathematics Teaching Questionnaire used by An, Kulm, and Wu (2004) is shown in Table 3 below and continued on the following page.

Table 3

Teaching Questionnaire (An, Kulm, & Wu, 2004)

Problem 1	
Adam is a 10-year-old student in 5th grade who has average ability. His grade on the last test was an 82 percent. Look at Adam's written work for these problems: $\frac{3}{4}+\frac{4}{5}=\frac{7}{9}$ $\quad 2\frac{1}{2}+1\frac{1}{2}=3\frac{2}{5}$	a. What prerequisite knowledge might Adam not understand or be forgetting? b. What questions or tasks would you ask Adam, in order to determine what he understands about the meaning of fraction addition? c. What real world example of fractions is Adam likely to be familiar with that you could use to help him?

Problem 2

A fifth-grade teacher asked her students to write the following three numbers in order from smallest to largest:

$$\frac{3}{8}, \frac{1}{4}, \frac{2}{3}$$

Latoya, Robert, and Sandra placed them in order as the follows.

Latoya: $\frac{1}{4}, \frac{2}{3}, \frac{3}{8}$

Robert: $\frac{2}{3}, \frac{1}{4}, \frac{3}{8}$

Sandra: $\frac{1}{4}, \frac{3}{8}, \frac{2}{3}$

a. What might each of the students be thinking?
b. What question would you ask Latoya to find out if your opinion of her thinking is correct?
c. How would you correct Robert's misconception about comparing the size of fractions?

Problem 3

You are planning to teach procedures for doing the following type of fraction multiplication.

$$\frac{1}{4} \times \frac{1}{14}$$

$$\frac{3}{4} \times \frac{2}{3}$$

a. Describe an introductory activity that would engage and motivate your students to learn this procedure.
b. Multiplication can be represented by repeated addition, by area, or by combinations.
c. Which one of these representations would you use to illustrate fraction multiplication to your students? Why?
d. Describe an activity that would help your students understand the procedure of multiplying fractions.

For Problem 1, 46% of U.S. teachers believed that Adam "forgot" the prerequisite knowledge of finding common denominators for addition of fractions, while 55% of Chinese teachers believed that Adam did not understand the prerequisite knowledge. Chinese teachers made statements supporting their beliefs. For example, teachers said that only like units can be added, as in the case of three books plus five books equals eight books. But, unlike units cannot be added, as in the case of three books and four desks. The Chinese teachers further stated that Adam did not understand the concept of like units and may not be thinking of fractions as numbers. An, Kulm, and Wu (2004) suggested that the U.S. teachers appeared to believe that learning consists of knowing or not knowing, remembering or forgetting. On the contrary, Chinese teachers showed evidence of understanding students' thinking more deeply and seemed to convey their belief that learning encompasses understanding with connections.

Although concrete models can help students to visualize fractions, understanding fractions as quantity is important (Sowder et al., 1998). An, Kulm, and Wu (2004) found that 93% of U.S. teachers used concrete models when teaching fractions, whereas only 42% of Chinese teachers used them. Instead, Chinese teachers focused on developing students' understanding of the unit and procedural fluency through following the rules. The authors also found that only 29% of U.S. teachers focused on developing fractions conceptually, while 51% of Chinese teachers did so. An, Kulm, and Wu (2004) also suggested that the way fractions are introduced as parts of a whole in U.S. textbooks may lead to misconceptions regarding the numerator and denominator.

An, Kulm, and Wu (2004) asserted that teachers should be able to identify students' misconceptions and correct those misconceptions by using questioning and tasks. They found that 86% of U.S. teachers used a variety of models to address and correct students' misconceptions in Problem 2. Chinese teachers also used models; however, Chinese teachers "focused on developing the explicit connection between the various models and abstract thinking" (An, Kulm, & Wu, 2004, p. 161). NCTM (2000) suggested that it is important for teachers to facilitate the linking of the concrete to the abstract when using concrete models. However, the authors found that U.S. teachers in their study “often ignored developing the connection between the manipulative activities and abstract thinking” (An, Kulm, & Wu, 2004, p. 165).

Another marked difference An, Kulm, and Wu (2004) found in the analysis of U.S. and Chinese teachers’ responses to Problem 2 involved questioning. That is, 100% of Chinese teachers were able to use questioning and tasks to correct students’ misconceptions, while only 61% of U.S. teachers could. Furthermore, 79% of U.S. teachers were not able to pose appropriate questions to help them identify student thinking, while only 39% of Chinese teachers could not do so. Effective questioning strategies are necessary for students to be able to further their understanding of concepts and make connections in mathematics (NCTM, 2000; NCTM, 2014).

As An, Kulm, and Wu (2004) analyzed data for Problem 3, they found that although many U.S. teachers suggested a variety of ways to learn multiplication, most U.S. teachers (64%) used only an area representation for multiplying fractions, whereas 67% of Chinese teachers used an area model and repeated addition. Furthermore, Chinese teachers were able to differentiate between when an area model was more appropriate and when repeated addition was more appropriate. Using multiple representations increases the chance for students to develop deeper, more connected understanding of fractions (NCTM, 2000).

For Problem 4, An, Kulm, and Wu (2004) found that 100% of Chinese teachers asked probing questions to explore students’ thinking and encouraged students to think deeply and critically, whereas 43% of U.S. teachers provided only general questions that would not provide them insight into students’ thinking. Moreover, Chinese teachers tended to use an algebraic approach to build students’ abstract thinking using procedures, while U.S. teachers tended to use charts and tables, concrete or pictorial models, and manipulatives.

An, Kulm, and Wu (2004) concluded, “Teaching for understanding includes a convergent process in which teachers build students’ mathematics ideas by connecting prior knowledge and concrete models to new knowledge, focusing on conceptual understanding and procedure development” (p. 169). In addition, the authors stated that although the use of manipulatives helps students to develop conceptual understanding, teachers need to balance instruction so that procedural skill is achieved as well. “Without developing firm understanding and skill with procedures, students will not be able to solve problems efficiently and confidently” (An, Kulm, & Wu, 2004, p. 169).

Prospective teacher literature. In this portion of the literature regarding teacher knowledge, literature about prospective teachers is presented. Studies on the effects of reform-oriented curricula; implementing mathematical tasks; the common and specialized content knowledge that Taiwanese prospective elementary teachers possess; the possible inadequacies of prospective middle school teachers’ mathematical knowledge for teaching fraction division; the obstacles and challenges prospective teachers experience learning fractions; and prospective teachers’ anxiety toward mathematics in a methods course focused on the use of manipulatives in learning are summarized.

Effects of reform-oriented curricula. Lloyd and Frykholm (2000) examined the effects of using reform-oriented middle school curricula with prospective elementary teachers. Their study included 50 prospective teachers, mostly female White sophomores, learning Geometry for prospective teachers in a one-semester math course. The classroom environment focused on prospective teachers' opportunities afforded by the curriculum materials, engagement both as learners and teachers of mathematics, small-group work, and sharing of work through class presentations. Course activities provided opportunities for prospective teachers to think about the nature of mathematics, make connections between their learning and that of their future students, and develop a vision for their future classroom practice.

Lloyd and Frykholm (2000) found that many prospective teachers (PTs) expressed apprehension about taking a Geometry course and described their previous learning experiences in negative terms. However, the PTs also expressed a desire to "fill the gaps" in their knowledge. As the PTs worked, they expressed that the activities "posed significant mathematical difficulties to them" (Lloyd & Frykholm, 2000, p. 577). Many PTs were unfamiliar with "open questions" and "experiments in math," which caused the PTs to experience frustration because of the misalignment with their previous learning experiences. One PT commented, "I must admit I found junior high math easier to understand the first time around" (Lloyd & Frykholm, 2000, p. 577). On the other hand, some PTs thrived in the reform-oriented environment as evidenced by a statement from one of the PTs, "I am learning how to look for reasons and explanations as opposed to simply believing 'the rules' that some really ancient dead guy came up with. I prefer being able to use my own mind in solving problems" (Lloyd & Frykholm, 2000, p. 577).

The authors also found that "some of the teachers who reacted most positively to the middle school work were those who struggled the most with it" (Lloyd & Frykholm, 2000, p. 578).

Prospective teachers were also tasked with teaching their classmates. As a result, PTs began to recognize that teaching demands extensive subject matter knowledge and that the reform-oriented curriculum materials required a different teaching strategy than they had previously seen. PTs also began to understand connections between their knowledge of Geometry and their ability to teach it (Lloyd & Frykholm, 2000). Furthermore, "one of the most fascinating aspects of teachers' reactions was the way they made connections between their own learning and their ideas about their future students' learning" (Lloyd & Frykholm, 2000, p. 580). Lloyd and Frykholm (2000) suggested that mathematics teacher educators consider integrating mathematics and pedagogy to help PTs make critical connections between their learning and teaching experiences.

The PTs in Lloyd and Frykholm's (2000) study reacted differently to their learning experiences. For example, one PT commented that learning junior high math was easier the first time; another PT commented that it was nice to look for reasons and be able to use his/her own mind. This difference in attitudes among elementary PTs raises the question of whether middle grades prospective teachers would react similarly to reform-oriented instruction.

Implementing mathematical tasks. Rathouz and Rubenstein (2009) discussed their analysis of the orchestration of a fractions task to supporting prospective elementary teachers' learning. The authors studied video records of prospective teachers attempting to resolve their confusions and justify their mathematical claims of generating two different story problems for $3\frac{1}{2}-\frac{2}{3}$. Instruction prior to this fractions task focused on whole numbers, the meaning of operations, and the importance of the referent whole.

Prospective teachers (PTs) generated a story problem about a person running three and one-half miles, stopping for a rest break after either "two-thirds of the way" or "two-thirds of a mile," and determining how much farther the person must run. The PTs could not decide whether "two-thirds of the way" or "two-thirds of a mile" was appropriate. The instructor facilitated a debate situation where PTs had to justify their reasoning. One PT generated a number line (shown in *Figure 2* below) to represent the three and one-half miles, and indicated where two-thirds of a mile and where two-thirds of the way were relative to the referent whole three and one-half miles.

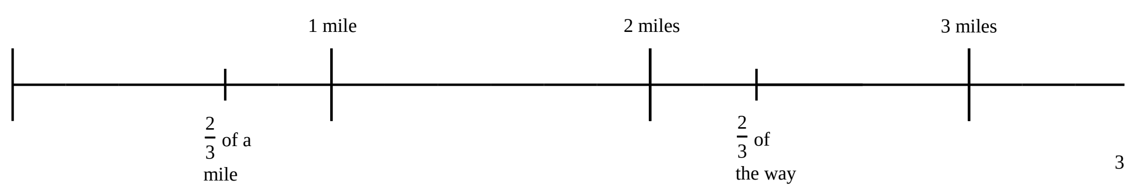


Figure 2. Two-thirds of a mile versus two-thirds of the way (Rathouz & Rubenstein, 2009)

Some PTs had difficulty understanding the referent whole: three and one-half miles versus one mile (Rathouz & Rubenstein, 2009). During the discussion, one PT remarked that when subtracting, it is important to have "like" units, e.g. three and one-half *miles* and two-thirds *mile*. Lamon (2007) suggested that a lack of understanding of the referent whole is common.

Once PTs determined that the appropriate wording for their story problem was “two-thirds of a mile,” the instructor challenged them to create an expression for a similar story problem in which the walker rested after “two-thirds of the way.” The instructor capitalized on the language used by one PT, “two-thirds of three and one-half miles,” and asked the PTs to generate an expression to represent the wording even though instruction had not yet addressed fraction multipliers. By addressing this aspect of the task, the instructor was preparing PTs for later instruction addressing fraction multiplication.

Rathouz and Rubenstein (2009) conjectured that the progress made by PTs in understanding this task was a result of highlighting and incorporating student misconceptions as learning opportunities; encouraging active debate that focused on reasoning; and building relationships among words, diagrams, meanings, and symbols. Rathouz and Rubenstein (2009) provided five characteristics of why the fractions task was worthwhile for prospective teachers:

- The fractions task addressed multiple goals;
- The fractions task was situated in the practice of teaching by having PTs generate the story problems;
- The fractions task afforded the opportunity for PTs to experience disequilibrium (Ball & Foranzi, 2009);
- The fractions task afforded PTs an opportunity to create and interpret representations; and
- The task and its justification elicited many common and mathematical language issues (Rathouz & Rubenstein, 2009, p. 98-99).

Since this task was enacted with elementary teachers, it would be interesting to integrate a task like this into a middle grades methods course and compare the results to those found by Rathouz and Rubenstein (2009). In addition, since PTs generated pictorial representations, it would be interesting to observe PTs using hands-on or virtual manipulatives like colored length rods, pattern blocks, etc., to model the task.

Understanding of fraction division. Lo and Lou (2012) used Ma's (1999) knowledge package for understanding the meaning of fraction division (see *Figure 3*) and classification schemes for posing and modeling fraction division word problems to study Taiwanese prospective elementary teachers' knowledge of fraction division.

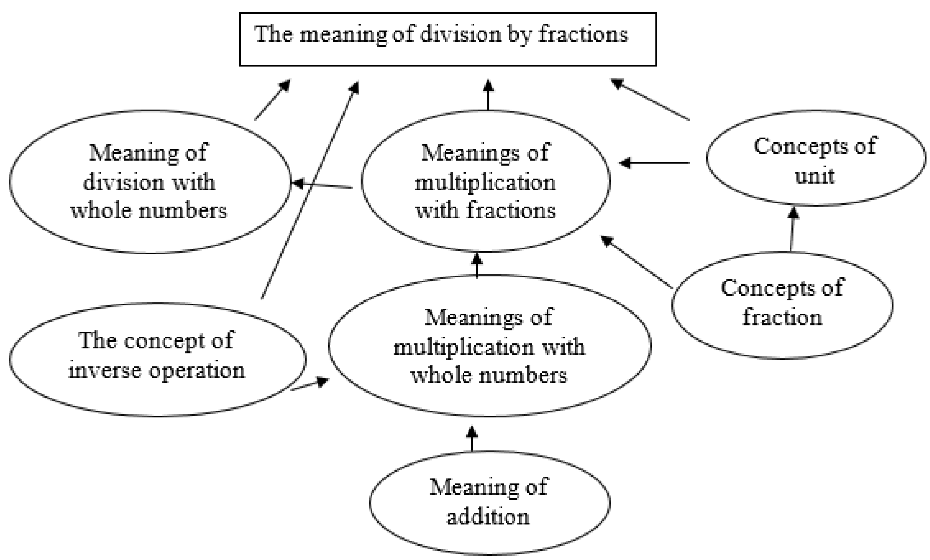


Figure 3. Ma's (1999) knowledge package for understanding the meaning of division by fractions (p. 77)

As evidenced by Ma's (1999) model, deep conceptual knowledge of fraction division is built upon a network of prior knowledge of whole numbers, inverse operations, knowledge of multiplication, knowledge of addition, and fraction knowledge.

Lo and Lou (2012) also outlined two meanings of division: measurement division and partition division. In measurement division, the quotient represents the number of groups of the divisor that are contained in the dividend, i.e. using the divisor as the multiplicand results in the quotient representing the multiplier. In partition division, the quotient represents the group size when the dividend is partitioned into a number of groups according to the divisor, i.e. using the divisor as the multiplier results in the quotient as the multiplicand.

Lo and Lou (2012) chose a word problem involving a traveling situation and a pictorial problem based on a number line. The authors indicated that their choice of the number line model was influenced by the emphasis of this model in recent U.S. reform documents (Common Core State Standards Initiative, 2010; National Mathematics Advisory Panel, 2008).

Lo and Lou (2012) were also interested in whether Taiwanese prospective elementary teachers would use the concept of unitizing or division as the inverse of multiplication. The authors provided an example of using a unitizing approach for the following: Determine the amount of candy needed for the whole class if $1\frac{1}{3}$ lb. is enough for $\frac{2}{3}$ of the class. Applying unitizing procedures would mean first thinking of how much candy is necessary for "$\frac{1}{3}$" of the class. Then, one would determine that one-half of the $1\frac{1}{3}$ lb. is needed for $\frac{1}{3}$ of the class, which equates to $\frac{2}{3}$ lb. of candy. Finally, since there are three-thirds in one whole, one needs $3 \times \frac{2}{3}$ lb. or 2 lb. of candy for the whole class.

The classification schemes for fraction division Lo and Lou (2012) used were:

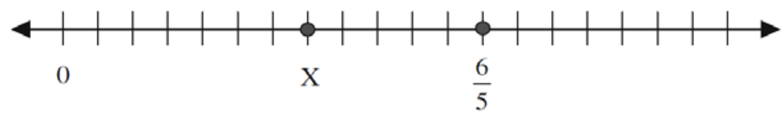
- Equal-group measurement division,
- Equal-group partition division,
- Comparison measurement division,
- Comparison partition division, and
- Rectangular area division.

Division problems of the first two structures deal with a certain number of groups, all of equal size as mentioned earlier under measurement and partition division. Within the third and fourth structures, comparison problems deal with multiplication comparison situations, where one set involves copies of the other. In rectangular area problems, the product is given as the area of a rectangle, and one of the dimensions, length or width, is given. Respondents are then prompted to determine the missing dimension. To accommodate for word problems generated by prospective teachers that did not fit the above constructs, Lo and Lou (2012) also included equal-group multiplication, comparison multiplication, and rectangular area multiplication in their coding. Lo and Lou (2012) also outline the most common types of pictorial models used to represent fractions: area, length, and set.

In their study, Lo and Lou (2012) administered a 16-item instrument to collect data regarding prospective elementary teachers' Common Content Knowledge (12 items) and Specialized Content Knowledge (4 items) (Ball, Thames, & Phelps, 2008). Their sample of participants initially included 28 special education majors, 7 art education majors, and 10 counseling education majors; however, only 36 participants were present for both data collection procedures. Although these participants had areas of specialization other than mathematics, all of them would be responsible for teaching all subjects once they completed their teacher education preparation and began teaching. In Lo and Lou (2012), the authors discussed the following three problems in Table 4.

Table 4

An excerpt from a 16-item test used by Lo and Lou (2012) to measure prospective elementary teachers' CCK and SCK

Problem 1	 What is the value of x? (a) $\frac{7}{12}$ (b) $\frac{3}{5}$ (c) $\frac{7}{10}$ (d) $\frac{7}{15}$ (e) None of the above
Problem 2	Jim jogged $1\frac{1}{2}$ miles yesterday. This is $\frac{3}{8}$ of his weekly goal. How many miles does he plan to run each week? Explain your solution.
Problem 3	Write a word problem that can be solved by using $8\frac{2}{3} \div \frac{1}{4}$ and model how to solve it by a drawing (p. 489).

In the problems shown above, the first two measure Common Content Knowledge; the third measures Specialized Content Knowledge. Lo and Lou (2012) found that all but one student answered Problem 1 correctly using one of three different strategies (unit-fraction approach, part-whole approach, and ratio approach), while 100% of participants answered Problem 2 correctly using one of four different strategies (unit-fraction approach, part-whole approach, ratio approach, and partition division approach).

Of the 40 participants who answered Problem 3, only 26 were able to correctly pose a word problem that was meaningful and develop a correct pictorial representation for solving their own word problem. The authors noted that their findings were similar to that of Ball (1990a, b) and Ma (1999) with practicing elementary teachers and that the majority of incorrect problems were symbolically representative of $8\frac{2}{3} \div 4$ or $8\frac{2}{3} \times 4$.

Of the 26 prospective teachers who successfully posed a word problem, 22 were able to provide an appropriate pictorial diagram. Lo and Lou (2012) determined that 15 out of the 22 drawings did not fully illustrate how to solve the word problem, but that the prospective teachers had relied on their verbal and symbolic reasoning to provide a correct solution. Another noteworthy statement by Lo and Lou (2012) was that Taiwanese prospective elementary teachers appeared to be able to move fluently between the arithmetic-based reasoning and algebraic-based reasoning. “The development of such ability is a critical goal of mathematics curriculum transition from the elementary to middle school level as students move from the pictorial representation typically used to solve word problems with fraction division to symbolic representation” (Lo & Lou, 2012, p. 496).

As a result of their study, Lo and Lou (2012) suggested that improving K-12 mathematics teaching and learning, as well as mathematics courses for prospective teachers that support the development of mathematical knowledge for teaching, may help to narrow the achievement gap between Taiwanese and U.S. students on international assessments such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). This sentiment is similar to that of Schmidt, Blömeke, and Tatto (2011) regarding prospective teacher knowledge.

Lo and Lou (2012) suggested that prospective elementary teachers need opportunities to develop general proficiency for posing word problems and using diagrams effectively to illustrate the reasoning behind a given solution. Since this research was done with prospective elementary teachers, it may be of interest to evaluate prospective middle school teachers' ability to pose appropriate fraction division word problems and use pictorial diagrams or concrete manipulatives as representations.

Prospective teachers' conceptual understanding. Li and Kulm (2008) studied middle grades prospective teachers' perceptions about their knowledge preparation and the extent of their mathematics knowledge on the topic of fraction division by developing assessments that would inform the researchers about the prospective teachers' knowledge in mathematics and pedagogy for teaching. The 46 participants in the study were either juniors or seniors taking a middle grades methods course at a university. Li and Kulm (2008) stated that "helping students learn mathematics with conceptual understanding was one major theme in the methods course that helps these pre-service teachers learn how to teach" (p. 40), as well as the use of a variety of representations in teaching school mathematics.

Data collected during the last class meeting of the methods course included a confidence survey regarding prospective teachers' perceptions about their knowledge preparation in curriculum and instruction in general. With respect to the surveys, 99% of prospective teachers rated their knowledge of the mathematics curriculum framework of their state as either high or proficient; 96% of prospective teachers correctly identified the appropriate grade level for teaching fraction division; and 98% of prospective teachers indicated that they were either ready or very ready to teach "Number—Representing and explaining computations with fractions using words, numbers, or models." In addition, 100% of prospective teachers either agreed or agreed a lot that multiple representations should be used in teaching mathematics topics; 96% either agreed or agreed a lot that teachers need to know students' common misconceptions/difficulties in teaching a mathematics topic; and 100% either agreed or agreed a lot that modeling real-world problems is essential to teaching mathematics.

In addition to the confidence survey, a mathematics test was developed to assess prospective teachers' mathematics knowledge for teaching (Li & Kulm, 2008). The mathematics test included items to assess prospective teachers' Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT). Sample items are shown in Table 5 below and continued on the following page.

Table 5

Sample items used by Li and Kulm (2008) to measure CCK, KCT, SCK, and KCS

Knowledge assessed	Sample Items
CCK	Find the value of $\frac{7}{9} \div \frac{2}{3}$. How many $\frac{1}{2}$'s are in $\frac{1}{3}$? Tell whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without evaluating. Explain your reasoning.

CCK	Johnny's Pizza Express sells several different flavors of large-size pizza. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was $\frac{3}{4}$ of the number of pepperoni pizzas sold, and was $\frac{2}{3}$ of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?
SCK	You are discussing operations with fractions in your class. During this discussion, John says, "It is easy to multiply fractions; you just multiply the numerators and the denominators. I think that we should define the other operations on fractions in a similar way: Addition $\frac{a}{b}+\frac{c}{d}=\frac{(a+c)}{(b+d)}$ Subtraction $\frac{a}{b}-\frac{c}{d}=\frac{(a-c)}{(b-d)}$ Division $\frac{a}{b}\div\frac{c}{d}=\frac{(a\div c)}{(b\div d)}$ How will you respond to John's suggestions? (Deal with each separately.)
KCS	For each of the following two word problems (a) write an expression that will solve the problem (do not calculate the expression), (b) write common incorrect responses, and (c) describe possible sources of these incorrect responses: Problem (1) A 7-meter-long rope was divided into 12 equal pieces. What was the length of each piece? Problem (2) Six pounds of sugar were packed in boxes, each box containing $\frac{3}{4}$ pound. How many boxes were needed to pack all the sugar?
KCT	How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$? Why is $\frac{2}{3} \div \frac{1}{6} = 4$?

Li and Kulm (2008) found that results on the mathematics tests were not in alignment with prospective teachers' confidence. That is, while 93% of the prospective teachers were able to correctly answer $\frac{7}{9} \div \frac{2}{3}$ procedurally, only 52% were able to correctly answer, "How many $\frac{1}{2}$'s are in $\frac{1}{3}$?" Similarly, only 59% were able to correctly answer, "Tell whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without evaluating. Explain your reasoning." Furthermore, only 39% of prospective teachers were able to solve the pizza problem about Johnny's Pizza Express. Regarding the item designed to measure specialized content knowledge, although 90% of the prospective teachers identified the generalizations for addition and subtraction as incorrect, only 2 of the 46 prospective teachers indicated that the generalization for division was correct. The majority of others cited the "flip and multiply" algorithm.

For the two items assessing knowledge of curriculum and teaching, 26% drew and used pictorial representations to explain why $\frac{2}{3} \div 2 = \frac{1}{3}$, while 22% used the "flip and multiply" algorithm to explain. The remaining prospective teachers were not able to provide a complete explanation for either $\frac{2}{3} \div 2 = \frac{1}{3}$ or $\frac{2}{3} \div \frac{1}{6} = 4$. Li and Kulm (2008) noted that none of the prospective teachers attempted to explain why the "flip and multiply" algorithm works.

Li and Kulm (2008) found that although prospective teachers had developed general pedagogical understanding for mathematics classroom instruction, their mathematics knowledge for teaching fraction division was procedurally sound, yet conceptually weak. However, the authors suggested, "Teachers can do a relatively better job when their thinking and explanation are aided by drawing pictorial representations" (Li & Kulm, 2008, p. 841). This statement about the use of pictorial representations raises the question of how prospective middle school teachers would express their understanding if they were using concrete or virtual manipulatives as representations as opposed to pictorial representations.

Li and Kulm (2008) concluded that because students (in grade 7 of their state) are expected to learn fraction division using various models and solve problems involving fraction division, “the development of students’ basic understanding and application of fraction division is an essential requirement in mathematics classroom instruction” (p. 841). Li and Kulm (2008) further contended, “These pre-service teachers’ insufficient mathematics knowledge would likely limit their capability of teaching this content topic conceptually in the future” (p. 841). Finally, the authors suggested that mathematics teacher educators need to “think deeply about what knowledge in mathematics and pedagogy pre-service teachers need to learn through their program of study, and how to help pre-service teachers develop a sound and deep conceptual understanding of mathematics they will teach” (p. 841).

Prospective teachers' obstacles and challenges. Much of the literature about rational numbers communicates that teaching and learning about rational numbers is a challenge. Osana and Royea (2011) examined the challenges and obstacles that prospective elementary teachers experienced during fraction instruction, and consequentially developed an intervention based on the principle of Progressive Formalization (Freudenthal, 1991) that focused on problem solving and on progressively formalizing prospective teachers' intuitive knowledge of fractions. Osana and Royea (2011) wanted to examine the potential effects of the intervention and uncover specific difficulties experienced by the prospective teachers during instruction.

Osana and Royea (2011) used Ball, Thames, and Phelps' (2008) conception of Mathematical Knowledge for Teaching (MKT) as a basis for their study. More specifically, they were interested in understanding the development of the prospective teachers' Specialized Content Knowledge (SCK), a component of MKT. Their research focused on "preparing teachers to develop the types of mathematical knowledge that will assist them to engage in tasks such as interpreting student thinking, providing clear, conceptual explanations for mathematical concepts and procedures, and creating tasks to mobilize specific mathematical ideas" (p. 333). Osana and Royea (2011) contended that little research has been done to examine specific instructional interventions that help prospective teachers to develop SCK. They hypothesized that their intervention had the potential to foster an inter-connected knowledge of school mathematics, which they stated is the core of SCK.

As a part of their study, Osana and Royea (2011) revised a fraction unit in the second of two required methods courses at a large Canadian university. In the previous unit, the instructional approach was teacher-led where pictorial models were used to explain the fraction concepts and procedures. The reason cited for the revision of the unit was the fact that the prospective teachers (PTs) admitted they memorized the models used by the instructors and replicated them on assignments. During instruction of the revised unit, 40 prospective elementary teachers worked in small groups creating their own representations of problem situations. The instructor (first author, Osana) used Progressive Formalization as the pedagogical approach by using solutions that the prospective teachers themselves created and making conceptual links between their strategies and more formal representations.

Due to the large number of PTs in the class and the inability of the researchers to determine precise struggles the PTs experienced, the researchers conducted one-on-one instructional interviews with eight PTs recruited at the university prior to taking either required methods course. During the interviews, Royea replicated the intervention, asking interviewees to first draw a picture to solve a problem, and then write a number sentence for the problem solution. During problem solving, Royea highlighted specific fundamental fraction concepts inherent in the PTs' solutions, as well as made explicit the connections between the model and the number sentence generated, the structure of the problem, or the symbolic representations of the procedures used. The fundamental fraction concepts used by Osana and Royea (2011) are listed in Table 6 on the following page.

Table 6

Fundamental fraction concepts used during the instructional interventions (Osana & Royea, 2011)

Concept number	Fundamental Fraction Concept
1	Wholes can be divided into parts
2	Parts have to be the same size
3	Part is smaller than the whole
4	The size of the part is based on the size of the unit
5	Fractions are expressed in terms of the original unit
6	Parts can be combined to form wholes
7	Parts (fractional units) can be combined no matter how many there are
8	Each fraction has many equivalent representations

In addition to videotaping the five individual one-hour instructional interventions, Osana and Royea (2011) administered three paper-and-pencil assessments of conceptual understanding and procedural knowledge to all students before and after the intervention. At the completion of the fraction assessment, each participant was presented with a "Problem Posing Task," such that they had to generate word problems. The following problems were used on the pre-test: $\frac{2}{6}-\frac{1}{4}$, $5\times\frac{1}{3}$, $8\div\frac{2}{3}$, and $7\frac{1}{2}\div 5$. The following problems were used on the post-test: $\frac{4}{5}-\frac{1}{2}$, $15\times\frac{1}{3}$, $6\div\frac{3}{4}$, and $9\frac{1}{3}\div 7$. The six types of fraction problems presented during the instructional intervention were equal sharing (fraction as quotient), addition with like denominators (part-whole concept), addition with unlike denominators (part-whole concept), multiplication—equal groups (part-whole concept), measurement division (unknown number of groups), and multiplication—fraction of a fraction (fraction as operator) (Osana & Royea, 2011).

A statistical analysis of the conceptual and procedural pre- and post-test and the problem-posing task showed no significant gains on the PTs' procedural knowledge and no significant gains on the problem-posing tasks. However, there was a significant improvement in PTs' conceptual knowledge on the post-test. An analysis of the data for the problem-posing task showed that the majority of errors were made on the subtraction ($\frac{2}{6}-\frac{1}{4}$ and $\frac{4}{5}-\frac{1}{2}$) and division ($8 \div \frac{2}{3}$) problems (Osana & Royea, 2011).

An example of an error made on the pre-test subtraction problem was treating the fractions as whole numbers. That is, a word problem generated was: "One boy has six pieces of pizza. He plans to eat only 2 pieces. However, his friend wants to eat one piece. How much of the pizza remains?" The most common error on the post-test subtraction problem ($\frac{4}{5}-\frac{1}{2}$) involved PTs generating a problem that involved multiplication instead of subtraction. For example, one word problem generated was: "Sandra has $\frac{4}{5}$ of a watermelon in her fridge. Her mom comes to visit and eats half of the watermelon in the fridge. How much did she eat?"

For the division problem ($8 \div \frac{2}{3}$), all but one prospective teacher made one or more errors. Six of the eight PTs generated a word problem that represented $\frac{2}{3} \div 8$ instead of the given problem. Osana and Royea (2011) speculated that this reversal of dividend and divisor occurred because participants were attempting to set up a partitive (partition) division model in which the number of groups was 8. They speculated that these PTs likely did not know how to use a measurement division model in which $\frac{2}{3}$ would represent the group size and the unknown would be the number of groups.

In order to analyze data regarding the obstacles PTs experienced during instruction, Osana and Royea (2011) created a descriptive model with paths that represented chronological trajectories of PTs' solutions. For example, PTs began by either pictorially representing the problem (as instructed) or symbolically representing the problem (even though they were prompted to provide a pictorial representation). The authors coded PTs' actions as Procedural or Meaningful. The code "Procedural" was used when drawings were not based on conceptual knowledge, but on known algorithms and procedures previously learned and applied by rote memory. The authors coded paths as "Meaningful" when they were based on intuitive understandings of fractions. A final aspect of the coding included a determination of whether PTs' number sentences reflected the mathematical structure of the problem or reflected the drawing and not the structure of the problem directly.

Osana and Royea (2011) found that one obstacle that occurred with PTs whose drawings were meaningful was that the number sentence produced was more closely tied to the drawing and less connected to the mathematical structure of the problem. Therefore, the PTs "had difficulty seeing the connection between the concepts used to create the drawing and the problem's structure" (p. 344). For example, the following problem was given: "Tom has 7 dog biscuits. His dog, Fido, eats $\frac{1}{4}$ biscuits a day. How many days will it take for Fido to eat all of the dog biscuits?" The authors discussed a prospective teacher who drew a meaningful model, which was clearly measurement division, but the prospective teacher was not able to see division in the problem likely due to the fact that the drawing she produced was more associated with repeated addition or multiplication.

Another example of difficulty experienced by a prospective teacher was in regard to the problem: "Aidan and his dad are making mini pizzas for a pizza party. It takes $\frac{3}{5}$ cup of grated cheese for one pizza. They need to make 15 mini pizzas. How many cups of grated cheese will they need?" Instead of being representative of multiplication, the prospective teacher's initial drawing was representative of $15 \div \frac{3}{5}$, which she wrote on her paper. She changed it to $\frac{3}{5} \div 15$, and then, finally to $\frac{3}{5} \times 15$. Osana and Royea (2011) suggested that this prospective teacher struggled with the concept of partitioning and the structure of the multiplication problem.

Many of the PTs created drawings as part of their problem solving. However, "rather than using their intuitive understandings of fractions to construct a drawing, the participants produced one that represented a procedure learned previously, either in high school or from earlier in the fractions instruction" (Osana & Royea, 2011, p. 346). PTs who performed in this manner experienced "considerable difficulty." Their number sentences were generated as a result of reading the problem instead of being based on their drawings. Therefore, they were not able to make connections between the concepts used and the drawings (Osana & Royea, 2011).

Osana and Royea (2011) discussed an example of PTs' procedural use of drawings. The prospective teacher, Salena, was presented a problem that required addition of two fractions with unlike denominators in order to solve it, $\frac{3}{4}+\frac{4}{6}$. She began by drawing a rectangle partitioned horizontally into four parts with three shaded and a different sized rectangle partitioned horizontally into six parts with four shaded. After a discussion with the instructor about the need for equal parts, Salena partitioned the fourths into sixteenths, but only considered them to be twelfths because she did not extend the vertical lines through the unshaded fourth. She attempted to partition the sixths in a similar way, but abandoned the process after the instructor discussed the notion of unit with her.

Salena then mentally found a common denominator of 12, drew two rectangles, and partitioned them into twelfths using all horizontal lines. At this point, Salena did not know how many parts to shade, which indicates a failure to connect the model and the number sentence (Osana & Royea, 2011). The authors suggested that her failure to connect the representations was due to her lack of conceptual knowledge of finding a common denominator.

Taking into consideration Osana and Royea's (2011) work, one might wonder whether similar results might be observed with PTs' meaningful or procedural use of manipulatives to represent problem situations. Moreover, this study was enacted with prospective elementary teachers, which leads one to question how the results may vary if prospective middle school teachers were involved in similar research.

Prospective teachers' anxiety toward mathematics. Teachers' mathematical knowledge for teaching is important; however, teachers' attitudes and beliefs are also important (Palardy & Rumberger, 2008; Vinson, 2001). Vinson (2001) studied prospective elementary teachers' anxiety toward mathematics for four consecutive quarters at an undergraduate institution. Vinson (2001) collected data about prospective teachers' anxiety by administering the *Mathematics Anxiety Rating Scale* (Richardson & Suinn, 1972) at the beginning and end of the quarter in which the prospective teachers (PTs) were enrolled in a methods course. Vinson (2001) also collected data from informal observations during the methods course and informal discussions and interviews with the prospective teachers.

A statistical analysis of the data showed that PTs' anxiety significantly decreased during three of the four quarters in which they were enrolled in a methods course that emphasized the use of manipulatives while learning mathematics. Vinson (2001) noted that the quarter in which there was not a significant decrease in prospective teachers' anxiety was the professor's first quarter teaching at the institution, as well as the professor's first time teaching a mathematics methods course. Vinson (2001) suggested that the professor might have exhibited more stress and uncertainty than in subsequent quarters, which might have affected PTs.

In addition to the statistical analysis, Vinson (2001) noted that some students experienced increased levels of anxiety, as revealed during interviews. Prospective teachers indicated that they believed their increased level of anxiety was due to the fact that they had not used manipulatives previously and that they were struggling to relearn the mathematics at the same time they were learning to use the manipulatives. On the other hand, some PTs indicated that they were better able to understand mathematics concepts and procedures when using pictorial and concrete representations (Vinson, 2001).

Vinson’s (2001) study was done with prospective elementary teachers. However, it may have implications on prospective middle grades teachers. That is, if the use of manipulatives to learn mathematics concepts is emphasized in a middle grades methods course, will prospective middle grades teachers experience an increase in anxiety due to unfamiliarity with manipulatives or a decrease in anxiety due to increased conceptual knowledge?

Table 7, shown below and continued on the following page, shows a summary of literature related to prospective teachers.

Table 7

Summary of literature related to prospective teachers

Date	Contributor	Contribution
2000	Lloyd & Frykholm	Studied the effects of using reform-oriented middle school curricula with prospective elementary teachers
2009	Rathouz & Rubenstein	Implementing mathematical tasks with prospective elementary teachers in a reasoning-based learning environment that generated discourse, representations, and disequilibrium

2012	Lo & Lou	Taiwanese prospective elementary teachers' Common Content Knowledge was stronger than their Specialized Content Knowledge of division of fractions
2008	Li & Kulm	Prospective middle grades teachers had limited conceptual understanding of fraction division, yet a positive (unfounded) self-perception of their mathematical knowledge
2011	Osana & Royea	Examined obstacles and challenges experienced by prospective elementary teachers while experiencing Progressive Formalization intervention during fraction instruction; significant increase in conceptual knowledge from pre- to post-test; examined procedural use versus meaningful use of pictorial representations
2001	Vinson	Some prospective elementary teachers' anxiety toward mathematics decreased after a methods class emphasizing manipulatives; some prospective elementary teachers experienced increases in anxiety due to their lack of previous experience with manipulatives and struggling to relearn mathematics at the same time they were learning to use the manipulatives

Summary of teacher knowledge. Because of the impact teaching can have on student achievement, having well-qualified teachers has been a concern of many (e.g., CBMS, 2012; NCTM, 2000; NCTM, 2014; Mathematical Sciences Education Board and National Research Council, 1989). Teachers need deep, connected knowledge of the mathematics they are required to teach, as well as understanding of possible student misconceptions and appropriate pedagogical practices (e.g., Tchoshanov, 2011; Kilpatrick et al., 2001; Suzuka et al., 2007; An et al., 2004). However, research has shown that U.S. teachers' knowledge of fractions may be deficient (e.g., Rathouz & Rubenstein, 2009; Li & Kulm, 2008).

There has been speculation about the possibility of a teacher "preparation gap" which may partially explain the student achievement gap (Schmidt, Blömeke, & Tatto, 2011). If a preparation gap has contributed to the student achievement gap, then perhaps further research into teacher preparation programs can address this concern.

As a result of reviewing literature on teacher knowledge, decisions were made to reflect aspects of the literature in this study. For example, deciding to study prospective teachers was the result of being interested in the possibility of a potential "teacher preparation gap" as suggested by Schmidt et al. (2011) and to examine the impact of engaging prospective teachers in constructivist learning experiences on those prospective teachers' developing conceptual knowledge (Kilpatrick et al., 2001).

Fractions as Rational Numbers

In this section, a rationale for studying fractions is presented. Confrey et al.'s (2009) learning trajectories for rational numbers are discussed next, followed by a discussion of Johanning's (2008) conception of fraction literacy. Then, teachers' conceptual understanding of fractions is examined. This section ends with a discussion about data from the National Assessment of Educational Progress (NAEP).

Rationale for studying fractions. Much research has been done on rational numbers (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Sowder, Phillip, Armstrong, & Schappelle, 1998; Moss & Case, 1999; Cramer, Post, & delMas, 2002; Olive & Vomvoridi, 2006). Researchers suggest that foundational understanding of rational numbers is necessary for students' later scholastic success. For example, proficiency with fractions is thought to be a predictor of success in algebraic reasoning (Booth, Newton, & Twiss-Garrity, 2014; National Mathematics Advisory Panel, 2008; Usiskin, 2007).

Usiskin (2007) asserted, "If algebra is to be 'for all,' then every student needs to have competence with fractions" (p. 370). Furthermore, Lesh, Post, Behr, and Silver (1983) suggested, "Many student difficulties in algebra can be traced back to an incomplete understanding of earlier fraction ideas" (p. 93). Finally, The National Mathematics Advisory Panel (2008) stated:

> Instruction focusing on conceptual knowledge of fractions is likely to have the broadest and largest impact on problem-solving performance, provided it is aimed at accurately solving problems that tap conceptual knowledge. Procedural knowledge is also essential and is likely to enhance conceptual knowledge and vice versa (p. 28).

Perhaps further research can continue to inform practice and provide insights into the complicated issues associated with rational number learning (Silver & Herbst, 2007).

Students often struggle with conceptual understanding of fractions (Cramer, Post, & delMas, 2002; Moss & Case, 1999; Kastberg & Norton, 2007). “An important part of learning about rational numbers is developing a clear sense of what they are” (Kilpatrick, Swafford, & Findell, 2001). However, children often have difficulty achieving mathematical proficiency involving rational numbers (Kilpatrick et al., 2001). Many times children apply poorly understood procedures for whole numbers when operating on fractions (Kilpatrick et al., 2001). The difficulties children experience with rational numbers is rooted in a lack of conceptual understanding (Kilpatrick et al., 2001). It is important for children to understand that rational numbers are numbers in the same way that whole numbers are numbers (Kilpatrick et al., 2001).

Children need to understand that rational numbers can be represented as fractions, decimals, or percentages, and are related to division, measurement, and ratio (Kilpatrick et al., 2001). Extensive time should be spent on helping students develop the concept of unit and connecting and understanding the representations associated with rational numbers (Kilpatrick et al., 2001). Kilpatrick et al. (2001) suggested that instructional approaches that use objects or contexts to help students make sense of the operations on rational numbers offer more promise for developing mathematical proficiency than rule-based approaches. Moreover, Kilpatrick et al. (2001) suggested, “Students’ learning opportunities should involve connecting symbolic representations and operations with physical or pictorial representations, as well as translating between various symbolic representations” (p. 416).

Learning trajectories for rational numbers. Research on rational numbers has been one of the most intensively research areas in mathematics (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). Recent research in mathematics education has focused on learning trajectories (Sarama & Clements, 2009; Confrey et al., 2009). Sarama and Clements (2009) suggested that mathematics learning trajectories are comprised of three parts: “a mathematical goal, a developmental path along which children’s math knowledge grows to reach that goal, and a set of instructional tasks for each level of children’s understanding along that path to help them become proficient in that level before moving on to the next level” (p. 64). Furthermore, Clements and Sarama (2004) posited that learning trajectories are grounded in constructivism and are emergent based on teachers’ interactions with children around the instructional tasks.

Sarama and Clements (2009) suggested that mathematics goals should include the big ideas. They also suggested that teachers consider the instructional tasks from a child’s perspective because children’s interpretations are different from that of adults. By assessing children’s level of understanding and being knowledgeable about the typical learning route that children follow in developing their understanding, teachers can provide instructional activities at the appropriate level for students that promote children’s growth to the next level in a developmental progression (Sarama & Clements, 2009).

Sarama and Clements’ (2009) specifically referred to teachers teaching children when discussing learning trajectories. However, it leads one to wonder how teacher educators might take these same ideas into consideration when teaching prospective teachers. That is, if prospective teachers are lacking conceptual knowledge, teacher educators need to consider how the lack of conceptual knowledge will affect the implementation of instructional activities intended to help them develop relational understanding. Moreover, similar to children, prospective teachers may not have the same interpretations of instructional tasks as that of the teacher educator.

Using a lens of learning trajectories, Confrey et al. (2009) reviewed the research on rational number reasoning "because of its potential to unpack complexity by revealing characteristics of gradual student learning over time" (p. 1-2). Confrey et al. (2009) identified seven major areas of research in rational number reasoning:

1) Equipartitioning/splitting;
2) Multiplication and division;
3) Ratio, proportion, and rate;
4) Fractions-as-numbers;
5) Area and volume;
6) Similarity and scaling; and
7) Decimals and percents.

The authors examined literature for the purpose of identifying areas of consensus about children's thinking about rational number reasoning (Confrey et al., 2009). Their syntheses had four goals: 1) identify common significant themes and findings; 2) introduce new distinctions to resolve differences or to integrate results; 3) discern the most compelling results among controversies; and 4) prepare to implement robust results into practice (p. 1-2).

As a result of their work, they constructed a working definition for Learning Trajectory: "A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey et al., 2009, p. 1-2).

Confrey et al. (2009) discussed three major meanings for *a/b*:

- *a/b* as a fraction (number),
- *a/b* as a ratio—a relation between two numbers in various contexts, and
- "*a/b* of" as an operator (p. 1-7).

The authors suggested that children's understanding of equipartitioning/splitting is foundational to understanding all three meanings for *a/b*. Confrey et al. (2009) further suggested that rational number reasoning should be developed in all three meanings of *a/b* in parallel and simultaneously build children's intuitive and explicit understanding of the links among the meanings. The authors also suggested that students must develop strong rational number reasoning for later success in algebraic reasoning, which resounded statements made by Usiskin (2007) and the National Mathematics Advisory Panel (2008).

Confrey et al. (2009) developed a Learning Trajectories map to provide a visual way of representing the connectedness of rational numbers. The authors used colors to signify a grouping schematic for rational number topics. They also used connecting segments to signify specific connections. On the following page, *Figure 4* shows the concept map that Confrey et al. (2009) developed.

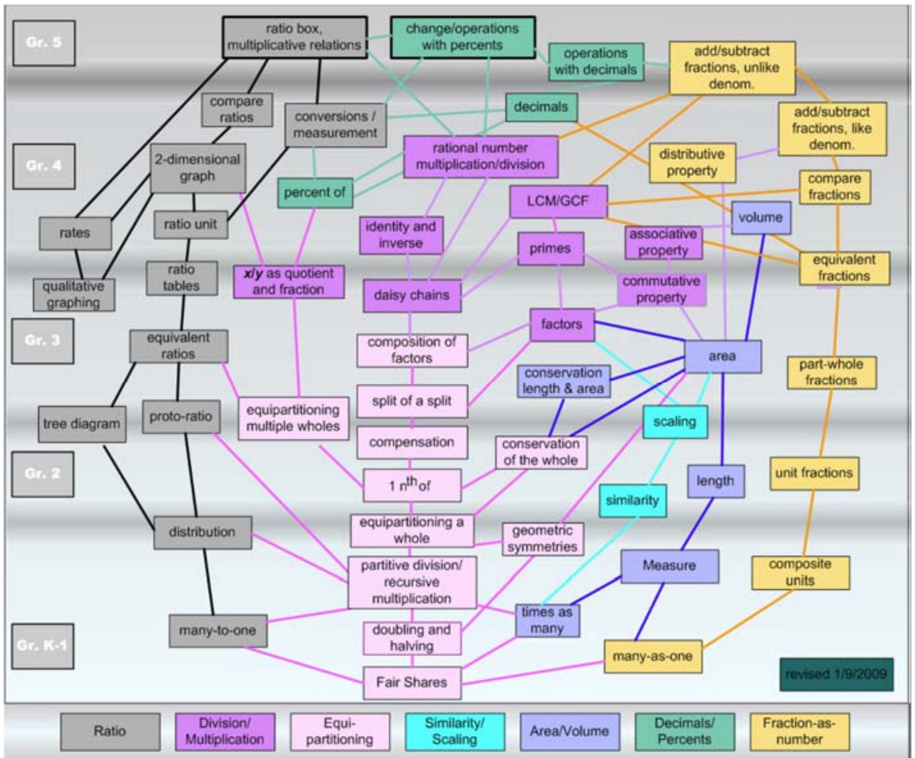


Figure 4. Confrey et al.'s (2009) Learning Trajectories Map for Rational Number Reasoning

In the map, the pink "spine" represents equipartitioning/splitting, pertaining to the creation of equal-sized groups. Confrey et al. (2009) stated that when equipartitioning/splitting is used as an operation, it leads to partitive division as well as to multiplication, with division most directly derived from equipartitioning followed by multiplication as its inverse. Furthermore, Confrey et al. (2009) suggested that equipartitioning is "inherently recursive, not iterative" and that reassembling the fair shares size *m*, by *n* times as many of that part, produces the original whole represented as *mn*.

Confrey et al. (2009) further discussed how equipartitioning is related to ratio, but distinguished between "many-to-one" and "many-as-one." That is, they stated that equipartitioning a collection into fair shares where both dimensions are maintained (e.g. coins and persons) produces a "many-to-one" grouping (e.g. coins per person), which leads to the ratio unit (n:1). When the process of equipartitioning is reversed, the number of objects formed by equal sharing covaries with the number of people. On the contrary, when only one dimension is kept, a "many-as-one" grouping is produced (e.g. *m* coins as a group). This "many-as-one" grouping leads to the idea of a composite unit (one *m* as the fair share, rather than *m* ones per person). Confrey et al. (2009) stated that reversing this process leads to the definition of multiplication as an iterative operation, which supports multiplication as repeated addition.

In explaining how equipartitioning is related to ratio, Confrey et al. (2009) continued with an example of multiple children sharing one cake. If two dimensions are considered (i.e. one-to-many instead of many-to-one), then one cake shared among *n* children results in a fair share of $1/n^{\text{th}}$ of the cake per child. When only one dimension is used, the fraction $1/n$ is defined as the unit fraction, which describes the part that results from partitioning the cake into *n* parts (Confrey et al., 2009). The authors suggested that this situation leads to *n* being the "splitter," which is subsequently viewed as a scalar or a divisor. Confrey et al. (2009) suggested that dropping the second dimension, whose use ensures covariation, requires that the whole and the part of the single dimension share a common unit of one. This requirement of a common unit of one extends to comparisons, addition, and subtraction, as well as representing fractions on the number line.

Confrey et al. (2009) reasoned that progressing through the "many-to-one" construct of equipartitioning leads to the learning trajectory for ratio, proportion, and rate (grey) because of the use of two different dimensions. They also claimed that the progression through “many-as-one” results in fraction-as-number (gold). Finally, progression through “many-as-one” by building upon “times as many” supports the development of measure for length, area, and volume (blue).

Confrey et al.’s (2009) work leads one to wonder to what extent prospective middle grades teachers understand the vastness of the connections among rational number concepts. The vastness of the connections is similar to the connectedness Hiebert and Carpenter (1992) and Hiebert and LeFevre (1986) discussed regarding conceptual knowledge, as well as Ball’s (1988) conceptual understanding. In addition, with Confrey et al.’s (2009) learning trajectories in mind, do middle grades prospective teachers themselves have any deficiencies in their own rational number understanding that may inhibit them as future teachers responsible for teaching rational number concepts? Finally, if middle grades prospective teachers do have deficiencies in their own rational number reasoning, what experiences in their middle grades methods can help them to develop more complete knowledge of rational numbers?

The Common Core Standards Writing Team (2013) addressed learning progressions with respect to the *Common Core State Standards for Mathematics*. They examined the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010) and discussed how the mathematics builds from one grade to the next. In their document examining number in grades 6-8 and the real number system for high school, the Common Core Standards Writing Team (CCSWT) explained that grade 6-8 students build on the representation of whole numbers and fractions as points on the number line and a firm understanding of the properties of operations on whole numbers and fractions. More specifically, the CCSWT (2013) noted that students begin with whole numbers as counting numbers, transition to corresponding points on the number line, and then connect whole numbers on the number line to measurement. By connecting whole numbers to measurement, students can then begin to make connections to fractions on the number line by partitioning the whole number unit into pieces (CCSWT, 2013). They also noted that partitioning the number line into tenths, hundredths, etc. helps students to transition to using measurements in the metric system.

The CCSWT (2013) also indicated that students learn whole number addition as concatenation (linking) and then represent addition of fractions similarly. Specifically, "$\frac{3}{5}+\frac{7}{5}$ can be seen as putting together a length that is 3 units of one fifth long with a length that is 7 units of one fifth long, making 10 units of one fifths in all" (CCSWT, 2013, p. 2). They noted that, "Representing sums as concatenated lengths on the number line is important because it gives students a way to think about addition that makes sense independently of how numbers are represented symbolically" (CCSWT, 2013, p. 2). They further noted that the concatenation model of addition extends to negative numbers in the seventh grade.

With respect to the properties of operations, the CCSWT (2013) stated that "building understanding of multiplication and division of rational numbers relies on a firm understanding of properties of operations" (p. 3). Specifically, the multiplicative identity can play an important role in students' understanding of equivalent fractions. The CCSWT (2013) explained the importance of the commutative property for multiplication through the example of $5\times\frac{1}{2}$ versus $\frac{1}{2}\times 5$. In $5\times\frac{1}{2}$, one can build on previous understanding of whole number multiplication as repeated addition, $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$, whereas $\frac{1}{2}\times 5$ leads to an understanding as "$\frac{1}{2}$ of 5" where *of* means multiplication.

In 5th grade, the *CCSSM* (NGA Center & CCSSO, 2010) includes dividing whole numbers by unit fractions and unit fractions by whole numbers. Dividing a whole number by a unit fraction is considered measurement division, while dividing a unit fraction by a whole number is partitive division. Beginning in 6th grade, division of a fraction by a fraction is required in the *CCSSM* (NGA Center & CCSSO, 2010), which can be more difficult to conceptualize (CCSWT, 2013). The CCSWT provided $\frac{8}{3}\div\frac{2}{3}$ as an example of using measurement division, indicating that the answer 4 is how many $\frac{2}{3}$ there are in $\frac{8}{3}$. They also discussed the fact that $\frac{2}{3}\div\frac{3}{4}$ could be interpreted as "$\frac{3}{4}$ of what amount is equal to $\frac{2}{3}$?" and that reasoning about this division in this manner can lead to understanding reciprocal relationships, proportional reasoning, one-step equations involving multiplication, and the invert-and-multiply rule for dividing fractions (CCSWT, 2013).

Fraction literacy. Johanning (2008) discussed her conception of fraction literacy, which she likened to reading literacy. Johanning (2008) conceived that fraction literacy entails knowing when to apply knowledge to specific contexts.

Johanning (2008) expounded on Mason and Spence's (1999) ideas of knowing-about a subject: (1) knowing-that, or factual knowledge; (2) knowing-how, or technique and skill; and (3) knowing-why, or having a story that reconstructs actions. Mason and

Spence (1999) argued, “The central problem of education is that knowing-about does not in itself guarantee knowing-to” (p.282). In other words, just because students know how to multiply fractions does not mean they know when to apply this knowledge to specific contexts (Johanning, 2008).

“The practices that students engage in when learning about fractions differ from the practices they engage in when learning to use fractions” (Johanning, 2008, p. 282-283). Sowder, Philipp, Armstrong, and Schappelle (1998) suggested that students need opportunities in a variety of situations in order for them to realize the many ways fractions can be used. This sentiment aligns with the suggestion made by Kastberg and Norton (2007) for teachers to provide experiences for students that will foster their understanding of rational number representations in a variety of mathematics domains.

Johanning (2008) used the terms “learning about” and “learning to use” based on her conception of mathematical literacy by blending parts of Scribner and Cole’s (1981) idea of literacy, Mason and Spence’s (1999) thoughts on knowing-about a subject, and Hiebert and Carpenter’s (1992) discussion of conceptual and procedural knowledge, which was the theoretical basis for her study. Johanning (2008) used this conception of mathematical literacy to study middle school students’ developing mathematical literacy. Johanning (2008) defined a literate user of fractions as one who would “not only know about fractions but be able to use knowledge about fractions to achieve goals in a variety of mathematical situations” (p. 285). The focus of her study was to identify common patterns of student behavior when students are learning to use fractions (Johanning, 2008).

Johanning (2008) conducted her study with sixth- and seventh-grade students who were taught mathematics through the use of the Connected Mathematics Project II curriculum. The study took place over two years and in several data collection phases. In the first year, 23 sixth-grade students participated; in the second year, 23 seventh-grade students participated. Of the 23 students in Year 1, there were eight who were also in the Year 2 class. Although some of the students in Year 2 had a different teacher than those in Year 1, they were all exposed to the same curriculum. Four students who were in both Year 1 and Year 2 were interviewed.

Johanning (2008) collected data through field notes and observations, interviews, and video-taping. Both of the teachers involved in the study were strong in their mathematical understanding and had an inquiry-based pedagogy. The selection of teachers was important because of the theoretical basis of Johanning's study. Inquiry-based learning allows for more discourse in the classroom than traditional pedagogy. By conducting the study in a classroom where discourse was encouraged, Johanning had a good vantage point from which to examine students' reasoning about the mathematics they were using.

As a result of her study, Johanning (2008) suggested that students do not naturally move from knowing the procedures of fractions to knowing when to apply them in certain mathematical contexts. Johanning (2008) stated, "Understanding how to use fractions is tied to understanding situations in which they can be used and the various ways that fractions and other mathematical content merge" (p. 291). This statement is similar to statements made by Hiebert and LeFevre (1986) in their discussion of conceptual and procedural knowledge. The statement is also related to Confrey et al.'s

(2009) learning trajectories for rational number reasoning and understanding the connections of rational numbers.

In addition, Johanning (2008) expressed that it is important to engage students in conversations about problems in which fractions are used in order to deepen their understanding of fractions. Johanning (2008) found that classroom discourse associated with the appropriateness of the use of fractions revealed ways in which students made connections that deepened their understanding. For example, students worked on a problem involving the area of a rectangular storm shelter. The area of the storm shelter was 24 square meters, with a length of $5\frac{1}{3}$ meters. The students were trying to find what number to multiply by $5\frac{1}{3}$ to get exactly 24. At least one student suggested that 5.3 could be used instead of $5\frac{1}{3}$, since the students could not use the repeating decimal $5.\bar{3}$ in their calculations. However, after showing that multiplying by 5.3 was not the same as multiplying by $5\frac{1}{3}$, it was determined that using a repeating decimal that has been truncated does not produce an exact answer like using a fraction does (Johanning, 2008). This meaningful discourse and investigation of the possible interchange of decimals with fractions deepened students' understanding of the use of fractions (Johanning, 2008). The author noted that it is commonplace for students to convert fractions to decimals when operating in contextual situations once they have learned how to do so.

Johanning (2008) found that what students learn when learning about fractions is different from what they learn when they have to use fractions in context. Furthermore, it is important for teachers to help students to develop situational understanding along with mathematical understanding (Johanning, 2008). Johanning (2008) stated that students need support when learning to use fractions, and that by using the fractions in context, they develop deeper understanding of the mathematics they are studying.

Taking Johanning's (2008) study into consideration, one might wonder to what degree prospective middle grades teachers have fraction literacy. In addition, if presented with a similar task of determining a missing dimension of a rectangle when given the area of the rectangle, how might teachers solve the task? Would prospective teachers use a truncated decimal for $5\frac{1}{3}$ like the students in Johanning's (2008) study? Finally, to what extent would prospective teachers know which procedures to apply in differing contexts?

Teachers' conceptual understanding of fractions. In order to facilitate students' conceptual learning of fractions and their ability to use fractions literately, to provide them with experiences that will promote fraction literacy, and to help students make connections across mathematical domains, teachers must first have conceptual understanding of fraction themselves (Kilpatrick et al., 2001).

While a graduate student at Michigan State University in 1989, Liping Ma was responsible for coding transcripts of teachers' responses to mathematics questions that were collected as part of the Teacher Education and Learning to Teach Study at the National Center for Research on Teacher Education (1991). As a result of studying these data, Ma (1999) found that "although 43% of the U.S. teachers in the study successfully calculated $1\frac{3}{4} \div \frac{1}{2}$, all failed to come up with a representation of division by fractions" (p. 64).

As part of the study, teachers were asked to create a representation as well as a story problem for the fraction division problem. Among the 23 teachers, six were not able to create a story problem and 16 created stories with misconceptions. “Only one teacher provided a conceptually correct but pedagogically problematic representation” (Ma, 1999, p. 64). In addition, Ma (1999) found that the teachers had various misconceptions about the meaning of division by fractions, including the notion that the answer to a division problem should be smaller than the dividend.

In their story problems, 10 U.S. teachers confused division by one-half with division by two, while six teachers confused dividing by one-half with multiplying by one-half. Ma (1999) noted that there seemed to be no difference to the teachers among division by one-half, division by two, and multiplication by one-half. There were two teachers, however, who did not confuse the operations even though they were not able to produce a representative story problem. Furthermore, their inadequate knowledge of the computational procedure interfered with them developing an understanding of the meaning of division of fractions. Ma (1999) stated that the lack of understanding in the meaning of division by fractions limited the teachers in being able to generate an appropriate representation.

Another finding by Ma (1999) was that the lack of conceptual understanding of fractions among the U.S. teachers was not present in a sample of 72 Chinese teachers who were studied. Among the 72 Chinese teachers studied, 90% of them solved the problem correctly based on procedural methods. A striking difference between the U.S. teachers and the Chinese teachers was that 65 of the 72 Chinese teachers created a total of more than 80 representative story problems, while only 8% were not able to create a story problem. Ma (1999) suggested that one of the reasons the U.S. teachers' fraction knowledge was inferior to that of the Chinese teachers was that most of the U.S. teachers' understanding was supported only by the partitive model of whole number division, whereas the Chinese teachers' understanding was connected to multiple conceptions of fraction division.

The conclusions by Ma (1999) regarding inservice teachers' lack of conceptual understanding of division of fractions have implications on student achievement. If Ma's (1999) sample is at all representative of teachers' knowledge, then a lack of student understanding in the rational number domain, as noted by Kastberg and Norton (2007), is relatively comprehensible, albeit undesirable. In addition, Ma's (1999) study examined data from 1991. Considering that these data were collected near the time of the new NCTM *Standards* documents, would similar data be present among prospective teachers who were likely in K-12 schools during the reform movement?

NAEP data related to fraction literacy. Kastberg and Norton (2007) examined and discussed student results on the 2003 National Assessment of Educational Progress (NAEP). Problems involving fractions on the NAEP included recognizing appropriately partitioned fractions; indicating a fractional location on a number line; solving time, length, and weight measurement problems involving fractions; recognizing relative extents of a whole; and forming a meaningful word problem requiring division by a fraction.

After studying the data, Kastberg and Norton (2007) found that improvement in students' procedural knowledge may account for increased overall performance instead of improvement in students' conceptual understanding. For example, both fourth-grade and eighth-grade students were asked to indicate the location for the fraction $\frac{3}{4}$ on a number line that had been partitioned into 8ths and already had the location for $\frac{1}{2}$ marked. Eighth-grade students outperformed fourth-grade students only by a margin of 7% (64% to 57%, respectively). Fourth-grade students typically chose $\frac{3}{8}$. Kastberg and Norton (2007) suggested that the basis for this error was that the location represented three out of four marks on the first half of the number line without taking into consideration the relative whole.

The typical incorrect response from eighth graders was to equate $\frac{1}{2}$ to $\frac{2}{4}$ and count one more mark to the right, which actually represents one more eighth, $\frac{5}{8}$. Kastberg and Norton (2007) reasoned that students were using their procedural knowledge when equating $\frac{1}{2}$ to $\frac{2}{4}$, but they lacked conceptual understanding of relative extents of a whole when they did not recognize the eighths in relationship to the given whole.

Kastberg and Norton (2007) also discussed students' performance on solving a word problem that required students to divide by a fraction. Based on the distracters students chose, the authors concluded that the increase in student performance was likely due to procedural fluency rather than conceptual understanding. Additionally, with respect to word problems, 8^{th}-grade students' performance on forming a meaningful word problem requiring division by a fraction has decreased (Kastberg & Norton, 2007).

In summary, Kastberg and Norton (2007) concluded, “What remains lacking among the majority of Grade 8 students is the concept of a fraction as a relative extent of a whole” (p. 87). Furthermore, the authors suggested that because of this inadequacy in students’ understanding, “students rely on rote procedures for which they have no meaningful interpretation” (p. 87). To address this inadequacy, Kastberg and Norton (2007) suggested that teachers provide students with opportunities to integrate mathematical knowledge and opportunities in a variety of mathematical contexts other than those focusing on numerical computation.

Summary of fractions as rational numbers. Fractions are a complex domain as evidenced by the lackluster achievement noted throughout research. Much research has been done to understand this complexity. However, research shows that U.S. students’ and teachers’ understanding of fractions is still lagging behind that of other countries.

Understanding the complex domain of fractions has implications on student achievement throughout schooling (National Mathematics Advisory Panel, 2008; Usiskin, 2007; Lesh, Post, Behr, & Silver, 1983). Perhaps further research can inform practice and bring about greater understanding of this complex domain and how to teach it in ways that impact student achievement in order to address these complex issues (Silver & Herbst, 2007).

Manipulatives as Representations

In this section, a rationale for studying manipulatives is provided. Next, manipulatives are examined from a historical perspective. Then, the teachers’ role in using manipulatives; how teachers’ use of manipulatives affects students’ learning; how teachers’ attitudes about manipulatives affect students’ learning; other factors affecting the use of manipulatives; the use of manipulatives with prospective elementary teachers to correct arithmetic misconceptions; and, briefly, the use of virtual manipulatives is considered.

Rationale for manipulatives. The National Council of Teachers of Mathematics (2000) stated in *Principles and Standards for School Mathematics*, "Representing ideas and connecting the representations to mathematics lies at the heart of understanding mathematics" (p. 136). Representations can be expressed through language (verbal), pictures (visual), manipulative models (physical), written symbols (symbolic), and real-word contextual situations (Lesh, Post, & Behr, 1987; Van de Walle, Karp, & Bay-Williams, 2013). *Figure 5* shows possible types of representations that can be used in mathematics instruction and their interconnectedness (NCTM, 2014, p. 25).

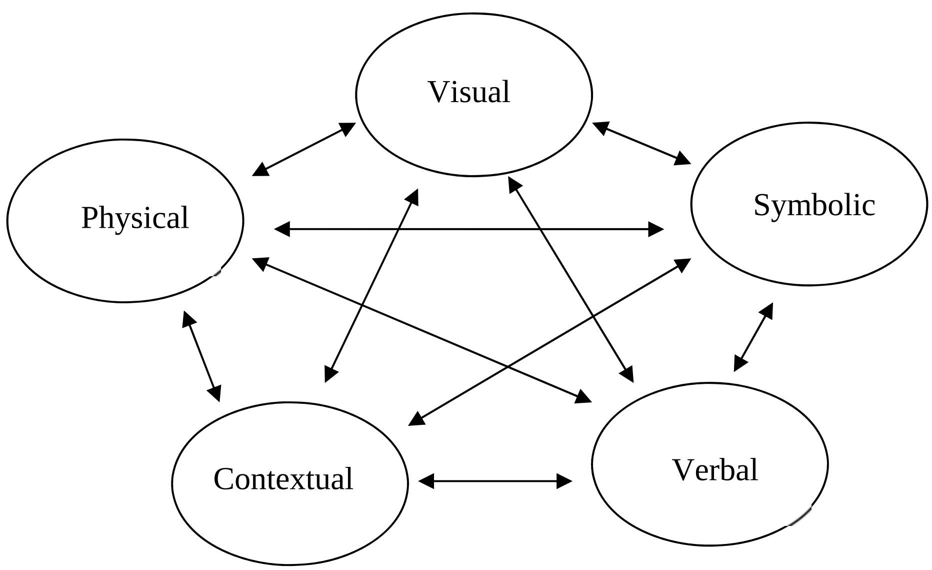


Figure 5. Types of representations and their interconnectedness

Cognitive psychology proposed by Piaget (1960), Bruner (1966), and Dienes (1969) suggested that children learn better when they have the opportunity to explore mathematics through the use of multiple representations and to make connections among these representations. Hiebert and Carpenter (1992) stated, “Mathematics is understood if its mental representation is part of a network of representations” (p. 67). In fact, “the degree of understanding is determined by the number and strength of the connections” [among the representations] (Hiebert & Carpenter, 1992, p. 67). This study focused on concrete representations through the use of manipulatives. However, pictorial representations as an extension of concrete models, verbal representations through written and spoken mathematical language, symbolic representations through facilitating the connection of the concrete models to the symbols, and real-world contexts were also examined as was appropriate.

Manipulatives are defined as “material objects designed to represent explicitly and concretely mathematical ideas that are abstract” (Moyer, 2001, p. 176). However, the manipulatives themselves are not carriers of the meaning and insight (Moyer, 2001). On the contrary, it is through the use of the manipulatives that students gain understanding (Moyer, 2001). However, some teachers may not be able to transform mathematical ideas into representations (Ball, 1992). Because of this inability, there is evidence that the mere use of manipulatives does not guarantee conceptual understanding for the students (Baroody, 1989).

Within the last two decades, virtual manipulatives became available via the World Wide Web that simulate the same process as can be used with concrete manipulatives but with greater repetition and precision available due to the nature of virtual manipulatives (e.g. National Library of Virtual Manipulatives http://nlvm.usu.edu/, Illuminations http://illuminations.nctm.org/, Math Playground http://www.mathplayground.com/). Virtual manipulatives are “interactive, Web-based visual representations of dynamic objects that present opportunities for constructing mathematical knowledge" (Moyer, Bolyard, & Spikell, 2002, p.373).

Virtual manipulatives are more than just pictures of manipulatives; students are able to slide, flip, and turn virtual manipulatives as if they were actual three-dimensional objects. The following are examples of manipulatives that exist in concrete and virtual form: colored length rods, pattern blocks, base-ten blocks, fraction bars, fraction circles, probability spinners, dice, algebra tiles, pentominoes, tangrams, coins, two-colored counters, geoboards, colored square tiles, and three-dimensional blocks (i.e. linking cubes). This list of concrete and virtual manipulatives is not meant to be an exhaustive list, but to draw attention to the fact that there are many concrete and virtual manipulatives available for teachers and students to use.

Although the use of manipulatives may be helpful for some students, the use of manipulatives as a pedagogical practice in and of itself is not sufficient (NCTM, 2000; Carbonneau, Marley, & Selig, 2013; Puchner, Taylor, O'Donnell, & Fick, 2008). If manipulatives are going to be used as an instructional strategy, teachers need to understand how the physical representation of manipulatives connects to the symbolic representation to realize the full potential of learning through using manipulatives (Ball, 1992; Moyer, 2001; NCTM, 2000), as well as which concrete representations are appropriate for particular content (Graeber, 1999; Cramer & Wyberg, 2009). Teachers also need to be knowledgeable about the appropriate use of manipulatives in instruction and use them as more than just diversions or as teacher-led activities (Ball, 1992; Moyer, 2001; Van de Walle, Karp, & Bay-Williams, 2013; Puchner, Taylor, O'Donnell, & Fick, 2008; NCTM, 2014).

In 2010, the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010) were adopted by forty-four states, the District of Columbia, four territories, and the Department of Defense Education Activity. The *Common Core State Standards for Mathematics* (*CCSSM*) is a set of standards that were developed based on "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (NGA Center & CCSSO, 2010, p. 4). The *CCSSM* articulates what students should know and be able to do at each grade level to be "college and career ready" upon graduation, but it does not prescribe a certain set of instructional interventions for teachers to use. The *CCSSM* (NGA Center & CCSSO, 2010) stresses conceptual understanding to undergird procedural fluency, which is in keeping with Kilpatrick et al.'s (2001) recommendation for mathematical proficiency. Although the *CCSSM* does not prescribe instructional strategies, there are 15 occurrences of "using visual fraction models" listed in the Standards when teaching fraction concepts (NGA Center & CCSSO, 2010).

"Mathematics is inherently representational in its intentions and methods" (Kaput, 1989, p. 169). Panasuk (2010) stated, "Mathematical relationships, principles, and ideas can be expressed in multiple representations including visual, verbal, and symbolic. Each type of representation articulates different meanings of mathematical concepts" (p. 238). Goldin and Shteingold (2001) suggested, "External systems of representation and internal systems of representation and their interaction are essential to mathematics teaching and learning" (p. 2). Goldin and Shteingold (2001) further stated that visual imagery, natural language, problem solving abilities, and attitudes toward mathematics all affect students' internal representations. These statements are similar to those made by Hiebert and Carpenter (1992) regarding representations.

Principles and Standards for School Mathematics (NCTM, 2000) stated that one of the purposes of using representations is for students to develop a mental image of concrete mathematical ideas upon which to reflect as they progress through their scholastic experiences. Mental images can be thought of as internal representations (Panasuk, 2010). Hiebert and Carpenter (1992) stated, “To think about mathematical ideas we need to represent them internally, in a way that allows the mind to operate on them” (p. 66). Internalization, however, requires a connection to external representations (Hiebert & Carpenter, 1992). Kaput (1989) used the term “fusion” to describe the internalization of an external representation. However, if students fail to develop these mental images, they may hold on to memorized rules instead of learning with comprehension (Saul, 2001).

Recently, NCTM released *Principles to Actions* (NCTM, 2014) which contained an updated set of Principles that are based on a decade of experience and research about excellent mathematics programs, as well as significant obstacles and unproductive beliefs that compromise progress. The updated Guiding Principles for School Mathematics presented in *Principles to Actions* (NCTM, 2014) are Teaching and Learning; Access and Equity; Curriculum; Tools and Technology; Assessment; and Professionalism. NCTM (2014) defined each Principle; discussed obstacles that influence school programs and strategies for overcoming the obstacles; provided illustrations of effectiveness related to each Principle; and proposed “specific actions for productive practices and policies that are essential for widespread implementation of Pre-K—12 mathematics programs with the power to ensure mathematical success for all students at last" (p. 4).

Historical perspective of manipulatives. In the 21st century, manipulatives are certainly not new to mathematics education. In the 1950s, Zoltan Dienes introduced what he called multi-base blocks (now known as base-ten blocks) as an embodiment of the place value system (Sriraman & Lesh, 2007). Dienes believed that the structure in the embodiment was important, as was recognizing structure across multiple embodiments. Dienes stated, "The structural features one recognizes from these multiple embodiments—this brings out the essence of abstraction" (Sriraman & Lesh, 2007, p. 67).

Also in the early 1950s, Weaver (1953) studied a teacher who allowed five students to use their choice of manipulatives (without direct instruction) to demonstrate the subtraction problem: 52 – 37. The teacher's purpose was to have students make the connection between the physical representation of regrouping and the symbolic representation associated with the algorithm (Weaver, 1953). One student, Billy, counted out fifty-two items and took away thirty-seven of them, one at a time. Although accurate, this strategy was not efficient, nor did it model the algorithm that the teacher wanted the students to eventually connect.

Weaver (1953) also discussed a student, Betty, who represented the number fifty-two using five bundles of ten and two singles in place-value pockets. Betty manipulated the bundles and singles in a way that represented the algorithm (Weaver, 1953). She unbundled a ten and placed them with the singles, and then removed three bundles of ten and seven singles. In contrast to Betty's model, Tommy used only five singles in the tens pocket and two singles in the ones pocket (Weaver, 1953). When he realized he would not have enough singles to subtract seven, Tommy removed one of the singles from the tens pocket and placed ten singles in the ones pocket. This manipulation is a physical representation of what the algorithm of subtraction with regrouping represents. Furthermore, it shows that Tommy had deep understanding of the process of subtraction when it requires regrouping (Weaver, 1953).

As time passed, manipulatives continued to be a popular topic among mathematics educators and researchers as evidenced by the following quote in the *Journal for Research in Mathematics Education*: "It is obvious from reading the articles or the advertisements in any recent mathematics teachers' journal on this continent or across the Atlantic that the use of manipulative activities in the teaching and learning of mathematics is in vogue" (Kieren, 1971, p. 228). Kieren (1971) went on to review Piaget's, Bruner's, and Dienes' contributions to the discussions about the learning theories around the use of manipulatives. He summarized by reminding readers that manipulatives "should best include a wide variety of concrete referents for a concept and can contribute a readiness foundation for later ideas" (Kieren, 1971, p. 232).

Research of manipulative use with students continued into the 1980s (Sowell, 1989). In the 1960s and 1970s, research findings were mixed (Sowell, 1989). Researchers continued to study teachers' and students' use of manipulatives in an effort to shed light on this complex issue (Sowell, 1989). Sowell (1989) examined 60 studies involving the use of concrete manipulatives where students worked directly with materials, pictorially where students watched demonstrations or used printed pictures, symbolically with pencil-and-paper work, or listened to lectures to learn mathematics. Although there was a difference in effectiveness between the concrete and symbolic instruction, the meta-analysis done by Sowell (1989) was inconclusive as to which manipulatives might be most appropriate.

Just prior to the turn of the 21st century, the National Council of Teachers of Mathematics (NCTM) released several documents advocating mathematics education reform. These documents included *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), and *Assessment Standards for School Mathematics* (NCTM, 1995).

NCTM (1989) stated in *Curriculum and Evaluation Standards for School Mathematics,* "The mathematics classroom envisioned in the *Standards* is one in which calculators, computers, courseware, and manipulative materials are readily available and regularly used in instruction" (p. 243). One example of using a model (i.e. manipulative materials) to represent fractions concepts found in the *Curriculum and Evaluation Standards for School Mathematics* involved demonstrating fractions using pattern blocks (NCTM, 1989, p. 224). The teaching suggestion was for students to be asked to use pattern blocks to demonstrate $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{2}{3}$, etc. in multiple ways, given that the yellow hexagon is one whole.

In NCTM's *Professional Standards for Teaching Mathematics* (1991), the focus shifted to standards for teaching mathematics, evaluating the teaching of mathematics, professional development for mathematics teachers, and support for developing teachers. Worthwhile tasks, discourse, and the learning environment as a mathematical community were emphasized as being central to improving mathematics instruction (NCTM, 1991). NCTM (1991) reiterated the need for tools such as calculators, computers, and concrete, pictorial, and metaphorical models. They further stated, “Knowing that students need opportunities to model concepts concretely and pictorially, for example, might lead a teacher to select a task that involves such representations” (NCTM, 1991, p. 27).

The release of the *Assessment Standards for School Mathematics* (NCTM, 1995) drew attention to the fact that curriculum, instruction, and assessment should be “seamless.” That is, if manipulative models are used during instruction, then students should have access to them during assessments. In addition, as an example of how teachers can use assessment evidence to plan the next day’s lesson, a vignette showed students using Cuisenaire™ rods to explore measurement (NCTM, 1995).

Next, NCTM’s monumental document *Principles and Standards for School Mathematics* (*PSSM*) was released in 2000. *PSSM* (NCTM, 2000) built upon the ideas presented in the three prior *Standards* documents. NCTM (2000) communicated the importance of representing mathematics by identifying it as one of the Process Standards. Furthermore, they stated, “Representing ideas and connecting the representations to mathematics lies at the heart of understanding mathematics” (NCTM, 2000, p. 136). This statement resounded the ideas of Zoltan Dienes regarding the importance of recognizing the structure of a representation and how mathematics is embodied in it (Sriraman & Lesh, 2007).

The *Common Core State Standards for Mathematics* (*CCSSM*) (NGA Center & CCSSO, 2010) advocates conceptual understanding to undergird procedural knowledge. The *CCSSM* (NGA Center & CCSSO, 2010) also suggested the use of concrete models, particularly in the Number domain. In fourth through sixth grades, concrete and visual models are referenced approximately 20 times, particularly in association with fractions.

For example, 6.NS.1 states, “Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem” (NGA Center & CCSSO, 2010, p. 42).

Building upon the Process Standards advocated by NCTM (2000) and the five strands of mathematical proficiency noted in the National Research Council’s *Adding It Up* document (Kilpatrick, Swafford, & Findell, 2001), the *CCSSM* (NGA Center & CCSSO, 2010) introduced the Standards for Mathematical Practice. These Standards describe what mathematically proficient students do. Standard 5 states, “Mathematically proficient students consider the available tools when solving a mathematical problem” (NGA Center & CCSSO, 2010, p. 7). The *CCSSM* summarizes this standard as “Use appropriate tools strategically” (NGA Center & CCSSO, 2010, p. 7). Concrete models are among the tools listed. These models can include colored length rods, base-ten blocks, fraction circles, pattern blocks, and more. However, concrete “model” should not be confused with mathematical “model” in Standard 4, which states that students can model using mathematics. The term *model with mathematics* means that students can generate mathematical models like equations or functions that describe a real-world situation or phenomenon (NCTM, 2000).

Recently, *Principles to Actions* (NCTM, 2014) established eight Mathematics Teaching Practices that represent “a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics” (p. 9). These practices are:

1) Establish mathematics goals to focus learning.
2) Implement tasks that promote reasoning and problem solving.

3) Use and connect mathematical representations.

4) Facilitate meaningful mathematical discourse.

5) Pose purposeful questions.

6) Build procedural fluency from conceptual understanding.

7) Support productive struggle in learning mathematics.

8) Elicit and use evidence of student thinking.

NCTM (2014) provided a sample high-leverage task of "Procedures with Connections" from Stein and Smith (1998) which involves the use of pattern blocks. The task was to find $\frac{1}{6}$ of $\frac{1}{2}$ using two hexagons as the referent whole. In addition, the following high-leverage task classified as "Doing Mathematics" was cited from Stein and Smith (1998): *"Create a real-world situation for the following problem:* $\frac{2}{3} \times \frac{3}{4}$. *Solve the problem you created without using the rule, and explain your solution."* A possible student response, shown below in *Figure 6*, was given as: "For lunch Mom gave me three-fourths of a pizza that we ordered. I could only finish two-thirds of what she gave me. How much of the whole pizza did I eat?"

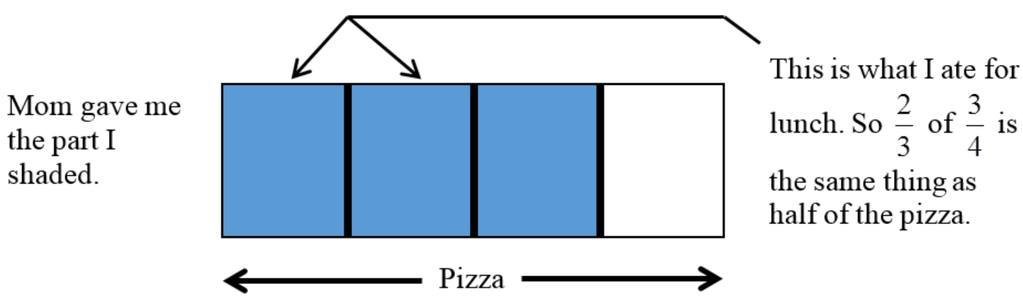


Figure 6. Possible pictorial student response to $\frac{2}{3} \times \frac{3}{4}$ (NCTM, 2014, p. 19)

In addition to establishing the Mathematics Teaching Practices, NCTM (2014) also updated the Principles established in *PSSM* (NCTM, 2000). The previous Technology Principle was revised to include tools other than electronic and renamed as the Tools and Technology Principle. Not only did NCTM (2014) include non-electronic tools such as manipulatives, measurement tools, and geometric models, they also advocated the use of such throughout K-12 mathematics programs. They noted that, although some people may view manipulatives as juvenile or unnecessary, students of all ages may benefit from the use of physical and virtual manipulatives because manipulatives can provide visual models of a range of mathematical ideas.

In addition, NCTM (2014) highlighted Orlich's (2000) research that indicated that many junior and high school students are still operating at a concrete level of thinking. NCTM (2014) reiterated the importance of using multiple representations such as contextual, visual, verbal, physical, and symbolic (Lesh, Post, & Behr, 1987) and emphasized that effective teaching involves not only using representations, but also making connections among multiple representations.

Principles to Actions (NCTM, 2014) emphasizes the importance of learning with understanding through the use of tools and technology to make sense of mathematical concepts; engaging in mathematical reasoning using tools and technology in an investigative and exploratory manner; and incorporating tools and technology that facilitate students' mathematical communication. Furthermore, *Principles to Actions* (NCTM, 2014) encouraged teachers to reflect upon how students may use tools and technology, to make instructional decisions based on goals and objectives, and to thoughtfully consider how to incorporate tools and technology in meaningful ways as an integral part of the curriculum.

NCTM (2014) also reminded teachers that there is still a potential for tools and technology to be used in unproductive, ineffective ways. They stated, “Effective use of technology and other tools requires careful planning; teachers need appropriate professional development to learn how to use them effectively” (NCTM, 2014, p. 82). If teachers provide step-by-step instructions on using tools and technology instead of providing students with opportunities to reason and explore, then the tools and technology will likely be unproductive (NCTM, 2014). In addition, tools and technology should not be used only as a “fun activity,” diversions from the norm, or for skills practice to develop procedural fluency without the development of conceptual understanding (NCTM, 2014). Because of the potential for unproductive use of tools and technology, NCTM (2014) stated that it is important for teachers to “have a deep knowledge of mathematics and understand how such tools and technology can be used strategically in ways that support meaningful learning” (p. 81).

Table 8, shown below and continued on the next page, contains a historical timeline for literature regarding manipulatives.

Table 8

Historical timeline regarding manipulatives

Date	Contributor	Contribution
1950s	Dienes	Introduced base-ten blocks to represent place value
1953	Weaver	Teacher using straws and place value pockets to teach regrouping

1971	Kieren	Advocated a wide variety of concrete referents as foundation for later topics
1989	Sowell	Meta-analysis of research on manipulatives was inconclusive
1989	NCTM	Suggested regular use of manipulatives in instruction and provided example of using pattern blocks
1991	NCTM	Suggested that teachers' knowledge of children's need to model concepts concretely should guide instructional choices
1995	NCTM	Instruction and assessment should be aligned; if manipulatives are used during instruction, then they should be available during assessment
2000	NCTM	Teachers need to facilitate the linking of multiple representations
2010	NGA Center & CCSSO	Suggested the use of concrete models when teaching fractions and other concepts like volume
2014	NCTM	Reiterated the importance of using multiple representations and making connections among the representations; provided an example of high-leverage tasks from Stein & Stein (1998) involving the use of pattern blocks and a real-world context to find a fraction of a fraction; advocated use of manipulatives throughout K-12

Summary for historical perspective of manipulatives. Manipulatives have been used in K-12 education for many years and highly researched. Advocates of manipulatives have touted their benefits (NCTM, 2000, 2014), while others have noted that research on the efficacy of manipulatives is mixed (Sowell, 1989).

NCTM (2014) reminded teachers that careful planning must precede the use of tools such as manipulatives during instruction if students are to benefit from their use. In addition, although some may have a perception of manipulatives as only for young children, NCTM (2014) conveyed their support of the use of manipulatives in an investigative and exploratory manner throughout K-12 schooling.

Teachers' role in using manipulatives. "The abilities to recognize, create, interpret, make connections and translate among representations are powerful communication tools for mathematical thinking" (Panasuk, 2010, p. 239). While *Principles and Standards for School Mathematics* advocated teachers modeling conventional ways of representing mathematics, they also stressed that "representations do not show the mathematics to the students" (NCTM, 2000, p. 209). *Principles and Standards* (NCTM, 2000) further emphasized that students need to know which representations are appropriate for particular situations and to move fluidly among multiple representations because, as students link different representations, it deepens their understanding of mathematics (NCTM, 2000).

Facilitating the linking of multiple representations as well as understanding and recognizing the mathematics embodied in the representations is a responsibility that lies with teachers (NCTM, 2000). Concrete materials being used need to closely match the significance of the mathematical relationships in order for them to be effective tools for students' learning (Hiebert & Carpenter, 1992). Hiebert and Carpenter (1992) stated:

> Differences in how physical materials are used to develop understanding are not only grounded in the nature of those physical materials: There also are important differences in how materials may be used to make connections explicit. As noted earlier, the effects of using materials on students' understanding may have as much to do with the context in which they are used and the way in which students interact with materials as with the materials themselves. We have argued that an important variable to consider is the explicitness with which the connections are drawn for students between features of the materials and symbolic representations (Hiebert & Carpenter, 1992, p. 88).

Moss and Case (1999) voiced concern about the use of the circle model as the only mode of instruction because it limits students to think only about the part-whole relationship of fractions. Cramer et al. (2002) echoed Moss and Case's (1999) concern regarding the use of the circle model and suggested compensating for it by intermittently using fractional pieces as the unit rather than always using the whole circle as the unit. Along the same lines, after studying the 1996 NAEP rational number data, Wearne and Kouba (2000) concluded that students lack an understanding of the unit, similar to Kastberg and Norton's (2007) findings for the 2003 NAEP. Finally, *Principles and Standards for School Mathematics* (NCTM, 2000) cautioned teachers not to introduce conventional representations to students before they can use them meaningfully because it could be counterproductive.

Teachers' uses affecting the efficacy of manipulatives. Students "are influenced both by what the teacher understands about what he or she is teaching, and by what he or she discerns about what students know and about how students might build productively upon that knowledge" (Thompson & Saldanha, 2003, p. 96). Teachers need to understand the manipulative and how it embodies the mathematics that they are attempting to represent (Ball, 1992; Moyer, 2001). Furthermore, how the manipulative is used is of great importance.

Van de Walle, Karp, and Bay-Williams (2013) cautioned teachers not to have students "do as I do" when using manipulative models. They contended that doing so opens up the possibility of students mimicking teachers' actions in a way that is mindless and leads to memorization of procedures. Instead, Van de Walle et al. (2013) suggested that students be allowed to use the manipulatives in ways that promote thinking and aid in the development of concepts.

Teachers should also be cognizant of how students are affected by the teachers' instruction. Izsák, Tillema, and Tunç-Pekkan (2008) studied one sixth-grade teacher who was implementing the Connected Mathematics Project curriculum (CMP). The authors discussed the negative effects of students' and teachers' different interpretations when representing fractions using drawings, specifically iterative and recursive partitioning compared to left-to-right partitioning. Although the sixth-grade teacher involved in the study had previously used paper folding to illustrate recursive partitioning as part of her implementation of the Connected Mathematics Project curriculum, she did not make the transition of using recursive partitioning when representing fractions on the number line (Izsák et al., 2008). Instead, when the teacher demonstrated how to partition a number line into eighths that was marked from zero to one, she used a left-to-right partitioning scheme and changed the location of the "1" based on her inaccurate partitioning (Izsák et al., 2008). In doing so, she inadvertently undermined one student's understanding of a fixed unit, which subsequently caused the student problems (Izsák et al., 2008). It should be noted, however, that the teacher's strategy of changing the fixed unit did not affect another student who had a stronger understanding of fractions (Izsák et el., 2008). As a final point, the teacher not only neglected to model recursive partitioning, but also did not capitalize on the fact that one of her highest achieving students used recursive partitioning and iterating when calculating $\frac{1}{3}+\frac{2}{9}$; she merely commented that what he had done was interesting (Izsák et al., 2008).

Although modeling fractions is important, teachers should exercise caution when representing concrete models through pictorial representations (Olive & Vomvoridi, 2006). In their study, Olive and Vomvoridi (2006) observed that a teacher who attempted to cut a circle into tenths had drawn it in such a way that it appeared that two of the tenths were the size of eighths (in a circle model). One of the students in her class who was interviewed did not have an understanding of the part-whole relationship of fractions, nor did he have the understanding that fractional pieces need to be the same size (e.g., each one-tenth of the circle model needs to be the same size as all the other one-tenth pieces). The teacher's poor attempt at drawing did not aid the student in developing a better understanding of the need for the fraction pieces to be of equal size (Olive & Vomvoridi, 2006). Also, when asked to draw a circle model representing fifths, the teacher did not draw it so that all the pieces were of equal size.

Incorporating manipulatives into instructional strategies does not guarantee effective use (Puchner, Taylor, O'Donnell, & Fick, 2008). Puchner et al. (2008) studied inservice elementary and middle school mathematics lessons in which manipulatives were used as an instructional strategy. The teachers in their study had participated in professional development and reported "an increased understanding of the importance of using manipulatives" (Puchner et al., 2008, p. 313). However, an analysis of lessons taught by these teachers showed a pattern of ineffective manipulative use and misuse. Puchner et al. (2008) suggested that effective use of manipulatives is more difficult than most realize.

In their study, Puchner et al. (2008) analyzed data collected from 23 different teachers during a lesson study that followed a summer professional development institute. In one particular sixth-grade lesson, students were shown how to use non-traditional methods (decomposition, arrays, and base-ten blocks) to demonstrate a multiplication problem. Many students used the base-ten blocks, but when they could not count accurately, they reverted to using an algorithm to get the right answer. In addition, some students first used the algorithm to get the right answer and then attempted to make the non-traditional methods fit. This "answer-first, model-second" method is similar to the procedural use of drawings described by Osana and Royea (2011) in their study of prospective elementary teachers. Puchner et al. (2008) pointed out, "Teachers noticed that, for some students, the answer and the non-traditional procedure to obtain the answer became two separate processes" (p. 319). Instead of developing conceptual knowledge of multiplication and linking the traditional algorithm to non-traditional strategies, the lesson became focused on students figuring out how to use the manipulative to come out with the answer they already knew (Puchner et al., 2008).

Another example of ineffective use of manipulatives highlighted by Puchner et al. (2008) was in an eighth-grade class using linking cubes. Observers noted that students seemed very confused about what they were supposed to do with the cubes and focused on color, counting, and rearranging the cubes instead of using them in an investigative way. Puchner et al. (2008) indicated that a major problem with this particular lesson was that the teacher neglected to “think carefully about how pedagogy will support the specific content goals of the lesson” (p. 322). Puchner et al. (2008) related this failure to link pedagogy and content to Shulman’s (1986) pedagogical content knowledge, suggesting that teachers had not carefully analyzed how the content would actually be learned using the manipulative.

Puchner et al. (2008) observed, “Manipulative use turned into an end in and of itself, rather than a tool leading to better understanding” (p. 321). Some students in their study used manipulatives in a procedural manner that mimicked the teacher’s actions, which is what Van de Walle et al. (2013) cautioned against. Puchner et al. (2008) suggested, “Student understanding through manipulatives occurs when students are motivated to use a manipulative as a tool to obtain the answer to a challenging problem” (p. 321). The authors suggested that professional development activities include analyzing inappropriate scenarios involving manipulatives and scenarios that do not involve manipulatives in order to help teachers understand that the mere use of manipulatives does not guarantee student learning. Puchner et al.’s (2008) research and the authors' suggestions may be of interest to teacher educators as they prepare prospective teachers to incorporate manipulatives in their future classes.

Teachers' attitudes affect the use of manipulatives. Teachers' attitudes about the use of manipulatives also affect how they use manipulatives and whether students benefit from the use of manipulatives. Izsák, Tillema, and Tunç-Pekkan (2008), mentioned earlier, were also interested in teachers' attitudes toward the use of manipulatives.

The teacher in Izsák, Tillema, and Tunç-Pekkan's (2008) study considered herself a predominately traditional teacher who focused on algorithms and seldom used manipulatives, although she reported using picture drawings when either introducing a topic or when there was confusion. By the nature of the Connected Mathematics Project curriculum that was being used, students were expected to use representations to understand what they were learning about fraction addition. However, Izsák et al. (2008) noted that the teacher often asked her students to visualize, not draw, fraction strips when they were having difficulties even though the class had previously used representations during instruction. Through interviews, the teacher shared her beliefs that students should rely less on pictures as they progressed.

As a result of examining their data, Izsák et al. (2008) interpreted the teacher's statements and actions as a desire for students to move *quickly* from representations to numeric methods. However, if the transition from the use of concrete representations to the use of mental images is expected too quickly, students may not make necessary connections and thus not benefit from the representation to the fullest extent possible (Hiebert & Carpenter, 1992).

As evidenced in Izsák et al.'s (2008) study, teacher attitudes have an impact on the use of representations in learning. For representations to benefit the students' learning, teachers not only need to understand how to represent mathematical concepts, but also need to consider students' needs and the usefulness of a representation (Moyer, 2001). Piaget (1952) asserted, "Children do not have the mental maturity to grasp abstract mathematical concepts presented in words or symbols alone and need many experiences with concrete materials and drawings for learning to occur" (p. 175). However, if teachers have the attitude that concrete materials are just for fun or just for use after a concept has been learned, then the usefulness of such materials are not being maximized (Moyer, 2001).

Beliefs about how students learn mathematics influences teachers as to how and why they use manipulatives (Moyer, 2001). Moyer (2001) studied the beliefs and attitudes of a group of teachers involved in a summer institute to see how their beliefs affected their use of manipulatives in instruction. Ten of the 18 participants (from the summer institute) volunteered to be a part of the study conducted by Moyer. Moyer collected self-report data from the teachers on a monthly basis regarding the most recent class they taught and what they employed to teach it. Classroom observations were conducted, as well as interviews with the teachers about how they used manipulatives in their classroom.

As a result of her study, Moyer (2001) found that although some type of mathematics tool was used in 79% of the mathematics lessons, half of the instances of manipulative use involved students observing the teacher using the manipulative. Moreover, Moyer (2001) found that there were very few instances of students participating in “hands-on” use of the manipulatives for learning. In general, the manipulatives were used as rewards for good behavior or to play games after classwork had been done (Moyer, 2001).

Moyer (2001) found the following reasons for using manipulatives were prevalent among the teachers: as a reward (or punishment by withholding); to give a break in the routine; to provide a visual model when introducing concepts; for student use in problem solving; to make it more fun; and to reinforce and provide enrichments for concepts. During interviews, attitudes emerged as teachers differentiated between ‘fun math,’ where manipulatives were used, and ‘real math,’ where pencil-and-paper techniques were used. Teachers routinely gave tests on Thursdays and let students use manipulatives on “fun” Fridays. One reason teachers cited for not using manipulatives was a lack of time due to the need to focus on criterion-referenced tests. When one student asked to use the manipulatives, the teacher said, “Go ahead if you think it will help, but it is quicker to do it mathematically” (Moyer, 2001, p. 189). Then, the teacher suggested the student solve the problem first without the manipulative.

Other factors affecting the use of manipulatives. Choice of curriculum may be another factor in how teachers and students utilize representations. Cramer, Post, and delMas (2002) conducted a quantitative study in a suburban school district south of Minneapolis with 66 fourth- and fifth-grade classrooms comparing the use of a conventional curriculum with the use of the Rational Number Project (RNP) to teach fractions.

Students who were in the classrooms that used the RNP curriculum were given the opportunity to use manipulative materials such as fraction circles, chips, paper folding, and pictures, as well as story problems and written symbols to explore fraction concepts (Cramer et al., 2002). Although the commercial curricula that were being used also incorporated representations of fractions like fraction bars, counters, paper folding, and fraction circles, the focus of the commercial curricula was more on developing the students' procedural fluency at the symbolic level (Cramer et al., 2002).

Cramer et al. (2002) found that students who used the RNP curriculum realized higher achievement on the post-test and retention test than students who used commercial curricula. Moreover, the students in the RNP curriculum displayed greater conceptual knowledge of fractions, were better able to judge the relative sizes of two fractions, and were better able to transfer their knowledge of fractions to problems they had not previously been taught (Cramer et al., 2002).

Additionally, even though the students using the commercial curricula spent more instructional time on adding and subtracting fractions than did students using the RNP, no significance difference was found between their achievements in that area (Cramer et al., 2002). Cramer et al. (2002) noted two significant differences in the two curricula: the emphasis on the use of hands-on manipulative models in the RNP versus on pictorial and symbolic representations and on procedural skill development in the commercial curricula. The authors attributed the greater gain of students in the RNP curriculum to instruction that involved multiple concrete models. Cramer et al. (2002) also suggested that because of the emphasis on students' use of concrete models, classroom discourse as a result of using the models could also have been a factor in the students' gains.

Not all manipulatives are equally beneficial with certain concepts. Cramer and Wyberg (2009) studied the efficacy of different concrete models for teaching the part-whole construct for fractions. “To be effective, manipulatives should mirror the structure of the concept and enable learners to use that structure to construct a mental model for that concept” (Cramer & Wyberg, 2009, p. 228). However, Cramer and Wyberg (2009) found that not all models were of the same usefulness in helping students to develop mental models for fraction understanding.

In their study of fourth and fifth graders, Cramer and Wyberg (2009) compared the use of paper folding strips, a fraction chart based on the paper folding strips, dot paper, chip models, and pattern blocks to help students develop mental images for fraction understanding, as well as the use of a number line to construct common benchmarks like $\frac{1}{2}$. Additionally, students explored mathematical concepts related to fractions through the inclusion of probability in the unit.

Cramer and Wyberg (2009) found that although lessons included pattern blocks, students subsequently had difficulty representing the fraction $\frac{3}{4}$ using the pattern blocks. They surmised that students did not develop a mental image that would aid them in modeling $\frac{3}{4}$ using the pattern blocks. The authors speculated that it is possible that, unlike fraction circles, because the pieces of the pattern blocks are not similar in shape, students had difficulty seeing the relationships between the pieces. As a result, Cramer and Wyberg (2009) asserted that the pattern blocks have serious limitations to support students’ development of mental images for fractions.

Additionally, Cramer and Wyberg (2009) found that neither the chip model, nor dot paper supported the need for finding common denominators when adding fractions, and furthermore, supported students' errors of adding numerators and adding denominators. Instead, these two models supported students' whole number thinking.

On the other hand, Cramer and Wyberg (2009) found that students' understanding of fractions was most supported by the fraction chart model that was related to paper folding. In fact, although fourth-grade students were not taught procedurally how to compare fractions, most were able to use mental images to determine which of two fractions was the larger. However, Cramer and Wyberg (2009) stated that there were limitations in using the fraction chart, particularly when fractions were greater than one and in developing students' estimation skills.

Finally, Cramer and Wyberg (2009) advised teachers to examine the following before selecting a manipulative model when working with fractions:

- Does the model build mental images for fractions and what happens when you operate on them so that students are able to construct effective estimation skills?
- Does the model show the need for finding common denominators when adding and subtracting?
- Does the model show the action of adding and subtracting clearly?
- Does the model too easily support the common incorrect strategies for adding and subtracting fractions?
- Can the students see the connections between the model and symbolic procedure for adding and subtracting fractions? (p. 254)

In summary, Cramer and Wyberg (2009) suggested that using "multiple models in learning fractions may be the most effective strategy for ensuring that students develop meaningful understanding of the complex domain of fractions" (p. 255).

Recovery through the use of manipulatives. Although students usually do not overcome the interference of rote knowledge independently, Mack (1993) suggested that by moving back and forth between students' informal knowledge of and symbolic representations of fractions, it is possible to overcome the interference of rote knowledge. In addition to using informal knowledge, the use of concrete manipulatives may be a viable way to address faulty understanding due to instrumental learning (focus on procedures) that precedes relational learning (Green, Piel, & Flowers, 2008).

Green et al. (2008) studied the effects of using manipulatives to correct two groups of prospective elementary teachers' arithmetic misconceptions. Green et al. (2008) used "guided constructivism," which they suggested was conceptually similar to Freudenthal's (1991) guided reinvention or guided reconstruction. The authors stated, "our intent was to guide students through a hierarchy of manipulatives that represent increasingly abstract objects—from concrete to representational to transitional to symbolic—by using manipulatives as tools for examining, constructing, seeing, and testing the performance of arithmetic operations and studying their interrelationships" (p. 236).

Study 1 consisted of 50 participants, while Study 2 consisted of 39 participants. The studies took place while prospective teachers (PTs) were enrolled in a required course for elementary education majors usually taken in the junior year. Most of the PTs were female with an average age of 26 years and an average SAT mathematics score of 487. During four classes ($1\frac{1}{3}$ hours each) of the 30-class semester, PTs used manipulatives to solve problems posed by the instructors. Students first compared solutions, and then the instructors demonstrated the solutions using overhead manipulatives or whiteboard illustrations. Instructors did not directly answer prospective teachers' questions; instead, they redirected the questions to the class or to the manipulative arrangement.

Initially, they found that when asked to pictorially represent $1\frac{1}{2} \div \frac{3}{4}$, only 15% of the students in their study could accurately do so (Green et al., 2008). However, after working with the subjects for only $1\frac{1}{3}$ hours using Cuisenaire™ rods, 66% of them were able to accurately represent the problem on a post-test (Green et al., 2008). The authors attributed the success to the fact that by using the Cuisenaire™ rods, students could see the pieces. The authors also reported a statistically significant decrease between the pre-test and post-test for arithmetic misconceptions (Green et al., 2008).

Green et al. (2008) believed that the success of their intervention was also related to prospective teachers' active use of manipulatives to solve problems. Green et al. (2008) indicated that no direct explicit instruction about misconceptions was given. Furthermore, the authors believed that if they had merely demonstrated the use of manipulatives to solve problems without prospective teachers' direct problem solving, the PTs would not have realized the gains they did, nor would they have decreased their arithmetic misconceptions. Although Green et al.'s (2008) study was enacted using prospective elementary teachers, its results are encouraging and leads one to question whether similar results could be realized with middle school prospective teachers.

Virtual manipulatives. As mentioned earlier, virtual manipulatives have been available for approximately twenty years now. One benefit of virtual manipulatives over pictorial representations of fractions is precision (Moyer-Packenham & Westenskow, 2013). Students are able to manipulate the unit, cutting it into halves, thirds, fourths, etc. with greater precision than is probable when drawn by hand. By using virtual manipulatives, students who have not yet grasped the concept of equal size pieces may be able to develop mental images that will help them to further their understanding of fraction concepts.

Increasing numbers of studies on virtual manipulatives are being published (Moyer-Packenham & Westenskow, 2013). In 2005, Reimer and Moyer indicated that some research on computer-based manipulatives has been inconclusive due to design and sampling that may affect student achievement results. In fact, Reimer and Moyer (2005) stated, "Although those results are mixed, the amount of research on high-quality dynamic virtual manipulative is so limited that a judgment about their potential uses in mathematics instruction is entirely speculative" (p. 8).

Since that time, Moyer-Packenham and Westenskow (2013) did a meta-analysis of the effects of virtual manipulatives on student achievement and mathematics learning by examining 66 studies on virtual manipulatives. They based their decision to do a meta-analysis on the fact that although research on virtual manipulatives spans two decades, "to date, there has been no attempt to synthesize this research base" (p. 36). Moyer-Packenham and Westenskow (2013) began by searching the literature through electronic data bases and found 150 publications about virtual manipulatives, some of which were opinion articles, theory papers, and articles suggesting instructional strategies. For their meta-analysis, Moyer-Packenham and Westenskow (2013) included only studies in which student achievement data were collected and for which threats to internal validity were not significant.

As a result of their meta-analysis, Moyer-Packenham and Westenskow (2013) found that “virtual manipulatives have a moderate effect on student achievement when compared with other instructional treatments” (p. 45). They also identified five “interrelated affordances of virtual manipulatives that promote student learning: focused constraint, creative variation, simultaneous linking, efficient precision, and motivation” (p. 46). Moyer-Packenham and Westenskow (2013) indicated that limited research on virtual manipulatives has been done with students beyond Grade 6. More specifically, the authors only included two studies at the university level in their meta-analysis. Interestingly though, the effect size Moyer-Packenham and Westenskow (2013) found for the use of virtual manipulatives at the university level was 1.17.

One example of a study that shows the potential effects of virtual manipulatives on student achievement was done by Reimer and Moyer (2005). Reimer and Moyer (2005) conducted a study in Reimer’s third-grade class using applets found on the National Library of Virtual Manipulatives (http://nlvm.usu.edu/) to determine what impact the use of virtual manipulatives had on students’ fraction learning. Nineteen of the 25 students in Reimer’s class participated in the study. Students were previously taught a unit on fractions that involved the use of physical manipulatives and other strategies. After instruction, students were given a test on their conceptual knowledge and on their procedural knowledge of fractions. Students were allowed to use physical manipulatives on the test due to the fact that they were used during instruction (Reimer & Moyer, 2005).

This practice is in alignment with instruction advocated by NCTM’s *Assessment Standards for School Mathematics* (1995).

After the test on fractions, students were taught using the virtual manipulatives applets (Reimer & Moyer, 2005). Prior to using the virtual manipulatives in the fractions lessons, the teacher allowed students time to use an applet on base-10 blocks found on the National Library of Virtual Manipulatives website so that they would be familiar with virtual manipulatives, but not the fractions applets. The teacher led instruction and discussions, and then the students used the applets in the computer lab for one hour each day.

The statistical analysis showed that students’ test scores on the conceptual knowledge tests were significantly different, but not the test scores on the procedural test (Reimer & Moyer, 2005). The class average on the conceptual pre-test was 60% and on the post-test, 69%. After looking at the scores individually, 53% of the students improved in their conceptual knowledge of fractions, four students showed no change (because their scores were 94%, 100%, 100%, & 100%), and five students’ scores decreased (Reimer & Moyer, 2005). The scores on the pre-test and post-test for the procedural knowledge were 90% and 96%, respectively. Because procedural test scores were so high initially, this left little room for improvement, unlike the conceptual knowledge pre-test. However, when looking at the differences in scores on both pre-tests (60% on conceptual versus 90% on procedural), questions could be raised regarding how the concrete manipulatives were used during instruction, as well as what other strategies were used. Table 9, shown on the following two pages, contains a summary of literature regarding concrete and virtual manipulatives.

Table 9

Summary of literature related to manipulatives

Contributor(s)	Level	Contribution
Van de Walle, Karp, & Bay-Williams (2013)	K-8	Cautioned against having students mimicking teachers' actions with manipulatives
Moss & Case (1999)	4th grade	Concern regarding over-use of circle model, limiting students to part-whole understanding
Izsák, Tillema, & Tunç-Pekkan (2008)	6th grade	Negative effects on students' understanding as a result of teacher's use of partitioning and attitudes toward representations
Olive & Vomvoridi (2006)	6th grade	Teachers' incorrect partitioning of circles interfered with students' understanding of need for unit fractions to be the same size
Puchner, Taylor, O'Donnell, & Fick (2008)	K-8	Ineffective or misuse of manipulatives in classroom activities after teachers participated in professional development
Moyer (2001)	In-service teachers	Teachers attitudes toward manipulatives affects classroom use of the manipulatives
Cramer, Post, & delMas (2002)	4th & 5th grades	Choice of curriculum affects student achievement through implementing manipulatives
Cramer & Wyberg (2009)	4th & 5th grade	Studied efficacy of different concrete models; found that pattern blocks did not help students

		create mental representations of fourths; found dot paper and chip model reinforced students' misconceptions of adding numerators and denominators
Green, Piel, & Flowers (2008)	Prospective elementary teachers	Used manipulatives to correct prospective elementary teachers' fraction misconceptions
Moyer-Packenham & Westenskow (2013)	K-university	Meta-analysis of research showed virtual manipulatives have a moderate effect size on student achievement
Reimer & Moyer (2005)	3rd grade	Used virtual manipulatives after concrete manipulatives to teach fraction concepts; students' conceptual knowledge significantly increased from pre-test to post-test

Summary of manipulatives literature. Using manipulatives can be a complex issue. If manipulatives are going to be used productively, students need to have engaging, investigative experiences using them, rather than being provided step-by-step direct instruction (NCTM, 2014; Van de Walle, Karp, & Bay-Williams, 2013).

Teachers must understand the appropriateness of manipulative materials with respect to the mathematics they are teaching (Cramer & Wyberg, 2009; Moss & Case, 1999; Puchner, Taylor, O'Donnell, & Fick, 2008), as well as an awareness of how students may be affected by the misuse of such materials (Olive & Vomvoridi, 2006; Izsák, Tillema, & Tunç-Pekkan, 2008). Finally, teachers need to be cognizant that their attitudes toward manipulatives may compromise the benefits that manipulatives can provide to students (Moyer, 2001; Izsák, Tillema, & Tunç-Pekkan, 2008).

A review of the literature on manipulatives informed this study by generating in the researcher a concern regarding: over-use of fraction circle model; efficacy of certain concrete models; and use of manipulatives to correct prospective teachers' misconceptions.

Summary of literature review. Teaching is a complex endeavor, and having well-qualified teachers is important. Teachers need to deeply understand the mathematics they are tasked to teach in order to be able to provide clear, conceptual explanations to their students and address students' misconceptions. Teachers need to be able to provide multiple representations of mathematical content and facilitate the linking of these representations. Furthermore, if teachers are going to incorporate concrete models as a form of representation, they will need experiences with these materials and will need to recognize that incorporating manipulatives into instruction is not a simple matter.

A variety of studies have shown that prospective teachers' conceptual knowledge is lacking even though their procedural knowledge is not (Lloyd & Frykholm, 2000; Rathouz & Rubenstein, 2009; Li & Kulm, 2008). If prospective teachers' conceptual knowledge is lacking, then they will need significant learning opportunities during their teacher education courses to develop conceptual knowledge, and thereby increase the likelihood that they will have relational understanding of fractions. At least one study has shown promise in addressing elementary prospective teachers' misconceptions regarding fractions through the use of concrete manipulatives (Green et al., 2008). Considering that much research has been done on the effectiveness of using manipulatives in the K-12 school setting, perhaps additional research done at the university level can further inform the field.

Chapter 3: Methods

In Chapter 2, literature was presented emphasizing the importance of having conceptual understanding to support procedural fluency (NCTM, 2014; Kilpatrick et al., 2001); relational understanding was defined as having both conceptual and procedural knowledge and knowing the connections between them (Skemp, 1987); the vast inter-connectedness of fraction concepts and the importance of teachers possessing relational understanding of fractions themselves was considered (e.g., Confrey et.al, 2009; Lesh, Post, Behr, & Silver, 1983; Kilpatrick et al., 2001; National Mathematics Advisory Panel, 2008; Usiskin, 2007); and literature related to manipulatives was reviewed (e.g., Moyer, Bolyard & Spikell, 2002; NCTM, 2000; Van de Walle, Karp, & Bay-Williams, 2013), including a historical look at the use of manipulatives in K -12 settings (e.g., Moyer, 2001; Ball, 1992). In each section, concerns were highlighted that current literature has not addressed.

The purpose of this study was to gather and analyze data through qualitative research methods to attempt to answer the following question:

> What is the impact of a manipulatives-intensive fractions unit in a middle grades methods course on prospective teachers' relational understanding of fractions?

Philosophical Beliefs

"All research is interpretive: guided by a set of beliefs and feelings about the world and how it should be understood and studied" (Denzin & Lincoln, 2011, p. 13). Within a set of beliefs, there may be beliefs about the nature of reality (ontology), the nature of knowledge (epistemology), and how we know the world or gain knowledge of it (methodology) (Denzin & Lincoln, 2011). Additionally, there may be beliefs about how to communicate findings (rhetorical) and biases that impact interpretations (axiological) (Creswell, 2007).

I will now present the beliefs that shaped this study. With respect to ontology, Lincoln, Lynham, and Guba (2011) suggested that qualitative researchers construct knowledge through their lived experiences and interactions, and therefore must participate in the research process in order to produce knowledge that is reflective of the participants' reality. During this study, I interacted with participants through one-on-one interviews and conducted observations during class in order to observe participants in the natural environment. Throughout this study, I constructed knowledge as a result of the interactions between the participants and me. Additionally, I used quotes in the words of participants as evidence of the participants' perspectives.

With respect to epistemology, Lincoln, Lynham, and Guba (2011) noted that people are shaped by their experiences, and the knowledge we generate is a result of our experiences. In practice, researchers collaborate with participants and spend time with participants in the field in order to become an "insider" (Creswell, 2007). Although it was not likely I could become an insider because of my status at the research site, I attempted to lessen the distance between the participants and me by responding to participants in a friendly manner to establish a positive rapport with them.

My beliefs about how knowledge is acquired influenced me in the methodological decisions that I chose for this study (Denzin & Lincoln, 2011). Because I believe knowledge is constructed, I chose to conduct one-on-one, task-based interviews as part of the research process in hopes that I could be privy to participants' knowledge. In practice, I described situations in detail in order to provide the reader with as much context of the experiences as possible. In addition, I examined details prior to making generalizations and continually revised my interview questions based on participants' responses (Lincoln & Guba, 1985).

The rhetorical practice of qualitative research allows the use of personal voice and writing in an informal, literary style, which allows me to refer to myself in first person (Creswell, 2007). However, because writing in first person presented the potential for the reader to focus more on my actions and less on the participants' knowledge, I chose to write in passive voice to communicate about the participants. When I communicated about my analysis and interpretations, I used first person. Furthermore, I used the language of qualitative research to convey each aspect of this study. For example, I chose wording that was tentative like *may*, *possible*, and *seemed* in lieu of definitive words.

Finally, the axiological practice of qualitative research is that the researcher acknowledges that research biases are present (Creswell, 2007). In practice, this requires me to openly discuss values that shape the narrative and to include my own interpretation in conjunction with the interpretations of participants. Therefore, I provided a section in this chapter disclosing my biases and the values that shaped this study.

Theoretical Basis for the Study

"Learning is a constructive process that occurs while participating in and contributing to the practices of the local community" (Cobb & Yackel, 2004, p. 220). Ernst von Glasersfeld (1995) claimed that people construct knowledge based on their own experiences, and that what one person experiences may or may not be like that of another. Ernst von Glasersfeld (1995) used the term “theory of knowing” to communicate his ideas of constructivism, and indicated that although each person individually makes meaning of their world, each person is influenced by the social interactions in which they are involved. Naylor and Keogh (1999) stated, “The central principles of this approach are that learners can only make sense of new situations in terms of their existing understanding. Learning involves an active process in which learners construct meaning by linking new ideas with their existing knowledge” (p. 93). A classroom is a social environment in which the participants bring unique backgrounds upon which they can build. Although multiple characters are engaged in this social environment at the same time, not all will be affected in the same way.

Constructivists posit knowing as a process (Ültranir, 2012). von Glasersfeld (1995) suggested that knowledge starts “in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience” (p. 1). Furthermore, knowledge is not passively received but is built up through a cognitive process (von Glasersfeld, 1995). As part of that process, teachers are tasked with providing students with opportunities and experiences for constructing knowledge instead of merely transmitting information (von Glasersfeld, 1995). However, von Glasersfeld (1995) stated that many teachers “convey what counts as accepted knowledge, rather than help students to build it up for themselves” (p. 185). He also suggested that students might be less likely to develop an aversion to mathematics if students are given the opportunity to understand that what they are expected to learn involves mental operations and abstractions, rather than actions and objects of the everyday world (p. 185).

Brooks and Brooks (1999) stated, “Deep understanding occurs when the presence of new information prompts the emergence or enhancement of cognitive structures that enable us to rethink our prior ideas” (p. 15). They further shared the following regarding constructivism:

> The constructivist vista, however, is far more panoramic and, therefore, elusive. Deep understanding, not imitative behavior, is the goal. But, capturing another person’s understanding is, if anything, a paradoxical enterprise. Unlike the repetition of prescribed behaviors, the act of transforming ideas into broader, more comprehensive images, escapes concise description. We see neither the transformed concept nor the process of construction that preceded its transformation. The only discernible aspect is, once again, the student’s behavior, but a different type of behavior. In the constructivist approach, we look not for what students can repeat, but for what they can generate, demonstrate, and exhibit (p. 16).

Ernest (1996) presented the following pedagogical implications of constructivism: sensitivity to the learners’ previous knowledge upon which to build; allowing learners to experience cognitive conflicts in order for them to have opportunities to resolve and remedy misconceptions; presenting learners with opportunities for metacognition and self-regulation; use of multiple representations with which to connect prior knowledge; learners’ awareness of the importance of learning goals; and an awareness of differences in social contexts (p. 346).

Brooks and Brooks (1999) listed the following characteristics of constructivist classrooms:

- Curriculum is presented whole to part with emphasis on big concepts;
- Pursuit of student questions is highly valued;
- Curricular activities rely heavily on primary sources of data and manipulative materials;
- Students are viewed as thinkers with emerging theories about the world;
- Teachers generally behave in an interactive manner, mediating the environment for students;
- Teachers seek the students' points of view in order to understand students' present conceptions for use in subsequent lessons;
- Assessment of student learning is interwoven with teaching and occurs through teacher observations of students at work and through students' exhibitions and portfolios; and
- Students primarily work in groups (p. 17).

In this study, the instructor of the middle grades methods course focused on providing opportunities and experiences for prospective teachers to construct their own knowledge instead of passive reception of information. Moreover, the instructor attempted to create an environment in which participants felt part of a social community, free to share their thoughts and conceptions through discourse or other means of non-verbal communication. The instructor also used multiple representations with which to connect prospective teachers' prior procedural knowledge and conceptual knowledge of fractions. Lastly, the instructor presented opportunities for prospective teachers to experience and resolve cognitive conflicts, as well as opportunities for metacognition and self-regulation. The bulleted list below outlines constructivist-based activities that occurred during the fractions unit studied.

- Examining multiplication with an emphasis on the meaning of the multiplier and multiplicand;
- Examining partitive and measurement division;
- Examining root words of numerator and denominator to link to meaning of each;
- Investigating fraction magnitude and equivalence using on-line applets and recursive partitioning;
- Comparing two fractions by examining the relative sizes of fractions instead of procedural algorithms;
- Adding fractions using fraction strips, decomposition of fractions, and pattern blocks;
- Writing algebraic expressions by investigating relationships among Cuisenaire™ rods and pattern blocks;
- Multiplying fractions using length, area, and region models and the distributive property;
- Using a fraction as an operator;
- Generating conjectures about the product and quotient based on the size of the multiplier relative to one and divisor relative to the dividend (respectively);
- Dividing fractions using length and region models;
- Introducing proportional reasoning using set models;
- Recursive partitioning using length and area models;
- Analyzing tasks based on the level of cognitive demand;
- Conducting discussions regarding the Common Core Standards for Mathematical Practice (NGA Center & CCSSO, 2010) and the link to the Process Standards from NCTM (2000).

With respect to the theoretical underpinnings for my research inquiry, throughout this study, I gathered and analyzed data in an attempt to develop understanding. During data collection and analysis, I constructed my own understanding based on my prior experiences, beliefs, and background knowledge associated with the topics of my study and my experiences. For example, during interviews, I did not merely record what participants said and did; I attempted to make meaning of what they were saying and doing. I attempted to analyze participants' responses based on what they said and how they said it, which guided me through the interview process. However, throughout this process, it is possible that I was influenced by my beliefs and prior knowledge of mathematics, learning, use of manipulatives, and more. If other researchers were to attempt to replicate my study, they may or may not uncover findings similar to mine because their prior experiences, beliefs, and background knowledge, which may influence their data analysis, may be different than mine.

Because of my belief that people construct their own knowledge, I chose methods to generate data with the hope that data analysis would provide insight into the relationships of interest. Using other data collection methods, I might have arrived at different interpretations and conclusions. For example, I chose to conduct task-based interviews because of my belief that I could learn about my participants' knowledge by interacting with them in a one-on-one, mathematically-rich environment. Furthermore, if I had used only tests without interviews, I might not have been privy to participants' depth of knowledge and connectedness of knowledge because of the limitations due to the data-collection instrument. However, because the researcher is the instrument in qualitative research, it was important for my design to allow me to generate data that could possibly provide depth of insight.

As a researcher in this study, my focus was vastly different than it would have been as an educator or a teacher educator. As a researcher, I sought to understand; whereas, if I were acting as an educator or a teacher educator, I would have attempted to affect my participants' experiences and knowledge level. At times during interviews and observations, remaining a researcher was difficult because of my prior experiences teaching. When I recognized that participants had gaps in their knowledge, my first impulse was to help them. However, as a researcher, I had to resist the urge to affect my participants' knowledge. Although I attempted to remain in the role of researcher, it is possible that the sequence of questions I asked participants affected the outcome of the interviews.

Conceptual Framework

Lester (2005) stated, "A conceptual framework is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation" (p. 460). Lester (2005) further suggested that a conceptual framework serves as a scaffold to support arguments about what is relevant to study and why. Huberman and Miles (1994) suggested that a conceptual framework "lays out the key factors, constructs, or variables, and the presumed relationships among them" (p. 440). They also suggested that graphic displays of main variables connected by bidirectional arrows specifying relationships among the variables are useful in making frameworks clear.

In *Figure 7*, shown below, I initially proposed the Venn diagram among the variables conceptual and procedural knowledge and use of manipulatives. As I gained understanding about their interaction, I continually considered revising my representation. My final representation is presented in Chapter 5 of this paper.

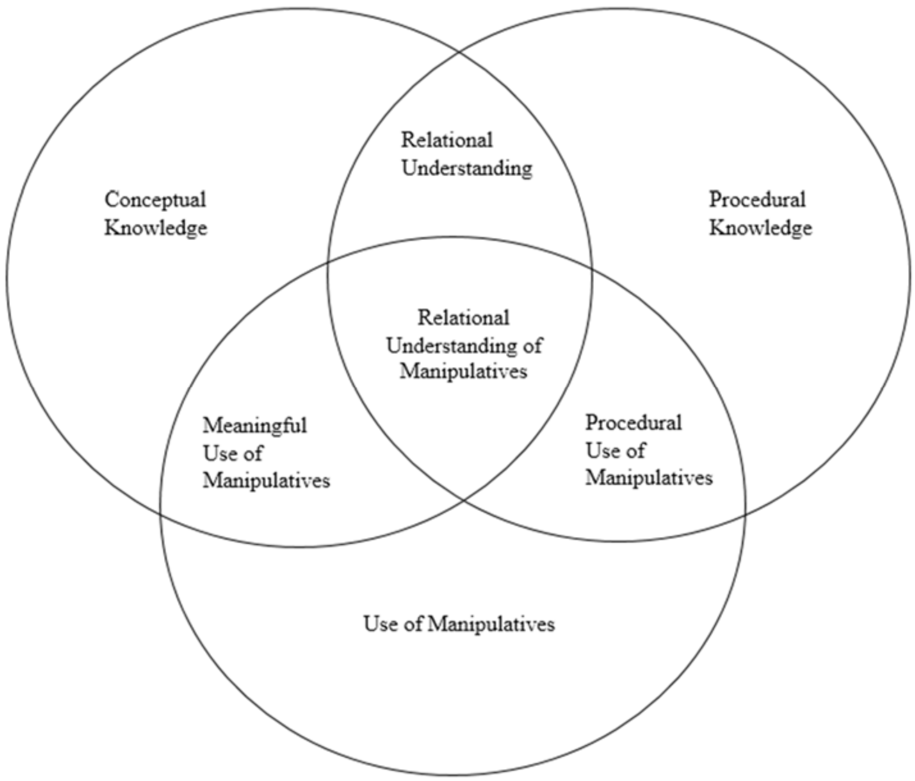


Figure 7. Initially Proposed Relationship among Conceptual and Procedural Knowledge and Use of Manipulatives

In the proposed diagram, conceptual knowledge means having knowledge of the relationship among pieces of information (Hiebert & Carpenter, 1992). Procedural knowledge refers to having knowledge of the formal language (i.e., symbol representation system) and rules, algorithms, or procedures used for completing mathematical tasks (Hiebert & LeFevre, 1986). Relational understanding is having both conceptual and procedural knowledge and understanding the connections between the two (Skemp, 1987). Relational understanding means having a connected understanding of mathematical concepts and procedures, knowing what procedures to perform and why to perform them, the ability to explain why a procedure works, and the ability to connect one mathematical concept to another or to make connections between mathematics and other domains (Skemp, 1987). Use of manipulatives refers to how participants used the manipulatives during classes and interviews. I viewed participants' use of manipulatives through a lens of procedural use of manipulatives versus meaningful use of manipulatives similar to that used by Osana and Royea (2011) in their work with prospective teachers using pictorial representations procedurally or meaningfully.

I posit that the intersection of the three variables (procedural knowledge, conceptual knowledge, and use of manipulatives) is relational understanding of manipulatives. If a person has relational understanding of manipulatives, they will be able to explain how the conceptual and procedural knowledge are connected to the use of the manipulatives. For example, using measurement division for $3 \div \frac{2}{3}$, the quotient is the number of two-thirds that can be made from three wholes. Each whole in the dividend is composed of $1\frac{1}{2}$ two-thirds, or $\frac{3}{2}$ two-thirds. Therefore, the quotient to the division problem can be calculated by multiplying $3 \times \frac{3}{2}$. The quotient is $4\frac{1}{2}$ groups of $\frac{2}{3}$ in 3 (1 yellow, 1 red, 1 purple, 1 green, and $\frac{1}{2}$ of a group pink). *Figure 8*, on the following page, shows a region model for this division problem.

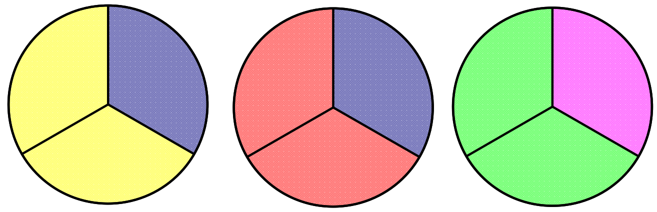


Figure 8. Region model for $3 \div \frac{2}{3}$

Another lens through which I viewed my study was the belief that students' learning should be grounded in the meaning of the mathematics, not just in the manipulation of the symbols of mathematics.

> What is inflicted on all too many children and older students is the manipulation of symbols with little or no meaning attached, according to a number of rote-memorized rules. This is not only boring (because meaningless); it is very much harder, because unconnected rules are much harder to remember than an integrated conceptual structure (Skemp, 1987, p. 18).

During the process of analyzing data, I examined the degree to which prospective teachers' responses (verbal or physical) provided evidence of comprehension of the connectedness among fractional conceptions, the meaning of operations on fractions, the meaning of the steps involved in procedures, and how and why procedures work. In analyzing participants' procedural knowledge, I examined the degree to which their responses showed evidence of knowledge of the symbols and ability to accurately perform the algorithms associated with procedures used to solve problems.

Because of my interest in prospective teachers' relational understanding of fractions, I examined my data through the lenses of Confrey et al.'s (2009) learning trajectories of fractions, as well as Lesh, Post, Behr, and Silver's (1983) conceptions of fractions: fraction as a ratio; fraction as an operator; fraction as division; and fraction as a measure. I also viewed participants' knowledge of fraction division using Ma's (1999) conception which indicates that prior knowledge of whole number multiplication and division is necessary for developing conceptual understanding of fraction division.

Another lens through which I viewed the data was the effects prior instrumental learning had on subsequent relational learning (Pesek & Kirshner, 2000). It stands to reason that all participants in my study had prior instrumental learning of mathematics. However, it may be the case that they had not experienced relational learning. I was particularly interested in how prospective teachers' prior instrumental learning affected the relational learning that they experienced in their middle grades methods course. If participants experienced interference due to prior instrumental learning, I was interested in discovering whether they were able to overcome the interference and what contributed to their ability to overcome the interference.

Positionality (Researcher Biases)

In 1990, I earned a Bachelor of Science degree in Mathematics from a four-year university in the southeastern United States. I knew that I wanted to teach mathematics; so, I worked various part-time jobs for one year before I began my graduate studies (at the same university). I was accepted into an alternative Master's program to earn a Master of Education degree in Secondary Mathematics Education. After earning my Master's degree in 1993, I began teaching Basic Math, Pre-Algebra, and Algebra I to eighth-grade students in a middle school in the southeastern U.S. One particular day during my second year teaching, I was teaching students the conventional algorithm for dividing fractions. The looks on their faces conveyed to me that they did not comprehend.

It was at that point, that I asked myself out loud, “Why do you multiply by the reciprocal?” I had procedural knowledge of how to perform the conventional algorithm but not conceptual knowledge of division by fractions, nor did I remember anyone ever expecting me to understand fraction division conceptually.

I spent the next few minutes processing the problem, trying to discern why the conventional algorithm worked and what it meant. I explored the problem using a set of fraction circles I borrowed from a colleague. After several minutes, I began to develop conceptual knowledge of dividing fractions, as well as understanding of the connections between the conventional algorithm and the manipulative. I explained my thinking to my class and began to challenge them to develop conceptual knowledge of dividing fractions. Some of the students in my class understood, which inspired me to begin teaching differently than I had previously. I began focusing more on the meaning of the mathematics that I was teaching, not just the procedures.

At that time in my career, I may or may not have been aware of the terms conceptual and procedural knowledge or relational understanding. Nonetheless, for the next 17 years, I strived to help my students to develop relational understanding (Skemp, 1987). Many students came to me without being able to demonstrate procedural fluency of the four basic operations on whole numbers and rational numbers. Each year I continued to integrate manipulatives into my instruction as a vehicle for conveying concepts to students in an effort to improve their relational understanding. During some lessons, students used the manipulatives, and some lessons I demonstrated using the manipulatives while I challenged students to think about why algorithms worked.

In January 2009, I resumed my graduate studies in an Education Specialist program in Secondary Mathematics Education in the southeastern U.S. I began reading research on the use of manipulatives, fractions, and conceptual and procedural knowledge. Concurrently, I began to use more cooperative learning groups, to differentiate instruction more, and to allow students to use manipulatives more themselves in an exploratory manner to solve non-routine problems. Over the years, I had been relatively successful helping students to improve their relational understanding of fractions. However, once I incorporated more reform-based methods of instruction, I noticed that my students were more enthusiastic about mathematics and demonstrated better gains than previously.

After having taught middle school and high school mathematics for 19 years, I became the director of an urban mathematics collaborative at a small four-year state university in the southeastern U.S. My responsibilities included, but were not limited to, conducting professional development sessions for practicing and prospective teachers, teaching middle grades methods courses for prospective teachers, and teaching graduate courses for a K-5 mathematics endorsement program at our university.

During professional development sessions with K-12 teachers, many verbally expressed their lack of knowledge about using manipulatives during instruction. In addition, when asked “why” a conventional algorithm worked, many admitted that they did not know why the procedure worked, just that they had always been told to “do it” that way. Often times, once the participants explored with manipulatives and were questioned about what they were doing, some were able to articulate some degree of relational understanding. Moreover, they began to verbally express a greater degree of comfort with using the manipulatives.

While teaching prospective methods courses, I found that my students had very few prior experiences with manipulatives. Taking into consideration the recent adoption of the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010), which advocates the use of concrete models particularly with fractions, I became interested in how mathematics educators prepare future teachers to become more proficient using manipulatives in their mathematics instruction. Furthermore, the *CCSSM* (NGA Center & CCSSO, 2010) emphasizes conceptual understanding to undergird students' procedural fluency. Because these standards were new then, my concern was that prospective teachers currently in colleges of education may have gaps in their conceptual knowledge of mathematics and may have limited experience using manipulatives to demonstrate their knowledge.

Prior to this study, I observed prospective teachers using manipulatives to explore fractions during their middle grades methods course. In addition, I conducted a task-based interviewed with a prospective teacher taking the methods course I was observing. The prospective teacher's reactions during the interview caused me to be even more interested in how prospective teachers' experiences with manipulatives affected their relational understanding of fractions.

Because of my aforementioned experiences, I developed a particular interest in the impact of the use of manipulatives to help prospective teachers develop conceptual and procedural knowledge of fractions (i.e., relational understanding). This study was my effort to better understand this phenomenon in order to inform the field and contribute to the ongoing conversations regarding these topics.

Finally, during this study, I may have paid more attention to particular aspects of prospective teachers' use of manipulatives because of my prior experiences with grade school students, prospective teachers, and practicing teachers. I may have also neglected to notice potentially important details or happenings because of my background and the biases I brought to this study. In addition, the interpretations, analyses, and conclusions that I made regarding data were likely affected by my previous experiences (Creswell, 2009).

Design of Study

Creswell (2009) stated, "Case studies are a strategy of inquiry in which the researcher explores in depth a program, event, activity, process, or one or more individuals" (p. 13). Yin (2009) defined case study as "an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 18).

Stake (1995) stated that instrumental case study is used when researchers have a need "for understanding the complex interrelationships among all that exists" (p. 37) and believe that they may gain insight by studying a particular case. "Case studies involve a detailed description of the setting or individuals, followed by analysis of the data for themes or issues" (Creswell, 2009, p. 184). Although some techniques for data collection are more commonly used than others, Merriam (2009) suggested that case study does not use a prescribed set of data collection or data analysis methods.

Stake (1995) indicated that “all research is a search for patterns, for consistencies” (p. 44). Stake (1995) also suggested that, as research progresses, the researcher continually refines his/her understanding about “the particular,” not a generalization. The researcher strives to understand a particular case, not just how it is different from others, but understanding the case itself (Stake, 1995). The researcher draws from a mix of his/her own understanding, prior personal experience, scholarship, and assertions from other researchers to arrive at an in-depth understanding (Stake, 1995). In addition, the researcher does not intervene or try to alter what participants know (Stake, 1995). Instead, the researcher attempts to understand the participants’ views and attempts to preserve the multiple realities, i.e. how people see things differently (Stake, 1995). In communication, the qualitative case researcher “uses narratives to optimize the opportunity of the reader to gain an experiential understanding of the case” (Stake, 1995, p. 40).

Yin (2009) stated that case study inquiry will likely involve more variables of interest than data points and will rely on multiple sources of evidence with a need for data to converge in a triangulating fashion. In addition, Yin (2009) stated that prior development of theoretical propositions should guide data collection and analysis. Yin (2009) advised researchers to consider the type of research question posed, the extent of control an investigator has over actual behavioral events, and the degree of focus on contemporary events. In particular, Yin (2009) stated that case studies are used in qualitative research when one wants to know answers to “how” and “why” questions about a phenomenon by collecting data through multiple sources and when the investigator does not require control over behavior events.

Merriam (2009) explained that for a study to be considered a case study, "one particular program or one particular classroom of learners (a bounded system) or one particular learner selected on the basis of typicality, uniqueness, success, and so forth, would be the unit of analysis" (p. 41). Merriam (2009) stated that qualitative case researchers "are interested in insight, discover, and interpretation rather than hypothesis testing" (p. 42).

Taking into consideration the aforementioned literature from Creswell (2009), Stake (1995), Yin (2009), and Merriam (2009), case study was an appropriate strategy for this study because I sought to develop an in-depth understanding of a phenomenon by collecting and analyzing data from multiple sources (e.g. interviews, observations, and assessments) over an extended period of time. By studying multiple participants, I attempted to understand the impact of a manipulatives-intensive fractions unit on prospective teachers' relational understanding. In addition, my prior understanding and experiences likely affected my interpretation of the data. I did not attempt to alter my participants' experiences, but to understand how their experiences were affected by multiple variables.

The context of the setting was an important factor in my research. If I only conducted interviews outside of the classroom environment without observing participants in the natural setting of the classroom, I would not have been able to examine participants acting naturally. As I designed my study, I took into consideration literature related to my topic and designed data collection procedures and protocols accordingly. Finally, in reporting data analysis, I used thick description in a narrative style to help the reader to gain an experiential understanding of the participants' experiences.

Trustworthiness

Lincoln and Guba (1985) stated that trustworthiness is related to one's ability to "persuade his or her audiences (and self) that findings of an inquiry are worth paying attention to, worth taking account of" (p. 290). They suggested that there are four constructs which need to be considered when establishing trustworthiness in naturalistic inquiry: truth value (credibility), applicability (transferability), consistency (dependability), and neutrality (confirmability).

With respect to truth value, Lincoln and Guba (1985) stated that "to demonstrate 'truth value,' the naturalist must show that he or she has represented those multiple constructions adequately, that the reconstructions (findings and interpretations) that have been arrived at via the inquiry are credible to the constructors of the original multiple realities" (p. 296). Lincoln and Guba (1985) suggested the following to increase credibility: prolonged engagement; persistent observation; triangulation; peer debriefing; and member checking.

In order to increase credibility through prolonged engagement and persistent observation, I observed participants in their EDMG 1 class five times over the course of a semester. I interviewed the participants prior to the beginning of the fractions unit and at the end of the semester as opposed to a shorter duration between interviews. Finally, I collected multiple forms of data for triangulation, which I outline in the Data Collection section of this document.

I also used the technique of peer debriefing suggested by Lincoln and Guba (1985). Throughout all phases of the study, I consulted with the chair of my committee, as well as other members of my committee, regarding decisions made for this study.

These decisions were related to the review of literature, development of data collection instruments, methods of data analysis, and interpretation of data. As each phase of the study occurred, I met with the chair of my committee regularly to evaluate my progress and make necessary revisions. In addition, Dr. Robert Hill (pseudonym) served as an additional peer debriefer. Dr. Hill has served on several dissertation committees and has been the director of a center responsible for evaluating grant projects.

The instructor of EDMG provided member checking during this study. Once I collected and analyzed observation and assessment data, I periodically consulted the instructor to verify that I correctly interpreted relevant data.

Lincoln and Guba (1985) suggested that naturalistic inquirers could satisfy referential adequacy by archiving “raw data.” Since I permanently deleted video recordings of interviews after transcription and analysis, I retained electronic copies of transcribed raw data throughout my study and used direct quotes from participants to support claims that I made.

Because generalizability is generally not the goal of qualitative research, Lincoln and Guba (1985) suggested that the best a naturalistic inquirer could hope for is transferability. They further suggested that the burden is on the original inquirer to provide context and sufficient descriptive data to make similar judgments possible for subsequent researchers. Therefore, I used rich, thick description in order to communicate as great a picture as possible to the reader.

Lincoln and Guba (1985) also stated that naturalist inquirers strive for dependability as opposed to reliability (aforementioned consistency). Qualitative researchers realize that interviews and observations cannot be repeated with exactly the same results due to effects such as time passing and participants' subsequent experiences affecting the context. In an effort to establish dependability, I used interview protocols so that I stayed focused on gathering relevant data that would help me answer my research questions.

Finally, to establish confirmability, I maintained an electronic audit trail that documented my progress, which was available to my committee upon request. In my audit trail, I maintained raw data to help me justify my findings; included notes on methodological decisions and trustworthiness notes; maintained materials related to the study's intentions (proposal); and included notes about instrument development and justifications.

In order to maintain confidentiality of participants, I used pseudonyms for all participants and maintained security of their identities by removing any identifying information from collected documents. I secured electronic documents on a password protected computer, including video recordings. After recordings were transcribed, analyzed, and reported on, I permanently deleted video recordings from all electronic devices.

Description of Sample

I collected data at a small four-year state university in the southeast, a site chosen as a convenience sample. At the time of the study, the university employed approximately 475 full-time and part-time faculty members and enrolled approximately 7,000 transient and residential undergraduate and graduate students, with approximately 56% White, 35% African American, 5% Hispanic, 2% Asian/Pacific Islander, 1% non-resident alien and 1% American Indian/Alaskan Native. At the time of data collection, approximately 40% of the university's student population were male; approximately 60% were female.

Middle-grades prospective teachers in the university's College of Education are required to take two middle grades mathematics methods courses as part of their middle grades teacher certification program designed for teachers of grades 4-8. The first of these two methods courses (pseudonym EDMG 1) focuses on upper elementary content such as operations on fractions and decimals, as well as an introduction to proportional reasoning. The second of these two methods courses furthers prospective teachers' experiences with proportional reasoning, as well as transformational geometry, developmental algebra, data analysis, statistics and probability. I chose to observe students in the first required course because the bulk of instruction on fraction concepts occurs in EDMG 1. In addition, concrete and virtual manipulatives are commonly used during fraction instruction in EDMG 1.

Within the College of Education over the last four years, an average of 33 undergraduate students enrolled in a middle grades teacher certification program and an average of 10 undergraduate students enrolled in the secondary mathematics teacher certification program. During the academic year of 2012-2013, the middle grades certification program began requiring students to declare two areas of specialization. For the year of 2012-2013 (the latest data available prior to the study), 19 students chose mathematics as one of their two areas of specialization.

Creswell (2009) suggested researchers "purposefully select" participants or sites that provide data collection that will help the researcher to understand the problem and the research question. Ms. Paige's EDMG 1 course was purposefully selected to study because of my knowledge of her methods of instruction in EDMG 1 and the content which is covered. Ms. Paige had taught the course as an adjunct instructor for three consecutive fall semesters prior to my study. She was a certified middle grades mathematics teacher who had worked at the university's mathematics professional development outreach center for 13 years, providing professional development experiences to area middle grades practicing and prospective mathematics teachers. Ms. Paige has seven years of experience teaching middle grades mathematics and was awarded an Education Specialist degree in Middle Grades Mathematics Education from the university in the spring of 2012.

Ms. Paige's instruction during EDMG 1 was focused on developing procedural and conceptual knowledge of middle grades mathematics, with opportunities for students to construct their own knowledge. She consistently provided opportunities for students to use concrete and virtual manipulatives during instruction, as well as opportunities for students to share their understanding with peers through presentations.

Ms. Paige used worthwhile tasks as a regular part of instruction. Many of the tasks she used are taken from NCTM publications like *Mathematics Teaching in the Middle School* and *Rich & Engaging Mathematical Tasks Grades 5—9* (NCTM, 2012), as well as *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle et al., 2013). She encouraged student-student discourse throughout classroom activities and invited students to present to their peers. In order to verify that Ms. Paige used reform-methods of teaching, I have included sample activities and class agendas from her course in the appendices (Appendix P).

Since the release of the *Common Core State Standards for Mathematics,* Ms. Paige has incorporated lessons on attending to the Standards for Mathematical Practice. Prior to the release of the *CCSSM* (NGA Center & CCSSO, 2010), Ms. Paige orchestrated her lessons around the Principles and Standards presented in *Principles and Standards for School Mathematics* (NCTM, 2000).

It is important for me to disclose that since September, 2012, Ms. Paige has been a part-time employee under my supervision. While I was taking a required qualitative research class in the fall of 2012, Ms. Paige allowed me to observe her EDMG 1 class on multiple occasions and helped me to solicit a prospective teacher to interview for my pilot study. When Ms. Paige decided to integrate a pre-test into her course, she asked me to examine the questions and give her feedback on the problems. During the fall of 2013, Ms. Paige also invited me to assist her prospective teachers with writing lesson plans. Ms. Paige's assistance with this study had no bearing on her employment status. I attempted to remain cognizant of my position as researcher and did not intentionally impose my ideas for EDMG 1 on Ms. Paige. Even though I attempted to remain in the roll of researcher during observations and interactions with Ms. Paige, it is possible that I may have influenced her instruction due to the nature of our working relationship. In addition, during data analysis, I may have drawn conclusions based on my previous experiences and not based on the data.

To obtain permission to collect data, I submitted the proper forms to the Institutional Review Board (IRB) at Auburn University and the university in which I collected data. The IRB granted approval for my study on October 24, 2014. After IRB approval, I asked a colleague in the college to administer the Instructor Consent Form

(Appendix N) to Ms. Paige. On October 28, 2014, Ms. Paige consented to observation of and data collection from her students as part of my study. Ms. Paige allowed me to speak with EDMG 1 students on October 28, 2014, about participation in my study. I gave each student in the class a consent form (Appendix O) and told them I would come back to class on October 30, 2014, to collect signed forms. On October 30, 2014, I returned to class and collected four signed consent forms out of 10 originally distributed. Table 10, shown below, contains information specific to the participants in the study. All participants, associated schools, and the course name referenced in this study are referred to by pseudonyms to ensure confidentiality. The term *traditional student* means that the student entered college immediately upon graduation from high school and pursues college studies on a continuous full-time basis.

Table 10

Participant Information

Pseudonym	Race	Traditional student	Major
Samantha	White American	No	Math/Science
Krystal	African American	Yes	Math/ELA
Jacob	White American	No	Math/Science
Matthew	Middle Eastern	No	Math/Science

Data Collection

Creswell (2007) suggested a compendium of data collection approaches for qualitative research including documents, records, interviews, observations, and physical artifacts. For triangulation purposes, data collection included pre-instruction surveys (tests) of conceptual and procedural knowledge of fractions; classroom observations;

assessment data for course assignments; individual task-based interviews; post-instruction tests of conceptual and procedural knowledge of fractions; and individual interviews regarding participants' perceptions of their understanding of fractions, use of manipulatives and other best practices in EDMG 1, and attitudes toward future classroom use of manipulatives and other best practices.

The term triangulation comes from surveying (Berg, 2009). The purpose of data triangulation in qualitative research is similar to that of surveying (Berg, 2009). By having multiple data points, one hopes to find a "true center" among the three points. For qualitative research, the researcher hopes to find mutual confirmation of measures and validation of findings through an analysis of multiple types of data (Berg, 2009). "An important feature of triangulation is not the simple combination of different kinds of data but the attempt to relate them so as to counteract the threats to validity identified in each" (Fielding & Fielding, 1986, p. 31, as cited in Berg, 2009).

Procedures for collecting data. Once all necessary approvals were met, I emailed each participant to schedule an interview time convenient to his/her schedule. As assessment documents were available, the instructor provided me with copies of each participant's documents. To schedule observation times, the instructor provided me with a list of potential observations dates and content to be covered. From that list, I chose dates that were convenient to my schedule and that would provide optimum information based on the content of the class. After the course concluded, I contacted each participant by email to schedule the final interview and data collection.

In order to answer my research question, I collected and analyzed assessment, interview, and observational data. Table 11, on the following page, delineates data-collection instruments and a timeline for data collection. In the sections following Table 11, I discuss each type of data-collection instrument in the order presented in the table.

Table 11

Data collection timeline

Instrument	Appendix	Time
Procedural Pre-Test	Appendix A	August 2014
Conceptual Pre-Test	Appendix E	August 2014
Initial Attitude Survey	Appendix J	November 5th & 7th, 2014
Pre-Instruction Task-based Interview	Appendix H	November 5th & 7th, 2014
Observations		November 4, 13, 18, 20 & December 2, 2014
Assessment data		October—December, 2014
Procedural Post-Test	Appendix A	December 15-18, 2014
Conceptual Post-Test	Appendix E	December 15-18, 2014
Post-Instruction Task-based Interview	Appendix H	December 15-18, 2014
Semi-Structured Interview	Appendix K	December 15-18, 2014

Procedural and conceptual knowledge tests. Pre-instruction tests of procedural and conceptual knowledge of fractions were done to establish a base-line for students' knowledge (Appendix A & Appendix E). These tests were administered to all students in the course as a regular part of instruction; however, I was privy only to the tests of the participants in my study. By obtaining these tests, I hoped to gain insight into students' thinking prior to observing them during instruction and to examine how participants' knowledge changed over the course of the semester.

Procedures for developing tests. Initially, Ms. Paige created a conceptual pre-test for the fall 2013 section of EDMG 1. At the time, she asked me to critique her pre-test. Ms. Paige made revisions based on my feedback and administered the revised pre-test to prospective teachers in EDMG 1.

While preparing the proposal for this study, I asked Ms. Paige about the possibility of revising her pre-test. She and I worked over the course of several months to refine the questions and include additional questions. Some of the problems we generated were based on literature, some were based on previous observations Ms. Paige made while she was teaching EDMG 1. My committee provided additional input into the development of the tests, as well as a mathematics teacher educator and a mathematician. Based on their feedback, I made final revisions to the conceptual test (Appendix E).

Based on input from Ms. Paige, I created the procedural test to compare students' procedural ability to answer the same questions on the conceptual test. However, I changed the decimal multiplication problem on the conceptual test so as not to risk participants remembering the answer from the procedural test while they were taking the conceptual test.

Procedures for administering tests. As a part of regular instruction, Ms. Paige administered the procedural knowledge test on the first day of class; she administered the conceptual knowledge test on the second day of class. Although the pre-tests were given by Ms. Paige at the beginning of the semester, I did not have access to the tests until after IRB approval.

Initial attitude survey. At the beginning of the pre-instruction interviews, I administered an attitude survey about participants' level of confidence in their conceptual and procedural knowledge of fractions, their familiarity with manipulatives, and their beliefs about students' use of manipulatives (Appendix J). I gave this survey based on Li and Kulm's (2008) work with prospective middle grades mathematics teachers in which they found that teachers' beliefs were not in alignment with their conceptual knowledge.

The purpose of the Initial Survey was to help me gain insight about participants' confidence and beliefs and to compare these data to the conceptual and procedural tests to determine whether participants had a realistic view of their ability. In lieu of a final attitude survey, I included prompts to address these same ideas in a final semi-structured interview.

Task-based interviews. As part of data collection, I engaged participants in two individual task-based interviews (Goldin, 2000). "Task-based interviews can serve as research instruments for making systematic observations in the psychology of learning mathematics and solving mathematical problems" (Goldin, 2000, p. 520). Task-based interviews can also be used for describing the participants' knowledge (Goldin, 2000).

Goldin (2000) suggested that the value of task-based interviews lies in the fact that they provide a structured mathematical environment that can be controlled to some extent. In addition, “task-based interviews make it possible to focus research attention more directly on the subjects’ processes of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce” (Goldin, 2000, p. 520). Finally, task-based interviews can help researchers infer and describe the deeper understandings in participants (Goldin, 2000).

Goldin (2000) made the following suggestions that may help with replicability and generalizability of research when using task-based interviews: 1) researchers should, whenever possible, create interview scripts including the questions posed and the major interview contingencies; 2) researchers should also make explicit their choices in script design; and 3) researchers should be as descriptive as possible about their population and methods so that other researchers can, at the least, closely replicate the tasks, questions, etc., and see similarities and differences across populations. I developed my interview protocol based on my literature review and prior experiences teaching middle school students, middle grades prospective teachers, and practicing teachers (Appendix H). A colleague who taught mathematics grades 7-12 in public schools for 30 years, college mathematics courses, and teacher education courses reviewed my interview protocol for the task-based interviews. Based on her feedback, I revised the protocol to what it is now.

Goldin (2000) also stated, “The goal of the comparability of research findings refers to the need for descriptions of conditions, observations, and inferences from the observations to be sufficiently precise that, when other observers under different conditions make observations and inferences, the findings of the studies can be compared with respect to some defined outcome variables” (p. 531). In addition, Goldin (2000) stated, “The concept of reliability includes measuring the consistency with which a task-based interview is conducted, observations are taken, and inferences are made from the observations using defined criteria" (p. 531).

I conducted all interviews in the resource room of my work department to have easy access to multiple types of manipulatives, as well as a computer for virtual manipulatives. The initial task-based interviews occurred on November 5 and November 7, 2014, several days prior to the beginning of the fractions unit. By the time the interviews occurred, students had already become familiar with the use of two-color counters; whole number and integer multipliers and multiplicands; and partitive and measurement division of whole numbers and integers. The second task-based interviews occurred December 15-18, 2014, after the course concluded. During each interview, participants attempted to solve two fraction tasks and generate a real-world scenario for each. The interviews lasted between approximately thirty minutes and a little more than an hour.

I video recorded each interview on a Sony HDR-PJ10 video camera I borrowed from a department within my college. I transferred the video recordings to a password-protected computer which belongs to my department at work. After transferring the videos to the password-protected computer, I permanently deleted the interviews from the camera. Once I transcribed, analyzed, and reported on my data, I permanently deleted the videos from the computer.

Observations. With respect to observations, Berg (2009) indicated the following possibilities for the researcher to act as he/she gathers data through observations: complete participant; participant as observer; observer as participant; and complete observer.

As a complete participant, the researcher acts as a hidden investigator. Since I conducted this study at an institution where I was likely to be known by some prospective teachers, it was less likely for me to be able to act as a *complete participant*. Acting as a *participant as observer* was not feasible because of potential difficulty in developing rapport with students. The role of *observer as participant* generally involves limited visits and more formal interactions. By collecting data as an *observer as participant*, I risked failing to understand the dynamics of the class and how the interactions among them affect the students' development. In addition, acting as an *observer as participant*, I may have inadvertently affected the students' knowledge by interacting with them. Therefore, I acted as a *complete observer* during this study. Generally speaking, a complete observer announces their role as a researcher and remains in the setting for a prolonged period of time as a passive observer (Berg, 2009). Acting as a complete observer was difficult for me during this study because each of the participants wanted to interact with me during the observations because of our familiarity with one another due to the pre-instruction individual interviews I conducted. I attempted to limit my interaction with the participants, but I did not completely ignore them because I did not want to risk offending them.

Observing for a prolonged period of time gave me the opportunity to gather data from which I could better understand the progression of students' conceptual and procedural knowledge of fractions as they used manipulatives. Another benefit of observing for a prolonged period of time was having the opportunity to gather data about how students reacted when using manipulatives while learning about the different operations on fractions: addition, subtraction, multiplication, and division. Finally, observing for a prolonged period of time also provided an opportunity for me to observe students using multiple types of concrete and virtual manipulatives, including but not limited to fraction bars, fraction circles, colored length rods, and pattern blocks.

I observed class five times during the fractions unit: November 4, 13, 18, and 20, and December 2. During observations, I took field notes of what transpired in class. I attempted to transcribe participants' actual wording without changing the words. However, in my notes, I inserted clarifiers of my perception about participants' responses. In my notes, I included drawings of the classroom setting (Bernard & Ryan, 2010) and what I observed as participants were using manipulatives, as well as how they interacted with one another and the instructor. During each observation, I attempted to seat myself so that I was not a distraction. However, by the nature of being a researcher and not one of their classmates, I may have inadvertently affected the environment even though I attempted to minimize my affect. In addition to taking field notes, I also kept a reflective journal. I wrote my reflective thoughts regarding observations as soon as possible after the observations, so that I documented my initial analysis as accurately as possible, as well as to minimize the amount of data lost due to memory loss (Bernard & Ryan, 2010).

I observed the following classroom activities in EDMG 1:

- Students working in pairs using manipulatives to explore fraction multiplication and division;
- Students presenting solutions to their peers pictorially and using manipulatives;
- The instructor facilitating classroom discourse;
- Students using pattern blocks during a fractions quiz;
- Students generating conjectures about the product of two fractions based on the size of the multiplier and multiplicand; and
- Students discussing how their class activities attended to the Standards for Mathematical Practice.

Assessment data. I collected assessment data for each participant to attempt to triangulate data (Berg, 2009), as well as to gain more insight of my participants. The three types of data were a math autobiography, a fractions quiz, and a problem report.

For three out of four participants, I collected a math autobiography from Ms. Paige that students wrote as an assignment. Ms. Paige could not locate a copy of Samantha's autobiography. Although I contacted Samantha multiple times by email, I was still not able to get a copy of her autobiography. In lieu of Samantha's math autobiography, I compiled information about Samantha that I collected through observations, interactions, and surveys, as well as from observations made by Ms. Paige. Students' math autobiography could include information about: mathematics courses that had an influence on the writer; memorable experiences in learning mathematics; attitudes toward mathematics; how experiences have influenced the writer's learning of mathematics and views about teaching mathematics; and why the writer wants to be a mathematics teacher.

The content of the fractions quiz was the meaning of fractions and fraction comparison schemas. Five of the problems required students to determine which fraction was larger or smaller based on fraction comparison schemas taught during the course. One problem on the fractions quiz required students to determine the value of a variety of pattern blocks based on a particular pattern block being assigned a value of one. Finally, students were to provide information about the meaning of the numerator and denominator of a fraction.

The problem report involved fraction multiplication. The problem report was an out-of-class homework assignment based on an in-class activity. The in-class activity involved students using manipulatives to model fraction multiplication. Problems included combinations of fractional, whole number, and mixed number multipliers and multiplicands. There was also a section that required students to reflect on their experiences.

Post-instruction procedural and conceptual knowledge tests. The Procedural and Conceptual Post-Tests were administered to each participant at the conclusion of the semester after each participated in the Post-Instruction Task-based Interview. The Post-Tests were administered only to the participants in my study.

The purpose of the post-instruction tests was to determine how students' conceptual and procedural knowledge of fractions changed. However, I was not focused on statistical significance of students' increase in knowledge since there were too few data to consider the tests a representative sample of a population. My interest was in the development of the participants' conceptual and procedural knowledge of fractions as a result of their experiences using manipulatives in their middle grades methods course. Therefore, I analyzed qualitative differences in students' understanding as demonstrated on the tests.

Semi-structured interview. At the end of the course, I conducted a semi-structured, individual interview with each participant to gather affective data on his/her beliefs regarding the development of their relational understanding of fractions, their use of manipulatives during their methods course, their anticipated use (or non-use) of manipulatives during their future classroom practice, and their anticipated use (or non-use) of best practices exemplified by the instructor.

I developed my interview protocol based on my literature review (Appendix K). After I constructed the interview protocol, a colleague familiar with qualitative research critiqued my protocol for alignment with my research question and for the potential my questions had to elicit useable data. Based on her feedback, I revised my protocol to what it is now.

Kvale and Brinkmann (2009) defined an interview as a conversation that has a structure and a purpose. Kvale and Brinkmann (2009) stated, "The qualitative research interview attempts to understand the world from the subjects' points of view, to unfold the meaning of their experiences, to uncover their lived world prior to scientific explanations" (p. 1). Kvale and Brinkmann (2009) indicated that the purpose of interviewing is to produce knowledge. They further stated that the research interview is a professional conversation "where knowledge is constructed in the inter-action between the interviewer and the interviewee" (p. 2).

Kvale and Brinkmann (2009) expounded on 10 intricacies of semi-structured research interviews: qualitative, descriptive, specificity, deliberate naiveté, focused, ambiguity, change, sensitivity, interpersonal situation, and positive experience. I will now explain how these apply to my research interview. “The qualitative interview seeks qualitative knowledge as expressed in normal language; it does not aim at quantification” (Kvale & Brinkmann, 2009, p. 30). Precision in description, use of language, and stringency in interpreting meaning are important for validity of research interviews. As I transcribed, analyzed, and reported on data from interviews, I used exact quotes from interviewees. I encouraged participants to describe as precisely as possible what they experienced and how they felt. For specificity, I attempted to word questions in ways that encouraged participants to give specific examples about what I was asking. For deliberate naiveté, although I have biases toward manipulatives, relational understanding, and fractions, I attempted to be curious and sensitive to participants’ views and obtain descriptions that were inclusive of the participants’ views and not only of the views that we have in common. I also included prompts about prospective teachers’ anticipated use of best practices in their future instruction to middle grades students.

Kvale and Brinkmann (2009) stated that interviews should be focused on themes and lead participants to discuss the themes of the interview, but not lead participants to specific opinions of the themes. Kvale and Brinkmann (2009) use the term “presuppositionlessness” for this. Because of my strong opinions about my research, I made a concerted effort not to lead participants to share in my opinions. With respect to ambiguity, Kvale and Brinkmann (2009) suggested that interviewees’ answers are sometimes ambiguous. However, it is the interviewers’ responsibility to clarify whether the ambiguity is a result of a failure to communicate in the interview setting or whether it is reflective of genuine inconsistences and contradictions in the interviewees' situation.

Although I was not attempting to affect the participants' views, the interview may have caused participants to discover new aspects of what they were describing and see relationships that they were not aware of previously. With respect to sensitivity, Kvale and Brinkmann (2009) indicated that different interviewers may get different results from interviewing. To minimize this, they suggested having a standardized interview form and keeping a presuppositionless attitude. To address these, I developed an interview protocol to follow and asked all participants each question (Appendix K).

As stated earlier, an interview produces knowledge as a result of the interactions between the interviewer and interviewee. With respect to interpersonal situation, Kvale and Brinkmann (2009) indicated that an interview can cause anxiety, as well as evoke defense mechanisms in either party. Therefore, I attempted to be sensitive to the participants' feelings and told each that we could end the interview at any time if he/she needed to do so. Because knowledge generated from an interview is the result of who is participating in the interview, an interview conducted by another interviewer may not produce the same knowledge.

Data Analysis Procedures

"Analysis is the search for patterns in data and for ideas that help explain why those patterns are there" (Bernard & Ryan, 2010, p. 109). Furthermore, data analysis involves interpreting the patterns and linking those findings to other research (Bernard & Ryan, 2010). Bernard and Ryan (2010) suggested that researchers begin with a small chunk of text and code it line-by-line. They suggested researchers then use the constant comparative method, followed by axial coding (Strauss & Corbin, 1998). Huberman and Miles (1996) stated that the constant comparative method involves examining how a sentence is similar to or different from the subsequent sentence. Bernard and Ryan (2010) also suggested that researchers keep running notes about the concepts they identify and include hypotheses about how the concepts may be related.

Bernard and Ryan (2010) suggested eight observational techniques to discover themes within data: repetitions; indigenous typologies or categories; metaphors and analogies; transitions; similarities and differences; linguistic connectors; missing data; and theory-related material. Additionally, Bernard and Ryan (2010) suggested four manipulative techniques to process texts: cutting and sorting; word lists and key-words-in-context; word co-occurrence; and metacoding. Bernard and Ryan (2010) suggested applying several techniques until no new themes are discovered.

To analyze my data, I used a priori codes taken from literature and read line-by-line for evidence of conceptual and procedural knowledge in participants' responses. As I analyzed my data, I looked for repetition; linguistic relationships between mathematical meaning and how participants' associated language revealed their understanding of the mathematics; and evidence of mathematical knowledge based on the definitions of mathematical knowledge found in literature. For example, when I analyzed fraction multiplication problems that were in the form $\frac{a}{b} \times c$, where *a*, *b*, and *c* were Natural numbers, I used Lesh, Post, Behr, and Silver's (1983) conception of fraction as an operator. Applying Lesh et al.'s (1983) conception of fraction as an operator on the problem $\frac{5}{8}$ of 2 means that the multiplicand 2 is partitioned into 8 equal parts and the value of 5 of the 8 equal parts is determined. A pictorial representation using a region model for this problem is shown in Figure 9. The symbolic representations that are aligned with this pictorial representation involve the traditional algorithm known as cross simplification and are represented as $\frac{5}{8}\times\frac{2}{1}=\frac{5}{4}\times\frac{1}{1}$, where a common factor of 2 has been divided from the denominator of $\frac{5}{8}$ and the numerator of $\frac{2}{1}$.

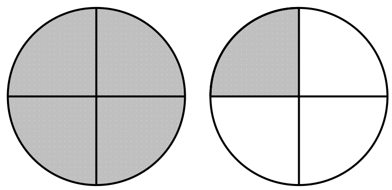


Figure 9. Region model representation for $\frac{5}{8}$ of 2

I analyzed participants' use of manipulatives using a coding scheme adapted from Osana and Royea (2011) in their work with prospective elementary teachers. A summary of Osana and Royea's (2011) use of the terms Meaningful and Procedural for coding participants' drawings of fraction problems is shown in Table 12 and continued on the following page. Then, Table 13, also on the following page, shows the coding schemas used for the participants' use of manipulatives and pictorial representations in this study.

Table 12

Summary of Osana and Royea's (2011) Coding Scheme

Coding Schemas for Participants' Use of Pictorial Representations	
Meaningful Use	Drawings based on intuitive understanding of fractions and the quantities expressed in the problem

Procedural Use	Drawings that were replications of a known standard procedure or were learned by rote from previous instruction If participants could not explain why they solved the problem using a particular picture or strategy

Table 13

Adapted Coding Schemas

Use of Manipulatives or Pictorial Representations	
Meaningful Use	Use of manipulatives or pictorial representations based on conceptual meaning of fractions
Procedural Use	Use of manipulatives or pictorial representations not based on conceptual meaning of fractions

In addition, I looked for evidence of students' conceptual knowledge of fractions using Lesh, Post, Behr, and Silver's (1983) definitions of fraction as part-whole, ratio, division, operator, and measure. I also looked for evidence of students' conceptual knowledge using Confrey et al.'s (2009) conceptions of equipartitioning as recursive partitioning, fraction iteration, and unit fractions. I looked for evidence of students' conceptual understanding of fractions using Lo and Lou's (2012) definitions of multiplication, partitive division, and measurement division. Finally, I looked for evidence of interference of prior instrumental learning on subsequent relational learning (Pesek & Kirshner, 2000).

With respect to validity, Bernard and Ryan (2010) contended that there is no ultimate demonstration of validity. That is, "the validity of a concept depends on the utility of the device that measures it and on the collective judgment of the scientific community that a construct and its measure are valid" (Bernard & Ryan, 2010, p. 72). However, reliability refers to agreement among coders and across methods and across studies (Bernard & Ryan, 2010). Strong inter-rater reliability increases the likelihood that a discovered theme is valid.

To establish reliability for scoring of the tests, two colleagues graded a sample of the procedural and conceptual tests. On samples of the procedural test, scores assigned among the three scorers only varied by one point overall. On the samples of the conceptual tests, scores assigned among the three scorers varied by an average of four points. To establish reliability for coding of text, I asked a colleague to code portions of text from the task-based interviews with participants.

Bernard and Ryan (2010) also suggested that researchers perform constant validity checks. That is, they advised researchers to watch for disagreements among knowledgeable informants and try to discover why; check for informant accuracy whenever possible; welcome negative evidence and determine the result of it; look for alternative explanations for phenomena; and try to fit negative cases into one's theory. In an effort to perform constant validity checks, I consulted with Ms. Paige regarding questions related to my participants' data.

Procedures for scoring tests. Ms. Paige and I developed the rubric shown below for grading the procedural knowledge test.

2 – Correct algorithm with correct answer
1 – Correct algorithm with some minor calculation errors
0 – No answer or incorrect procedure with incorrect answer

For the procedural test, the highest possible score was 34 points (17 problems). The range 0-26 represented low procedural, and 27-34 represented high procedural. These ranges were based on approximately 80% proficiency.

When analyzing prospective teachers' responses on the conceptual knowledge test, I used a coding scheme adapted from Forrester and Chinnappan (2010). The coding scheme (rubric) is shown below:

3 – Conceptual representation with clear reasoning and correct explanation
2 – Conceptual representation with unclear reasoning and partially correct explanation
1 – Representation provided with no evidence of conceptual knowledge
0 – No representation

Based on the rubric, the highest possible score on the conceptual test was 51 points (17 problems). For the conceptual test, the ranges were: 0-25 for low conceptual knowledge and 26-51 for high conceptual knowledge. I based the range of 0-25 on a person receiving more ones than twos (i.e., 9 ones and 8 twos). This range of scores also aligns with a person making less than 50% on the test. Since I only established low and high, the other range was 26-51 because it covers the remainder of possible scores.

Summary of Methods

The qualitative research methods and data collection methods were chosen for this study because of the literature I reviewed and their potential to provide data that would help me answer my research question, as well as based on my own prior experiences. The theoretical underpinning for this study, constructivism, guided my data collection and analyses, as well as the knowledge I gained as I conducted the study. Instruments were refined over the course of the study based on feedback I received from various university personnel I asked to review the instruments in light of my research question. Finally, the conceptual framework that guided this study was rooted in the literature associated with the content of this study.

Chapter 4: Research Findings

Introduction

In this chapter, data collected for the four participants in this study are reported. The participants were seniors enrolled in the first of two required math methods courses prior to student teaching. Each participant was a middle grades education major. At the university, all middle grades education majors were required to choose two disciplines in which to major. Three of the four participants were math/science majors, while the fourth participant was a math/English Language Arts major. Two of the participants were female, two were male. The four participants are referred to by the pseudonyms Samantha, Krystal, Jacob, and Matthew, and are reported on respectively.

Sample class activities are provided in Appendix P. Class activities during the fractions unit included:

- Examining multiplication with an emphasis on the meaning of the multiplier and multiplicand;
- Examining partitive and measurement division;
- Investigating fraction magnitude and equivalence using on-line applets and recursive partitioning;
- Comparing two fractions by examining the relative sizes of fractions instead of procedural algorithms;
- Adding fractions using fraction strips, decomposition of fractions, and pattern blocks;
- Multiplying fractions using length, area, and region models and the distributive property;

- Using a fraction as an operator;
- Generating conjectures about the product and quotient based on the size of the multiplier relative to one and divisor relative to the dividend (respectively);
- Dividing fractions using length and region models;
- Recursive partitioning using length and area models;
- Analyzing tasks based on the level of cognitive demand;
- Conducting discussions regarding the Common Core Standards for Mathematical Practice (NGA Center & CCSSO, 2010) and the link to the Process Standards from NCTM (2000).

Data were collected about each participant's knowledge of fractions through the following instruments:

- Procedural Pre-Test (Appendix A),
- Conceptual Pre-Test (Appendix E),
- Initial Attitudinal Survey (Appendix J),
- Pre-Instruction Task-based Interview (Appendix H),
- Classroom Observations,
- Assessments given by the EDMG 1 instructor,
- Procedural Post-Test (Appendix A),
- Conceptual Post-Test (Appendix E),
- Post-Instruction Task-based Interview (Appendix H), and
- Semi-Structured Interview (Appendix K).

In this chapter, I present data for each participant that helped to answer my research question. When appropriate, I included direct quotes from each participant. I did not change misspelled words because I believed that this information helps the reader have a better understanding of each participant's overall knowledge base. However, I removed filler words in order to make the text easier to read (e.g., *like*, *to where*, and *you know*). Additionally, when appropriate, I included images of actual student work.

Summary of the instruments. The problems on the Procedural Tests included operations on whole numbers, fractions, and decimals; fraction equivalence; and fraction comparison. All problems on the Procedural Tests were middle grades problems. For each problem on the Procedural Tests, there was a related problem on the Conceptual Test. The only problem that was numerically different was the decimal multiplication problem. It was different on the Procedural and Conceptual Tests because of the concern that students might remember the answer and not provide justification.

The Initial Survey asked students to rate their knowledge of and attitudes toward mathematics and manipulatives. There was also a prompt asking participants to share additional information as they wished.

The Individual Task-based Interviews involved interacting with participants in a one-on-one setting using manipulatives to solve fraction tasks. A variety of manipulatives were available for participants to use, if they chose to do so (See Appendix L). The problems given during the Task-based Interviews were chosen based on related literature about fractions.

Five classroom observations were done beginning November 4, 2014, and ending December 2, 2014. During observations, I made notes about what participants were doing and saying. As much as was possible, I recorded what the participants said instead of my interpretations. After each observation, I wrote a reflection on my initial interpretation of what I observed. I varied my seating in the room from one observation to the next as seating was available in the room.

The various assessment documents I collected for the participants included a math autobiography (except for Samantha), a fractions quiz, and a problem report. The problem report was a homework assignment that followed a classroom activity. Essentially, students were to restate in their own words and provide a picture for tasks that were done in a previous classroom activity.

Finally, the Semi-Structured Interview, conducted at the end of the semester, involved questioning participants in a one-on-one setting about their experiences in the course. There were questions about the participants' perception of their understanding, instructional practices of the instructor, their attitudes toward manipulatives, and specific experiences that occurred during the course.

Samantha

During the fall of 2014, Samantha, a White American female in her mid-twenties, participated in this study while she was a student in EDMG 1. Samantha, originally from the Northeast, was a non-traditional student enrolled in the middle grades education program at a mid-sized, four-year university in the Southeast. Samantha's declared areas of specialty were mathematics and science. In addition to attending classes at the university, Samantha worked to support herself and her daughter.

Beginning in third grade, Samantha was placed in advanced math classes (Initial Survey, Appendix J). Early in her education, Samantha's teachers "were hands on" and allowed her to "ask many questions" (Initial Survey, Appendix J). Samantha remembered using manipulatives for algebraic equations in her fifth-grade program for high achieving students. On the contrary, Samantha stated, “Not many manipulatives were used in [her] secondary education” (Initial Survey, Appendix J). According to Samantha, she “struggled in college math” because “teaching styles were very different and relied on a lot of independent learning” (Initial Survey, Appendix J). Samantha stated that she experienced difficulty because “if [she] did not understand something, [she] had no one to walk [her] through the steps” (Initial Survey, Appendix J).

At the beginning of the study, Samantha rated herself as somewhat confident in her procedural ability in the operations on fractions and very confident in her understanding of the meaning behind the operations on fractions (Initial Survey, Appendix J). Samantha also rated herself as somewhat confident in understanding the ‘why’ behind the conventional algorithms for fractions. Although Samantha indicated that she was only somewhat familiar with manipulatives during her prior learning experiences, she felt that it was very important for middle grades students to use manipulatives to learn mathematics (Initial Survey).

Samantha demonstrated an increase on her Conceptual Tests from Pre-Test to Post-Test, but demonstrated a decrease on her Procedural Tests from Pre-Test to Post-Test. Table 14, shown below, conveys her scores on all four tests.

Table 14

Samantha’s Procedural and Conceptual Knowledge

	Pre-Test	Post-Test	Change
Procedural	82.4%	79.4%	– 2.9%
Conceptual	82.4%	92.2%	+ 9.8%

Fraction multiplication. Data regarding Samantha's procedural and conceptual knowledge of fraction multiplication were collected through Pre- & Post-Instruction Tests, Pre- & Post-Instruction Task-based Interviews, assessment, and observational data. In the next sections, these data are expounded upon.

Pre-instruction data. The problems that best demonstrate Samantha's initial procedural and conceptual knowledge of fraction multiplication were $\frac{1}{2}\times\frac{1}{3}$, $\frac{5}{8}$ of 2, and $2\frac{1}{6}\times 3\frac{3}{5}$. For $\frac{1}{2}\times\frac{1}{3}$, Samantha's pictorial representation did not show clear evidence of conceptual knowledge of a fractional multiplier, i.e, fraction as an operator (Lesh, Post, Behr, & Silver, 1983). Based on Samantha's work on the Conceptual Pre-Test, shown in Figure 10, it was not clear whether Samantha knew that $\frac{1}{2}\times\frac{1}{3}$ means to take one-half *of* one-third.

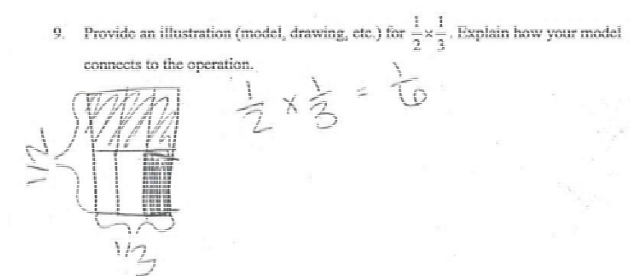


Figure 10. Samantha's pictorial representation for $\frac{1}{2}\times\frac{1}{3}$

Similarly, Samantha's pictorial representations for $\frac{5}{8}$ of 2 on both Pre-Tests, shown in Figure 11 on the following page, appeared to be aligned with $2\times\frac{5}{8}$, or two groups of $\frac{5}{8}$.

Procedural Pre-Test Conceptual Pre-Test

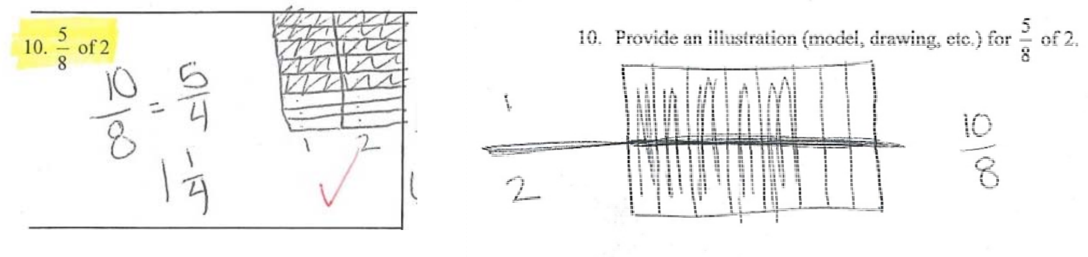


Figure 11. Samantha’s representations for $\frac{5}{8}$ of 2

For the mixed number multiplication problems, Samantha did not correctly calculate $2\frac{1}{6}\times 3\frac{3}{5}$, nor did she identify errors in $2\frac{1}{6}\times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$. On both Pre-Tests, Samantha appeared to obtain a common denominator to multiply mixed numbers. Some of Samantha’s work on the Procedural Pre-Test, shown in Figure 12, appeared to be somewhat aligned with the incorrect procedure provided on the Conceptual Pre-Test, $2\frac{1}{6}\times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$.

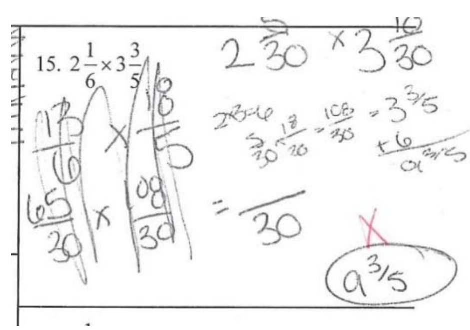


Figure 12. Samantha’s procedural knowledge of mixed number multiplication

During Samantha’s Pre-Instruction Task-based Interview, she was asked to demonstrate $\frac{3}{4}\times 6$ using manipulatives. Initially, Samantha considered using the yellow hexagon as the whole. However, Samantha decided to use the pink double hexagon because she stated that she did not want to use two different pattern block pieces to represent three-fourths, the red isosceles trapezoid for one-half and the brown right trapezoid for one-fourth. To model $\frac{3}{4}\times 6$, Samantha used the pink double hexagon as the whole and the red isosceles trapezoid as one-fourth (Appendix L). Samantha created six sets of three red isosceles trapezoids, shown in Figure 13, which seemed to represent $6\times\frac{3}{4}$ instead of $\frac{3}{4}\times 6$. When Samantha was asked to clarify whether she demonstrated $\frac{3}{4}\times 6$ or $6\times\frac{3}{4}$, she said, “Three-fourths times six because my group size was three-fourths, and I had six groups of three-fourths.”

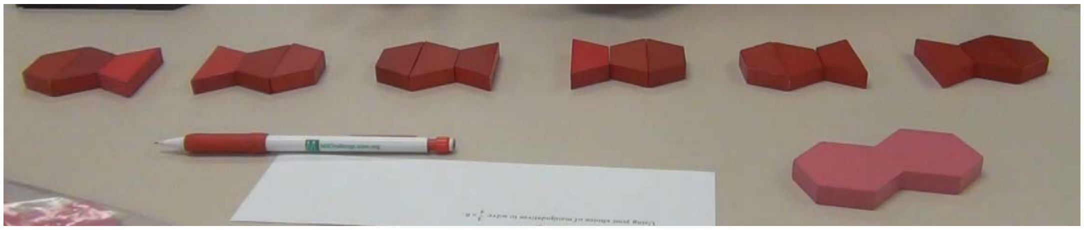


Figure 13. Samantha used pattern blocks to model 6 groups of $\frac{3}{4}$

Samantha made statements that provided insight into her conceptual knowledge. Initially, Samantha stated that the multiplier was “the action of what [is done] to the multiplicand.” Then, she stated that “the multiplicand is supposed to [be] representative of how many groups there are.” Samantha also said, “If I were to do six times three-fourths or three-fourths times six, it would still be the same value, but it would have a different meaning.”

After Samantha determined the answer to be $4\frac{1}{2}$, she expressed confidence in her answer but offered to "think about it in reverse." To do so, Samantha had "to think what one-fourth of six would be," which was language associated with using a fraction as an operator (Lesh, Post, Behr, & Silver, 1983). Samantha first expressed the desire to use a calculator. Since a calculator was not provided, she used mental calculations to determine that one-fourth of six was one and one-half. Then, to calculate $\frac{3}{4} \times 6$, Samantha stated, "I'm multiplying one and one-half times three, yeah, four and a half." After Samantha performed the calculations mentally, she provided pictorial and symbolic representations (shown in Figure 14) for her calculations.

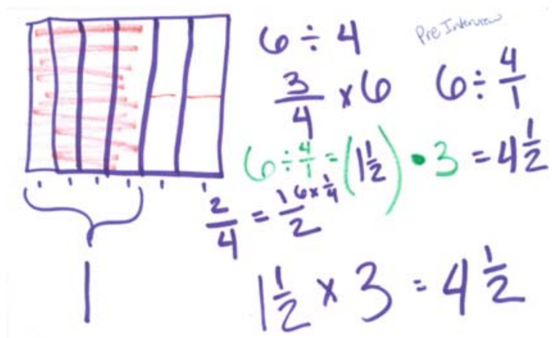


Figure 14. Samantha's work for calculating three-fourths of six

After Samantha demonstrated fraction multiplication, she was asked to generate a real-world scenario for $\frac{3}{4} \times 6$. Samantha stated, "Let's say I have seventy-five cents. Each one of these [red isosceles trapezoids] is a quarter. But, there is six kids. How much money do we have all together?" To provide clarification, Samantha continued, "So, if each kid has seventy-five cents or three quarters, which is three-fourths because it's four quarters to the dollar, how much money do they have collectively?" Samantha concluded by providing the answer of four and a half dollars and saying that she used money because it is relative to everyone. Prior to the end of the interview, Samantha generated another real-world scenario for fraction multiplication and provided an answer. She stated,

> I'm doing a bake sale. Instead of making my normal one cake, I want to make six cakes. If it takes three-fourths cups of sugar for one cake, but I'm making six cakes, so I'm now going to have three-fourths cups of sugar times six cakes, how much sugar will that be for the total amount of cakes? Four and a half cups of sugar.

Samantha's pre-instruction data seems to point her flexibility in using concrete, pictorial, and symbolic representations, as well as in her ability to generate real-world scenarios. However, Samantha's use of the *multiplier* and *multiplicand* in a fraction multiplication problem did not appear to be aligned with the conceptual meaning of those terms.

Observational and assessment data. After Samantha participated in the initial task-based interviews, some assessment and observational data were collected pertaining to her progressing conceptual knowledge of fraction multiplication. However, there was limited information available through observation and assessment data about how her procedural knowledge of fraction multiplication progressed.

During a class discussion about fraction multiplication, Samantha voiced a general statement that seemed to be aligned with conceptual knowledge of using a fraction as an operator. That is, a student in the class stated that the fraction multiplication problem, $\frac{1}{2} \times -\frac{3}{4}$, was representative of "taking one part of the multiplicand." Samantha followed this statement with, "You are taking a part of a part," seemingly to generalize multiplying by a proper fraction.

During a class discussion, the instructor asked students to generate a real-world scenario for $16 \times \frac{3}{4}$. Samantha responded, "We're going on a road trip. It's going to be 16 days. We've traveled three-fourths of the time so far. How many days have we traveled?" However, Samantha's scenario seems to be conceptually aligned with $\frac{3}{4} \times 16$. The instructor, Ms. Paige, drew a rectangle on the board and partitioned it into fourths. The class discussed dividing out a common factor of 4 from the 4 and the 16 to simplify the problem to 3×4. Ms. Paige then asked the class to generalize dividing out a common factor. Samantha explained that $\frac{a}{b} \times bc$ represented a generalization for being able to divide a common factor from the denominator of the multiplier and the numerator of the multiplicand.

During fraction multiplication instruction, the class was asked to use manipulatives to model the problem $\frac{1}{2} \times \frac{2}{3}$. Samantha began with two orange one-third pieces; then, she placed a light blue one-sixth piece on each orange one-third. Next, Samantha put the two one-sixth pieces together, compared them to the one-third, and stated that they were the same. Samantha's use of the manipulatives seemed to be aligned with the symbolic representation of $\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right)=\frac{1}{6}+\frac{1}{6}$, which uses the distributive property.

The manner in which Samantha modeled fraction multiplication for $\frac{1}{2} \times \frac{2}{3}$ was different from what she did on her Problem Report for $\frac{2}{3} \times \frac{3}{4}$. The Problem Report was an out-of-class assignment that was based on an in-class activity. In her Problem Report, Samantha noted that $\frac{3}{4}$ needed to be "divided into three equal parts," which was "simple because it [was] already partitioned into three equal pieces of $\frac{1}{4}$."

An area in which Samantha seemed to have made progress during the fractions unit was connecting the meaning of fraction multiplication to the manipulatives. The instructor asked students to model $1\frac{1}{2} \times 1\frac{1}{2}$ as $\frac{3}{2} \times \frac{3}{2}$ because another student questioned whether the concrete representation would be the same even though the symbolic representations were different. Samantha stated that she needed "one group and half of a group." Using fraction circles, Samantha began with one red whole and one pink half to represent one group of $1\frac{1}{2}$. Then, she used one pink half and one yellow fourth to represent half of a group of $1\frac{1}{2}$. Samantha's representation did not appear to be aligned with $\frac{3}{2} \times \frac{3}{2}$. In contrast to Samantha's representation, Jacob, another student in EDMG 1 who participated in this study, demonstrated $\frac{3}{2} \times \frac{3}{2}$ using fraction circles by beginning with three pink halves. Then, Jacob partitioned three-halves into two equal groups that each contained one pink half and one yellow fourth. Next, Jacob assembled three groups of one pink half and one yellow fourth to represent $\frac{3}{2} \times \frac{3}{2}$. Once Jacob finished his demonstration, Samantha said, “I like that we can switch the representation. I never really thought about that.”

There were two fraction multiplication problems on Samantha’s Problem Report that were somewhat related to problems on the Pre-Tests, $\frac{2}{3} \times 2$ and $1\frac{1}{4} \times 2$. For $\frac{2}{3} \times 2$, Samantha used a number line model and fraction circles shown in Figure 15. It seemed that Samantha used the idea of fraction as an operator on the number line model. For the circle model, Samantha’s pictorial representation seems to be more aligned with $2 \times \frac{2}{3}$, or two groups of two-thirds.

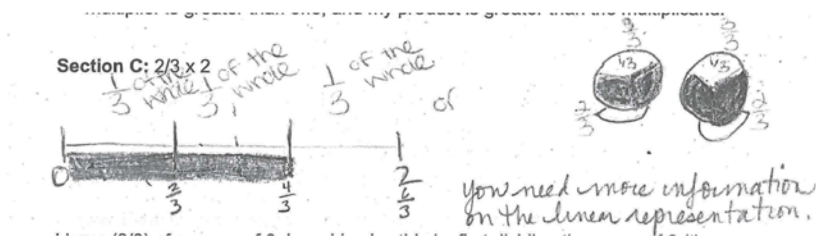


Figure 15. Samantha’s representations for $\frac{2}{3} \times 2$

The explanation Samantha provided about $\frac{2}{3} \times 2$ was conceptually aligned with using a fraction as an operator. She wrote,

> I would solve this first by dividing the group of 2 (the multiplicand) into three equal parts. I am dividing the multiplicand into three because it is the denominator of the fraction in the multiplier. The denominator represents how many parts my whole is partitioned into. In this case, the whole is 2

(the multiplicand). Out of the equal parts, I would only have 2 parts. I only have 2 parts of the three because the numerator of the fraction in the multiplier represents the parts of the whole that I have. The answer is $\frac{4}{3}$. This makes sense because the multiplier was less than one, and my product was less than the multiplicand.

For $1\frac{1}{4}\times 2$, Samantha first stated that she "decomposed the multiplier to be $\left(1+\frac{1}{4}\right)$." Then, she noted that she was "solving for $(1\times 2)+\left(\frac{1}{4}\times 2\right)$." Next, Samantha "had one whole group of two so it would be $2+\left(\frac{1}{4}\times 2\right)$." Samantha noted, "2 divided by 4 is $\frac{1}{2}$. Each part is $\frac{1}{2}$, and I only have the one part. I then have $2+\frac{1}{4}=2\frac{1}{4}$ as the product." Samantha's pictorial representation for this problem is shown in Figure 16.

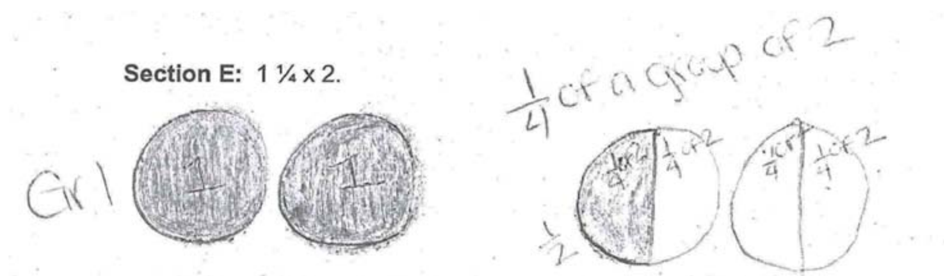


Figure 16. Samantha pictorially represented $1\frac{1}{4}\times 2$

In the reflective part of the Problem Report, Samantha made several statements about how completing the assignment helped her. She stated, "By using manipulatives, my understanding of mathematics was deepened because it gave me the representation as to why the answer is the way it is. The visual helped me distinguish the steps of solving and provided justification as to why my answers make sense."

Another comment Samantha made in her reflection was, "Students need more than just the algorithm to understand what they are doing. This investigation allowed me to do the work behind the algorithm. It is important to use this approach in teaching mathematics to build connections and provide the meaning behind what they [students] are doing." Although Samantha mentioned 'the algorithm,' she did not mention which algorithm.

Observational and assessment data seemed to indicate that Samantha refined her use of the *multiplier* and *multiplicand* in fraction multiplication problems. She continued to demonstrate flexibility in using multiple representations for fraction multiplication, including length and region models.

Post-instruction data. At the conclusion of the semester-long course, Samantha completed the Procedural and Conceptual Post-Tests and participated in two interviews, the Post-Instruction Task-based Interview and the Semi-Structured Interview. Samantha's Procedural Post-Test score was 79.4%, a drop of approximately 3%, while her Conceptual Post-Test score was 92.2%, an increase of 9.8%.

Although Samantha improved her pictorial representation for $\frac{1}{2}\times\frac{1}{3}$ from Pre-Test to Post-Test, she did not provide a precise enough representation to earn full credit. In contrast to her Pre-Test, Samantha communicated that $\frac{1}{2}\times\frac{1}{3}$ meant "to take $\frac{1}{2}$ of $\frac{1}{3}$." However, Samantha did not partition her circle equally into sixths to fully communicate her answer of $\frac{1}{6}$, as shown on the following page in Figure 17.

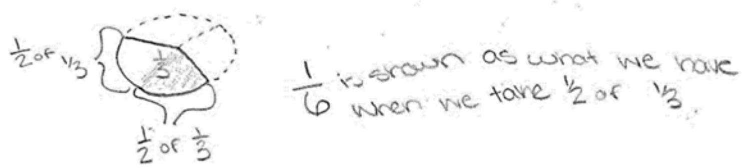


Figure 17. Samantha's representation for $\frac{1}{2} \times \frac{1}{3}$

On the Procedural Post-Test, Samantha provided a correct answer of $1\frac{1}{4}$ for $\frac{5}{8}$ of 2, but did not provide a pictorial representation as she did on the Procedural Pre-Test, nor did she show any work. On the Conceptual Post-Test, it seemed that Samantha attempted to use a fraction as an operator for $\frac{5}{8}$ of 2, but she did not clearly communicate her answer, as shown in Figure 18.

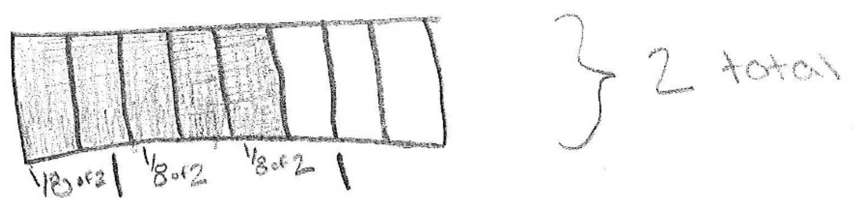


Figure 18. Samantha's representation for $\frac{5}{8}$ of 2

For $2\frac{1}{6} \times 3\frac{3}{5}$ on the Procedural Post-Test, Samantha used the multiplier as the number of groups and calculated the answer to be $7\frac{4}{5}$. On the Conceptual Post-Test, Samantha did not provide a complete description of the error in the work given, as shown in Figure 19 on the following page.

Procedural Post-Test Conceptual Post-Test

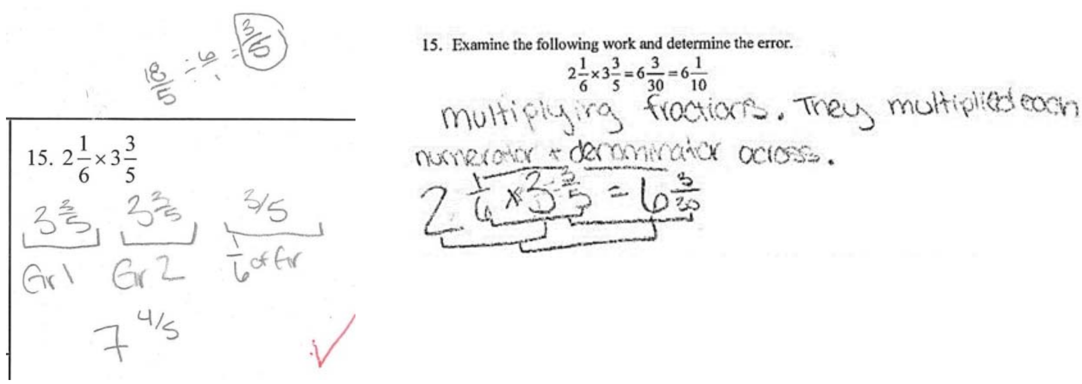


Figure 19. Samantha’s work for mixed number multiplication

During the Post-Instruction Task-based Interview, Samantha was asked to use manipulatives to model $\frac{2}{3} \times \frac{3}{4}$. One notable difference from her Pre-Instruction Interview was the fact that she began by estimating her answer. She stated, “My multiplier isn’t one, so I don’t have one whole group of the three-fourths.” When Samantha was asked whether she knew that before her methods course, she replied,

> Actually, no. I don’t think anyone ever broke it down. Now I have more reasoning into deciding the reasonableness of my answers. I liked how we broke that down and had a huge discussion. Now it’s something I look to before I even do anything. I’m like, ‘Alright, so my answer should be somewhere in this range.’

As Samantha explained $\frac{2}{3} \times \frac{3}{4}$, she used fraction circles to demonstrate her understanding, as shown in Figure 20 on the following page. Samantha explained,

> If I have three-fourths, I have 3 pieces. Now, with my multiplier, because it’s two-thirds, it’s not one whole one, so I don’t even have one whole

group of these [three-fourths]. But because it's a fraction, what I have to do is divide it by the numerator [sic. denominator] into three equal parts, which is perfect, 'cause three-fourths has three, one-fourth equal parts. And then, out of these, I have one-third of three-fourths, one-third of three-fourths, one-third of three-fourths. Out of that, I have two [moved two of the fourths away from the third and put them together] as determined by the numerator of the multiplier. So, my answer, which I knew was going to be less than three-fourths, I have two, one-fourths [points individually to each fourth that she placed together] which is equivalent to one-half [overlaid the pink one-half onto the yellow two-fourths].

Each one-fourth is one-third of three-fourths

Two one-fourths is equivalent to one-half

Figure 20. Samantha demonstrated knowledge of fraction as an operator

After Samantha finished explaining her work using the manipulatives, she was asked to explain how the manipulative connects to a conventional algorithm. Samantha said,

I've actually really enjoyed using the manipulatives 'cause it gives you the visualization as to how the pieces are related. It's just a model. That's how it connects. It's modeling that algorithm. Before you'd just multiply across, but I never knew what it was doing; I just knew what to do.

As Samantha continued, she said that she multiplied across. In other words, she multiplied the numerators, multiplied the denominators, and simplified the product. Specifically, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$, followed by $\frac{6}{12} = \frac{1}{2}$. Then, she added, "Now, the manipulative explains the story of the algorithm, showing me what is going on and why my answer is the answer it is." The algorithm that appeared to be aligned with the work Samantha did with the manipulatives was cross simplification. Samantha indicated that she would partition the three pieces into three separate groups, which meant that a common factor of three could be divided from the denominator of the multiplier and the numerator of the multiplicand. If the cross simplification algorithm were used, then the problem $\frac{2}{3} \times \frac{3}{4}$ would have symbolically transformed into $\frac{2}{1} \times \frac{1}{4}$.

After Samantha demonstrated $\frac{2}{3} \times \frac{3}{4}$, she generated a real-world scenario for this problem. Samantha stated that, since it was Christmastime, she planned to buy her daughter stocking stuffers. She said, "I only have three-fourths of the stocking [filled up]. I wanna give her two-thirds of it. So I need to figure out how much of a stocking should I fill it up on? Half."

Post-instruction data showed that Samantha refined her understanding of the meaning of the *multiplier* and *multiplicand* within the context of fraction multiplication. In addition, Samantha appeared to have refined her understanding of a fraction multiplier, i.e. fraction as an operator, as well as her conceptual knowledge of mixed number multiplication.

Summary of fraction multiplication. The assessment, observational, and interview data collected regarding Samantha's procedural and conceptual knowledge of fraction multiplication showed that Samantha may have lacked procedural knowledge of the traditional algorithm associated with mixed number multiplication, but she seemed to compensate by applying conceptual knowledge to calculate her answers. The data also showed that Samantha believed that the algorithm was evident in her use of manipulatives even though she never explicitly explained how it was evident. The data also indicated that Samantha used her conceptual knowledge of fraction multiplication more than her procedural knowledge. Finally, Samantha was able to generate real-world scenarios for fraction multiplication, although not all of them seemed aligned with the symbolic representation that Samantha believed she was representing.

Fraction division. Pre- and post-instruction data related to Samantha's procedural and conceptual knowledge of fraction division were collected and examined. Limited observational data were available related to fraction division. These data are reported in the next section.

Pre-instruction data. Although Samantha's answers on her Procedural Pre-Test were correct for all four fraction division problems, she showed limited work for these problems. For example, for $2\frac{1}{3} \div 3$, Samantha wrote $\frac{7}{3} \div \frac{3}{1} = \frac{7}{9}$. Because Samantha did not invert the divisor and change division to multiplication, it cannot be assumed that she did or did not know the standard algorithm associated with fraction division.

On the Conceptual Pre-Test, Samantha did not symbolically represent her answer for each division problem, as shown on the following page in Figure 21. Samantha's pictorial representations for the fraction division problems appear to be aligned with measurement division.

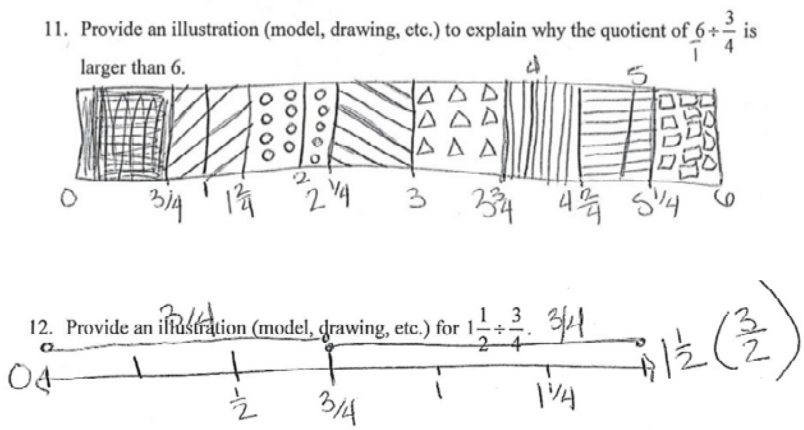


Figure 21. Samantha’s representations for fraction division

Between the times that Samantha completed the Pre-Tests and participated in the Pre-Instruction Task-based Interview, the EDMG 1 class learned about measurement division with whole numbers and integers, but not with fractions. Additionally, the class used pattern blocks during an activity that required students to determine the value of each of the pattern blocks in comparison to a specified pattern block that was assigned a value of one (Appendix L).

During the Pre-Instruction Task-based Interview, Samantha was asked to use manipulatives to represent $3 \div \frac{2}{3}$. Samantha used the yellow hexagon as one whole, the black chevron as two-thirds, and the blue rhombus as one-third. As Samantha modeled the division problem (shown in Figure 22 on the following page), she indicated that measurement division would be best and explained that dividing by two-thirds meant that she was taking out two-thirds to see how many groups of two-thirds [she had]. This explanation seemed aligned with Lo and Lou (2012) conception of the measurement interpretation of division. Another name for this model of division is the repeated subtraction model of division (Van de Walle, Karp, & Bay-Williams, 2013), which Samantha alluded to when she stated, “take away another two-thirds.”

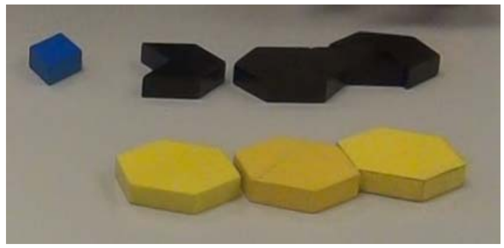


Figure 22. Samantha used pattern blocks to demonstrate $3 \div \frac{2}{3}$

Next, Samantha explained the answer and interpreted the remainder, which was represented by the blue rhombus. She explained:

> I have four ‘two-thirds’, but then I still have leftovers so that could either be the remainder, which is equivalent to one-third. So the four groups of two-thirds that I have…I have four plus, I guess one-half. It would be four and one-half because this [blue rhombus] is one-half of two-thirds. So two-thirds fits into three, four complete times and then one half time.

When she was asked whether she could confirm her answer procedurally, Samantha laughed and said, “With a calculator.” Samantha then followed up by saying, “I guess I could try.”

To calculate the answer to $3 \div \frac{2}{3}$, Samantha converted $3 \div \frac{2}{3}$ to $\frac{9}{3} \div \frac{2}{3}$ and divided the numerators and divided the denominators. Samantha explained,

> So with me having nine-thirds which is equivalent to three, and I’m dividing it by two-thirds, I can kind of just go across the top, I’m pretty

sure, because it's all the same parts. So two fits into nine four times but then I have ... Yeah, yeah, okay, sorry I got lost. So, I have four, but then I have…okay, yeah. So then I have, so if I subtract four, if I'm using like, I guess, I guess my model that means I have one-third after like, I guess, taking away the four, divided by two-thirds. Well one-third is half of two-thirds, so that's how I guess I got my one-half. I don't know if that's correct, but it makes sense to me.

When Samantha finished her explanation and was asked what made her question her answer, she responded, "Algebra rules, fraction rules, the rules. So I don't know if I'm following all the rules correctly I guess." When she was asked if she knew another way to calculate $3 \div \frac{2}{3}$, Samantha immediately responded, "With a calculator. My first thought was to convert this fraction [two-thirds] into a decimal, but it's a repeating decimal. So, it's hard to do, I guess, without…there's going to be slight error if I do a decimal by hand because I can't keep the repeating part of it." Samantha's reference to changing two-thirds to a decimal was language aligned with fraction as division (Lesh, Post, Behr, & Silver, 1983). In conclusion, Samantha said, "I feel confident in my answer. I'm not quite sure if I explained it properly or if I followed the correct procedure, but it makes sense to me."

Pre-instruction data indicated that Samantha's conceptual knowledge of the measurement interpretation of fraction division was consistent. Based on statements that Samantha made, it appeared that she was more confident in her conceptual knowledge of fraction division than her procedural knowledge.

Observational data. During fraction division instruction, the class was asked to generate a word problem for $\frac{1}{4} \div 4$. Samantha stated, “I have $\frac{1}{4}$ cup of bleach. I want to make 4 containers of sanitizer. How much bleach would be in each container of sanitizer?” The class did not discuss whether the scenario was aligned with the measurement interpretation of division or the partitive interpretation of division (Lo & Lou, 2012).

The class also explored and discussed a problem related to flower beds and fertilizer. The task prompted students to determine how many bags of fertilizer were necessary to cover five flower beds if each bag of fertilizer covered three-fourths of a flower bed. Symbolically, the problem was represented by $5 \div \frac{3}{4}$. During the class discussion, EDMG-1 students discussed the reason for inverting and multiplying as part of the conventional fraction division algorithm. However, Samantha never contributed to the discussion.

Limited observation and assessment data were available to examine how Samantha’s procedural and conceptual knowledge of fraction division progressed between the pre-tests and post-tests. With such limited information, it is not appropriate to make conclusions at this point about Samantha’s procedural and conceptual knowledge of fraction division.

Post-instruction data. Samantha correctly answered two of the four fraction division problems on the Procedural Post-Test, whereas she previously answered all four fraction division problems correctly on the Procedural Pre-Test. Similar to her Procedural Pre-Test, Samantha showed very limited work on her Procedural Post-Test. For example, with no work shown to indicate how she calculated her answer, Samantha provided the answer for $6 \div \frac{3}{4}$ as 4.5, which is in stark contrast to her answer on the Conceptual Post-Test shown in Figure 23.

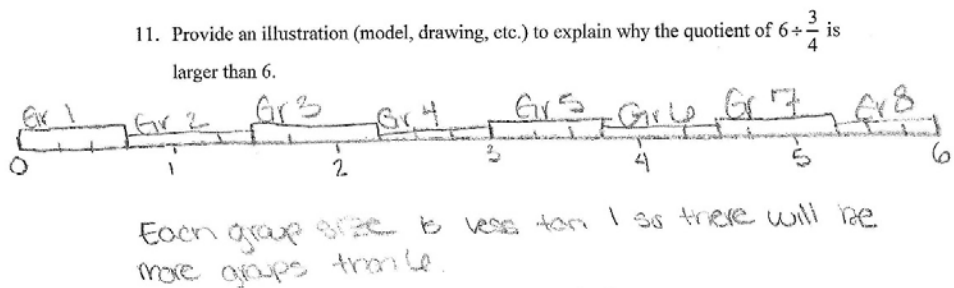


Figure 23. Samantha’s representation for $6 \div \frac{3}{4}$

For the problem $1\frac{1}{2} \div \frac{3}{4}$ on the Procedural Post-Test, Samantha answered “2” without showing work. On the Conceptual Post-Test for this problem, Samantha initially appeared to show fraction multiplication instead of division, but she marked out her work and generated another representation, which is shown in Figure 24.

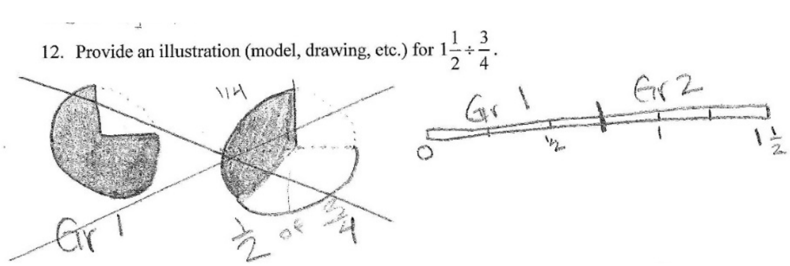
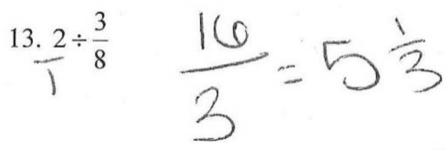


Figure 24. Samantha initially confused multiplication and division

The problem $2 \div \frac{3}{8}$ was the final noteworthy division problem on the Procedural Post-Test. Figure 25 shows the contrast between Samantha's work on the Pre-Test and Post-Test.

Procedural Pre-Test



Procedural Post-Test

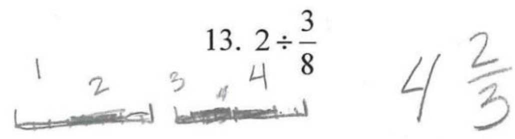


Figure 25. A comparison of Samantha's procedural knowledge of fraction division

On her Procedural Post-Test, it appeared that Samantha shaded $\frac{3}{8}$ five times. However, instead of numbering them as "1, 2, 3, 4, 5," Samantha numbered them "1, 2, 3, 4, 4." In addition, Samantha did not shade all of the eighths because there were $\frac{2}{8}$ left unshaded instead of $\frac{1}{8}$, as there should have been. Moreover, it was difficult to see whether there were actually $\frac{3}{8}$ shaded every time or whether she partitioned the second whole into nine parts instead of eight. The mistakes that she made and an understanding of measurement division helps one to make sense of her answer of $4\frac{2}{3}$. It is worth mentioning that, although Samantha did not answer $2 \div \frac{3}{8}$ correctly, she used the correct denominator for her answer. A common mistake for fraction division problems like this is to indicate that the answer is $\frac{2}{8}$ because each remaining piece is one-eighth of the original whole.

During the Post-Instruction Task-based Interview, Samantha was asked to use manipulatives to represent $\frac{5}{6} \div \frac{1}{2}$. Samantha began, “Okay. Um, so I’m gonna grab manipulatives in a minute, but already I know that my answer is going to be larger than one, ‘cause five-sixths is greater than one-half so I can definitely get one-half in there.” In response to being asked which type of division she was using, Samantha responded, “Partition? No, no, no, no, measurement model where the repeated subtraction ... where I’m taking each part out.” Her response in the post-instruction interview was similar to that of her pre-instruction interview.

Samantha used fraction circles to model her answer for $\frac{5}{6} \div \frac{1}{2}$ (shown in Figure 26 on the following page) as she explained:

> So, here’s my five-sixths. I need to divide it by one-half. I’m able to get one whole half in there, but then I have this part [the remaining two-sixths]. So, I’m gonna keep that [pink one-half] just to say it’s accounted for. And I have to figure out what part of this [remaining two-sixths] ... Or, not what part but like, what portion of the half this is. So, I have two-sixths left over, but it doesn’t make up a whole half. But, if I put one more one-sixth, I do have one whole half. So, since there’s three one-sixths [that] will make up the half, two ‘one-sixths’ out of the half is two-thirds. So, I’d say my answer would be one and two-thirds [points to the pink one-half as she says ‘one’ and points to the two-sixths as she says ‘two-thirds’].

Figure 26. Samantha's representation for $\frac{5}{6} \div \frac{1}{2}$ using fraction circles

When Samantha finished her explanation, she was asked whether she was confident in her answer. She responded,

> No, not necessarily. Um, I had a lot of difficulty with this in class 'cause changing the referent whole, like what I noticed that I kept doing um, was I kept kind of like, I guess doing multiplication, or I couldn't change ... I had difficulty changing the whole. And, but I'm gonna go with it.

Next, Samantha was asked to confirm her answer using her procedural knowledge. Samantha explained,

> Yeah. So, it'd be difficult for me to divide five into a half um, just 'cause 2.5 sixths is difficult to like, think about. So what I'm going to do is use an equivalent fraction, so I'm going to have ten-twelfths. To where if I could divide that in half, it would be five-twelfths (see Figure 27). Yeah. And even then I can't convert, honestly. I'm trying to think though. Two-thirds ... No, no, no, no, that's wrong. I don't know, I had a very difficult time doing the division part with the manipulatives.

$$\frac{5}{6} = \frac{10}{12} \overset{\div\frac{1}{2}}{=} \frac{5}{12}$$

Figure 27. Samantha's symbolic representation for $\frac{5}{6} \div \frac{1}{2}$

Samantha seemed to confuse *dividing by one-half* with *dividing in half*, which was a problem that Ma (1999) found common among US teachers in her study. Samantha's verbiage also varied as she explained. Samantha began by saying she was dividing *five into a half*; then, she stated she would *divide that in half*. Subsequently, she said, "I'm not quite sure. As far as dividing in half, um ... Oh, oh, and I'm doing it here, too. I keep multiplying by accident, and that's why I'm going wrong." It appeared by her statement that Samantha realized what she was doing wrong.

In conclusion, Samantha shared, "I liked what I did with the manipulatives because I can at least reason as to how I approached my answer. I know I definitely have at least one half. So this [procedural work] is wrong." Then, Samantha said, "I'm definitely more confident with manipulatives. That makes more sense as far as approaching my answer."

Post-instruction data showed that Samantha's procedural knowledge of fraction division was not consistent. These data also showed that Samantha's conceptual knowledge of fraction division was higher than her procedural knowledge. However, at times, Samantha seemed to confuse fraction multiplication with division. The data also showed that Samantha was generally able to resolve the confusion she experience by relying on her conceptual knowledge of fraction division.

Summary of fraction division. Throughout the study, Samantha consistently displayed conceptual knowledge of measurement division of fractions. On the contrary, Samantha's procedural knowledge of fraction division was not as consistent. Even though, at times, Samantha experienced some confusion related to her conceptual knowledge of fraction division, she was generally able to overcome the confusion. Based on observation and interview data, it is possible that Samantha experienced interference from her lack of procedural knowledge (Pesek & Kirschner, 2000).

Many of the times Samantha explained fraction division using manipulatives, she articulated her understanding clearly and, in general, accurately. However, when she was asked to perform calculations procedurally, Samantha was not as successful performing the associated procedures as she had been providing a conceptual explanation even though she claimed that she "understood the procedural part." The evidence regarding Samantha's procedural and conceptual knowledge of fraction division and using manipulatives contradicted what Samantha stated when she discussed her difficulties with division. Samantha believed that she understood the procedural part and that she got confused when she thought about the manipulatives. However, Samantha's story paints the opposite picture.

Summary for Samantha. Samantha's assessment, observational, and interview data all support that Samantha's conceptual knowledge of fractions was higher than her procedural knowledge. Although Samantha refined her conceptual knowledge of fractions, she did not appear to show progress in her procedural knowledge of fractions. Even after a manipulatives-intensive fractions unit in her middle grades methods course, Samantha appeared to have areas of weakness in both her procedural and conceptual knowledge of fraction multiplication and division. Her use of manipulatives and pictorial representations was, for the most part, meaningful with only a few possible occurrences of procedural use.

Although Samantha's conceptual knowledge was stronger than her procedural knowledge of fraction division, she experienced some confusion that may have been related to gaps in both types of knowledge. This confusion may have interfered with Samantha's sense of confidence in her knowledge as she used manipulatives to demonstrate fraction division.

With respect to Samantha's relational understanding of fraction multiplication and division, there was little evidence to support that Samantha knew the connections between the procedural and conceptual knowledge that she possessed. Evidently, Samantha had some awareness of a possible deficit in relational understanding because she rated her ability to help her future students to develop relational understanding of fractions as a seven out of ten. Samantha provided a specific example of her weakness: fraction division. She said, "I guess in order for me to teach [division] to somebody else, I would have to first fully master both the procedural knowledge and the conceptual knowledge, understand both thoroughly before I feel like I could pass that along."

Krystal

During the fall of 2014, Krystal, an African American female in her early twenties, participated in this study while she was an EDMG-1 student. Krystal was a traditional student enrolled in the middle grades education program at the university; her two chosen areas of specialty were mathematics and English Language Arts. Krystal was originally from the city in which the university was located and was educated in the local public school system, grades K-12. At the time of this study, Krystal was single with no children.

In her mathematics autobiography, Krystal conveyed several positive comments about mathematics and her experiences learning mathematics, both prior to high school graduation and during her college experience. Krystal stated that mathematics became her favorite subject in third grade when she learned about multiplication and saw patterns in the numbers of the multiplication table. Krystal also stated that she learned to do algebra in her honors classes in middle school and completed geometry, trigonometry, and pre-calculus in high school. Krystal indicated that she “had no problem with geometry and trigonometry” and that “algebra made complete sense” to her. Krystal mentioned other topics in mathematics as well. She said that she “got a thrill out of variables” and that she liked “simplifying equations and factoring.”

Krystal’s college experiences seemed mostly positive. She shared that statistics “was a breeze,” but that “Applied Statistics made the student dig deeper to fully evolve in their learning.” Krystal also commented, “Derivatives started off scary, but once understood, it was a breeze.” Krystal also discussed the difficulties she experienced learning mathematics; she wrote, “I am comfortable stating when I don’t understand something and know when to seek for help.”

Krystal also provided several examples of how she viewed mathematics as being necessary for success in life. Krystal stated that she works with children at a recreation center and has told the children that mathematics is useful for baking, building construction, working with money in retail, and getting paid wages.

Finally, Krystal shared her opinions on teaching. Krystal believed that teachers are important and that she “will be a great contribution” to the field. However, she expressed concern that “the passion that some teachers have now has disintegrated;” that “there isn’t a lot of spunk in some teachers anymore;” and that teachers may have become complacent. She also stated,

> Math can be learned in different ways by the student. I believe that it starts with the teacher. If the teacher knows different methods in how to teach and break down a math problem for each student to better understand, then every student will get what they need from the teacher to accomplish the work. That is why it is important to have different techniques and methods to present to a classroom because everyone learns differently. In one [college] class, it seemed as if we had to teach ourselves the text and come to class to go over homework and take quizzes and tests. Instead of the professor explaining how to work problems and why the answer is what it is, he just tested us on what we knew. Of course, the class grade average was below average but for the few of us that knew to study on our own, it made a difference.

Krystal's score on the Procedural Pre-Test was 76.5%. On her Initial Survey, Krystal rated her confidence in her procedural ability in the operations on fractions as "somewhat confident" (Appendix J). However, Krystal's score on her Conceptual Pre-Test score was 54.9%. For her conceptual understanding of the meaning behind the operations on fractions and in her understanding of the 'why' behind the conventional algorithms of adding, subtracting, and multiplying fractions, she rated her confidence as "slightly confident" (Initial Survey, Appendix J). For her conceptual understanding of fraction division, Krystal selected "not at all confident" (Initial Survey, Appendix J) in her understanding the 'why' behind the conventional algorithm.

During the study, Krystal gave accounts of her use of manipulatives that seemed somewhat at odds. On her Initial Survey, Krystal indicated that she was "very familiar" (Appendix J) with manipulatives during her own learning experiences. However, during her Pre-Instruction Task-based Interview, Krystal stated, "I don't recall using manipulatives when we were in school."

From Pre- to Post-Test, Krystal demonstrated positive gains on both the Procedural and Conceptual Tests. Table 15 shows Krystal's scores, as well as the changes from Pre-Tests to Post-Tests.

Table 15

Krystal's Procedural and Conceptual Knowledge

	Pre-Test	Post-Test	Change
Procedural	76.5%	85.3%	+8.8%
Conceptual	54.9%	56.9%	+2.0%

Fraction multiplication. Data regarding Krystal's procedural and conceptual knowledge of fraction multiplication were collected through Pre- & Post-Instruction Tests, Pre- & Post-Instruction Task-based Interviews, assessment, and observational data. In the next sections, these data are expounded upon.

Pre-instruction data. On her Procedural Pre-Test, Krystal correctly calculated $\frac{1}{2}\times\frac{1}{3}$ and $2\frac{1}{6}\times3\frac{3}{5}$ using conventional algorithms. However, Krystal did not calculate $\frac{5}{8}$ of 2 as a multiplication problem. Instead, Krystal interpreted the problem as division and calculated the answer as $\frac{5}{16}$.

On her Conceptual Pre-Test, Krystal provided a pictorial representation that did not seem to be representative of $\frac{5}{8}$ of 2, as shown in Figure 28 below.

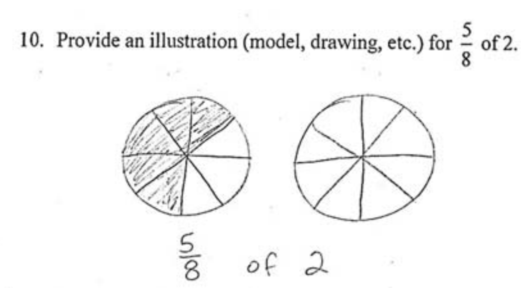


Figure 28. Krystal's representation for $\frac{5}{8}$ of 2

Similarly, Krystal represented $\frac{1}{2}\times\frac{1}{3}$ as two circles with one-half of the first circle shaded and one-third of the second circle shaded, as shown in Figure 29 on the following page. Krystal's representation did not appear connected to the conceptual meaning of one-half of one-third.

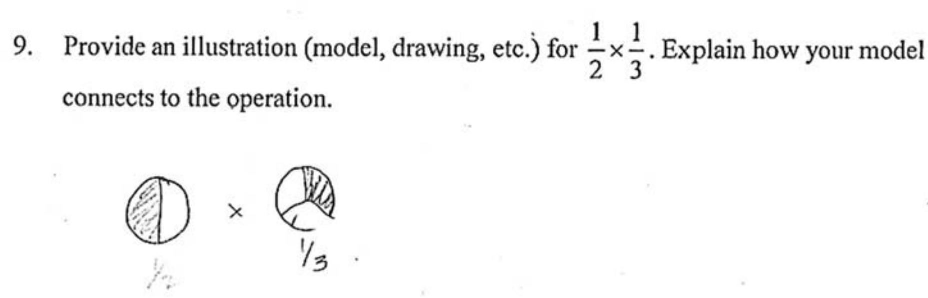

9. Provide an illustration (model, drawing, etc.) for $\frac{1}{2}\times\frac{1}{3}$. Explain how your model connects to the operation.

Figure 29. Krystal's representation of $\frac{1}{2}\times\frac{1}{3}$

On the Conceptual Pre-Test, Krystal was asked to determine the error in $2\frac{1}{6}\times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$. She converted both mixed numbers to improper fractions, renamed each using the common denominator of 30, added the numerators to obtain $\frac{173}{30}$, and then simplified to $5\frac{23}{30}$. It seemed that Krystal added the two mixed numbers instead of multiplying them.

During the Pre-Instruction Task-based Interview, Krystal was asked to demonstrate $\frac{3}{4}\times 6$ using her choice of manipulatives. Krystal responded,

> "Um, you can't have three-fourth of a group. So when we learn um, multiplication of fractions using manipulatives, we always started with the multiple… [self-corrected] multiplier, which is three-fourths. And that would've been three-fourth groups of six. But that's a lot, so I'm gonna just you know um, what property is that? Commutative? Or associative? I'm just gonna swap them and do six groups of three-fourth."

At this point in the course, Mrs. Paige already provided instruction on multiplication with whole numbers and integers and used the correct terminology and meaning associated with *multiplier* and *multiplicand*. However, use of a fraction multiplier had not been discussed or explored.

Instead of modeling $\frac{3}{4}\times 6$, Krystal seemed to capitalize on her knowledge of the commutative property to model $6\times\frac{3}{4}$ by making six groups of three-fourths using fraction circle pieces. She first determined the answer to be $\frac{18}{4}$ and explained that eighteen-fourths is equivalent to eighteen one-fourths or $18\left(\frac{1}{4}\right)$. She subsequently provided an equivalent answer of $4\frac{1}{2}$ by moving the fourths pieces to create four wholes with two remaining one-fourths. During the interview, Krystal did not write down any procedures to calculate her answer.

When Krystal was asked to generate a real-word scenario for $\frac{3}{4}\times 6$, she generated seven scenarios as a part of her problem-solving process. Below are the seven scenarios Krystal generated over the span of approximately seven minutes:

- I have six candy bars, and I'm going to give each friend three-fourth of each bar. How many candy bars are there total?
- Let's say if I have three-fourth of a bag of candy and I give them to six friends. How many do I have?
- So if I have three-fourth of a bag of candy, six *bags* of three-fourths.
- If I have six bags that hold three-fourth pieces of candy, how many… how much candy do I have?

- If I have three-fourth of a bag of candy, six three-fourth, how much do I have?
- If I have six bags of candy, which hold three-fourth pieces of candy, how much ... And if I put all the candy in one big bag, if I wanted to give them ... No, that will be too much, I'm going into too many operations. Okay then, if I put them in one big bag, how many ... how much candy do I have?
- Okay, so if I have six bags of candy which hold ... which measures ... which is three-fourth of a bag full. If I put all of my candy together, how many bags of candy can I make?

For the first scenario, Krystal was asked what the scenario implied. Krystal responded, "Maybe an addition problem." For the next five scenarios, Krystal was asked to clarify the appropriate unit of measure for each scenario. When Krystal voiced the final scenario and expressed her confidence in its correctness, she articulated the answer to be $4\frac{1}{2}$ bags of candy.

During this scenario-generating experience, Krystal referenced the importance behind the word *of* as it relates to multiplication. Krystal said, "Like especially because I said 'of'. I said three-fourth of the bags. So that implies multiplication." This reference may provide evidence of Krystal's progressing conceptual knowledge. On her Procedural Pre-Test for the problem $\frac{5}{8}$ of 2, Krystal used division instead of multiplication.

Pre-instruction data showed that Krystal's procedural knowledge of fraction multiplication was fairly consistent. Krystal also demonstrated conceptual knowledge of fraction multiplication in the form of $a \times \frac{b}{c}$, where *a*, *b*, and *c* are Natural numbers. However, there may be gaps in Krystal's conceptual knowledge of fraction multiplication in the form of $\frac{a}{b} \times c$, where *a*, *b*, and *c* are Natural numbers. Finally, Krystal's use of manipulatives to demonstrate her conceptual knowledge was generally meaningful.

Observational and assessment data. During course instruction addressing fraction multiplication, Ms. Paige asked students to model $1\frac{1}{2} \times 1\frac{1}{2}$ as $\frac{3}{2} \times \frac{3}{2}$ because another student questioned whether the concrete representations would be the same even though the symbolic representations were different.

Krystal began with three pink one-half pieces. To show that she halved the $\frac{3}{2}$, Krystal placed one yellow one-fourth piece on each of the pink halves. She then used the three pink one-half pieces and the three yellow one-fourth pieces to represent her final answer of $2\frac{1}{4}$.

Krystal's Problem Report provided further evidence of her developing conceptual knowledge of fraction multiplication. The Problem Report was an out-of-class assignment that was based on an in-class activity. The problems on the assignment that were similar to Pre-Test and interview problems were $\frac{2}{3} \times 2$, $\frac{1}{2} \times \frac{3}{4}$, $\frac{2}{3} \times \frac{3}{4}$ and $1\frac{1}{2} \times 1\frac{1}{2}$.

For the problem $\frac{2}{3}\times 2$, Krystal provided a region model representation, as shown in Figure 30. The phrases “I have shown $\frac{2}{3}$ of a group of 2” and “1 group of $\frac{2}{3}$” that Krystal used did not seem aligned.

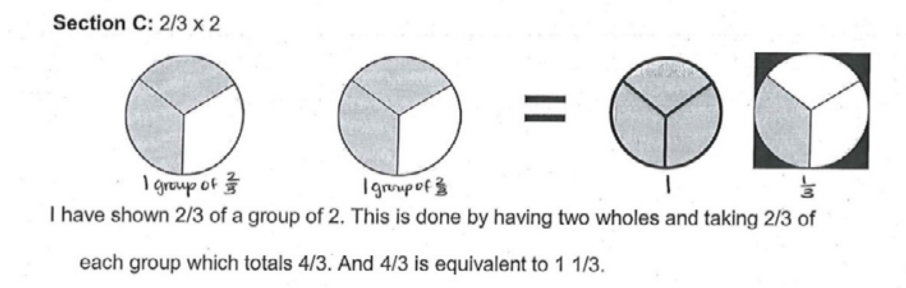


Figure 30. Krystal’s representation for $\frac{2}{3}\times 2$

To show $\frac{1}{2}\times\frac{3}{4}$, Krystal provided a region model and justification, as shown in Figure 31. Ms. Paige noted on the assignment that Krystal did not provide sufficient pictorial justification for her answer.

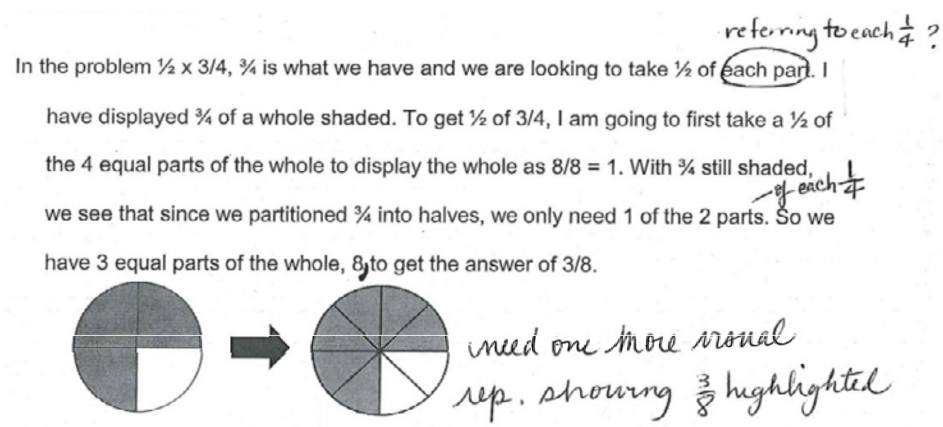


Figure 31. Krystal’s representation for $\frac{1}{2}\times\frac{3}{4}$

Krystal's pictorial representation and justification for $\frac{2}{3}\times\frac{3}{4}$ are shown in Figure 32. Krystal's explanation of "from each shaded $\frac{1}{4}$ we take account of 2 of the 3 parts or $\frac{2}{12}+\frac{2}{12}+\frac{2}{12}$" seemed indicative of the mathematics associated with using the distributive property and language associated with using a fraction as an operator. The symbolic representation that aligns with Krystal's verbal representation is $\frac{2}{3}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$, while the symbolic representation that aligns with Krystal's pictorial representation is $\frac{2}{3}\left(\frac{3}{12}+\frac{3}{12}+\frac{3}{12}\right)=\frac{2}{12}+\frac{2}{12}+\frac{2}{12}$. The writing on Krystal's paper was done by Ms. Paige as she graded Krystal's paper.

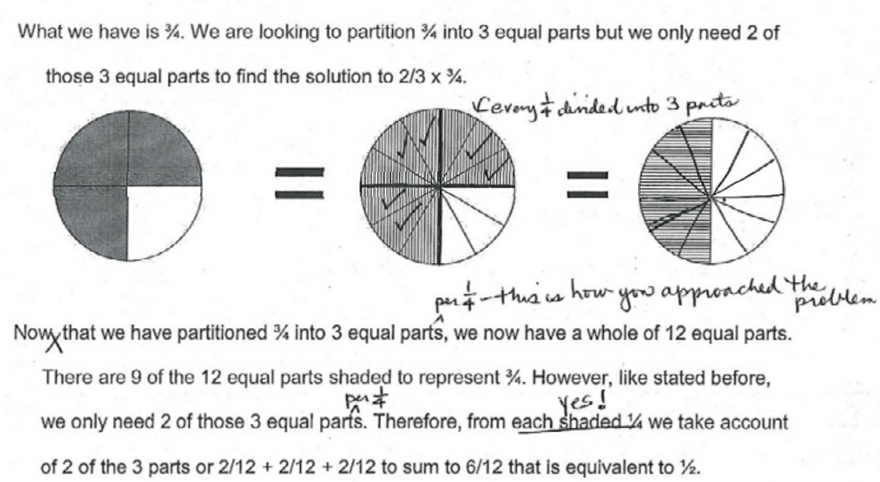


Figure 32. Krystal's representation for $\frac{2}{3}\times\frac{3}{4}$

For $1\frac{1}{2}\times 1\frac{1}{2}$, Krystal included a reference to the distributive property and decomposing fractions shown on the following page in Figure 33. She provided a pictorial representation that effectively models the problem and is aligned with her symbolic and written explanation.

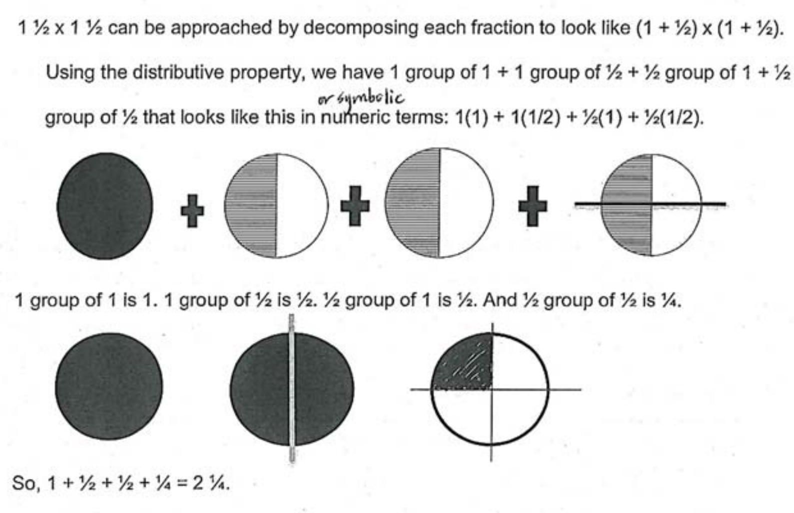


Figure 33. Krystal's representation for $1\frac{1}{2}\times1\frac{1}{2}$

In Krystal's reflective part of the assignment, she first referenced learning the "why about multiplying fractions." She indicated that the "investigation influenced [her] views on teaching fractions as a teacher." Krystal also stated that she learned what multiplying fractions means, that the manipulatives were a great help, and that her understanding deepened "tremendously." Krystal shared, "Although I thought I understood fractions, I really didn't have a deeper knowledge of fractions that I now have."

Observational and assessment data provided evidence of Krystal's developing conceptual knowledge of mixed number multiplication and the distributive property of multiplication. Krystal's use of manipulatives and pictorial representations to demonstrate her conceptual knowledge of fraction multiplication was generally meaningful.

Post-instruction data. Krystal demonstrated positive gains from the Pre-Tests to the Post-Tests. The larger of the two gains was on her Procedural Post-Test, a gain of 8.8%. Part of this gain can be attributed to Krystal's improvement on the problem $\frac{5}{8}$ of 2. Instead of dividing, as she did on the Pre-Test, Krystal wrote the problem as multiplication, $\frac{5}{8} \times \frac{2}{1}$, and provided the answer $\frac{10}{8}$.

On her Conceptual Post-Test, Krystal realized an overall gain of 1.96%. Part of Krystal's overall gain included improvement on two of the three fraction multiplication problems. For $\frac{1}{2} \times \frac{1}{3}$, Krystal appeared to demonstrate increased conceptual knowledge of using a fraction operator, as shown in Figure 34. However, Krystal did not partition the whole circle into sixths to demonstrate why one-half of one-third is $\frac{1}{6}$.

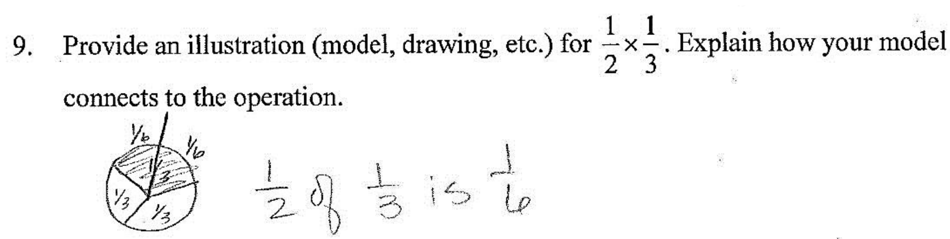

9. Provide an illustration (model, drawing, etc.) for $\frac{1}{2} \times \frac{1}{3}$. Explain how your model connects to the operation.

Figure 34. Krystal's representation of $\frac{1}{2} \times \frac{1}{3}$

Similarly, Krystal exhibited improvement from Conceptual Pre-Test to Post-Test on the fraction multiplication problem, $\frac{5}{8}$ of 2, shown in Figure 35 on the following page. As noted earlier, Krystal shaded five-eighths of one on the Conceptual Pre-Test instead of five-eighths of two. Krystal's work on this problem is similar to her work for $\frac{2}{3} \times 2$ on the Problem Report mentioned earlier.

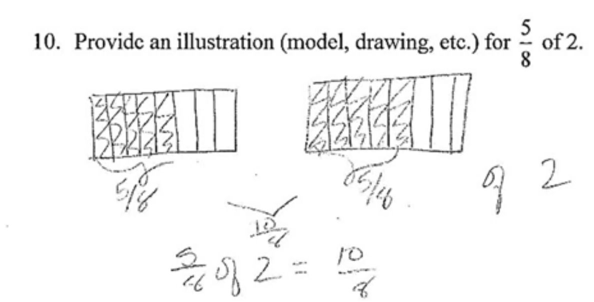


Figure 35. Krystal's representation for $\frac{5}{8}$ of 2

For the remaining multiplication problem, $2\frac{1}{6} \times 3\frac{3}{5}$, Krystal noted that the error was "distributing the whole numbers" and "distributing the fractions." While her meaning was not completely clear, it is possible that she was aware of the need for the partial products of $\frac{1}{6} \times 3$ and $2 \times \frac{3}{5}$ to be included in the product since she accurately used the distributive property to multiply two mixed numbers on her Problem Report.

During the Pre-Instruction Task-based Interview, Krystal was asked to use manipulatives to demonstrate $\frac{2}{3} \times \frac{3}{4}$. Krystal asked, "Do I have to *use* the manipulatives *first* before I do the algorithm?" When she was told that it was her choice, Krystal responded, "Okay, because that's going to kill me. I just want to simplify it right off the bat (Laughter). Because if we cross simplify [moved her hands to make an X as she said 'cross simplify'], the answer will be one-half." When she was asked to justify her answer with a manipulative, Krystal whispered, "I'll try my best." Krystal's question and responses may be signs that she was more confident in her procedural knowledge than in either her conceptual knowledge or her use of manipulatives.

Similar to her performance on the Conceptual Pre-Test for $\frac{1}{2} \times \frac{1}{3}$, Krystal literally modeled the multiplier $\frac{2}{3}$ and the multiplicand $\frac{3}{4}$ using fraction circles. Krystal began, “Three-fourths is what we have.” Then, she modeled three-fourths using three yellow one-fourths. As she stated, “We’re doing two-thirds of this [three-fourths],” Krystal overlaid the fourths with two orange one-third pieces with the beginning edges of the fraction pieces lined up with one another, as shown in Figure 36. Krystal then filled in the gap with a one-twelfth piece. Krystal’s actions seemed to be aligned with addition and subtraction of fractions instead of fraction multiplication.

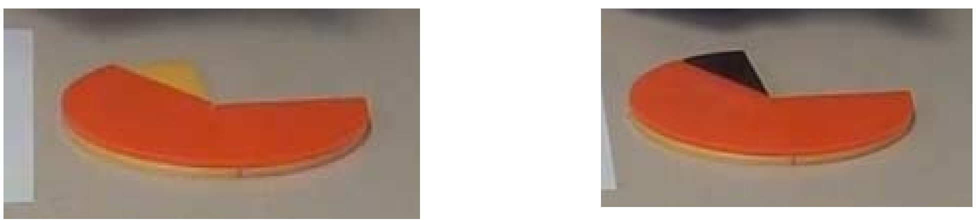


Figure 36. Krystal’s initial representation of $\frac{2}{3} \times \frac{3}{4}$

As Krystal demonstrated $\frac{2}{3} \times \frac{3}{4}$, she overlaid additional twelfths onto the orange thirds and stated, “That’s twelve-thirds.” When she was asked to clarify, she said, “Is that incorrect? Was I wrong? There are four-twelfths, four, whatever. (Laughs) There’s four of these twelfths that make up one-third.” Krystal continued,

> So, but three-thirds, we have twelve-thirds, twelve-fourths. We have twelve-fourths. Yeah, we have twelve of these fourths. Is that the correct

> one? Oh my lanta. Am I saying the right one, Jesus. Okay. Let's regroup. Three-fourths is what we have. I have two-thirds group of three-fourth. Two-thirds doesn't cover or make up the full three-fourths. So, I added this 12th here [pointing to the one black twelfth that she overlaid in the gap]. Oh, Jesus, I'm getting hot. (Laughter).

Throughout the interview, Krystal articulated $\frac{2}{3} \times \frac{3}{4}$ several different ways. She initially voiced the four expressions listed below but was unable to produce a representation for these expressions.

- Two-thirds group of three-fourths;
- Two-third groups of three-fourths;
- Two-thirds groups of three-fourths; and
- Two-thirds group of three-fourth.

After Krystal spent approximately six minutes working, I asked Krystal what $\frac{2}{3} \times \frac{3}{4}$ meant. Krystal responded, "I'm looking for two-thirds of a group of three-fourths." When Krystal articulated 'two-thirds of a group of three-fourths,' she pointed to the orange thirds. I then asked, "What has to be done to the group?" Krystal responded, "I have to take two parts of three." Immediately thereafter, Krystal demonstrated $\frac{2}{3} \times \frac{3}{4}$, stated that the answer was one-half, and expressed confidence in her work. Krystal's representations are shown in Figure 37 on the following page.

Figure 37. Krystal used twelfths to model $\frac{2}{3} \times \frac{3}{4}$

Krystal was then asked to generate a real-world scenario for $\frac{2}{3} \times \frac{3}{4}$. Krystal stated, “I have three-fourths of a pizza for lunch. I ate two-thirds of that pizza. How much pizza did I eat?” Once she stated the scenario, she used the fraction circles to demonstrate the scenario in a manner similar to her previous demonstration. Krystal explained, “I ate two thirds of that slice, two-thirds of that slice, and two-thirds of that. So I ate two-thirds of each one-fourth.” Krystal concluded, “You will see that I ate half of my pizza which was once a whole pizza.” She also commented, “You have to make sure that your language is correct. That’s what I’ve been working on since.”

Summary of fraction multiplication. Krystal's procedural knowledge appeared to be consistent from Pre-Test to Post-Test. From pre-instruction to post-instruction, Krystal seemed to make progress in her conceptual knowledge of fraction multiplication, but may still have gaps in her conceptual knowledge. On the six different fraction multiplication problems involving a fraction multiplier, Krystal tended to use alternative strategies to determine the answers instead using the fraction multiplier as an operator on the multiplicand. For example, Krystal appeared to use the commutative and distributive properties of multiplication on $\frac{3}{4} \times 6$ and $\frac{2}{3} \times \frac{3}{4}$. It also appeared that Krystal made progress in her ability to generate real-world scenarios for fraction multiplication.

Fraction division. Assessment, observational, and interview data were collected regarding Krystal's procedural and conceptual knowledge of fraction division. In this section, the data is expounded upon.

Pre-instruction data. Krystal correctly answered the four fraction division problems on the Procedural Pre-Test using the conventional algorithm of inverting the divisor and multiplying. For each of the three problems that involved a whole number, Krystal rewrote the whole numbers using equivalent fractions. For example, Figure 38 on the following page shows Krystal's work for $2 \div \frac{3}{8}$. Krystal also used the conventional algorithm of cross simplification once she changed the problem to multiplication.

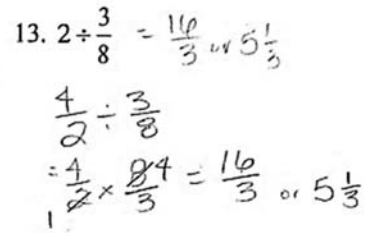


Figure 38. Krystal displayed procedural knowledge of fraction division

On the same four fraction division problems on the Conceptual Pre-Test, Krystal earned full credit on one of them, $2\frac{1}{3} \div 3$. For this problem, Krystal stated, “less than 1 because $2\frac{1}{3}$ is being divided by a number greater than its value.” For the remaining division problems, Krystal’s pictorial representations showed evidence of the measurement interpretation of division, as shown in Figure 39.

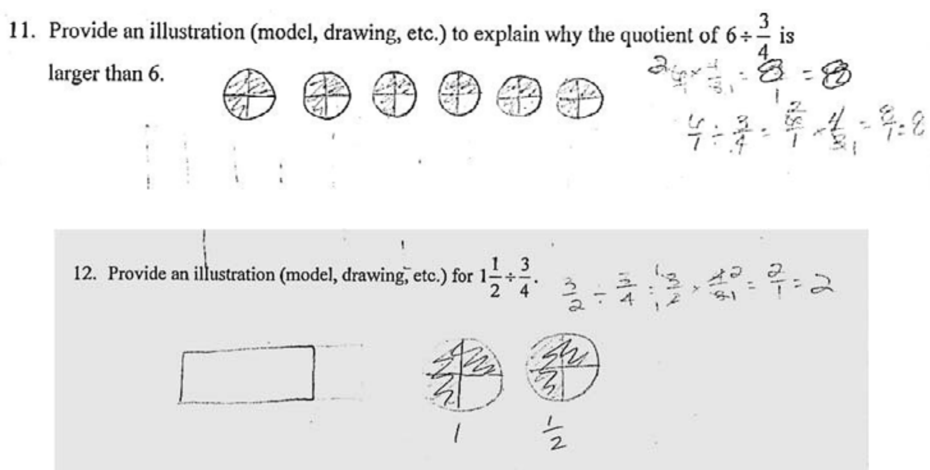


Figure 39. Krystal’s initial conceptual knowledge of fraction division

During the Pre-Instruction Task-based Interview, Krystal was asked to demonstrate $3 \div \frac{2}{3}$ using manipulatives. Her immediate reaction was to chuckle. When Krystal was questioned about her reaction, she stated, “Oh, really. I mean of course I *know* how to do it um, *written*. But…” After about one minute, Krystal said, “I don’t know how to start. Because you need three wholes and you need to divide it by two-thirds.”

Krystal used fraction circles to demonstrate $3 \div \frac{2}{3}$, shown in Figure 40. After she represented the fraction division, Krystal procedurally worked the problem, produced the answer $4\frac{1}{2}$, and commented that her manipulative model was not aligned with her procedural answer.

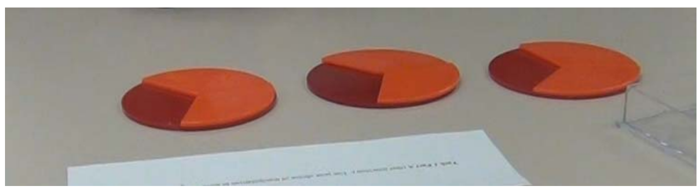


Figure 40. Krystal’s initial representation of $3 \div \frac{2}{3}$

After generating her first representation for $3 \div \frac{2}{3}$, Krystal stated, “Three divided by two-thirds. You’re trying to find out how many times two-thirds can go into three.” Krystal communicated that *go into* was not proper terminology as she generated the model shown in Figure 41.

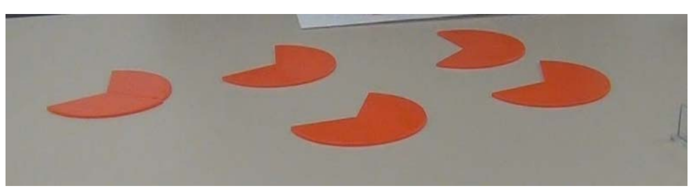


Figure 41. Krystal’s second representation of $3 \div \frac{2}{3}$

Subsequently, Krystal said, “If I have two-thirds, I’m trying to get it to make three wholes. Let’s see how many two-thirds to make three wholes.” Krystal moved the thirds to make three whole circles, as shown in Figure 42.

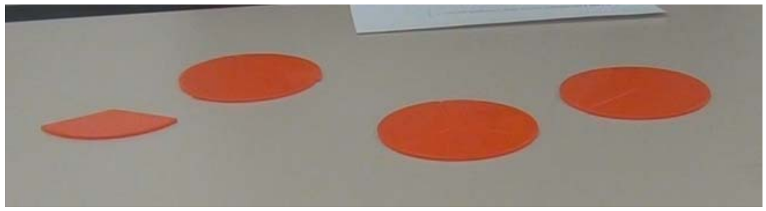


Figure 42. Krystal’s third representation of $3 \div \frac{2}{3}$

While this is an effective representation of $3 \div \frac{2}{3}$ using the measurement interpretation of division, Krystal had difficulty interpreting the remaining one-third. Krystal said, “I still don’t know what is the half. So, I’m still missing something. A lot in fact, quite a bit.” Krystal added, “And now despite this [referring to the extra one-third], which was just out here, I have three. So I’m closer. (chuckled) So I guess um, the missing link is how I interpret the reciprocal.” For approximately three more minutes, Krystal continued to work through the problem. She mentioned division, groups of two-thirds, and that she felt like she had done this previously. However, Krystal was not able to resolve her perceived misalignment with her procedural answer and her manipulative representation.

During the interview, Krystal was asked to generate a real-world scenario for $3 \div \frac{2}{3}$. Krystal said, “If I have three pizzas, and I give my friends two-thirds of each pizza, how many pieces do I have?” Krystal demonstrated her scenario using the orange fraction circle pieces (see Figure 40) as she said, “I take two-thirds of each pizza, then I will have one-third of each pizza left. However, that's not the result." Krystal continued, "The answer would be four and one-half. So, I only have three one-thirds. But I feel like that doesn't make sense." Krystal was not able to reconcile her answer of $4\frac{1}{2}$ with her real-world scenario and the fraction circles she used.

Pre-instruction data suggested that Krystal's procedural knowledge of fraction division appeared to be greater than her conceptual knowledge. During her interview, Krystal appeared to have exhibited more confidence in her procedural knowledge than in her conceptual knowledge. The pictorial, verbal, and concrete representations that Krystal provided for fraction division seemed aligned with the measurement interpretation of division.

Observational and assessment data. Krystal appeared to display knowledge of the measurement interpretation of division during a class discussion about fraction division. During the previous class meeting, the class solved the following problem:

> Rachel has 5 flower beds to fertilize. She knows the square footage of the flower beds and the "per square foot" information on the fertilizer bag. She figures out that a bag of fertilizer will cover $\frac{3}{4}$ of a flower bed. How many bags of fertilizer will she use?

At the beginning of class, the class revisited the solution and discussed why the traditional algorithm works and what it means. When Ms. Paige asked what type of division the problem was, Krystal stated, "Measurement." Ms. Paige responded, "I already know how much I am measuring out. What am I looking for?" Krystal answered, “Groups.” When Ms. Paige did not respond, another student in the class clarified, “Number of groups.”

Following the discussion about the flower bed/fertilizer problem, the class explored a real-world context problem about sharing a 4-ft. party sub-sandwich. In the scenario, each person received $\frac{1}{4}$ ft. of the sub-sandwich, for which the instructor wrote on the white board $4 \div \frac{1}{4}$. The class was tasked with using a number line to model the problem and determine the number of people who could be served. Krystal volunteered to demonstrate the problem at the white board. She used a number line representation and partitioned each foot into fourths. As she drew, Krystal used recursive partitioning (Confrey et al., 2009) as she halved each whole, and then halved the halves to get fourths. As Krystal carried out this action, she stated, “When we do that, we find that there are sixteen equal parts in our whole.”

After the discussion about the sandwich being shared, the class explored another fraction division problem using pattern blocks. As the class explored $\frac{3}{4} \div \frac{1}{2}$ using pattern blocks, Krystal seemed to confuse dividing *by one-half* and dividing *in half*. Ms. Paige suggested to students to use the hexagon pattern block as the whole. By using the hexagon as the whole, students had to use the brown right trapezoid as one-fourth (See Appendix L).

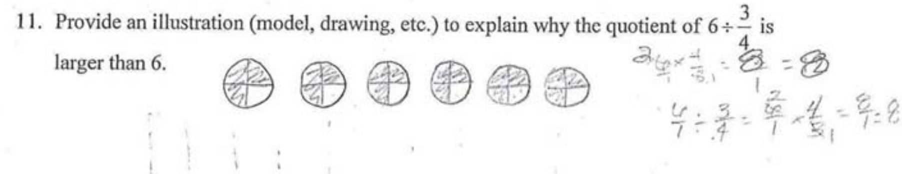
Krystal worked with another student to represent the problem. Once Krystal and her partner had three brown right trapezoids to show three-fourths, Krystal placed three orange square tiles vertically on top of the trapezoids. As she did this, Krystal said that each fourth needed to be divided in half. By dividing the fourths in half, Krystal may have confused dividing *by one-half* with dividing *in half*, which is equivalent to dividing by two. In addition, placing the square tiles vertically on the trapezoids did not partition each trapezoid into two equal parts. Krystal stated that her answer was six-fourths, which is correct but not aligned with the action she performed with the manipulative. It seems possible that Krystal's response was based on her procedural knowledge rather than her work with the manipulatives.

Observational data showed that Krystal appeared to have some conceptual knowledge of the measurement interpretation of fraction division, but she seemed to use that knowledge inconsistently. Her use of pictorial and concrete representations seemed generally meaningful.

Post-instruction data. In keeping with her performance on the Procedural Pre-Test, Krystal correctly answered all four of the fraction division problems on the Procedural Post-Test. Krystal improved from Conceptual Pre-Test to Post-Test on one of the four fraction division problems, $6 \div \frac{3}{4}$.

Krystal's Post-Test response for $6 \div \frac{3}{4}$, shown in Figure 43 on the following page, provided more insight into her conceptual knowledge than her Pre-Test response offered. Krystal explained multiplying $\frac{4}{3} \times 6$ by stating, "It takes $\frac{4}{3}$ to make 1."

Pre-Test



Post-Test

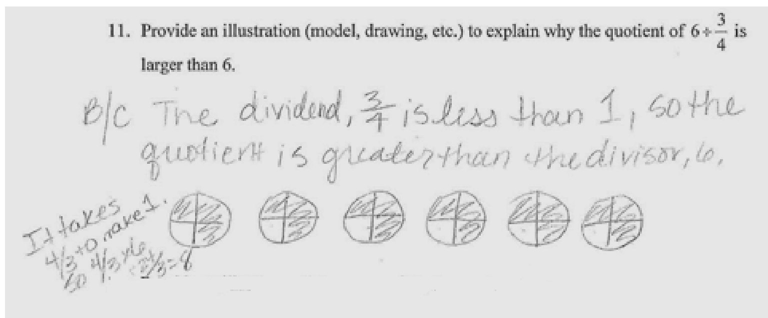


Figure 43. Krystal demonstrated some progress toward relational understanding

Krystal’s responses for $1\frac{1}{2} \div \frac{3}{4}$ and $2\frac{1}{3} \div 3$, shown in Figure 44, showed less evidence of an improvement in her conceptual knowledge of fraction division.

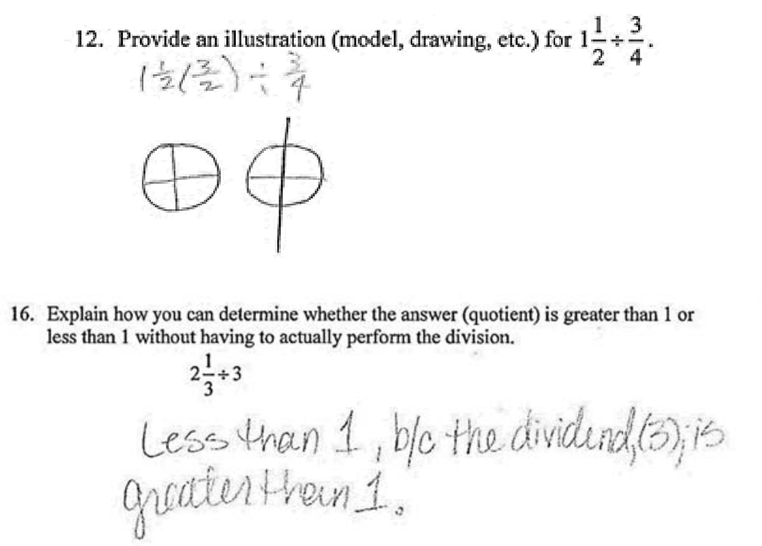


Figure 44. Krystal’s Conceptual Post-Test answers for fraction division

During her Post-Instruction Task-based Interview, Krystal was asked to model $\frac{5}{6} \div \frac{1}{2}$ using manipulatives. Krystal began, “This is what we have. It’s five-sixth divided by a half. So, we could either cut each, not cut, but um, partition each part into half and add them up.” As Krystal explained that she could partition each part into half, she used her index finger to imitate the action of cutting one of the sixths in half. Krystal then overlaid a one-tenth piece onto one of the sixths. Krystal immediately stated that the one-tenth was not half [of the sixth] and removed it. Krystal continued, “Or we could just, because we did it in class before, take half of the five-sixth.” To show what she meant, Krystal laid a pencil vertically such that it split the five-sixths in half, as shown below in Figure 45.

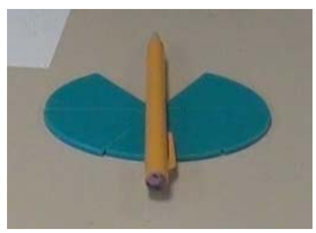


Figure 45. Krystal first representation for $\frac{5}{6} \div \frac{1}{2}$

After an approximate 10-second delay, Krystal was asked, “So, what would that answer be?” Krystal responded, “In my mind, I’m thinking two and a half. I’m just trying to make sure.” Up to this point, there was no evidence that Krystal attempted to work the problem procedurally. Krystal appeared to keep her attention focused on the manipulative to explain her answer.

To justify her answer of $2\frac{1}{2}$, Krystal explained that she would use one-half of the first sixth with one-half of the second sixth to make one; one-half of the third sixth with one-half of the fourth sixth to make one; and have one-half of the fifth sixth left. Taking her explanation into consideration, Krystal would have been correct for $\frac{1}{2} \times \frac{5}{6}$ if she had said "two and a half *sixths*" or $\frac{2\frac{1}{2}}{6}$ because that answer is equivalent to $\frac{5}{12}$.

When Krystal was asked which model of division she was using, she answered, "I am using…measurement." Krystal then continued to justify her answer by saying, "Because, um, you know, it will be like how many halves can, I don't want to say go into, but how many halves can we make from five-sixths. And that will be two and a half." Krystal's interpretation of measurement division was correct: the number of halves that can be made from five-sixths. However, her answer, two and a half, was not correct.

When Krystal was asked if there were any of the manipulatives that could justify her answer, Krystal picked up a black one-twelfth fraction piece and overlaid it onto a one-sixth piece, laid another twelfth adjacent to the previous twelfth. To verify that she had twelfths, Krystal quietly counted and decided that she was correct. Krystal overlaid more twelfths onto the sixths. Once she covered the five-sixths with twelfths, Krystal said, "Okay, five-sixths is equivalent to two, four, six, eight, ten-twelfths. Now, but we're only looking for half of each part of five-sixth. So, take that half, and put that half, that half, that half, that half." As Krystal explained, she removed every other twelfth from each of the sixths and put them beside the five-sixths fraction model, as shown in Figure 46 on the following page.

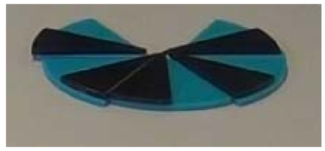


Figure 46. Krystal's second representation for $\frac{5}{6} \div \frac{1}{2}$

Krystal continued, "So, now we have half of each part and we have one half, two-half, three, four, and five-halves, which is equivalent to two and one-half." As she explained, Krystal pointed to the remaining twelfths that were left overlaid on the sixths. As Krystal said "which is equivalent to two and one-half," she rearranged the twelfths that she previously removed, as shown in Figure 47.

Figure 47. Krystal's representation for $2\frac{1}{2}$

When Krystal was asked to clarify her answer of two and one-half, she answered, "Two and a half parts." Shortly thereafter, Krystal stated, "It's two and a half of five-sixth." When Krystal was asked if she was confident in her answer, she responded, "I am." Krystal subsequently experienced confusion when she calculated the answer as $1\frac{2}{3}$, which was not aligned with her previous answer of $2\frac{1}{2}$.

Once Krystal realized her answers did not match, Krystal was asked what it meant to divide by one-half. She immediately responded, "To take half of it or to multiply by 2." Krystal's answer of "multiply by 2" seemed to point to her procedural knowledge of the algorithm for dividing fractions, which is to multiply by the reciprocal of the divisor. As Krystal explained that she had to multiply by two, she created two sets of five-sixths using fraction circles. She also used the manipulatives to demonstrate equivalence between two sets of five-sixths and her answer of $1\frac{2}{3}$, as shown in Figure 48.

$\frac{2}{1} \times \frac{5}{6}$ $\quad 1\frac{4}{6}$ $\quad 1\frac{2}{3}$

Figure 48. Krystal's final representation for $\frac{5}{6} \div \frac{1}{2}$

When Krystal was asked to explain her work, she said, "I multiplied by the reciprocal because [removes one-sixth from the whole] five-sixths is our referent whole, but we still have a missing part, and so we have to fill the missing part. So, we need to multiply by the reciprocal, which makes it, in this problem, a whole number." As Krystal said 'five-sixths is our referent whole,' she removed the one-sixth that she previously used to create six-sixths, or one whole. Then, as Krystal said 'we still have a missing part,' she pointed to the empty space that remained since she removed the one-sixth piece. As she said 'we have to fill the missing part,' Krystal replaced the one-sixth piece to make one whole. One final statement that Krystal made in her justification was, "So, it's like filling the whole, kind of." Although Krystal was able to use manipulatives to demonstrate the action happening because of the algorithm of inverting and multiplying (two groups of five-sixths), it was not clear that Krystal possessed conceptual knowledge of why inverting and multiplying is an appropriate algorithm for dividing by fractions.

Summary of fraction division. Krystal appeared to make progress in her procedural and conceptual knowledge of fraction division during the fraction unit. However, there appeared to be continued gaps in her conceptual knowledge. For example, Krystal seemed to confuse dividing *by one-half* with dividing *in half* during a class activity and during the Post-Instruction Task-based Interview. Although Krystal verbally expressed measurement division correctly during her Post-Instruction Interview, she did not consistently use the information to demonstrate her knowledge. In some cases, Krystal appeared to use her procedural knowledge to determine how to use the manipulative to demonstrate fraction division. For example, during the Post-Instruction Task-based Interview, Krystal seemed to use her procedural knowledge of the algorithm to represent $\frac{5}{6} \div \frac{1}{2}$.

Summary for Krystal. Krystal's procedural knowledge of fraction multiplication and division was generally consistent from Pre-Test to Post-Test. Krystal demonstrated an increase in her conceptual knowledge of fraction multiplication and division, yet her conceptual knowledge of these two fraction concepts was inconsistent. For example, at times, Krystal confused fraction multiplication with division. It also seemed that Krystal was more confident in her procedural knowledge than in her conceptual knowledge. At least once, Krystal appeared to use her procedural knowledge to guide her use of manipulatives. It also seemed that Krystal tended to use region models more than length models to demonstrate fraction multiplication and division.

With respect to relational understanding, there were few instances of Krystal connecting her procedural knowledge to her conceptual knowledge. Therefore, the depth of Krystal's relational understanding of fraction multiplication and division was unclear.

When Krystal was asked about her confidence in helping her future students develop relational understanding of fractions, she responded that she planned to “use tons of manipulatives.” Krystal expressed a desire to present fractions in ways that would engage students and “keep the classroom exciting,” like “ice cream, candy, books, games—nothing boring” (Semi-Structured Interview, p. 16).

Jacob

Jacob, a White American male in his late twenties, participated in this study in the fall of 2014 while he was a student in EDMG 1. Jacob was a non-traditional, full-time student enrolled in the middle grades education program. Jacob’s two chosen areas of specialty were mathematics and science. At the time of the study, he was married with one young son; his wife was an elementary teacher for a local school system. Jacob served in the military immediately after high school and completed two tours of duty in combat. He was originally from the northeast but lived in the southeast at the time of the study.

In Jacob’s mathematics autobiography, he made positive and negative comments about his mathematics experiences. Jacob also shared information about his desire to be a mathematics teacher, as well as his views on the usefulness of mathematics. Jacob indicated that he chose to become a middle grades mathematics teacher because he loves teaching and believes middle grades is where most students learn to love mathematics or to hate it. He also stated that he hoped to be able to get more students to love mathematics than to hate it.

Jacob referenced a specific experience that sparked his interest in becoming a mathematics teacher. Jacob shared that, until he took Trigonometry/Pre-Calculus, mathematics came very easy to him. He wrote, “I could not grasp the concepts of Trigonometry/Pre-Calculus it was totally foreign to me” (Mathematics Autobiography, p. 1). Jacob further stated, “The class made me rethink everything I thought I knew about math” (Mathematics Autobiography, p. 1). Jacob explained, “I had to go back to the basics and figure out the ‘how’ and the ‘why’ math worked and not just the memorization of math” (Mathematics Autobiography, p. 1). Consequently, Jacob decided, “If I could resolve this problem at the source, than [then] it would help a lot of students so they didn’t have to be in my shoes” (Mathematics Autobiography, p. 1).

Jacob also shared his views on the importance of learning math. He wrote, “Students today don’t fully understand the gravity of what math really does for us as a society and how it plays major roles in all aspects of our lives. Today, to [too] many students think that math does not need to be learned because it can just be done on a computer” (Mathematics Autobiography, p. 1). Jacob also noted that math is very important and is used in everyday life around the world. Jacob’s views on what math does for us as a society, how math plays major roles in all aspects of our lives, and is used in everyday life around the world may have been related to his life experiences in the military.

Jacob specifically named his favorite mathematics subjects as geometry and algebra. He stated that he liked doing proofs and enjoyed working with angles, sides, and different shapes in geometry. With regard to algebra, Jacob stated that he liked solving the equations and figuring out the unknown.

In Jacob’s Initial Survey, he rated his confidence in both his procedural ability in the operation on fractions and his understanding of the meaning of the operations on fractions as somewhat confident (Appendix J). Jacob indicated that he was somewhat confident in understanding the 'why' behind the conventional algorithms of fractions. Jacob also noted on his survey that he learned math through memorization and repetition.

On his Conceptual Pre-Test, Jacob wrote that he was not confident in any of his answers on the Pre-Test. Jacob also shared, "Drawing math problems examples is not something I have ever had to do. It [is] hard to put it in any other terms other than what it is." Table 16 shows Jacob's scores on all four tests as well as the change from Pre- to Post-Tests.

Table 16

Jacob's Procedural and Conceptual Knowledge

	Pre-Test	Post-Test	Change
Procedural	88.2%	91.1%	+2.9%
Conceptual	45.1%	82.4%	+37.3%

Fraction multiplication. In the next section, data for Jacob's procedural and conceptual knowledge of fractions are reported. Data collected included pre- and post-instruction tests; observational and assessment data; and pre- and post-instruction interview data.

Pre-instruction data. Jacob correctly answered two of the three fraction multiplication problems on the Procedural Pre-Test: $\frac{1}{2}\times\frac{1}{3}$ and $2\frac{1}{6}\times3\frac{3}{5}$. For the mixed number multiplication problem, Jacob used the conventional algorithm of converting the mixed numbers to improper fractions, cross simplifying, and multiplying to obtain the answer $\frac{39}{5}$. The problem Jacob missed on the Procedural Pre-Test was $\frac{5}{8}$ of 2. He wrote the problem as division, $2\div\frac{5}{8}$, and used the conventional algorithm of inverting the divisor and multiplying to get his answer of $\frac{16}{5}$.

Jacob did not receive credit for any of the three multiplication problems on the Conceptual Pre-Test. For $\frac{1}{2}\times\frac{1}{3}$, Jacob provided a pictorial representation that did not appear to be aligned with the conceptual meaning of fraction multiplication. Jacob's representation is shown in Figure 49.

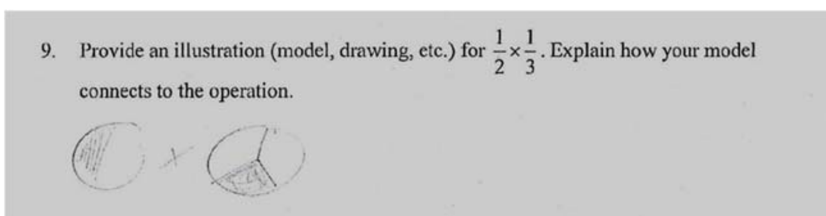


Figure 49. Jacob's representation for $\frac{1}{2}\times\frac{1}{3}$

Although Jacob did not get full credit on the Conceptual Pre-Test for $\frac{5}{8}$ of 2, the drawing he provided on the Pre-Test, shown in Figure 50 on the following page, appeared to be conceptually aligned with using a fraction as an operator. Using $\frac{5}{8}$ as a multiplier means to split the multiplicand into eight equal parts; then keep (or shade) five of those parts. Since Jacob did not provide any information as to what the numerical answer was, it was not clear as to the depth of his knowledge.

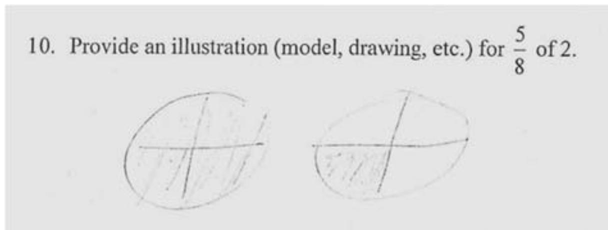


Figure 50. Jacob's correct model for $\frac{5}{8}$ of 2

For the final fraction multiplication problem, Jacob was asked to determine the error in the problem $2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$. Jacob did not provide an explanation about the error. Instead, Jacob worked out the problem using a conventional algorithm of converting the mixed numbers to improper fractions to obtain $\frac{13}{6} \times \frac{18}{5}$, multiplying the numerators and multiplying the denominators to obtain $\frac{234}{30}$, simplifying to obtain $\frac{39}{5}$, and converting his answer back to the mixed number $7\frac{4}{5}$.

During Jacob's Pre-Instruction Task-based Interview, he was asked to use manipulatives to represent $\frac{3}{4} \times 6$. Jacob immediately began to calculate the answer using cross simplification and multiplying the numerators and multiplying the denominators to obtain the answer $\frac{9}{2}$. As soon as he finished, Jacob reached for a set of fraction circles. Before Jacob opened the set, he put them to the side and began to draw six circles on paper, as shown in Figure 51. Although he stated that he did not remember the name of the manipulative, Jacob explained that his plan would be to use the fraction circles. Jacob continued by partitioning the six circles into fourths.

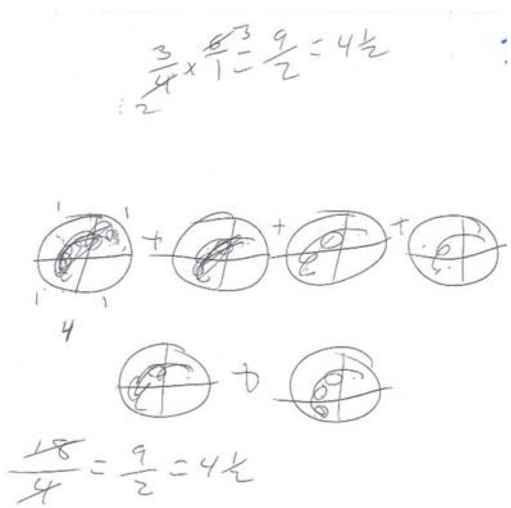


Figure 51. Jacob's pictorial representation for $\frac{3}{4}\times 6$

As Jacob pictorially represented his thinking, he also verbally explained. Jacob said,

> We'll just use like pizza and pie…you're going to have six of them [drawing six circles on the paper] make those into fourths, of course, equal parts, and then essentially it would be like three of each of the fourths, since multiplication is repetitive addition, and so on and so forth, so you'd just model it out to have, you'd have six pies, you take three-fourths of each, you'd essentially just add them together [counts by ones to eighteen

pointing to each fourth as he counts] you'd have eighteen-fourths, which would reduce down to… [mumbles, two, nine, four, half]

After Jacob finished explaining, he again reached for a set of fraction circles, but he said he changed his mind because he was trying to think if there might be an easier way rather than opening all of the packages just looking for the fourths. Instead, Jacob chose the two-colored counters and said, "Essentially it's going to be the same." Although Jacob indicated that he did not know that the fraction circles were a region model and the two-colored counters were a set model, he generated the manipulative representation shown in Figure 52 to represent $\frac{3}{4}\times 6$. Jacob's use of a set model and fraction circles was similar to the representations for three-fourths that he provided on his Conceptual Pre-Test.

Six groups of four-fourths

Six groups of three-fourths

Figure 52. Jacob's second representation of $\frac{3}{4}\times 6$

As Jacob demonstrated with the set model, he explained, "So you have six groups of essentially what is four-fourths so then you [removing one counter from each group] do that and you have six groups of three-fourths so then you would just count them all up [counts by ones to eighteen] so you have eighteen-fourths." Jacob was asked to reconcile his answers $4\frac{1}{2}$ and $\frac{9}{2}$ with the model he chose, the two-colored counters set model. Without hesitation, Jacob moved the 18 counters in order to have four counters in each group. As Jacob worked, he explained, “If you reconsolidate to make a whole…the groups whole of being the group of four, four in a group and you reconsolidate so that there is four in each group, then you have one, two, three, four [groups], and then half a group left.” Jacob’s representation for this explanation is shown in the image on the left in Figure 53.

Jacob also explained how to reconcile the model with his answer $\frac{9}{2}$. Jacob said, “If you break them into half groups, so if your group is consistent of four, so then half the group would be consistent of two, so then if you make them all into groups of two, then you’d have nine ‘half’ groups.” Jacob’s representation for this explanation is shown in the image on the right in Figure 53.

$4\frac{1}{2}$ $\frac{9}{2}$

Figure 53. Jacob’s representations for his answers of $4\frac{1}{2}$ and $\frac{9}{2}$

Pre-instruction data showed that Jacob demonstrated flexibility in representing fraction multiplication using a variety of representations. However, it appeared that Jacob’s use of the *multiplier* and *multiplicand* in a fraction multiplication problem was not aligned with the conceptual meaning associated with those terms. Pre-instruction data also showed that Jacob’s procedural knowledge of fraction multiplication was consistently demonstrated. Jacob's Conceptual Pre-Test answer to $\frac{1}{2}\times\frac{1}{3}$ may indicated that Jacob has gaps in his conceptual knowledge.

Observational and Assessment Data. Jacob's Problem Report, an out-of-class assignment based on an in-class activity, provided insight into Jacob's developing conceptual knowledge of fraction multiplication. There were three problems on the assignment that were similar to Pre-Test fraction multiplication problems: $\frac{2}{3}\times 2$, $1\frac{1}{4}\times 2$, and $1\frac{1}{2}\times 1\frac{1}{2}$. There was one problem that was also part of the Post-Instruction Task-based Interview: $\frac{2}{3}\times\frac{3}{4}$.

To demonstrate $\frac{2}{3}\times 2$, Jacob provided the region model shown in Figure 54, as well as an explanation to accompany his pictorial representation. Jacob interpreted $\frac{2}{3}\times 2$ as "two third groups of two." Jacob indicated that each circle should be divided into three equal parts. Jacob then stated, "You will then take two of 1/3 from each whole." Jacob concluded that his final answer was $1\frac{1}{3}$.

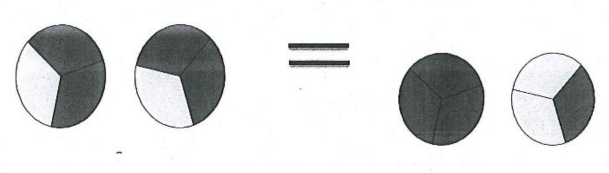


Figure 54. Jacob's representation for $\frac{2}{3}\times 2$

The second fraction multiplication problem on the Problem Report that was somewhat related to the Pre-Test problem of $\frac{5}{8}$ of 2 was $1\frac{1}{4} \times 2$. The main difference between these two problems is that the multiplier for $1\frac{1}{4} \times 2$ was a mixed number. Jacob wrote, “This problem is one and one fourth groups of two wholes.” Jacob began with two whole circles and then indicated, “You are going to want to take one fourth of the group of two whole. In order to do this you are going to dived [divide] the two wholes into four equal parts (this step will give you four one half).” Jacob provided a final answer of $2\frac{1}{2}$ along with the pictorial representation shown in Figure 55.

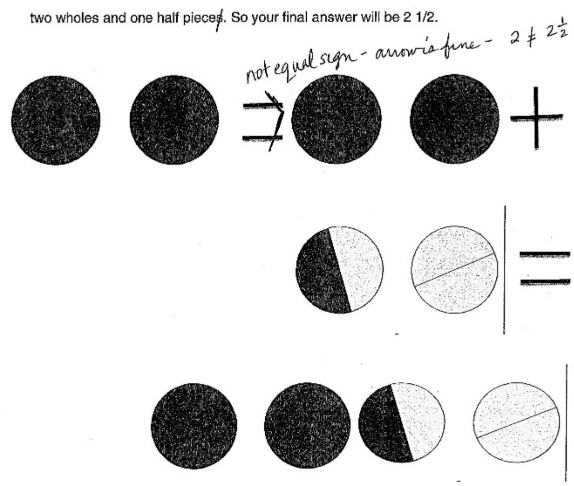


Figure 55. Jacob’s representation for $1\frac{1}{4} \times 2$

The final problem on the Problem Report related to the Pre-Tests was $1\frac{1}{2} \times 1\frac{1}{2}$, which Jacob determined to be $2\frac{1}{4}$. Jacob noted that $1\frac{1}{2} \times 1\frac{1}{2}$ means “one and one half groups of one and one half." Jacob's pictorial representation is shown in Figure 56. The notes on his solution were written by Ms. Paige.

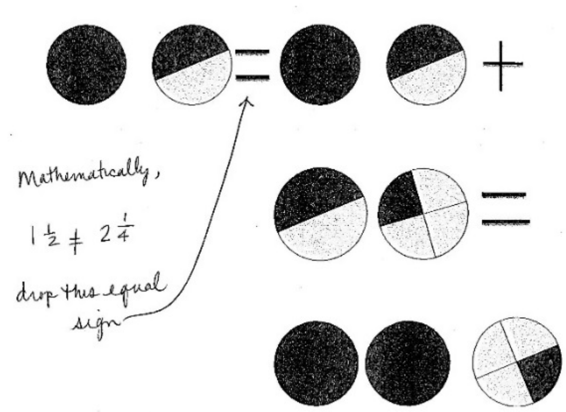


Figure 56. Jacob's representation for $1\frac{1}{2} \times 1\frac{1}{2}$

Jacob provided two methods for calculating the answer to $\frac{2}{3} \times \frac{3}{4}$. In his first explanation, Jacob wrote, "Since you have three parts of the whole that you are working with (the $\frac{3}{4}$) and the denominator (3 of the $\frac{2}{3}$) you can simply take two of the $\frac{1}{4}$ pieces from the whole of the $\frac{3}{4}$." Jacob's representation is shown below in Figure 57.

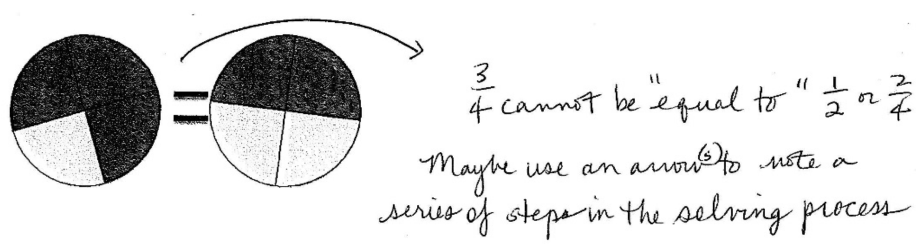


Figure 57. Jacob's first representation for $\frac{2}{3} \times \frac{3}{4}$

Secondly, Jacob renamed $\frac{3}{4}$ as $\frac{9}{12}$; then, he appeared to use the multiplier $\frac{2}{3}$ as a fraction operator on the $\frac{9}{12}$, as shown in Figure 58. Jacob explained,

> The second way to do it is when you have the whole divided and shaded into $\frac{3}{4}$. You then divide each of the four $\frac{1}{4}$ pieces into three more equal parts. This will make your whole divided into 12 equal parts. You will see that at this point you have a whole divided into 12 equal parts and 9 of those parts are shaded ($\frac{9}{12}$ of the whole is shaded this is because $\frac{9}{12}$ is equivalent to $\frac{3}{4}$). Now you take two [of] every three pieces are [and] partition them out. In the end you will have 6 of the 9 pieces portioned out. When you consolidate all your pieces you will see that you have six $\frac{1}{12}$ pieces which is $\frac{6}{12}$ and is equivalent to you[r] final answer of $\frac{1}{2}$.

Figure 58. Jacob's second representation for $\frac{2}{3} \times \frac{3}{4}$

In the reflective part of the assignment, Jacob indicated that using the manipulative helped him to think about multiplying fractions in a manner other than algorithmically. Jacob acknowledged that he usually performed the algorithm without thinking about what fraction multiplication means. He stated, “The manipulative made me think about what multiplication in and of itself really means (____ groups of ____) and how to be able to show that using manipulatives.” Jacob also communicated how his thoughts about teaching were impacted. He shared, “I came to realize that just teaching the basic algorithm would not be good enough.”

Jacob noted that a breakthrough for him was trying to figure out how to multiply improper fractions, specifically $\frac{3}{2}\times\frac{3}{2}$ instead of $1\frac{1}{2}\times1\frac{1}{2}$. After class concluded the day fraction multiplication was explored using manipulatives, Jacob questioned the instructor about how to model $\frac{3}{2}\times\frac{3}{2}$. They worked several minutes, trying various pictorial representations unsuccessfully. Next, they attempted to use manipulatives to represent the problem. After several minutes of the instructor and Jacob working, Jacob was able to demonstrate $\frac{3}{2}\times\frac{3}{2}$ using fraction circles and using a fraction as an operator. Therefore, the instructor asked him to present it to the class during the following class meeting.

At the next class meeting, Jacob used fraction circles to model $\frac{3}{2}\times\frac{3}{2}$. He began with three pink halves and told the class, “Your problems [manipulatives] are supposed to be representative of your numbers.” Jacob continued, “The denominator [of the fraction multiplier] is two. You want to divide into two equal parts.” As Jacob explained, he demonstrated that only one of the halves needed to be split in half, as shown in the second image in Figure 59 on the following page. Each group, then, contained one pink half and one yellow fourth, which was equivalent to three-fourths. Jacob counted the groups, “One group, two groups.” Next, Jacob said that they needed three groups of the three-fourths in order to show $\frac{3}{2} \times \frac{3}{2}$. Jacob finished by putting the halves together to make one whole, one-half with two-fourths to make one whole, and had one-fourth remaining. Jacob’s representations are shown in Figure 59.

Three-halves	Three-halves split into two equal parts	Three groups of three-halves	Two and one-half

Figure 59. Jacob’s representations for $\frac{3}{2} \times \frac{3}{2}$

Finally, during a class discussion, another student generated a real-world context that the student believed to be aligned with $16 \times \frac{3}{4}$. The context involved 16 pans of barbeque, three-fourths of which had been eaten. Jacob questioned the appropriateness of the scenario with respect to the symbolic representation. Jacob asked, “Wouldn’t that be $\frac{3}{4} \times 16$ instead of $16 \times \frac{3}{4}$?”

Observational and assessment data collected during the fractions unit showed that Jacob appeared to have refined his conceptual knowledge of fraction multiplication. Jacob also demonstrated flexibility in using multiple representations for fraction multiplication.

Post-instruction data. On the Procedural Post-Test, Jacob calculated $\frac{1}{2} \times \frac{1}{3}$ as $\frac{1}{6}$ and $2\frac{1}{6} \times \frac{3}{5}$ as $7\frac{4}{5}$ using the conventional algorithm he used on his Procedural Pre-Test. For the problem $\frac{5}{8}$ of 2, Jacob first treated the problem as division, changed the problem to $\frac{5}{8} \times \frac{1}{2}$, and wrote $\frac{5}{16}$ as the answer. However, Jacob erased his work and replaced it with the answer $\frac{10}{16}$ without showing work.

On the Conceptual Post-Test, for $\frac{5}{8}$ of 2, Jacob drew two circles, each partitioned into fourths, and shaded five parts, which was similar to his work on the Conceptual Pre-Test. Jacob duplicated this representation and interpreted his answer as $\frac{10}{16}$, as shown in Figure 60.

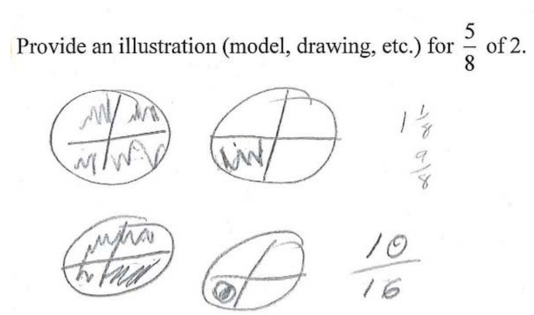


Figure 60. Jacob’s representation for $\frac{5}{8}$ of 2

Also similar to his Conceptual Pre-Test, Jacob did not communicate a specific error in $2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$. Instead, Jacob again applied his procedural knowledge to calculate the answer as $7\frac{4}{5}$ using the standard algorithm of converting each to improper fractions, cross simplifying, and multiplying.

Finally, Jacob's pictorial representation for $\frac{1}{2}\times\frac{1}{3}$ on his Conceptual Post-Test is shown in Figure 61. Jacob's representation appeared to be a mixture of a set model and a region model, both of which he used separately during his Pre-Instruction Task-based Interview.

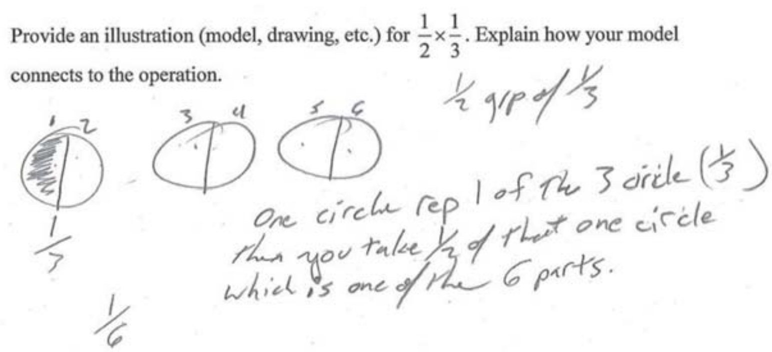


Figure 61. Jacob's representation for $\frac{1}{2}\times\frac{1}{3}$

During his Post-Instruction Task-based Interview, Jacob was asked to use manipulatives to represent $\frac{2}{3}\times\frac{3}{4}$. He began by interpreting $\frac{2}{3}\times\frac{3}{4}$ as "$\frac{2}{3}$ grp of $\frac{3}{4}$," which is what he wrote on his task paper. Jacob first modeled the multiplicand using three yellow one-fourth fraction-circle pieces and the multiplier using two orange one-third fraction-circle pieces. For several minutes, Jacob overlaid the orange thirds and the black twelfths onto the yellow fourths. When Jacob was asked about his thinking, he responded, "It would be two-thirds groups of three-fourths so I was modeling three-fourths and then I was looking at two-thirds compared to that." The series of pictures in

Figure 62, progressing from left to right, illustrate the process Jacob went through as he represented $\frac{2}{3}\times\frac{3}{4}$ using fraction-circle manipulatives.

Figure 62. Jacob's process as he represented $\frac{2}{3}\times\frac{3}{4}$

Immediately thereafter, Jacob was asked the reason why he was comparing two-thirds to three-fourths. As Jacob continued to overlay the yellow and orange fraction-circle pieces, Jacob began, "Because for multiplication it's 'two-thirds groups of three-fourths' so you're seeing how…" Next, Jacob was asked what it meant to have two-thirds groups. He responded, "It means you're not going to have a whole group, you're going to have two-thirds of the group." Jacob was then asked how the fact that two-thirds of a group of three-fourths is less than three-fourths aligned with the manipulatives. Jacob said, "So, you have your three-fourths, and you are trying to find out how much your two-thirds would take of that three-fourths." As he explained, Jacob overlaid two orange thirds onto the three yellow fourths. For approximately two more minutes, Jacob continued manipulating the fraction pieces but did not appear to determine an answer to $\frac{2}{3}\times\frac{3}{4}$.

Since Jacob appeared to stall in his progress, he was asked to model three-thirds of three-fourths because Jacob previously used that terminology. Once Jacob showed three yellow fourths, he was reminded that he stated earlier that two-thirds of that would be less than three-fourths. Immediately, Jacob said, “Got it.” He then explained and demonstrated how to use the fraction-circle pieces to arrive at one-half as the answer, as shown in Figure 63. Jacob said,

> Three-fourths. Your whole [red]. And this is three-fourths of it [first image from the left]. So three-fourths. So, your denominator of the two-thirds is three, and now you have three parts that you are working with because your numerator over here [$\frac{3}{4}$] is three, how many parts you’re working with. So, your three is your new whole, and if you take *two* of those three [second image from the left] which is your new working parts, you have two-fourths which compared to a whole [third image] is a half [fourth image] so it would be one-half.

Figure 63. Jacob’s second representation for $\frac{2}{3}\times\frac{3}{4}$

As he explained, Jacob referenced remembering the work he and the instructor had done to model $\frac{3}{2}\times\frac{3}{2}$ and how they used the numerator and denominator of the fraction multiplier to operate on the fraction multiplicand.

When Jacob was asked about his confidence in his answer, he said, “Pretty confident. Extremely confident.” Jacob confirmed his answer procedurally by multiplying the fractions and simplifying; he also indicated that he had already mentally calculated the answer. Jacob shared,

> Sometimes it’s easier for me to do it that way and work backwards, and sometimes, um, it’s easier to go the other…I think it probably would be easier to just go without it [the answer] and being able to show it [the problem] without knowing the final answer as far as *learning* how to do it. That way you learn the process rather than learning how to figure out the answer. But, knowing the answer allows me to check what I am doing because I am not fully confident in my abilities. As you can see, it takes a while to get to the final answer. If I knew 100% what I was doing, I’d be like, ‘oh, there you go’ [referring to the manipulative that showed the answer of one-half]. So knowing the answer allows me to figure out the right way to show it.

From the time Jacob was presented the task to the time he said ‘got it,’ almost eight minutes elapsed. Conversely, once Jacob stated that he ‘got it,’ it only took him one and one-half minutes, from beginning to end, to explain and demonstrate using the fraction-circle pieces.

Once Jacob demonstrated $\frac{2}{3} \times \frac{3}{4}$, he was asked to explain how the manipulative connects to a conventional algorithm. Jacob first reiterated what he shared earlier using two-thirds as a fraction operator on three-fourths. Then, Jacob was asked to explain how the manipulative connects to the answer $\frac{6}{12}$. Jacob first provided a pictorial representation, shown below in Figure 64. He stated that each of the shaded pieces were fourths. Based on the fact that he called them ‘fourths’ and the labels he provided for the circles, it appeared that Jacob considered each circle to be one whole.

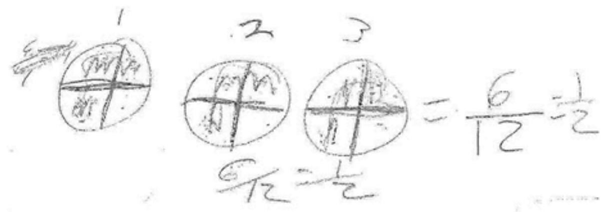


Figure 64. Jacob’s representation to justify $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$

After Jacob provided the pictorial representation in Figure 64, he decided to use the fraction-circle pieces to justify the answer $\frac{6}{12}$. Jacob reverted back to modeling the problem as he had originally by overlaying the orange and yellow fraction pieces and filling in the missing piece. The images (from left to right) in Figure 65 show how Jacob used the manipulatives as he explained the connection.

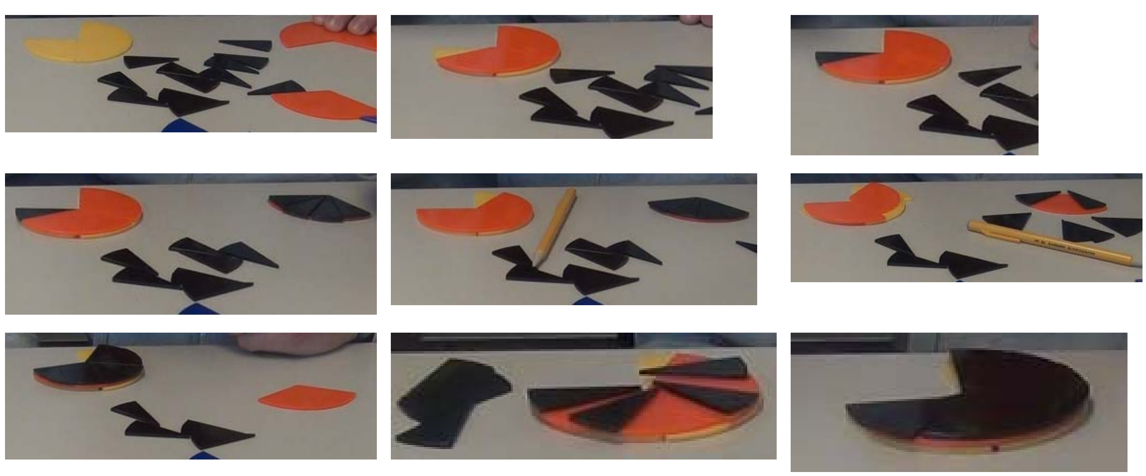


Figure 65. Jacob’s use of fraction-circle pieces as he explained the answer $\frac{6}{12}$

When Jacob was questioned about the use of the twelfths, Jacob said, "The twelfth is what the denominator is going to be up here [indicating procedural answer written on task paper], and it's also what is left over when you overlay with the two-thirds over the three-fourths." When he was asked what operation he modeled, Jacob said, "I don't really know. I don't know if I'm really making a procedural connection to it or if I'm just trying to see the representation, the comparison of two-thirds *to* three-fourths." An excerpt from the transcript is provided below.

Peppers: And what does your two-thirds represent in this problem?

Jacob: Um, the multiplier [sounds unsure]

Peppers: Which tells you…?

Jacob: Um, what you're doing to the multiplicand [starts manipulating the fraction pieces]

Peppers: Can you talk about what you are doing right now?

Jacob: It takes four twelfths to make one-third and since you have two-thirds that can fit onto…that fit with the three-fourths, um, that means that eight of the twelfths will fit on the two-thirds. So, then if you take two of every three, because of two-thirds… So, out of these three, take two; out of these three, you take two and then, what did I just do?

Although Jacob appeared to be using the fraction two-thirds as an operator, the difficulty he experienced was likely related to the fact that Jacob was operating on the multiplier two-thirds instead of the multiplicand three-fourths. When Jacob was asked whether the whole he was working with was two-thirds, he responded, "No, the whole would be the three-fourths. I think that's where I just screwed it up." Jacob clarified,

> So, [counting twelfths] nine-twelfths make up three-fourths, and eight-twelfths make up the two-thirds, so then if you take two of every three of the twelfths, you'd get six-twelfths. I'm just trying to think if that happens to be a coincidence or if it's actually a correct way of doing it. And the reason why the twelfths would come into play is because there is a twelfth left over of the two-thirds so then your new comparative whole would be the twelfths, finding out how many twelfths are in the three-fourths, and you're taking two of every three of the twelfths.

Although Jacob appeared to make progress toward justifying why the answer was six-twelfths, it seemed there were still problems with his justification. It is noteworthy that $\frac{2}{3} \times \frac{3}{4}$ was on Jacob's Problem Report and that Jacob provided two different correct methods for determining the answer. However, in the Problem Report, Jacob never seemed to connect the algorithm to the explanation he provided nor the pictorial representation.

Finally, Jacob was asked to generate a real-world scenario for $\frac{2}{3} \times \frac{3}{4}$. He generated the following scenario: "3 friends ate $\frac{3}{4}$ of a whole pizza that were all exactly the same size. How much did 2 of the three friends eat?" Jacob then revised his original scenario. His revised scenario was: "3 friends each had their own pizza. They each ate $\frac{3}{4}$ of the pizza. How much pizza did 2 of the 3 friends eat of all 3 pizzas?"

Post-instruction data revealed that Jacob refined his conceptual knowledge of a fraction operator but still seemed to have gaps in his conceptual knowledge. The data also pointed to Jacob's flexibility in using multiple representations.

Summary of fraction multiplication. Jacob's procedural knowledge of fraction multiplication was consistent throughout the data collected. His conceptual knowledge of fraction multiplication was not as consistent as his procedural knowledge. During the fractions unit, Jacob used manipulatives to explore fraction multiplication and, many times, successfully demonstrated a fraction operator in a multiplication problem using manipulatives. However, by the conclusion of the course, Jacob continued to experience some difficulty demonstrating a fraction operator using manipulatives. The difficulty Jacob experienced may have been due to interference of his procedural knowledge on his conceptual knowledge. Jacob's use of manipulatives was a mixture of procedural and meaningful use. For example, during his Post-Instruction Task-based Interview, after Jacob demonstrated $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$ meaningfully, his use of manipulatives to justify the algorithm for $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ appeared to be procedural.

Fraction division. Data collected regarding Jacob's procedural and conceptual knowledge of fraction division are discussed in this section. These data were collected through pre- and post-instruction tests, pre- and post-instruction interviews, classroom observations, and assessments given by the instructor of the course.

Pre-instruction data. On the Procedural Pre-Test, Jacob correctly answered all four of the fraction division problems: $6 \div \frac{3}{4}$, $1\frac{1}{2} \div \frac{3}{4}$, $2 \div \frac{3}{8}$, and $2\frac{1}{3} \div 3$. Jacob's work included the use of the conventional algorithm of fraction division, multiplying by the reciprocal of the divisor. His work also included the use of the conventional algorithm of cross simplification associated with fraction multiplication. In addition, below Jacob's work for $1\frac{1}{2} \div \frac{3}{4}$, he wrote, "Keep change flip."

Jacob earned partial credit on one of the four fraction division problems on his Conceptual Pre-Test, the real-world context that was aligned with $2 \div \frac{3}{8}$. The real-world context was: "One serving of pizza is $\frac{3}{8}$ of a pizza. How many servings will be available in 2 pizzas?" Jacob's work for this problem is shown in Figure 66.

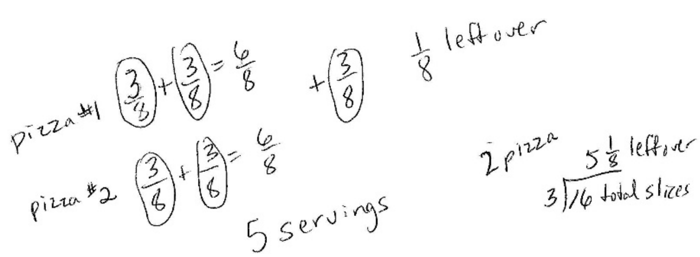


Figure 66. Jacob's representation for the context problem for $2 \div \frac{3}{8}$

Jacob left two of the fraction division problems blank: justifying why $6 \div \frac{3}{4}$ is greater than 6 and providing a pictorial representation to illustrate $1\frac{1}{2} \div \frac{3}{4}$. To justify whether the quotient of $2\frac{1}{3} \div 3$ was greater than one or less than one without performing the division, Jacob answered, "If what your dividing by is 1 or greater."

During his Pre-Instruction Task-based Interview, Jacob was presented with the task of modeling $3 \div \frac{2}{3}$ using manipulatives. After approximately one minute, Jacob began to work the problem procedurally but stopped prior to completing it. He turned his attention toward the manipulatives that were available but hesitated and said, “I’m not a big fan of fractions or manipulatives.” He continued,

> I’ve just never really been good at it, you know, like, as far as…as far as, like doing them [fractions] without using a calculator or anything like that. And as far as manipulatives, it was just never used growing up in math. Like the first time I used manipulatives was in the class I’m in now. So it’s kind of a new thing.

Next, Jacob was asked what it meant to divide three by two-thirds. Jacob responded with a laugh, “You multiply it by its inverse: three-halves. Keep, change, flip, because I like multiplication better. I honestly don’t even know.”

After one and one-half minutes, Jacob verbalized, “You have three wholes and you are dividing it by two-thirds.” Jacob clarified, “If you have three, and you’re breaking it up into two-thirds, then you have two-thirds of three.” During the interview, Jacob generated a pictorial representation for $3 \div \frac{2}{3}$, shown in Figure 67 on the following page, but he did not attempt to use a manipulative to model the problem. Even though Jacob partitioned each circle into thirds, he stated that he was not sure why he did that. In addition, Jacob mentioned the two types of division, partitive and “multiple subtracting;” yet, it did not appear that he was able to capitalize on his knowledge of these types of division to model the fraction division problem.

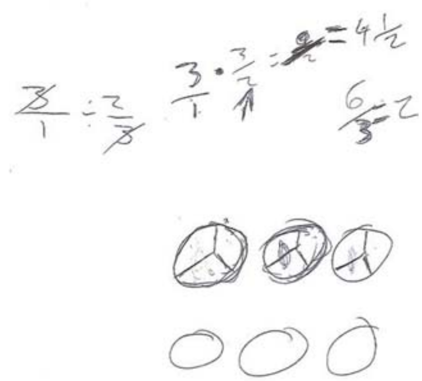


Figure 67. Jacob's representation for $3 \div \frac{2}{3}$

During the interview, Jacob expressed his discomfort with using manipulatives, as well as some frustration at not knowing how to answer the questions he was being asked. Some of the possible signs of frustration that Jacob displayed during the interview were exhaling sharply; saying "I don't even know anymore" and "I'm at a loss;" raising his hands in the air, which seemed to convey surrender; tapping and fidgeting with his pencil; laughter that did not seem indicative of humor; and swiveling his chair back and forth.

Jacob's pre-instruction data indicated that his procedural knowledge of fraction division was higher than his conceptual knowledge. Although he provided a representation for a real-world fraction division context problem, Jacob was not able to generate a real-world context for $3 \div \frac{2}{3}$.

Observational and assessment data. Limited data were collected about Jacob's progressing knowledge of fraction division during the fractions unit. However, during a class discussion about fraction division, Jacob inquired about the reason for the conventional algorithm of invert and multiply.

The class was discussing a real-world context about people sharing a 4-foot sub-sandwich, where each serving was $\frac{1}{4}$ ft. long. Jacob asked, "What is your justification for telling someone to flip that over?" In response, Ms. Paige used her fingers and began counting fourths. She held up one finger to represent the first foot and counted off four servings, each $\frac{1}{4}$ ft. in size. Likewise, Ms. Paige counted for each of the four feet and restated that there were four groups of four servings per group. To reinforce what she was doing, Ms. Paige wrote 4 groups × $\frac{4\,\text{servings}}{1\,\text{group}}$ = 16 servings on the board and simplified the unit *group* using dimensional analysis. Jacob indicated that he understood, and the class proceeded to another problem.

Based on the limited data available through observations and assessments, it is not appropriate to draw any conclusions about Jacob's developing procedural and conceptual understanding. It may be appropriate to state that Jacob seemed interested in understanding the reasons behind the conventional algorithm for fraction division, multiplying by the reciprocal of the divisor.

Post-instruction data. On his Procedural Post-Test, Jacob gained 2.9%, which was not surprising considering his high Pre-Test score of 88.2%. Jacob realized the largest gain (+37.3%) on his Conceptual Post-Test of all four participants in the study. Of Jacob's 37.3% gain, 13.7% can be attributed to his answers on the fraction division problems. In this section, post-instruction data related to Jacob's procedural and conceptual knowledge of fraction division are reported.

As Jacob had done on his Procedural Post-Test, he received full credit for all four fraction division problems: $6 \div \frac{3}{4}$, $1\frac{1}{2} \div \frac{3}{4}$, $2 \div \frac{3}{8}$, and $2\frac{1}{3} \div 3$. The work shown on his Post-Test was similar to that of his Pre-Test, which included conventional algorithms associated with fraction division.

In contrast to his Conceptual Pre-Test, Jacob received full credit on three of the four fraction division problems: $6 \div \frac{3}{4}$, $1\frac{1}{2} \div \frac{3}{4}$, and $2 \div \frac{3}{8}$. Noteworthy with respect to Jacob's possible preference toward procedures was the fact that Jacob provided a representation for $6 \div \frac{3}{4}$ without showing any procedural work. Also notable was the fact that Jacob wrote 'grp' beside the first three groups of three-fourths, which likely meant that he knew the answer represented the number of groups. Jacob's work for $6 \div \frac{3}{4}$ is shown in Figure 68 on the following page.

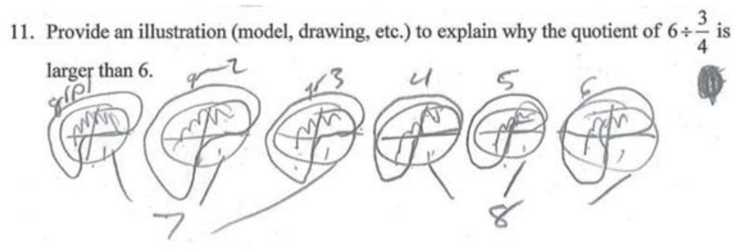


Figure 68. Jacob's representation for $6 \div \frac{3}{4}$

Although Jacob earned partial credit for his response to $2\frac{1}{3} \div 3$, shown in Figure 69, it seemed that Jacob was comparing the sizes of the dividend and divisor to each other.

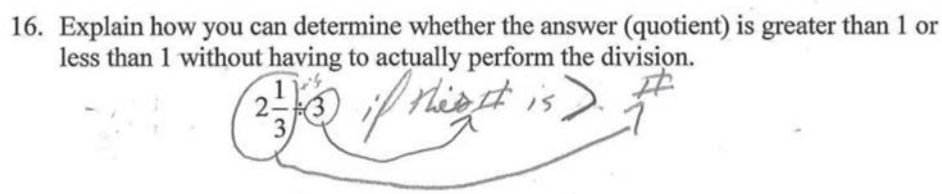


Figure 69. Jacob's response for estimating the quotient for $2\frac{1}{3} \div 3$

Jacob began the Post-Instruction Task-based Interview by interpreting $\frac{5}{6} \div \frac{1}{2}$ as, "How many groups of a half can you make from five-sixths?" To demonstrate, Jacob used fraction circles, as shown in Figure 70 on the following page. Initially, Jacob explained, "You have one, and two-sixths left over and that's a third." However, he then calculated the answer to be $1\frac{2}{3}$ and revised his explanation. Jacob said,

> So, you have your five-sixths. You have, 'How many groups of one-half can you make from five-sixths?' So, you have a half…that's one half, and

> you have some left over. Then you compare to the other half, and it's not enough…not enough left…so you have two of the three parts left to make the other half [fourth image from the left], and since these are thirds, you have two-thirds, so you have one whole group of a half and two-thirds of a group of a half, so you have one and two-thirds.

Figure 70. Jacob's process as he modeled $\frac{5}{6} \div \frac{1}{2}$

Once Jacob complete his explanation, he was asked to generate a real-world context for $\frac{5}{6} \div \frac{1}{2}$. Jacob considered filling up bottles, as well as the amount of gas in a tank. However, he did not follow through with those scenarios. Instead, Jacob generated the following scenario to represent $\frac{5}{6} \div \frac{1}{2}$: If you have $\frac{5}{6}$ cup of flour and you need $\frac{1}{2}$ of a cup to make a cake, how many cakes can you make? He provided the answer, one and two-thirds but added, "You can make one whole cake and then you'd be able to make two-thirds of another cake, not taking into account the old theory 'can you really dig half a hole', can you really make half a cake?"

A marked difference between Jacob's Pre-Instruction Interview and his Post-Instruction interview was the manner in which he reacted toward using manipulatives. Initially, Jacob expressed not being a fan of manipulatives; subsequently, Jacob did not hesitate to reach for the manipulatives when he was presented the task. Another marked difference from the Pre-Instruction Interview to the Post-Instruction Interview was

Jacob's demeanor during the interview. When Jacob was asked about his confidence, he first said, "I'm more confident now than I was walking out of our first session together [laughing]. Our first session, I was like, 'I got a lot of work to do.' Jacob also shared,

> Going into it, I went in with the overconfidence aspect of 'it's just fractions, it's just division, it's just multiplication, like it's not that hard, it's middle school. I was like, 'Too easy.' But, all this new light, I'm like, 'Wow, I didn't know that.' So, I thought I knew it all, but now I am realizing I don't know it all, and I have so much more work to do. So, I'm less confident about it to where I have to keep striving to learn more at this point because I know that I *don't* know it all like I thought I did, and I'm like, 'Wow, I apparently had no clue.' You know? I knew the algorithm, but I didn't know all of this about it, so I am less confident.

Post-instructional data showed that Jacob's procedural knowledge of fraction division was consistent and was higher than his conceptual knowledge. Although Jacob increased his conceptual knowledge of fraction division, the data indicated that there are still gaps in his conceptual knowledge.

Summary of fraction division. Although Jacob progressed in his conceptual knowledge of fraction division, it appeared that he continued to prefer using his procedural knowledge. As evidenced by his Conceptual Post-Test and Post-Instruction Interview, there continued to be gaps in his conceptual knowledge of fraction division. Finally, even though Jacob indicated that he was less confident than previously, he appeared somewhat confident in using manipulatives to model fraction division.

Summary for Jacob. After experiencing constructivist teaching during a manipulatives-intensive fractions unit in a middle grades methods course for prospective teachers, Jacob demonstrated gains in his conceptual knowledge of fraction multiplication and division as evidenced by data collected from tests, observations, and interviews. Additionally, Jacob demonstrated greater confidence using manipulatives than before the fractions unit. The increased confidence Jacob demonstrated might have been due to the increase in his conceptual knowledge of fractions.

With respect to relational understanding, there was little data to confirm that Jacob achieved relational understanding in any of the areas examined. Jacob was not able to explain connections between his procedural and conceptual knowledge of fraction multiplication and division. Part of this inability might be attributed to the gaps that likely existed in his conceptual knowledge. When asked about his confidence in his ability to help his future students develop relational understanding, Jacob responded, “I am confident in my ability *not* to steer someone into left field, um, but I’m not a hundred percent confident of ‘I am the greatest teacher alive, this is how you do this.’ But I am more confident in my ability than I was before.”

Matthew

Matthew, a male student of Middle Eastern descent enrolled in the middle grades education program at the university, participated in this study while he was an EDMG-1 student. Matthew’s two chosen areas of specialty were mathematics and science. Matthew was a non-traditional student in his mid- to late-twenties and single with no children. He was a student athlete at the university and worked part-time at a local athletic facility teaching lessons in his area of athletic expertise. Matthew previously served in his country’s military, but not in a combat position. Instead, he served in a teaching capacity.

In his mathematics autobiography, Matthew cited several reasons for wanting to become a mathematics teacher and shared multiple experiences that contributed to his desire to be a mathematics teacher. These experiences included family relationships, language difficulties, and life situations.

Matthew’s stated that his grandmother was a mathematics teacher and was very influential in his understanding of mathematics. However, when his family relocated to another country, language barriers prevented his grandmother from continuing to teach mathematics. Although Matthew’s grandmother did not continue formally teaching mathematics, she continued helping him and, according to Matthew, passed her passion along to him. In fact, because of Matthew’s limited knowledge of the language in his new country of residence, he said that he depended on his grandmother and father for help understanding mathematics and other subjects when he started elementary school.

Matthew shared a memory about using Cuisenaire rods to learn addition and subtraction that he said felt like occurred yesterday. Matthew wrote, “Using these bars was one of my favorite activities because it felt like I was playing with a puzzle especially in that period. The period where words did not really make sense to me and the only thing I could relate to was numbers because it was a universal thing” (Mathematics Autobiography, p. 2).

In seventh grade, Matthew was accepted into a school that offered advanced math and science programs. He noted that geometry was one of his favorite subjects and that his grandmother helped to simplify the content. Matthew also shared that as he progressed in school he began to analyze his teachers' techniques and compare their techniques to how he would teach the content.

After Matthew finished school, he served in his country's army. He stated that his country's army had an education corp that was responsible for educating the public about the army to try to connect the two parties. Matthew indicated that he was sent into schools to tutor students who were struggling with mathematics. He shared that, at first, tutoring was a challenge because he did not know what to expect. However, he subsequently felt more comfortable, and it became clear to Matthew that he wanted to be a teacher.

Matthew's indicated that his decision to become a math teacher was the result of a long process. He noted, "Teaching is an honorable profession and is one of the most important professions there are" (Mathematics Autobiography, p. 3). Matthew also wrote, "Teachers have the capability to change and improve people's lives, and I want to become a teacher for one sole purpose, making a difference in other people's lives" (Mathematics Autobiography, p. 3). In summary, Matthew wrote,

> My own thoughts of how a good teacher should be, my goals, along with the life experiences I picked up from my grandmother, my father, and my teachers all have an impact on why I want to be a teacher. Throughout my whole life, I have always had teachers who motivate me to succeed and accomplish my dreams. These great inspirational individuals have helped me shape my decision to pursue a teaching career (Mathematics Autobiography, p. 3).

In his Initial Survey, Matthew indicated that he was somewhat confident in his procedural ability in the operations on fractions and his understanding of the meaning of the operations on fractions. In addition, Matthew rated himself as somewhat confident in understanding the 'why' behind the conventional algorithms of fractions. Finally, despite recalling using Cuisenaire rods during elementary school, Matthew indicated he was only slightly familiar with using manipulatives.

Matthew demonstrated an increase on his Conceptual Tests from Pre-Test to Post-Test, but demonstrated a decrease on his Procedural Tests from Pre-Test to Post-Test. Table 17 communicates Matthew's scores on all four of the tests, as well as the changes from Pre-Tests to Post-Tests.

Table 17

Matthew's Procedural and Conceptual Knowledge

	Pre-Test	Post-Test	Change
Procedural	94.1%	91.2%	– 2.9%
Conceptual	74.5%	86.3%	+ 11.8%

Fraction multiplication. Data were collected regarding Matthew's procedural and conceptual knowledge of fraction multiplication prior to, during, and after fraction instruction. In this section, these data are reported.

Pre-instruction data. On his Procedural Pre-Test, Matthew demonstrated his procedural knowledge of fraction multiplication by correctly answering two of the three problems: $\frac{1}{2}\times\frac{1}{3}$ and $2\frac{1}{6}\times 3\frac{3}{5}$. In Matthew's work, he used of the conventional algorithm associated with fraction multiplication. For example, Matthew converted each mixed number to an improper fraction and then used cross simplification to determine the answer $7\frac{4}{5}$.

On his Procedural Pre-Test, Matthew interpreted the problem $\frac{5}{8}$ of 2 on his as a division problem and calculated the answer as $\frac{5}{16}$. One notable difference between Matthew's work and his peers' work was the symbolic notation used. Instead of using the obelus ($\div$), Matthew used a colon (:) to indicate division. Matthew used a colon on two other fraction division problems on the Conceptual Pre-Test as well. In the United States, colons are commonly used when writing ratios, a comparison of two numbers. Matthew's use of a colon could be related to his early educational experiences in his country of origin.

Matthew's representation on the Conceptual Pre-Test for the fraction multiplication problem $\frac{1}{2}\times\frac{1}{3}$ contained an area model. Matthew also provided a verbal interpretation to accompany his pictorial representation, as shown in Figure 71 on the following page.

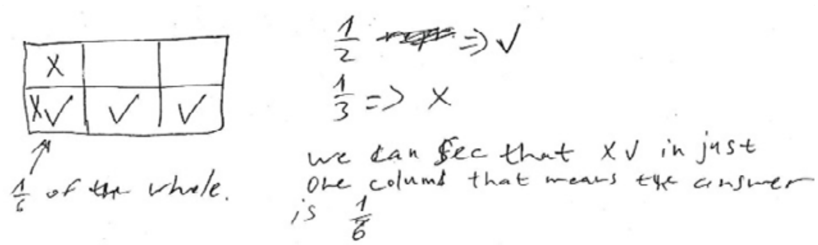


Figure 71. Matthew's representation for $\frac{1}{2} \times \frac{1}{3}$

For the problem $\frac{5}{8}$ of 2, Matthew seemed to begin with a length model representation but provided a region model instead. Matthew's work for $\frac{5}{8}$ of 2 is shown in Figure 72.

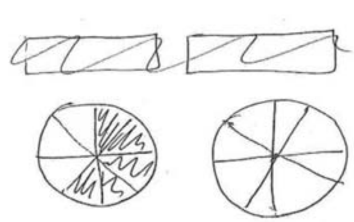
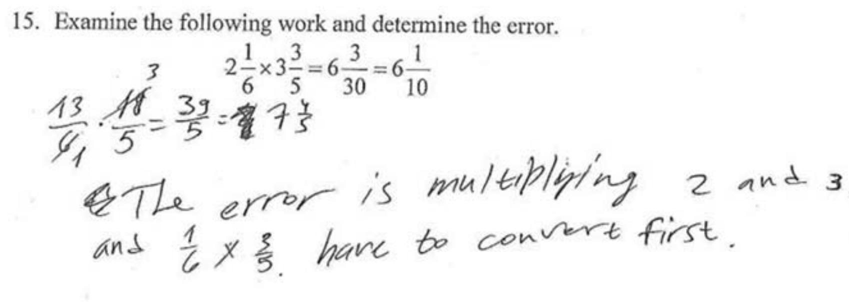


Figure 72. Matthew's representation for $\frac{5}{8}$ of 2

To address the error in $2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$, Matthew first noted that the error was multiplying the whole numbers and multiplying the fractions. Then, he referenced the need to convert first. Although he did not specify what he meant by convert, given that he provided procedural work alongside his explanation, it is likely that he meant that the mixed numbers needed to be converted to improper fractions. Matthew's work is shown in Figure 73 on the following page.



15. Examine the following work and determine the error.

$2\frac{1}{6}\times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$

$\frac{13}{\cancel{6}_1}\cdot\frac{\cancel{18}^{3}}{5} = \frac{39}{5} = 7\frac{4}{5}$

The error is multiplying 2 and 3. and $\frac{1}{6} \times \frac{3}{5}$. have to convert first.

Figure 73. Matthew's justification for the error associated with $2\frac{1}{6}\times 3\frac{3}{5}$

During his Pre-Instruction Task-based Interview, Matthew first used two-color counters as a set model to represent $\frac{3}{4}\times 6$ even though he stated that he was not aware of the type of model he was using. He also expressed concern about students thinking that the two-color counters represented positives and negatives instead of fractions. As Matthew arranged the counters, he stated that $\frac{3}{4}\times 6$ meant three-fourth groups of six. Matthew also indicated that the 6 was the multiplier and $\frac{3}{4}$ was the multiplicand. Next, Matthew shared that he did not remember the meanings associated with the terms *multiplier* and *multiplicand*, but he clarified that his model represented, "Six of three-fourths." Matthew's representation is shown in Figure 74.

Figure 74. Matthew's first representation for $\frac{3}{4}\times 6$

When Matthew was asked what his answer was, he worked the problem on paper using a standard algorithm of multiplying the numerators and the denominators and stated that the answer was $\frac{18}{4}$ or $4\frac{1}{2}$. I asked Matthew, “Can you show me that in your model?” Once he considered it, he said, “No, I cannot. It’s not a good model.” He then decided to use fraction circles instead. Matthew placed four sets of three yellow one-fourth fraction-circle pieces in front of him and said, “I have four times three-fourths.” He continued,

> I want to show, uh, six groups of, of three-fourths. *Ah* [says it as if he has just realized something]... six groups of three fourths. And here is three-fourth groups of six. You see what I am doing is, I am using the property of multiplication, that you can, uh, it doesn’t matter if it’s six by three or three by six, but kids, they don’t know it.

As Matthew verbalized six groups of three-fourths and three-fourth groups of six, he wrote the symbolic representations for each on his task paper: $6 \times \frac{3}{4}$ and $\frac{3}{4} \times 6$. Matthew subsequently provided the region model representation for $6 \times \frac{3}{4}$ shown in Figure 75.

$6 \times \frac{3}{4}$

$4\frac{1}{2}$

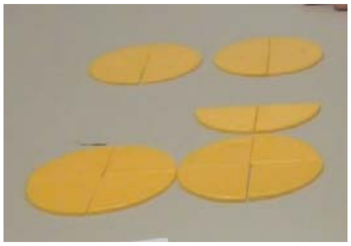


Figure 75. Matthew’s representation for $6 \times \frac{3}{4} = 4\frac{1}{2}$

Next, Matthew was asked to reconsider the set model that he generated earlier and explain how the answer $\frac{18}{4}$ was evident in the set model (see Figure 74). Matthew stated that each of the counters represented one-fourth of one whole, where four counters represented four-fourths or one whole group.

Once Matthew was asked to generate a real-world problem for $\frac{3}{4}\times 6$, he indicated that he could generate a real-world problem if he knew how to model $\frac{3}{4}\times 6$. Therefore, he again attempted to use the fraction circles to model $\frac{3}{4}\times 6$. Matthew stated that he would be able to determine *three-fourths of six* by calculating *two-fourths* [of six] and *one-fourth* [of six]. Then, Matthew stated that he knew that one-half of six was three. Once Matthew calculated one-fourth of six by multiplying $6\times\frac{1}{4}=1\frac{1}{2}$, he demonstrated his answer using the fraction circles, as shown in Figure 76.

Matthew began with six, the given number or multiplicand.

Matthew showed that three-fourths of six is four and one-half.

Figure 76. Matthew's representation for $\frac{3}{4}\times 6$

Once Matthew demonstrated $\frac{3}{4}\times 6$, he generated the following real-world context for the problem: "We have six pizzas. The dad of the family always eats three-fourth of no matter what the order. We need to figure out how many pizzas he will eat." Then, Matthew stated that the answer was $4\frac{1}{2}$ pizzas.

Pre-instructional data showed that Matthew progressed in his conceptual knowledge of fraction multiplication. He also demonstrated flexibility in using multiple representations.

Observational and assessment data. During the in-class fraction multiplication activity, data were collected regarding Matthew's use of both pictorial representations and manipulatives, primarily fraction circles. Matthew began the class activity by drawing pictorial representations. For $\frac{1}{2} \times \frac{1}{2}$, Matthew stated that he should begin by drawing a representation of the first number. Matthew transitioned to verbally expressing the problem as taking one-half of one-half. As a result, Matthew drew a circle, halved it, shaded in half of it, and said that that was how much he had. Then, he verbalized that he needed to take one-half of that. To demonstrate, Matthew drew a line across the shaded one-half, which made it the equivalent fraction two-fourths. Finally, Matthew shaded darker one of the already-shaded fourths and showed it to his classmate Jacob to find out if Jacob agreed.

Matthew transitioned to using fraction circle manipulatives because his poorly-drawn representations caused him some confusion. After using the fraction circle manipulatives to model $\frac{2}{3} \times \frac{3}{4}$ as 'three-fourths of a group of two-thirds' and as 'two-thirds of a group of three-fourths,' Matthew realized that the respective processes were different from each other even though the resulting products were equivalent. Once Matthew was aware of how the process was affected by the use of the multiplier and multiplicand, he began using these terms appropriately as he modeled the problems with the manipulatives. As he continued modeling fraction multiplication and mixed number multiplication with the fraction circle manipulatives and confirming his answers procedurally, Matthew became very excited about what he had done.

Although Matthew used fraction circle manipulatives during the class activity, he used a length model pictorial representation on his Problem Report that he submitted for a grade. The work that Matthew represented for $\frac{2}{3} \times \frac{3}{4}$ is shown in Figure 77.

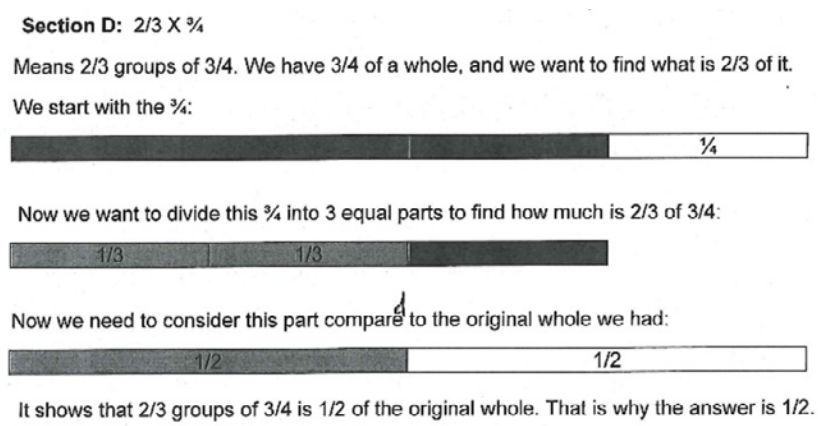


Figure 77. Matthew's length model representation for $\frac{2}{3} \times \frac{3}{4}$

In the reflective part of the assignment, Matthew shared several positive comments related to the use of manipulatives to demonstrate fraction multiplication.

- Working with manipulatives helped me to see how mathematics actually makes sense.
- Teaching this topic to young adolescents with manipulatives will improve their sense making mechanism because they can see and even touch the answer.

- This investigation helped me understand the procedure, and it convinced me not to teach students procedures right away.
- I learned during this investigation that even though we have the commutative property of multiplication, it matters which number is the multiplicand and which is the multiplier when we are trying the represent the problem with manipulatives and when we add context to the problem.

Observational and assessment data showed that Matthew continued to refine his understanding of fraction multiplication. He also continued to demonstrate flexibility in his use of multiple representations.

Post-instruction data. Matthew's performance on fraction multiplication on the Procedural Post-Test was not like his performance on the Procedural Pre-Test on two problems, $\frac{5}{8}$ of 2 and $2\frac{1}{6}\times 3\frac{3}{5}$. On the Pre-Test problem $\frac{5}{8}$ of 2, Matthew interpreted it as division. However, on the Post-Test, Matthew interpreted the problem as multiplication and calculated the answer $\frac{10}{8}=1\frac{1}{4}$ using the standard algorithm of multiplying the numerators and the denominators and simplifying.

Matthew calculated $2\frac{1}{6}\times 3\frac{3}{5}$ on the Procedural Pre-Test as $7\frac{4}{5}$ but calculated it as $13\frac{4}{5}$ on the Procedural Post-Test. Matthew first converted each mixed number to an improper fraction, cross simplified, and then multiplied the numerators and the denominators. His error occurred when he multiplied the numerators, 13×3, and produced 69 instead of 39. As a result, his answer was $13\frac{4}{5}$. This error was part of the reason Matthew's Procedural Post-Test was 2.9% less than his Pre-Test.

On his Conceptual Post-Test, Matthew's pictorial representation for $\frac{1}{2}\times\frac{1}{3}$ was a length model, whereas previously he used an area model. As shown in Figure 78, Matthew used solid lines to denote the thirds and dashed lines in *each* third to denote that the whole was partitioned into sixths.

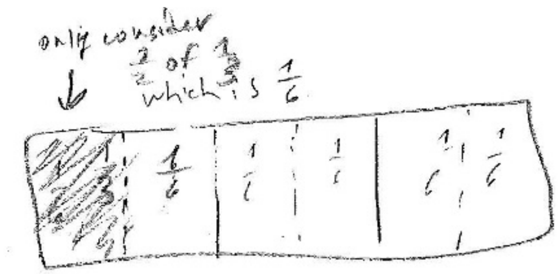


Figure 78. Matthew's length model representation for $\frac{1}{2}\times\frac{1}{3}$

Conversely, Matthew's pictorial representation for $\frac{5}{8}$ of 2 was somewhat difficult to interpret, as shown in Figure 79. Furthermore, Matthew did not provide a numerical answer to accompany his pictorial representation.

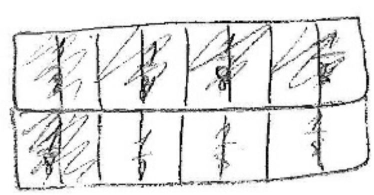


Figure 79. Matthew's representation for $\frac{5}{8}$ of 2

For the problem $2\frac{1}{6} \times 3\frac{3}{5}$, Matthew's Conceptual Post-Test differed very little from his Pre-Test answer. He again noted that the error in $2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$ was that the whole numbers and fractions cannot be multiplied separately; however, he did calculate the answer on the Post-Test as he had on the Pre-Test.

To demonstrate $\frac{2}{3} \times \frac{3}{4}$ as part of his Post-Instruction Task-based Interview, Matthew began with three yellow fourths overlaid on the red one whole. Then, Matthew overlaid two orange thirds onto the three-fourths, as shown in Figure 80. Matthew said, "We have to see how much is two-thirds out of…," but he did not complete his sentence. Once Matthew was questioned about how he was using the manipulative, he said, "I want to see what part of two-thirds from three-fourths."

Figure 80. Matthew's first representation for $\frac{2}{3} \times \frac{3}{4}$

As Matthew considered how to progress, the following exchange took place between him and the researcher:

Matthew:	Well, I want to see what's, what part of um, two-thirds from three-fourth. Fourths. So yeah I, I would say that probably I compared it.
Peppers:	Okay. How does that connect to multiplication, what you just said?
Matthew:	Well, because it's two-thirds of three-fourth. Two-thirds, um, two-thirds groups. Two-thirds groups of three-fourths.

Peppers:	Okay.
Matthew:	So ... well, I have to use the (laughs). I have to use um ... two-thirds and three-fourths.
Peppers:	What does it mean when you say two-thirds groups?
Matthew:	I don’t know. I mean ... what does it mean?
Peppers:	To say you have two-thirds groups? I believe that’s what you said, isn’t it?
Matthew:	Yeah.
Peppers:	Okay.
Matthew:	Okay, so let’s say I have um, that’s the part of what I have after that, so let’s say two-thirds of six, so that’s the part of s- the two-thirds that’s part of six. Um ...
Peppers:	Is that what you’re showing me with your manipulatives right now?
Matthew:	Yes. [sounds unsure of himself]
Peppers:	How?
Matthew:	That’s a better question. Um ...
Peppers:	What is the orange two-thirds of?
Matthew:	(sighs) Two-thirds of that ... mmm, okay. Two-thirds is the.. of the whole.
Peppers:	Okay.
Matthew:	Here we need to find the two-thirds of the three-fourths and yeah. So we need to see. We need to divide each one of them by three. [picks up two of the purple tenths and places them on one of the fourths, removes them] Is it this one? [picks up three black twelfths and places them on one of the fourths, continues by placing two more twelfths on the next fourth] Well, wait, we don’t need to do that [removes all of the twelfths], because it’s already divided by three. We have three parts and we consider only two. So if we consider only two, it’s gonna be half of the whole. So the answer is half.

The images in Figure 81 show Matthew's process of representing $\frac{2}{3} \times \frac{3}{4}$.

Figure 81. Matthew's process of representing $\frac{2}{3} \times \frac{3}{4}$

After Matthew explained and demonstrated $\frac{2}{3} \times \frac{3}{4}$ using the fraction circle manipulatives, he was asked about his initial difficulty. Matthew stated that, because the first task involved fraction division, he had difficulty transitioning from division to multiplication. In addition, Matthew stated that he knew the answer was one-half and that, as a result, overlaying the thirds onto the fourths did not make sense because of the gap between the thirds and the fourths.

Next, Matthew generated the following real-world scenario for $\frac{2}{3} \times \frac{3}{4}$: "Three-fourths of the students in the classroom are boys. Two-thirds of them are playing tennis. How many students are playing tennis in the classroom?" Matthew stated that the answer was half of the students. When he was asked to clarify with respect to his original question of *how many students*, Matthew said, "That's *how much* because we can't say *how many*. We will have to give actually the number of students and not the ratio."

Post-instruction data showed that Matthew's procedural and conceptual knowledge of fraction multiplication was generally consistent. At times, Matthew experience some confusion, but he seemed to be able to resolve it. He also continued to demonstrate flexibility using multiple representations for fraction multiplication.

Summary of fraction multiplication. Considering that Matthew's procedural knowledge of fraction multiplication was high at the onset of the course, the data collected showed limited evidence of improvement in this area, except for the problem $\frac{5}{8}$ of 2. Conversely, the data showed that Matthew made more progress in his conceptual knowledge of fraction multiplication than in his procedural knowledge. Even though Matthew demonstrated gains in his conceptual knowledge of fraction multiplication, the data collected showed that he continued to have areas of weakness. In general, Matthew demonstrated flexibility in his use of multiple representations for fraction multiplication, including length, area and set models as well as real-world contexts. Overall, Matthew's use of manipulatives was a mixture of procedural and meaningful.

Fraction division. Data collected regarding Matthew's procedural and conceptual knowledge of fraction division are discussed in this section. These data were collected through pre- and post-instruction tests, pre- and post-instruction interviews, and classroom observations.

Pre-instruction data. Matthew demonstrated his initial procedural knowledge of fraction division on the Procedural Pre-Test by correctly answering all four fraction division problems: $6 \div \frac{3}{4}$, $1\frac{1}{2} \div \frac{3}{4}$, $2 \div \frac{3}{8}$, and $2\frac{1}{3} \div 3$. As he procedurally calculated answers, Matthew used the standard algorithms associated with fraction division, including multiplying by the reciprocal of the divisor and cross simplifying.

On his Conceptual Pre-Test, Matthew provided a pictorial representation for three of the problems, as well as the procedural calculations for each of those three, as shown in Figure 82. Matthew's representation for the context problem provided more insight into his conceptual knowledge of fraction division.

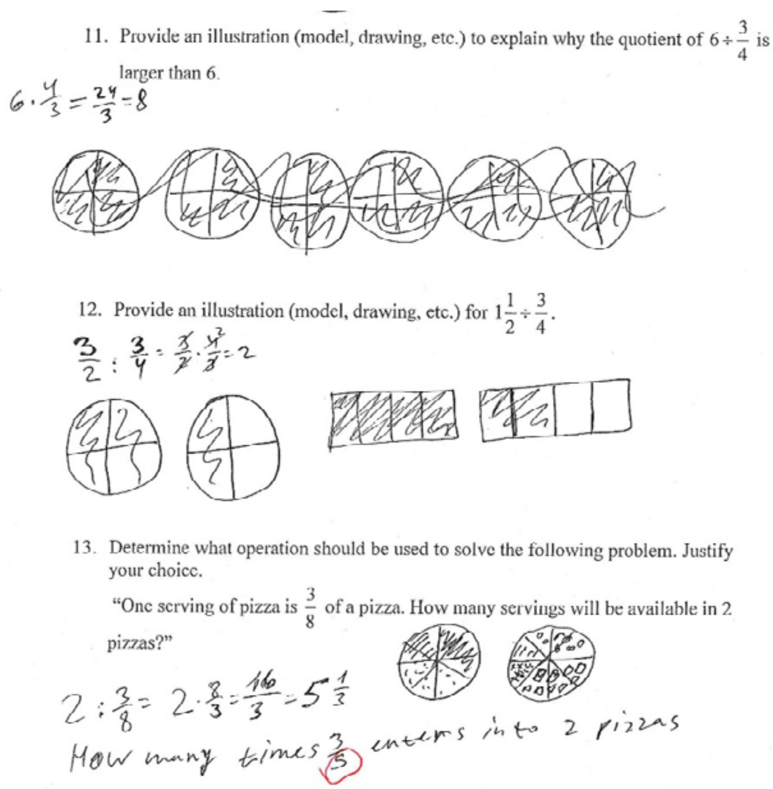


Figure 82. Matthew's initial conceptual knowledge of fraction division

By the time Matthew's Pre-Instruction Task-based Interview occurred, class instruction had already addressed the measurement interpretation of division using whole numbers and integers but not fractions. To demonstrate $3 \div \frac{2}{3}$, Matthew used three red one whole circles from fraction circle sets and overlaid them with orange thirds, shown in Figure 83 on the following page. As Matthew worked to make sense of his answer, he provided four different answers: $4\frac{1}{3}$, $3\frac{1}{3}$, $1\frac{1}{2}$, and $4\frac{1}{2}$ (respectively).

Figure 83. Matthew's demonstration of $3 \div \frac{2}{3}$

As Matthew explained why he believed the answer was $4\frac{1}{2}$, he indicated that he mentally calculated the answer using the conventional algorithm of multiplying by the reciprocal of the divisor. Once he was asked to explain further, Matthew said, "Because we're dividing it by two-thirds, what we [have] left is one-third, which is a half of two-thirds."

Next, Matthew attempted to generate a real-world scenario for $3 \div \frac{2}{3}$. The scenario was about having three pizzas that would be shared. Each person was supposed to eat two-thirds of a pizza. However, Matthew did not formulate a question for the scenario. He considered whether the answer could be $4\frac{1}{2}$ people but decided the unit *people* was not appropriate. Then, Matthew altered his scenario such that two-thirds of a pizza was eaten each day. He asked, "How many days it's going to take me to finish that pizza?" Matthew interpreted that the *one-half* in his answer represented one-half of a day. Matthew stated that he was not confident in his scenario, but he thought it was the best he could do at the time.

Pre-instructional data showed that Matthew's procedural knowledge of fraction division was consistent, while his conceptual knowledge was inconsistent. It appeared that Matthew had some gaps in his conceptual knowledge of fraction division.

Observational and assessment data. Limited observational and assessment data were collected regarding Matthew's progressing fraction division knowledge. During the fractions unit, the class used manipulatives to model the measurement interpretation of fraction division in a real-world context, as well as without context. The real-world context problem was about flower beds being fertilized. Five flower beds were to be fertilized using bags of fertilizer that would cover three-fourths of each flower bed. The class modeled the problem using pattern blocks, represented it symbolically as $5 \div \frac{3}{4}$, and discussed the reason for the standard algorithm of changing division to multiplication and inverting the divisor.

Another context problem that was discussed was about a four-foot party sandwich that would be shared among guests. Each guest was to receive one-fourth of a foot of the sandwich, which was represented symbolically as $4 \div \frac{1}{4}$. A pictorial length model representation was drawn on the white board and discussed.

After the discussion concluded, Matthew inquired as to why inverting and multiplying was an appropriate algorithm. The instructor explained that each foot provided four servings and related that to the multiplication problem 4×4. In addition, the instructor wrote the following on the white board: $4\,\text{groups} \times \frac{4\,\text{servings}}{1\,\text{group}} = 16\,\text{servings}$.

The instructor noted that the unit *group* simplified and the unit *servings* remained as the unit for the answer. After the instructor provided the explanation, Matthew did not comment any further.

Based on the limited data available, it is not appropriate to draw any conclusions about Matthew's progressing procedural and conceptual knowledge of fraction division. In addition, no data related to Matthew's use of representations was collected. Therefore, no conclusions can be made at this point.

Post-instruction data. Despite the fact that Matthew correctly answer all four fraction division problems on his Procedural Pre-Test, he only answered three of the four correctly on the Procedural Post-Test. For the problem Matthew answered incorrectly, $1\frac{1}{2} \div \frac{3}{4}$, he converted the mixed number to an improper fraction, inverted the divisor, changed division to multiplication, and cross simplified. However, instead of getting the answer 2 as he had on the Pre-Test, Matthew wrote $\frac{1}{2}$ as his answer on the Post-Test. It was not clear as to why Matthew gave this answer.

Matthew's answers on the Conceptual Post-Test for fraction division also contained procedural knowledge, as shown in Figure 84.

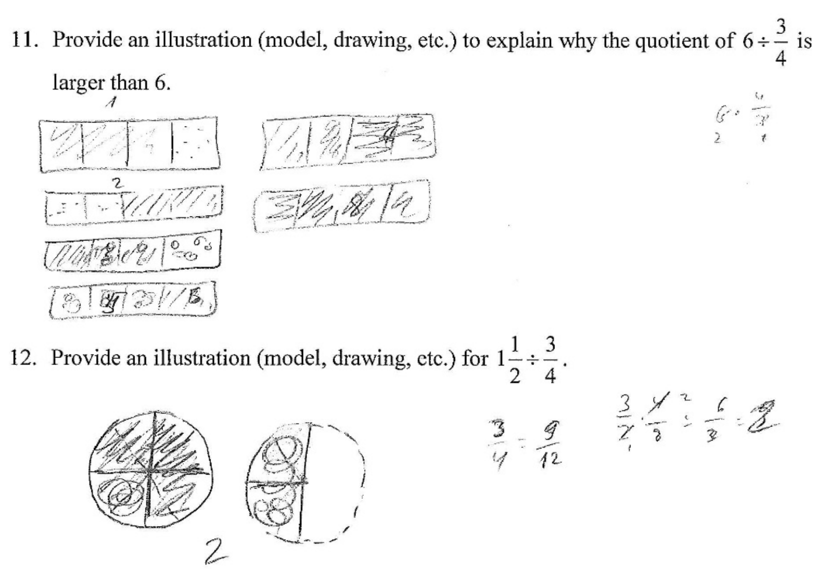


Figure 84. Matthew's Conceptual Post-Test representations for fraction division

One particular difference between Matthew's Post-Test representations for $6 \div \frac{3}{4}$ in comparison to his Pre-Test representation was the fact that he darkened in all of the fourths, whereas previously he left one-fourth of each whole unshaded on his Pre-Test.

For the remaining fraction division problem, Matthew previously indicated on his Conceptual Pre-Test that the answer to $2\frac{1}{3} \div 3$ should be less than one because "3 does not fit into $2\frac{1}{3}$ not even one time." On his Conceptual Post-Test, Matthew noted that if the number used to divide by (divisor) is larger than the number being divided (dividend), the answer will be less than one. Matthew also provided the converse, that if the divisor is smaller than the dividend, the answer is larger than one. Matthew was able to articulate his generalization better on the Post-Test answer than he did on the Pre-Test.

During his Post-Instruction Task-based Interview, for the problem $\frac{5}{6} \div \frac{1}{2}$, Matthew began by stating that he needed to determine how many groups of one-half can be made from five-sixths. Matthew continued by saying that the answer should be one and that he needed to determine how much the remaining part was when compared to the half. As Matthew compared the remaining part ($\frac{2}{6}$) to one-half, as shown in Figure 85, he said, "It's two-thirds of the half." He also stated that the answer was $1\frac{2}{3}$.

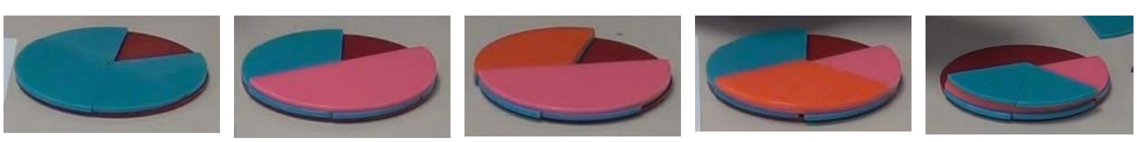


Figure 85. Matthew's process modeling $\frac{5}{6} \div \frac{1}{2}$

Once Matthew provided his answer of $1\frac{2}{3}$, he was asked to justify the algorithm of inverting the divisor and multiplying. Initially, he said laughingly, “Because I was taught like this all my life.” Then, Matthew explained, “Our whole [red] contained two parts of the original whole [pink].” As he explained, he hovered the pink one-half over the five-sixths that he used previously, which is shown in Figure 86. However, Matthew did not explain what function $\frac{5}{6}$ has in the procedure.

Figure 86. Matthew demonstrating his explanation of multiplying by the reciprocal

Finally, Matthew generated a real-world scenario similar to one the class explored during fraction division instruction. Matthew stated, “I have five-sixths of a bed. It takes two bags [of fertilizer] to fill up one bed. How many bags I’m going to use for this?”

Post-instruction data showed that Matthew’s procedural knowledge of fraction division was fairly consistent. The data also showed that Matthew refined his conceptual knowledge of fraction division from what it was initially. He also effectively demonstrated the measurement interpretation of division using multiple representations. His use of manipulatives appeared to be generally meaningful.

Summary of fraction division. As evidenced by the data collected, Matthew's procedural knowledge of fraction division was, for the most part, consistent. Initially, Matthew demonstrated at least some degree of conceptual knowledge of measurement division of fractions, but he was not confident in his knowledge. Data showed that Matthew's conceptual knowledge of fraction division increased. Overall, Matthew's use of manipulatives was generally meaningful. After the conclusion of the fractions unit, Matthew expressed greater confidence in his conceptual knowledge of fraction division than he had initially.

Summary for Matthew. The data collected showed that Matthew's procedural knowledge of fraction multiplication and division were both generally consistent prior to and after the fractions unit. The data also showed that Matthew's conceptual knowledge of these concepts increased and/or was refined after the fractions unit, but there were still gaps in his knowledge. Even though Matthew experienced difficulty transitioning from fraction division to fraction multiplication, he was aware of this difficulty and seemed to resolved it.

Matthew may have experienced some interference of his procedural knowledge on his developing conceptual knowledge or on his ability to demonstrate his knowledge using manipulatives. Generally, Matthew procedurally worked out problems in his head and then attempted to use the manipulative to show his answer. At times, Matthew experienced some confusion as he attempted to model his answer using the manipulatives. It was not clear whether the confusion was interference due to procedural knowledge.

With respect to relational understanding, Matthew demonstrated both procedural and conceptual knowledge of fraction multiplication, but there was limited evidence that he understood the connections between them. In addition, although Matthew demonstrated procedural and conceptual knowledge of fraction division, he only demonstrated limited evidence that he understood the connections between them. Finally, even though Matthew attempted to explain the connections between the procedural and conceptual meaning of fraction multiplication and division and the manipulative models, there was not enough data to draw any conclusions about his relational understanding of manipulatives.

Summary of Research Findings

The four prospective teachers in this study showed varying levels of procedural and conceptual knowledge of fraction multiplication and division as evidenced by the observational, interview, and assessment data. Initially, all four participants demonstrated gaps in their conceptual knowledge and some inconsistencies in their procedural knowledge. All of the participants demonstrated some progress in filling the gaps in their conceptual knowledge, but it is possible that not all of the demonstrated gaps were filled.

The four participants also had limited prior learning experiences using manipulatives, and, at times, experienced some level of confusion as they attempted to use the manipulatives to demonstrate mathematical concepts. The four participants displayed varying levels of procedural and meaningful use of manipulatives and pictorial representations as they attempted to demonstrate their conceptual knowledge. In addition, representations provided by all four of the participants lacked precision in some manner, whether pictorial or verbal.

With respect to relational understanding, none of the participants clearly demonstrated relational understanding of the fraction concepts being investigated. Even though none of the prospective teachers were able to articulate a clear connection between the manipulative representation and the associated symbolic algorithms, they all indicated that they intended to use manipulatives in their future classrooms to help students develop conceptual understanding of fraction concepts.

Chapter 5: Conclusions, Discussions, and Suggestions for Future Research

This chapter first presents limitations to this study, followed by conclusions of the study. Next, findings related to the literature are presented, and implications for prospective-teacher methods courses are discussed. Finally, further areas of study are suggested based on the findings of the study.

Limitations

Prior to discussing the conclusions of this study, it is necessary to point out specific limitations of the study. The limitations of this study were sample location, size, time constraints, and data collection. These limitations do not compromise the usefulness of this study, but they do limit its generalizability. This study was conducted in a one-semester, middle grades methods course in a mid-sized, four-year university in the southeastern United States. Data were collected for only four of the 10 students in the class, each with varying academic backgrounds. It was not clear as to how the differences in their academic backgrounds impacted the results. With respect to data collection, there were no interviews conducted related to the Procedural and Conceptual Tests. If interviews had been conducted, clarifications might have been made for incomplete answers which would have impacted the analysis and interpretation of the data. If this study were to be replicated in a different size university, in a different region of the U.S., and with a different population, the results of such a study may or may not be similar to the findings of this study.

Conclusions

The purpose of this study was to determine the impact of a manipulatives-intensive fractions unit during a middle grades methods course on prospective teachers' relational understanding of fractions. In Chapter 1, I suggested that prospective teachers in current colleges of education may have gaps in their conceptual knowledge of mathematics and limited experiences using manipulatives to learn mathematics. In Chapter 2, I asserted that relational understanding is having both conceptual and procedural knowledge and knowing the connections between them (Skemp, 1987). As a result of this study, I found that prospective teachers seemed to have gaps in their conceptual knowledge of fractions and were quite unfamiliar with using manipulatives. Additionally, the prospective teachers in this study demonstrated a variety of levels of procedural and conceptual knowledge of fractions. However, there was limited evidence that showed that these prospective teachers were able to make connections between their procedural and conceptual knowledge.

It is possible that prospective teachers did not demonstrate relational understanding of fractions because of the gaps in their conceptual knowledge of fractions at the beginning of the course and the limited time available during the course to develop necessary connections. It stands to reason that before one can possess relational understanding of a concept, one must first possess procedural and conceptual knowledge of the concept. Considering that there is limited time available during a one-semester middle grades methods course, it is possible that the gaps in their knowledge limited them from having ample time to fully develop their relational understanding.

In Chapter 3, I proposed a Venn diagram among three variables: procedural knowledge, conceptual knowledge and use of manipulatives (see *Figure 7*). Based on Skemp's (1987) conception of relational understanding as having procedural and conceptual knowledge and knowing the connections between them, I proposed that the intersection of the aforementioned three variables could be thought of as having relational understanding of manipulatives.

As I examined participants' data through a lens of procedural use or meaningful use of manipulatives, I realized that I should revise my proposed diagram to include all representations. This realization was based on the fact that, at times, participants' used pictorial representations in lieu of manipulatives. Some pictorial representations were based on participants' procedural knowledge, and some were meaningful because they were based on participants' conceptual knowledge. For example, Jacob used pictorial representations during his Post-Instruction Task-based Interview to model $\frac{2}{3} \times \frac{3}{4}$. As Jacob represented the fraction multiplication pictorially, it seemed that he may have been using the representation procedurally to produce the answer that he already calculated.

In considering Hiebert and Carpenter's (1992) claim that "mathematics is understood if its mental representation is part of a network of representations (p. 67)," I focused on the phrase *network of representations*. I also considered literature that suggested that teachers need to understand how the physical representation of manipulatives connects to the symbolic representation to realize the full potential of learning through using manipulatives (Ball, 1992; Moyer, 2001; NCTM, 2000). If mathematics is going to be a useful tool to solve problems, its users need to understand the connections among procedural knowledge, conceptual knowledge, and representations to realize its maximum usefulness. Therefore, I posit that one has relational understanding of representations when one knows which representations to use in order to best represent a mathematical idea conceptually and is able to choose the appropriate procedure to apply and why to apply it. Furthermore, having relational understanding of representations means the user of the representations will also recognize connections among the various types of representations: symbolic, pictorial, concrete, verbal, and real-world context. Figure 87, shown below, is my revised version of this idea.

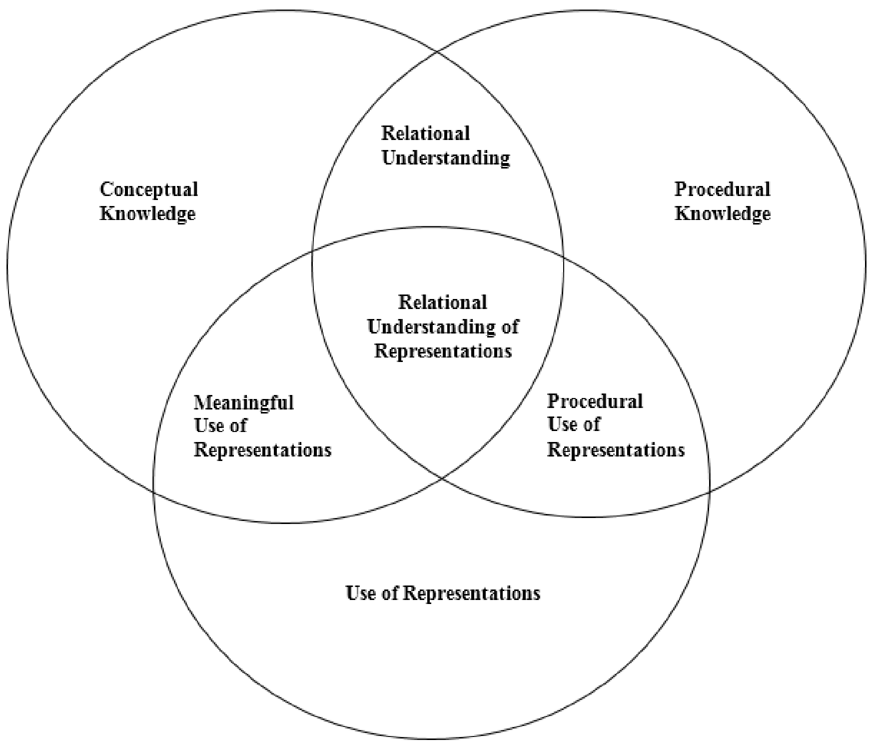


Figure 87. Final Diagram of Relationship among Procedural and Conceptual Knowledge and Use of Representations

During the fractions unit, a class activity (mentioned earlier in this paper) provided an opportunity for prospective teachers (PTs) to possibly develop relational understanding of representations. The class was presented with a task about determining the number of bags of fertilizer needed for a specified number of flower beds. The real-word context is a type of representation. Prospective teachers were required to represent the context either concretely with manipulatives or pictorially. Additionally, PTs were responsible for expressing the problem symbolically, explaining why the context implied division, and explain why the traditional algorithm of invert and multiply related to the context. If PTs were able to make connections among multiple representations and recognize the connections among the representations and their procedural and conceptual knowledge, then I suggest that they would have demonstrated relational understanding of representations.

Since it was not clear that any of the four participants demonstrated relational understanding of fraction concepts, I was not able to determine the extent to which they achieved relational understanding of representations. Further study in this area may reveal more about this idea.

Findings Related to Literature

In comparison to Ma's (1999) findings, two of the four participants in this study, Samantha and Krystal, experienced confusion between *dividing by one-half* and *dividing in half*. Samantha seemed able to resolve her confusion by focusing on the conceptual meaning of division by one-half and her demonstration using fraction circles. Krystal seemed to rely more on her procedural knowledge to clarify that dividing by one-half meant to multiply by the reciprocal, 2.

With respect to Pesek and Kirshner's (2000) interference of procedural knowledge on conceptual knowledge, all four participants seemed to experience some degree of confusion as they attempted to demonstrate fraction multiplication and division. This confusion could have been related to the gaps in their conceptual knowledge, their limited experience using manipulatives, or interference from their existing or lacking procedural knowledge. For example, as Jacob attempted to model $\frac{2}{3} \times \frac{3}{4}$ during his Post-Instruction Task-based Interview, he first used fraction circles in a manner related to fraction comparison or fraction addition and subtraction instead of focusing on the conceptual meaning of multiplication. At the beginning of the task, Jacob procedurally calculated the answer even though he did not communicate it until the end of the interview.

Moss and Case (1999) expressed concern about the over-use of the circle model because its over-use potentially limits students' fraction knowledge to part-whole relationship. The participants in this study were provided a variety of manipulatives (See Appendix L) with which to demonstrate their knowledge during the Pre- and Post-Instruction Interviews. During their Pre-Instruction Task-based Interviews, Samantha used pattern blocks and pictorial area representations for multiplication and pattern blocks for division; Krystal used fraction circles for multiplication and division; Jacob used a set model for multiplication and drew circles for division; and Matthew used a set model for fraction multiplication but discarded it for fraction circles, and he used fraction circles for division. On the contrary, all four participants exclusively used fractions circles during their Post-Instruction Task-based Interview.

If these prospective teachers use manipulatives once they begin teaching, the question arises as to whether they will exclusively use the fraction circles or whether they will recognize the usefulness of other manipulatives. If they recognize the usefulness of other manipulative models, the further question will be whether they chose an appropriate manipulative for a specific problem/concept.

Johanning (2008) provided an example of students using decimals in lieu of fractions or mixed numbers and the teacher helping the students to determine that doing so would not provide equivalence. The example dealt specifically with using the truncated decimal 5.3 instead of the mixed number $5\frac{1}{3}$. During Samantha's interview, she noted that she could attempt to use a decimal for $\frac{2}{3}$, but she recognized that it would not provide an equivalent answer. This example provides insight into Samantha's ability to guide her future students to greater conceptual knowledge of the inappropriateness of using truncated decimals in lieu of fractions.

In Chapter 2, research by Puchner, Taylor, O'Donnell, and Fick (2008) and Olive and Vomvoridi (2006) were highlighted regarding poorly-drawn pictorial representations that undermined students' understanding that fractional pieces need to be the same size. In this study, all four prospective teachers displayed a lack of precision of drawing fractions using a circle as the whole. For example, in Krystal's pictorial representation of $\frac{1}{2}\times\frac{1}{3}$ on her Post-Instruction Conceptual Test, not all of the thirds in her circle model were equivalent.

Puchner et al. (2008) also expressed concern about the "answer-first, model-second" method of using manipulatives. Some of the prospective teachers also displayed the "answer-first, model-second" method of using manipulatives. For example, Jacob consistently calculated answers procedurally prior to attempting to model the problems conceptually. Although it was not clear whether this method interfered with Jacob's ability to conceptually model problems, it is possible that interference did occur. For example, during his Post-Instruction Task-based Interview, Jacob experienced difficulty modeling $\frac{2}{3}\times\frac{3}{4}$. However, once Jacob focused on the conceptual meaning of the problem, he was successful modeling the problem. These findings possibly point to potential problems in these prospective teachers' future classrooms. If these practices continue, students' appropriate use of manipulatives may be adversely impacted.

Finally, in Chapter 2, literature by Cramer and Wyberg (2009) was discussed regarding the efficacy of concrete models. The difference in the concrete models used by Samantha and Krystal to model $3\div\frac{2}{3}$ may have contributed to Samantha's correct solution and may have interfered with Krystal generating a complete solution. Samantha used a black chevron to model the fraction $\frac{2}{3}$, whereas Krystal used two individual $\frac{1}{3}$ fraction circle pieces. Krystal experienced difficulty making sense of the remaining $\frac{1}{3}$ fraction piece but realized that she was closer than she had been during a previous attempt. It is possible that the use of the two individual $\frac{1}{3}$ fraction pieces interfered with Krystal counting the groups of $\frac{2}{3}$ that were needed to make 3 because Krystal counted the thirds in increments of $\frac{1}{3}$. In contrast, Samantha counted each chevron and recognized that the remaining rhombus ($\frac{1}{3}$) was half of a chevron.

Implications for Prospective Teacher Methods Courses

This study may be of interest to colleges of education as they attempt to prepare prospective teachers to successfully implement the new standards. Generally, the primary purpose for methods courses is to familiarize prospective teachers with appropriate, research-based pedagogical methods for facilitating experiences that lead to their students' mathematical learning, not for the prospective teachers to learn the mathematics that they will be teaching. However, teachers cannot teach beyond their own level of knowledge. All four of the participants in this study demonstrated gaps in their procedural and conceptual knowledge of fractions as assessed through tests, observations, and interviews. Furthermore, although all four participants demonstrated some degree of procedural and conceptual knowledge of fractions, none of them clearly demonstrated relational understanding of fractions.

Since there is limited time available in methods courses and much to accomplish, perhaps teacher educators should consider integrating instruction through a focus on Ball et al.'s (2008) Mathematical Knowledge for Teaching, as well as on Ma's (1999) Pedagogical Content Knowledge. By integrating pedagogical issues with mathematical content in this manner, perhaps prospective teachers will have opportunities to develop specialized content knowledge, common content knowledge, knowledge of the mathematical horizon, knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.

Another possible implication of this study for teacher educators is how to address gaps in prospective teachers' knowledge. Middle grades prospective teachers at the university in which this study was conducted are not required to take a mathematics course related to middle grades mathematics like elementary prospective teachers are required to take for their program, i.e. "math for elementary teachers." Instead, they are required to take the following courses: college algebra, pre-calculus, applied calculus, introductory statistics, discrete mathematics, and geometry & measurement. Perhaps a course solely focused on "math for middle grades teachers" could be instituted to address prospective teachers' relational understanding of middle grades mathematics content rather than addressing the issue during the methods course. Requiring a mathematics course specifically for middle grades teachers to explore the mathematics content of grades 5-8 would be in alignment with recommendations by the Conference Board of Mathematical Sciences (2012). They recommended a mathematics course for prospective middle grades teachers that provides the opportunity to explore middle grades mathematics content, but specified that instructors for the course model appropriate pedagogy aligned with the Standards for Mathematical Practice (National Governors Association Center for Best Practices, & Council of Chief State School Officers [NGA Center & CCSSO], 2010).

Taking into consideration that the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010) suggests that teachers use concrete models to facilitate fraction understanding, teachers must possess deep understanding of the manipulative models they use, the appropriateness of the model, and how to connect the attributes of the manipulative model to the symbolic representation. Of the four participants in this study, none of them explicitly explained the connections between the manipulative model and the related symbolic representation. If these prospective teachers are to successfully use manipulatives to facilitate their students' procedural and conceptual knowledge, it is likely that they will first need to spend additional time using the materials and experience for themselves the connections between the concrete and symbolic representations before they will be able to facilitate such experiences for their students.

During this study, all four participants provided forms of representation that lacked attention to precision. One of the Standards of Mathematical Practice advocated in the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010) is to attend to precision. A lack of attention to precision by teachers may negatively impact students' understanding. Therefore, teacher educators may find this study helpful in considering the types of representations and whether prospective teachers attend to precision in the various types of representations they employ.

Further Areas of Study

This study focused on the impact that a manipulatives-intensive fractions unit in a middle grades methods course had on the relational understanding of fractions of four specific prospective teachers. As with any research, the process of attempting to answer questions related to a study gives birth to new questions.

First of all, the question arises as to what other aspects of the course impacted prospective teachers' relational understanding. Secondly, what, if any, aspects of the course hindered prospective teachers' development of relational understanding? What part did the difference in participants' mathematics education backgrounds play in their propensity to persevere?

Considering that all four participants in this study used fraction circles during their Post-Instruction Task-based Interviews, the question also arises as to whether they will be knowledgeable enough to choose other appropriate manipulatives to model particular concepts. Therefore, further research could be done to examine prospective teachers' ability to choose an appropriate manipulative when given a specific scenario or concept.

Because this study was done prior to the prospective teachers' student teaching experience, further research could examine whether prospective teachers' correctly implemented manipulatives during their student teaching experience. In addition, further research could examine whether the prospective teachers' allowed students to explore concepts using manipulatives or whether the prospective teachers *showed* students how to use the manipulatives.

Further longitudinal research could be done to examine the development of prospective teachers' relational understanding of fractions beginning at the onset of their first methods course and concluding after their student teaching experience. In addition, since this research studied prospective teachers' relational understanding, research could also be done to study practicing teachers' relational understanding of fractions.

Finally, teachers' relational understanding of representations could also be studied within the rational number domain, as well as other domains. By studying the extent of prospective and practicing teachers' relational understanding of representations, research may inform practice and provide strategies to improve student achievement in facilitating the use of representations, as well as connecting the representations.

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Appendices

Appendix A: Procedural Knowledge Test

Calculate each answer. Show all associated work.

1. 4×3
2. $8 \div 2$
3. Determine which fraction is larger: $\frac{10}{11}$ or $\frac{9}{10}$
4. Determine which fraction is larger: $\frac{15}{31}$ or $\frac{10}{19}$
5. Provide 3 values equivalent to three-fourths.
6. Show numerically why $\frac{12}{15}$ is equivalent to $\frac{28}{35}$.
7. $\frac{2}{3} + \frac{3}{4}$
8. $5\frac{2}{3} - 1\frac{3}{4}$
9. $\frac{1}{2} \times \frac{1}{3}$
10. $\frac{5}{8}$ of 2
11. $6 \div \frac{3}{4}$
12. $1\frac{1}{2} \div \frac{3}{4}$
13. $2 \div \frac{3}{8}$
14. $4.21 + 135 + 69.7$
15. $2\frac{1}{6} \times 3\frac{3}{5}$

16. $2\frac{1}{3} \div 3$

17. 4.25×9.24

Appendix B: Answer Key to Procedural Test

1. $4 \times 3 = 12$

2. $8 \div 2 = 4$

3. Determine which fraction is larger: $\frac{10}{11}$ or $\frac{9}{10}$ Answer: $\frac{10}{11}$

   Possible answer: Each fraction is one unit fraction from being one whole. The fraction which is missing the smaller unit fraction is $\frac{10}{11}$. That is, $\frac{10}{11}$ is $\frac{1}{11}$ from 1, while $\frac{9}{10}$ is $\frac{1}{10}$ from 1. Since $\frac{1}{11} < \frac{1}{10}$, $\frac{10}{11}$ is not as far from 1 as $\frac{9}{10}$.

   Possible answer: Get a common denominator and compare numerators. That is,

   $\frac{100}{110} > \frac{99}{110}$

   Possible answer: Convert each to a decimal and compare. That is, $\frac{10}{11} \approx 0.\overline{90}$, while

   $\frac{9}{10} = 0.9$

4. Determine which fraction is larger: $\frac{15}{31}$ or $\frac{10}{19}$ Answer: $\frac{10}{19}$

   Possible answer: Compare to benchmark of $\frac{1}{2}$. $\frac{10}{19}$ is greater than $\frac{1}{2}$, while $\frac{15}{31}$ is less than $\frac{1}{2}$.

   Possible answer: Compare decimals. $\frac{15}{31} \approx 0.4839$; $\frac{10}{19} \approx 0.5263$

   Possible answer: Get a common denominator and compare numerators.

   $\frac{285}{589} < \frac{310}{589}$

5. Provide 3 values equivalent to three-fourths.

Possible answers: 0.75, 75%, $\frac{3}{4}$ or any equivalent fraction

6. Show numerically why $\frac{12}{15}$ is equivalent to $\frac{28}{35}$.

   Possible answer: Both simplify to $\frac{4}{5}$ or 0.8

   Possible answer: Both can be converted to $\frac{84}{105}$ or an equivalent fraction.

7. $\frac{2}{3}+\frac{3}{4}$

   Possible work: $\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$ or $1\frac{5}{12}$

8. $5\frac{2}{3}-1\frac{3}{4}$

   Possible work: $\frac{17}{3}-\frac{7}{4} \rightarrow \frac{68}{12}-\frac{21}{12} \rightarrow \frac{47}{12} \rightarrow 3\frac{11}{12}$

   Possible work: $5\frac{8}{12}-1\frac{9}{12} \rightarrow 4\frac{20}{12}-1\frac{9}{12} \rightarrow 3\frac{11}{12}$

9. $\frac{1}{2}\times\frac{1}{3}= \frac{1}{6}$

10. $\frac{5}{8}$ of 2 = $1\frac{1}{4}$

   Possible work: $\frac{5}{8}\times 2 \rightarrow \frac{5}{8}\times\frac{2}{1} \rightarrow \frac{10}{8} \rightarrow \frac{5}{4} \rightarrow 1\frac{1}{4}$ or $\frac{10}{8} \rightarrow 1\frac{2}{8} \rightarrow 1\frac{1}{4}$

   Possible work: Cross simplification $\frac{5}{8}\times\frac{2}{1} \rightarrow \frac{5}{4}\times\frac{1}{1}$ by dividing a common factor of 2

11. $6\div\frac{3}{4}$

   Possible work: $\frac{6}{1}\div\frac{3}{4} \rightarrow \frac{6}{1}\times\frac{4}{3} \rightarrow \frac{24}{3} \rightarrow 8$

Possible: Cross simplification $\frac{6}{1}\times\frac{4}{3}\rightarrow\frac{2}{1}\times\frac{4}{1}$ by dividing a common factor of 3

12. $1\frac{1}{2}\div\frac{3}{4}$

Possible work: $\frac{3}{2}\div\frac{3}{4}\rightarrow\frac{3}{2}\times\frac{4}{3}\rightarrow\frac{12}{6}\rightarrow 2$

Also possible: Cross simplification $\frac{3}{2}\times\frac{4}{3}\rightarrow\frac{1}{1}\times\frac{2}{1}$ by dividing common factor of 3

and a common factor of 2

13. $2\div\frac{3}{8}$

Possible work: $\frac{2}{1}\div\frac{3}{8}\rightarrow\frac{2}{1}\times\frac{8}{3}\rightarrow\frac{16}{3}\rightarrow 5\frac{1}{3}$

14. $4.21+135+69.7=208.91$

15. $2\frac{1}{6}\times 3\frac{3}{5}=7\frac{4}{5}$

Possible work: $\frac{13}{6}\times\frac{18}{5}\rightarrow\frac{234}{30}\rightarrow 7\frac{24}{30}\rightarrow 7\frac{4}{5}$

Also possible: Cross simplification $\frac{13}{6}\times\frac{18}{5}\rightarrow\frac{13}{1}\times\frac{3}{5}\rightarrow\frac{39}{5}\rightarrow 7\frac{4}{5}$

16. $2\frac{1}{3}\div 3=\frac{7}{9}$

Possible work: $\frac{7}{3}\div\frac{3}{1}\rightarrow\frac{7}{3}\times\frac{1}{3}\rightarrow\frac{7}{9}$

17. $4.25\times 9.24=39.27$

Appendix C: Rubric for Procedural Knowledge Test

2 – Correct algorithm with correct answer

1 – Correct algorithm with some minor calculation errors

0 – No answer or incorrect procedure with incorrect answer

Appendix D: Conceptual Knowledge Test (including justifications for use)

1. Provide an illustration (model, drawing, etc.) to represent $4 \times 3 = 12$ and a model (drawing) to represent $3 \times 4 = 12$. Explain the meaning of the 4, the 3, and the 12 in each representation.

   I chose to use this problem to determine whether prospective teachers (PTs) understand the purpose of a multiplier (number of groups) and a multiplicand (number in each group)(Tsay & Hauk, 2009).

2. Provide illustrations (models, drawings, etc.) to represent $8 \div 2 = 4$ interpreted in two different ways. Explain the meaning of the 8, the 2, and the 4 for each interpretation.

   I chose this problem to determine whether prospective teachers understand the difference between the partitive model of division and the measurement model of division (Osana & Royea, 2011). This will help me to understand their background knowledge as I am analyzing their responses to the fraction division problems which involve the measurement model.

3. Without converting each fraction to a decimal or using a conventional algorithm, determine which fraction is larger: $\frac{10}{11}$ or $\frac{9}{10}$. Explain your reasoning.

   I chose this problem because I want to find out whether PTs will recognize that each fraction is 2 unit fractions away from one whole and use the knowledge of which unit fraction is larger to determine magnitude (Cramer & Wyberg, 2009).

4. Without converting each fraction to a decimal or using a conventional algorithm, determine which fraction is larger: $\frac{15}{31}$ or $\frac{10}{19}$. Explain your reasoning.

   I chose this problem because I want to find out whether PTs will use $\frac{1}{2}$ *as a benchmark or will procedurally find common a denominator or convert each fraction to a decimal (Cramer & Wyberg, 2009).*

5. Provide 4 different representations to express “three-fourths.”

   I chose this problem based on an article I read in Teaching Children Mathematics (Watanabe, 2002) focusing on multiple representations.

6. Explain why $\frac{12}{15}$ is equivalent to $\frac{28}{35}$.

I chose this problem because I want to see whether PTs will simplify each fraction, obtain a common denominator, convert each to decimals, or think in terms of percentages.

7. Explain why $\frac{2}{3}$ cannot be added to $\frac{3}{4}$ without finding a common denominator.

I chose this problem because I want to find out if PTs understand the concept of a unit fraction and the requirement that the unit must be the same in order to add. For example, An, Kulm, and Wu (2004) cited an example given by Chinese teachers that books plus books equals books, but one cannot add books plus desks. Chinese teachers also stated that students who make the mistake of adding across are not thinking of fractions as numbers.

8. Student A and Student B solved the following problem: $5\frac{2}{3} - 1\frac{3}{4}$

Student A

$$5\frac{2}{3} \rightarrow 5\frac{8}{12} \rightarrow \overset{4}{\cancel{5}}\,\frac{\overset{18}{\cancel{8}}}{12}$$

$$-1\frac{3}{4} \rightarrow -1\frac{9}{12} \rightarrow -1\frac{9}{12}$$

$$3\frac{9}{12} = 3\frac{3}{4}$$

Student B

$$5\frac{2}{3} \rightarrow 5\frac{8}{12} \rightarrow \overset{4}{\cancel{5}}\,\frac{\overset{20}{\cancel{8}}}{12}$$

$$-1\frac{3}{4} \rightarrow -1\frac{9}{12} \rightarrow -1\frac{9}{12}$$

$$3\frac{11}{12}$$

Which student's response is correct?

Determine the error in the incorrect response.

I chose this problem based on my experience teaching middle school students. Students often misapplied the regrouping strategy for whole numbers when regrouping with mixed numbers.

9. Provide an illustration (model, drawing, etc.) for $\frac{1}{2} \times \frac{1}{3}$. Explain how your model connects to the operation.

I am interested in how PTs model this problem: area, number line, circular region, or not model it at all.

10. Provide an illustration (model, drawing, etc.) for $\frac{5}{8}$ of 2.

This problem was illustrated in Ball, Thames, & Phelps (2008) as an example of specialized content knowledge. Ball et al. (2008) showed this using fraction circles.

11. Provide an illustration (model, drawing, etc.) to explain why the quotient of $6 \div \frac{3}{4}$ is larger than 6.

A version of this problem was on the original pre-test for EDMG 1 administered for the first time in the fall 2013; however, the instructor did not ask the prospective teachers to provide a model. I am using it to find out if PTs understand "How many groups of $\frac{3}{4}$ *" or "* $\frac{3}{4}$ *of what number is 6?" (Lamon, 2001)*

This problem was also used for measurement division by Osana and Royea (2011) with prospective elementary teachers.

12. Provide an illustration (model, drawing, etc.) for $1\frac{1}{2} \div \frac{3}{4}$

I chose this problem because Green, Piel, and Flowers (2008) used this problem in their study of using manipulatives to correct PTs' misconceptions regarding fractions.

13. Determine what operation should be used to solve the following problem. Justify your choice.

"One serving of pizza is $\frac{3}{8}$ of a pizza. How many servings will be available in 2 pizzas?"

I am using this problem to determine PTs' fraction literacy (Johanning, 2008).

14. Explain why it is necessary to line up the decimals when adding the following numbers: $4.21 + 135 + 69.7$

I chose this problem because of my experience teaching middle grades. My former students often lined up digits instead of lining up common place values. The students lined up the digits 1, 5, and 7 instead of lining up the digits in the ones place, the 4, 5, and 9. Conceptual understanding of like place values is similar to understanding of like units (Hiebert & LeFevre, 1986).

15. Examine the following work and determine the error.

$$2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$$

I chose this problem because I want to find out whether PTs will explain the error using the distributive property or procedural knowledge of multiplying mixed numbers, that is, whether they mention changing the mixed numbers to improper fractions.

16. Explain how you can determine whether the answer (quotient) is greater than 1 or less than 1 without having to actually perform the division.

$$2\frac{1}{3} \div 3$$

I am using this problem to determine PTs' procedural knowledge of dividing fractions. (Osana & Royea, 2011). Since the dividend is smaller than the divisor, one may choose the partitive model or think in terms of proportional reasoning that stems from part-total.

17. Without using a calculator or performing the standard algorithm, determine where the decimal should be placed in the product (answer): $8.25 \times 6.34 = 52305$
Explain your reasoning.

I chose this problem for two reasons:
1) During TEAM-Math summer institute, a similar problem was given, and a teacher I knew personally applied a "counting decimal places" procedure to the problem and got it wrong. Even after a discussion, the teacher still did not seem to understand what she had done wrong.
2) In my own experience teaching at CSU, I gave this exact problem to my students who were enrolled in a K-5 mathematics endorsement class. A graduate-level student who is also an in-service fourth grade teacher applied a "counting decimal places" procedure and did not get it right. A middle grades science teacher enrolled in the class explained why 5.2305 was wrong by stating that 8 × 6 equals 48 so the answer must be more than 48. She then stated that it must be 52.305.

List the problem number(s) with which you felt the MOST confident.

This request was on the original pre-test given to EDMG 1 in the fall of 2013.

List the problem number(s) with which you felt the LEAST confident.

I added this request after talking with a faculty member at the university who works as an evaluator for grants. He questioned why I had a prompting for MOST confident only and not LEAST confident as well.

Appendix E: Conceptual Knowledge Test

1. Provide an illustration (model, drawing, etc.) to represent $4 \times 3 = 12$ and an illustration (model, drawing, etc.) to represent $3 \times 4 = 12$.

$4 \times 3 = 12$	$3 \times 4 = 12$

Explain the meaning of the 4, the 3, and the 12 in each representation.

2. Provide illustrations (models, drawings, etc.) to represent $8 \div 2 = 4$ interpreted in two different ways.

1st interpretation of $8 \div 2 = 4$	2nd interpretation of $8 \div 2 = 4$

Explain the meaning of the 8, the 2, and the 4 for each interpretation.

3. Without converting each fraction to a decimal or using a conventional algorithm, determine which fraction is larger: $\frac{10}{11}$ or $\frac{9}{10}$. Explain your reasoning.

4. Without converting each fraction to a decimal or using a conventional algorithm, determine which fraction is larger: $\frac{15}{31}$ or $\frac{10}{19}$. Explain your reasoning.

5. Provide 4 different representations to express “three-fourths.”

6. Explain why $\frac{12}{15}$ is equivalent to $\frac{28}{35}$.

7. Explain why $\frac{2}{3}$ cannot be added to $\frac{3}{4}$ without finding a common denominator.

8. Student A and Student B solved the following problem: $5\frac{2}{3} - 1\frac{3}{4}$

Student A

$$5\frac{2}{3} \rightarrow 5\frac{8}{12} \rightarrow \overset{4}{\cancel{5}}\frac{\overset{18}{\cancel{8}}}{12}$$

$$-1\frac{3}{4} \rightarrow -1\frac{9}{12} \rightarrow -1\frac{9}{12}$$

$$3\frac{9}{12} = 3\frac{3}{4}$$

Student B

$$5\frac{2}{3} \rightarrow 5\frac{8}{12} \rightarrow \overset{4}{\cancel{5}}\frac{\overset{20}{\cancel{8}}}{12}$$

$$-1\frac{3}{4} \rightarrow -1\frac{9}{12} \rightarrow -1\frac{9}{12}$$

$$3\frac{11}{12}$$

Which student's response is correct?

Determine the error in the incorrect response.

9. Provide an illustration (model, drawing, etc.) for $\frac{1}{2} \times \frac{1}{3}$. Explain how your model connects to the operation.

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10. Provide an illustration (model, drawing, etc.) for $\frac{5}{8}$ of 2.

11. Provide an illustration (model, drawing, etc.) to explain why the quotient of $6 \div \frac{3}{4}$ is larger than 6.

12. Provide an illustration (model, drawing, etc.) for $1\frac{1}{2} \div \frac{3}{4}$

13. Determine what operation should be used to solve the following problem. Justify your choice.
    "One serving of pizza is [illegible] of a pizza. How many servings will be available in 2 pizzas?"

14. Explain why it is necessary to line up the decimals when adding the following numbers: $4.21 + 135 + 69.7$

15. Examine the following work and determine the error.

$$2\frac{1}{6} \times 3\frac{3}{5} = 6\frac{3}{30} = 6\frac{1}{10}$$

16. Explain how you can determine whether the answer (quotient) is greater than 1 or less than 1 without having to actually perform the division.

$$2\frac{1}{3} \div 3$$

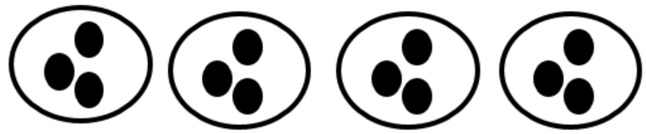
17. Without using a calculator or performing the standard algorithm, determine where the decimal should be placed in the product (answer): $8.25 \times 6.34 = 52305$
Explain your reasoning.

List the problem number(s) with which you felt the MOST confident and explain why.

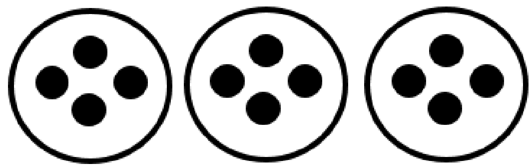
List the problem number(s) with which you felt the LEAST confident and explain why.

Appendix F: Answer Key for Conceptual Knowledge Test

1. The multiplier is the number of groups; the multiplicand is the group size. Sample correct answer for 4×3 (4 groups of 3)

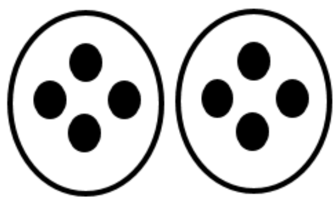


Sample correct answer for 3×4 (3 groups of 4)

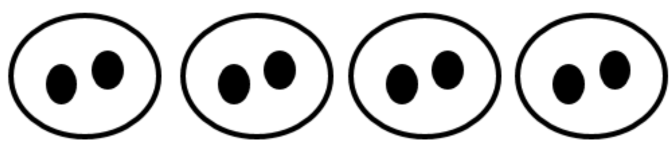


2. $8 \div 2$ shown as partitive model (split into given # of groups) versus measurement model (split based on given # in each group)

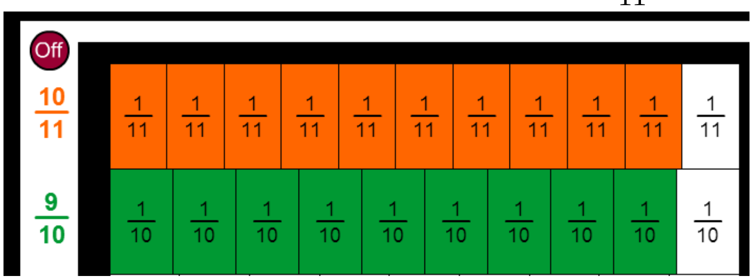
Partitive division model



Measurement division model



3. To determine whether students understand that each fraction is 1 unit fraction away from a whole and that the fraction with the smaller unit is “missing less” than the fraction with the larger unit; therefore, $\frac{10}{11}$ is closer to one than $\frac{9}{10}$.

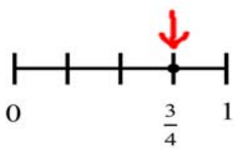


4. $\frac{15}{31}$ versus $\frac{10}{19}$

Using $\frac{1}{2}$ as a benchmark, $\frac{15.5}{31}$ is exactly one-half, so $\frac{15}{31}$ is less than $\frac{1}{2}$.
Using $\frac{1}{2}$ as a benchmark, $\frac{9.5}{19}$ is exactly one-half, so $\frac{10}{19}$ is more than $\frac{1}{2}$.

5. Possible answers:

Location on a number line



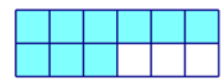
Symbolic Equivalents

Decimals: 0.75, 0.750

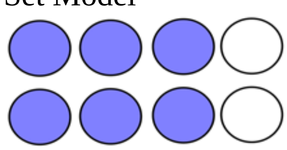
Fractions: $\frac{3}{4}$, $\frac{6}{8}$, etc

Percentage: 75%

Area Model

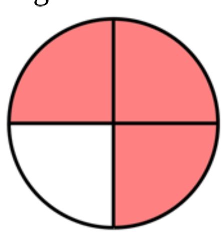


Set Model

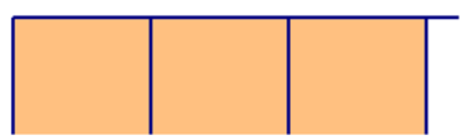


Real-World Context
Example: I worked 8 hours Friday; 6 hours was spent working in my yard. What fraction of my time working Friday was spent working in the yard?

Region Model



Length Model



6. Both fractions are equivalent to $\frac{4}{5}$ or 0.8. Also, the numerator is 80% of the denominator.

7. You cannot add $\frac{2}{3}$ to $\frac{3}{4}$ because they do not have the same size unit fraction. The unit fraction $\frac{1}{3}$ is larger than the unit fraction $\frac{1}{4}$. Fractions must be of the same size unit from the same whole in order to add them.

8. $5\frac{2}{3} - 1\frac{3}{4}$

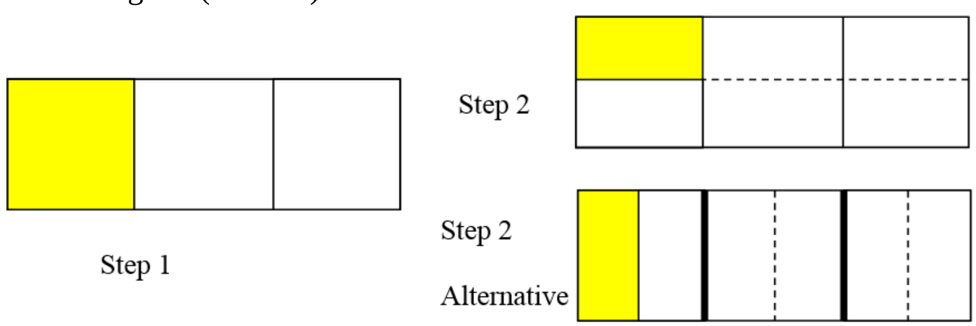
Student A shows a misconception of regrouping. Student A erroneously adds 10 to the numerator 8 as if he/she were working in base ten.

Student B shows the proper procedure for regrouping, that is, $1 = \frac{12}{12}$ which is added to the existing $\frac{8}{12}$ to get $\frac{20}{12}$.

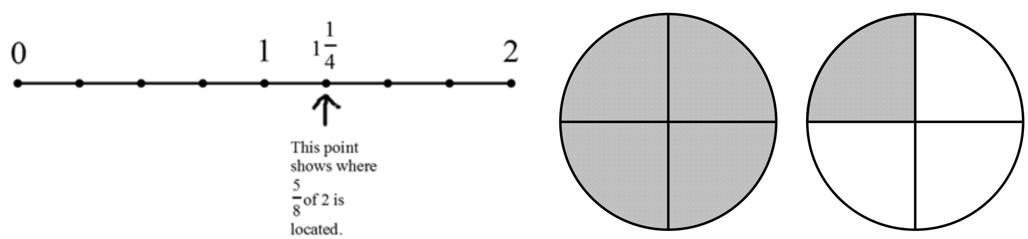
9. Model $\frac{1}{2} \times \frac{1}{3}$. First, the yellow shaded part is $\frac{1}{3}$ of the rectangle.

Step 1: $\frac{1}{2}$ of $\frac{1}{3}$ means to split the $\frac{1}{3}$ into two equal pieces and use one of those.

Step 2: In order to determine the answer, you must compare the final shaded part to the original (referent) whole.

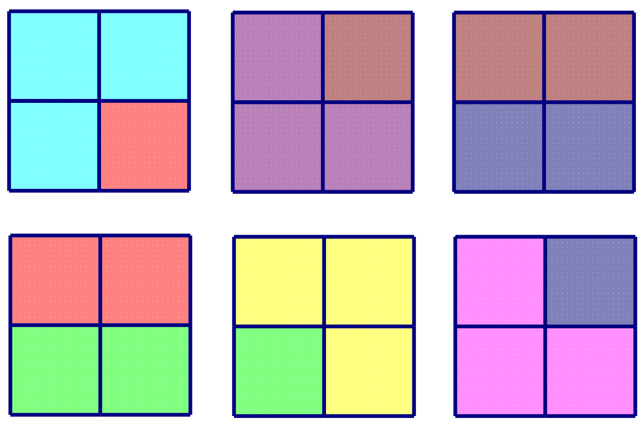


10. Provide an illustration (model, drawing, etc.) for $\frac{5}{8}$ of 2.

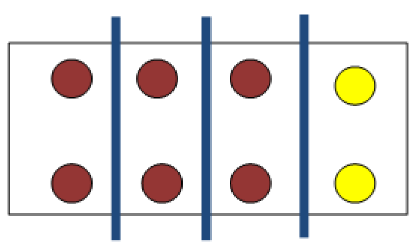


11. Two different possibilities for $6 \div \frac{3}{4}$:

Possibility 1: Using the measurement model of division such that the divisor is the "group size," then the quotient tells how many groups of size $\frac{3}{4}$ can be made from 6. Each whole will provide at least one $\frac{3}{4}$, with something remaining. Therefore, there will be at least 6 groups of size $\frac{3}{4}$. The remaining pieces can be used to make more groups of $\frac{3}{4}$. Therefore, the answer will be larger than 6.

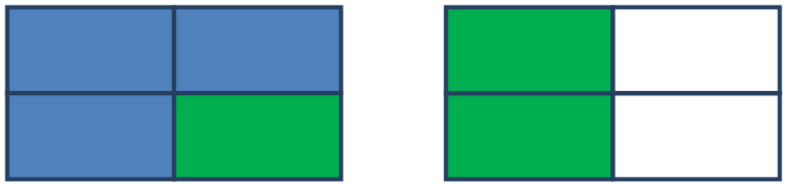


Possibility 2: Partitive division. 6 represents $\frac{3}{4}$ of the "unknown" total. So one can conceive that $\frac{3}{4}$ of the total (i.e. $\frac{3}{4}$ × total) would equal 6. Since $\frac{3}{4}$ is not the "entire" total and the $\frac{3}{4}$ part is 6, then the entire total must be larger than 6.

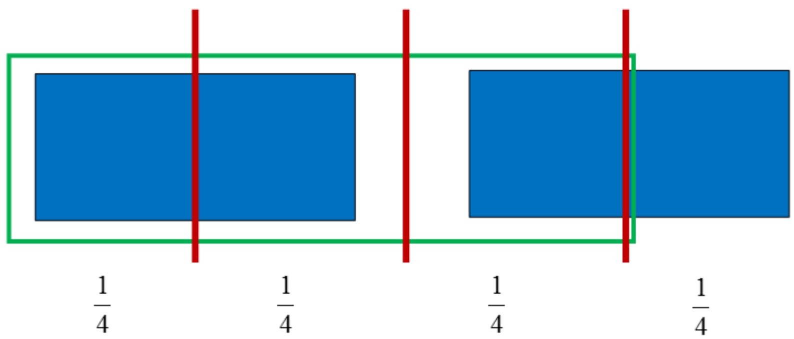


$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

12. Model $1\frac{1}{2} \div \frac{3}{4} = 2$ (1 blue $\frac{3}{4}$ and 1 green $\frac{3}{4}$) (Measurement model)

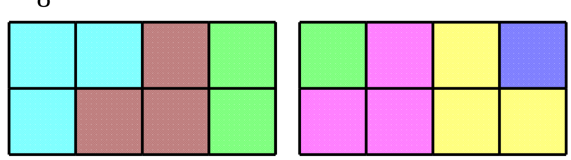


Another possibility: $1\frac{1}{2}$ is $\frac{3}{4}$ of what quantity? (Partitive model)



$1\frac{1}{2}$ (outlined in green) is $\frac{3}{4}$ of 2 (the blue rectangles).

13. The word problem requires division because you are asking how many $\frac{3}{8}$ can be made from 2 wholes (measurement model). There are five- $\frac{3}{8}$'s (dark blue, yellow, green, red, and purple) and one-third of a $\frac{3}{8}$ (light blue) in 2 wholes. The light blue can be thought of as a complex fraction $\left(\frac{\frac{1}{8}}{\frac{3}{8}}\right)$ which simplifies to $\frac{1}{3}$ (of a group the size $\frac{3}{8}$).



14. $4.21 + 135 + 69.7$

This problem is similar to adding fractions. In order to add anything, one must have like units to add. "Tenths" must be added to "tenths," "hundredths" must be added to "hundredths," etc.

15. Procedurally $2\frac{1}{6} \times 3\frac{3}{5} \rightarrow \frac{13}{6} \times \frac{18}{5} \rightarrow \frac{234}{30} \rightarrow 7\frac{24}{30} \rightarrow 7\frac{4}{5}$

or

$$2\frac{1}{6} \times 3\frac{3}{5} \rightarrow \frac{13}{\cancelto{1}{6}} \times \frac{\cancelto{3}{18}}{5} \rightarrow \frac{39}{5} \rightarrow 7\frac{4}{5}$$

Using partial products from the distribute property

$$2\frac{1}{6} \times 3\frac{3}{5} \rightarrow \left(2 + \frac{1}{6}\right)\left(3 + \frac{3}{5}\right) \rightarrow 2 \cdot 3 + 2 \cdot \frac{3}{5} + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot \frac{3}{5} \rightarrow 7\frac{4}{5}$$

Only the 2×3 and $\frac{1}{6} \times \frac{3}{5}$ were done in the work shown on the test.

16. Procedurally $2\frac{1}{3} \div 3 \rightarrow \frac{7}{3} \div \frac{3}{1} \rightarrow \frac{7}{3} \times \frac{1}{3} \rightarrow \frac{7}{9}$

Conceptually: When a dividend is smaller than its divisor, the quotient is less than one whole because there is not one whole divisor which can be made from the dividend (measurement model). Using the partitive division model, you have $2\frac{1}{3}$ to be partitioned into 3 equal groups. There is not enough of the dividend for 1 whole to be in each group.

17. $8.25 \times 6.34 = 52305$

If prospective teachers only use a procedural algorithm of "counting decimal places" commonly associated with multiplying decimals, they will get 5.2305
If prospective teachers use number sense or estimation, they may note that 8×6 is 48, so the answer must be larger than 48, but relatively close to 48. Therefore, the answer must 52.305

Appendix G: Rubric for Conceptual Knowledge Test

3 – Conceptual representation with clear reasoning and correct explanation

2 – Conceptual representation with unclear reasoning and partially correct explanation

1 – Representation provided with no evidence of conceptual understanding

0 – No representation

This rubric was adapted from Forrester and Chinnappan (2010).

Appendix H: Interview Protocol

Thank you for being willing to allow me to interview you. If at any time during the interview you feel uncomfortable and wish to end the interview, please let me know. The purpose of these interviews is to study how prospective teachers use fraction manipulatives and how that relates to their procedural knowledge and conceptual understanding of fractions. I am video recording these interviews. However, only the chair of my committee and I will have access to these videos. After I transcribe the videos, I will permanently delete them from my files. When referring to you in my study, I will not use your real name nor compromise your identity in any way.

During your interviews, I will present you with fraction tasks and a variety of manipulatives from which to choose that you can use to model the tasks. If you prefer to use virtual manipulatives instead of concrete, you are welcome to do so. During the tasks, if you want to change which manipulative you are using, you are welcome to do so. While you are working on the tasks, I may ask you questions about your thinking, why you are doing what you are, and what something means. When I ask you questions, I am not indicating that what you are doing is right or wrong. I am simply attempting to make sure that I understand and do not make assumptions. As I ask you questions, I am not attempting to help you to get a right answer or to teach you anything. That is not the purpose of this interview. Do you have any questions for me before we get started?

Task 1 Part A (first interview):

Use your choice of manipulatives to solve $3 \div \frac{2}{3}$.

Possible interview questions:

What does it mean to divide?
What purpose does the dividend serve in a division problem?
What purpose does the divisor serve in a division problem?
What does a remainder tell you in a division problem?
What does it mean to divide by $\frac{2}{3}$?
Can you tell whether the answer will be larger or smaller than the dividend before you actually solve it? How do you know?
Are you confident in your answer?
Can you explain how you determined your answer using the manipulatives?
Can you interpret the meaning of the remainder in this problem?
Does your answer make sense?
Is there any other way to interpret the meaning of this division problem?
Would you get the same answer if you reversed the order of the numbers?
Using procedures you are comfortable with, can you perform the calculations for this problem to get an answer?
What caused you to begin the problem the way you did?
How did you decide which manipulative to use?
Explain how the manipulative connects to a conventional algorithm.

Task 1 Part B (first interview):

Generate a real-world story problem that would be solved by the previous problem: $3 \div \frac{2}{3}$. Explain how you know that your scenario is appropriate.

Possible questions:

How does the story problem you generated relate to what you modeled with manipulatives?
How do you know that the story problem you generated is appropriate for this problem?
Are there any other real-world story problems that you could generate for this problem?
Are you confident in your answer? How confident are you in your answer?

Task 2 Part A (first interview):

Use your choice of manipulatives to solve $\frac{3}{4} \times 6$.

Possible Questions:

What does it mean to multiply by a fraction?
How do you know whether the answer will be larger or smaller than the numbers in your problem?
Can you relate multiplying by $\frac{3}{4}$ to anything other idea in mathematics?
Would you get the same answer if you reversed the order of the numbers?
Does $\frac{3}{4} \times 6$ mean the same thing as $6 \times \frac{3}{4}$?
Would the manipulative be used in the same way if you reversed the order?
How did you decide which manipulative to use?
Explain how the manipulative connects to a conventional algorithm.

<u>Task 2 Part B</u> (first interview):

Generate a real-world story problem that would be solved by the previous problem: $\frac{3}{4} \times 6$

Explain how you know that your scenario is appropriate.

Possible Questions:

How does the story problem you generated relate to what you modeled with manipulatives?
How do you know that the story problem you generated is appropriate for this problem?
Are there any other real-world story problems that you could generate for this problem?
Are you confident in your answer? How confident are you in your answer?

<u>Task 3 Part A</u> (second interview): Use your choice of manipulatives to solve $\frac{5}{6} \div \frac{1}{2}$ and interpret the answer with respect to the manipulative.

Possible Questions:

What does it mean to divide?
What purpose does the dividend serve in a division problem?
What purpose does the divisor serve in a division problem?
What does a remainder tell you in a division problem?
What does it mean to divide by $\frac{1}{2}$?
Can you tell whether the answer will be larger or smaller than the dividend before you actually solve it? How do you know?
Are you confident in your answer?

Can you explain how you determined your answer using the manipulatives?
Can you interpret the meaning of the remainder in this problem?
Does your answer make sense?
Is there any other way to interpret the meaning of this division problem?
Using procedures you are comfortable with, can you perform the calculations for this problem to get an answer?
What caused you to begin the problem the way you did?
Would you get the same result if you divided $\frac{1}{2} \div \frac{5}{6}$?
How did you decide which manipulative to use?
Explain how the model connects to a conventional algorithm.

<u>Task 3 Part B</u> (second interview): Generate a real-world story problem that would be solved by the previous problem: $\frac{5}{6} \div \frac{1}{2}$. Explain how you know that your scenario is appropriate.

Possible Questions:

How does the story problem you generated relate to what you modeled with manipulatives?
How do you know that the story problem you generated is appropriate for this problem?
Are there any other real-world story problems that you could generate for this problem?
Are you confident in your answer? How confident are you in your answer?

<u>Task 4 Part A</u> (second interview): Use your choice of manipulatives to solve $\frac{2}{3} \times \frac{3}{4}$. Explain how the manipulative connects to a conventional algorithm.

Possible questions:

What does the first number in a multiplication problem tell you?
What does the second number in a multiplication problem tell you?
Do you know the names for each number in a multiplication problem?
What does it mean to multiply by a fraction?
How will your answer compare to each of the numbers in your problem? That is, will your answer be larger or smaller than the numbers you multiply with? Can you explain how you know?
How do you know when to "cross simplify" or "cross multiply?"
Are you confident in your answer? How confident are you in your answer?
How did you decide which manipulative to use?

Task 4 Part B (second interview): Generate a real-world story problem that would be solved by the previous problem: $\frac{2}{3} \times \frac{3}{4}$. Explain how you know that your scenario is appropriate.

Possible Questions:

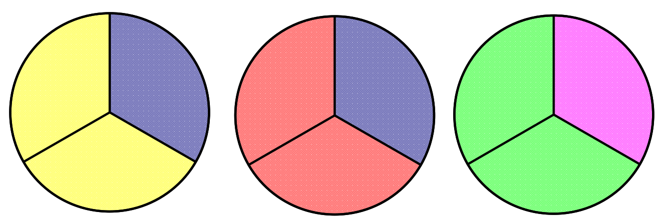
How does the story problem you generated relate to what you modeled with manipulatives?
How do you know that the story problem you generated is appropriate for this problem?
Are there any other real-world story problems that you could generate for this problem?
Are you confident in your answer? How confident are you in your answer?

Appendix I: Solutions for Task-Based Interviews

Task 1 Part A (first interview):

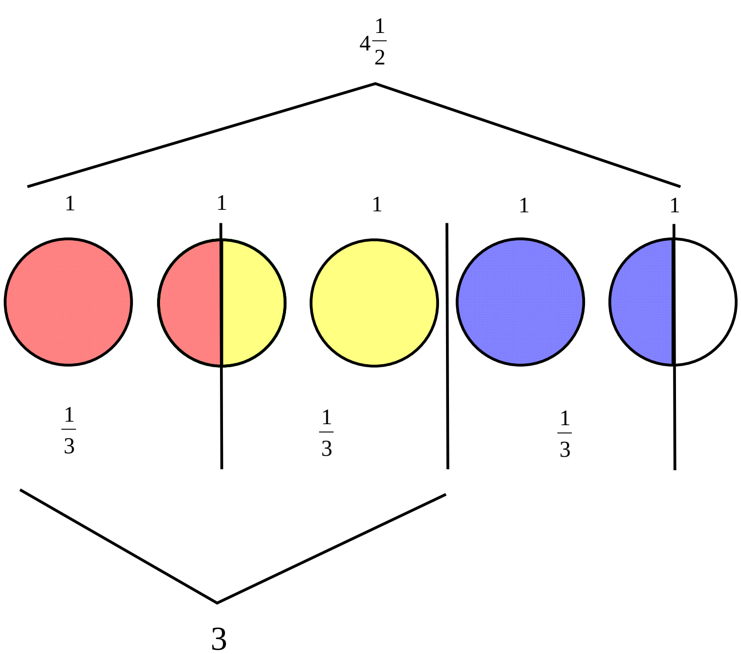
Use your choice of manipulatives to solve $3 \div \frac{2}{3}$.

This measurement-division-model task could be done using pattern blocks, colored length rods, fraction circles, or colored square tiles.



The answer is $4\frac{1}{2}$ groups of $\frac{2}{3}$ in 3 (1 yellow, 1 red, 1 purple, 1 green, and $\frac{1}{2}$ of a group pink).

Another possibility: Partitive division: $\frac{2}{3}$ of what number is 3?



Task 1 Part B (first interview):

Generate a real-world story problem that would be solved by the previous problem: $3 \div \frac{2}{3}$

Explain how you know that your scenario is appropriate.

Possible answers:

You have \$3, which is $\frac{2}{3}$ of what I have. How much money do I have?

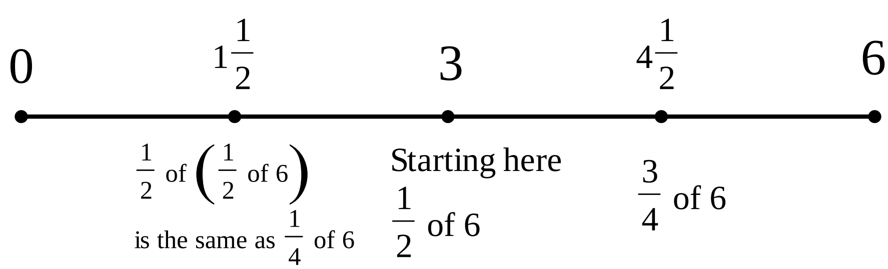
While walking, I drink $\frac{2}{3}$ of a bottle of water every mile that I walk. How many miles will I have walked after I drink 3 bottles of water?

Task 2 Part A (first interview):

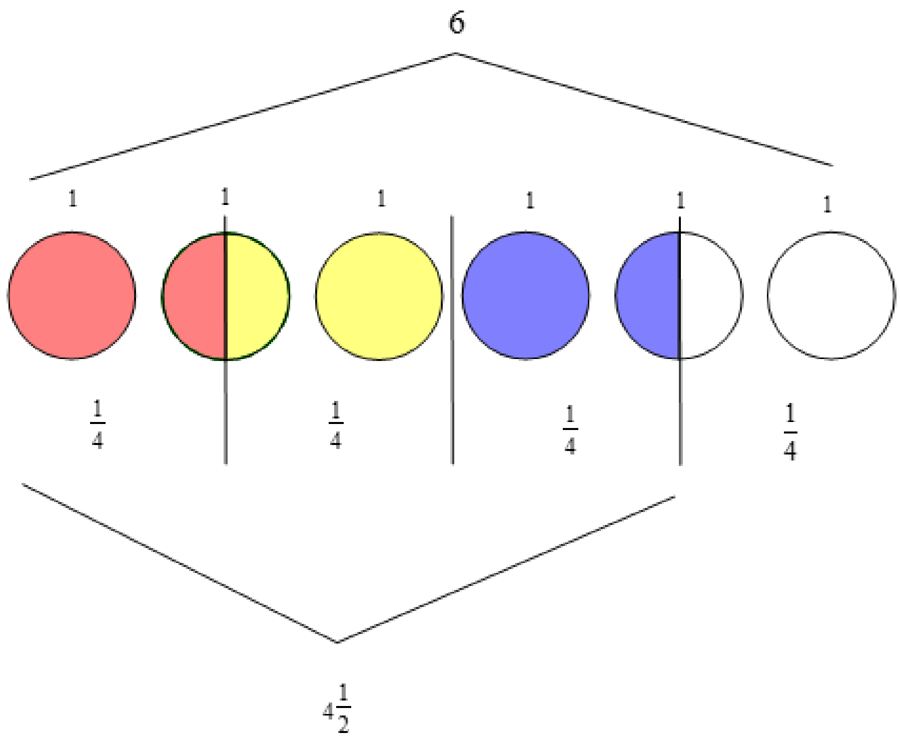
Use your choice of manipulatives to solve $\frac{3}{4} \times 6$.

Explain how the manipulative connects to a conventional algorithm.

This problem was on the original pre-test for EDMG 1 administered for the first time in the fall of 2013. However, the instructor did not ask the prospective teachers to model it. I chose to keep this problem because I want to analyze whether prospective teachers understand fraction-as-multiplier (Tsay & Hauk, 2009). This task is also in alignment with Lamon's (2001) idea of fraction as an operator, i.e. "three-fourths of something."



Could use circles.



Task 2 Part B (first interview):

Generate a real-world story problem that would be solved by the previous problem: $\frac{3}{4} \times 6$

Explain how you know that your scenario is appropriate.

Possible answers:

The price I pay for a shirt on sale is 75% of the original cost $6, which means I will pay less than $6.

Task 3 Part A (second interview):

Use your choice of manipulatives to solve $\frac{5}{6} \div \frac{1}{2}$.

Explain how the manipulative connects to a conventional algorithm and interpret the answer with respect to the manipulative.

I chose this problem because it was on the TEAM-Math post-test I took after summer institute. One of the teachers at the school where I was teaching at the time had no idea how to model this problem. He was very frustrated about it.

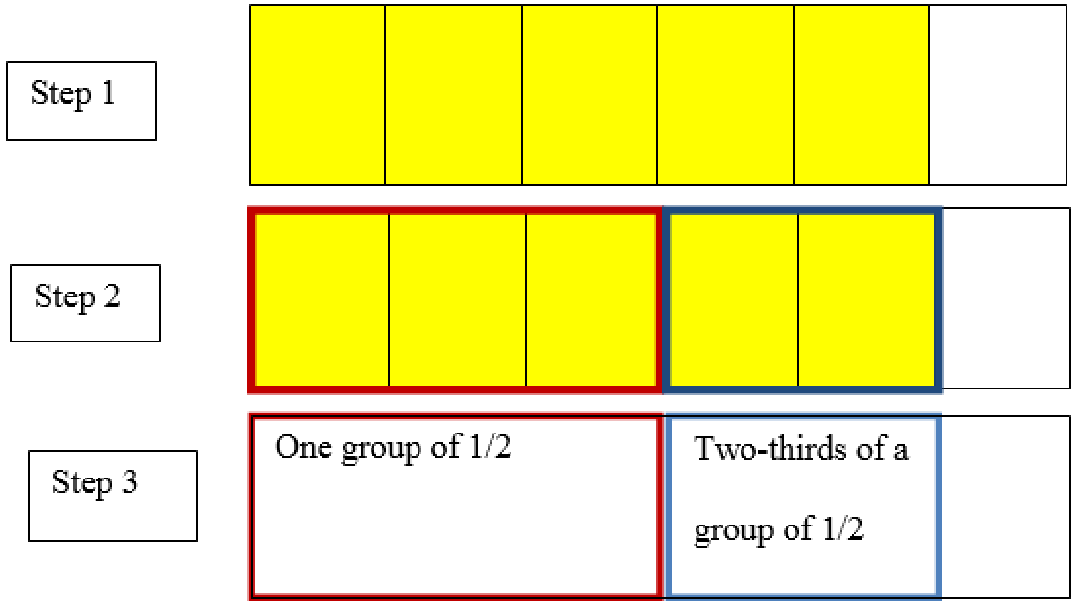
The yellow represents $\frac{5}{6}$.

Step 1: To divide $\frac{5}{6}$ by $\frac{1}{2}$ can mean, how many groups of size $\frac{1}{2}$ can be made from $\frac{5}{6}$ of one whole. (measurement division model)

Step 2: There is one group of $\frac{1}{2}$ in $\frac{5}{6}$ (outlined in red), with $\frac{2}{6}$ of the whole remaining (outlined in blue).

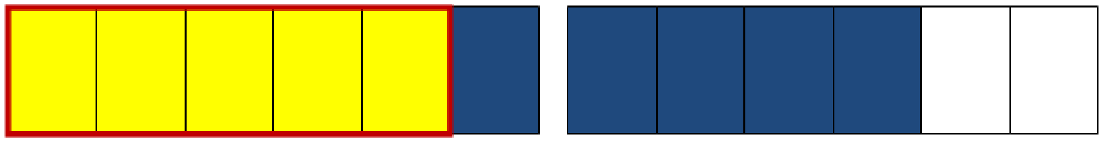
Step 3: It requires $\frac{3}{6}$ to make $\frac{1}{2}$, but there are only $\frac{2}{6}$ remaining of the necessary $\frac{3}{6}$.

Therefore, it takes one and two-thirds groups of $\frac{1}{2}$ to make $\frac{5}{6}$ (of one whole).

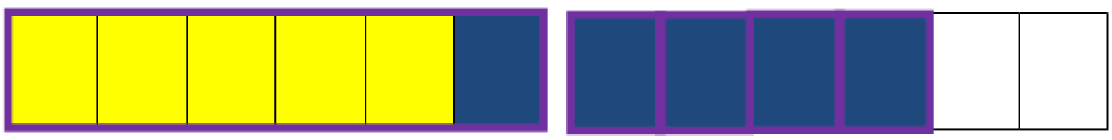


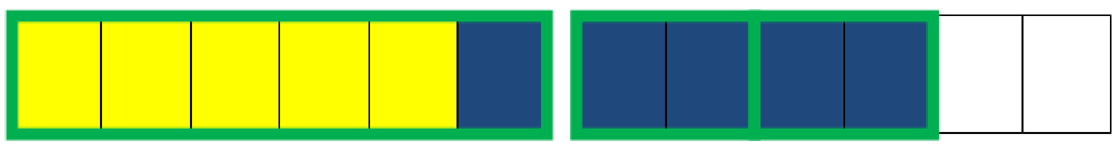
Task 3 Part A Another possibility: Partitive division

$\frac{5}{6}$ is $\frac{1}{2}$ of what quantity?



$\frac{5}{6}$ (outlined in red) is $\frac{1}{2}$ of $\frac{10}{6}$ (yellow and blue together). To get that, I have to double my quantity $\frac{5}{6}$.





$\frac{10}{6}$ is equivalent to $1\frac{4}{6}$ (outlined in purple) or $1\frac{2}{3}$ (outlined in green)

Task 4 Part A (second interview)

Use your choice of manipulatives to solve $\frac{2}{3} \times \frac{3}{4}$.

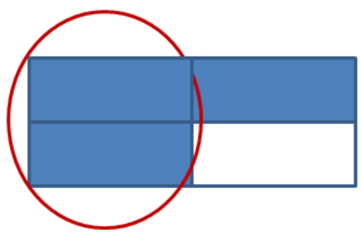
Explain how the model connects to a conventional algorithm.

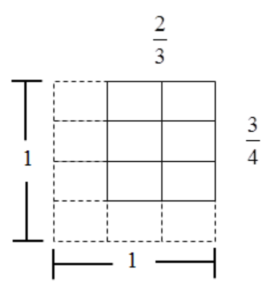
A similar problem was in Stein and Smith (1998). "Create a real-world situation for the following problem: $\frac{2}{3} \times \frac{3}{4}$. *Solve the problem you created without using the rule, and explain your solution." Stein and Smith provide a rectangular illustration of a possible solution similar to what I have illustrated in my answers to the Pre-Assessment. In addition, I want to analyze prospective teachers' understanding of fraction multiplier operating on a fraction multiplicand (Tsay & Hauk, 2009).*

To model $\frac{2}{3} \times \frac{3}{4}$, begin by shading $\frac{3}{4}$ of a whole; then, select two out of the three shaded pieces. This could be modeled using colored square tiles, fraction circles, colored length rods, or pattern blocks.

Answer shows $\frac{2}{4}$

cross simplify 3





Additional possible model (Flores, 2002, p. 244)

Flores' model shows $\frac{6}{12}$

which would be in alignment with multiplying straight across

Task 4 Part B (second interview):

Generate a real-world story problem that would be solved by the previous problem: $\frac{2}{3} \times \frac{3}{4}$

Explain how you know that your scenario is appropriate.

Possible answer:

Answer from Stein and Smith (1998): “My mom gave me three-fourths of a pizza to eat. I ate two-thirds of what she gave me. Therefore, I ate one-half of the original (whole) pizza.”

Appendix J: Initial Survey

For the statements below, use the following scale:

Not at all confident	Slightly confident	Somewhat confident	Very confident
1	2	3	4

1) Rate your overall level of confidence in your procedural ability in the operations on fractions.

   1 2 3 4

2) Rate your overall level of confidence in your understanding of the meaning behind the operations on fractions.

   1 2 3 4

3) Rate your overall level of confidence in understanding the "why" behind the conventional algorithms of fractions.

   a. For addition of fractions: 1 2 3 4

   b. For subtraction of fractions: 1 2 3 4

   c. For multiplication of fractions: 1 2 3 4

   d. For division of fractions: 1 2 3 4

For the statements below, use the following scale:

not at all familiar	slightly familiar	somewhat familiar	very familiar
1	2	3	4

4) Rate your familiarity with manipulatives during your own learning experiences:

1 2 3 4

5) Rate your familiarity with the use of manipulatives:

a. To teach mathematics in general: 1 2 3 4

b. To teach fraction concepts: 1 2 3 4

6) Rate your familiarity with virtual manipulatives:

1 2 3 4

For the statement below, use the following scale:

not at all important	slightly important	somewhat important	very important
1	2	3	4

7) Rate your belief about the importance of middle grades students using manipulatives to learn mathematics:

1 2 3 4

Please briefly describe your own school experiences learning math.

If you would like to provide any additional information, please feel free to write it in the space provided.

Thank you for your willingness to share with me

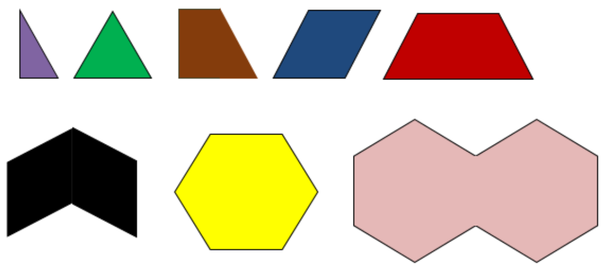
Appendix K: Semi-Structured Follow-Up Interview Questions

1. Tell me about a specific example of your fraction understanding that has changed during your methods course. (e.g., fraction equivalence, meaning of numerator and denominator, operations on fractions, etc.)
2. Tell me about what aspect of the course was the most helpful to you in improving your fraction understanding. (e.g. collaborative group work, peer demonstrations, use of multiple representations, use of manipulatives, etc.)
3. Tell me about what aspect of the course was the least helpful to you in improving your fraction understanding.
4. Tell me about a specific misconception(s) about fractions that you had when you began your methods course. (If needed) What do you think helped you to correct that misconception?
5. Tell me about a specific fraction concept that you are still confused about. What interfered with you being able to resolve your understanding of that concept?
6. Relational understanding is having both conceptual and procedural knowledge and an understanding of the connections between the two. Tell me about your confidence in being able to help your future students to develop relational understanding of fractions; and of mathematics in general.
7. Tell me about which instructional practices from your methods course you are likely to incorporate into your future teaching and why.
8. Tell me about which instructional practices from your methods course you are not likely to incorporate into your future teaching and why.

9. Tell me about a specific example of using manipulatives that helped you gain fraction understanding.
10. Tell me about a specific example of using manipulatives that did not help you gain fraction understanding.
11. Tell me about a specific instance when you were comfortable using manipulatives in class. (If needed) What do you think contributed to your comfort?
12. Tell me about a specific instance when you were uncomfortable using manipulatives in class. (If needed) What do you think contributed to your discomfort?
13. Tell me about the likelihood that you will use manipulatives with your students when you become a teacher. What influences that decision?
14. Tell me about a specific way you plan to use manipulatives in your future classroom. (e.g., teacher directed, student focused, “fun Friday,” etc.)
15. Tell me about a specific concrete or virtual manipulative you plan to use and a specific fraction concept you will teach using that manipulative. (e.g., fraction circles, number lines, colored length rods, etc. to teach equivalence, operations, etc.)
16. Tell me about a specific concrete manipulative you are more likely to use than virtual and why.
17. Tell me about a specific virtual manipulative you are more likely to use than concrete and why.

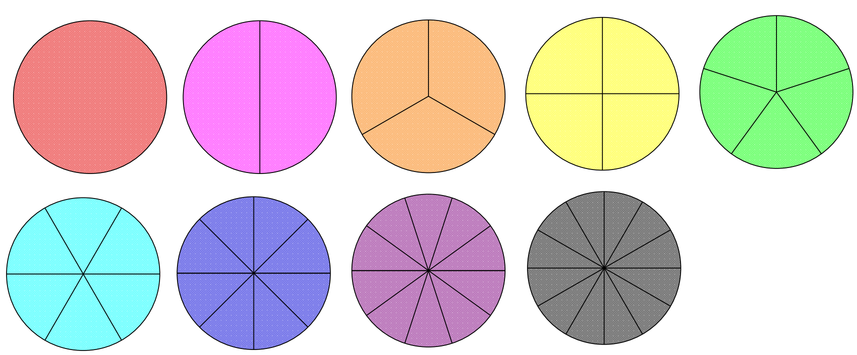
Appendix L: Manipulatives

Pattern Blocks

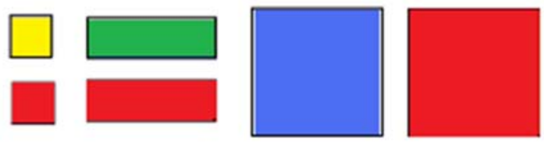


Cuisenaire Rods

Fraction Circles



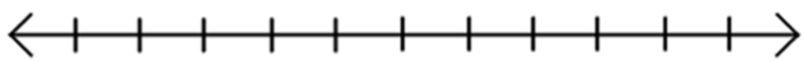
Algebra Tiles



The yellow square is 1; the red square is -1.
The green rectangle is x; the red rectangle is $-x$.
The large blue square is x^2; the large red square is $-x^2$.

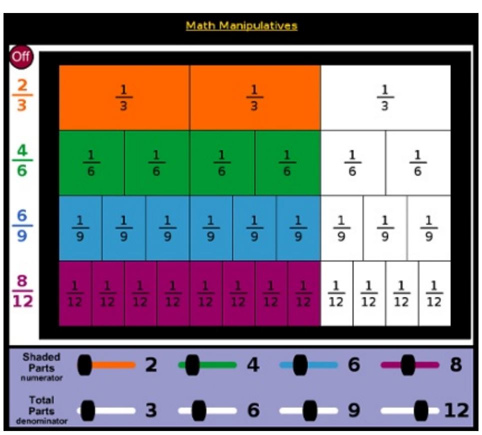
Ruler to show fractions of one foot

Number lines



Two-color counters

Virtual Manipulatives-Fraction Bars
(http://www.mathplayground.com/Fraction_bars.html)



Appendix M: Codebook for Analyzing Data

Concept	Example	Operational Definition (reference)	Additional information
Attitudinal interference (AttdInf)		Experiencing difficulty due to attitudes toward a new experience or due to the new experience causing uncomfortableness which is in contrast to the comfort of a prior experience (Pesek & Kirshner, 2000)	May say things like "Why can't we just do it the old way of doing things?"
Conceptual Knowledge (ConcptKnw)		Conceptual knowledge is achieved when one recognizes the relationships between pieces of information. (Hiebert & LeFevre, 1986)	
Procedural Knowledge (PrcdKnw)		Knowledge of the formal language (i.e., symbol representation system) and rules, algorithms, or procedures used for completing mathematical tasks (Hiebert & LeFevre, 1986)	
Relational understanding (RelUnd)		Understanding both what procedures to perform and why to perform them (Skemp, 1987)	The term "understanding" focuses on a person's cognitive processes.
Instrumental understanding (InstUnd)		Possessing the knowledge of "rules" and procedures associated with symbols of mathematics (Skemp, 1987)	The term "understanding" focuses on a person's cognitive processes.

Relational learning (RelLrn)		learning that focuses on the meaning of mathematical concepts (Pesek & Kirshner, 2000)	The term “learning” focusing on the type of instruction being provided by the teacher.
Instrumental learning (InstLrn)		“learning rules without reasons” (Skemp, 1987, p. 9); learning that focuses on recall and procedural-skill development (Pesek & Kirshner, 2000)	The term “learning” focusing on the type of instruction being provided by the teacher.
Fraction as part-whole comparisons (FrPartWh)		The *part-whole interpretation* of rational number depends directly on the ability to partition either a continuous quantity or a set of discrete objects into equal-sized subparts or sets. (Lesh, Post, Behr, & Silver, 1983)	Uses one dimension
Fraction as a ratio (FrRatio)	3 girls to 4 boys	*Ratio* is a relation that conveys the notion of relative magnitude; Ratio is a comparison of two quantities’ (Lesh, Post, Behr, & Silver, 1983)	Uses two dimensions
Fraction as division (FrDiv)	$\frac{8}{2}$ as $8 \div 2$	(Lesh, Post, Behr, & Silver, 1983)	Related to decimals
Fraction as an operator (FrOper)	$\frac{2}{3}$ of $12 = 8$	When operating on continuous object like length, the fraction acts as a stretcher-shrinker combination similar to scale factors. (Lesh, Post, Behr, & Silver, 1983)	Related to multiplier-divider; denominator acts as divider; numerator acts as multiplier

Fraction as a measure of discrete or continuous quantities (FrMeas)	$\frac{3}{4}$ hr. = 45 mins	Associated with area, length, and volume (Lesh, Post, Behr, & Silver, 1983)	Related to part-whole construct of fractions; can use number line, geometric regions, or sets of discrete objects
Equipartitioning / splitting as recursive partitioning (ReCurPart)	One-half of one-half is one-fourth	Successive partitioning of a whole to create unit fractions; Begin with one whole, partitioning and successive partition creates a multiplicative number of parts (based on multiplicative reasoning) (Confrey et al., 2009)	Specific example: Begin with one whole. Split it in two. You have twice as many parts as previously. Then split each part in two. You have twice as many parts as the previous partition. In comparison to your original whole, you have 2*2 as many parts. The whole is represented by $1\times\frac{2}{2}\times\frac{2}{2}$ which is equivalent to $\frac{4}{4}$. Reassembling the parts leads to $4\times\frac{1}{4}$ as $m\times n$.
Iterative (Iter)		Building from the unit fraction to the whole based on additive reasoning (Confrey et al., 2009)	$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$
Unit fraction (UntFr)	$\frac{1}{n}$	A fraction with a numerator of 1; A fair share of 1 whole to be shared among *n* (Confrey et al., 2009)	Related to ratio, but only one dimension is maintained; 1 whole shared among *n* means that each share will be $\frac{1}{n^{th}}$ of the whole
Knowledge of multiplication (KnwMult)	4×3	The first number is the multiplier; it tells you how many "groups" or "sets" you have. The	4 sets of 3 objects in each set is 12 objects

		second number is the multiplicand; it tells you what is in each “group” or “set.” (Lo & Lou, 2012)	
Knowledge of partitive division (KnwPartDiv)	8 ÷ 2	The first number is the dividend; it is what is to be “shared.” The second number is the divisor; it tells you how many sets you will partition the dividend into. The quotient represents what will be in each set. (Lo & Lou, 2012)	I have 8 objects that I will share between 2 people. Each person will get 4 objects.
Knowledge of measurement division (KnwMeasDiv)	8 ÷ 2	The first number is the dividend; it is what is to be “shared.” The second number is the divisor; it tells you the number of objects in each set to be shared. The quotient represents the number of sets shared. (Lo & Lou, 2012)	I have 8 objects; I will share 2 objects with each person. I will be able to share with 4 people.
Procedural use of manipulatives (PrcdUseManp)		Use of manipulatives is a replication of a known standard procedure or ones that were learned by rote from a previous problem during instruction; participants cannot explain why they solve the problem using a particular picture or strategy (adapted from Osana & Royea, 2011)	Participant may explain that they are doing “the same thing as before.”
Meaningful use of manipulatives (MnUseManp)		Use of manipulatives is based on intuitive understandings of fractions and the quantities expressed in	

	the problem (adapted from Osana & Royea, 2011)	
Mathematical Knowledge for Teaching (MKT)	The mathematical knowledge used to carry out the work of teaching mathematics (Hill, Rowan, & Ball, 2005)	
Common Content Knowledge (CCK)	The mathematical knowledge and skill used in settings other than teaching (Ball, Thames, & Phelps, 2008)	Questions that would be answerable by persons other than mathematics teachers
Specialized Content Knowledge (SCK)	The mathematical knowledge and skill unique to teaching (Ball, Thames, & Phelps, 2008)	Not typically needed for purposes other than teaching
Knowledge of Content and Students (KCS)	Combines knowledge about students and knowledge about mathematics; teachers must be able to anticipate students' thinking and what they are likely to find confusing, including common conceptions and misconceptions (Ball, Thames, & Phelps, 2008)	

Knowledge of Content and Teaching (KCT)	Knowing about teaching and knowing about mathematics; includes knowing about the design of instruction. (Ball, Thames, & Phelps, 2008)	For example, sequencing content for instruction, choosing appropriate examples with specific goals in mind, evaluating advantages and disadvantages of particular representations for specific content, identifying advantages and disadvantages of specific instructional methods and strategies, responding to student contributions, and posing questions

Appendix N: Instructor Consent Form

Auburn University
Auburn University, Alabama 36849-5212

Curriculum and Teaching
College of Education
5040 Haley Center

Telephone: (334)844-4434
Fax: (334)844-6789

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WITH CURRENT DATES HAS BEEN APPLIED TO THIS DOCUMENT.)

INSTRUCTOR CONSENT FORM
For a Research Study entitled

An Exploratory Study of the Impact of a Manipulatives-Intensive Fractions Unit during a Middle Grades Methods Course on Prospective Teachers' Relational Understanding of Fractions

This letter is to request your approval to conduct a research study to investigate the development of prospective teachers' relational understanding of fractions through the use of concrete and virtual manipulatives during the middle grades mathematics methods course you are teaching, EDMG 1. This study is being conducted by Denise S. Peppers, a graduate student at Auburn University, under the supervision of Professor Dr. Marilyn E. Strutchens. Your course was selected as a possible course for the study because it is a middle grades math methods course for prospective teachers, and you plan to use concrete and virtual manipulatives during instruction.

Pre-instruction and post-instruction assessment data will be collected and analyzed for qualitative changes, not for statistical significance. Data regarding students' attitudes toward mathematics, fractions, and manipulatives will be collected. Based on the results of the pre-tests and attitude data, Mrs. Peppers will choose two to four participants for the study to observe and interview. Observational data will be collected a minimum of five times during the semester. Interview data will be collected from selected students during interviews conducted outside of normal class time. Additional assessment data will be collected such as class reflections, homework tasks, and standardized test scores.

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Instructor's Initials
Page 1 of 3

With respect to the interviews, the first interview will take place during the first four weeks of class; the second interview will take place during the last four weeks of class; and the final interview will take place within two weeks of the class ending. The interviews will take place at times other than during class; efforts will be made to schedule interviews at times that are most convenient to participants. Interviews will take place at the Columbus Regional Mathematics Collaborative resource room on the third floor of the Cunningham Center. During interviews, participants will be asked to solve fraction tasks using manipulatives and about their experiences using manipulatives in the EDMG 1 class. Mrs. Peppers will make an effort to keep the interviews to no more than 75 minutes. Interviews will be video recorded and transcribed so that data from the interview can be analyzed.

The purpose of this letter is to request your permission to use data gathered from pre- and post-tests, surveys, observations, interviews, and other assessment data as part of this research study.

The only risk that your students may encounter would be a breach in confidentiality. All identifying information will be removed from any documents that are collected. Once data has been collected, analyzed, and reported on, all information pertaining to your students will be destroyed. Video recordings will not be shared with anyone other than members of Mrs. Peppers' dissertation committee and will be permanently erased upon the completion of this research study. All data will be stored on a password protected computer; only Mrs. Peppers will have the password. Furthermore, in the process of writing up the analysis, all participants will be referred to by pseudonyms (names other than their real names).

There will be no cost to you or your students associated with participating in this study.

Allowing access to your students is completely voluntary. Your decision about whether or not to allow access to your students or to not allow access to your students will not jeopardize your standing or relations with Columbus State University (CSU), the College of Education and Health Professions, the Department of Teacher Education at CSU, the Columbus Regional Mathematics Collaborative, Auburn University (AU), the College of Education, or the Curriculum and Teaching Department at AU.

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Instructor's Initials

Any information obtained in connection with this study will remain confidential. Information obtained through this study may be used to fulfill educational requirements, published in professional journals, and presented at professional meetings.

If you have any questions about this study, please ask them now or contact Mrs. Peppers at the Columbus Regional Mathematics Collaborative at (706)568-2480 or by email at dsp0003@auburn.edu or peppers_denise@columbusstate.edu. You may also contact Dr. Strutchens. Her phone number is (334)844-6838, and her email address is strutme@auburn.edu. A copy of this document will be given to you to keep.

If you have questions about your rights as a research participant, you may contact the Auburn University Office of Human Subjects Research of the Institutional Review Board by phone at (334)844-5966 or by email at hsubjec@auburn.edu or IRBChair@auburn.edu or IRBadmin@auburn.edu

HAVING READ THE INFORMATION PROVIDED, YOU MUST DECIDE WHETHER OR NOT YOU WISH TO PARTICIPATE IN THIS RESEARCH STUDY. YOUR SIGNATURE INDICATES YOUR WILLINGNESS TO PARTICIPATE.

Course Instructor's signature Date	Investigator obtaining consent Date
Printed Name	Printed Name

Page 3 of 3

Appendix O: Participant Consent Form

Auburn University
Auburn University, Alabama 36849-5212

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PARTICIPANT CONSENT FORM
For a Research Study entitled

An Exploratory Study of the Impact of a Manipulatives-Intensive Fractions Unit during a Middle Grades Methods Course on Prospective Teachers' Relational Understanding of Fractions

You are invited to participate in a research study that will be used to investigate the development of prospective teachers' relational understanding of fractions through the use of concrete and virtual manipulatives during a middle grades mathematics methods course. This study is being conducted by Denise S. Peppers, a graduate student at Auburn University, under the supervision of Professor Dr. Marilyn E. Strutchens. You were selected as a possible participant because you are currently enrolled in a middle grades math methods course for prospective teachers and are age 19 or older.

As a regular part of instruction, all students will take pre-tests to assess their conceptual understanding and procedural knowledge of fractions. If you decide to participate in this research study, you will be asked to take an initial attitude survey. Based on the results of the pre-test and attitude survey, Mrs. Peppers will choose one to four participants for the study. Mrs. Peppers will observe EDMG 1 class while students use manipulatives during a fractions unit. Additionally, Mrs. Peppers will individually interview the chosen participants three times over the course of the semester in order to gain insight into participants' developing relational understanding of fractions through the use of manipulatives while taking the EDMG 1 course. Additional assessment data will be collected such as class reflections, homework tasks, and standardized test scores. Lastly, participants will take post-tests to assess their conceptual and procedural knowledge of fractions.

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Participant's Initials
Page 1 of 3

With respect to the interviews, the first interview will take place during the first four weeks of class; the second interview will take place during the last four weeks of class; and the final interview will take place within two weeks of class concluding. The interviews will take place at times other than during class; efforts will be made to schedule interviews at times that are most convenient to participants. Interviews will take place at the Columbus Regional Mathematics Collaborative resource room on the third floor of the Cunningham Center. During interviews, participants will be asked to solve fraction tasks using manipulatives and about their experiences using manipulatives in the EDMG 1 class. Mrs. Peppers will make an effort to keep the interviews to no more than 75 minutes. Interviews will be video recorded and transcribed so that data from the interview can be analyzed. Your total time commitment to this study will be approximately four hours.

The purpose of this letter is to request your permission to use data gathered from pre- and post-tests, observations, interviews, and other assessment data as part of this research study.

The only risk that you may encounter would be a breach in confidentiality. All identifying information will be removed from any documents that are collected. Once data has been collected, analyzed, and reported on, all information pertaining to you will be destroyed. Video recordings will not be shared with anyone other than members of Mrs. Peppers' dissertation committee and will be permanently erased upon the completion of this research study. All data will be stored on a password protected computer; only Mrs. Peppers will have the password. Furthermore, in the process of writing up the analysis, all participants will be referred to by pseudonyms (names other than their real names).

To thank you for your time, you will receive a $40 gift card. There will be no cost to you associated with participating in this study.

As part of normal class instruction in EDMG 1, all students will be required by the instructor to participate in the instructional activities. However, your participation in the study is completely voluntary. If you change your mind about your participation, you can be withdrawn from the study at any time without penalty. If you choose to withdraw, your data can be withdrawn as long as it is identifiable. Your decision about whether or not to participate or to stop participating will not jeopardize your standing in EDMG 1 nor relations with the instructor, Columbus State University (CSU), the College of Education and Health Professions, the Department of Teacher Education at CSU, the Columbus Regional Mathematics Collaborative, Auburn University (AU), the College of Education, or the Curriculum and Teaching Department at AU.

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Participant's Initials
Page 2 of 3

Any information obtained in connection with this study will remain confidential. Information obtained through your participation may be used to fulfill an educational requirement, published in a professional journal, presented at a professional meeting, etc.

If you have any questions about this study, please ask them now or contact Mrs. Peppers at the Columbus Regional Mathematics Collaborative at (706)568-2480 or by email at dsp0003@auburn.edu or peppers_denise@columbusstate.edu. You may also contact Dr. Strutchens. Her phone number is (334)844-6838, and her email address is strutme@auburn.edu. A copy of this document will be given to you to keep.

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Participant's signature Date	Investigator obtaining consent Date
Printed Name	Printed Name

Page 3 of 3

Appendix P: Samples of Activities and Class Agendas for EMDG 1

Activity: "Don't Fence Me In"

- Math concepts—fixed perimeter; maximize area; definition of rectangle; systematic approach to problem solving; creation of definitions of area & perimeter; reference dimensions as rows by columns; discuss circle as shape with most area given static perimeter; 1 dimension versus 2 dimensions—hard for students
- Create table of rectangle dimensions; focus on systematic reasoning; How do you know when you have all rectangular possibilities?
- Use of tools—inch grid paper; square tiles; string; If you didn't use them; how might a student use them? How might they help you conceive of the mathematics?
- School students struggle with area/perimeter; context helps; what is length; what are square units?

Activity: Examine NCTM Process Standards and Common Core Standards for Mathematical Practice

Class Agenda—September 8, 2014

- Introduce lesson—multiplying integers with manipulatives (i.e., two-color counters)
- 7^{th}-grade lesson that has its foundation in 3^{rd} grade
- Importance of building new learning from prior learning

- Teacher's job is to introduce new information by connecting it to prior learning/understanding
    - Takes planning
    - Analysis of to-be-taught concepts
    - Deep understanding of to-be-taught concepts
    - Teachers cannot JUST START as lesson
- Put your student/teacher hats on; will toggle between both
- PowerPoint
- Groups according to Process Standards activity from last week

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PARTICIPANT CONSENT FORM
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You are invited to participate in a research study that will be used to investigate the development of prospective teachers' relational understanding of fractions through the use of concrete and virtual manipulatives during a middle grades mathematics methods course. This study is being conducted by Denise S. Peppers, a graduate student at Auburn University, under the supervision of Professor Dr. Marilyn E. Strutchens. You were selected as a possible participant because you are currently enrolled in a middle grades math methods course for prospective teachers and are age 19 or older.

As a regular part of instruction, all students will take pre-tests to assess their conceptual understanding and procedural knowledge of fractions. If you decide to participate in this research study, you will be asked to take an initial attitude survey. Based on the results of the pre-test and attitude survey, Mrs. Peppers will choose two to four participants for the study. Mrs. Peppers will observe the EDMG 4121 class a minimum of five times during the semester. Additionally, Mrs. Peppers will individually interview the chosen participants three times over the course of the semester in order to gain insight into participants' developing relational understanding of fractions through the use of manipulatives while taking the EDMG 4121 course. Additional assessment data will be collected such as class reflections, homework tasks, and standardized test scores. Lastly, participants will take post-tests to assess their conceptual and procedural knowledge of fractions.

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Protocol # 14-333 EP1410

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With respect to the interviews, the first interview will take place during the first four weeks of class; the second interview will take place during the last four weeks of class; and the final interview will take place within two weeks of the class ending. The interviews will take place at times other than during class; efforts will be made to schedule interviews at times that are most convenient to participants. Interviews will take place at the Columbus Regional Mathematics Collaborative resource room on the third floor of the Cunningham Center. During interviews, participants will be asked to solve fraction tasks using manipulatives and about their experiences using manipulatives in the EDMG 4121 class. Mrs. Peppers will make an effort to keep the interviews to no more than 75 minutes. Interviews will be video recorded and transcribed so that data from the interview can be analyzed. Your total time commitment to this study will be approximately four hours.

The purpose of this letter is to request your permission to use data gathered from pre- and post-tests, observations, interviews, and other assessment data as part of this research study.

The only risk that you may encounter would be a breach in confidentiality. All identifying information will be removed from any documents that are collected. Once data has been collected, analyzed, and reported on, all information pertaining to you will be destroyed. Video recordings will not be shared with anyone other than members of Mrs. Peppers' dissertation committee and will be permanently erased upon the completion of this research study. All data will be stored on a password protected computer; only Mrs. Peppers will have the password. Furthermore, in the process of writing up the analysis, all participants will be referred to by pseudonyms (names other than their real names).

To thank you for your time, you will receive a $40 gift card. There will be no cost to you associated with participating in this study.

As part of normal class instruction in EDMG 4121, all students will be required by the instructor to participate in the instructional activities. However, your participation in the study is completely voluntary. If you change your mind about your participation, you can be withdrawn from the study at any time without penalty. If you choose to withdraw, your data can be withdrawn as long as it is identifiable. Your decision about whether or not to participate or to stop participating will not jeopardize your standing in EDMG 4121 nor relations with the instructor, Columbus State University (CSU), the College of Education and Health Professions, the Department of Teacher Education at CSU, the Columbus Regional Mathematics Collaborative, Auburn University (AU), the College of Education, or the Curriculum and Teaching Department at AU.

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Participant's signature Date	Investigator obtaining consent Date
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Printed Name	Printed Name

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Protocol # 14-333 EP 1410

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INSTRUCTOR CONSENT FORM
For a Research Study entitled

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This letter is to request your approval to conduct a research study to investigate the development of prospective teachers' relational understanding of fractions through the use of concrete and virtual manipulatives during the middle grades mathematics methods course you are teaching, EDMG 4121. This study is being conducted by Denise S. Peppers, a graduate student at Auburn University, under the supervision of Professor Dr. Marilyn E. Strutchens. Your course was selected as a possible course for the study because it is a middle grades math methods course for prospective teachers, and you plan to use concrete and virtual manipulatives during instruction.

Pre-instruction and post-instruction assessment data will be collected and analyzed for qualitative changes, not for statistical significance. Data regarding students' attitudes toward mathematics, fractions, and manipulatives will be collected. Based on the results of the pre-tests and attitude data, Mrs. Peppers will choose two to four participants for the study to observe and interview. Observational data will be collected a minimum of five times during the semester. Interview data will be collected from selected students during interviews conducted outside of normal class time. Additional assessment data will be collected such as class reflections, homework tasks, and standardized test scores.

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Instructor's Initials

With respect to the interviews, the first interview will take place during the first four weeks of class; the second interview will take place during the last four weeks of class; and the final interview will take place within two weeks of the class ending. The interviews will take place at times other than during class; efforts will be made to schedule interviews at times that are most convenient to participants. Interviews will take place at the Columbus Regional Mathematics Collaborative resource room on the third floor of the Cunningham Center. During interviews, participants will be asked to solve fraction tasks using manipulatives and about their experiences using manipulatives in the EDMG 4121 class. Mrs. Peppers will make an effort to keep the interviews to no more than 75 minutes. Interviews will be video recorded and transcribed so that data from the interview can be analyzed.

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There will be no cost to you or your students associated with participating in this study.

Allowing access to your students is completely voluntary. Your decision about whether or not to allow access to your students or to not allow access to your students will not jeopardize your standing or relations with Columbus State University (CSU), the College of Education and Health Professions, the Department of Teacher Education at CSU, the Columbus Regional Mathematics Collaborative, Auburn University (AU), the College of Education, or the Curriculum and Teaching Department at AU.

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____
Course Instructor's signature Date

____
Investigator obtaining consent Date

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Printed Name

____
Printed Name