Revisiting the Intersection Problem for Maximum Packings of K_{6n+5} with Triples

by

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Abstract

In 1989, Gaetano Quattrocchi gave a complete solution of the intersection problem for maximum packings of K_{6n+5} with triples when the leave (a 4–cycle) is the same in each maximum packing. Quattrocchi showed that $I[2] = 2$ and $I[n] = \{0, 1, 2, \ldots, \frac{(\binom{n}{2}-4)}{3} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ for all $n \equiv 5 \pmod 6 \geq 11$. We extend this result by removing the exceptions $\{x-1, x-2, x-3, x-5\}$ when the leaves are not necessarily the same. In particular, we show that $I[n] = \{0, 1, 2, \ldots, \frac{(\binom{n}{2}-4)}{3}\}$ for all $n \equiv 5 \pmod 6$.

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Chapter 1

Maximum Packings on K_n, $n \equiv 5 \pmod 6$

1. Introduction

A *Steiner Triple System* of order n, STS(n), is a pair $(S, \mathcal{T})$, where $\mathcal{T}$ is a set of edge-disjoint *triangles* (or *triples*) which partitions the edge set of K_n (the complete undirected graph on n vertices) with vertex set S. It is well known that the spectrum for Steiner triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$, and that if $(S, \mathcal{T})$ is a triple system of order n then $|\mathcal{T}| = \frac{n(n-1)}{6}$. Define $I(n)$ and $J(n)$ as follows:

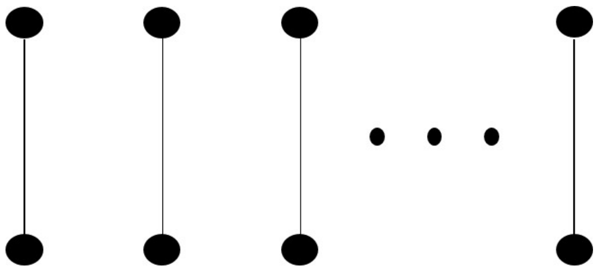
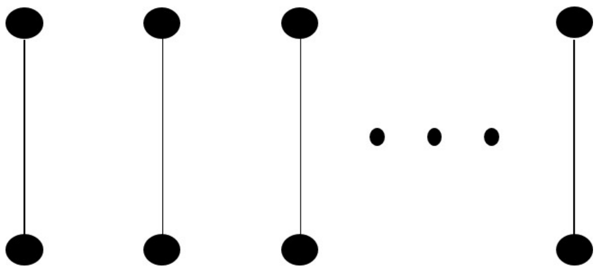
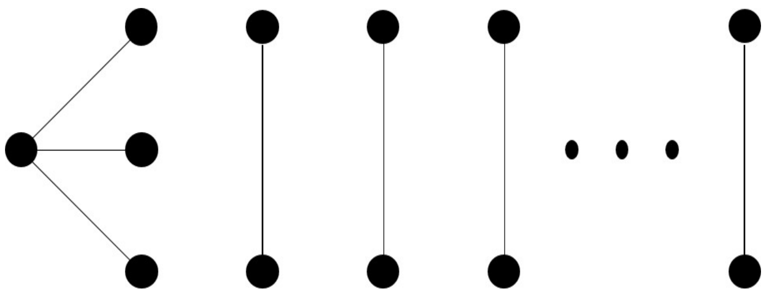
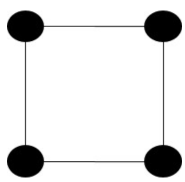
$$\begin{cases} I(n) & = \{0,\, 1,\, 2,\, \ldots,\, x = \frac{n(n-1)}{6}\} \backslash \{x-1,\, x-2,\, x-3,\, x-5\}, \text{ and} \\ J(n) & = \{k \mid \text{there exists a pair of triple systems of order } n \text{ having exactly } k \text{ triples} \\ & \quad \text{in common}\}. \end{cases}$$

A natural question to ask is the following: *for which* $k \in \{0,\, 1,\, 2,\, \ldots,\, \frac{n(n-1)}{6}\}$ *does there exist a triple system of order* n *having* k *triples in common?* The following theorem gives a complete solution of the intersection problem for triple systems.

Theorem 1.1 (C.C. Lindner, A. Rosa[3])**.** Let $n \equiv 1$ or $3 \pmod 6$. Then $J(n) = I(n)$, if $n \neq 9$ and $J(9) = I(9)/\{5,\, 8\}$. ■

Now when $n \equiv 1$ or $3 \pmod 6$ there does not exist a triple system and so the intersection problem for maximum packings of K_n with triples is immediate. A *packing* of K_n with triples in a pair (S, P) where P in a collection of edge disjoint triples of K_n with vertex set S. If P is as large as possible, then (S, P) is said to be a *maximum packing* of K_n with triples.

The set of unused edges is called the *leave*. The following easy to read table gives the leave for a maximum packing of K_n with triples for each $n \equiv 0,\, 1,\, 2,\, 3,\, 4,\, 5 \pmod 6$.

$n \equiv \pmod 6$	leave
0	*1-factor* 
1	*Steiner Triple System*
2	*1-factor* 
3	*Steiner Triple System*
4	*tripole* 
5	*4-cycle* 

Theorem 1.1 gives a complete solution for Steiner Triple Systems ($n \equiv 1$ or $3 \pmod 6$). The intersection problem for maximum packings of K_n with triples has been completely solved for $n \equiv 0, 2, 4, 5$ when the leave is the SAME in the following two papers (best described with a table).

$n \equiv \pmod 6$	Intersection spectrum
0 or 2	*D. Hoffman and C. C. Lindner*[2] $I(6) = \{0, 4\}$, $I(8) = \{0, 2, 8\}$ and for all $n \equiv 0$ or $2 \geq 12$, $I(n) = \{0, 1, 2, \ldots, \frac{n(n-2)}{6} = x\} \backslash \{x-1, x-2, x-3, x-5\}$
4	*G. Quattrocchi*[4] $I(4) = \{1\}$ and for all $n \equiv 4 \pmod 6 \geq 10$, $I(n) = \{0, 1, 2, \ldots, \frac{\left(\binom{n}{2} - \frac{(n+2)}{2}\right)}{3} = x\} \backslash \{x-1, x-2, x-3, x-5\}$
5	*G. Quattrocchi*[4] $I(5) = 2$ and for all $n \equiv \pmod 6 \geq 11$, $I(n) = \{0, 1, 2, \ldots, \frac{\left(\binom{n}{2} - 4\right)}{3} = x\} \backslash \{x-1, x-2, x-3, x-5\}$

As mentioned, the leaves in the above table are always the same. The object of this thesis in the extension of the intersection problem for maximum packings of K_{6n+5} with triples when the leaves ($= 4-$ cycles) are not necessarily the same. In particular we remove all of the exceptions for K_{6n+5} by showing that $I(n) = \{0, 1, 2, \ldots, \left(\binom{n}{2} - 4\right)/3\}$ for all $n \equiv 5 \pmod 6$.

2. Three Examples

In everything that follows $J^*(n) = \{0, 1, 2, \ldots, \left(\binom{n}{2} - 4\right)/3\}$. We will need examples for $n = 5, 11,$ and 17. In each case we will show that $I(n) = J^*(n)$, thereby removing the exceptions in Quatrocchi's constructions for $n \equiv 5 \pmod 6$.

Example 2.1 ($n = 5$) Define three maximum packings of order 5 $(X,\ P_1)$, $(X,\ P_2)$, and $(X,\ P_3)$ as follows:

$$\begin{cases} 1.\ X = \{1,\ 2,\ 3,\ 4,\ 5\},\ P_1 = \{\{1,\ 2,\ 3\}, \{1,\ 4,\ 5\}\} \text{ with leave } (2,\ 4,\ 3,\ 5); \\ 2.\ X = \{1,\ 2,\ 3,\ 4,\ 5\},\ P_2 = \{\{1,\ 4,\ 5\}, \{2,\ 3,\ 4\}\} \text{ with leave } (1,\ 2,\ 5,\ 3); \\ 3.\ X = \{1,\ 2,\ 3,\ 4,\ 5\},\ P_3 = \{\{1,\ 2,\ 4\}, \{1,\ 5,\ 3\}\} \text{ with leave } (2,\ 3,\ 4,\ 5). \end{cases}$$

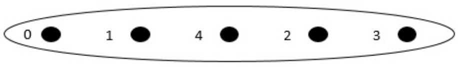
Then $|P_1 \cap P_3| = 0$, $|P_1 \cap P_2| = 1$, and $|P_1 \cap P_1| = 2$. It follows that $I(5) = J^*(5) = \{0,\ 1,\ 2\}$.

Example 2.2 ($n = 11$) Let (X, F) be a 1-factorization of K_6 with vertex set X and (Y, P_1) and (Y, P_2) any two maximum packings of K_5 with triples in *Example 2.1*. Define a pair of maximum packings C_1 and C_2 of K_{11} with triples with vertex set $X \cup Y$ as follows:

F =

F_0	F_1	F_4	F_2	F_3
8, 9	9, 10	7, 10	8, 10	7, 8
7, 10	6, 8	6, 9	6, 7	6, 9
5, 6	5, 7	5, 8	5, 9	5, 10

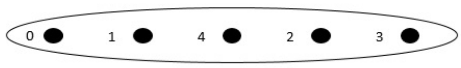
P_1 =



F =

F_0	F_1	F_4	F_2	F_3
8, 9	9, 10	7, 10	8, 10	7, 8
7, 10	6, 8	6, 9	6, 7	6, 9
5, 6	5, 7	5, 8	5, 9	5, 10

P_2 =



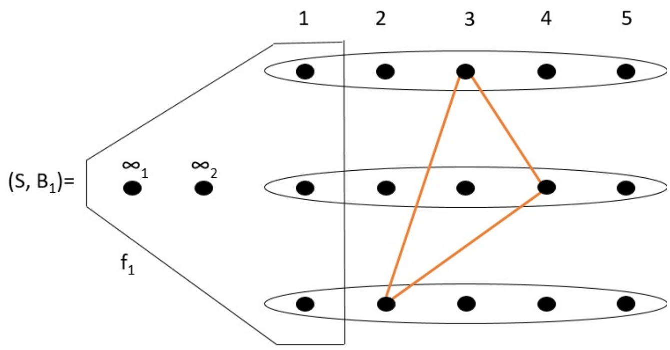
$$\begin{cases} 1.\ \{i,\ x,\ y\} \in C_1 \text{ and } C_2 \text{ for each } i \in \{0,\ 1,\ 2,\ 3,\ 4\} \text{ and } \{x,\ y\} \in F_i, \\ 2.\ P_1 \subseteq C_1 \text{ and } P_2 \subseteq C_2. \text{ The leave in each case are the leaves in } P_1 \text{ and } P_2. \end{cases}$$

By permuting the columns of F and using the examples in *2.1* independently we obtain the intersection numbers 0, 1, 2, ..., 9, 10, 11, 15, 16, 17. So it remains to obtain the intersection numbers 12, 13, 14. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the following two mutually balanced configurations consisting of a 4–cycle and 3–triples.

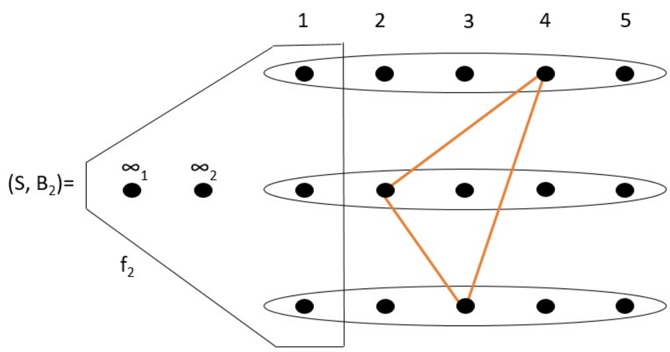
$$Z_1 = \left\{ \begin{array}{cccc} (1, & 2, & 3, & 4) \\ & \{2, & 8, & 10\} \\ & \{3, & 5, & 10\} \\ & \{4, & 5, & 8\} \end{array} \right. \quad Z_2 = \left\{ \begin{array}{cccc} (1, & 4, & 8, & 2) \\ & \{3, & 4, & 5\} \\ & \{2, & 3, & 10\} \\ & \{5, & 8, & 10\} \end{array} \right.$$

None of the triples in $\mathcal{Z}_1$ and $\mathcal{Z}_2$ belong to P_1 or P_2. So removing $\mathcal{Z}_1$ from C_1 and replacing it with Z_2 reducing the number of type (1) triples by 3. Taking P_1 and P_2 to have 0, 1, or 2 triples in common gives intersection numbers 12, 13, and 14.

Example 2.3 **(**$n = 17$**)** Let $\mathcal{Q} = \{1, 2, 3, 4, 5\}$ and let $(\mathcal{Q}, \circ_1)$ and $(\mathcal{Q}, \circ_2)$ be two quasigroups such that $1 \circ_1 1 = 1 \circ_2 1 = 1$. Set $\mathcal{S} = \{\infty_1, \infty_2\} \cup (\{1, 2, 3, 4, 5\} \times \{1, 2, 3\})$ and define PBDs $(\mathcal{S}, B_1)$ and $(\mathcal{S}, B_2)$ of order 17 as follows:



$\circ_1$	1	2	3	4	5
1	1				
2					
3				2	
4					
5					



$\circ_2$	1	2	3	4	5
1	1				
2					
3					
4		3			
5					

1. $f_1 = f_2 = \{\infty_1, \infty_2, 11, 12, 13\} \in B_1 \cap B_2$. We can define copies of *Example 2.1* independently on f_1 and f_2 so that $| f_1 \cap f_2 | \in \{0, 1, 2\}$.

2. For each $i, j \in \{1, 2, 3, 4, 5\}$, let $\{i1, j2, (i \circ_1 j, 3)\} \in B_1$ and $\{i2, j2, (i \circ_2 j, 3)\} \in B_2$. (Note that $\{11, 12, 13\} \in B_1 \cap B_2$.) Since the intersection numbers for quasigroups of order 5 are $\{0, 1, 2, \ldots, 25\} \backslash \{24, 23, 22, 20\}$[1] and since in each of the quasigroups $(\mathcal{Q}, \circ_1)$ and $(\mathcal{Q}, \circ_2)$ $1 \circ_1 1 = 1 \circ_2 1 = 1$ and the triple $\{11, 21, 31\} \in f_1 \cap f_2$ the type (2) intersection numbers are $\{0, 1, 2, \ldots, 17, 18, 20, 24\}$.

3. For each $i \in \{1, 2, 3\}$ set $X(i) = \{\infty_1, \infty_2\} \cup \{\{1, 2, 3, 4, 5\} \times \{i\}\}$ and define a triple system $(X(i), T(i))$ where $\{\infty_1, \infty_2, 1i\} \in T(i)$. Since the intersection numbers for triple systems of order 7 are 0, 1, 3, 7; the intersection numbers for $T(i) \backslash \{\infty_1, \infty_2, 1i\}$, $T(j) \backslash \{\infty_1, \infty_2, 1j\}$ for each i and j are 0, 2, and 6.

The intersection numbers in (1), (2), and (3) are independent of each other and so the intersection numbers for (S, B_1) and (S, B_2) consists of $x + y + z$, where $x = | f_1 \cap f_2 | \in \{0, 1, 2\}$, $y \in \{0, 1, 2, \ldots, 17, 18, 20, 24\}$, and $z \in \{0, 2, 6\} + \{0, 2, 6\} + \{0, 2, 6\}$. A straightforward computation shows that $x+y+z \in \{0, 1, 2, \ldots, 44\} \backslash \{41\}$. So all that remains is to show that $41 \in J^*(17) = \{0, 1, 2, \ldots, 44\}$ (no exceptions). Take (S, B_1) and (S, B_2) to be the same. Define $T(1)$ in B_1 to be

$$T(1) = \begin{cases} \infty_1 & \infty_2 & 11 \\ 11 & 21 & 31 \\ 11 & 41 & 51 \\ \infty_1 & 21 & 51 \\ \infty_1 & 31 & 41 \\ \infty_2 & 21 & 41 \\ \infty_2 & 31 & 51 \end{cases}$$

We can assume in f_1 that the leave is the $4-$ cycle $(\infty_1, \infty_2, 11, 12)$. Then the configuration

$$Z_1 = \begin{cases} (\infty_1, & \infty_2, & 11, & 12) \\ & \{21, & 31, & 11\} \\ & \{31, & 41, & \infty_1\} \\ & \{21, & 41, & \infty_2\} \end{cases}$$

belongs to B_1. If we replace Z_1 in B_1 with

$$Z_2 = \begin{cases} (\infty_1, & 12, & 11, & 31) \\ & \{\infty_1, & \infty_2, & 41\} \\ & \{\infty_2, & 11, & 21\} \\ & \{21, & 31, & 41\} \end{cases}$$

we reduce the intersection number between B_1 and B_2 from 44 to 41. This completes *Example 2.3.*

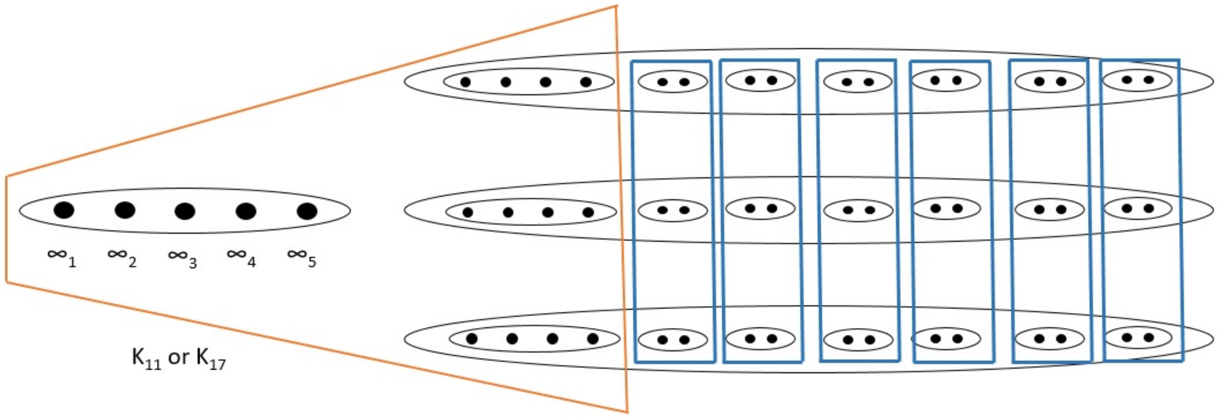
3. The $6n + 5$ Construction

With the three examples in *Section 2* in hand we can proceed to the main construction showing that $I(6n+5) = J^*(6n+5) = \{0, 1, 2, \ldots, \frac{(\binom{n}{2}-4)}{3}\}$ for all n.

<u>The $6n+5$ Construction:</u> Let $6n + 5 \geq 23$ and let (X, G, B) be a $GDD(2n, 2, 3)$ or $GDD(2n, \{2, 4^*\}, 3)$, where $\{2, 4^*\}$ means there is exactly one group of size 4 and the rest have size 2. Set $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (X \times \{1, 2, 3\})$ and define a maximum packing, P of K_{6n+5} as follows:

1. Place an example of order 11 or 17 on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (g \times \{1, 2, 3\})$ where g is a block of size 2 if all blocks have size 2; or 4 if g in the unique block of size 4.

2. For all other blocks (which necessarily have size 2) place a copy of *Example 2.2* or 2.3 on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (g \times \{1, 2, 3\})$ minus the block $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ of size 5.



3. For each triple $\{a, b, c\} \in B$ decompose $K_{3,3,3}$ into 9 triples with parts $a \times \{1, 2, 3\}$, $b \times \{1, 2, 3\}$ and $c \times \{1, 2, 3\}$.

Then (S, P) is a maximum packing of K_{6n+5} with triples with leave a 4–cycle. Now take two copies of (S, P). We need construct only the intersection numbers $x-1$, $x-2$, $x-3$, and $x-5$ since Quattrocchi has taken care of everything else. But this is easily done by defining a pair of maximum packings of order 11 or 17 on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (g \times \{1, 2, 3\})$ intersecting in $x-1$, $x-2$, $x-3$, or $x-5$ triples, where $x = 17$ or 44 as the case may be. This completes the proof. We have the following theorem:

Theorem 3.1. $I(6n+5) = J^*(6n+5)$ for all $6n+5$. ■

References

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