Compactifications of indecomposable topological spaces

by

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Abstract

A *continuum* is a compact and connected topological space. A continuum that is not the union of any two of its proper subcontinua is said to be *indecomposable*. We examine topological spaces which are closely related to indecomposable continua, specifically, widely-connected spaces and the hyperspace of the Stone-Čech remainder of the half-line. A *widely-connected* space is a connected space all of whose non-trivial connected subsets are dense in the entire space. We answer questions of Erdős, Bellamy, and Mioduszewski with the following examples: a completely metrizable widely-connected subset of the plane; a widely-connected subset of Euclidean 3-space that is not indecomposable upon the addition of a single limit point; a widely-connected subset of Euclidean 3-space that is contained in a composant of each of its compactifications; widely-connected spaces of large cardinalities. Then, we construct the maximum possible number of non-homeomorphic subcontinua of $C(\mathbb{H}^*)$, each of which is a union of two order arcs. We also characterize when βX is indecomposable and study the structure of $C(\mathbb{H}^*)$ using the property of Kelley.

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Chapter 1

Introduction

A topological space that is compact and connected is called a continuum. Usually, a continuum can be decomposed into two proper subcontinua – for instance, $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $[0, \frac{1}{3}] \cup [\frac{1}{4}, 1]$ are decompositions of the unit interval $[0, 1]$. A continuum which admits no such decomposition is said to be indecomposable. By a standard result of continuum theory, every connected subset of an indecomposable continuum is either dense or nowhere dense. This document addresses connected spaces with similar properties.

Widely-connected spaces are connected topological spaces all of whose non-degenerate connected subsets are dense in the original space. Examples are usually constructed as dense subsets of indecomposable continua by carefully eliminating all of the nowhere dense subcontinua. Techniques for constructing widely-connected sets in this manner are provided in Chapter 2. In Section 2.2 we describe a new method for producing widely-connected subsets of a planar indecomposable continuum known as the bucket-handle continuum. An example is obtained in Section 2.2.4 by deleting only countably many compact sets from the bucket-handle. A metrizable space is completely metrizable if and only if it is a G_δ-subset of one (each) of its compactifications. Thus, the problem of finding a completely metrizable widely-connected space, due to Paul Erdős and Howard Cook, is solved. The original technique for constructing widely-connected spaces is revived in Section 2.3. Applying it to the large indecomposable continua of Michel Smith produces widely-connected spaces of arbitrarily large cardinality, answering a question of David Bellamy.

Now there is the question of whether every widely-connected space is a dense subset of an indecomposable continuum. Equivalently: *Is the Stone-Čech compactification of a widely-connected space necessarily an indecomposable continuum?* This question is made interesting by the fact that the indecomposability of a widely-connected set can be destroyed by adding limit points. In fact, Example 2.26 shows that the addition of a single limit point to a widely-connected subset of Euclidean 3-space can destroy indecomposability. The existence of such an example was suggested by Mary Ellen Rudin, who proved that this scenario is impossible in the plane but never published her example. The special limit point glues together many disjoint closed subsets of the original space, and is therefore not representative of a point in the Stone-Čech compactification.

In Section 2.4.1 we describe a property of X that is necessary and sufficient in order for βX to be indecomposable. Our characterization shows that some extreme types of widely-connected spaces have indecomposable compactifications. Also, when X is a separable metric space, βX is indecomposable if and only if X has a *metric* indecomposable compactification – Section 2.4.2. In the language of continuum theory the question above is equivalent to: *If W is a connected separable metric space that is irreducible between every two of its points, then does W have an irreducible compactification?* Perhaps surprisingly, W may not have a compactification that is irreducible between two points in W. A counterexample is given in Section 2.4.3; it implies a negative answer to a question of Jerzy Mioduszewski.

The Stone-Čech remainder $\mathbb{H}^* := \beta[0,\infty) \setminus [0,\infty)$ is a non-metric indecomposable continuum. The subject of Chapter 3 is the hyperspace of subcontinua of $\mathbb{H}^*$, denoted $C(\mathbb{H}^*)$. An order arc in the hyperspace is a linearly ordered continuum that is obtained by continuously increasing from one continuum to another. In Section 3.2, a method of Alan Dow and Klaas Pieter Hart is used to find $2^{\mathfrak{c}}$ mutually non-homeomorphic subcontinua of $C(\mathbb{H}^*)$, each of which is a union of two order arcs. If the Continuum Hypothesis fails, then there is the maximum possible number of $2^{\mathfrak{c}}$ non-homeomorphic order arcs in $C(\mathbb{H}^*)$, in sharp contrast to the metric case (if X is a metric continuum then every order arc in $C(X)$ is homeomorphic to the unit interval $[0,1]$). On the other hand, if the Continuum Hypothesis holds then there are at least 3 different order arcs, but the exact number is still unknown. Finally, in Section 3.3 we use the property of Kelley to learn more about the general structure of $C(\mathbb{H}^*)$.

1.1 Preliminaries

Let X be a topological space. Then X is *connected* if it cannot be partitioned into two non-empty disjoint open subsets. A subset of X that is both closed and open is said to be *clopen*. Since open and closed sets are complementary, X is connected if and only if every clopen subset of X is equal to $\varnothing$ or all of X. Some ways of creating larger connected spaces from smaller ones are described by the following.

Theorem 1.1 ([11] Theorem 6.1.10). *If the collection $\{C_s : s \in S\}$ of connected subspaces of a topological space has a non-empty intersection, then $\bigcup_{s\in S} C_s$ is connected.*

Theorem 1.2 ([11] Theorem 6.1.11). *If a subspace C of X is connected then every subspace A of X which satisfies $C \subseteq A \subseteq \overline{C}$ is also connected.*

Theorem 1.3 ([18] §46 II Theorem 4). *If C is a connected subset of a connected space X, and if U and V are disjoint relatively clopen subsets of $X \setminus C$, then $C \cup U$ and $C \cup V$ are connected.*

The *component* of a point $x \in X$ is the union of all connected subsets of X which contain x. The *quasi-component* of x is the intersection of all clopen subsets of X which contain x. The component of x is always contained in the quasi-component of x, but they are not necessarily equal. Consider for instance $X = (\{0\} \times 2) \cup (\{1/n : n \geq 1\} \times [0, 1])$. The quasi-component of $\langle 0, 0\rangle$ is $\{0\} \times 2$, but the component of $\langle 0, 0\rangle$ is $\{0\} \times 1$.

Following [11], X is *hereditarily disconnected* if all of its components are singletons, and *totally disconnected* if all of its quasi-components are singletons. Every totally disconnected space is hereditarily disconnected, but the converse is not necessarily true – see Examples X_1 and X_3 in Sections 2.2.2 and 2.2.4.

A *continuum* (plural form *continua*) is a connected compact Hausdorff space.

Theorem 1.4 ([11] Theorem 6.1.18). *Let $\{C_s : s \in S\}$ be a collection of subspaces of a topological space X each of which is a continuum. If $C_s \subseteq C_t$ or $C_t \subseteq C_s$ for each s and t in S, then $\bigcap_{s\in S} C_s$ is a continuum.*

Theorem 1.5 ([11] Theorem 6.1.23). *In a compact space X the component of a point $x \in X$ coincides with the quasi-component of the point x.*

Theorem 1.6 ([11] Theorem 6.1.25). *If A is a closed subset of a continuum X such that $\varnothing \neq A \neq X$, then for every component C of the space A we have $C \cap \partial A \neq \varnothing$.*

If X is a continuum, then P is a *composant* of X if there exists $p \in X$ such that P is the union of all proper subcontinua of X that contain p. By Theorems 1.1 and 1.6, each composant of X is connected and dense in X.

If X is a continuum, then X is *decomposable* if there are two proper subcontinua $H, K \subsetneq X$ such that $X = H \cup K$. Otherwise, X is *indecomposable*.

Theorem 1.7 ([18] §48 V Theorem 2). *Let X be a continuum. The following are equivalent:*

(i) X is indecomposable;

(ii) Every proper subcontinuum of X is nowhere dense in X.

Theorem 1.8 ([18] §48 VI Theorem 5). *The composants of an indecomposable continuum are disjoint.*

Examples of indecomposable continua:

- Knaster bucket-handle continuum – [18] §48 V Example 1
- Stone-Čech remainder of the half-line – [32] Theorem 9.13
- Examples with one and two composants – David Bellamy [3]
- Examples with large numbers of composants – Michel Smith [30]

A *compactification* of a space X is a compact Hausdorff space in which X is densely embedded. Recall that a space is T_1 if all of its singletons are closed, and completely regular if for any closed $A \subseteq X$ and $x \in X \setminus A$ there is a continuous function $f : X \to [0,1]$ such that $f[A] = \{0\}$ and $f(x) = 1$. Every T_1 completely regular space, sometimes called a Tychonoff space or a $T_{3.5}$ space, has a particular compactification called the Stone-Čech compactification, which we describe now.

Let X be Tychonoff. $Z \subseteq X$ is a *zero set* if there is a continuous function $f : X \to \mathbb{R}$ such that $Z = f^{-1}\{0\}$. Zero sets are obviously closed. The opposite is true if X is metrizable. Indeed, if d is a metric for X and A is a closed subset of X, then A is the zero set of the mapping $d_A : X \to [0, \infty)$ defined by $d_A(x) = \inf\{d(x, y) : y \in A\}$. Let $\mathcal{Z}(X) = \{Z \subseteq X : Z \text{ is a zero set}\}$. Say that $p \subseteq \mathcal{Z}(X)$ is a *filter* if

i. $p \neq \varnothing$ and $\varnothing \notin p$,

ii. $Z_1 \cap Z_2 \in p$ whenever $Z_1, Z_2 \in p$, and

iii. $Z \in p$ whenever $Z \in \mathcal{Z}(X)$ is a superset of a member of p.

A filter $p \subseteq \mathcal{Z}(X)$ is an *utrafilter* if no other filter properly contains p. Equivalently, p is an ultrafilter if p is a filter and $Z \in p$ whenever $Z \in \mathcal{Z}(X)$ intersects each member of p. Every filter has the finite intersection property, and every subset of $\mathcal{Z}(X)$ with the finite intersection property is contained in an ultrafilter by Zorn's Lemma. Let

$$\beta X = \big\{p \subseteq \mathcal{Z}(X) : p \text{ is an ultrafilter}\big\}.$$

For each $U \subseteq X$ such that $X \setminus U \in \mathcal{Z}(X)$, let $\mathrm{ex}_{\beta X} U = \{p \in \beta X : (\exists Z \in p)(Z \subseteq U)\}$. The set of all $\mathrm{ex}_{\beta X} U$'s is a basis for a topology on βX; the topology of βX is the topology generated by this basis.

Theorem 1.9. *βX is compact Hausdorff, and $e : X \hookrightarrow \beta X$ by $x \mapsto \{Z \in \mathcal{Z}(X) : x \in Z\}$ is a well-defined dense homeomorphic embedding*

Thus βX is a compactification of X (the *Stone-Čech compactification*). Up to homeomorphism, it is the unique compactification of X with the following extension property.

Theorem 1.10. *If Y is compact Hausdorff and $f : X \to Y$ is a continuous function, then there is a unique continuous function $\beta f : \beta X \to Y$ such that $\beta f \restriction X = f$.*

In particular, every continuous function from X into $[0,1]$ extends to βX. βf is called the *Stone-Čech extension* of f. Additional properties of βX:

- $\mathrm{cl}_{\beta X} Z = \{p \in \beta X : Z \in p\}$ for each $Z \in \mathcal{Z}(X)$
- $\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_1 = \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$ for all $Z_1, Z_2 \in \mathcal{Z}(X)$
- If γX is a compactification of X, $\iota : X \to X$ is the identity map, and $\beta\iota : \beta X \to \gamma X$ is the Stone-Čech extension of ι, then $\beta\iota[\beta X \setminus X] = \gamma X \setminus X$ ($\beta X \setminus X$ is called the *Stone-Čech remainder of* X). In particular, $\beta\iota$ maps onto γX.
- If X is compact Hausdorff, then $X \simeq \beta X$.
- βX is connected if and only if X is connected.

Chapter 2

Widely-connected spaces

2.1 Background

A topological space X is *punctiform* if X does not contain a continuum with more than one point. The earliest examples of connected punctiform spaces are due to Mazurkiewicz [21], Kuratowski and Sierpinski [17], and Knaster and Kuratowski [15]. Each of these examples is the graph of a hereditarily discontinuous function. For any space X and real-valued function $f : X \to \mathbb{R}$, let $\text{disc}(f) = \{x \in X : f \text{ is not continuous at } x\}$ be the set of discontinuities of f, and let $\text{Gr}(f) = \{\langle x, f(x)\rangle : x \in X\}$ be the graph of f. By *f is hereditarily discontinuous*, we mean $\overline{\text{disc}(f)} = X$.

The example of Kuratowski and Sierpinski exploits the fact that a function with a single discontinuity can have a connected graph. Let $\varphi(x) = \sin(1/x)$ for $x \neq 0$ and put $\varphi(0) = 0$. Then $\text{disc}(\varphi) = \{0\}$, but $\text{Gr}(\varphi)$ is connected as it consists of two rays with a common limit point at the origin. Now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi(x - q_n)}{2^n},$$

where $\mathbb{Q} = \{q_n : n < \omega\}$ is an enumeration of the rationals. f satisfies the conclusion of the Intermediate Value Theorem and $\text{disc}(f) = \mathbb{Q}$. These properties imply that $\text{Gr}(f)$ is a punctiform connected G_δ-set in the plane.

Two types of punctiform connected spaces exclude graphs of functions from the real line. A topological space X is *biconnected* if X is connected and X is not the union of two disjoint and non-degenerate connected subsets. A topological space W is *widely-connected* if W is connected and every non-degenerate connected subset of W is dense in W.

Theorem 2.1. *Let X be a connected space. The following are equivalent:*

(i) X is biconnected;

(ii) X contains no pair of disjoint and non-degenerate connected sets.

Proof. Clearly (ii)⇒(i). We prove ¬(ii)⇒¬(i). Suppose that A and B are disjoint connected subsets of X each with more than one point. Let C be the component of B in $X \setminus A$. Then $X \setminus C$ is connected. Indeed, suppose that U and V are disjoint relatively clopen subsets of $X \setminus C$ such that $X \setminus C = U \cup V$. Since A is a connected subset of $X \setminus C$, we can assume that $A \subseteq U$. By Theorem 1.3, $C \cup V$ is connected. By maximality of C we have $C \cup V \subseteq C$, thus $V = \varnothing$. In summary, C and $X \setminus C$ are disjoint connected sets whose union is X. Each has more than one point because $A \subseteq X \setminus C$ and $B \subseteq C$. □

Theorem 2.2. *Let W be a connected space. The following are equivalent:*

(i) W is widely-connected;

(ii) Every subset of W is either connected or hereditarily disconnected.

Proof. (i)⇒(ii): Suppose that W is widely-connected and $A \subseteq W$ is not hereditarily disconnected. Then there is a connected $C \subseteq A$ with $|C| > 1$. Then $\overline{C} = W$. Thus $C \subseteq A \subseteq \overline{C}$, whence A is connected. (ii)⇒(i): Suppose that W is not widely-connected. Then there is a connected $C \subseteq W$ with $|C| > 1$ and $\overline{C} \neq W$. Let $p \in W \setminus \overline{C}$. Then $C \cup \{p\}$ is a subset of W that is neither connected nor hereditarily disconnected. □

Theorem 2.3. *Biconnected and widely-connected Hausdorff spaces are punctiform.*

Proof. Let X he a Hausdorff space, and suppose that X contains a continuum $K, |K| > 1$. Let p and q be distinct points in K. There are two relatively open subsets of K, say U and V, such that $p \in U$, $q \in V$, and $\overline{U} \cap \overline{V} = \varnothing$. Let P be the component of p in $\overline{U}$, and let Q be the component of q in $\overline{V}$. Then P and Q are disjoint closed connected subsets of X, and each has more than one point by Theorem 1.6. By Theorem 2.1, X is not biconnected. Since $\overline{P} \neq X$, X is not widely-connected. □

Theorem 2.4. *If $f : \mathbb{R} \to \mathbb{R}$, then $Gr(f)$ is neither widely-connected nor biconnected.*

Proof. Assuming that $\mathrm{Gr}(f)$ is connected, we show that $\mathrm{Gr}(f \restriction [0,\infty))$ is connected. Suppose that A and B are non-empty closed subsets of $\mathrm{Gr}(f \restriction [0,\infty))$ and $A \cup B = \mathrm{Gr}(f \restriction [0,\infty))$. Assume that $\langle 0, f(0)\rangle \in A$. Then $\mathrm{Gr}(f \restriction (-\infty,0]) \cup A$ and B are non-empty closed subsets $\mathrm{Gr}(f)$ whose union is $\mathrm{Gr}(f)$. Since $\mathrm{Gr}(f)$ is connected, $(\mathrm{Gr}(f \restriction (-\infty,0]) \cup A) \cap B \neq \varnothing$. The only way this happens is if $A \cap B \neq \varnothing$. Thus $\mathrm{Gr}(f \restriction [0,\infty))$ is connected, so $\mathrm{Gr}(f)$ is not widely-connected. By similar arguments, $\mathrm{Gr}(f \restriction (-\infty,0)) = \bigcup_{n=1}^{\infty} \mathrm{Gr}(f \restriction (-\infty,-1/n])$ is connected, so $\mathrm{Gr}(f)$ is not biconnected. $\square$

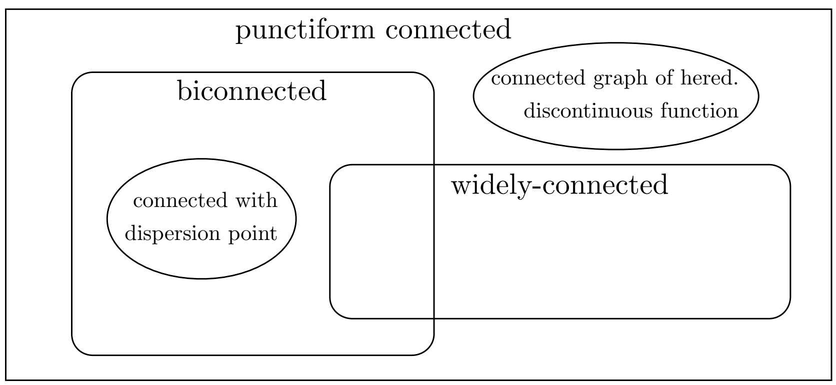

Figure 2.1: Classes of some T_2 connected spaces

Knaster and Kuratowski [16] gave the first example of a biconnected space by constructing a connected space with a dispersion point. A point p in a connected space X is called a *dispersion point* if $X \setminus \{p\}$ is hereditarily disconnected. The Knaster-Kuratowski fails to be topologically complete as it contains closed copies of the rationals. However, Knaster and Kuratowski later constructed a completely metrizable dispersion point space by deforming their punctiform connected graph in [15]. Erdős [12] constructed a totally disconnected closed subspace of the Hilbert space ℓ^2 with a remarkable property, namely, that of Theorem 2.10. Roberts [25] embedded the Erdős space into the plane so that the addition of a single point produces a connected set. The augmented embedding is a completely metrizable *explosion point* space ($X \setminus \{p\}$ is *totally* disconnected).

Miller [19] used the Continuum Hypothesis (CH) to construct the first example of a biconnected set without a dispersion point. Miller's biconnected set has no cut points precisely because it is widely-connected (apply Theorem 1.3). The first widely-connected space slightly predates Miller's example, and was constructed by Swingle [31].

Rudin [26] also used CH to construct a connected subset of the plane, each of whose connected subsets has countable complement (in the entire set). Her example is easily seen to be both widely-connected and biconnected. Every connected metric space has a connected subset with infinite complement, so in a sense Rudin's example is the most extreme type of widely-connected metric space. Consistently, there are completely regular and perfectly normal connected spaces all of whose connected subsets have finite complement. These are due to Gruenhage [13]. Most recently, Rudin [27] showed that under Martin's Axiom there is a biconnected subset of the plane which has no dispersion point and is *not* widely-connected. It is still unknown if a metric biconnected set without a dispersion can be constructed without additional set-theoretic axioms.

We will see in Section 2.2.3 that the examples of Swingle, Miller, and Rudin fail to be completely metrizable, hence the question of Erdős and Cook: *Is there a completely metrizable widely-connected space?* This question appears as Problem G3 in [29] and [24], and as Problem 123 in [6].

2.2 In the bucket-handle

In this section we describe a method for constructing widely-connected subsets of the bucket-handle continuum. Our method produces three examples, one of which is completely metrizable. The building blocks are called connectible sets. Let C denote the middle-thirds Cantor set in the interval $[0,1]$. If X is a subset of $C\times(0,1)$, then X is *connectible* if $\langle c,0\rangle\in A$ whenever A is a clopen subset of $X\cup(C\times\{0\})$, $c\in C$, and $A\cap(\{c\}\times(0,1))\neq\varnothing$. There is a one-to-one correlation between connectible subsets of $C\times(0,1)$ and connected subsets of the Cantor fan – this is the intuition behind the examples in Section 2.2.2.

2.2.1 Assembly of connectible sets

Let K be the Knaster bucket-handle continuum. The rectilinear version of K depicted in Figure 2.2 is formed by successively removing open middle-third canals (white regions of the figure), beginning with the unit square $[0,1]^2$. $K \setminus \{\langle 0,0\rangle\}$ is locally homeomorphic to $C \times (0,1)$. Moreover, if $\Delta = \{\langle x,x\rangle \in K : x \in [0,1]\}$ is the diagonal Cantor set in K, then $K \setminus \Delta$ is the union of ω-many copies of $C \times (0,1)$, to wit, $K \setminus \Delta = \bigcup_{i<\omega} K_i$ where the K_i's are shown in Figure 2.3.

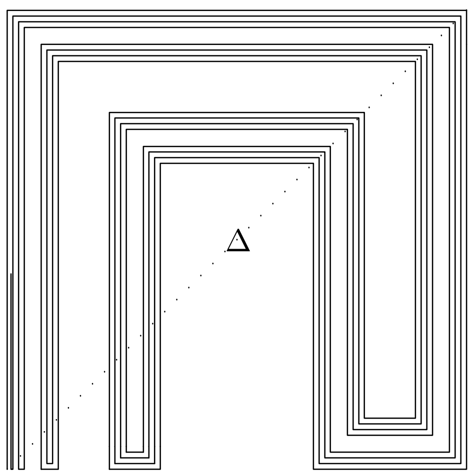

Figure 2.2: K

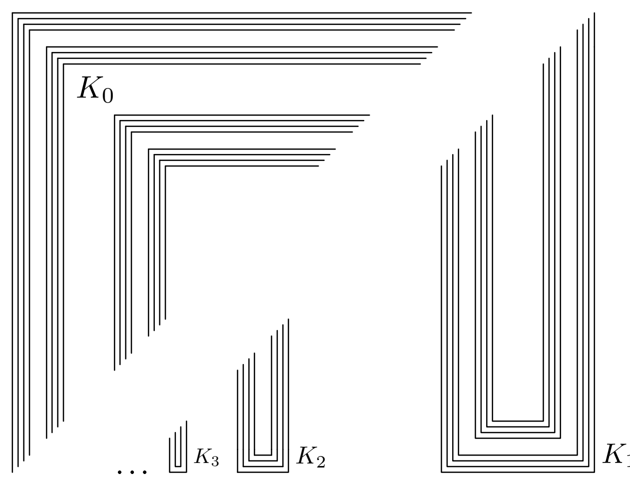

Figure 2.3: $K \setminus \Delta$

For each $i < \omega$ let $\varphi_i : C \times (0,1) \to K_i$ be a simple bending homeomorphism that witnesses $K_i \simeq C \times (0,1)$. If X is a subset of $C \times (0,1)$ then define

$$W[X] = \Delta \cup \bigcup_{i<\omega} \varphi_i[X].$$

Proposition 2.5. *If X is a dense connectible subset of $C \times (0,1)$, then $W[X]$ is connected. If X is also hereditarily disconnected, then $W[X]$ is widely-connected.*

Proof. Let X be a dense connectible subset of $C \times (0,1)$.

Suppose for a contradiction that $W := W[X]$ is not connected. Let A and B be non-empty closed subsets of W such that $W = A \cup B$ and $A \cap B = \varnothing$. Then $K = \overline{A} \cup \overline{B}$ because

W is dense in K. There exists $p \in \overline{A} \cap \overline{B}$ because K is connected. $p \notin \Delta$ because $\Delta \subseteq W$, so there exists $i < \omega$ such that $p \in K_i$. Thinking of $\overline{K_i}$ as $C \times [0,1]$ with $C \times 2 \subseteq \Delta$, give local coordinates $\langle p(0), p(1) \rangle$ to p. Assume that $p' := \langle p(0), 0 \rangle \in A$. Let (b_n) be a sequence of points in $K_i \cap B$ converging to p. Then $b_n' := \langle b_n(0), 0 \rangle \in B$ for each $i < \omega$ because X is connectible. The sequence (b_n') converges to p'. Since B is closed in W, $p' \in B$. We have $p' \in A \cap B$, a contradiction.

Suppose that $W[X]$ is not widely-connected. Let A be a non-dense connected subset of W with more than one point. Every non-degenerate proper subcontinuum of K is an arc, therefore $\overline{A}$ is an arc. Let $e : [0,1] \hookrightarrow K$ be a homeomorphic embedding such that $e([0,1]) = \overline{A}$. Let $a, b \in A$ with $a \neq b$, and let $r, s \in [0,1]$ such that $e(r) = a$ and $e(s) = b$. Assume that $r < s$. $e^{-1}[A]$ is connected, so $[r,s] \subseteq e^{-1}[A]$. Clearly $e([r,s]) \not\subseteq \Delta$, so there exists $i < \omega$ such that $e([r,s]) \cap K_i \neq \varnothing$. Then $e^{-1}[K_i] \cap [r,s]$ is a non-empty open subset of $[r,s]$ and thus contains a non-degenerate interval I. $e[I]$ is a non-degenerate connected subset of $W \cap K_i = \varphi_i[X]$. Thus X is not hereditarily disconnected. □

Remark 2.6. A widely-connected set cannot contain an interval, so every widely-connected subset of K is dense in K.

2.2.2 Connectible sets X_1 and X_2

Here we describe two connectible sets X_1 and X_2. Both sets will be hereditarily disconnected and dense in $C \times (0,1)$, so that $W[X_1]$ and $W[X_2]$ will be widely-connected.

Example $\mathbf{X_1}$. Let C' be the set of all endpoints of intervals removed from $[0,1]$ in the process of constructing C, and let $C'' = C \setminus C'$. Let

$$X_1 = \big(C' \times \mathbb{Q} \cap (0,1)\big) \cup \big(C'' \times \mathbb{P} \cap (0,1)\big),$$

where $\mathbb{Q}$ and $\mathbb{P}$ are the rationals and irrationals, respectively. Clearly X_1 is hereditarily disconnected and dense in $C \times (0,1)$. The reader may recognize X_1 as the Knaster-Kuratowski fan [16] minus its dispersion point. In proving that X_1 is connectible, we essentially prove that the Knaster-Kuratowski fan is connected.

Proposition 2.7. X_1 *is connectible.*

Proof. Let A be a non-empty clopen subset of $X_1 \cup (C \times \{0\})$ and let $\langle c, r\rangle \in A$. Suppose for a contradiction that $\langle c, 0\rangle \in B := X_1 \setminus A$. There are open sets $U \subseteq C$ and $V \subseteq (0,1)$ such that $\langle c, 0\rangle \in U \times \{0\} \subseteq B$ and $\langle c, r\rangle \in X_1 \cap (U \times V) \subseteq A$. Enumerate $\mathbb{Q} \cap (0,1) = \{q_i : i < \omega\}$. For each $i < \omega$ let $C_i = \{c \in C : \langle c, q_i\rangle \in \overline{A} \cap \overline{B}\}$. Each C_i closed and nowhere dense in C. By the Baire Category Theorem, $C \setminus (C' \cup \bigcup_{i<\omega} C_i)$ is dense in C, so there exists $d \in U \cap C'' \setminus \bigcup_{i<\omega} C_i$. Then $\overline{A}$ and $\overline{B}$ form a non-trivial clopen partition of the connected set $\{d\} \times [0,1)$, a contradiction. □

Example $\mathbf{X_2}$. Define $\|\cdot\| : \mathbb{R}^\omega \to [0,\infty]$ by

$$\|x\| = \sqrt{\sum_{i=0}^{\infty} x_i^2}.$$

The Hilbert space ℓ^2 is the set $\{x \in \mathbb{R}^\omega : \|x\| < \infty\}$ with the norm topology generated by $\|\cdot\|$. The subspace $\mathfrak{E} := \{x \in \ell^2 : x_i \in \{0\} \cup \{1/n : n \in \mathbb{N}\}$ for each $i < \omega\}$ of ℓ^2 is called the *complete Erdős space* ($\mathfrak{E}$ is closed subset of the completely metrizable ℓ^2, and is therefore complete). Let $f : [0,\infty) \to [0,1]$ be the function with the following graph.

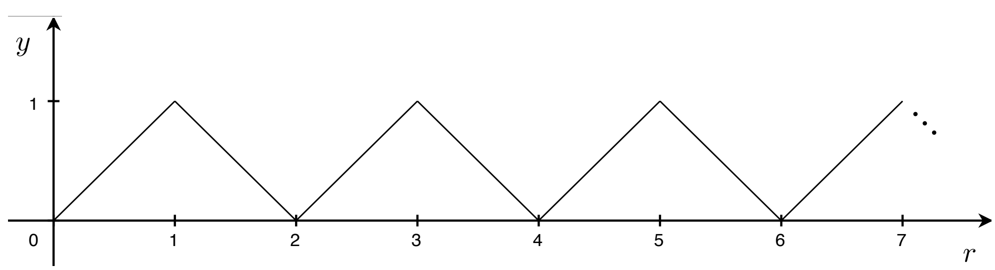

Figure 2.4: $y = f(r)$

The Cantor set is the unique zero-dimensional compact metric space without isolated points, therefore $(\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega \simeq C$. We will think of these spaces as being equal.

Define $\xi : \mathfrak{E} \to C \times [0,1]$ by $x \mapsto \langle x, f\|x\|\rangle$. Note that ξ is continuous, as the inclusion $\mathfrak{E} \hookrightarrow (\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega$ is continuous and f and $\|\cdot\|$ are both continuous. Let

$$X_2 = \xi[\mathfrak{E}] \setminus (C \times 2).$$

X_2 has at most one point from each fiber $\{c\} \times (0,1)$, so it is totally disconnected.

Remark 2.8. If instead $f : [0,\infty) \to [0,1]$ is defined by $f(r) = 1/(1+r)$, then ξ is a homeomorphic embedding (see [25] and [10] 6.3.24). By adding $C \times \{0\}$ to $\xi[\mathfrak{E}]$ and then contracting it to a point, one obtains a completely metrizable dispersion point space. However, in this case $\xi[\mathfrak{E}]$ is nowhere dense in $C \times (0,1)$. We will see that the sinusoidal f makes ξ a dense embedding, but destroys completeness.

Proposition 2.9. *X_2 is dense in $C \times (0,1)$.*

Proof. Let $U \times (a,b)$ be a non-empty open subset of $C \times (0,1)$. Assume that U is a basic open subset of $(\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega$; $U = \prod_{i<\omega} U_i$ and $n = \max\{i < \omega : U_i \neq X\}$. For each $i \leq n$ choose $x_i \in U_i$. Let $c \in (a,b)$. There exists $p \in f^{-1}\{c\}$ such that $p^2 > \sum_{i \leq n} x_i^2$. Let $r = p^2 - \sum_{i \leq n} x_i^2$. There is an increasing sequence of rationals (q_i) such that $q_0 = 0$ and $q_i \to r$ as $i \to \infty$. Each $q_{i+1} - q_i$ is a positive rational number $\frac{a_i}{b_i}$ for some $a_i, b_i \in \mathbb{N}$. For each $i < \omega$ let $y^i \in \{\frac{1}{b_i}\}^{a_i \cdot b_i}$ be the finite sequence of $a_i \cdot b_i$ repeated entries $\frac{1}{b_i}$. Let $y = y^0 {}^\frown y^1 {}^\frown y^2 {}^\frown ...$ be the sequence in $\{1/n : n \in \mathbb{N}\}^\omega$ whose first $a_0 \cdot b_0$ coordinates are $\frac{1}{b_0}$, whose next $a_1 \cdot b_1$ coordinates are $\frac{1}{b_1}$, etc. Note that $\sum_{i=0}^\infty y_i^2 = \sum_{i=0}^\infty q_{i+1} - q_i = r$. Now put $z := \langle x_0, ..., x_n, y_0, y_1, ...\rangle$. We have $z \in U$ and $\|z\| = p$, so that $\xi(z) = \langle z, c\rangle \in U \times (a,b)$. □

The following is due to Erdős [12].

Proposition 2.10. *$\{\|x\| : x \in A\}$ is unbounded if A is a non-empty clopen subset of $\mathfrak{E}$.*

Proof. Let A be a non-empty subset of $\mathfrak{E}$ such that $\{\|x\| : x \in A\}$ is bounded. We show that A has non-empty boundary.

Let $N \in \omega$ such that $\|x\| < N$ for each $x \in A$, and let $a^0 \in A$. Define a^1 as follows. There is a least $j \in [1, N]$ such that $\langle 1, 1, ..., 1, a^0_j, a^0_{j+1}, ...\rangle \in \mathfrak{E} \setminus A$ (replacing a^0_i with 1 for each $i < j$). Let $a^1_i = 1$ if $i < j-1$ and $a^1_i = a^0_i$ otherwise. Then $a^1 \in A$ and $d(a^1, \mathfrak{E} \setminus A) \leq 1$. Let $j_0 = 0$ and $j_1 = j$.

Suppose $k > 1$ and $a^n \in A$ and increasing integers j_n have been defined, $n < k$, such that $d(a^n, \mathfrak{E} \setminus A) \leq 1/n$ and $a^n_i = a^0_i$ for $i \geq j_n$. There exists $j' > j_{k-1}$ such that $a^0_i < 1/k$ whenever $i \geq j'$. There is a least $j \in [1, kN]$ such that

$$\langle a^{k-1}_0, ..., a^{k-1}_{j'-1}, 1/k, 1/k, ..., 1/k, a^{k-1}_{j'+j}, ...\rangle \in \mathfrak{E} \setminus A.$$

Let $j_k = j' + j$. Define a^k by letting $a^k_i = 1/k$ if $j' - 1 < i < j_k - 1$ and $a^k_i = a^{k-1}_i$ otherwise.

Finally, define a by setting it equal to a^k on $[j_{k-1}, j_k]$, $k \in \mathbb{N}$. $\|a\| \leq N$, otherwise there is a finite sum $\sum_{i=0}^n a_i^2$ greater than N^2, but then $\|a^k\| > N$ if $k > n$. Thus $d(a^0, a) \leq 2N$, and so $\sum_{i=n}^\infty (a^0_i - a_i)^2 \to 0$ as $n \to \infty$. So $d(a^k, a) \to 0$ as $k \to \infty$. Therefore a is a limit point of A. Also, $d(a, \hat{a}^k) \leq d(a, a^k) + d(a^k, \hat{a}^k)$, where $\hat{a}^k$ is equal to a^k with $a^k_{j_k-1}$ increased to $1/k$ (so $\hat{a}^k \in \mathfrak{E} \setminus A$). By construction $d(a^k, \hat{a}^k) < 1/k$, so it follows that $d(a, \hat{a}^k) \to 0$ as $k \to \infty$. Therefore a is also a limit point of $\mathfrak{E} \setminus A$. □

Proposition 2.11. *X_2 is connectible.*

Proof. Let A be a clopen subset of $X_2 \cup (C \times \{0\})$. Suppose that $\langle a, f\|a\|\rangle \in A$ and $\langle a, 0\rangle \in \xi[\mathfrak{E}] \setminus A$ for some $a \in C$. There is a clopen $U \subseteq C$ such that $\langle a, 0\rangle \in \xi[\mathfrak{E}] \cap (U \times \{0\}) \subseteq \xi[\mathfrak{E}] \setminus A$, so that $\{\langle x, f\|x\|\rangle : x \in U \text{ and } \|x\| \text{ is even}\} \subseteq \xi[\mathfrak{E}] \setminus A$. Let n be an even integer greater than $\|a\|$.

$$\xi^{-1}\big[A \cap U \times [0,1]\big] \cap \{x \in \mathfrak{E} : \|x\| < n\} = \xi^{-1}\big[A \cap U \times [0,1]\big] \cap \{x \in \mathfrak{E} : \|x\| \leq n\}$$

is a non-empty (it contains a) clopen subset of $\mathfrak{E}$. Its set of norms is bounded above (by n). This contradicts Proposition 2.10. □

2.2.3 Necessary conditions for completeness

Proposition 2.12. *$W[X]$ is completely metrizable if and only if the same is true of X.*

Proof. Suppose that X is completely metrizable. Then each $\varphi_i[X]$ is a G_δ-set in K. Further, the sets $\varphi_i[X]$ lie in pairwise disjoint open regions of K, and so their union is G_δ in K. Thus $W[X]$ is the union of two G_δ's, Δ and $\bigcup_{i<\omega} \varphi_i[X]$, which is again a G_δ (in K). The converse is true because $W[X]$ contains an open copy of X. $\square$

Remark 2.13. X_1 is not complete because it contains closed copies of the rationals, e.g. in the fibers above C'. So $W[X_1]$ is not complete. In fact, every G_δ-superset of X_1 contains an arc – Exercise 1.4.C(c) in [10].

Theorem 2.14. *If G is a dense G_δ in $C \times \mathbb{R}$, then*

$$D := \{c \in C : G \cap \{c\} \times \mathbb{R} \textit{ is a dense } G_\delta \textit{ in } \{c\} \times \mathbb{R}\}$$

is a dense G_δ in C.

Proof. Let $\{G_i : i < \omega\}$ be a collection of open sets in $C \times \mathbb{R}$ such that $G = \bigcap_{i<\omega} G_i$, and let $\{V_j : j \in \omega\}$ be a countable basis for $\mathbb{R}$ consisting of non-empty sets. For each i and j, define $F(i, j) = \{c \in C : G_i \cap (\{c\} \times V_j) = \varnothing\}$.

$D = C \setminus \bigcup_{i,j\in\omega} F(i, j)$: Suppose that $d \in D$ and $i, j \in \omega$. Then $G \cap \{d\} \times V_j \neq \varnothing$ by density of $G \cap \{d\} \times \mathbb{R}$ in $\{d\} \times \mathbb{R}$. As $G \subseteq G_i$, we have $G_i \cap \{d\} \times V_j \neq \varnothing$, so that $d \notin F(i, j)$. Thus $D \subseteq C \setminus \bigcup_{i,j\in\omega} F(i, j)$. Now let $c \in C \setminus \bigcup_{i,j\in\omega} F(i, j)$. Fix $i < \omega$. As $G_i \cap \{c\} \times V_j \neq \varnothing$ for each $j \in \omega$, we have that $G_i \cap \{c\} \times \mathbb{R}$ is dense in $\{c\} \times \mathbb{R}$. Thus $G \cap \{c\} \times \mathbb{R} = \bigcap_{i<\omega} G_i \cap \{c\} \times \mathbb{R}$ is a countable intersection of dense open subsets of $\{c\} \times \mathbb{R}$. By the Baire property of $\mathbb{R}$, $G \cap \{c\} \times \mathbb{R}$ is a dense G_δ in $\{c\} \times \mathbb{R}$, whence $c \in D$.

Each $F(i, j)$ is closed and nowhere dense in C: Fix $i, j \in \omega$ and let $F = F(i, j)$. F is closed in C: Let $c \in C \setminus F$. There exists $r \in \mathbb{R}$ such that $\langle c, r\rangle \in G_i \cap \{c\} \times V_j$. There is an open set $U \times V \subseteq G_i$ with $c \in U$ and $r \in V \subseteq V_j$. Then $F \cap U = \varnothing$. So $C \setminus F$ is open. F

is nowhere dense in C: If $U \subseteq C$ is non-empty and open, then by density of G there exists $\langle c, r\rangle \in G \cap U \times V_j$. Then c witnesses that $U \nsubseteq F$.

It now follows that from the Baire property of C that D is a dense G_δ in C. □

Remark 2.15. By Theorem 2.14, every dense G_δ in $C \times [0,1]$ has an uncountable intersection with some arc $\{c\} \times [0,1]$. $W[X_2]$ is not complete because X_2 has at most one point from each arc. Moreover, we find that every dense G_δ in K has an uncountable intersection with some composant. The original widely-connected space by Swingle [31] has only one point from each composant, while Miller's biconnected set [19], also widely-connected, has only countably many points in any given composant. Both spaces are of course dense in K, and so they fail to be complete.

Theorem 2.16. *If W is a completely metrizable widely-connected subset of K, then there is a closed $F \subseteq K$ such that $W \cap F = \varnothing$, $K \setminus F$ is connected, and $F \cap P \neq \varnothing$ for each composant P of K.*

Proof. Since W is completely metrizable, there is a collection $\{F_n : n < \omega\}$ of closed subsets of K such that $W = K \setminus \bigcup_{n<\omega} F_n$. Let φ_0 and K_0 be as defined in Section 2.2.1. By the widely-connected property of W, for each $c \in C$ there exists $n < \omega$ such that $F_n \cap \varphi_0[\{c\} \times (0,1)] \neq \varnothing$. Thus $C = \bigcup_{n<\omega} \pi[\varphi_0^{-1} F_n]$ where π is the first coordinate projection in $C \times (0,1)$. By the Baire Category Theorem there exists $N \in \omega$ and an open $V \subseteq C$ such that $V \subseteq \pi[\varphi_0^{-1} F_N]$.

$F := F_N$ is as desired: By the remark following Proposition 2.5, W is dense in K. Thus W is a dense connected subset of $K \setminus F$, so $K \setminus F$ is connected. Let P be a composant of K. P is dense in K, so there is a point $\varphi_0(\langle c, r\rangle) \in P \cap K_0$. Then $\varphi_0[\{c\} \times (0,1)] \subseteq P$. By design $F \cap \varphi_0[\{c\} \times (0,1)] \neq \varnothing$, so that $F \cap P \neq \varnothing$. □

2.2.4 Solution to the Erdős-Cook problem

The main result of [7] is that K has a closed subset F with the properties stated in Theorem 2.16. Considered as a subset of $C \times [0,1]$, F is the closure of the graph of a certain hereditarily discontinuous function. Specifically, let $D = \{d_n : n < \omega\}$ be a dense subset of the non-endpoints in C, let (a_n) be a sequence of positive real numbers such that $\sum_{n=0}^{\infty} a_n = 1$, and define $f : C \to [0,1]$ by

$$f(c) \;=\; \sum_{\{n<\omega : d_n < c\}} a_n.$$

Then $F = \overline{\mathrm{Gr}(f)}$.

Example $\mathbf{X_3}$. Enumerate the rationals $\mathbb{Q} = \{q_n : n < \omega\}$, and let

$$Y = C \times \mathbb{R} \setminus \bigcup_{n<\omega} \overline{\mathrm{Gr}(f + q_n)}.$$

Define a homeomorphism $\Xi : C \times \mathbb{R} \to C \times (0,1)$ by

$$\Xi(\langle c, r\rangle) = \left\langle c, \frac{\arctan(r)}{\pi} + \frac{1}{2}\right\rangle.$$

Finally,

$$X_3 = \Xi[Y].$$

Proposition 2.17. *If U is open in C and $V = (a,b)$ is an open interval in $\mathbb{R}$, then for every $c \in U$, $(\{c\} \times V) \setminus \overline{\mathrm{Gr}(f)}$ is a quasi-component of $(U \times V) \setminus \overline{\mathrm{Gr}(f)}$.*

Proof. We first show that if $d_n \in U \cap D$ $(n < \omega)$, then $(\{d_n\} \times V) \setminus \overline{\mathrm{Gr}(f)}$ is contained in a quasi-component of $(U \times V) \setminus \overline{\mathrm{Gr}(f)}$.

Fix $n < \omega$ such that $d_n \in U$. Let $r = f(d_n)$ and $s = f(d_n) + a_n (= \inf\{f(c) : d_n < c\})$. Since f is a non-decreasing function with set of (jump) discontinuities D, $F := \overline{\mathrm{Gr}(f)}$ meets a vertical interval $\{c\} \times [0,1]$ in exactly one point $\langle c, f(c)\rangle$ if $c \in C \setminus D$, and meets $\{d_n\} \times [0,1]$ in exactly two points $\langle d_n, r\rangle$ and $\langle d_n, s\rangle$. Thus $(\{d_n\} \times V) \setminus F$ is equal to $\{d_n\} \times (a,b)$, $\{d_n\} \times (a,r) \cup (r,b)$, $\{d_n\} \times (a,s) \cup (s,b)$, or $\{d_n\} \times (a,r) \cup (r,s) \cup (s,b)$, depending on whether neither, one, or both of r and s are in V.

Because d_n is a non-endpoint of C, it is the limit of increasing and decreasing sequences of points in $C \backslash D$. Suppose that $r \in V$. Then the intervals $\{c\} \times (a, \min\{b, f(c)\})$, $d_n < c \in U$, are well-defined subsets of $(U \times V) \setminus F$. They limit to points vertically above and below $\langle d_n, r\rangle$, so that $\{d_n\} \times (a, r) \cup (r, \min\{b, s\})$ is contained in a quasi-component of $(U \times V) \setminus F$. Similarly, if $s \in V$ then in $(U \times V) \setminus F$ the intervals $\{c\} \times (\max\{a, f(c)\}, b)$, $d_n > c \in U \setminus D$, bridge the gap in $\{d_n\} \times (\max\{a, r\}, s) \cup (s, b)$ (see Figure 2.5). Thus $(\{d_n\} \times V) \setminus F$ is contained in a quasi-component of $(U \times V) \setminus F$.

Since D is dense in U and $(\{d_n\} \times V) \setminus F$ is dense in $\{d_n\} \times V$ for each $n < \omega$, $(\{c\} \times V) \setminus F$ is contained in a quasi-component of $(U \times V) \setminus F$ for each $c \in U$. The opposite inclusion holds because C has a basis of clopen sets. □

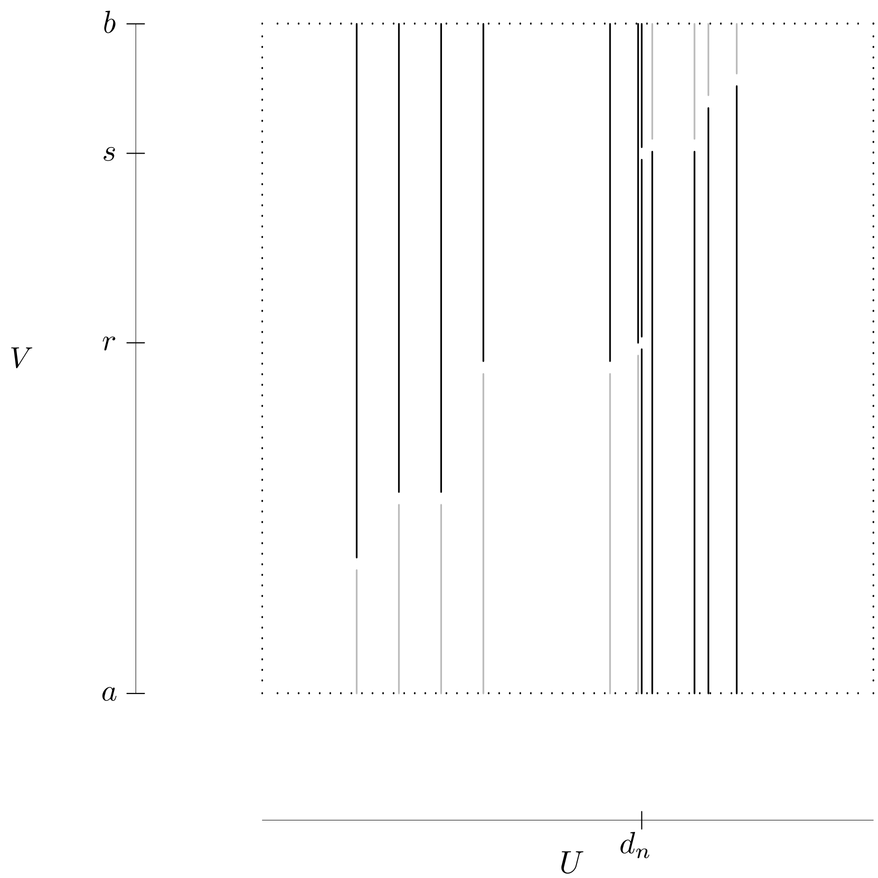

Figure 2.5: $(U \times V) \setminus \overline{\mathrm{Gr}(f)}$ when $d_n \in U$ and $r, s \in V$

Proposition 2.18. *For each $c \in C$, $(\{c\} \times \mathbb{R}) \cap Y$ is a quasi-component of Y.*

Proof. We need to show each fiber $(\{c\} \times \mathbb{R}) \cap Y$ is contained in a quasi-component of Y. Suppose not. Then there exists $c \in C$ and a partition of Y into disjoint clopen sets A and B such that $A \cap (\{c\} \times \mathbb{R}) \neq \varnothing$ and $B \cap (\{c\} \times \mathbb{R}) \neq \varnothing$. Density of Y in $C \times \mathbb{R}$ implies that $\overline{A} \cup \overline{B} = C \times \mathbb{R}$. In particular, $\{c\} \times \mathbb{R} \subseteq \overline{A} \cup \overline{B}$, so $\overline{A} \cap \overline{B} \neq \varnothing$ by connectedness of $\{c\} \times \mathbb{R}$. Now apply the Baire Category Theorem in $\overline{A} \cap \overline{B}$. As $\overline{A} \cap \overline{B} \subseteq \bigcup_{n<\omega} \overline{\mathrm{Gr}(f + q_n)}$, there exists $N \in \omega$, an open $U \subseteq C$, and an open interval $V \subseteq \mathbb{R}$ such that

$$\varnothing \neq \overline{A} \cap \overline{B} \cap (U \times V) \subseteq F_N := \overline{\mathrm{Gr}(f + q_N)}.$$

Note that $\overline{A} \cap (U \times V) \setminus F_N = ((C \times \mathbb{R}) \setminus \overline{B}) \cap (U \times V) \setminus F_N$; equality also holds if the roles of A and B are reversed. So $\overline{A} \cap (U \times V) \setminus F_N$ and $\overline{B} \cap (U \times V) \setminus F_N$ are open. They are also disjoint, and their union is equal to $(U \times V) \setminus F_N$. By Proposition 2.17, $(\{c\} \times V) \setminus F_N$ is contained in either $\overline{A}$ or $\overline{B}$ whenever $c \in U$. So

$$U_1 := \pi\big[\overline{A} \cap (U \times V) \setminus F_N\big] \text{ and } U_2 := \pi\big[\overline{B} \cap (U \times V) \setminus F_N\big]$$

are disjoint open subsets of C, π being the first coordinate projection (π is an open mapping). Further, $U_1 \cup U_2 = U$ because $\big|F_N \cap (\{c\} \times V)\big| \leq 2$ for each $c \in C$. Hence $U_1 \times V$ and $U_2 \times V$ form a clopen partition of $U \times V$. As $A \cap (U \times V) \subseteq U_1 \times V$ and $B \cap (U \times V) \subseteq U_2 \times V$, we have $\overline{A} \cap \overline{B} \cap (U \times V) = \varnothing$, a contradiction. □

Proposition 2.19. *X_3 is connectible.*

Proof. Suppose that A is a clopen subset of $X_3 \cup (C \times \{0\})$, $c \in C$, and $A \cap (\{c\} \times (0,1)) \neq \varnothing$. The homeomorphism Ξ preserves the form of quasi-components. Therefore, Proposition 2.18 implies that $X_3 \cap (\{c\} \times (0,1)) \subseteq A$. Since X_3 is dense in $\{c\} \times [0,1)$ and A is closed, we have $\langle c, 0\rangle \in A$. □

Theorem 2.20. $W[X_3]$ *is widely-connected and completely metrizable.*

Proof. Each fiber $(\{c\} \times \mathbb{R}) \cap Y$ is a real line minus one or two shifted copies of $\mathbb{Q}$, and is therefore hereditarily disconnected and dense in $\{c\}\times\mathbb{R}$. It follows that X_3 is hereditarily disconnected and dense in $C \times (0,1)$. By Propositions 2.5 and 2.19, $W[X_3]$ is widely-connected. X_3 is a G_δ-subset of $C \times (0,1)$ by design, so it is completely metrizable. By Proposition 2.12, $W[X_3]$ is completely metrizable. $\square$

E.W. Miller [19] and M.E. Rudin [26] showed that, consistently, a widely-connected subset of the plane can be biconnected. This is not the case with $W[X_3]$.

Proposition 2.21. $W[X_3]$ *is not biconnected.*

Proof. We first show that if S is a countable subset of $W := W[X_3]$, then $W \setminus S$ is connected. To that end, let $S = \{x_n : n < \omega\}$ be a countable subset of W. Note that Proposition 2.17 holds if $\overline{\mathrm{Gr}(f)}$ is replaced by $\overline{\mathrm{Gr}(f)} \cup \{x\}$ for any $x \in C \times \mathbb{R}$. By the proofs of Propositions 2.18 and 2.19, it follows that for each $i < \omega$,

$$K_i \setminus \Big(\bigcup_{n<\omega} \big(\varphi_i \circ \Xi\big[\overline{\mathrm{Gr}(f + q_n)}\big]\big) \cup \{x_n\}\Big)$$

is connectible (considered as a subset of $C \times (0,1)$). By Proposition 2.5, $W' := \Delta \cup (W \setminus S)$ is connected.

Let $W'' = W \setminus S$. We want to show that W'' is connected. Suppose to the contrary that $\{A, B\}$ is a non-trivial clopen partition of W''. Then $\mathrm{cl}_{W'} A \cap \mathrm{cl}_{W'} B$ is a non-empty subset of Δ. Let $p \in \mathrm{cl}_{W'} A \cap \mathrm{cl}_{W'} B$. There exists $j \in \omega$ such that $p \in \overline{K_j}$. There is a K-neighborhood of p that identifies with $C \times [-1, 1]$ in such a way that $C \times [-1, 0] \subseteq \overline{K_0}$, $C \times [0, 1] \subseteq \overline{K_j}$, and $p \in C \times \{0\}$. We think of $C \times [-1, 1]$ as being an actual subset of K, and give local coordinates $\langle d, 0\rangle$ to p.

Note that $W'' \cap \{c\} \times [-1, 1]$ is dense in $\{c\} \times [-1, 1]$ for each $c \in C$. Since $W'' \cap K_0$ and $W'' \cap K_j$ are connectible, for each point $\langle c, 0\rangle \in A$ we have $W'' \cap (\{c\} \times [-1, 1]) \subseteq A$, and similarly for B. If p is the limit of sequences of points in A and B that are in $C \times \{0\}$, then points in $W'' \cap (\{d\} \times (0,1])$ would be limit points of fibers in A and B. This cannot happen. Therefore there is an open $U \subseteq C$ such that $p \in W'' \cap (U \times \{0\}) \subseteq A$, without loss of generality. Since $p \in \overline{B}$, there exists $b \in B$ such that $b(0)$, the first local coordinate of b, is in U. There is an open set $T \times V$ with $b \in W'' \cap (T \times V) \subseteq B$. Since $W'' \cap \Delta$ is dense in Δ, there exists $a \in A \cap [(U \cap T) \times \{0\}]$. We have a contradiction: $W' \cap \{a(0)\} \times [-1,1]$ meets both A and B. We have shown that W'' is connected.

Thus every separating subset of W is uncountable. Let $\mathscr{S} = \mathscr{S}(W)$ be the set of closed separators of W, defined in Lemma 2.22. If Z is a Bernstein set in W, then $Z \cap S \neq \varnothing$ and $(W \setminus Z) \cap S \neq \varnothing$ for each $S \in \mathscr{S}$, implying that both Z and $W \setminus Z$ are connected. A set like Z can be constructed as follows, using the fact that W is separable and complete. Let $\{S_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathscr{S}$. Every member of $\mathscr{S}$ is closed and uncountable, and thus has cardinality $\mathfrak{c}$. Let y_0 and z_0 be distinct points in S_0. If $\alpha < \mathfrak{c}$ and y_β and z_β have been defined for $\beta < \alpha$, then there are two distinct points $y_\alpha, z_\alpha \in S_\alpha \setminus (\{y_\beta : \beta < \alpha\} \cup \{z_\beta : \beta < \alpha\})$. Then $Z = \{z_\alpha : \alpha < \mathfrak{c}\}$ is as desired. W is the union of two non-degenerate disjoint connected sets Z and $W \setminus Z$, so W is not biconnected. $\square$

Completely metrizable biconnected subsets of the plane are given in [15] and [25]. The quotient $X_3/(C \times \{\frac{1}{2}\})$ is another such example. These spaces are biconnected because they have dispersion points. The examples of E.W. Miller [19] and M.E. Rudin [26] are biconnected subsets of the plane without dispersion points. As an analogue to the question of Erdős and Cook, we would like to know:

Question 1. Is there a completely metrizable biconnected space without a dispersion point?

An example would not necessarily have to be widely-connected – see M.E. Rudin [27]. By the proof of Proposition 2.21, a separable example would have to contain a countable closed separator. Miller's biconnected set has many countable closed separators, but, as previously shown, is not complete. M.E. Rudin's examples [26, 27] fail to be complete for essentially the same reason.

2.3 Large cardinality

In [4], David Bellamy asked whether there are widely-connected sets of arbitrarily large cardinality. The answer is yes – Theorem 2.25.

Lemma 2.22. *Let X be an indecomposable continuum. Let $\mathscr{C}(X)$ be the set of composants of X, and let*

$$\mathscr{S}(X) = \big\{S \subseteq X : S \textit{ is closed and } X \setminus S \textit{ is not connected}\big\}.$$

If $|\mathscr{S}(X)| \leq |\mathscr{C}(X)|$ then X has a dense widely-connected subspace of size $|\mathscr{C}(X)|$.

Proof. Write $\mathscr{S}$ for $\mathscr{S}(X)$ and $\mathscr{C}$ for $\mathscr{C}(X)$. Since each member of $\mathscr{C}$ is connected and dense in X (Theorems 1.1 and 1.6), we have $C \cap S \neq \varnothing$ for each $C \in \mathscr{C}$ and $S \in \mathscr{S}$. Let $\Psi : \mathscr{C} \to \mathscr{S}$ be a surjection, and for each $C \in \mathscr{C}$ let $\psi(C) \in C \cap \Psi(C)$. Let $W = \{\psi(C) : C \in \mathscr{C}\}$. Note that $|W| = |\mathscr{C}|$ since the members of $\mathscr{C}$ are disjoint (Theorem 1.8).

W is dense in X: Let U be a non-empty open subset of X. Let $V \subseteq X$ be non-empty and open such that $X \neq \overline{V} \subseteq U$. Then $\partial V \in \mathscr{S}$. There exists $C \in \mathscr{C}$ such that $\Psi(C) = \partial V$. Then $\psi(C) \in W \cap \overline{V} \subseteq W \cap U$.

W is connected: Supposing that W is not connected, there are non-empty open subsets U and V of X such that $U \cap W \neq \varnothing$, $V \cap W \neq \varnothing$, and $W = (U \cap W) \cup (V \cap W)$. By density of W we have $U \cap V = \varnothing$. So $X \setminus (U \cup V) \in \mathscr{S}$. By definition $W \cap [X \setminus (U \cup V)] \neq \varnothing$, contrary to $W \subseteq U \cup V$.

W is *widely*-connected: If A is a non-dense connected subset of W, then $\overline{A}$ is a proper subcontinuum of X and is therefore contained in a composant $C \in \mathscr{C}(X)$. By the definition of W and the fact that the composants of X are disjoint, $|W \cap C| = 1$, so A is degenerate. □

Remark 2.23. The assumption $|\mathscr{S}(X)| \leq |\mathscr{C}(X)|$ is actually equivalent to $|\mathscr{S}(X)| = |\mathscr{C}(X)|$, and so Ψ can be taken to be a bijection. Indeed, let $\lambda = |\mathscr{C}(X)|$ and let U and V be non-empty open subsets of X with $\overline{U} \cap \overline{V} = \varnothing$. $|X| \geq \lambda$ implies that $|X \setminus \partial U| \geq \lambda$ or $|X \setminus \partial V| \geq \lambda$. Assuming that $|X \setminus \partial U| \geq \lambda$, let $\mathscr{S}'(X) = \{\partial U \cup \{x\} : x \in X \setminus \partial U\}$. Then $\mathscr{S}'(X) \subseteq \mathscr{S}(X)$ implies $|\mathscr{S}(X)| \geq \lambda$.

Lemma 2.24. *Let κ be an infinite cardinal. If $\{X_\alpha : \alpha < \kappa\}$ is a collection of spaces each with a basis of size $\leq \kappa$, then $\prod_{\alpha<\kappa} X_\alpha$ and*

$$X := \varprojlim \{X_\alpha : \alpha < \kappa\}$$

have bases of size $\leq \kappa$, and X has $\leq 2^\kappa$ closed subsets.

Proof. For each $\alpha < \kappa$ let $\mathscr{B}_\alpha$ be a basis for X_α with $|\mathscr{B}_\alpha| \leq \kappa$. For each $f \in \left[\bigcup_{\alpha<\kappa} \{\alpha\} \times \mathscr{B}_\alpha\right]^{<\omega}$ let $U_f = \{x \in \prod_{\alpha<\kappa} X_\alpha : x(f(i)(0)) \in f(i)(1) \text{ for each } i \in \operatorname{dom}(f)\}$. Then

$$\mathscr{B} = \left\{U_f : f \in \left[\bigcup_{\alpha<\kappa} \{\alpha\} \times \mathscr{B}_\alpha\right]^{<\omega}\right\}$$

is a basis for $\prod_{\alpha<\kappa} X_\alpha$ with $|\mathscr{B}| \leq \kappa^{<\omega} = \kappa$. The inverse limit X also has a basis of size $\leq \kappa$, namely $\{B \cap X : B \in \mathcal{B}\}$, because it is a subspace of the product.

Define $\varphi : \mathcal{P}(\mathscr{B}) \to \tau$ by $\varphi(\mathscr{U}) = X \cap \bigcup \mathscr{U}$, where τ is the topology of X. If $U \in \tau$ then letting $\mathscr{U} = \{B \in \mathscr{B} : B \cap X \subseteq U\}$ we have $\varphi(\mathscr{U}) = U$, so that φ is surjective. Therefore $\left|\{X \setminus U : U \in \tau\}\right| = |\tau| \leq \left|\mathcal{P}(\mathscr{B})\right| \leq 2^\kappa$. □

Theorem 2.25. *For each infinite cardinal κ there is a completely regular widely-connected space of size 2^κ.*

Proof. Let κ be given. By Theorem 2 of [30], there is an indecomposable continuum M which has 2^κ composants. The continuum M is constructed as an inverse limit of κ-many continua, each of which has a basis of size $\leq \kappa$. To be more specific, $M = \varprojlim \{M_\alpha : \alpha < \kappa\}$ where M_α is a subcontinuum of $[0,1]^{\alpha+1}$ for each $\alpha < \kappa$ (see the second and third paragraphs of the proof in [30] for the successor and limit cases of α, respectively). By Lemma 2.24, $[0,1]^{\alpha+1}$ has a basis of size $\leq \kappa$ for each $\alpha < \kappa$, so $|\mathscr{S}(M)| \leq 2^\kappa = |\mathscr{C}(M)|$. By Lemma 2.22, M contains a widely-connected space of size 2^κ. □

Mazurkiewicz [20] proved that every metric indecomposable continuum has $\mathfrak{c} = 2^\omega$ composants. Thus, if X is a metric indecomposable continuum then $\mathscr{C}(X) = \mathscr{S}(X)$, and so by Lemma 2.22 X has a dense widely-connected subset. This technique does not apply in general, as there are (non-metric) indecomposable continua with very few composants. The Stone-Čech remainder of $[0,\infty)$ is consistently an indecomposable continuum with only one composant [22]. There are also indecomposable continua with one and two composants assuming only the standard ZFC axioms [3].

Question 2. Does every indecomposable continuum have a dense widely-connected subset?

2.4 Indecomposability of compactifications

This section addresses the following multi-part question from the 2004 Spring Topology and Dynamics Conference report, due to Jerzy Mioduszewski. Part (a) of the question is also posed in [4] by David Bellamy, who conjectured a positive answer to part (b).

Question 3 (Mioduszewski [23])**.** Let W be a widely-connected space.

(a) Is βW an indecomposable continuum?

(b) If W is metrizable and separable, does W have a metric compactification which is an indecomposable continuum?

(c) If W is separable and metrizable does W have a metric compactifacation γW such that for every composant P of γW, $W \cap P$ is (i) hereditarily disconnected? (ii) finite? (iii) a singleton?

2.4.1 A property of quasi-components

By Theorem 1.7, the term *indecomposable* can be consistently defined for arbitrary topological spaces – say that X is *indecomposable* if every connected subset of X is either dense or nowhere dense in X. Thus, widely-connected spaces are indecomposable connected spaces whose nowhere dense connected subsets are trivial. Compactifying a widely-connected space means adding limit points to it. This process can easily destroy indecomposability.

Example 2.26. Let K, K_0, $W := W[X_3]$, and $A := \varphi_0 \circ \Xi[\overline{\mathrm{Gr}(f)}]$ be as defined in Section 2.2; A is a copy of Dębski's set in $K_0 \setminus W$. Consider $\widehat{W} := W \cup A$ to be on the surface of the unit sphere $S^2 \subseteq \mathbb{R}^3$. If d is a metric on $\mathbb{R}^3$, then the mapping $q : \widehat{W} \to \mathbb{R}^3$ defined by $x \mapsto d_A(x) \cdot x$ is a homeomorphic embedding of W (by compactness of A) with a limit point at the origin (q shrinks A to the point $\langle 0, 0, 0\rangle$). Note that A intersects each quasi-component of $(W \cup A) \cap K_0$. So $q[(W \cup A) \cap K_0]$ is a connected subset of $q[\widehat{W}]$ that is neither dense nor nowhere dense; $q[\widehat{W}]$ is not indecomposable. In summary, $q[W]$ is a widely-connected subset of $\mathbb{R}^3$ that fails to be indecomposable when a single limit point is added to it.

The disaster in Example 2.26 is caused by many quasi-components of $W \cap K_0$ limiting to a connected set. Generally speaking, the quasi-component structure is critical to the existence of indecomposable compactifications.

Property $\mathcal{Q}$. *X has Property $\mathcal{Q}$* means that for every two non-empty disjoint open sets U and V there are two closed sets A and B such that $X = A \cup B$, $A \cap U \neq \varnothing$, $B \cap U \neq \varnothing$, and $A \cap B \subseteq V$.

Remark 2.27. If X has no isolated points, then X has Property $\mathcal{Q}$ if and only if $\mathrm{int}_X Q = \varnothing$ whenever Q is a quasi-component of a non-dense subset of X. Every perfect totally disconnected space has Property $\mathcal{Q}$, and every space with Property $\mathcal{Q}$ is indecomposable.

Lemma 2.28. *If Y has Property $\mathcal{Q}$, $X \subseteq Y$, and $\overline{X} = Y$, then X has Property $\mathcal{Q}$.*

Proof. Let U and V be non-empty disjoint open subsets of X. There are open subsets U' and V' of Y such that $U = U' \cap X$ and $V = V' \cap X$. Let A' and B' be closed subsets of Y

such that $Y = A' \cup B'$, $U' \cap A' \neq \varnothing$, $U' \cap B' \neq \varnothing$, and $A' \cap B' \subseteq V'$. Let $A = A' \cap X$ and $B = B' \cap X$. Clearly A and B are closed in X, $X = A \cup B$, and $A \cap B \subseteq V$. Note that $U' \cap A' = U' \cap Y \setminus B'$ and $U' \cap B' = U' \cap Y \setminus A'$ are non-empty open subsets of Y. $\overline{X} = Y$ implies that $U \cap A \neq \varnothing$ and $U \cap B \neq \varnothing$. □

The following is implicit in the proof of Lemma 9.8 in [32].

Lemma 2.29. *If U and V are disjoint open subsets of X and W is open in βX such that $W \cap X = U \cup V$, then $W_0 := W \cap \operatorname{cl}_{\beta X} U$ and $W_1 := W \cap \operatorname{cl}_{\beta X} V$ are disjoint open sets in βX and $W_0 \cup W_1 = W$.*

Proof. By density of X in βX we have $W \subseteq \operatorname{cl}_{\beta X} U \cup \operatorname{cl}_{\beta X} V$. Thus $W_0 \cup W_1 = W$. We need to show that $W \cap \operatorname{cl}_{\beta X} U \cap \operatorname{cl}_{\beta X} V = \varnothing$. Suppose for a contradiction that there exists $p \in W \cap \operatorname{cl}_{\beta X} U \cap \operatorname{cl}_{\beta X} V$. By Urysohn's Lemma there is a mapping $F : \beta X \to [0, 1]$ such that $F(p) = 0$ and $F[\beta X \setminus W] = 1$. Define $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ F(x) & \text{if } x \notin U. \end{cases}$$

If T is open in $[0, 1]$ then

$$f^{-1}[T] = \begin{cases} \big(F^{-1}[T] \cap X\big) \cup U & \text{if } 1 \in T \\ F^{-1}[T] \cap V & \text{if } 1 \notin T, \end{cases}$$

so f is continuous. Since $p \in \operatorname{cl}_{\beta X} V$ and $f \restriction V = F \restriction V$, we have $\beta f(p) = F(p) = 0$. On the other hand, $p \in \operatorname{cl}_{\beta X} U$ implies that $\beta f(p) = 1$, a contradiction. □

Theorem 2.30. *βX is indecomposable if and only if X has Property $\mathcal{Q}$.*

Proof. Suppose that βX is indecomposable. To show that X has Property $\mathcal{Q}$, by By Lemma 2.28 it suffices to show that βX has Property $\mathcal{Q}$. Let U and V be non-empty disjoint open subsets of βX. Let T be non-empty and open in βX such that $\overline{T} \subseteq V$. Since U is not contained in a component of $\beta X \setminus T$, by Theorem 1.5 there are disjoint closed sets A and B such that $A \cup B = \beta X \setminus T$, $A \cap U \neq \varnothing$, and $B \cap U \neq \varnothing$. Then $A \cup \overline{T}$ and $B \cup \overline{T}$ satisfy Property $\mathcal{Q}$ for U and V.

Suppose that X has Property $\mathcal{Q}$. Let K be a proper closed subset of βX with non-empty interior; we show that K is not connected. Let S and T be non-empty open subsets of βX such that $S \subseteq K$ and $K \cap \mathrm{cl}_{\beta X} T = \varnothing$. Let A and B be disjoint closed subsets of X such that $A \cup B = X \setminus T$, $A \cap S \neq \varnothing$, and $B \cap S \neq \varnothing$. Note that $U := A \setminus \mathrm{cl}_{\beta X} T$ and $V := B \setminus \mathrm{cl}_{\beta X} T$ are open in X, $U \cap S \neq \varnothing$, and $V \cap S \neq \varnothing$. By Lemma 2.29, $W := \beta X \setminus \mathrm{cl}_{\beta X} T$ is the union of the two disjoint open sets $W_0 := W \cap \mathrm{cl}_{\beta X} U$ and $W_1 := W \cap \mathrm{cl}_{\beta X} V$. Since $K \subseteq W$ and $W_i \cap K \neq \varnothing$ for each $i < 2$, K is not connected. □

Corollary 2.31. *βX is an indecomposable continuum if and only if X is connected and has Property $\mathcal{Q}$.*

Proof. βX is a continuum if and only if X is connected. □

Corollary 2.32. *If X is compact Hausdorff, then X is indecomposable if and only if X has Property $\mathcal{Q}$.*

Proof. $\beta X \simeq X$ when X is compact Hausdorff. □

Widely-connected spaces are usually constructed as dense subsets of indecomposable continua. Gary Gruenhage [13] constructed completely regular and perfectly normal examples by a different method, assuming Martin's Axiom and the Continuum Hypothesis, respectively. Their co-infinite subsets are totally disconnected (it's worth noting that there is no connected *metric* space with this property), thus both examples have Property $\mathcal{Q}$, and so their Stone-Čech compactifications are indecomposable.

If W is widely-connected and βW is *decomposable*, then there is a perfect hereditarily disconnected $Y \subseteq W$ which fails to have Property $\mathcal{Q}$: Let U and V witness that W does not have Property $\mathcal{Q}$. Let T be a non-empty open set with $\overline{T} \subseteq V$. $\overline{T}$ is the union of two relatively clopen sets A and B. Then $Y := W \setminus A$ is as desired. No clopen partition of $Y \setminus B$ divides U.

Example 2.33. A perfect hereditarily disconnected set without Property $\mathcal{Q}$.

Let C, C', and $X_1 \subseteq C \times (0,1)$ be as defined in Section 2.2.2. Let $X_1' = \{\langle -y, x\rangle : \langle x, y\rangle \in X_1\}$ be the copy of X_1 rotated 90° about the origin. Then $Y := X_1' \cup X_1$ is hereditarily disconnected. We show that X_1' is contained in a quasi-component of $Y \setminus (C \cap (1/2, 1]) \times (0, 1]$. Suppose that A is clopen in $Y \setminus (C \cap (1/2, 1]) \times [0, 1]$ and $A \cap X_1' \neq \varnothing$. Since A is open, there exists $c \in C' \setminus 2$ such that $A \cap ((-1, 0) \times \{c\}) \neq \varnothing$. Since $c \in \mathbb{Q}$ and X_1' is connectible, $\langle 0, c\rangle \in A$. Then $X_1 \cap (\{0\} \times (0, 1)) \subseteq A$ because $X_1 \cap ([0, 1/2) \times (0, 1))$ is connectible. So $X_1' \cap ((-1, 0) \times C' \setminus 2) \subseteq A$. Since A is closed, we have $X_1' \subseteq A$.

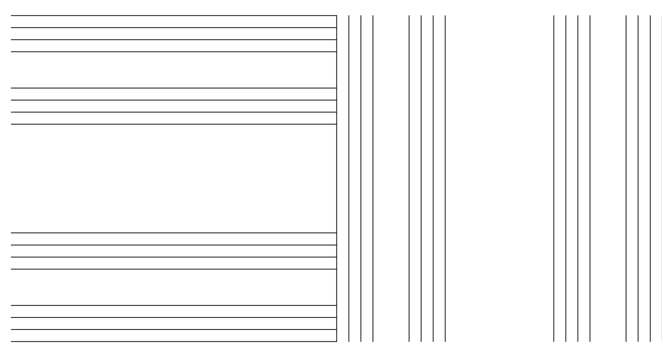

Figure 2.6: $Y \subseteq \big([-1, 0] \times C\big) \cup \big(C \times [0, 1]\big)$

2.4.2 The separable metric case

Throughout, assume that X is a separable metric space. The standard metric on the Hilbert cube $[0,1]^\omega$ is given by $\rho(y, y') = \sum_{n<\omega} \big|y(n) - y'(n)\big| \cdot 2^{-n}$. The space $([0,1]^\omega)^X$ is the set of functions from X into $[0,1]^\omega$, endowed with the complete metric $\varrho(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}$. The following results will be used to show that Questions 3(a) and 3(b) are equivalent if W is metrizable and separable.

Lemma 2.34 (§46 V Theorem 3 in [18])**.** *There is a continuous $\varphi : X \to 2^\omega$ such that the quasi-components of X coincide with the non-empty point inverses of φ.*

Lemma 2.35 (§44 V Corollary 4a and §44 VI Lemma in [18]). *If A and B are disjoint closed subsets of X, then*

$$\left\{g \in \left([0,1]^\omega\right)^X : \overline{g[A]} \cap \overline{g[B]} = \varnothing\right\}$$

is a dense open subset of $([0,1]^\omega)^X$.

Lemma 2.36 (§44 VI Theorem 2 in [18]). *The set of homeomorphic embeddings from X into $[0,1]^\omega$ is a dense G_δ-set in* $([0,1]^\omega)^X$.

Theorem 2.37. *βX is indecomposable if and only if X has an indecomposable metric compactification.*

Proof. Suppose that βX is indecomposable. Let $\{U_n : n < \omega\}$ be a countable basis for X. For each $n < \omega$ let $\varphi_n : X \setminus U_n \to 2^\omega$ be given according to Lemma 2.34. Let $\{C_i : i < \omega\}$ be the canonical clopen basis for 2^ω. For each $\langle n, i\rangle \in \omega^2$, let $A_{\langle n,i\rangle} = \varphi_n^{-1}[C_i]$ and $B_{\langle n,i\rangle} = \varphi_n^{-1}[2^\omega \setminus C_i]$. For each $\langle n, i\rangle \in \omega^2$,

$$\left\{g \in \left([0,1]^\omega\right)^X : \overline{g[A_{\langle n,i\rangle}]} \cap \overline{g[B_{\langle n,i\rangle}]} = \varnothing\right\}$$

is a dense open subset of $([0,1]^\omega)^X$ by Lemma 2.35. By Lemma 2.36 and the fact that $([0,1]^\omega)^X$ is complete, there is a homeomorphic embedding $e : X \hookrightarrow [0,1]^\omega$ such that $\overline{e[A_{\langle n,i\rangle}]} \cap \overline{e[B_{\langle n,i\rangle}]} = \varnothing$ for each $\langle n, i\rangle \in \omega^2$.

We now show that the metric compactification $\overline{e[X]}$ is indecomposable. Let K be a proper closed subset of $\overline{e[X]}$ with non-empty interior. There exists $n < \omega$ such that $K \cap \overline{e[U_n]} = \varnothing$. Since K has non-empty interior, by Property $\mathcal{Q}$ (in X) there exists $i < \omega$ such that $K \cap e[A_{\langle n,i\rangle}] \neq \varnothing$ and $K \cap e[B_{\langle n,i\rangle}] \neq \varnothing$. As $\overline{e[X]} = \overline{e[U_n]} \cup \overline{e[A_{\langle n,i\rangle}]} \cup \overline{e[B_{\langle n,i\rangle}]}$, we have $K \subseteq \overline{e[A_{\langle n,i\rangle}]} \cup \overline{e[B_{\langle n,i\rangle}]}$. Finally, $\overline{e[A_{\langle n,i\rangle}]} \cap \overline{e[B_{\langle n,i\rangle}]} = \varnothing$, so K is not connected.

Now suppose that γX is an indecomposable compactification of X. By Corollary 2.32, γX has property $\mathcal{Q}$. X also has Property $\mathcal{Q}$ by Lemma 2.28, thus βX is indecomposable by Theorem 2.30. ☐

Remark 2.38. The main result of [5] is that every nowhere locally compact separable metric space has a dense embedding into the Hilbert space $\ell^2 \simeq (0,1)^\omega$. Widely-connected Hausdorff spaces have no compact neighborhoods by Theorem 1.5. Thus, every widely-connected separable metric space has a decomposable metric compactification equal to the Hilbert cube $[0,1]^\omega$. The Hilbert cube has only one composant – by Corollary 2.39 this is not a coincidence.

If X is a connected space and $p, q \in X$, then X is *irreducible between p and q* if no proper closed connected subset of X contains both p and q. A continuum is said to be *irreducible* if it is irreducible between some pair of its points, i.e., if it has more than one composant.

Corollary 2.39. *If X is an indecomposable connected space, then βX is indecomposable if and only if X has an irreducible compactification (which is necessarily indecomposable).*

Proof. If βX is indecomposable, then by Theorem 2.37 X has a metric indecomposable compactification which necessarily has $\mathfrak{c}$ composants.

Now suppose that βX is decomposable. Let γX be a compactification of X. If $\iota : X \to X$ is the identity map, then the Stone-Čech extension $\beta\iota : \beta X \to \gamma X$ maps proper subcontinua to proper subcontinua and maps onto γX. Thus to prove γX has only one composant, it suffices to show that βX has only one composant. To that end, let $p, q \in \beta X$. Let H and K be proper subcontinua of βX such that $\beta X = H \cup K$. Assume that $p \in K$. If $q \in K$, then p and q are in the same composant. Otherwise, $q \in U := \beta X \setminus K$. Since X is indecomposable and $\overline{U \cap X} \neq X$, $U \cap X$ is not connected, whence U is not connected by Lemma 2.29. Let T be a proper clopen subset of U with $q \in T$. Then $K \cup T$ is a proper subcontinuum of βX containing p and q. Again p and q are in the same composant of βX.

Suppose that γX is irreducible (between two points p and q) and decomposable. There is a proper subcontinuum $K \subseteq \gamma X$ with non-empty interior such that $p \in K$. Then $\beta\iota^{-1}[K]$ is a proper closed subset of βX with non-empty interior. Thus, if $q' \in \beta\iota^{-1}\{q\}$ then there is a proper subcontinuum $L \subseteq \beta X$ such that $q' \in L$ and $L \cap \beta\iota^{-1}[K] \neq \varnothing$ (composants are dense). Then $\beta\iota[L] \cup K$ is a subcontinuum of γX containing p and q, so $\beta\iota[L] \cup K = \gamma X$. It follows that $\beta X \setminus \beta\iota^{-1}[K] \subseteq L$, so βX is decomposable. $\square$

Observe that a widely-connected space is a connected space that is irreducible between every two of its points. By Theorem 2.37 and Corollary 2.39, Question 3(b) is equivalent to:

Question 3(b)$'$**.** If W is a connected separable metric space that is irreducible between every two of its points, then does W have an irreducible compactification?

2.4.3 A composant-locked widely-connected set

In this section we show that W may not have a compactification that is irreducible between two points in W. This provides a complete negative answer to Question 3(c) (see the statement of Theorem 2.44). By Corollary 2.39, a positive answer to any part of Question 3(c) would have implied a proof of Bellamy's conjecture (a positive answer to Question 3(b)). Our counterexample demonstrates that a different line of attack is needed.

Let $e \in 2^{\mathbb{Z}}$ be the point defined by concatenating all finite binary sequences in the positive and negative directions of $\mathbb{Z}$. That is, if $\{b_i : i < \omega\}$ enumerates $\bigcup_{n<\omega} 2^n$, then let $e \restriction \mathbb{Z}^+ = b_0{}^\frown b_1{}^\frown b_2{}^\frown ...$, and define $e(n) = e(-n)$ for $n \in \mathbb{Z}^-$. Let $\eta : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the shift map $\eta(x)(n) = x(n + 1)$; clearly η is a homeomorphism. The backward and forward orbits of e, $E_0 = \{\eta^n(e) : n \in \mathbb{Z}^-\}$ and $E_1 = \{\eta^n(e) : n \in \mathbb{Z}^+ \setminus \{0\}\}$, are dense in $2^{\mathbb{Z}}$ by construction. Finally, $E_0 \cap E_1 = \varnothing$. For if $n, m \in \omega$ such that $\eta^{-n}(e) = \eta^m(e)$, then $\eta^{n+m}(e) = e$. Density of E_0 implies that e is not a periodic point. So $n + m = 0$, whence $n = m = 0$. We have $\eta^{-n}(e) = \eta^m(e) = e \in E_0 \setminus E_1$.

Let $K \subseteq [0, 1]^2$ and $\Delta \subseteq K$ be as defined in Section 2.2.1. $K \setminus \Delta$ is the union of two triangular open sets $K_0 = \{\langle x, y\rangle \in K : x < y\}$ and $K_1 = \{\langle x, y\rangle \in K : x > y\}$. Let X be any widely-connected subset of K, and for each $i < 2$ put $W_i = (X \cap K_i) \cup \Delta$. Note that each W_i is hereditarily disconnected, but $W_0 \cup W_1 = X \cup \Delta$ is connected since X is connected and (necessarily) dense in K. Define $W \subseteq 2^{\mathbb{Z}} \times K$ by

$$W = \bigcup_{i<2} (E_i \times W_i);$$

W is hereditarily disconnected, but it will become widely-connected when certain pairs of its points are identified. Let $p, q \in \Delta$ be such that K is irreducible between p and q. Those familiar with the Cantor set and the composants of K may choose $p = \langle 1, 1 \rangle$ and $q = \langle \frac{1}{4}, \frac{1}{4} \rangle$, for instance. Define a relation on $2^{\mathbb{Z}} \times K$ by $\sim = \big\{ \big\langle \langle x, p \rangle, \langle \eta(x), q \rangle \big\rangle : x \in 2^{\mathbb{Z}} \big\}$, and finally, put $\widetilde{W} = W/\sim$.

Remark 2.40. Recall that the quotient of a compact metric space is metrizable whenever Hausdorff. The entire quotient $(2^{\mathbb{Z}} \times K)/\sim$ is easily seen to be Hausdorff, and is therefore metrizable (and compact). Together with $\dim((2^{\mathbb{Z}} \times K)/\sim) = 1$, this implies via the Menger-Nöbeling Theorem (1.11.4 in [10]) that $(2^{\mathbb{Z}} \times K)/\sim$ embeds into Euclidean 3-space $\mathbb{R}^3$, and so of course $\widetilde{W}$ also embeds into $\mathbb{R}^3$.

Lemma 2.41. *Let X be a space. If $U \subseteq X$ is open, A_0 is clopen in $\overline{U}$, A_1 is clopen in $X \setminus U$, and $A_0 \cap \partial U = A_1 \cap \partial U$, then $A_0 \cup A_1$ is clopen in X.*

Proof. Clearly $A_0 \cup A_1$ is closed in X and $(A_0 \cup A_1) \setminus \partial U$ is open in X. We show that $A_0 \cup A_1$ is open in X by showing that it is an X-neighborhood of each of its points in ∂U. Suppose that $a \in (A_0 \cup A_1) \cap \partial U$. Then $a \in A_0 \cap A_1$. There are two open subsets of X, V_0 and V_1, such that $a \in V_0 \cap \overline{U} \subseteq A_0$ and $a \in V_1 \cap (X \setminus U) \subseteq A_1$. Then $a \in V_0 \cap V_1 \subseteq A_0 \cup A_1$; to prove the inclusion, suppose $x \in V_0 \cap V_1$ and consider the two cases $x \in \overline{U}$ and $x \in X \setminus U$. □

Lemma 2.42. *For each $i < 2$, $\{0\} \times W_i$ is a quasi-component of*

$$Y := (\{0\} \times W_i) \cup \bigcup_{n \geq 1} (\{1/n\} \times W_{1-i}).$$

Proof. Fix $i < 2$. Clearly $\{0\} \times W_i$ contains a quasi-component of Y. Now let A be a clopen subset of Y such that $A \cap \{0\} \times W_i \neq \varnothing$; we show that $\{0\} \times W_i \subseteq A$. For each $a \in A \cap \Delta$ there is an integer $n(a)$ and an open $U(a) \subseteq \Delta$ such that $a \in [0, 1/n(a)] \times U(a) \subseteq A$ (by $[0, 1/n]$ we mean $\{0\} \cup \{1/k : k \geq n\}$). By compactness of $A \cap \Delta$, there is a finite $F \subseteq A$ such that $A \cap \Delta \subseteq \bigcup\{[0, 1/n(a)] \times U(a) : a \in F\}$. Similarly, there is a finite $E \subseteq Y \setminus A$ and a collection of open sets $\{[0, 1/n(b)] \times U(b) : b \in E\}$ such that $(Y \setminus A) \cap \Delta \subseteq \bigcup\{[0, 1/n(b)] \times U(b) : b \in E\}$. Let $m = \max\{n(y) : y \in F \cup E\}$.

Let $A_0 = A \cap (\{0\} \times W_i)$ and $A_1 = \pi[A \cap (\{1/m\} \times W_{1-i})]$, where $\pi : \mathbb{R}^3 \to \{0\} \times \mathbb{R}^2$ is the projection onto the y-z-plane. $A_0 \cap (\{0\} \times \Delta) = A_1 \cap (\{0\} \times \Delta)$ by the choice of m. Applying Lemma 2.41 with $U = \{0\} \times (W_i \setminus \Delta)$, we have $A_0 \cup A_1$ is clopen in $\{0\} \times (W_0 \cup W_1)$. Since $\{0\} \times (W_0 \cup W_1) = \{0\} \times (X \cup \Delta)$ is connected, $A_0 \cup A_1 = \{0\} \times (W_0 \cup W_1)$. Thus $\{0\} \times W_i \subseteq A_0 \subseteq A$. □

Theorem 2.43. *$\widetilde{W}$ is widely-connected.*

Proof. For each $i < 2$ and $e' \in E_i$, $\{e'\} \times W_i$ is a quasi-component of W. This follows immediately from Lemma 2.42 since there is a sequence of points $(e_n) \in (E_{1-i})^\omega$ such that $e_n \to e'$ as $n \to \infty$. Thus, the relation $\sim$ links together the quasi-components of W, so that $\widetilde{W}$ is connected.

To prove that $\widetilde{W}$ is *widely*-connected, we let C be a non-empty connected subset of $\widetilde{W}$ with $\overline{C} \neq \widetilde{W}$, and show that $|C| = 1$. Let $U \times V$ be a non-empty open subset of $2^{\mathbb{Z}} \times K$ such that $\{p, q\} \cap V = \varnothing$ and $C \cap (U \times V) = \varnothing$. Let $x \in C$. Let $x' \in W$ such that $x = x'/\sim$, and let e' be the first coordinate of x'. Since E_0 and E_1 are dense in $2^{\mathbb{Z}}$, there exist $n, m > 0$ such that $\{\eta^{-n}(e'), \eta^m(e')\} \subseteq U$. Since η^{n+m} is a homeomorphism and $2^{\mathbb{Z}}$ has a basis of clopen sets, there is a clopen $A \subseteq U$ such that $\eta^{-n}(e') \in A$ and $\eta^{n+m}[A] \subseteq U$. Let L and M be disjoint closed subsets of K such that $K \setminus V = L \cup M$, $p \in L$, and $q \in M$. Then

$$T := \Big([W \setminus (U \times V)] \cap \big[(A \times L) \cup (\eta^{n+m}[A] \times M) \ \cup \bigcup_{0<i<n+m} (\eta^i[A] \times K)\big] \Big) / \sim$$

is a clopen subset of $\widetilde{W} \setminus (U \times V)$.

Let

$$Q = T \cap \Big(\bigcup_{-n \le i \le m} \{\eta^i(e')\} \times K \Big) / \sim .$$

Note that $x \in Q$. If $y \in T \setminus Q$, then by the construction of T there is a clopen $S \subseteq T$ such that $y \in S$ and $S \cap Q = \varnothing$. Thus $C \subseteq Q$, which implies $|C| = 1$ (see Figure 2.7). □

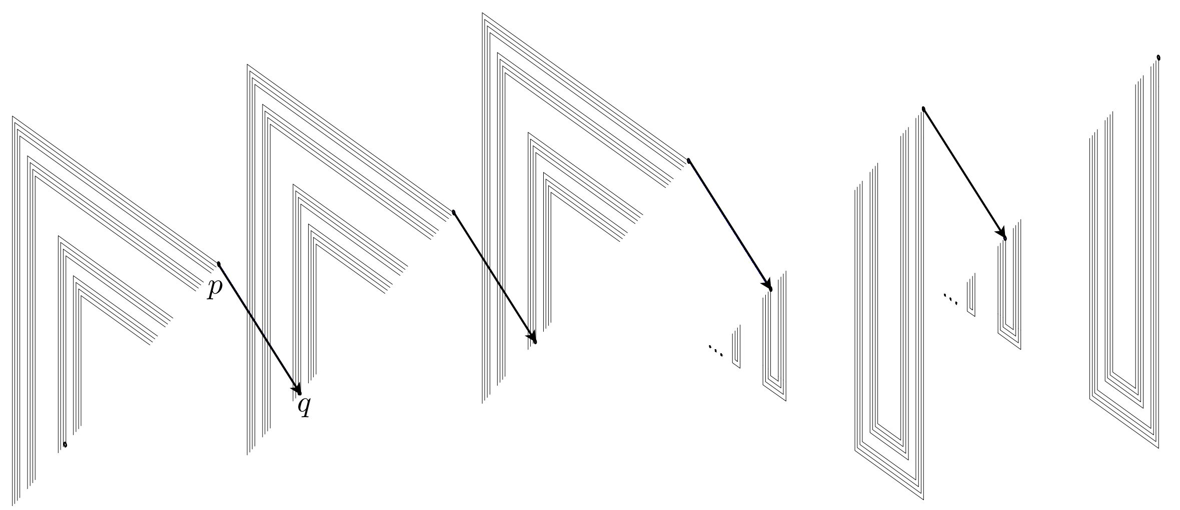

Figure 2.7: Superset of Q if $n = m = 2$ and $e' = e$

Theorem 2.44. *If $\gamma\widetilde{W}$ is a compactification of $\widetilde{W}$, then there is a composant P of $\gamma\widetilde{W}$ such that $\widetilde{W} \subseteq P$.*

Proof. It suffices to show that for every two points $x, y \in \widetilde{W}$ there is a non-empty open $T \subseteq \widetilde{W}$ such that x and y are in the same quasi-component of $\widetilde{W} \setminus T$. This would imply that if $\gamma\widetilde{W}$ is a compactification of $\widetilde{W}$ and T' is open in $\gamma\widetilde{W}$ such that $T' \cap \widetilde{W} = T$, then x and y are contained in a quasi-component of $\gamma\widetilde{W} \setminus T'$. By Theorem 1.5, x and y are contained in a component of $\gamma\widetilde{W} \setminus T'$, a proper subcontinuum of $\gamma\widetilde{W}$.

Let x and y be given. Pick two points $x', y' \in W$ such that $x = x'/\sim$ and $y = y'/\sim$, and let $n, m \in \mathbb{Z}$ such that $\eta^n(e)$ and $\eta^m(e)$ are the first coordinates of x' and y', respectively. Assume that $n \le m$. There is a non-empty clopen $A \subseteq 2^{\mathbb{Z}}$ such that $A \cap \{\eta^i(e) : n-1 \le$

$i \leq m+1\} = \varnothing$. For $i \in \mathbb{Z}$, set $\delta(i) = 0$ if $i \leq 0$ and $\delta(i) = 1$ if $i > 0$. Then

$$R := \Big(\bigcup_{n \leq i \leq m} \{\eta^i(e)\} \times W_{\delta(i)} \Big) / \sim$$

is a subset of $\widetilde{W} \setminus T$ where $T = (A \times K)/\sim$. And $\{x, y\} \subseteq R$. Each fiber $\{\eta^i(e)\} \times W_{\delta(i)}$, $n \leq i \leq m$, is a quasi-component of $W \setminus (A \times K)$ by density of $E_{1-\delta(i)}$ in $2^{\mathbb{Z}} \setminus A$ (Lemma 2.42). The fibers are linked together by $\sim$, so R is contained in a quasi-component of $\widetilde{W} \setminus T$. $\square$

Chapter 3

Hyperspace of $\mathbb{H}^*$

3.1 Order arcs in $C(X)$

If X is a continuum, then let $C(X)$ be the set of all subcontinua of X. The basis for the (Vietoris) topology on $C(X)$ consists of the sets

$$\langle U_1, ..., U_n \rangle = \left\{ A \in C(X) : A \subseteq \bigcup \{U_i : 1 \leq i \leq n\} \text{ and } A \cap U_i \neq \varnothing \text{ for each } i \right\},$$

where $U_1, ..., U_n$ are open subsets of X. It is well-known that if X is a continuum then so is $C(X)$. Suppose that $\mathcal{N} \subseteq C(X)$ and $A, B \in C(X)$ with $A \subseteq B$. Then $\mathcal{N}$ is a *nest in* $C(X)$ *from* A *to* B if

i. $A, B \in \mathcal{N}$,

ii. $A \subseteq N \subseteq B$ for all $N \in \mathcal{N}$, and

iii. $N \subseteq N'$ or $N' \subseteq N$ for all $N, N' \in \mathcal{N}$.

Say that $\mathcal{N}$ is a *maximal* nest in $C(X)$ from A to B if $\mathcal{N}$ is a nest in $C(X)$ from A to B and there is no nest in $C(X)$ from A to B which properly contains $\mathcal{N}$. More generally, $\mathcal{N}$ is a (maximal) nest in $C(X)$ if there exist $A, B \in C(X)$ such that $\mathcal{N}$ is a (maximal) nest in $C(X)$ from A to B.

Lemma 3.1. *If $\mathcal{M}$ is a maximal nest in $C(X)$, then for any $M_1, M_2 \in \mathcal{M}$ with $M_1 \subsetneq M_2$ there exists $M \in \mathcal{M}$ such that $M_1 \subsetneq M \subsetneq M_2$.*

Proof. There is a non-empty relatively open $U \subseteq M_2$ such that $\overline{U} \cap M_1 = \varnothing$. Let C be the component of M_1 in $M_2 \setminus U$. Clearly $M_1 \subseteq M \subsetneq M_2$. By Theorem 1.6, $M \cap \partial U \neq \varnothing$, whence $M_1 \subsetneq M$. ∎

Lemma 3.2. *If $\mathcal{M}$ is a maximal nest in $C(X)$, then the subspace and order (induced by $\subseteq$) topologies on $\mathcal{M}$ coincide.*

Proof. Let (M_1, M_2) be open in $(\mathcal{M}, \subseteq)$, and suppose $M \in (M_1, M_2)$. Let U_1 be open in X containing M but missing a point in M_2. Let $U_2 = U_1 \setminus M_1$. Then $M \in \langle U_1, U_2 \rangle \subseteq (M_1, M_2)$.

Let $\mathcal{U} = \langle U_1, ..., U_n \rangle$ be a basic open set in $\mathcal{M}$ and suppose $M \in U$. Assume $A \neq B$, and assume for the moment that $M \neq A, B$. By maximality of $\mathcal{M}$, we have

$$\overline{\bigcup [A, M)} = M = \bigcap (M, B].$$

If $\mathcal{U} \cap [A, M) \neq \varnothing$, then there exists i such that $U_i \cap M' = \varnothing$ for each $M' \in [A, M)$. Then U_i is a neighborhood of some point in M missing $\bigcup [A, M)$, a contradiction. So there exists $M_1 \in \mathcal{U} \cap [A, M)$. If $\mathcal{U} \cap (M, B] = \varnothing$, then the sets $M'' \setminus \bigcup U_i$, $M'' \in (M, B]$, are non-empty closed and decreasing, but have empty intersection, contrary to compactness of X. So there exists $M_2 \in \mathcal{U} \cap (M, B]$. We have $M \in (M_1, M_2) \subseteq \mathcal{U}$.

If $M = A$ or $M = B$ then replace (M_1, M_2) with $[A, M_2)$ or $(M_1, B]$, respectively. □

Theorem 3.3. *Let $\mathcal{N}$ be a nest in $C(X)$. The following are equivalent:*

(i) $\mathcal{N}$ is maximal

(ii) $\mathcal{N}$ is continuum (in the subspace topology).

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{N}$ is a maximal nest. We have already discovered that $(\mathcal{N}, \subseteq)$ is dense. Now we show it is complete. If $\mathcal{S} \subseteq \mathcal{N}$ then $H = \overline{\bigcup \mathcal{S}}$ is the smallest member of $C(X)$ containing all members of $\mathcal{S}$, and $K = \bigcap \mathcal{S}$ is the largest member of $C(X)$ contained in all members of $\mathcal{S}$. $\mathcal{N} \cup \{H, K\}$ is a nest, so by maximality $H, K \in \mathcal{N}$. We have $H = \sup_{(\mathcal{N}, \subseteq)} \mathcal{S}$ and $K = \inf_{(\mathcal{N}, \subseteq)} \mathcal{S}$. This finishes our proof that $(\mathcal{N}, \subseteq)$ is complete. If a linear order with first and last elements is dense and complete, then it is a continuum (in the order topology). So $(\mathcal{N}, \subseteq)$ is a continuum. By the lemma above, $\mathcal{N}$ is a continuum.

$(ii) \Rightarrow (i)$: Suppose $\mathcal{N}$ is not maximal. Let $\mathcal{M}$ be a proper extension of $\mathcal{N}$, and let $M \in \mathcal{M} \setminus \mathcal{N}$. For each $N \in \mathcal{N}$ there is a basic open set $\mathcal{U}_N$ in $C(X)$ containing N, missing

M. Note that if $N \subseteq M$ then everything in $\mathcal{N} \cap \mathcal{U}_N$ is properly contained in M, and if $M \subseteq N$ then everything in $\mathcal{N} \cap U_N$ properly contains M. Thus $\mathcal{N} \cap \bigcup\{\mathcal{U}_N : N \subseteq M\}$ and $\mathcal{N} \cap \bigcup\{\mathcal{U}_N : M \subseteq N\}$ are disjoint. As these sets are both open and their union covers $\mathcal{N}$, we have that $\mathcal{N}$ is not connected. □

An *order arc* in $C(X)$ is a nest in $C(X)$ satisfying one (both) of the conditions in the previous theorem. For any $A, B \in C(X)$ with $A \subseteq B$, there is an order arc in $C(X)$ from A to B. This follows from applying Zorn's Lemma to the set of nests from A to B, partially ordered by inclusion.

3.2 Non-homeomorphic continua

A *standard subcontinuum* of $\mathbb{H}^*$ is constructed with a sequence of closed intervals in $\mathbb{H}$, together with a free ultrafilter on ω. Formally, let $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ be unbounded sequences of numbers in $\mathbb{H}$ such that $a_n < b_n \leq a_{n+1}$ for each $n < \omega$, and let $u \in \omega^*$. Define

$$[a_u, b_u] = \bigcap_{A \in u} \operatorname{cl}_{\beta\mathbb{H}} \bigcup_{n \in A} [a_n, b_n];$$

$[a_u, b_u]$ is a subcontinuum of $\mathbb{H}^*$. In the special case that $a_n = n$ and $b_n = n + 1$ for each $n < \omega$, we will write $\mathbb{I}_u = [0_u, 1_u]$ for $[a_u, b_u]$.

There is a dense subset of $\mathbb{I}_u$ that naturally identifies with the ultrapower $[0,1]^\omega/u$. If $(x_n) \in [0,1]^\omega$ then $x_u := \{\{x_n + n : n \in A\} : A \in u\} \in \mathbb{I}_u$ corresponds to $x/u \in [0,1]^\omega/u$. The set $P_u := \{x_u : x \in [0,1]^\omega\}$ is dense in $\mathbb{I}_u$, and the subspace topology on P_u is the same as the linear order topology on $[0,1]^\omega/u$. If $\overline{[0,1]^\omega/u}$ denotes the Dedekind completion of $[0,1]^\omega/u$ with first and last elements, then there is a continuous $\varphi : \mathbb{I}_u \to \overline{[0,1]^\omega/u}$ such that:

- $\varphi^{-1}\{x/u\} = \{x_u\}$ for each $x \in [0,1]^\omega$;
- $L_x := \varphi^{-1}\{x\}$ is an indecomposable continuum for each $x \in \overline{[0,1]^\omega/u}$;
- $[L_x, L_y] := \varphi^{-1}[x, y]$ is a continuum for each $x < y \in \overline{[0,1]^\omega/u}$.

Thus, the completion of $[0,1]^\omega$ can be viewed as a linearization of $\mathbb{I}_u$. The continua L_x and $[L_x, L_y]$ are called *layers* and *subintervals* of $\mathbb{I}_u$, respectively.

Lemma 3.4. *$C_u := \big\{[0_u, L_x] : x \in \overline{[0,1]^\omega/u}\big\}$ is an order arc.*

Proof. Clearly C_u is a nest; we show that C_u is a continuum. Since $(C_u, \subseteq) \simeq \overline{[0,1]^\omega/u}$ is a continuum, it suffices to show the inclusion order topology on C_u is finer than its subspace topology. Let $\mathcal{U} := C_u \cap \langle U_1, ..., U_n\rangle$ be a basic open set in C_u and suppose that $[0_u, L_x] \in \mathcal{U}$. By compactness there exists $b_u \in P^u$ such that $x < b_u$ and $[0_u, b_u] \subseteq \bigcup_{i=1}^n U_i$. For each $i \in \{1, ..., n\}$ there exists $a_u^i \in P^u$ with $a_u^i < x$ and $a_u^i \in U_i$. Let $a_u = \max a_u^i$. Then $[0_u, L_x] \in ([0_u, a_u], [0_u, b_u]) \subseteq \mathcal{U}$. □

The structure of $\mathbb{I}_u$ easily generalizes. In fact, for any standard subcontinuum $[a_u, b_u]$ there is a natural homeomorphism between $\mathbb{I}_u$ and $[a_u, b_u]$ that is an isomorphism between the P_u points of each – by a P_u point of $[a_u, b_u]$, we mean a sequence of points (x_n) mod u, where $x_n \in [a_n, b_n]$ for each $n < \omega$.

Lemma 3.5. *If A is a standard subcontinuum of $\mathbb{H}^*$ and L is a non-trivial layer of A, then there are two order arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ in $C(\mathbb{H}^*)$ from L to A such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \{L, A\}$.*

Proof. If $A = [a_u, b_u]$ and $L = L_y$, then by the arguments in Lemma 3.4,

$$\mathcal{A}_1 = \big\{[L_x, L_y] : x \le y\big\} \cup \big\{[a_u, L_x] : y < x\big\} \text{ and}$$
$$\mathcal{A}_2 = \big\{[L_y, L_x] : y \le x\big\} \cup \big\{[L_x, b_u] : x < y\big\}$$

are order arcs. $\mathcal{A}_1 \cap \mathcal{A}_2 = \{L, A\}$ since L is not and end layer of A. □

Lemma 3.6 (Section 2.1 in [9]). *If L is a linear order with $|L| \le \omega_1$, then there is an order-preserving embedding $\varphi : L \to (C(\mathbb{H}^*), \subseteq)$ such that for each $l \in L$:*

(i) $\varphi(l)$ is a standard subcontinuum of $\mathbb{H}^*$;

(ii) $l < l' \in L$ implies $\varphi(l)$ is a subset of a layer of $\varphi(l')$.

A point p is *triodic* if some connected neighborhood of p has exactly three connected components upon the removal of p.

Theorem 3.7. *There is a collection of 2^{ω_1} pairwise non-homeomorphic subcontinua of $C(\mathbb{H}^*)$, each of which is a union of two order arcs.*

Proof. Let $(\omega, 0]$ be the first infinite ordinal ω with its usual ordering reversed. For each subset X of limit ordinals less than ω_1, let

$$L_X = (\omega_1 \times \{0\}) \cup \bigcup_{\alpha \in X} \{\alpha + 1\} \times (\omega, 0]$$

with the lexicographic ordering.

Fix X and let $\varphi : L_X \to C(\mathbb{H}^*)$ be given by Lemma 3.6. Write A^l for $\varphi(l)$. For each $l' \in L_X$ let $L^{l'}$ be the layer of $A^{l'}$ which contains $\bigcup_{l<l'} A^l$.

For each pair of successors $l < l' \in L_X$ (i.e., there exist $\alpha < \omega_1$ and $n < \omega$ such that $l = \langle \alpha, 0 \rangle$ and $l' = \langle \alpha + 1, 0 \rangle$, or $l = \langle \alpha, n+1 \rangle$ and $l' = \langle \alpha, n \rangle$) we define a continuum $K^{l,l'}$ as follows. Let $\mathcal{A}_0$ be any order arc from A^l to $L^{l'}$, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be order arcs from $L^{l'}$ to $A^{l'}$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \{L^{l'}, A^{l'}\}$ (Lemma 3.5). Let $K^{l,l'} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.

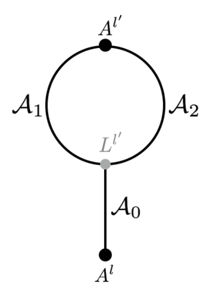

Figure 3.1: $K^{l,l'}$

Suppose that $\alpha < \omega_1$ is a limit. Let $\mathcal{A}_0$ be an order arc from $\mathrm{cl}_{\mathbb{H}^*} \bigcup_{l<\langle \alpha,0 \rangle} A^l$ to $L^{\langle \alpha,0 \rangle}$ ($\mathcal{A}_0$ is possibly trivial), let $\mathcal{A}_1$ and $\mathcal{A}_2$ be order arcs from $L^{\langle \alpha,0 \rangle}$ to $A^{\langle \alpha,0 \rangle}$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \{L^{\langle \alpha,0 \rangle}, A^{\langle \alpha,0 \rangle}\}$, and let $K_\alpha = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.

For each $\alpha \in X$ let K^α be an order arc from $A^{\langle \alpha,0 \rangle}$ to $\bigcap_{n<\omega} A^{\langle \alpha+1,n \rangle}$. Note that if $\alpha \in X$ then $A^{\langle \alpha,0 \rangle} \subseteq \bigcap_{n<\omega} A^{\langle \alpha+1,n \rangle} = \bigcap_{n<\omega} L^{\langle \alpha+1,n \rangle}$ which is indecomposable (Lemma 6.4 in [14]), so the inclusion is proper and K^α is non-trivial. Let

$$K_X = \bigcup \left\{K^{l,l'} : l' \text{ is the successor of } l \text{ in } L_X\right\} \cup \bigcup_{\text{limit } \alpha<\omega_1} K^\alpha \cup \bigcup_{\alpha\in X} K_\alpha \cup \left\{ \mathrm{cl}_{\mathbb{H}^*} \bigcup_{l\in L_X} A^l \right\}.$$

Clearly K_X is the union of two order arcs from $A^{\langle 0,0\rangle}$ to $\mathrm{cl}_{\mathbb{H}^*} \bigcup_{l\in L_X} A^l$, so it is a continuum.

Suppose that $h : K_X \to K_Y$ is a homeomorphism. We show (1) $h(A_X^{\langle\alpha,0\rangle}) = A_Y^{\langle\alpha,0\rangle}$ for each limit $\alpha < \omega_1$, and then (2) $X = Y$.

(1): Base case: $\alpha = \omega$. $h(A_X^{\langle 0,0\rangle}) = A_Y^{\langle 0,0\rangle}$, $h(L_X^{\langle 1,0\rangle}) = L_Y^{\langle 1,0\rangle}$, $h(A_X^{\langle 1,0\rangle}) = A_Y^{\langle 1,0\rangle}$. $h(A_X^{\langle n,0\rangle}) = A_Y^{\langle n,0\rangle}$ for each $n < \omega$. $h(\mathrm{cl}_{\mathbb{H}^*} \bigcup_{l<\langle\omega,0\rangle} A_X^l) = \mathrm{cl}_{\mathbb{H}^*} \bigcup_{l<\langle\omega,0\rangle} A_Y^l$. $h(A_X^{\langle\omega,0\rangle}) = A_Y^{\langle\omega,0\rangle}$.

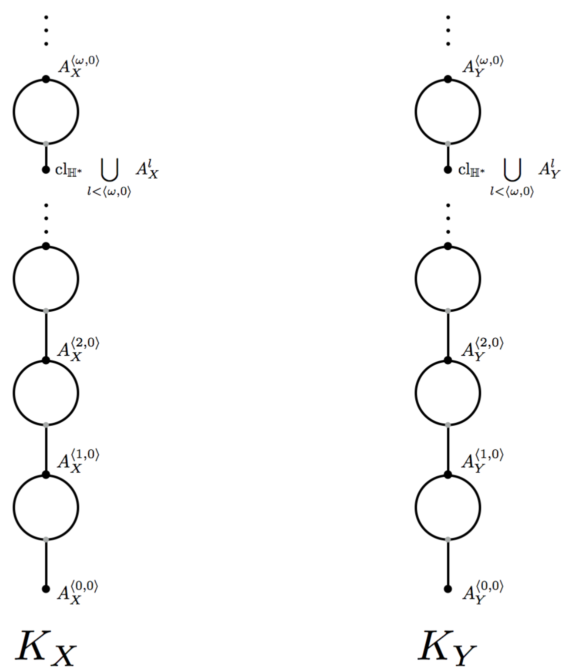

Figure 3.2: Initial segments of K_X and K_Y

Suppose that $\alpha < \omega_1$ is a limit and $h(A_X^{\langle\beta,0\rangle}) = A_Y^{\langle\beta,0\rangle}$ for each limit $\beta < \alpha$. If α is a limit of successors then $\alpha = \beta + \omega$ for some $\beta < \alpha$, and $h(A_X^{\langle\alpha,0\rangle}) = A_Y^{\langle\alpha,0\rangle}$ as in the base case. If α is a limit of limits, then by the induction hypothesis $h(\mathrm{cl}_{\mathbb{H}^*} \bigcup_{l<\langle\alpha,0\rangle} A_X^l) = \mathrm{cl}_{\mathbb{H}^*} \bigcup_{l<\langle\alpha,0\rangle} A_Y^l$, and then it follows that $h(A_X^{\langle\alpha,0\rangle}) = A_Y^{\langle\alpha,0\rangle}$.

(2): Let $\alpha \in X$ and suppose for a contradiction that $\alpha \notin Y$. We know that $h(A_X^{\langle\alpha,0\rangle}) = A_Y^{\langle\alpha,0\rangle}$. Every neighborhood of $\bigcap_{n<\omega} A_X^{\langle\alpha+1,n\rangle}$ contains infinitely many triodic points, but it cannot map to a point with this property (contradiction). Similarly, $\alpha \in Y$ implies $\alpha \in X$. Thus $X = Y$. $\square$

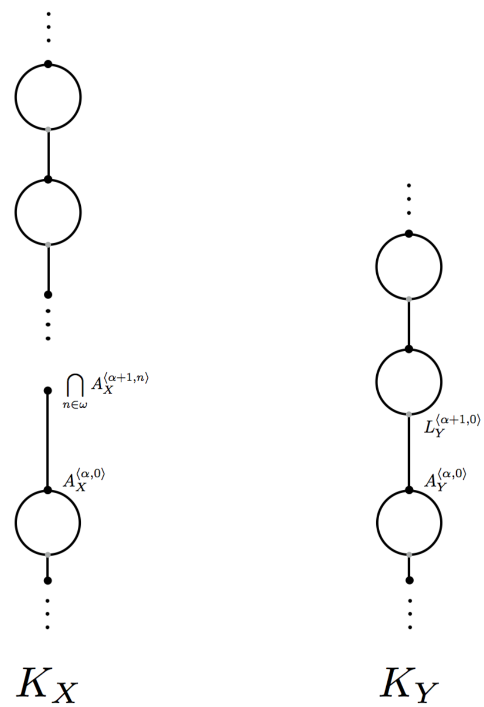

Figure 3.3: $\alpha \in X \setminus Y$

Remark 3.8. Every decomposable subcontinuum of $\mathbb{H}^*$ is a non-trivial interval of some standard subcontinuum, and thus contains non-trivial layers. The K_X's are hereditarily decomposable, so they are not subspaces of $\mathbb{H}^*$.

Recall that $C_u \simeq (C_u, \subseteq) \simeq \overline{[0,1]^\omega/u}$. Assuming $\neg$CH, there are $2^{\mathfrak{c}}$ non-isomorphic orders $\overline{[0,1]^\omega/u}$ – see [8]. Thus, if CH fails then there are $2^{\mathfrak{c}}$ mutually non-homeomorphic order arcs in $C(\mathbb{H}^*)$. In general, it is easy to construct three different order arcs in $C(\mathbb{H}^*)$. In any standard subcontinuum there are intervals I_1, I_2, I_3 such that

1. $\mathrm{cf}(I_1) = \mathrm{coi}(I_1) = \omega$;

2. $\mathrm{cf}(I_2) = \mathrm{coi}(I_2) = \omega_1$;

3. $\mathrm{cf}(I_3) = \omega$ and $\mathrm{coi}(I_3) = \omega_1$.

For each $n \in \{1,2,3\}$, I_n is homeomorphic to an order arc that increases from one of its end layers up to I_n. These order arcs are different because of their endpoints have different cofinalities/coinitialities.

Question 4 (CH)**.** Are there more than three non-homeomorphic order arcs in $C(\mathbb{H}^*)$? Infinitely many? The maximum possible number (2^{ω_1})?

3.3 Property of Kelley

If X is a continuum and $K \in C(X)$, then X has the *property of Kelley at K* provided that for each open $C(X)$-neighborhood $\mathcal{U}$ of K there is an open X-neighborhood V of K such that for each $x \in V$ there exists $L \in \mathcal{U}$ with $x \in L$. Say that X has the *property of Kelley* if it has the property of Kelley at each member of $C(X)$.

Lemma 3.9. *If K is a subcontinuum of $\mathbb{H}^*$ and W is a neighborhood of K, then there is a standard subcontinuum $[a_u, b_u]$ such that $K \subseteq [a_u, b_u] \subseteq W$.*

Proof. Separate K and $\beta\mathbb{H} \setminus W$ with open sets U and V which have disjoint closures in $\beta\mathbb{H}$. Assume $\inf U < \inf V$. Let

$$\begin{aligned} a_0 &= \inf U & b_0 &= \sup\{x \in U : (a_0, x) \cap V = \varnothing\} \\ c_0 &= \inf V \cap (b_0, \infty) & d_0 &= \sup\{x \in V : (c_0, x) \cap U = \varnothing\} \\ a_1 &= \inf U \cap (d_0, \infty) & &\text{etc.} \end{aligned}$$

The process never ends since U and V both meet $\mathbb{H}^*$. Let $U_1 = \bigcup_{n<\omega}(a_n, b_n)$ and $V_1 = \bigcup_{n<\omega}(c_n, d_n)$. Since U and V have disjoint closures, the sequence $a_0, b_0, c_0, d_0, a_1, \ldots$ is strictly increasing and converging to infinity, so that $\mathrm{cl}_{\mathbb{H}} U_1 = \bigcup_{n<\omega}[a_n, b_n]$ and similarly for $\mathrm{cl}_{\mathbb{H}} V_1$. In particular, $\mathrm{cl}_{\mathbb{H}} U_1 \cap \mathrm{cl}_{\mathbb{H}} V_1 = \varnothing$.

Note that $\mathrm{cl}_{\beta\mathbb{H}} \mathrm{ex}_{\beta\mathbb{H}} U_1 \subseteq \mathrm{cl}_{\beta\mathbb{H}} \mathrm{cl}_{\mathbb{H}} U_1$. For suppose that $p \notin \mathrm{cl}_{\beta\mathbb{H}} \mathrm{cl}_{\mathbb{H}} U_1$. Then $\mathrm{cl}_{\mathbb{H}} U_1 \notin p$ so there exists $A \in p$ with $A \cap \mathrm{cl}_{\mathbb{H}} U_1 = \varnothing$. Thus $\mathrm{ex}_{\beta\mathbb{H}}(\mathbb{H} \setminus \mathrm{cl}_{\mathbb{H}} U_1)$ is an open subset of $\beta\mathbb{H}$

containing p missing $\mathrm{ex}_{\beta\mathbb{H}} U_1$. So $p \notin \mathrm{cl}_{\beta\mathbb{H}} \mathrm{ex}_{\beta\mathbb{H}} U_1$. Similarly, $\mathrm{cl}_{\beta\mathbb{H}} \mathrm{ex}_{\beta\mathbb{H}} V_1 \subseteq \mathrm{cl}_{\beta\mathbb{H}} \mathrm{cl}_{\mathbb{H}} V_1$. Thus $\mathrm{cl}_{\beta\mathbb{H}} \mathrm{ex}_{\beta\mathbb{H}} U_1 \cap \mathrm{cl}_{\beta\mathbb{H}} \mathrm{ex}_{\beta\mathbb{H}} V_1 = \varnothing$.

Note that $U \cap \mathbb{H} \subseteq U_1$ and $V \cap \mathbb{H} \subseteq V_1$, so we have $K \subseteq U \subseteq \mathrm{ex}_{\beta\mathbb{H}} U_1$ and $\beta\mathbb{H} \setminus W \subseteq V \subseteq \mathrm{ex}_{\beta\mathbb{H}} V_1$. Putting everything together, $K \subseteq \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n<\omega} [a_n, b_n] \subseteq W$. Now let $u = \{A \subseteq \omega : K \subseteq \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n \in A} [a_n, b_n]\}$. It is easy to see that u is a filter. Because K is connected, for each $A \subseteq \omega$ we must have $A \in u$ or $\omega \setminus A \in u$, so that u is in fact an ultrafilter. We have $K \subseteq [a_u, b_u] \subseteq W$. $\square$

Lemma 3.10. *If $\mathscr{B}$ is a basis for the topology on X, then $\{\langle B_1, ..., B_n\rangle : B_i \in \mathscr{B}$ and $n = 1, 2, ...\}$ is a basis for the (Vietoris) topology of $C(X)$.*

Proof. Let $\langle U_1, ..., U_n\rangle$ be a basic neighborhood of a point $A \in C(X)$. For each $a \in A$, if $I(a) = \{i \in \{1, ..., n\} : a \in U_i\}$ then there exists $B(a) \in \mathscr{B}$ such that $a \in B(a) \subseteq \bigcap_{i \in I(a)} U_i$. Then $\{B(a) : a \in A\}$ is an open cover of A, and thus has a finite subcover $\{B_i : 1 \leq i \leq m\}$. Then $A \in \langle B_1, ..., B_m\rangle \subseteq \langle U_1, ..., U_n\rangle$. $\square$

Theorem 3.11. *$\mathbb{H}^*$ has the property of Kelley.*

Proof. Let $K \in C(\mathbb{H}^*) \setminus \{\mathbb{H}^*\}$ and let $\mathcal{U}$ be open in $C(\mathbb{H}^*)$ with $K \in \mathcal{U}$. Assume that $\mathcal{U} = \langle \mathrm{ex}^*_{\beta\mathbb{H}} U_1, ..., \mathrm{ex}^*_{\beta\mathbb{H}} U_n\rangle$, where each U_i is open in $\mathbb{H}$ and $\mathrm{ex}^*_{\beta\mathbb{H}} U_i = (\mathrm{ex}_{\beta\mathbb{H}} U_i) \cap \mathbb{H}^*$. By Lemma 3.9 there is a standard subcontinuum $[a_u, b_u]$ with $K \subseteq [a_u, b_u] \in \mathcal{U}$. Using the fact that $\bigcup_{i=1}^n \mathrm{ex}^*_{\beta\mathbb{H}}(U_i) = \mathrm{ex}^*_{\beta\mathbb{H}} \bigcup_{i=1}^n U_i$, we have

$$D := \left\{n < \omega : [a_n, b_n] \subseteq \bigcup U_i \text{ and } [a_n, b_n] \cap U_i \neq \varnothing \text{ for each } i\right\} \in u.$$

There is a sequence $(V_n)_{n \in D}$ of disjoint open intervals in $\mathbb{H}$ such that $[a_n, b_n] \subseteq V_n \subseteq \bigcup U_i$ for each $n \in D$. Let $V = \mathrm{ex}^*_{\beta\mathbb{H}} \bigcup_{n \in D} V_n$. We have $K \subseteq [a_u, b_u] \subseteq V$. Now let $q \in V$. There exists $L \in q$ such that $L \subseteq \bigcup_{n \in D} V_n$. For each $n \in D$ there is a closed interval $[c_n, d_n]$ with $[a_n, b_n] \cup (L \cap V_n) \subseteq [c_n, d_n] \subseteq V_n$. Let $v = \{A \subseteq \omega : \bigcup_{n \in A \cap D} [c_n, d_n] \in q\}$. Then $v \in \omega^*$ and $q \in [c_v, d_v] \in \mathcal{U}$. $\square$

Corollary 3.12. *Let $A \in C(\mathbb{H}^*) \setminus \{\mathbb{H}^*\}$. (i) If $\mathcal{U}$ is a non-empty open subset of $C(\mathbb{H}^*)$ then there exists $B \in \mathcal{U}$ such that $A \cap B = \varnothing$. (ii) The arc component of $C(\mathbb{H}^*) \setminus \{A\}$ containing $\mathbb{H}^*$ is the only dense arc component of $C(\mathbb{H}^*) \setminus \{A\}$.*

Proof. (i): Let $K \in \mathcal{U}$. Let W be open in $\mathbb{H}^*$ with $K \subseteq W \subseteq \overline{W} \neq \mathbb{H}^*$. Let $\mathcal{V} = \mathcal{U} \cap \langle W \rangle$. Then $K \in \mathcal{V}$. By the property of Kelley at K there is a non-empty open $V \subseteq \mathbb{H}^*$ such that for each $q \in V$ there exists $L_q \in \mathcal{C}(\mathbb{H}^*)$ with $q \in L_q \in \mathcal{V}$. If $L_q \cap A \neq \varnothing$ for each $q \in V$, then $\overline{A \cup \bigcup_{q \in V} L_q}$ is a proper subcontinuum of $\mathbb{H}^*$ with non-empty interior. Therefore there exists $q \in V$ such that $A \cap L_q = \varnothing$; let $B = L_q$.

(ii): Let $\mathcal{E}$ be the arc component of $C(\mathbb{H}^*) \setminus \{A\}$ containing $\mathbb{H}^*$. Suppose that $\mathcal{F}$ is an arc component of $C(\mathbb{H}^*) \setminus \{A\}$ such that $\mathcal{F} \neq \mathcal{E}$. Then $\mathbb{H}^* \notin \mathcal{F}$, so each member of $\mathcal{F}$ is a subset of A. So $\langle \mathbb{H}^* \setminus A \rangle$ is a non-empty open subset of $C(\mathbb{H}^*)$ missing $\mathcal{F}$, thus $\mathcal{F}$ is not dense in $C(\mathbb{H}^*)$. Now we show $\mathcal{E}$ is dense. Let $\mathcal{U}$ be a non-empty open subset of $C(\mathbb{H}^*)$. By (i) there exists $B \in \mathcal{U}$ such that $A \cap B = \varnothing$. Then an order arc from B to $\mathbb{H}^*$ witnesses $B \in \mathcal{E} \cap \mathcal{U}$. □

Lemma 3.13 ([14] Theorem 5.7)**.** *If K is a decomposable subcontinuum of $\mathbb{H}^*$, then K is a non-degenerate subinterval of a standard subcontinuum.*

Lemma 3.14 ([14] Theorem 5.9)**.** *If K and L are subcontinua of $\mathbb{H}^*$, L is indecomposable, and $K \cap L \neq \varnothing$, then $K \subseteq L$ or $L \subseteq K$.*

Theorem 3.15. *Let $p \in \mathbb{H}^*$,*

$$\mathcal{K} = \left\{ K \in C(\mathbb{H}^*) : K \in \alpha \textit{ for every order arc } \alpha \textit{ in } C(\mathbb{H}^*) \textit{ from } \{p\} \textit{ to } \mathbb{H}^* \right\}$$

and $\mathcal{L} = \{L \in C(\mathbb{H}^*) : L$ is indecomposable and $p \in L\}$, then $\mathcal{K} = \mathcal{L}$ is a compact totally disconnected subspace of $C(\mathbb{H}^*)$.

Proof. $\mathcal{L} \subseteq \mathcal{K}$ by Lemma 3.14. Now we prove $\mathcal{K} \subseteq \mathcal{L}$ by showing that each member of $\mathcal{K}$ is indecomposable. Let D be a decomposable subcontinuum of $\mathbb{H}^*$ with $p \in D$. By Lemma 3.13,

D is an interval of a standard subcontinuum $[a_u, b_u]$. Without loss of generality, there exists $c_u \in D$ with $c_u < p$ and $[a_u, c_u) \cap D \neq \varnothing$. Then $[c_u, b_u] \cup [a_{u+1}, b_{u+1}]$ continuum containing p that is $\subseteq$-incomparable with D. Thus an order arc from $\{p\}$ to $[c_u, b_u] \cup [a_{u+1}, b_{u+1}]$ to $\mathbb{H}^*$ does not contain D, so $D \notin \mathcal{K}$.

$\mathcal{K}$ is closed in $C(\mathbb{H}^*)$: Suppose that $B \in C(\mathbb{H}^*) \setminus \mathcal{K}$. There exists an order arc α from $\{p\}$ to $\mathbb{H}^*$ such that $B \notin \alpha$. There exists $A \in \alpha$ such that $A \not\subseteq B$ and $B \not\subseteq A$. Let $a \in A \setminus B$ and $b \in B \setminus A$. Let U be open in $\mathbb{H}^*$ with $a \in U$ and $\overline{U} \cap B = \varnothing$. Let V be open in $\mathbb{H}^*$ with $b \in V$ and $V \cap A = \varnothing$. Then $B \in \langle \mathbb{H}^* \setminus \overline{U}, V \rangle \subseteq C(\mathbb{H}^*) \setminus \mathcal{K}$.

$\mathcal{L}$ is totally disconnected: Let A and B be two points in $\mathcal{L}$. Without loss of generality there exists $b \in B \setminus A$. By Lemma 3.9 there is a standard subcontinuum C such that $A \subseteq C$ and $b \notin C$. Then $p \in C \cap L$ for each $L \in \mathcal{L}$. By Lemma 3.14, $\mathcal{L}$ is the union of the two disjoint closed sets $\{L \in \mathcal{L} : L \subseteq C\}$ and $\{L \in \mathcal{L} : L \supseteq C\}$. The first set contains A and the second set contains B. $\square$

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