Dedekind Domains and the P-rank of Ext

by

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Abstract

We address what can be said of torsion-free finite rank modules A and B over a Dedekind domain R when their Ext's are isomorphic, extending an answer to Fuchs' Problem 43 and its dual by Goeters. We obtain a result for the covariant case when $\hat{R_P}$ has infinite rank over R, noting that A and B are quasi-isomorphic iff the P-rank of their Hom sets match. In the contravariant case, we see A and B are quasi-isomorphic implies their extension groups are isomorphic, with the converse holding when again $\hat{R_P}$ has infinite rank over R. Along the way, we find equivalent conditions that hold for Noetherian domains whose completions are not complete in the P-adic topology.

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Dedicated to my grandmother, LeeAnn Sweatt, whose perseverance and affability through great struggle will forever inspire me. I hope to pay forward everything you've done for me in my life ahead.

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Chapter 1

Introduction

Problem 43 in [4] asks to characterize the relation between abelian groups A and B such that $Ext_{\mathbb{Z}}(A,C) \cong Ext_{\mathbb{Z}}(B,C)$ for all Abelian groups C. A solution to this problem and its dual was given by Goeters in the case that A, B and C are torsion free Abelian groups of finite rank in [6] and [8] respectively.

In the 80's, it was commonly believed that results about Abelian groups extend canonically to modules over Dedekind domains. Lee Lady suggested that Abelian group theorists should work directly in the context of modules over Dedekind domains R. He showed the feasibility of such an approach in [10] for countable Dedekind domains of characterization 0. However, Nagata showed that there exits an uncountable discrete valuation domain R of characteristic 0 whose P-adic completion $\hat{R}$ of R has finite rank. In particular, this provides an example of a Dedekind domain to which Goeters's solution of Fuchs' Problem cannot readily be extended.

First, we will begin to motivate our problem by discussing Dedekind domains. We will work to present several equivalent conditions Dedekind domains satisfy at the end of this section.

Proposition 1.0.1. *[9] Let T be a multiplicative subset of an integral domain R such that $0 \notin T$. If R is integrally closed, then $T^{-1}R$ is integrally closed as well.*

Proof. $T^{-1}R$ is an integral domain, and R may be identified with a subring of $T^{-1}R$. Extending this identification, the quotient field Q of R may be considered as a subfield of the quotient field Q' of $T^{-1}R$, so that $Q = Q'$.

Let $u \in Q^{'}$ be integral over $T^{-1}R$. Then for some $r_i \in R$ and $t_i \in T$,

$$u^n + (r_{n-1}/t_{n-1})u^{n-1} + \cdots + (r_1/t_1)u + (\text{rank}/t_0) = 0.$$

Multiplying by t^n where $t = t_0 t_1 \cdots t_{n-1} \in T$ shows that tu is integral over R. Since $tu \in Q^{'} = Q$ and R is integrally closed, $tu \in R$. Therefore, $u = tu/t \in T^{-1}R$, whence $T^{-1}R$ is integrally closed. □

Proposition 1.0.2. *[10] If M and N are R-modules and S is a multiplicative set, then*

i. $\text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) = \text{Hom}_R(S^{-1}M, S^{-1}N) = \text{Hom}_R(M, S^{-1}N)$.

ii. $S^{-1}M \otimes_{S^{-1}R} S^{-1}N = S^{-1}M \otimes_R S^{-1}N$.

Proof. (*i.*) For $\varphi \in \text{Hom}_R(S^{-1}M, S^{-1}N)$, $m \in M$, $r \in R$, and $s, s^{'} \in S$,

$$\varphi\left(\frac{r}{s}\frac{m}{r^{'}}\right) = \frac{s}{s}\varphi\left(\frac{rm}{ss^{'}}\right) = \frac{r}{s}\varphi\left(\frac{sm}{ss^{'}}\right) = \frac{r}{s}\varphi\left(\frac{m}{s^{'}}\right).$$

Thus every R-linear map from $S^{-1}M$ to $S^{-1}N$ is in fact $S^{-1}R$-linear. Furthermore, every R-linear map from M to $S^{-1}N$ extends uniquely to a map from $S^{-1}M$ to $S^{-1}N$.

(*ii.*) This follows from the fact that for $m \in S^{-1}M$, $n \in S^{-1}N$, $r \in R$, and $s \in S$, the following holds in $S^{-1}M \otimes_R S^{-1}N$:

$$m \otimes \frac{rn}{s} = \frac{sm}{s} \otimes \frac{rn}{s} = \frac{rm}{s} \otimes \frac{sn}{s} = \frac{rm}{s} \otimes n.$$

□

Proposition 1.0.3. *[10] Let M be a finitely generated module over a Noetherian ring R. For every R-module N and multiplicative set S, we have $S^{-1}\text{Hom}_R(M, N) \cong \text{Hom}_R(S^{-1}M, S^{-1}N)$.*

Proof. Let $\frac{\varphi}{s} \in S^{-1}\text{Hom}_R(M, N)$, and define $\psi : S^{-1}\text{Hom}_R(M, N) \to \text{Hom}_R(S^{-1}M, S^{-1}N)$ by

$$\psi\left(\frac{\varphi}{s}\right)\left(\frac{m}{s^{'}}\right) = \frac{\varphi(m)}{ss^{'}}.$$

ψ is clearly an isomorphism when $M = R$, and thus when $M = R^t$ for finite t. Generally, because M is finitely generated, there exists a surjection $\varepsilon : R^t \to M$ for some finite t. Since R is Noetherian, $\ker \varepsilon$ is also finitely generated, and we thus get an exact sequence $R^s \to R^t \to M \to 0$. Since localization preserves exactness, applying $\operatorname{Hom}(\_, N)$ and localizing with respect to S yields a commutative diagram.

□

Proposition 1.0.4. *[10] Let M, N, P be modules over a commutative ring R.*

i. If $m_1, m_2 \in M$, then $m_1 = m_2$ if and only if $m_1/1 = m_2/1 \in M_I$ for all maximal ideals I.

ii. $M = 0$ if and only if $M_I = 0$ for all maximal ideals I.

iii. Suppose that $N, P \subseteq M$. Then $N = P$ if and only if $N_I = P_I$ for all maximal ideals I.

iv. If $\varphi \in \operatorname{Hom}_R(M, N)$, then φ is monic [epic] if and only if $\varphi_I : M_I \to N_I$ is monic [epic] for all maximal ideals I.

v. A sequence $M \to N \to P$ is exact if and only if the induced sequence $M_I \to N_I \to P_I$ is exact for all maximal ideals I.

vi. If M is a submodule of a vector space over the quotient field F of R, then $M = \bigcap_P M_P$.

Next, we present Nakayama's Lemma - a useful tool when dealing with finitely generated modules.

Lemma 1.0.5 (Nakayama's Lemma)**.** *[9] If J is an ideal in a commutative ring R with identity, then the following conditions are equivalent.*

(a) J is contained in every maximal ideal of R.

(b) $1_R - j$ is a unit for every $j \in J$.

(c) If A is a finitely generated R-module such that $JA = A$, then $A = 0$.

(d) If B is a submodule of a finitely generated R-module A such that $A = JA + B$, then $A = B$.

Proof. $(a \to b)$ If $j \in J$ and $1_R - j$ is not a unit, then the ideal $(1_R - J)$ is not R itself, and therefore is contained in a maximal ideal $M \neq R$. But $1_R - j \in M$ and $j \in j \subseteq M$ imply that $1_R \in M$, which is a contradiction. Therefore, $1_R - j$ is a unit.

$(b \to c)$ Since A is finitely generated, there must be a minimal generating set $X = \{a_1, \ldots, a_n\}$ of A. If $A \neq 0$, then $a_1 \neq 0$ by minimality. Since $JA = A$, $a_1 = j_1a_1 + \cdots + j_na_n$ for some $j_i \in J$, whence $1_Ra_1 =_1$ so that

$$(1_R - j_1)a_1 = 0 \text{ if } n = 1$$

and

$$(1_R - j_1)a_1 = j_2a_2 + \cdots + j_na_n \text{ if } n > 1.$$

Since $1_R - j_1$, $a_1 = (1_R - j_1)^{-1}a_1$. Thus, if $n = 1$, then $a_1 = 0$, which is a contradiction. If $n > 1$, then a_1 is a linear combination of $a_2, \ldots, a_n$. Consequently, $\{a_2, \ldots, a_n\}$ generates A, which contradicts the choice of X.

$(c \to d)$ The quotient module A/B is such that $J(A/B) = A/B$, whence $A/B = 0$ and $A = B$ by assumption.

$(d \to a)$ If M is any maximal ideal, then the ideal $JR+M$ contains M. But $JR+M \neq R$, otherwise $R = M$ by assumption. Consequently, $JR + M = M$ by maximality. Therefore, $J = JR \subseteq M$. □

Corollary 1.0.6. *[10] Let M be a finitely generated module over a local ring. M is projective if and only if it is free.*

Proposition 1.0.7. *[10] A finitely generated projective module M over a local ring R is free. In fact, if I is the maximal ideal in R and $m_1, \ldots, m_t \in M$ are such that the cosets $\bar{m}_1, \ldots, \bar{m}_t$ are a basis for M/IM as a vector space over R/I, then $m_1, \ldots, m_t$ are a basis for M.*

Proof. Let I be the unique maximal ideal in R. Choose $m_1, \dots, m_t \in M$ so that the cosets $\bar{m}_1, \dots, \bar{m}_t$ are a basis for the vector space M/IM over the field R/I. Let $\varphi : R^t \to M$ be defined by $\varphi(r_1, \dots, r_t) = \sum r_i m_i$. It follows easily from Nakayama's Lemma that φ is surjective. Since M is a projective module, φ splits, so $R^t = K \oplus L$ with $K = \mathrm{Ker}\varphi$ and $L \cong M$. Then K is finitely generated. Since φ induces an isomorphism from R^t/IR^t to M/IM, it follows that $K/IK \oplus L/IL \cong M/IM$ These are finite dimensional vector spaces over the field R/I and comparing dimensions yields $K/IK = 0$. Thus $K = 0$ by Nakayama's Lemma. Thus φ is monic and hence an isomorphism. □

Proposition 1.0.8. *[10] A finitely generated module M over a Noetherian ring R is projective if and only if M_P is a free R_P-module for all prime ideals P.*

Proof. ($\to$) Using the criterion that projective modules are just the direct summands of free modules, it is easy to see that the localization of a projective R-module at P is a projective module over R_P. It then follows from 1.0.7 that this localization is a free R_P-module.

($\leftarrow$) Suppose now that M is finitely generated and for all P, M_P is a free R_P-module. To show that M is projective one must show that for every surjection $\varphi : X \to Y$, the induced map $\varphi_* : \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y)$ is surjective. By 1.0.4, it suffices to prove that for all maximal ideals P, the localized map $(\mathrm{Hom}_R(M, X))_P \to (\mathrm{Hom}_R(M, Y))_P$ is surjective. But since M is finitely generated, by 1.0.2 and 1.0.3 there are natural isomorphisms yielding the following commutative diagram:

$$\begin{array}{ccccc}
(\mathrm{Hom}_R(M,X))_P & \longrightarrow & (\mathrm{Hom}_R(M,Y))_P & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_{R_P}(M_P, X_P) & \longrightarrow & \mathrm{Hom}_{R_P}(M_P, Y_P) & \longrightarrow & 0
\end{array}$$

where the bottom map is surjective since M_P is a projective R_P-module. Thus $(\mathrm{Hom}_R(M, X))_P \to (\mathrm{Hom}_R(M, Y))_P$ is a surjection proving the result. □

Remark 1.0.9. *[10] The hypothesis that M be finitely generated is essential here. There are many examples of non-finitely generated non-projective modules M such that M_P is a free R_P-module for all prime ideals P - they are called* locally free.

Proposition 1.0.10. *[10] Let R be an integral domain with quotient field F and let P be an R-submodule of F. Then the following conditions are equivalent:*

(a) P is projective.

(b) There exist elements $p_1, \ldots, p_n \in P$ and $f_1, \ldots, f_n \in F$ such that $f_i P \subseteq R$ for all i and $\sum f_i p_i = 1$.

(c) There exists a submodule M of F such that $MP = R$.

Furthermore in this case P is generated by $p_1, \ldots, p_n$.

Proof. $(a \to b)$ Since P is projective, it is a summand of a free module $R^{(f)}$, and there exist maps $\sigma : P \to R^{(I)}$ and $\pi : R^{(I)} \to P$ such that $\pi\sigma = 1_P$. Localizing at the zero ideal, σ extends to a map $\sigma_0 : F \to F^{(I)}$ and π to a map $\pi_0 : F^{(I)} \to F$. For each $i \in I$, let f_i be the i^{th} coordinate of $\sigma_0(1$ and let $p_i = \pi_0(e_i)$, where e_i is the canonical i^{th} basis vector of $F^{(I)}$. Then the composition of σ_0 with the projection of $F^{(I)}$ onto the i^{th} coordinate is given by $x \mapsto f_i x$. Since this composition maps P into R, it follows that $f_i P \subseteq R$. Furthermore, since π is given by $\sum y_i e_i \mapsto \sum y_i\_i$, the equation $\pi_0\sigma_0(1) = 1$ translates to $\sum f_i p_i = 1$. This sum can have only finitely many non-trivial terms, and at this point we can replace I by the finite set of $i \in I$ such that $f_i p_i \neq 0$.

$(b \to a)$ Map P onto R^n by $\sigma : p \mapsto (f_1 p, \ldots, f_n p)$ and map R^n to P by $\pi : (r_1, \ldots, r_n) \mapsto \sigma r_i p_i$. Then $\pi\sigma(p) = \sigma p f_i p_i = p1 = p$. Thus σ is a split monomorphism and P is a summand of a free module, hence is projective.

$(b \to c)$ Let M be the submodule of F generated by $f_1, \ldots, f_n$. Then clearly $MP \subseteq R$. But $1 = \sum f_i p_i \in MP$ so $MP = R$. Note also that $p_1, \ldots, p_n$ generate P since for $p \in P$, we have $p = p \sum f_i p_i = \sum (f_i p) p_i$ and all $f_i p \in R$.

$(c \to b)$ If $MP = R$, then $1 \in MP$ so there exist $f_i \in M$, $p_i \in P$ with $\sum f_i p_i = 1$. Furthermore, for all i, $f_i P \subseteq MP = R$. □

Lemma 1.0.11. *[10] A commutative ring R is integrally closed if and only if R_P is integrally closed for all prime ideals p.*

Proof. $(\to)$ $S^{-1}R$ is integrally closed for every multiplicative set S. Let Q denote the quotient field of R, let $q \in Q$ be integral over $S^{-1}R$, and let $f \in S^{-1}R[X]$ be a monic polynomial satisfied by q. Let d be the degree of f and let $s \in S$ be a common denominator for the coefficients of f. Then $s^d f(q) = 0$, and sq satisfies some monic polynomial in $R[X]$. Thus, $sq \in R$ by assumption, whence $q \in S^{-1}R$.

$(\leftarrow)$ Let $q \in Q$ be integral over R. Then q is integral over each R_P. If all R_P are integrally closed, then $q \in \bigcap R_P = R$. □

Remark 1.0.12. *Note that by the proposition, projective ideals are finitely generated. Hence, if every ideal in an integral domain is projective, then that integral domain is also Noetherian.*

Definition 1.0.13. *[9] Let R be an integral domain with quotient field Q. A* ***fractional ideal*** *of R is a nonzero R-submodule M of Q such that $rM \subseteq R$ for some nonzero $r \in R$.*

Example 1.0.14. *[9] Every nonzero finitely generated R-submodule M of Q is a fractional ideal. For if M is finitely generated by $q_1, \ldots, q_n \in Q$, then $M = Rq_1 + \cdots + Rq_n$ and for each i, $q_i = r_i/s_i$ with $0 \neq s_i, r_i \in R$. Let $s = s_1 \cdots s_n$. Then $s \not= 0$ and $sM = Rs_2 \cdots s_n r_1 + \cdots + Rs_1 \cdots s_{n-1} t_n \subseteq R$.*

Remark 1.0.15. *[9] If I is a fractional ideal of a domain R and $aI \subseteq R$ for some nonzero element a of R, then aI is an ordinary ideal in R and the map $I \to aI$ given by $x \mapsto ax$ is an R-module isomorphism.*

Lemma 1.0.16. *[9] Let $I_1, I_2, \ldots, I_n$ be ideals in an integral domain R.*

i. The ideal $I_1 I_2 \cdots I_n$ is invertible if and only if each I_j is invertible.

ii. *If $P_1 \cdots P_m = I = P_1' \cdots P_n'$ where P_i and P_j' are prime ideals in R with every P_i invertible, then $m = n$ and $P_i = P_i'$ for each $i = 1, \ldots, m$ after reindexing.*

Proof. (i.) If J is a fractional ideal such that $J(I_1 \cdots I_n) = R$, then for each $j = 1, \ldots, n$ we have $I_j(I_1 \cdots I_{j-1}I_{j+1} \cdots I_n = R$, whence I_j is invertible. Conversely, if each I_j is invertible, then $(I_1 \cdots I_n)(I_1^{-1} \cdots I_n^{-1} = R$, whence $I_1 \cdots I_n$ is invertible.

(ii.) We proceed by induction on m. If $m > 1$, choose one of the P_i, say P_1 such that tP_1 oes not properly contain P_i fo $I = 2, \ldots, m$. Since $P_1' \cdots P_n' = P_1 \cdots P_m \subset P_1$ and P_1 is prime, some P_j', say P_1', is contained in P_1. Similarly, we have $P_i \subseteq P_1'$ for some i. Because $P_i \subseteq P_1' \subseteq P_1$, by the minimality of P_1 we have $P_i = P_1' = P_1$. Since $P_1 = P_1'$ is invertible, then we have $P_2 \cdots P_m = P_2' \cdots P_n'$. By the induction hypothesis, $m = n$ and $P_i = Q_i$ for $i = 1, \ldots, m$ after reindexing. □

Lemma 1.0.17. *[9] Every invertible fractional ideal of an integral domain R with quotient field Q is a finitely generated R-module.*

Proof. Let I be such an ideal. Since $I^{-1}I = R$, there exist $a_i \in I^{-1}$ and $b_i \in I$ such that $1_R = \sum_1^n a_i b_i$. If $c \in I$, then $c = \sum_1^n (ca_i)b_i$. Furthermore, each $ca_i \in R$ since $a_i \in I^{-1} = \{q \in Q | qI \subseteq R\}$. Therefore, I is generated as an R-module by $b_1, \ldots, b_n$. □

Definition 1.0.18. *A* ***discrete valuation ring*** *is a principal ideal domain that has exactly one nonzero prime ideal.*

Lemma 1.0.19. *[9] If R is a Noetherian, integrally closed integral domain and R has a unique nonzero prime ideal P, then R is a discrete valuation ring.*

Proof. We need only show that every proper ideal in R is principal.

Claim 1: Let Q be the quotient field of R. For every fractional ideal I of R, the set $\bar{I} = \{q \in Q | qI \subseteq I\}$ is R.

Proof. Clearly $R \subseteq \bar{I}$. Because $\bar{I}$ is a subring of Q and a fractional ideal of R, $\bar{I}$ is isomorphic as an R-module to an ideal of R. Thus since R is Noetherian, $\bar{I}$ is finitely generated, whence every element of $\bar{I}$ is integral over R. Therefore $\bar{I} \subseteq R$ since R is integrally closed. We conclude $\bar{I} = R$. $\square$

Claim 2: R is properly contained in P^{-1}.

Proof. Let $\mathcal{F}$ be the set of all ideals J in R such that R is properly contained in J^{-1}. Since P is a proper ideal, every nonzero element of P is a nonunit. If $J = (a)$ for some nonzero $a \in P$, then $1_r/a \in J^{-1}$, but $1_r/a \notin R$, whence R is properly contained in J^{-1} and $\mathcal{F}$ is nonempty. Since R is Noetherian, $\mathcal{F}$ contains a maximal element M. We claim M is a prime ideal of R. If $ab \in M$ with $a, b \in R$ and $a \notin M$, choose $c \in M^{-1} \setminus R$. Then $c(ab) \in R$, whence $bc(aR + M) \subseteq R$ and $bc \in (aR + M)^{-1}$. Therefore, $bc \in R$, else $aR + M \in \mathcal{F}$ contradicting maximality of M. Consequently $c(bR+M) \subseteq R$, and thus $c \in (bR+M)^{-1}$. Since $c \notin R$, the maximality of M implies that $bR + M = M$, whence $b \in M$. Therefore M is prime, whence $P = M$ by uniqueness. We conclude $R \subsetneq M^{-1} = P^{-1}$. $\square$

Claim 3: P is invertible.

Proof. Clearly $P \subseteq PP^{-1} \subseteq R$. By the argument following the claims, P is the unique maximal ideal in R, so that $P = PP^{-1}$ or $PP^{-1} = R$. If $P = PP^{-1}$, then $P^{-1} \subseteq \bar{P}$ and by claims 1 & 2, $R \subsetneq P^{-1} \subseteq \bar{P} = R$, a contradiction. Therefore, $PP^{-1} = R$ and P is invertible. $\square$

Claim 4: $\bigcap_{n \in N} P^n = 0$.

Proof. If $\bigcap_{n \in N} P^n \neq 0$, then $\bigcap_{n \in N}$ is a fractional ideal of R. But by claims 1 & 2, $R \subsetneq P^{-1} \subseteq \overline{\bigcap_{n \in N} P^n} = R$. So $\bigcap_{n \in N} P^n = 0$. $\square$

Claim 5: P is principal.

Proof. There exists $a \in P$ such that $a \notin P^2$ by claim 4. Then aP^{-1} is a nonzero ideal in R such that $aP^{-1} \not\subseteq P$, otherwise $a \in aR = aPP^{-1} \subset P^2$. The argument following this claim shows that every proper ideal in R is contained in P, whence $aP^{-1} = R$. Therefore by claim 3, $(a) = (a)R = (a)P^{-1}P = (aP^{-1})P = RP = P$, and P is principal. $\square$

Now, let I be any proper ideal of R. Then I is contained in a nonzero maximal ideal M of R, which is necessarily prime. By uniqueness, $M = P$, whence $I \subseteq P$. Since $\bigcap_{n \in N} P^n = 0$, there is a largest integer m such that $I \subseteq P^m$ and $I \not\subseteq P^{m+1}$. Choose $b \in I \setminus P^{m+1}$. Since $P = (a)$ for some $a \in R$, $P^m = (a)^m = (a^m)$. Since $b \in P^m$, $b = ua^m$. Furthermore, $u \notin P = (a)$, otherwise $b \in P^{m+1} = (a^{m+1})$. Therefore, $P^m = (a^m) = (ua^m) = (b) \subseteq I$, whence I is the principal ideal $P^m = (a^m)$.

□

Theorem 1.0.20. *[9][10] The following conditions on an integral domain R are equivalent.*

(a) Every proper ideal in R is a product of a finite number of prime ideals.

(b) Every proper ideal in R is uniquely a product of a finite number of prime ideals;

(c) Every nonzero ideal in R is invertible;

(d) Every fractional ideal of R is invertible;

(e) the set of all fractional ideals of R is a group under multiplication;

(f) every ideal in R is projective;

(g) every fractional ideal of R is projective;

(h) R is Noetherian, integrally closed, and every nonzero prime ideal is maximal;

(i) R is Noetherian, and for every nonzero prime ideal P of R, the localization R_P of R at P is a discrete valuation ring.

Proof. The equivalence $(d) \leftrightarrow (e)$ is trivial. $(a) \rightarrow (b)$ and $(b) \rightarrow (c)$ follows from 1.0.16. $(c) \leftrightarrow (f)$ and $(g) \leftrightarrow (d)$ are immediate consequences of 1.0.10. $(f) \rightarrow (g)$ follows from 1.0.15.

$(c) \rightarrow (i)$ The ideals in R_P have the form I_P where I is an ideal in R. By hypothesis, I is projective, so by 1.0.8 I_P is a free R_P-module. Thus all ideals of R_P are free, so that R_P is a local principal ideal domain, hence a discrete valuation ring.

$(i) \to (h)$ To see that R is integrally closed, it suffices by 1.0.11 to see that R_P is integrally closed for all primes P, which is true if R_P is a discrete valuation ring since principal ideal domains are integrally closed. Now, let P be a prime ideal in R. The prime ideals contained in P correspond to the prime ideals of W_P. Since W_P is a discrete valuation ring, its only prime ideals are PW_P and 0. Thus there are no non-trivial prime ideals strictly contained in P, so P has height one. It follows that all prime ideals of R are maximal.

$(h) \to (f)$ Let I be an ideal in R. Since R is Noetherian, I is finitely generated. Hence, by 1.0.8, it suffices to show that I_P is a free R_P-module for all primes P. But since R_P is a principal ideal domain, I_P is in fact free.

$(d) \to (h)$ Every ideal of R is invertible by (d) and hence finitely generated by 1.0.17. Therefore R is Noetherian. Let K be the quotient field of R. If $u \in K$ is integral over R, then $R[u]$ is a finitely generated R-submodule of K. Consequently, 1.0.14 shows that $R[u]$ is a fractional ideal of R. Therefore, $R[u]$ is invertible by (d). Thus since $R[u]R[u] = R[u]$, we have $R[u] = RR[u] = (R^{-1}[u]R[u])R[u] = R^{-1}[u]R[u] = R$, whence $u \in R$. Therefore, R is integrally closed. Finally, if P is a nonzero prime ideal in R, then there is a maximal ideal M of R that contains P. M is invertible by (d). Consequently $M^{-1}P$ is a fractional ideal of R with $M^{-1}P \subseteq M^{-1}M = R$, whence $M^{-1}P$ is an ideal in R.

$(h) \to (i)$ R_P is an integrally closed integral domain by 1.0.1. Every ideal in R_P is of the form $I_P = \{i/s | i \in I, s \notin P\}$, where I is an ideal of R. Since every ideal of R is finitely generated by (h), it follows that every ideal of R_P is finitely generated. Therefore, R_P is Noetherian. Every nonzero prime ideal of R_P is of the form I_P, where I is a nonzero prime ideal of R contained in P. Since every nonzero prime ideal of R is maximal by (h), P_P must be the unique nonzero prime ideal in R_P. Therefore, R_P is a discrete valuation ring by 1.0.19.

$(i) \to (a)$ We first show that every nonzero ideal I is invertible. II^{-1} is a fractional ideal of R conained in R, whence II^{-1} is an ideal in R. Suppose $II^{-1} \neq R$. Then there is a maximal ideal M containing II^{-1}. Since M is prime, the ideal I_M in R_M is principal by (i); say $I_M = (a/s)$ where $a \in I$ and $s \in R \setminus M$. Since R is Noetherian, I is finitely generated, say $I = (b_1, \ldots, b_n)$. For each i, $b_i/1_R \in I_M$, whence in R_M, $b_i/1_R = (r_i/s_i)(a/s)$ for some $r_i \in R, s_i \in R \setminus M$. Therefore $s_I s b_i = r_i a \in I$. Let $t = s s_1 s_2 \cdots s_n$. Since $R \setminus M$ is multiplicative, $t \in R \setminus M$. In the quotient field of R, we have for every t, $(t/a)b_i = tb_i/a = s s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n r_i \in R$, whence $t/a \in I^{-1}$. Consequently, $t = (t/a)a \in I^{-1}I \subseteq M$, which contradicts that $t \in R \setminus M$. Therefore $I^{-1}I = R$ and I is invertible.

For each proper ideal I of R, choose a maximal ideal M_I of R such that $I \subseteq M_I \subsetneq R$. If $I = R$, then let $M_R = R$. Then IM_I^{-1} is a fractional ideal of R with $IM_I^{-1} \subseteq M_I M_I^{-1} \subseteq R$. Therefore, IM_I^{-1} is an ideal of R that clearly contains I. Also, we have $I \subsetneq IM_I^{-1}$ since otherwise

$$M_I = RM_I = I^{-1}IM_I = I^{-1}(IM_I^{-1})M_I = RR = R,$$

which contradicts our choice of M_I. Let S be the set of all ideals of R and define a function $f : S \to S$ by $I \mapsto IM_I^{-1}$.

Let J be a proper ideal in R. We now show J is the product of maximal (hence prime) ideals. There exists by the Recursion Theorem a function $\phi : \mathbb{N} \to S$ such that $\phi(0) = J$ and $\phi(n+1) = f(\phi(n))$. If we denote $\phi(n)$ by J_n and M_{J_n} by M_n, then we have an ascending chain of ideals $J = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$ such that $J = J_0$, and $J_{n+1} = f(J_n) = J_n M_n^{-1}$. Since R is Noetherian and J is proper, there is a least integer k such that

$$J = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_k = J_{k+1}.$$

Thus $J_k = J_{k+1} = f(J_k) = J_k M_k^{-1}$, which can occur only if $J_k = R$. Consequently, $R = J_k = f(J_{k-1}) = J_{k-1}M_{k-1}^{-1}$, whence

$$J_{k-1} = J_{k-1}R = J_{k-1}M_{k-1}^{-1}M_{k-1} = RM_{k-1} = M_{k-1}.$$

Since $M_{k-1} = J_{k-1} \subsetneq J_k = R$, M_{k-1} is a maximal ideal. The minimality of k insures that each of $M_0, \ldots, M_{k-2}$ is also maximal, otherwise $M_i = R$ so that $J_{i+1} = J_i M_I^{-1} = J_i R^{-1} = J_i R = J_i$. We have

$$M_{k-1} = J_{k-1} = J_{k-2} M_{k-2}^{-1} = J_{k-2} M_{k-3}^{-1} M_{k-2}^{-1} = \cdots = J M_0^{-1} \cdots M_{k-2}^{-1}.$$

Since each M_i is invertible,

$$J = M_{k-1}(M_0 \cdots M_{k-2}).$$

Thus J is the product of maximal (hence prime) ideals. □

Definition 1.0.21. *[9] A **Dedekind domain** is an integral domain R satisfying any of the conditions of the previous theorem.*

Remark 1.0.22. *Evidently, every principal ideal domain is Dedekind, but the converse is false. For example, $\mathbb{Z}[\sqrt{10}]$ is Dedekind but not principal. We will see later that every Dedekind domain is Noetherian.*

Definition 1.0.23. *[10] A module M over a ring R is said to have **finite length** if and only if it has a composition series*

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_l = M$$

where each quotient M_i/M_{i-1} is a simple module. In this case, we define length(M) to be the length l of this composition series.

Remark 1.0.24. *[9] The Jordan-Hölder Theorem asserts that any two compositions series of a module M are equivalent, so length(M) is well-defined. Another standard result is that a module has finite length if and only if it is both Noetherian and Artinian.*

Definition 1.0.25. *[10][7] For any prime ideal P and torsion-free module A of the Dedekind domain R, we define the **P-rank** $r_P(A)$ of A to be the length of A/PA. Equivalently, we may define $r_P(A)$ as the dimension of A/PA as a vector space over R/P.*

Proposition 1.0.26. *[10] Let A and B be torsion-free modules over Dedekind domain R, and P a prime ideal in R.*

i. $r_P(A \oplus B) = r_P(A) \oplus r_P(B)$.

ii. If B is an essential submodule of A, then their ranks are the same and $r_P(B) \geq r_P(A)$.

iii. $r_P(A) = r_P(A_P)$.

iv. $r_P(A) \leq rank(A)$.

v. $rank(A \otimes B) = (rank(A))(rank(B))$ *and* $r_P(A \otimes B) = r_P(A) r_P(B)$.

vi. $rank(A) = 0$ *if and only if* $A = 0$.

vii. $r_P(A) = 0$ *if and only if A is P-divisible.*

viii. $r_P(A)$ *is the same as the number of summands QA/A isomorphic to $R(P^\infty)$.*

Definition 1.0.27. *[10] Let R be a Dedekind domain, let M be a finite rank torsion free R-module, and let p be a prime ideal of R. The* ***p-adic filtration*** *on M is the family of submodules*

$$M \supseteq pM \supseteq p^2M \supseteq \dots.$$

The topology generated by taking the p-adic filtration on M as a neighborhood basis at 0 is called the ***p-adic topology*** *on M. The* ***p-adic completion*** *of M is the submodule $\hat{M}$ of $\prod_1^\infty M/p^kM$ consisting of those sequences $m_1, m_2, \ldots \in \prod_1^\infty M/p^kM$ such that $m_{k+1} \equiv m_k$ (mod p^kM) for all k.*

Proposition 1.0.28. *[10] Let R, M, and p be as in the previous definition. The topology inherited by $\hat{M}$ is the same as the inverse limit topology.*

Proof. The neighborhood system at 0 in the inverse limit topology has a basis consisting of those submodules U_n consisting of elements whose first n coordinates are zero. Since the first n coordinates live in M/p^kM for $k \geq n$, it follows that $p^n\hat{M} \subseteq U_n$. On the other hand, since the sequences in $\hat{M}$ satisfy the condition $m_{n+k} \equiv m_k$ (mod p^n, it follows that if $m_r = 0$ for $r \leq n$, then $m_r \in p^n\hat{M}$ for all r. Thus $U_n \subseteq p^k\hat{M}$. We conclude the inverse limit topology and the p-adic topology are the same. □

In the following, we assume that the Dedekind domain R with field of quotients Q is not complete in the R-adic topology. As observed in [7], non-complete Dedekind domains fall into two distinct cases [7]):

Type I For each maximal ideal P, the P-adic completion $\hat{R_P}$ of the localization R_P has infinite R_P-module rank.

Type II R is local and the completion of R, $\hat{R}$ has finite rank.

Theorem 1.0.29. *[7] A Dedekind domain R is not complete in the R-adic Topology if and only if R is a type I or a type II domain*

Since R is a domain, multiplication by $r \in R$ on A or B induces multiplication by r on $Ext^1_R(A,B)$. Moreover, $Ext^1_R(A,B)$ is a divisible module whenever R is Dedekind. Therefore, it is of the form $\oplus_{P \in spec(R)} D_P \oplus D_0$ with P-primary component $D_P \cong \oplus_{I_P} E(E/P)$ and torsion-free component $D_0 = \oplus_{I_0} Q$.

Given any maximal ideal P of R, the *P-rank* of a module A is denoted by $r_P(A)$ and is defined as $r_P(A) = dim_{R/P} A/PA$. If $a \in P \setminus P^2$, then $aR_P = PR_P$ since R_P is a discrete valuation domain. The sequence $0 \to B \overset{\alpha}{\to} B \to B/aB \to 0$ induces

$$0 \to Hom(A,B) \overset{a}{\to} Hom(A,B) \to Hom(A/aA, B/aB) \to S \to 0$$

where S is the submodule of $Ext(A,B)$ annihilated by a. Localizing at P, gives the formula;

$$r_P(A)r_P(B) - r_P(Hom(A,B)) = \{\epsilon \in Ext(A,B) \mid P\epsilon = 0\}.$$

Theorem 1.0.30. *[7] Let R be a Dedekind domain and $P \in spec(R)$. If A and C are torsion-free R-modules of finite rank, then*

$$Ext_R^1(A, C) \cong \oplus_P D_P \oplus D_0$$

with $D_P \cong (Q/R_P)^{e_P}$ and D_0 torsion-free such that

$$e_P = r_P(A) r_P(B) - r_P(Hom_R(A, B)).$$

While this result appears to be independent of the type of the Dedekind domain, we want to point out that D_0 has finite rank exactly when R has type II. Thus the structure of Ext actually varies according to D_0 in the Type II case which, in turn, depends upon the rank of $\hat{R}$ [7]. A formula used to determine the rank of D_0 will be given later,

Corollary 1.0.31. *[7] A Dedekind R which is not complete satisfies exactly one of the following;*

i) For all $P \in spec(R)$, the completion of R_P has infinite rank, or

ii) R is local with maximal ideal P, and the completion of R, $\hat{R}$, has finite rank over R.

For the rest of this chapter, R is a Dedekind domain with quotient field Q, unless otherwise indicated.

Definition 1.0.32. *[10] Define an equivalence relation $\star$ on the set of all submodules of Q by $A \star B$ if and only if A and B are isomorphic to a submodule of the other. The* ***type*** $\mathbf{t}(A)$ *of A is the equivalence class of A under $\star$, and we write $\mathbf{t}(A) \leq \mathbf{t}(B)$ when A is isomorphic to a submodule of B. We say $\mathbf{t}(A)$ and $\mathbf{t}(B)$ are* ***incomparable*** *if $\mathbf{t}(A) \not\leq \mathbf{t}(B)$ and $\mathbf{t}(B) \not\leq \mathbf{t}(A)$.*

Definition 1.0.33. *[10] Let A be a torsion-free finite rank module over a Dedekind domain R. The* ***typeset*** *of A, denoted by $\mathbf{T}(\mathbf{A})$, is the set of types of all non-trivial elements of A, or equivalently the set of types of all pure rank-one submodules of A. Dually, we define the*

cotype set *of A, denoted by* $\mathbf{CT(A)}$*, to be the set of types of all rank-one homomorphic images of A.*

Proposition 1.0.34. *[10] Let* $A = H \oplus K$*.* $\mathbf{T(A)}$ *consists of* $\mathbf{T(H)} \cup \mathbf{T(K)}$ *together with all types* $\mathbf{s} \wedge \mathbf{t}$ *with* $\mathbf{s} \in \mathbf{T(H)}$ *and* $\mathbf{t} \in \mathbf{T(K)}$*.*

Corollary 1.0.35. *[10] If* $A = A_1 \oplus \cdots \oplus A_n$ *where the* A_i *are rank 1 modules and* $\boldsymbol{t}_i = \boldsymbol{t}(A_i)$*, then* $\boldsymbol{T}(A)$ *consists of* $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n\}$ *and all types obtained from this set by taking greatest lower bounds.*

Definition 1.0.36. *[10] The* ***inner type*** *of a torsion-free finite rank module A over a Dedekind domain is* $\mathbf{IT(A)} = \mathbf{infT(A)}$*, and the* ***outer type*** *of A is* $\mathbf{OT(A)} = \mathbf{supCT(A)}$*.*

Proposition 1.0.37. *[10] A torsion-free finite rank module A over a Dedekind domain R is projective if and only if* $\mathbf{OT(A)} = \mathbf{t(R)}$*.*

Proof. Note that $\mathbf{t} \leq \mathbf{t(R)}$ is equivalent to $\mathbf{t} = \mathbf{t(R)}$. By corollary 1.0.35 if A is projective then $\mathbf{CT(A)} = \mathbf{t(R)}$, so $\mathbf{OT(A)} = \mathbf{t(R)}$. Conversely, if $\mathbf{OT(A)} = \mathbf{t(R)}$, then $\mathbf{CT(A)} = \mathbf{t(R)}$ so every rank-one homomorphic image of A is projective. By induction on the rank of A, we conclude that A is projective. $\square$

Chapter 2

Torsion-Free Modules of Finite Rank

Throughout this chapter, let R be an integral domain with field of quotients Q. The endomorphism ring $E(A) = E_R(A)$ of a R-module M is the R-module $\mathrm{Hom}(A, A) = \mathrm{Hom}_R(A, A)$ with composition of maps as multiplication. The quasi-endomorphism ring is $QE(A) = Q \otimes_R E(A)$. Our first results will explore some of the basic properties of torsion-free modules of finite rank over an integral domain. Because there are striking similarities to the situation in case of Abelian groups, we refer to that case instead of giving details whenever possible.

If A is a torsion-free module of finite rank n over R, then $A \subseteq Q^n$. Thus, $E(A)$ can be viewed as a subring of $Mat_n(Q)$, and the quasi-endomorphism ring of R is Artinian as a subring of $Mat_n(Q)$. In particular, there are primitive idempotents $e_1, \ldots, e_n$ of $QE(A)$ such that $1_A = e_1 + \ldots + e_n$. Thus,

$$A \doteq A_1^{k_1} \oplus \ldots \oplus A_n^{k_n}$$

where each A_i is a strongly indecomposable R-module and $A_i \sim A_j$ only if $i = j$. We refer the reader to the case of Abelian groups, observing that Jónsson's arguments about quasi-decompositions of torsion-free groups of finite rank carry over literally to our setting.

Theorem 2.0.1. *[10, Theorem 3.25] Let A be a torsion-free module of finite rank over an integral domain R. If $\gamma \in E(A)$, then $A \doteq H \oplus K$, where H and K are invariant under γ. Moreover, the restriction of γ to H is a quasi-automorphism of H, and $\gamma^n(K) = 0$ for some $n \geq 1$.*

Proof. The ascending chain

$$\ker\gamma \subseteq \ker\gamma^2 \subseteq \dots$$

of pure submodules of A has to stationary for some $n < \omega$. Let $K = \ker\ \gamma^n$ and $H = \gamma^n(A)$. It is easy to see that H and K are invariant under γ, and $\gamma^n(K) = 0$.

Furthermore, if $h \in H \cap \ker\ \gamma$, then there exists $a \in A$ such that $h = \gamma^n(a)$, and $\gamma^{n+1}(a) = \gamma(h) = 0$ so that $a \in \ker\ \gamma^{n+1} = \ker\ \gamma^n$. Thus, $h = \gamma^n(a) = 0$; and $H \cap \ker\gamma = 0$. Since the restriction of γ to H is monic, it is a left regular element of $QE(A)$. However, $QE(H)$ is a right and left Artinian ring, and left regular elements in such rings are units. Thus, $\gamma|H$ is a quasi-automorphism of H. If $\theta \in QE(H)$ is an inverse to the restriction of γ^n to H, then $\theta\gamma^n \in Q\mathrm{Hom}_R(A, H)$ and $\theta\gamma^n$ restricts to the identity on H. Hence, $\theta\gamma^n$ is a quasi-projection, and $A \doteq H \oplus \ker(\theta\gamma^n)$. But $\ker(\theta\gamma^n) = \ker\gamma^n = K$ as desired. □

Corollary 2.0.2. *A torsion-free module A of finite rank over an integral domain R is strongly indecomposable if and only if every (quasi-)endomorphism of A is either monic or belongs to $N(QE(A))$. In particular, $QE(A)$ is a local ring in this case.*

Proof. If every endomorphism of A is either monic or belongs to $N(QE(A))$, then $QE(A)$ cannot have non-trivial idempotents. If e were a non-trivial idempotent, then there would exist a nonzero $r \in R$ such that $re \in E(G)$ and re is neither monic nor nilpotent.

Conversely, consider $\gamma \in E(A)$. By the previous theorem, $A \doteq H \oplus K$, where H and K are invariant under γ, the restriction of γ to H is a quasi-automorphism of H, and $\gamma^n(K) = 0$ for some $n \geq 1$. Since A is strongly indecomposable, either γ is a quasi-automorphism of A or $A = \ker\ \gamma^n$. In the latter case, γ is nilpotent.

Furthermore, if γ is not a quasi-automorphism, then it cannot not be monic since left regular elements in left and right Artinian rings are units. Therefore, $\beta\gamma$ is not a quasi-automorphism for every $\beta \in QE(A)$. Hence, it must also be nilpotent. Thus, the left ideal generated by γ contains only nilpotent elements, and $\gamma \in N(QE(A))$. This shows that

$N(QE(A))$ is a maximal left ideal (and necessarily the unique one), since any element not in the nilradical is invertible in $QE(A)$. Thus $QE(A)$ is a local ring. □

We continue with a result which originates in [2, Theorem 9.10], but needs to be slightly modified to fit our setting.

Theorem 2.0.3. *Let R be an integral domain, and let*

$$A \doteq A_1^{n_1} \oplus \ldots \oplus A_m^{n_m}$$

be a torsion-free R-module of finite rank where each A_i is strongly indecomposable, and A_i is quasi-isomorphic to A_j iff $i = j$. Suppose that N is the nilradical of $E(A)$ and that J is the Jacobson radical of $QE(A)$.

a) $N = J \cap E(A)$; and N is nilpotent.

b) If T_i denotes the endomorphism ring of A_i for $i = 1, \ldots, m$, then

$$E(A)/N \doteq \prod_i Mat\,(T_i/N(T_i))\,.$$

c) If $A = A_1^{k_1} \oplus \ldots \oplus A_r^{k_r}$, then

$$N = \oplus_i N_i \oplus [\oplus_{j \neq i} \mathrm{Hom}_R(A_i^{k_i}, A_j^{k_j})]$$

where N_i denotes the nilradical of $E(A_i^{k_i})$.

Proof. a) Since $QE(A)$ is Artinian, J is nilpotent and $J \cap E(A) \subseteq N$. On the other hand, every nilpotent right ideal I of $E(A)$ gives rise to a nilpotent right ideal QI of $QE(A)$. Thus, $I \subseteq J \cap E(A)$; and $N \subseteq J \cap E(A)$. In particular, $N = J \cap E(A)$ is nilpotent.

b) Our arguments follow those of [2, Theorem 9.10]. Let $B = A_1^{n_1} \oplus \cdots \oplus A_m^{n_m}$. Note that $E(A) \doteq E(B)$ when viewed as subrings of $QE(A)$. Hence, $E(A)/N(E(A)) \doteq E(B)/N(E(B))$

as subrings of $QE(A)/J(QE(A))$. Thus, it is sufficient to prove

$$E(B)/N(E(B)) = \prod_i Mat_{n_i}\left(T_i/N(T_i)\right).$$

Represent $E(B)$ as a matrix ring of the form $\left(\mathrm{Hom}_R(A_i^{n_i}, A_j^{n_j})\right)_{i,j}$, and consider

$$I = (\oplus_i\{N(E(A_i^{n_i}))\}) \oplus \left(\oplus\{\mathrm{Hom}_R(A_i^{n_i}, A_j^{n_j})|i \neq j\}\right) \subseteq E(B).$$

It suffices to prove that I is an ideal of $E(B)$ and $I \subseteq N(E(B))$. In this case,

$$E(B)/I \simeq \prod_i \left(E(A_i^{n_i})/N(E(A_i^{n_i}))\right)$$

so that $I = N(E(B))$ since $N(E(B)/I) = 0$ and

$$\begin{aligned} E(B)/N(E(B)) &\simeq \prod_i \left(E(A_i^{n_i})/N(E(A_i^{n_i}))\right) \\ &\simeq \prod_i Mat_{n_i}\left(E(A_i)/N(E(A_i))\right) \\ &\simeq \prod Mat_{n_i}\left(T_i/N(T_i)\right), \end{aligned}$$

as needed.

To show that I is an ideal of $QE(A)$, let

$$f \in \mathrm{Hom}_R(A_i^{n_i}, A_j^{n_j}),$$

$$x \in N(E(A_k^{n_k})) \subseteq I$$

and

$$y \in \mathrm{Hom}_R(A_r^{n_r}, A_s^{n_s})$$

where $r \neq s$. Then, $fx \in I$, except possibly for the case that $i = j = k$. In this case, $fx \in N(E(A_i^{n_i})) \subseteq I$. Also, $fy \in I$, except possibly for the case that $s = i$ and $r = j$. In this case, $y : A_j^{n_j} \to A_i^{n_i}$ and f induces maps $A_j \to A_i \to A_j$. If this latter composite is nonzero, it must be an element of $N(T_j)$. Otherwise, the composite is a monomorphism, since A_j is strongly indecomposable so that A_j is a quasi-summand of A_i, contradicting the choice of the A_i's. Consequently,

$$fy \in \mathrm{Mat}_{n_j}(N(T_j)) = N(\mathrm{Mat}_{n_j}(T_j)) = N(E(A_j^{n_j})) \subseteq I.$$

Similarly, $xf \in I$ and $yf \in I$.

To show that $I \subseteq N(E(B))$, it suffices to prove that $I^{(l)} = 0$ for some l. If $x \in I^{(l)}$, then x is the sum of elements which are the composition of l morphisms $A_i^{n_i} \to A_j^{n_j}$ for $i \neq j$ and morphisms in $N(E(A_i^{n_i}))$. Choose k with $N(E(A_i^{n_i}))^(k) = 0$ for $1 \leq i \leq m$. Choose l large enough so that any composition of l morphisms, as described above, has some subscript repeated at least k times. If $A_i^{n_i} \to A_j^{n_j} \to \ldots \to A_i^{n_i}$ is a repetition of the subscript i then, as above, the composition must be in $N(E(A_i^{n_i}))$. Consequently, $I^{(l)} = 0$.

c) is a direct consequence of b). □

Chapter 3

The Covariant Case

In this section we present a numerical set of invariants that serve as complete set of quasi-isomorphism invariants between modules A and B. These invariants are closely linked to the structure of *Ext* ([7]). A module is called *reduced* when its maximal divisible submodule is zero. If A and M are R-modules, then the *A-socle $S_A(M)$ of M* is $S_A(C) = \Sigma_{f \in Hom(A,C)} f(A)$. Finally, if U is a submodule of a torsion-free module M, then $U_* = \{x \in M \mid xr \in U \text{ for some } 0 \neq r \in R\}$.

Proposition 3.0.1. *[10] Let A and B be torsion-free finite rank modules over a Dedekind domain R.*

(a) If $A \subseteq B$, then $\mathbf{OT(A)} \leq \mathbf{OT(B)}$.

(b) For any $\mathbf{t} \in \mathbf{T(A)}$, $\mathbf{t} \leq \mathbf{OT(A)}$.

(c) If $B \lhd A$, then $\mathbf{OT(A/B)} \leq \mathbf{OT(A)}$.

(d) $\mathbf{OT(A)}$ is P-divisible if and only if $r_P(A) < rank(A)$.

Proof. (a) If φ maps B onto a rank-one module M, then φ extends to a map $\varphi' : A \to QM$ and $\mathbf{t(A)} = \mathbf{t}(\varphi(\mathbf{B})) \leq \mathbf{t}(\varphi'(\mathbf{A})) \in \mathbf{CT(A)}$. Thus every type in $\mathbf{CT(B)}$ is less than or equal to a type in $\mathbf{CT(A)}$, so $\mathbf{OT(B)} \leq \mathbf{OT(A)}$.

(b) If M is a rank 1 pure submodule of A, then $\mathbf{t(M)} = \mathbf{OT(M)} \leq \mathbf{OT(A)}$ by (a).

(c) If M is a homomorphic image of A/B, then M is also a homomorphic image of A, so $\mathbf{CT(A/B)} \subseteq \mathbf{CT(A)}$ and $\mathbf{OT(A/B)} = \mathbf{sup}(\mathbf{CT(A/B)}) \leq \mathbf{sup}(\sup \mathbf{CT(A)}) = \mathbf{OT(A)}$.

(d) $r_P(A) \leq rank(A)$ if and only if A has a homomorphic image which is P-divisible. This is the case if and only if A has a rank 1 P-divisble homomorphic image (since the homomorphic image of a P-divisble module is P-divisble) if and only if $\mathbf{CT(A)}$ contains a P-divisble type. $\mathbf{OT(A)}$ is the least upper bound of a finite subset of $\mathbf{CT(A)}$, hence $\mathbf{OT(A)}$ is P-divisble if and only if some element of $\mathbf{CT(A)}$ is P-divisible. Thus $r_P(A) < rank(A)$ if and only if $\mathbf{OT(A)}$ is P-divisble. □

Proposition 3.0.2. *[10] Let A be a torsion-free finite rank module over a Dedekind domain R, and let $r = rank(A)$. If $\mu_1, \ldots, \mu_n$ is a maximal linearly independent set in* $\operatorname{Hom}(A, Q)$*, then* $\mathbf{OT(A)} = \mathbf{sup}\{\mathbf{t}(\mu_\mathbf{1}(\mathbf{A})), \ldots, \mathbf{t}(\mu_\mathbf{n}(\mathbf{A}))\}$.

Proof. We first have $\mathbf{OT(A)} \geq \mathbf{sup}\{\mathbf{t}(\mu_\mathbf{1}(\mathbf{A})), \ldots, \mathbf{t}(\mu_\mathbf{n}(\mathbf{A}))\}$. Since $\mu_1, \ldots, \mu_n$ form a basis for $\operatorname{Hom}(A, Q)$, for any $\mu \in \operatorname{Hom}(A, Q)$, there exist $r, r_1, \ldots, r_n \in R$ such that $r\mu = \sum_{i=1}^n r_i\mu_i$. Then $r\mu(A)$ is contained in the submodule of Q generated by $\mu_1(A), \ldots, \mu_n(A)$, and we have $\mathbf{t}(\mu(\mathbf{A})) = (\mathbf{r}\mu(\mathbf{A})) \leq \mathbf{sup}\{\mathbf{t}(\mu_\mathbf{i}(\mathbf{A}))|\mathbf{i} = \mathbf{1}, \ldots \mathbf{n}\}$. Thus

$$\mathbf{OT(A)} \leq \mathbf{sup}\{\mathbf{t}(\mu_\mathbf{1}(\mathbf{A})), \ldots, \mathbf{t}(\mu_\mathbf{n}(\mathbf{A}))\}.$$

The result follows. □

Lemma 3.0.3. *Let A and B be torsion-free finite rank modules over a Dedekind domain R. If $B/S_A(B)$ and $A/S_B(A)$ are torsion, then A and B share a nonzero quasi-summand.*

Proof. To simplify our notation, denote $E(A)$ by E and $Nil(E)$ by N. By Theorem 2.0.3, the Jacobson radical J of QE satisfies $N = J \cap E$. Moreover, J and N are nilpotent since QE is left Artinian. Let $N_* = \langle NA \rangle_*$ be the pure submodule of A generated by $g(A)$ for all $g \in N$. If $N^n = 0$ and $N^{n-1} \neq 0$, then $N^{n-1}N_* = 0$ implies $A/N_* \neq 0$. Since A/N_* is torsion-free and $(S_B(A) + N_*)/N_*$ is full in A/N_* by hypothesis, $S_B(A) \not\subseteq N_*$. Therefore, there is an $f : B \to A$ with $f(B) \not\subseteq N_*$.

We may write $B \sim B_1 \oplus \cdots \oplus B_k$ with each B_i strongly indecomposable by Theorem 2.0.3. Clearly,

$$S_A(B) \sim S_A(B_1) \oplus \cdots \oplus S_A(B_k).$$

For some i, $f(S_A(B_i)) \not\subseteq N_*$ since otherwise $\sum_{i=1}^k f(S_A(B_i)) \subseteq N_*$ implying $rf(S_A(B)) \subseteq N_*$ for some nonzero $r \in R$. Therefore, we would obtain $f(S_A(B)) \subseteq N_*$. But, for any $b \in B$, there is a $0 \neq \ell \in R$ with $\ell b \in S_A(B)$ by the hypothesis, so that $f(\ell v) = \ell f(v) \in N_*$. By the purity of N_*, this implies $f(v) \in N_*$, contradicting that $f(B) \not\subseteq N_*$. We may assume there is a map $g : A \to B_1$ such that $fg(A) \not\subseteq N_*$.

From the definition of N_*, we have $N \leq \mathrm{Hom}(A, N_*)$, so that $fg \notin N$. Now E/N is a full subring of the semi-simple ring QE/J so there are $h, h^{'} \in E$ such that $e = (hf)(gh^{'})$ is not nilpotent mod N because QE/J is a direct product of matrix rings. Relabel hf and $gh^{'}$ as f and g respectively, and consider the restriction $f : B_1 \to A$.

We now have $gf \in E(B_1)$. As in Section 3, we use the fact that B_1 is strongly indecomposable to obtain that $\alpha = gf$ is invertible in $QE(B_1)$ or α is nilpotent. If $(gf)^n = 0$, then $e^{n+1} = f(gf)^n g = 0$, a contradiction. So α must be invertible. Consequently, there is $0 \neq s \in R$ such that $s\alpha^{-1} \in E(B_1)$ and $s1_{B_1} = s\alpha^{-1} gf$. Call $g^{'} = s\alpha g$.

Any $a \in A$ satisfies

$$sa = sa - f(g^{'}(a)) + f(g^{'}(a)).$$

Because $sa - f(g^{'}(a)) \in \ker(g^{'})$, we have $A \sim A^{'} \oplus \ker(g^{'})$. Since f is a monomorphism, $A^{'} \cong B_1$. □

The following result was originally shown by Beaumont and Pierce in the case that A is a subring of a finite dimensional $\mathbb{Q}$-algebras, but it carries over to torsion-free finite rank rings over integral domains:

Theorem 3.0.4. *([3, Theorem 1.4] and [10, Proposition 7.21]) Let R be an integral domain whose field of quotients Q is a perfect field. Let A be a torsion-free free R-algebra which has finite rank as an R-module. Let $QA = S_1 \oplus N_1$ where $N_1 = N(QA)$ and S_1 is a subring of QA. Then, $N(A) = N_1 \cap A$ and $A \doteq (S_1 \cap A) \oplus (N_1 \cap A)$.*

Observe that Q is a perfect field whenever R^+ is torsion-free as an Abelian group.

Theorem 3.0.5. *Let A and B be reduced torsion-free finite rank modules over a Dedekind domain R with field of quotients Q.*

a) If $A \sim B$, then $r_P(\mathrm{Hom}(A, C)) = r_P(\mathrm{Hom}(B, C))$ for all primes P and all finite rank modules C.

b) If Q is perfect, then $A \sim B$ whenever $r_P(\mathrm{Hom}(A, C)) = r_P(\mathrm{Hom}(B, C))$ for all primes P and all finite rank modules C.

Proof. a) Since $\mathrm{Hom}(A, C)$ is quasi-isomorphic to $\mathrm{Hom}(B, C)$ and r_P is quasi-isomorphism invariant, the result follows.

b) We will show that $S_A(B)$ is full in B; the result follows from Lemma 3.0.3 via induction on the rank of A.

Let B_1 be a pure, strongly indecomposable quasi-summand of B and $S_1 = S_A(B_1)_*$. To simplify the argument, we may assume that B_1 is a summand of B, say $B = B_1 \oplus K$, since we can replace B by a module quasi-isomorphic to it. Consider the exact sequences

$$0 \to \mathrm{Hom}(A, S_1) \to \mathrm{Hom}(A, B_1) \xrightarrow{\alpha} \mathrm{Hom}(A, B_1/S_1)$$

and

$$0 \to \mathrm{Hom}(B, S_1) \to \mathrm{Hom}(B, B_1) \xrightarrow{\beta} \mathrm{Hom}(B, B_1/S_1).$$

By the definition of $S_A(B_1)$, we have $im\ \alpha = 0$. By the hypothesis, $r_P(im\ \beta) = r_P(\mathrm{Hom}(B, B_1)) - r_P(\mathrm{Hom}(B, S_1)) = r_P(\mathrm{Hom}(A, B_1)) - r_P(\mathrm{Hom}(A, S_1)) = r_P(im\alpha) = 0$ for all P. Thus, $im\ \beta$ is divisible by [10]. Moreover, $\ker\ \beta = \mathrm{Hom}(B_1, S_1) \oplus \mathrm{Hom}(K, S_1)$ is a pure submodule of $\mathrm{Hom}(B, B_1) = \mathrm{Hom}(B_1, B_1) \oplus \mathrm{Hom}(K, B_1)$ with a divisible cokernel. Hence, $\mathrm{Hom}(B_1, B_1)/\mathrm{Hom}(B_1, S_1)$ is divisible as a direct summand of a divisible module.

Let R_1 denote the nilradical of $E(B_1)$ and $N_1 = (R_1 C_1)_* \leq B_1$. As in the previous lemma, $N_1 \neq B_1$. As mentioned before, every endomorphism of B_1 is either in R_1 or else is a monomorphism. Hence, $R_1 = \mathrm{Hom}(B_1, N_1)$. Therefore, if $\mathrm{Hom}(B_1, S_1) \not\subseteq \mathrm{Hom}(B_1, N_1) =$

R_1, then there is a monomorphism $f : B_1 \to B_1$ with $im\ f \leq S_1$. In this case, $rank(B_1) = rank(S_1)$ implies $S_1 = B_1$ since S_1 is pure in B_1. We now show that the case $\mathrm{Hom}(B_1, S_1) \subseteq R_1$ is not possible.

Suppose $I = \mathrm{Hom}(B_1, S_1) \subseteq \mathrm{Hom}(B_1, N_1) = R_1$. From above, we have that $E(B_1)/I$ is divisible. Consequently, $E(B_1)/R_1$ is divisible. By Theorem 3.0.4, $E(B_1)/R_1$ is a quasi-summand of $E(B_1)$. But $E(C_1)$ is reduced, a contradiction. Thus, $I \not\subseteq R_1$.

Write $B \sim B_1 \oplus \cdots \oplus B_k$ for strongly indecomposable B_i. Then $\langle S_A(B_i)\rangle_* = C_i$ implies $\langle S_A(B)\rangle_* = B$. Therefore, $S_A(B)$ is full in B and by symmetry $S_C(A)$ is full in A. From Lemma 3.0.3, A and B share a nonzero quasi-summand, say $A \sim M \oplus A'$ and $B \sim M \oplus B'$ with $M \neq 0$. Then

$$\begin{aligned} r_P(\mathrm{Hom}(A, C)) &= r_P(\mathrm{Hom}(G, C)) + r_P(\mathrm{Hom}(A', C)) \\ &= r_P(\mathrm{Hom}(G, C)) + r_P(\mathrm{Hom}(B', C)) \\ &= r_P(\mathrm{Hom}(B, C)) \end{aligned}$$

for all P and all C. Thus

$$r_P(\mathrm{Hom}(A', C)) = r_P(\mathrm{Hom}(B', C))$$

for all P and C. The result follows by induction on $rank(A)$. □

Lemma 3.0.6. *Let R be a Dedekind domain, $P \in spec(R)$, and A a torsion-free R-module of finite rank. Then, $Ext(A, R_P) = 0$ if and only if $OT(A) \leq type(R_P)$.*

Proof. Suppose $\mathrm{Ext}(A, R_P) = 0$. For a pure corank 1 submodule U of A, consider the exact sequence

$$0 \to U \to A \to A/U \to 0.$$

We apply the functor $\mathrm{Hom}_R(\_, R_P)$ to get the induced sequence

$$0 \to \mathrm{Hom}(A/U, R_P) \to \mathrm{Hom}(A, R_P) \to \mathrm{Hom}(U, R_P)$$

$$\to Ext(A/U, R_P) \to Ext(A, R_P) = 0.$$

Since $\mathrm{Hom}(U, R_P)$ is a finite rank R-module, $\mathrm{Ext}(A/U, R_P)$ has finite rank too. We want to show $\mathrm{Hom}(A/U, R_P) \neq 0$ from which we obtain $\mathbf{t}(A/U) \leq \mathbf{t}(R_P)$. Consequently, $OT(A) \leq \mathbf{t}(R_P)$.

If $\mathrm{Hom}(A/U, R_P) = 0$, then A/U is P-divisible by [10]. We consider the sequence $0 \to A/U \to Q \to D \to 0$ where D is divisible and torsion with $D[P] = 0$ since A/U is P-divisible. It induces

$$0 = \mathrm{Ext}(D, R_P) \to \mathrm{Ext}(Q, R_P) \to \mathrm{Ext}(A/U, R_P) \to 0.$$

Observe that $\mathrm{Ext}(Q, R_P) = \mathrm{Ext}_{R_P}(Q, R_P)$. Considering the sequence $0 \to R_P \to Q \to Q/R_P \to 0$ of R_P-modules, we obtain the sequences

$$0 = \mathrm{Hom}_{R_P}(Q, R_P) \to \mathrm{Hom}_{R_P}(R_P, R_P)$$

$$\to \mathrm{Ext}_{R_P}(Q/R_P, R_P) \to Ext_{R_P}(Q, R) \to 0$$

and

$$0 = \mathrm{Hom}_{R_P}(Q/R_P, Q) \to \mathrm{Hom}_{R_P}(Q/R_P, Q/R_P)$$

$$\to \mathrm{Ext}_{R_P}(Q/R_P, R_P) \to \mathrm{Ext}_{R_P}(Q/R_P, Q) = 0.$$

Since R_P is a discrete valuation domain such that $\hat{R_P}$ has infinite rank as an R_P-module, we obtain that

$$\mathrm{Ext}_{R_P}(Q, R_P) = \mathrm{Ext}(Q, R_P) \cong \mathrm{Ext}(A/U, R_P)$$

has infinite rank contradicting what has already been shown.

Conversely, observe that $OT(A) \leq \mathbf{t}(R_P)$ implies that there exists an exact sequence

$$0 \to A \to \oplus_n R_P \ (E).$$

To see this, let $F = \oplus_{i=1}^n x_i R \subseteq A$ be a free submodule such that A/F is torsion. For $i = 1, \ldots, n$, consider the pure corank 1 submodule $U_i = (\oplus j \neq i x_i R)_*$ of A. Since $\mathbf{t}(A/U_i) \leq \mathbf{t}(R_P)$, there exists a map $\varphi_i : A \to R_P$ with $\varphi_i(U_i) = 0$ and $\varphi(x_i) \neq = 0$. Define $\varphi : A \to R_P^n$ by $\varphi(a) = (\varphi_1(a), \ldots, \varphi_n(a))$. If $\varphi(a) = 0$, then

$$ra = r_1 x_1 + \ldots + r_n x_n \in F$$

for some nonzero $r \in R$ since A/F torsion. But, $\varphi(a) = 0$ implies $\varphi_i(a) = 0$ for all i. Hence $0 = \varphi_i(ra) = \varphi_i(r_1 x_1 + \ldots + r_n x_n) = \varphi_i(r_i x_i)$ But, $\varphi_i(x_i) \neq 0$ yields $r_i = 0$. Hence φ is a monomorphism.

The sequence E induces

$$0 = \mathrm{Ext}(\oplus_n R_P, R_P) \to \mathrm{Ext}(A, R_P) \to 0$$

because

$$\mathrm{Ext}(\oplus_n R_P, R_P) \cong \mathrm{Ext}_{R_P}(\oplus_n R_P, R_P) = 0.$$

□

Lemma 3.0.7. *Let R be a Dedekind domain, and X a rank 1 R-module. A torsion-free R-module A of finite rank satisfies $OT(A) \leq \mathbf{t}(X)$ if and only if* $\mathrm{rank}(\mathrm{Hom}(A, X)) = \mathrm{rank}(A)$.

Proof. Suppose $OT(A) \leq \mathbf{t}(X)$, and consider a free submodule $F = \oplus x_i R$ of A such that A/F is torsion. If $U_i = (\oplus_{j \neq i} x_i R)_*$, then

$$\mathbf{t}(A/U_i) \leq OT(A) \leq \mathbf{t}(X)$$

yields that there exists $0 \neq \varphi_i : A \to R_{p_i}$ with $\varphi_i(U_i) = 0$. To see that $\{\varphi_i, \ldots, \varphi_n\}$ is R-independent, suppose $r_1\varphi_1 + \cdots + r_n\varphi_n = 0$. For $i = 1, \ldots, n$, we have $0 = r_1\varphi_1(x_i) + \cdots + r_n\varphi_n(x_i) = r_i\varphi_i(x_i)$. Since $\varphi_i(x_i) \neq 0$, we have $r_i = 0$.

Conversely, suppose that $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(A, X)$ linearly independent where $n = \mathrm{rank}(A)$. Since $\mathrm{rank}(\mathrm{Hom}(A, Q)) = n = \mathrm{rank}(\mathrm{Hom}(A, X)$, we obtain

$$OT(A) = \{\mathbf{t}(\alpha_1(A)), \ldots, \mathbf{t}(\alpha_n(A))\} \leq \mathbf{t}(X)$$

by Proposition 3.0.2. $\square$

Theorem 3.0.8. *Let A and B finite rank torsion-free modules over a Dedekind domain R such that $\hat{R}_P$ has infinite rank over R for all $P \in spec(R)$.*

a) If $\mathrm{Ext}(A, C) \cong \mathrm{Ext}(B, C)$ for all torsion-free R-modules C of finite rank, then $A \cong P_1 \oplus A_1 \oplus D_1$ and $B \cong P_2 \oplus B_1 \oplus D_2$ where $OT(A) = OT(B)$, P_1 and P_2 are finitely generated projective, D_1 and D_2 are torsion-free divisible of finite rank, and A_1 and B_1 are reduced with $r_P(A_1) = r_P(B_1)$ and $r_P(\mathrm{Hom}_R(A_1, C)) = r_P(\mathrm{Hom}_R(B_1, C))$ for all $P \in Spec(R)$ and all torsion-free finite rank modules C. Moreover, if Q is a perfect field, then $A_1 \sim B_1$.

b) If $OT(A) = OT(B)$ and $A \cong P_1 \oplus A_1 \oplus D_1$ and $B \cong P_2 \oplus B_1 \oplus D_2$ where P_1 and P_2 are finitely generated projective, D_1 and D_2 are torsion-free divisible of finite rank, and $A_1 \sim B_1$ are reduced, then $\mathrm{Ext}(A, C) \cong \mathrm{Ext}(B, C)$ for all torsion-free R-modules C of finite rank.

Proof. a) Since every finite rank torsion-free R-module X can be written as $X = P \oplus Y$ with P projective and $\mathrm{Hom}_R(Y, P) = 0$, it suffices to consider the case that

$$\mathrm{Hom}_R(A, R) = \mathrm{Hom}_R(B, R) = 0.$$

In particular, $\text{Ext}(A,C) \cong \text{Ext}(B,C)$ for all C of rank at most n. Write $A \cong A_1 \oplus D_1$ and $B \cong B_1 \oplus D_2$ where D_1 and D_2 are torsion-free divisible of finite rank. Since $\text{Ext}_R(D_i, C)$ is torsion-free divisible, we have $r_P(\text{Ext}_R(A_1,C)) = r_P(\text{Ext}_R(A,C)) = r_P(\text{Ext}_R(B,C)) = r_P(\text{Ext}_R(B_1,C))$. Since we only consider the P-ranks of Ext_R in the following, we may assume that A and B are reduced.

We know that, for all torsion-free modules X and Y of finite rank,

$$r_p(\text{Ext}(X,Y)) = r_p(X)r_p(Y) - r_p(\text{Hom}(X,Y))$$

as was shown shown in another paper. Using this for A and B and a module C having rank $\leq n$, we get

$$r_p(A)r_p(C) - r_p(\text{Hom}(A,C)) = r_p(B)r_p(C) - r_p(\text{Hom}(B,C)).$$

Because $\text{Hom}(A,R) = \text{Hom}(B,R) = 0$, we have

$$r_p(\text{Hom}(A,R)) = r_p(\text{Hom}(B,R)) = 0$$

and

$$r_p(A) = r_p(A)r_p(R) = r_p(B)r_p(R) = r_p(B).$$

But then, $r_p(\text{Hom}(A,C)) = r_p(\text{Hom}(B,C))$ using the above formula. By Theorem 3.0.5, A and B are quasi-isomorphic.

Assume not both $OT(A)$ and $OT(B)$ are the type of Q. Without loss of generality, we may assume $\tau = OT(A) \neq \mathit{type}(Q)$. Observe that there has to be $P \in \mathit{Spec}(R)$ such that no rank 1 quotient of A is P-divisible. If we could find a rank 1 quotient of A for every P which is P-divisible, then its type at that prime would be infinite. Then the sup of the types of the rank 1 quotients of A would be infinite for all primes, and $OT(A) = \mathit{type}(Q)$, a contradiction. In particular, $A/U \subseteq R_P$ for all pure corank 1 submodules U of A.

Observe that $OT(X) \leq type(R_P)$ if and only if $X \neq PX$ for all finite rank torsion-free modules X. By Lemma 3.0.6, $\text{Ext}(A, R_p) = 0$ if and only if $OT(A) \leq type(R_P)$. Since the Ext-modules are isomorphic, $\text{Ext}(B, R_p) = 0$, and we obtain $OT(B) \leq type(R_p)$ using Lemma 3.0.6 once more. Moreover , $r_P(A) = rank(A)$ and $r_P(B) = r_0(B)$ by [10, Proposition 2.34]. If X is a rank 1 module of type τ, then $\text{Hom}(A, X)$ can be embedded into a finite direct sum of copies of X and $r_0(\text{Hom}(A, X)) = r_0(A)$ by Lemma 3.0.7. By [10, Proposition 2.34], $OT(\text{Hom}_R(A, X) \leq \tau$. Another application of [10, Proposition 2.34] yields

$$r_p(Hom(A, X)) = r_0(\text{Hom}(A, X)) = rank(A) = r_p(A).$$

Since

$$OT(\text{Hom}_R(B, X) \leq \tau \leq type(R_P),$$

we obtain $r_p(\text{Hom}_R(B, X) = r_0(\text{Hom}(B, X))$. On the other hand

$$r_P(A)r_P(X) - r_P(\text{Hom}(A, X)) = r_P(B)r_P(X) - r_P(\text{Hom}(B, X))$$

and $r_P(A) = r_P(B)$ yield $r_P(\text{Hom}(A, X)) = r_P(\text{Hom}(B, X))$. Thus,

$$\begin{aligned} r_P(B) &= r_P(A) = r_P(\text{Hom}_R(A, X)) \\ &= r_P(\text{Hom}_R(B, X)) = r_0(\text{Hom}_R(B, X)) \\ &\leq r_0(B) = r_P(B). \end{aligned}$$

In particular, $r_0(\text{Hom}(B, X)) = r_0(B)$. Another application of Lemma 3.0.7 yields $OT(B) \leq \tau = OT(A) < type(Q)$. By symmetry, $OT(A) = OT(B)$.

b) Standard homological arguments show that

$$r_0(\text{Ext}_R(A, C)) = r_0(\text{Ext}(Q, C)$$

is infinite for all reduced torsion-free groups C of finite rank if $OT(A) = type(Q)$. To see this, observe that there exists an exact sequence $0 \to U \to A^n \to Q \to 0$ for some $n < \omega$. It induces

$$\mathrm{Hom}(U, C) \to \mathrm{Ext}_R(Q, C) \to \mathrm{Ext}_R(A^n, C).$$

Since $\hat{R_P}$ has infinite rank for all $P \in spec(R)$ and C is not algebraically compact, we have $0 < r_0(\mathrm{Ext}(Q, C))$ is infinite. Since $\mathrm{Hom}(U, C)$ has finite rank,

$$r_0(\mathrm{Ext}_R(A, C)) = r_0(\mathrm{Ext}(Q, C).$$

Since $OT(A) = OT(B)$, $r_0(\mathrm{Ext}(B, C)) = r_0(\mathrm{Ext}(Q, C)$ is also infinite. Since $r_0(\mathrm{Ext}(D_i, C)) = r_0(\mathrm{Ext}(Q, C)$ is infinite, we obtain that $\mathrm{Ext}(A, C)$ and $\mathrm{Ext}(B, C)$ have the same infinite torsion-free rank in this case too. On the other hand, the P-ranks of the Ext-modules are determined completely by A_1 and B_1. Since $A_1 \sim B_1$, the P-ranks have to coincide. On the other hand, if $OT(A) = OT(B) < type(Q)$, then $D_1 = D_2 = 0$. Since the Ext-modules are divisible, their structure is completely determined by their torsion-free and their P-ranks. $\square$

Chapter 4

The Contravariant Case

If R_P is complete in the P-adic topology for some $P \in spec(R)$, then $\mathrm{Ext}^1_{R_P}(A, R_P) \cong \mathrm{Ext}_{R_P}(B, R_P) = 0$ for all torsion-free R_P-modules A and B. In particular A and B need not be quasi-isomorphic. We continue our discussion by showing that the discussion of the isomorphism of Ext-modules restricts to the case that R is a Dedekind domain such that $\hat{R_P}$ has infinite rank for all $P \in spec(R)$:

Proposition 4.0.1. *The following conditions are equivalent for a Noetherian integral domain R with field of quotients Q such that R_P is not complete in the P-adic topology for any $P \in spec(R)$:*

- *a) R is a Dedekind domain such that $\hat{R_P}$ has infinite rank for all $P \in spec(R)$.*
- *b) If M and N are quasi-isomorphic torsion-free R-modules of finite rank and D_1 and D_2 are torsion-free divisible of finite rank, then $\mathrm{Ext}(M \oplus D_1, A) \cong \mathrm{Ext}(N \oplus D_2, A)$ for all torsion-free R-modules A.*

Proof. $a) \rightarrow b)$: Observe that $\mathrm{Ext}(M, A)$ is divisible if M is torsion-free and R is a Dedekind domain. However, quasi-isomorphic divisible modules over Dedekind domains are isomorphic. Moreover, consider the exact sequence

$$0 \rightarrow \mathrm{Hom}(R, R) \rightarrow \mathrm{Ext}(Q/R, R) \rightarrow \mathrm{Ext}(Q, R) \rightarrow 0.$$

If we can show that $\mathrm{Ext}(Q/R, R)$ has infinite torsion-free rank, then $\mathrm{Ext}(Q^n, A) \cong \mathrm{Ext}(Q^m, A)$ for all $n, m < \omega$. However, the Ext-module fits into the exact sequence

$$0 \rightarrow \mathrm{Hom}(Q/R, Q/R) \rightarrow \mathrm{Ext}(Q/R, R) \rightarrow 0$$

from which we obtain

$$\text{Ext}(Q/R, R) \cong \Pi_{P \in spec(R)} End_R(E(R/P)).$$

However, $End_R(E(R/P)) \cong \hat{R_P}$ by [10, Proposition 0.83]. By a), $\text{Ext}(Q/R, R)$ has infinite rank.

b) $\to$ *a)*: Let I be a nonzero ideal of R. Since $I \sim R$, we obtain $\text{Ext}(I, M) \cong \text{Ext}(R, M) = 0$ for all torsion-free modules R-modules M of finite rank. We consider an exact sequence $0 \to U \to F \to I \to 0$ in which F is finitely generated free. Since U has finite rank, $\text{Ext}(I, U) \cong \text{Ext}(R, R) = 0$. Thus, the sequence splits, and I is projective.

Let $P \in spec(R)$, and assume that $\text{rank}(\hat{R_P}) < \infty$. Arguing as in *a)* $\to$ *b)* with R_P replacing R, we obtain that $\text{Ext}(Q, R_P)$ is an epimorphic image of $\text{Ext}(Q/R_P, R_P) \cong End(Q/R_P) \cong \hat{R_P}$. Thus, $0 < \text{rank}(\text{Ext}(Q, R_P)) < \infty$ observing that R_P is no complete in the P-adic topology. But then

$$\text{rank}(\text{Ext}^1_R(Q, R_P)) < \text{rank}(\text{Ext}^1_R(Q \oplus Q, R_P))$$

contradicting b). □

If A and B are torsion-free finite rank R-modules over an integral domain, then $A[B] = \cap\{ker(f) | f \in \text{Hom}_R(A, B)\}$ denotes the B-radical of A. In particular, if $A[B] = 0$, then A can be viewed as a submodule of B^n for some n.

Theorem 4.0.2. *Let A and B be torsion-free modules of finite rank over an integral domain R. If $A[B] = 0$ and $B[A] = 0$, then $A \doteq A_1 \oplus A_2$ and $B \doteq B_1 \oplus B_2$ such that A_1 and B_1 are nonzero and strongly indecomposable and $A_1 \sim B_1$.*

Proof. We consider the two-sided ideal $S = \text{Hom}(B, A)\text{Hom}(A, B)$ of $E(A)$. For $0 \neq a \in A$, there exists $f : A \to B$ with $f(a) \neq 0$ since $A[B] = 0$. Similarly, $B[A] = 0$ yields that we can find $g : B \to A$ with $gf(a) \neq 0$. In particular, S cannot be contained in N. To see this, choose $k > 0$ such that $N^k = 0$ but $N^{k-1} \neq 0$, and select $0 \neq x \in N^{k-1}A$. By what has been shown, there is $s \in S$ with $0 \neq sx$. If $S \subseteq N$, then $sx \in SN^{k-1} \subseteq N^k A = 0$, a contradiction.

By Theorem 2.0.3, $J \cap E(A) = N$ so that

$$E(A)/N = E(A)/J \cap E(A) \cong [E(A) + J]/J$$

can be viewed as a subring of the semi-simple ring $QE(A)/J$. As mentioned before, $A \sim A_1^{k_1} \oplus \cdots \oplus A_r^{k_r}$ such that each A_i strongly indecomposable and A_i is not quasi-isomorphic to A_j if $i \neq j$. Without loss of generality, we may assume $A = A_1^{k_1} \oplus \ldots \oplus A_r^{k_r}$. By Theorem 2.0.3, we obtain

$$N = \oplus_i N_i \oplus [\oplus_{j \neq i} \mathrm{Hom}_R(A_i^{k_i}, A_j^{k_j}]$$

where N_i denotes the nilradical of $E(A_i^{k_i})$. Hence,

$$E(A)/N \cong \prod_i \mathrm{Mat}_{k_i}(R_i) = T$$

where $R_i = E(A_i)/N(E(A_i))$.

Let $\sigma \in S \backslash N$, and write $\sigma = \beta\alpha$ for $\alpha \in \mathrm{Hom}_R(A, B)$ and $\beta \in \mathrm{Hom}_R(B, A)$. Identifying $\sigma + N$ with its image in the ring T under the previous ring isomorphism, we obtain that one of the components of $\sigma + N$ in T is nonzero. Without loss of generality, we may assume that the numbering of $\{A_1, \ldots, A_n\}$ has been chosen in such a way that say the first component is nonzero.

We write $C = A_1^{k_1}$,and let $\delta : C \to A$ and $\pi : A \to C$ be the natural maps associated with the given decomposition of A. Then, we obtain that $\pi\sigma\delta = (\pi\beta)(\alpha\delta) \in \mathrm{Hom}_R(B, C)\mathrm{Hom}_R(C, B)$ is not an element of $N(E(C))$ by what just has been shown. Since $E(C)) = \mathrm{Mat}_{k_1}(A_1)$, some (i, j)-entry of $\pi\beta\alpha\delta$ is a non-nilpotent endomorphism of A_1. Let γ_j be the projection onto the j^{th}-coordinate and λ_i be the embedding into the i^{th}-coordinate.

Then

$$\iota = (\gamma_j \pi \beta)(\alpha \delta \lambda_i) \in \mathrm{Hom}_R(B, A_1)\mathrm{Hom}_R(A_1, B)$$

does not belong to $NE(A_1)$. As in the proof of Theorem 2.0.3, this means that $\iota \notin J(QE(A_1))$. Hence, ι is invertible in $QE(A_1)$ since the latter is a local ring. We can find $0 \neq r \in R$ and $\eta \in E(A_1)$ such that $\eta\iota$ and $\iota\eta$ are multiplication by r on A_1. Thus, $\eta\gamma_j\pi\beta : B \to A_1$ and $\alpha\delta\lambda_i : A_1 \to B$ satisfy

$$(\eta\gamma_j\pi\beta)(\alpha\delta\lambda_i) = r1_{A_1}.$$

Thus, B has a quasi summand isomorphic to A_1 □

Observe that all nonzero prime ideals of a Dedekind domain R are maximal. We let *spec*(R) denote the collection of maximal ideals of R in this case. In particular, we can define the P-rank of a torsion-free R-module A as the composition length of the module A/PA [10]. We refer the reader to [10] and [1] for details on the P-rank of a module.

Lemma 4.0.3. *If A and B are quasi-isomorphic torsion-free modules of finite rank over a Dedekind domain R, then $r_P(A) = r_P(B) < \infty$ for all $P \in spec(R)$.*

Proof. [10, Proposition 1.26] yields $r_P(A) \leq \mathrm{rank}(A) < \infty$ and $r_P(A) \leq r_P(U)$ whenever A is a finite rank module over a Dedekind domain R and U is an essential submodule of A. Hence, $r_P(A) \leq r_P(B)$ and vice-versa. □

Theorem 4.0.4. *Let A and B be torsion-free reduced modules of finite rank over a Dedekind domain R. Then*

$$r_P(\mathrm{Hom}_R(C, A)) = r_P(\mathrm{Hom}_R(C, B))$$

for all $P \in spec(R)$ and all torsion-free modules C of finite rank if and only if $A \sim B$.

Proof. Suppose that A and B are quasi-isomorphic. Since the P-rank of a module is a quasi-isomorphism invariant, and $\mathrm{Hom}_R(C, A)$ and $\mathrm{Hom}_R(C, B)$ are quasi-isomorphic modules, their P-ranks are the same for all $P \in spec(R)$ and torsion-free modules C of finite rank.

Conversely, we show that A is quasi-isomorphic to B if

$$r_P(\mathrm{Hom}_R(C, A)) = r_P(\mathrm{Hom}_R(C, B))$$

for all $P \in spec(R)$ and any torsion-free homomorphic image C of A or B. For such modules C, we obtain that

$$r_P(\mathrm{Hom}_R(C, A)) = r_P(\mathrm{Hom}_R(C, B))$$

is finite for all P since the homomorphism modules have finite torsion-free rank.

Let $A^{'} = A[B]$ and consider the exact sequence

$$0 \to A^{'} \to A \to A/A^{'} \to 0.$$

It induces the exact sequences

$$0 \to \mathrm{Hom}_R(A/A^{'}, B) \to \mathrm{Hom}_R(A, B) \xrightarrow{\alpha} \mathrm{Hom}_R(A^{'}, B)$$

and

$$0 \to \mathrm{Hom}_R(A/A^{'}, A) \to \mathrm{Hom}_R(A, A) \xrightarrow{\beta} \mathrm{Hom}_R(A^{'}, A).$$

By the definition of $A[B]$, the map

$$0 \to \mathrm{Hom}_R(A/A^{'}, B) \to \mathrm{Hom}_R(A, B)$$

is an isomorphism so that $im\ \alpha = 0$. By our hypothesis, we obtain

$$\begin{aligned} r_P(im\ \beta) &= r_P(\mathrm{Hom}_R(A, A)) - r_P(\mathrm{Hom}_R(A/A^{'}, A)) \\ &= r_P(\mathrm{Hom}_R(A, B)) - r_P(\mathrm{Hom}_R(A/A^{'}, B)) \\ &= r_P(im\ \alpha) = 0 \end{aligned}$$

for all prime ideals P. Therefore, $im\ \beta$ is divisible.

If $im\ \beta \neq 0$, then $\mathrm{Hom}_R(A^{'}, A)A^{'} \subseteq A$ contains a nonzero divisible submodule which is not possible. Thus, $im\ \beta = 0$. In particular, $A' = \beta(id_A)A' = 0$. Therefore, $A[B] = 0$. Similarly, we show $B[A] = 0$.

By Theorem 4.0.2, there is a nonzero K such that A is quasi-isomorphic to $K \oplus A_1$ and B is quasi-isomorphic to $K \oplus B_1$ for some A_1 and B_1. If C is an epimorphic image of B_1 or A_1, then C is quasi-isomorphic to an epimorphic image of B or A respectively. Therefore

$$r_P(\mathrm{Hom}(C, K)) + r_P(\mathrm{Hom}(C, A_1)) = r_P(\mathrm{Hom}(C, A))$$

$$= r_P(\mathrm{Hom}(C, B)) = r_P(\mathrm{Hom}(C, K)) + r_P(\mathrm{Hom}(C, B_1)),$$

which implies $r_P(\mathrm{Hom}(C, A_1)) = r_P(\mathrm{Hom}(C, B_1))$ for all P. Inducting on the rank of $A+B$, we obtain that A_1 is quasi-isomorphic to B_1. □

Our next result shows that Warfield's formula for the P-rank of Hom holds for modules over Dedekind domain. Observe that $\mathrm{Ext}(A, -)$ is divisible whenever R is Dedekind and A is a torsion-free R-module. Thus, the P-rank of the module $\mathrm{Ext}(A, -)$ as defined before would be 0. If D is a divisible module, then we replace the notion of P-rank by that of the R/P-dimension of the P-socle $D[P] = \{x \in D | Px = 0\}$.

Proposition 4.0.5. *Let R be a Dedekind domain, and M and N torsion-free R-modules of finite rank. For all $P \in spec(R)$,*

$$r_P(\mathrm{Hom}_R(M, N)) = r_P(M)r_P(N) - dim_{R/P}(\mathrm{Ext}^1_R(M, N)[P]).$$

Proof. The result is a direct consequence of Theorem 1.0.30. □

Corollary 4.0.6. *Let A and B be torsion-free modules of finite rank over a Dedekind domain R. If A and B are quasi-isomorphic, then* $\mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, B)$ *for all torsion-free finite* *rank modules C. Moreover, if the P-adic completion of R_P has infinite rank for all $P \in$ spec(R), then the converse holds.*

Proof. Since R is a Dedekind domain, $\mathrm{Ext}(M, -)$ is divisible whenever M is torsion-free. Since divisible quasi-isomorphic modules are isomorphic, we obtain $\mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, B)$ for all torsion-free finite rank modules C if A and B have the desired form.

b) Write $A = D_A \oplus A'$ and $B = D_B \oplus B'$, with D_A and D_B divisible, and A' and B' reduced. Then,

$$\mathrm{Ext}(C, A') = \mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, B) \cong \mathrm{Ext}(C, B').$$

Thus, we may assume that A and B are reduced. For any finite rank torsion-free R-module C, we obtain

$$r_P(\mathrm{Hom}(C, A)) = r_P(C) r_P(A) - dim_{R/P}(\mathrm{Ext}(C, A)[P])$$

and

$$r_P(\mathrm{Hom}(C, B)) = r_P(C) r_P(B) - dim_{R/P}(\mathrm{Ext}(C, B)[P])$$

from which we get

$$r_P(C) r_P(A) - r_P(\mathrm{Hom}(C, A)) = r_P(C) r_P(B) - r_P(\mathrm{Hom}(C, B)).$$

We fix $P \in spec(R)$, and consider the P-adic completion $\hat{R_P}$ of the module R_P as in [10]. Since $\hat{R_P}$ has infinite rank as an R-module, we can find a pure submodule C of $\hat{R_P}$ containing R_P with

$$\mathrm{rank}(C) = \mathrm{rank}(A) + \mathrm{rank}(B) + 1.$$

If $\alpha : C \to A$, then $ker\ \alpha \neq 0$, and $C/\ker\ \alpha$ is divisible since $r_P(C) = 1$ and $r_Q(C) = 0$ for $P \neq Q \in spec(R)$. Hence, $\mathrm{Hom}(C, A) = 0$. In the same way, $\mathrm{Hom}(C, B) = 0$. Hence,

$$\begin{aligned} r_P(A) &= r_P(C) r_P(A) - r_P(\mathrm{Hom}_R(C, A)) \\ &= r_P(C) r_P(B) - r_P(\mathrm{Hom}_R(C, B)) = r_P(B). \end{aligned}$$

By Theorem 4.0.4, A and B are quasi-isomorphic. □

If R is a maximal discrete valuation domain, then all torsion-free R-modules C of finite rank are projective, so $\mathrm{Ext}(C, M) = 0$ for all M. In particular, $\mathrm{Ext}(C, M) \cong \mathrm{Ext}(C, N)$ does not yield that M and N need to be quasi-isomorphic. Thus, the condition on the rank of $\hat{R_P}$ cannot be omitted.

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