

TRIANGLE CENTERS AND KIEPERT'S HYPERBOLA

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TRIANGLE CENTERS AND KIEPERT'S HYPERBOLA

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Charla Nicole Baker was born on December 30, 1977 in Louisville Alabama to Charles and Ruseda Baker. She graduated from George W. Long High School and attended George C. Wallace Community College. She entered Troy State University where she pursued Bachelor of Science degrees in both Computer Science and Mathematics. After graduating from Troy State University she entered graduate school at Auburn University to acquire a Master of Science degree in Mathematics.

THESIS ABSTRACT

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In this paper, we discuss the proofs of the primary classical triangle centers and Kiepert's Hyperbola as a solution to Lemoine's Problem. The definitions of terms which will be used throughout the paper are presented. A brief description of well-known triangle centers as well as complete proofs of the remaining classical triangle centers is provided. Many of the proofs of the classical triangle centers require the use of Ceva's Theorem. Ceva's Theorem is proven in the beginning prior to the introduction of the triangle centers. We also explore the proof of Kiepert's Hyperbola as a solution to a problem posed by Lemoine in 1868. A proof of the Nine-Point Circle is provided since the center of Kiepert's Hyperbola lies on the Nine-Point Circle. The trilinear coordinate system provides the basis for the proof of Kiepert's Hyperbola. A brief description of the system and the proofs of its primary theorems are given. The proof of Kiepert's Hyperbola is given along with its properties.

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CHAPTER 1

INTRODUCTION

The field of Geometry is subdivided into many areas such as Euclidean, Non-Euclidean, Convex, Discrete, Hyperbolic, and Algebraic. This paper explores the subject of triangle centers as studied in Euclidean Geometry. We consider a triangle center in a general sense meaning that a triangle center is the point of concurrence of three "special" lines of a reference triangle. For many centuries, mathematicians have been discovering triangle centers. The most well-known centers, the incenter, the centroid, the circumcenter, and the orthocenter were discovered by the ancient Greeks thus classifying them as "classical". However, the height of the study of triangle centers occurred during the 1800's, where mathematicians such as Fermat and Lemoine continued the research on triangle centers. The subject was revisited in the early 1990's. The study of triangle centers provides a solid foundation for the concept of triangles and a great teaching tool for Euclidean Geometry.

In this paper, thirteen triangle centers are examined. In Chapter 2, terms and definitions which will be used throughout the paper are presented. Chapter 3 provides descriptions and proofs for the triangle centers. A review of the properties and proofs for the incenter, centroid, circumcenter, and orthocenter are given. Ceva's Theorem plays an important role in many of the existence proofs of the triangle centers. The theorem is proven and provides a framework for the existence proofs of the centroid, Nagel Point, Gergonne Point, and the Symmedian or Lemoine Point. The Mittenpunkt or Middles-point is presented following the Symmedian Point along with the Spieker Center and Steiner Point. The Napoleon and Fermat Points are a result from the proof of Kiepert's Hyperbola.

In 1868, Lemoine posed a problem concerning the coordinates of a given triangle if the coordinates of the peaks of equilateral triangles raised on the sides are known. In 1869, Ludwig Kiepert presented a solution to the problem known as Kiepert's Hyperbola. Kiepert's Hyperbola passes through several of the triangle centers presented in Chapter 3. In Chapter 4, Kiepert's Hyperbola is examined. The chapter begins with the proof of the Nine-Point or Feuerbach Circle since the center of Kiepert's Hyperbola lies on the Nine-Point Circle. The asymptotes of Kiepert's Hyperbola are Simson Lines. Therefore, the existence proof of Simson Lines is provided. The proof for Kiepert's Hyperbola involves the use of trilinear coordinates. The coordinate system is introduced in Chapter 4 and many theorems regarding the system are proven. Finally, the proof of Kiepert's Hyperbola is given along with its relationship to many of the triangle centers. Kiepert's Hyperbola not only passes through the vertices of the given triangle, but also passes through the centroid, orthocenter, Spieker Center, Fermat Point, and Napoleon Point[2].

CHAPTER 2
TERMS AND THEIR DEFINITIONS

Altitude: The line passing through a vertex of a given triangle and perpendicular to the opposite side.

Angle Bisector: The line which passes bisects a vertex of a given triangle.

Brocard Circle: The circle with the symmedian point and the circumcenter as its diameter for a given triangle.

First Brocard Triangle: The triangle constructed by connecting the points on the Brocard Circle where the perpendiculars through the circumcenter meet the Brocard Circle.

Centroid: The point of concurrence of the medians of a given triangle.

Circumcenter: The point of concurrence of the perpendicular bisectors of a given triangle.

Cyclic Quadrilateral: A quadrilateral in which all vertices lie on the same circle.

Euler Line: The line passing through the centroid, circumcenter, and orthocenter of a given triangle.

Euler Points: The midpoints of the segments connecting the orthocenter to the vertices of a given triangle.

Euler Triangle: The triangle connecting the three Euler Points.

Fermat Point: The point of concurrence of three lines, each passing through a vertex and the peak of an equilateral triangle raised on the opposite side of a given triangle.

Feuerbach Circle: See Nine-Point Circle.

Gergonne Point: The point of concurrence of three lines, each passing through a vertex

and the point of tangency of the opposite side and the incircle of a given triangle.

Incenter: The point of concurrence of the angle bisectors of a given triangle.

Kiepert's Hyperbola: The rectangular hyperbola formed by the point of concurrence of three lines, each connecting a vertex of a given triangle and the peak of a similar isosceles triangle raised on the opposite side, as the base angle varies.

Lemoine Point: See Symmedian Point.

Medial Triangle: The triangle whose vertices are the midpoints for a given triangle.

Median: The line passing through the vertex and the midpoint of the opposite side of a given triangle.

Mittelpunkt or Middles-point: The point of concurrence of three lines, each passing through the center of the excircle for a given side and the midpoint of that side for a given triangle.

Nagel Point: The point of concurrence of three lines, each passing through a vertex and the point of tangency of the opposite side and the opposite excircle of a given triangle.

Napoleon Point: The point of concurrence of three lines, each passing through a vertex and the centroid of an equilateral triangle raised on the opposite side of a given triangle.

Nine-Point or Feuerbach Circle: The circle on which the medians, the feet of the altitudes, and the Euler points of a given triangle lie.

Orthocenter: The point of concurrence of the altitudes of a given triangle.

Perpendicular Bisector: A line which bisects a side of a given triangle and is perpendicular to that side.

Simson Line: A line which passes through the three feet of the altitudes from any point on the circumcircle.

Spiker Center: The point of concurrence of the angle bisectors of the medial triangle of a given triangle.

Steiner Point: The point of concurrence of three lines, each passing through a vertex which parallel to the opposite side of the First Brocard Triangle for a given triangle.

Symmedian Line: For a vertex of a given triangle it is the reflection of the medial line at that vertex across the angle bisector at that vertex.

Symmedian or Lemoine Point: The point of concurrence of the three symmedian lines of a triangle.

Trilinear coordinates: An set of ordered triples, $\alpha : \beta : \gamma$, which are proportional to the signed distances from a point to the sidelines of a given triangle.

Notation

Triangle ABC will be denoted as $\triangle ABC$.

Given $\triangle ABC$, a, b, and c will denote the sides BC, CA, and AB respectively.

$\angle ABC$ will denote the angle with point B as its vertex as well as the measure of that angle.

The line, segment, and segment length between two points A and B will be denoted as AB.

The symbol " \overrightarrow{AB} " will be used for vector AB.

The symbol " \cong " will be used for congruence.

The symbol " \sim " will be used for similarity.

An altitude is understood to be the line passing through a vertex of a triangle which is perpendicular to the opposite side.

CHAPTER 3
TRIANGLE CENTERS

In this chapter, we focus on the classical triangle centers. However, a list of both classical and recent triangle centers can be found on Clark Kimberling's website. The website also includes an encyclopedia of all triangle centers. Kimberling also provides more information on the centers presented in Chapter 3 including their trilinear coordinates and properties [3]. Before proving the existence of several triangle centers, a theorem must be introduced. Ceva's Theorem is useful in determining whether three lines are concurrent. Many of the proofs in this chapter utilize Ceva's Theorem.

3.1 Ceva's Theorem

Theorem 3.1 (*Ceva's Theorem*) *If D , E , and F are points on the sides c , a , and b respectively of a given triangle, $\triangle ABC$, then the lines AE , BF , and CD concur at a point K if and only if $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$.*

Proof.

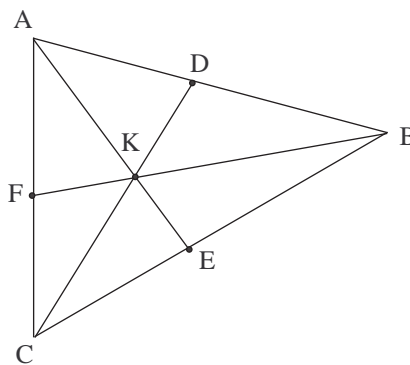


Figure 1: Ceva's Theorem

First, assume that the segments AE, BF, and CD concur at a point K. (Figure 1) Triangles CKE and BKE share altitude h_K . Therefore

$$\text{area}(\triangle CKE) = CE \cdot \frac{h_K}{2} \text{ and } \text{area}(\triangle BKE) = EB \cdot \frac{h_K}{2}.$$

Therefore

$$\frac{CE}{EB} = \frac{\text{area}(\triangle CKE)}{\text{area}(\triangle BKE)}.$$

In the same manner,

$$\frac{CE}{EB} = \frac{\text{area}(\triangle BAE)}{\text{area}(\triangle CAE)}.$$

Since $\triangle AKB = \triangle BAE - \triangle BKE$ and $\triangle AKC = \triangle CAE - \triangle CKE$ then

$$\frac{CE}{EB} = \frac{\text{area}(\triangle CKA)}{\text{area}(\triangle AKB)}.$$

Likewise,

$$\frac{AF}{FC} = \frac{\text{area}(\triangle AKB)}{\text{area}(\triangle CKB)} \text{ and } \frac{BD}{DA} = \frac{\text{area}(\triangle CKB)}{\text{area}(\triangle CKA)}.$$

Therefore,

$$\frac{CE}{EB} \cdot \frac{AF}{FC} \cdot \frac{BD}{DA} = \frac{\text{area}(\triangle CKA)}{\text{area}(\triangle AKB)} \cdot \frac{\text{area}(\triangle AKB)}{\text{area}(\triangle CKB)} \cdot \frac{\text{area}(\triangle CKB)}{\text{area}(\triangle CKA)} = 1.$$

Now, assume that for $\triangle ABC$ with any points D, E, and F on its sides,

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1.$$

Let the lines AE and BF intersect at a point K. Then there is some point, X, on side AB such that CX passes through K.

Then

$$\frac{AX}{XB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1.$$

Therefore, by setting the two equations equal to each other,

$$\frac{AD}{DB} = \frac{AX}{XB}.$$

So $D = X$. Thus, the lines AE , BF , and CD are concurrent at a point. \square

3.2 Review of Centroid, Orthocenter, Circumcenter, and Incenter

The centroid, orthocenter, circumcenter, and incenter are perhaps the most well known triangle centers. Euler found that the incenter, circumcenter, and orthocenter all lie on the same line known as the Euler Line. This section provides a brief review of their proofs and properties.

Centroid: The centroid is the point of concurrence of the medians of a given triangle. The centroid is the center of mass for the given triangle. The existence of the centroid is easily proven using Ceva's Theorem.

Orthocenter: The orthocenter of a given triangle is the point of concurrence of the altitudes. We will provide the proof which utilizes simple vector calculus.

Theorem 3.2 *Given $\triangle ABC$, the three altitudes of the triangle concur at a point called the orthocenter.*

Proof.

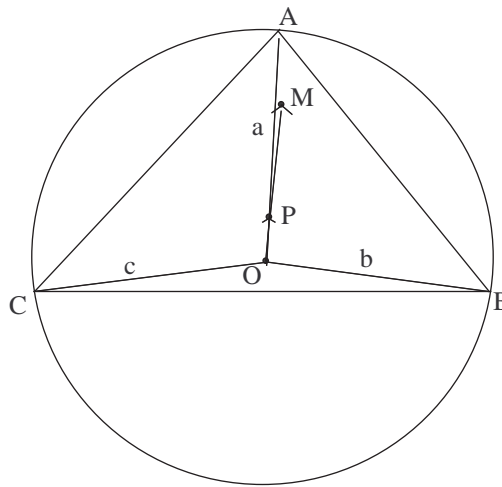


Figure 2: Orthocenter

Given $\triangle ABC$ let vectors a , b , and c be vectors from the circumcenter O to the vertices A , B , and C respectively. (Figure 2) Let the circumcenter be O . Now Let OP be the vector from the circumcenter O to the centroid P . Then

$$P = \frac{a + b + c}{3}.$$

Let \overrightarrow{OM} be the vector with same direction which is three times the length of \overrightarrow{OP} . Then

$$\overrightarrow{OM} = a + b + c.$$

Now $\overrightarrow{AB} = b - a$ and

$$\overrightarrow{MC} = c - (a + b + c) = -a - b.$$

Therefore,

$$\overrightarrow{AB} \cdot \overrightarrow{MC} = (b - a) \cdot (-a - b) = -b^2 + a^2 = 0$$

since $b = a$. Therefore, \overrightarrow{MC} is perpendicular to \overrightarrow{AB} . In the same manner, \overrightarrow{MA} is perpendicular to \overrightarrow{CB} and \overrightarrow{MB} is perpendicular to \overrightarrow{AC} . Thus, M is the point of concurrence of the altitudes of triangle ABC. \square

Circumcenter: The circumcenter of a given triangle is the point of concurrence of the perpendicular bisectors of the triangle. The existence proof for the circumcenter follows from the fact that the point lies equal distance from each vertex.

Incenter: The incenter of a given triangle is the point of concurrence of the angle bisectors of the triangle. The existence proof shows that the incenter is equal distance from the sidelines of the triangle.

3.3 Gergonne Point

Theorem 3.3 (*Gergonne's Theorem*) Given $\triangle ABC$, let the incircle have center I and be tangent to the sides c , a , and b at the points D , E , and F respectively. Then the lines AE , BF , and CD concur at a point K called the Gergonne Point. (Figure 3)

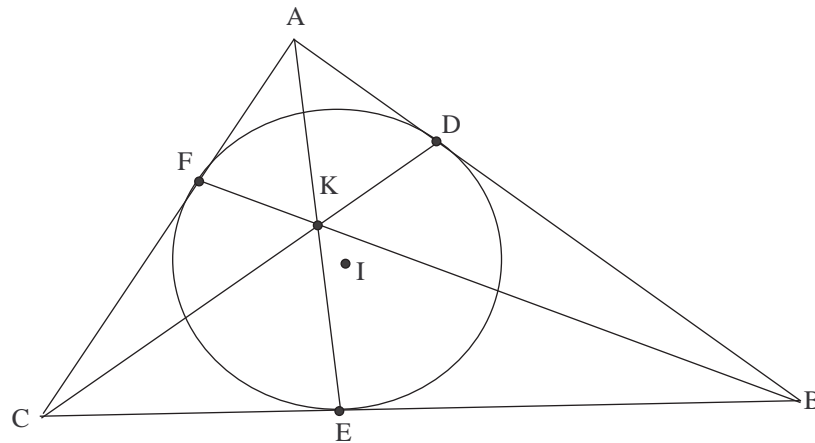


Figure 3: Gergonne Point

Proof.

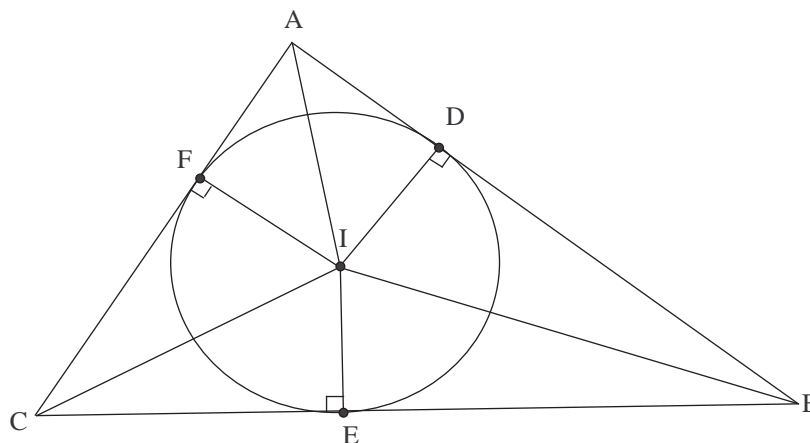


Figure 4: Radii and Angle Bisectors

I lies on the angle bisectors of the triangle and the radii of the incircle, ID, IE, and IF are perpendicular to the sides AB, BC, and CA.(Figure 4) Therefore, three pairs of congruent triangles are formed such that $BE = BD$, $CF = CE$, and $AD = AF$ then

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1.$$

By Ceva's Theorem the lines AE, BF, and CD are concurrent.

□

3.4 Nagel Point

Theorem 3.4 (*Nagel's Theorem*) Given $\triangle ABC$ let E , I , and L be the points on the sides c , a , and b respectively which are tangent to the side's excircles. Then the lines AI , BL , and CE concur at a point called the Nagel Point.(Figure 5)

Proof.

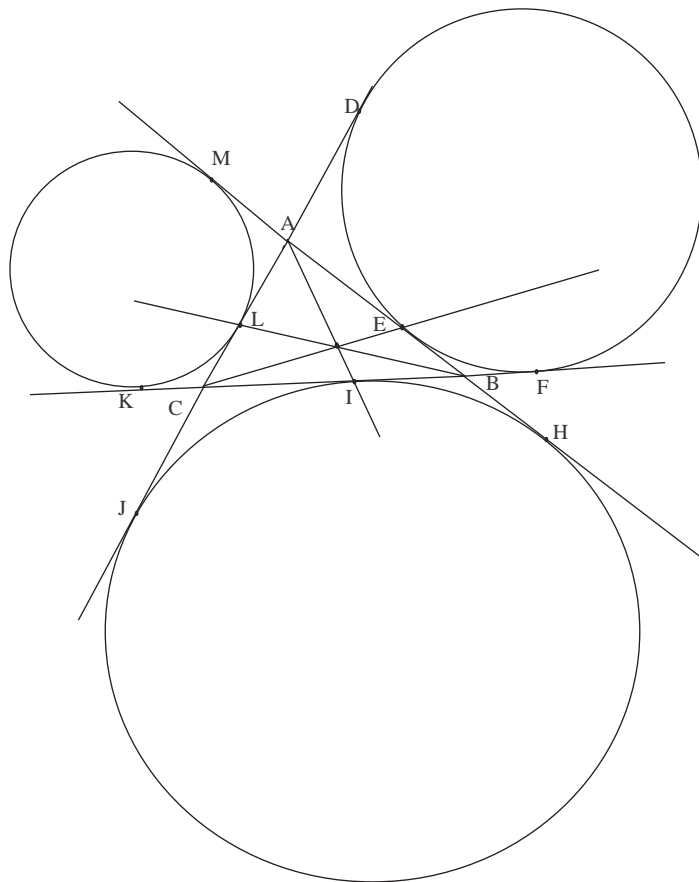


Figure 5: Nagel Point

Since $BM = BK$, $AM = AL$, and $CL = CK$ then $BK = s$ where s is the semiperimeter of $\triangle ABC$. Likewise $CF = s$. Therefore,

$$CL = EB = s - a.$$

Also

$$AE = CI = s - b \text{ and } AL = IB = s - c.$$

Therefore,

$$\frac{AE}{EB} \cdot \frac{IB}{CI} \cdot \frac{CL}{AL} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1.$$

So by Ceva's Theorem the lines AI, BL, and CE concur. □

3.5 Symmedian or Lemoine Point

Another application of Ceva's Theorem is a proof involving the symmedian lines of a triangle. A symmedian line of a given triangle at a vertex is the line reflection of the median of that vertex across the angle bisector. The three symmedian lines of a given triangle are concurrent by Ceva's Theorem.

Theorem 3.5 *Given $\triangle ABC$, the symmedian lines AS , BS' , and CS'' are concurrent at a point called the Symmedian or Lemoine Point. (Figure 6)*

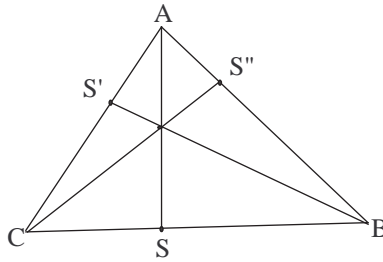


Figure 6: Symmedian Point

Proof.

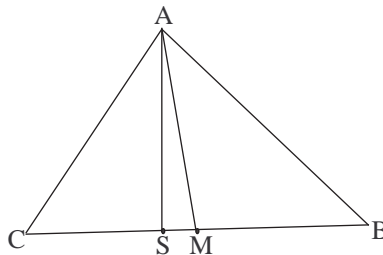


Figure 7: Symmedian Line

For $\triangle ABC$ let the median from vertex A intersect BC at the point M. Let the symmedian line from vertex A intersect BC at the point S.(Figure 7) Then by computing the areas of triangles CAM and BAS,

$$\frac{\text{area}(\triangle BAS)}{\text{area}(\triangle CAM)} = \frac{BS}{CM}$$

since triangles BAS and CAM share a common height h .

Then using an alternative area formula,

$$2 \cdot \text{area}(\triangle BAS) = AB \cdot AS \sin(\angle SAB)$$

and

$$2 \cdot \text{area}(\triangle CAM) = AM \cdot AC \sin(\angle CAM).$$

Since $\angle SAB = \angle CAM$ then,

$$\frac{\text{area}(\triangle BAS)}{\text{area}(\triangle CAM)} = \frac{AB \cdot AS}{AM \cdot AC} = \frac{BS}{CM}.$$

For triangles ASC and AMB,

$$\frac{\text{area}(\triangle ASC)}{\text{area}(\triangle AMB)} = \frac{AC \cdot AS}{AM \cdot AB} = \frac{CS}{BM}.$$

Since $\overline{CM} = \overline{BM}$, dividing the two equations gives

$$\frac{\frac{BS}{CM}}{\frac{CS}{BM}} = \frac{BS}{CS} = \frac{AB^2}{AC^2}.$$

For the other vertices,

$$\frac{CS'}{AS'} = \frac{BC^2}{AB^2} \text{ and } \frac{AS''}{BS''} = \frac{AC^2}{BC^2}$$

with S' and S'' being the point of intersection of the symmedian lines from vertex B and vertex C respectively.

Now multiplying the three equalities yields

$$\frac{BS}{CS} \cdot \frac{CS'}{AS'} \cdot \frac{AS''}{BS''} = \frac{AB^2}{AC^2} \cdot \frac{BC^2}{AB^2} \cdot \frac{AC^2}{BC^2} = 1.$$

By Ceva's Theorem the symmedian lines AS, BS', and CS'' are concurrent.

□

3.6 Mittenpunkt or Middles-point

Theorem 3.6 (*Mittenpunkt or Middles-point*) Given $\triangle ABC$ with O_a , O_b , and O_c as the excenters and A' , B' , and C' as the midpoints of sides a , b , and c respectively, the lines O_aA' , O_bB' , and O_cC' are concurrent at a point called the Mittenpunkt. (Figure 8)

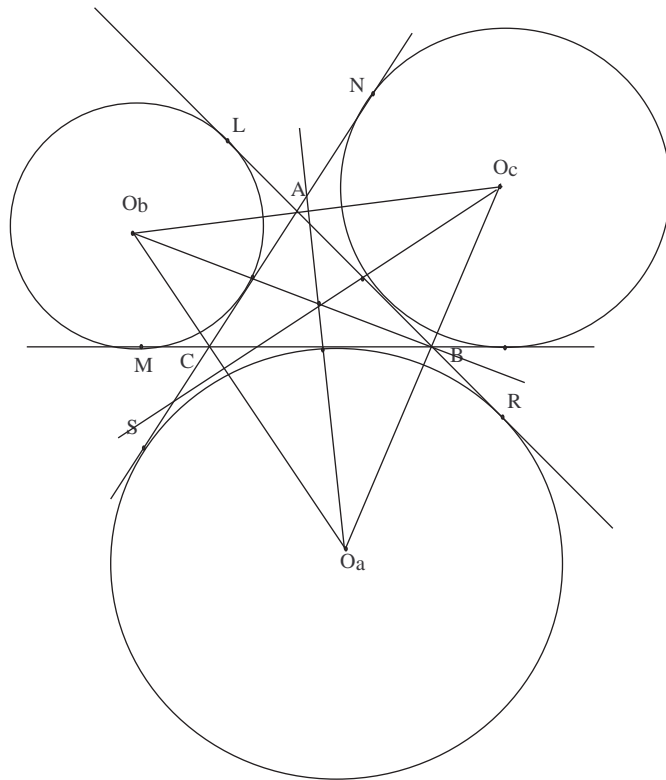


Figure 8: Mittenpunkt

Proof. Let L and M be points on the excircle with center O_b tangent to BA and BC respectively. Let N be the point on the excircle with center O_c tangent to CA . (Figure 9) The lines O_bA and O_cA bisect the external angles of vertex A .

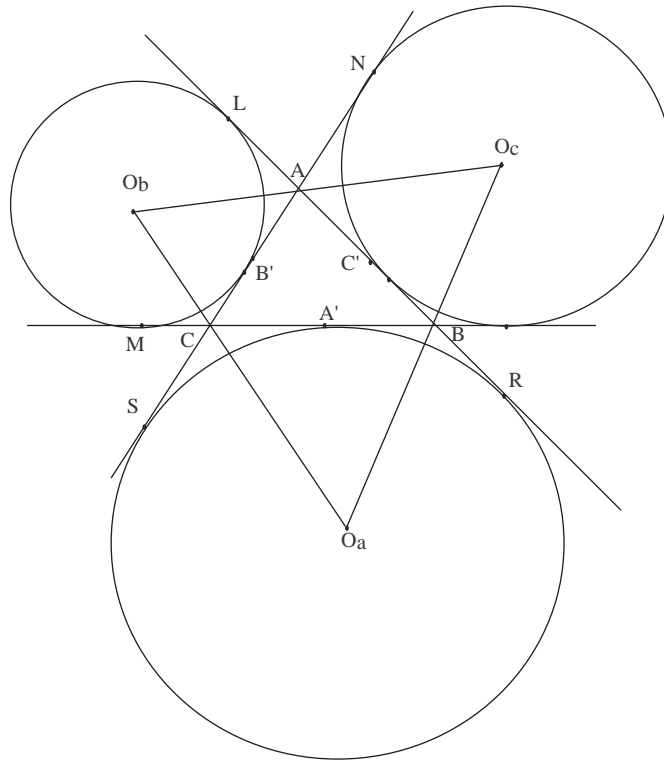


Figure 9: Excircles and Excenters

Then

$$\angle O_b A C = \angle L A O_b = \alpha.$$

In the same manner,

$$\angle B C O_a = \angle O_a C S = \gamma$$

where S is the point on tangency of the line AC with the excircle with center O_a .

Also

$$\angle C B O_a = \angle O_a B R = \beta$$

where R is the point of tangency of the line AB to the excircle with center O_a .

So

$$\angle CAB = \pi - 2\alpha, \angle ABC = \pi - 2\beta, \text{ and } \angle BCA = \pi - 2\gamma.$$

Therefore,

$$\alpha = \frac{\pi - \angle CAB}{2}, \beta = \frac{\pi - \angle ABC}{2}, \text{ and } \gamma = \frac{\pi - \angle BCA}{2}.$$

Then

$$\angle BO_aC = \pi - \left(\frac{\pi - \angle BCA}{2} \right) - \left(\frac{\pi - \angle ABC}{2} \right) = \frac{\angle BCA}{2} + \frac{\angle ABC}{2} = \frac{\pi - \angle CAB}{2} = \alpha.$$

Likewise,

$$\angle CO_bA = \beta$$

and

$$\angle BO_cA = \gamma.$$

Therefore, $\triangle AO_cB$ is similar to $\triangle O_cO_bO_a$. Let line O_cT be the angle bisector of $\angle AO_cB$. Reflecting $\triangle AO_cB$ about line O_cT we get that $\triangle AO_cB$ and $\triangle O_cO_bO_a$ are homothetic. Thus, the image of the median O_cC' is the median $O_cC'_r$ of $\triangle O_cO_bO_a$. Therefore, line O_cC' is a symmedian line for $\triangle O_aO_cO_b$. (Figure 10)

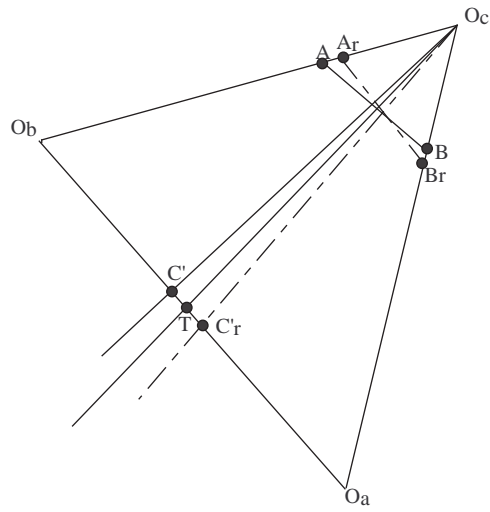


Figure 10: Symmedian Line

Likewise the lines O_bB' and O_aA' are symmedian lines for $\triangle O_aO_cO_b$. Therefore, by Theorem 3.5 the lines O_aA' , O_bB' , and O_cC' are concurrent.

□

3.7 Spieker Center

Theorem 3.7 *Given $\triangle ABC$, let D , E , and F be the midpoints of the sides c , a , and b respectively. Then the angle bisectors of the medial triangle, $\triangle DEF$ concur at a point K known as the Spieker Center. (Figure 11)*

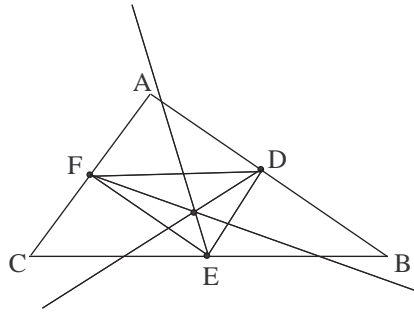


Figure 11: Spieker Center

Proof. Since the angle bisectors of any given triangle concur at a point, the angle bisectors of medial triangle, $\triangle DEF$, concur at a point. \square

3.8 Steiner Point

Theorem 3.8 *Given $\triangle ABC$ with circumcircle having center X and Brocard Triangle, $\triangle a'b'c'$, then the lines through the vertices $\triangle ABC$ which are parallel to the "opposite" sides of the Brocard triangle concur at a point known as the Steiner Point. (Figure 12)*

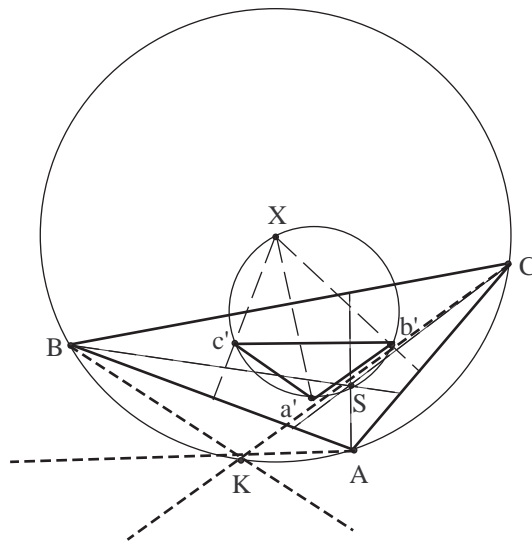


Figure 12: Steiner Point

Proof. Let the line through vertex B and parallel to $a'c'$ and the line through vertex C and parallel to $a'b'$ concur at a point K. (Figure 13)

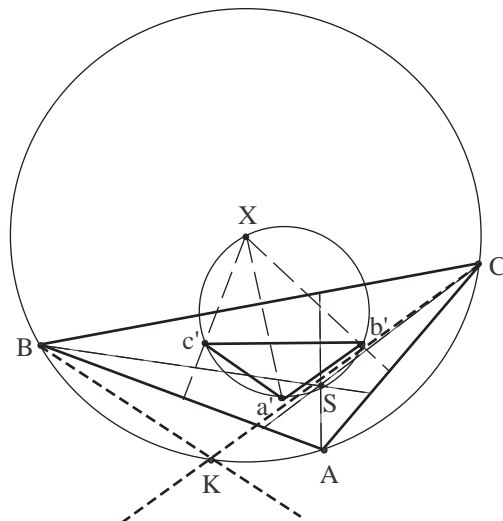


Figure 13: K on the circle

Then,

$$\angle BKC = \angle BAC$$

since both angles cut off the same chord of the circumcircle with center X. Therefore, K lies on the circumcircle of $\triangle ABC$. In the same manner, let the line through vertex B and parallel to ac and the line through vertex A and parallel to cb concur at a point M.

Then,

$$\angle BMA = \angle ACB$$

since both angles cut off the same chord of the circumcircle with center X. Therefore, M also lies on the circumcircle.

Since line BK can only pass through a circle in at most two points, then

$$K = M.$$

Thus the lines BK, CK, and AK are concurrent at the point K on the circumcircle \square

3.9 Napoleon Point

Theorem 3.9 *Given $\triangle ABC$ with equilateral triangles raised on it's sides, the lines ,each passing through a vertex and the centroid of the opposite equilateral triangle, concur at a point called the Napoleon Point. (Figure 14)*

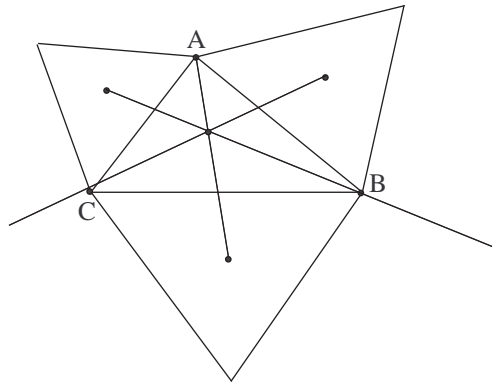


Figure 14: Napoleon Point

Proof. Refer to the proof of Kiepert's Hyperbola in Chapter 4. □

3.10 Fermat Point

Theorem 3.10 *Given $\triangle ABC$ with equilateral triangles raised on its sides, the lines, each passing through a vertex and the peak of the opposite equilateral triangle, concur at a point called the Fermat Point.*

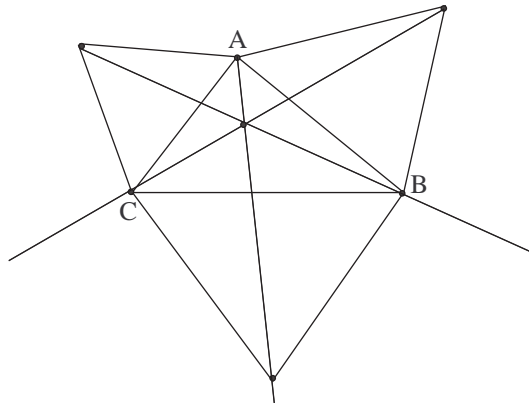


Figure 15: Fermat Point

Proof. Refer to the proof of Kiepert's Hyperbola in Chapter 4. □

CHAPTER 4

KIEPERT'S HYPERBOLA

In 1868, Lemoine posed the following problem [2]: *Construct a triangle, given the peaks of the equilateral triangles constructed on the sides (p. 188).* Ludwig Kiepert posed a solution to the problem in 1869. Kiepert generalized the problem to isosceles triangles. Kiepert's solution is stated as follows[2]: *If three triangles, $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$, with equivalent base angles are constructed on the sides of a given triangle, $\triangle ABC$, then the lines AA' , BB' , and CC' concur. The locus of the point P as the base angle of the isosceles triangles varies forms the equation*

$$\frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0,$$

or equivalently,

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - b^2)}{\beta} + \frac{ab(a^2 - b^2)}{\gamma} = 0(p.189).$$

He showed that the lines, each passing through a vertex of the given triangle and a peak of the constructed isosceles triangle, concur at a point. As the base angle of the isosceles triangles vary the locus of the point of concurrence creates a rectangular hyperbola. The hyperbola passes through the vertices of the given triangle and several triangle centers including the centroid, orthocenter, Spieker center, Napoleon Point, Fermat Point, and the Brocard point [5]. The Simson Lines of the given triangle are the asymptotes for the rectangular hyperbola. The existence proof for Simson Lines is given in section two. The proof of Kiepert's Hyperbola requires the use of trilinear coordinates. Several theorems involving trilinear coordinates will be proven prior to the completion of Kiepert's Hyperbola

theorem. The center of Kiepert's Hyperbola lies on the Nine-Point Circle. The Nine-Point Circle proof is well known. However, for completeness the proof is given in the first section of this chapter.

4.1 Nine-Point Circle

The Nine-Point Circle or Feuerbach Circle passes through the medians, the feet of the altitudes, and the Euler points of a given triangle. A lemma is needed to prove that all nine points lie on the same circle.

Lemma 1 *Let X be any point on a circle with center O and radius r . Let T be any fixed point. Then the locus of M , the midpoint of segment TX , is on a circle with radius $\frac{r}{2}$ and center N which is the midpoint of segment OT .*

Proof.

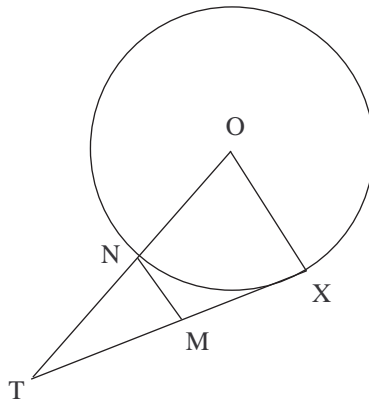


Figure 16: Midpoint Lemma

By Figure 16, $\triangle OXT$ is similar to $\triangle NMT$. Therefore $MN = \frac{1}{2} XO$ or $MN = \frac{r}{2}$. Thus M lies on a circle with radius $\frac{r}{2}$ and center N . \square

Theorem 4.1 For a given triangle the midpoints, the feet of the altitudes, and the Euler points lie on a circle called the Nine-Point Circle.

Proof. Let a , b , and c be sides BC , CA , and AB respectively. For a given triangle, $\triangle ABC$, let the midpoints of the sides a , b , and c be D , E , and F respectively. Let the feet of the altitudes to sides a , b , and c be G , H , and I respectively. Let J be the orthocenter of $\triangle ABC$ and let K , L , and M be the midpoints of AJ , BJ , and CJ where K , L , and M are the Euler Points of the triangle. Also, let O be the circumcenter of $\triangle ABC$ with a circumcircle of radius r . Let X be the midpoint of OJ . (Figure 17)

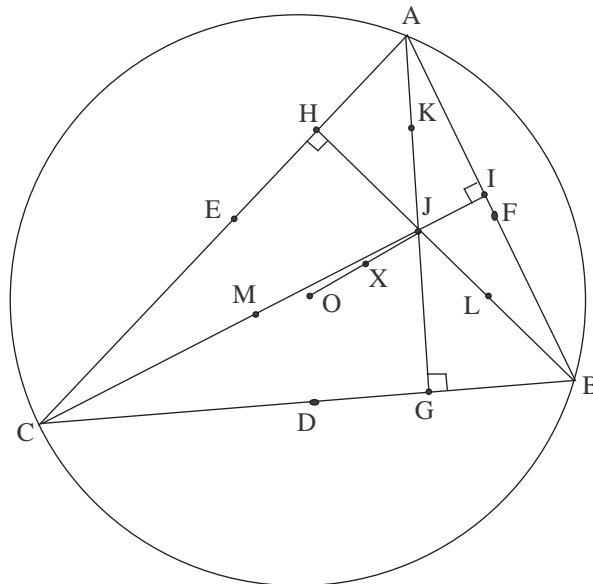


Figure 17: Circumcircle, Medians, Feet of Altitudes, and Euler Points

$\angle ACJ \cong \angle ABH$ since both angles are complements of $\angle CAB$. Let J be the point on which the line BH touches the circle. Then $\angle ABN \cong \angle ACN$ since they cut off arc NA .

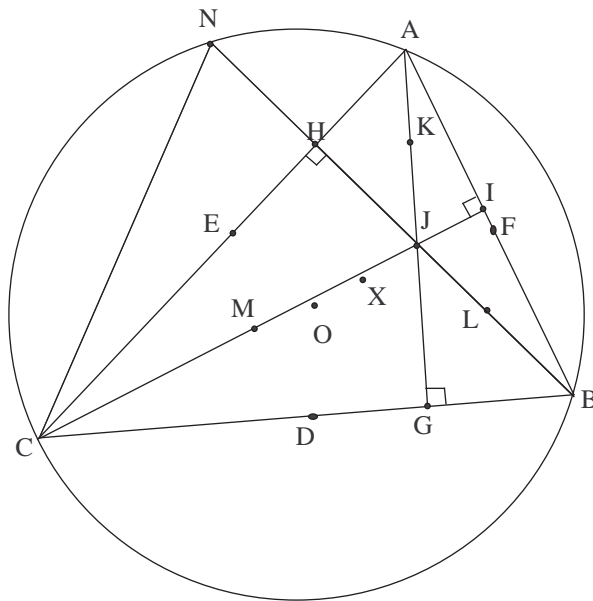


Figure 18: H, I, G, K, L, and M are on the same circle

Therefore, $\angle ACN \cong \angle ACJ$ and $\angle ABN \cong \angle ACJ$. Since $\angle JHC \cong \angle NHC$ then $\triangle HCN \cong \triangle HCJ$. Thus $JH = HN$. By Lemma 1 J lies on the circle with X as the center and radius $\frac{r}{2}$. Likewise H and I lie on the circle. L is the midpoint of JB so L is on the circle with center X and radius $\frac{r}{2}$ by Lemma 1. Likewise M and K lie on the circle. Therefore H, I, J, L, M, and K lie on the circle with center X and radius $\frac{r}{2}$. (Figure 18)

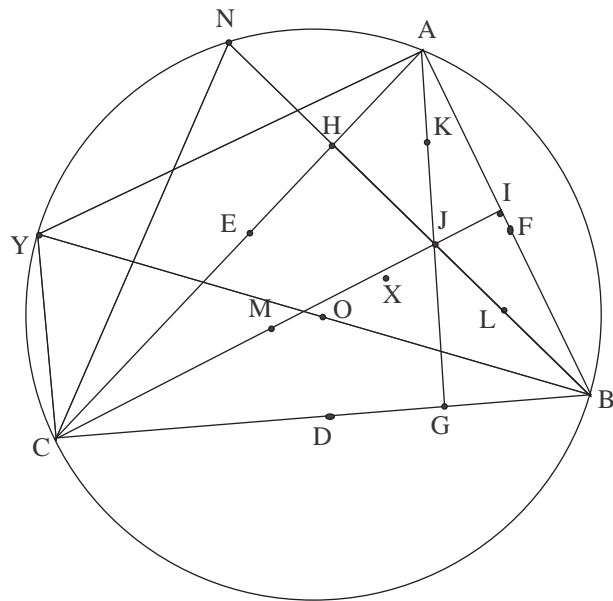


Figure 19: E, F, and D lie on the circle

Let line BO meet the circumcircle at the point Y. (Figure 19) Then YCJA is a parallelogram. Therefore YJ and CA intersect at E since E is the midpoint of CA. Therefore E is also the midpoint of YJ. By Lemma 1, E lies on the circle with center X and radius $\frac{r}{2}$ as do D and F. (Figure 20)

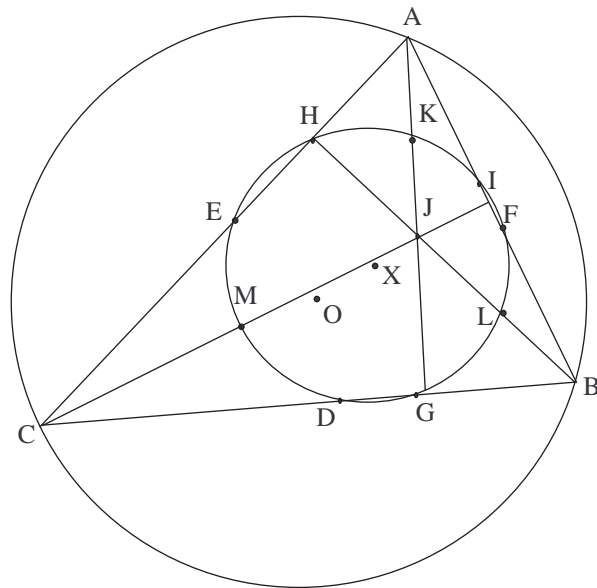


Figure 20: Nine-Point Circle

□

4.2 Simson Line

Since the asymptotes for Kiepert's Hyperbola are Simson Lines, the existence proof of Simson Lines will be given.

Theorem 4.2 *Given $\triangle ABC$ and a point P on its circumcircle then the feet of the perpendiculars, W , U , and V , to the sides BC , AC , and AB respectively are collinear. The line passing through the feet of the altitudes is called a Simson Line. (Figure 21)*

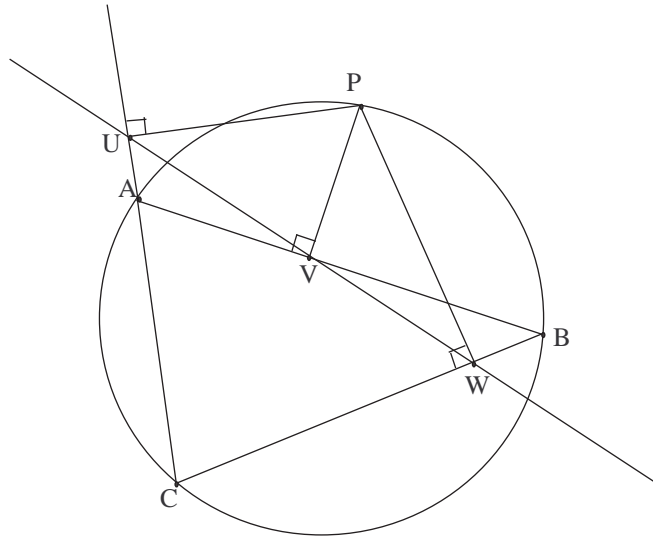


Figure 21: Simson Line

Proof. Quadrilateral CAPB is cyclic therefore

$$\angle WCA + \angle WPB + \angle APW = 180^\circ.$$

Likewise, quadrilateral CUPW is cyclic so

$$\angle WCA + \angle UPA + \angle APW = 180^\circ.$$

So

$$\angle WPB = \angle UPA.$$

And since quadrilateral PVWB is also cyclic then

$$\angle WPB = \angle WVB$$

since they cut off the same angle. Quadrilateral PVAU is cyclic therefore,

$$\angle UPA = \angle UVA.$$

Therefore,

$$\angle WVB = \angle UVA.$$

Since $\angle WVB$ and $\angle UVA$ are vertical angles the U, V, and W are collinear. □

4.3 Trilinear Coordinates

Many coordinate systems are used to determine triangle properties. In **cartesian coordinates** the distances from two given perpendicular axes are assigned to each point. Polar coordinates are more useful for certain analytic computations. If the signed distance from the origin O to a point P and the signed angle OP to the x-axis are assigned to P then **polar coordinates** are being used. **Barycentric coordinates** are a set of ordered triples of masses, $w_1 : w_2 : w_3$, defined for a point P inside a given triangle such that P lies on the centroid of the triangle. Trilinear coordinates can be very useful when exploring various properties of triangles and their centers. **Trilinear coordinates** are a set of ordered triples, $\alpha : \beta : \gamma$, of numbers which are proportional to the signed distances from a point to the sidelines of a given triangle.(Figure 22)

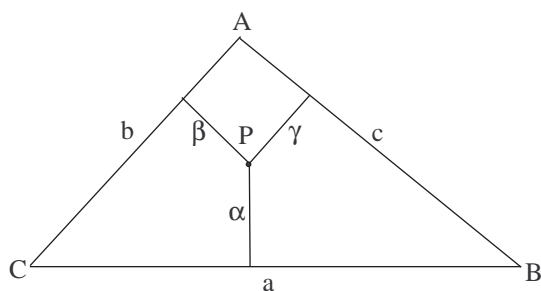


Figure 22: Trilinear Coordinates

In trilinear coordinates the ratio of the distances are important so that

$$\alpha : \beta : \gamma = k \alpha : k \beta : k \gamma$$

for any nonzero constant k .

Actual trilinear coordinates give the actual directed distances from a point to the sides of a given triangle. Actual trilinear coordinates are given as

$$k\alpha : k\beta : k\gamma$$

where $k = \frac{2 \Delta}{a\alpha + b\beta + c\gamma}$ [3]. More information on trilinear coordinates can be found in the works of Kimberling[3][4] and Coxeter[1].

To determine collinearity of three points in trilinear coordinates a determinant must be evaluated.

Theorem 4.3 *Given three points $\alpha_1 : \beta_1 : \gamma_1$, $\alpha_2 : \beta_2 : \gamma_2$, and $\alpha_3 : \beta_3 : \gamma_3$, the three points are collinear if*

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0.$$

Proof. Let $\triangle ABC$ be the reference triangle. Let P and P' be points such that $P = 0$ and $P' = 1$ and let t be a point on the line passing through P and P' such that $0 < t < 1$.(Figure 23)

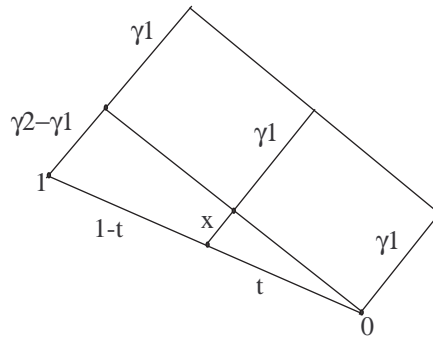


Figure 23: Function $\gamma(t)$

From Figure 23, the distance from any point t between P and P' is given by the function

$$\gamma = t\gamma_1 + (1 - t)\gamma_2.$$

In the same manner,

$$\alpha = t\alpha_1 + (1 - t)\alpha_2,$$

$$\beta = t\beta_1 + (1 - t)\beta_2.$$

Therefore,

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (1 - t) \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \alpha_2 \end{bmatrix} + t \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}.$$

Thus, $[\alpha, \beta, \gamma]$ is a linear combination of $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$.

Therefore,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

□

A line in trilinear coordinates is defined in terms of three parameters: l , m , and n . Using the theorem for collinearity, we can derive the equation of a line in trilinear coordinates.

Theorem 4.4 *Given two points $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ a line through these two points in trilinear coordinates has the form*

$$l \alpha + m \beta + n \gamma = 0$$

with

$$l = \beta_1 \gamma_2 - \gamma_1 \beta_2$$

$$m = \gamma_1 \alpha_2 - \alpha_1 \gamma_2$$

$$n = \alpha_1 \beta_2 - \beta_1 \alpha_2.$$

Proof. Using the collinearity theorem and replacing one set of coordinates with $\alpha : \beta : \gamma$ we get,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} =$$

$$(\beta_1\gamma_2 - \gamma_1\beta_2)\alpha + (\gamma_1\alpha_2 - \alpha_1\gamma_2)\beta + (\alpha_1\beta_2 - \beta_1\alpha_2)\gamma = 0.$$

Therefore, the equation of the line passing through $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ is given by

$$l\alpha + m\beta + n\gamma = 0$$

where

$$l = \beta_1\gamma_2 - \gamma_1\beta_2$$

$$m = \gamma_1\alpha_2 - \alpha_1\gamma_2$$

$$n = \alpha_1\beta_2 - \beta_1\alpha_2.$$

□

Now that we have determined the equation for a line in trilinear coordinates, a theorem is introduced regarding line concurrence.

Theorem 4.5 *Three trilinear lines*

$$l_1 \alpha + m_1 \beta + n_1 \gamma = 0,$$

$$l_2 \alpha + m_2 \beta + n_2 \gamma = 0,$$

$$l_3 \alpha + m_3 \beta + n_3 \gamma = 0$$

concur at one point if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

Proof. If three lines,

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$

$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$

$$l_3\alpha + m_3\beta + n_3\gamma = 0$$

are concurrent, then they meet at one point. Therefore, the system of equations has a solution. Since trilinear coordinates define distances to the sides of a reference triangle, then at most two of the coordinates can be zero. However, the trivial solution $0 : 0 : 0$ is also a solution of the system. Thus, two solutions exist.

Therefore,

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

Since

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

then

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} =$$

$$l_1 \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix} + m_1 \begin{vmatrix} l_2 & n_2 \\ l_3 & n_3 \end{vmatrix} + n_1 \begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} =$$

$$(m_2n_3 - m_3n_2)l_1 + (l_2n_3 - l_3n_2)m_1 + (l_2m_3 - l_3m_2)n_1 = 0$$

The point of concurrence is given as

$$P = m_2n_3 - m_3n_2 : l_2n_3 - l_3n_2 : l_2m_3 - l_3m_2.$$

□

Remark: The line at infinity in trilinear coordinates is given as

$$a\alpha + b\beta + c\gamma = 0$$

since the distances from the line at infinity to the vertices of the reference triangle become equal in the limit.

4.4 Kiepert's Hyperbola

For the proof of Kiepert's Hyperbola, we assume that the given triangle is scalene. We will, however, discuss the cases in which the given triangle is an equilateral or isosceles triangle.

Equilateral Triangle: If the given triangle, $\triangle ABC$, is equilateral then as the base angle of the three isosceles triangles raised on the sides of the given triangle varies the lines, each passing through a vertex and the peak of the isosceles triangle raised on the opposite side, always concur at the centroid of the triangle.

Isosceles Triangle: If the given triangle, $\triangle ABC$, is an isosceles triangle with side AB equal to side CA, then the point of concurrence lies on the perpendicular bisector of side BC. But in the limit case where ϕ is approaching $\frac{\pi}{2}$ the lines, each passing through a vertex and the peak of the isosceles triangle raised on the opposite side, are parallel.

Theorem 4.6 *If three triangles, $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$, with equivalent base angles are raised on the sides of a given triangle, $\triangle ABC$, then the lines AA' , BB' , and CC' concur. The locus of the point P as the base angle of the isosceles triangles varies forms a hyperbola with equation*

$$\frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0,$$

or equivalently,

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - b^2)}{\beta} + \frac{ab(a^2 - b^2)}{\gamma} = 0.$$

Before we prove Theorem 4.6, note that some of the special cases have already been proven.

Case 1: If $\phi = 0$ then the lines AA' , BB' , and CC' are the medians of $\triangle ABC$. Thus, the lines concur at the centroid of $\triangle ABC$.(Figure 24)

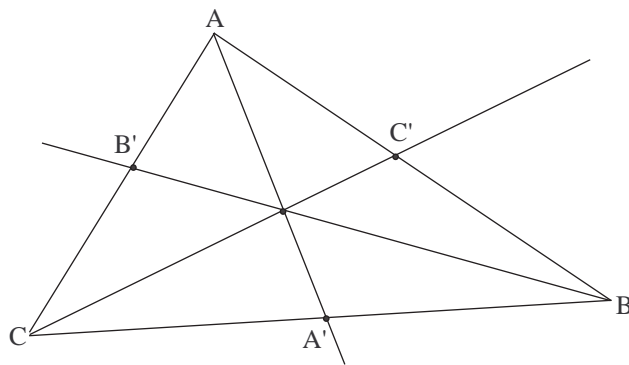


Figure 24: Centroid

Case 2: In the limit case, as ϕ approaches $\frac{\pi}{2}$, the lines AA' , BB' , and CC' get closer to the altitudes of $\triangle ABC$. Therefore, in the limit case, the hyperbola passes through the orthocenter but the orthocenter, is not a point of concurrence.(Figure 25)

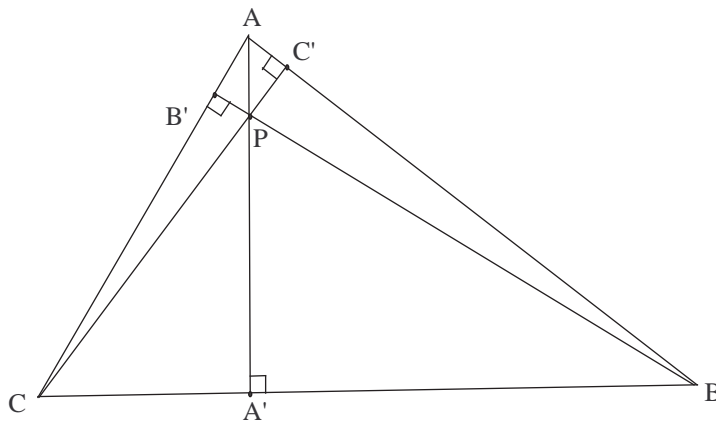


Figure 25: Orthocenter

Proof. In the general case, let $\triangle ABC$ be acute and let the constructed isosceles triangles lie outside the given triangle. Let ϕ represent the base angle of the constructed triangles.

Also let A , B , and C represent the angle at vertex A , vertex B , and vertex C respectively.

In trilinear coordinates, let

$$A = 1 : 0 : 0,$$

$$B = 0 : 1 : 0,$$

$$C = 0 : 0 : 1.$$

Letting x be the length of side $A'C$ then the distance from vertex A' to side BC is $x \sin(\phi)$. (Figure 26)

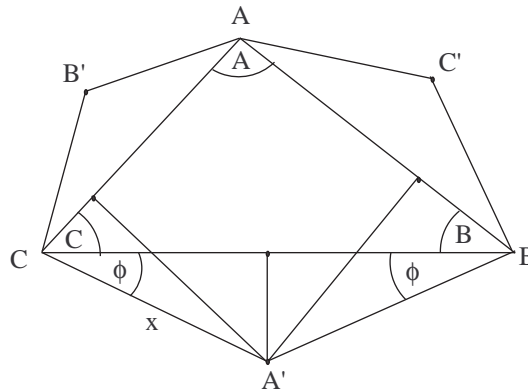


Figure 26: Trilinear Coordinates

Since the incenter of $\triangle ABC$ and A' lie on opposite sides of BC , the trilinear coordinates for the distance is $-x \sin(\phi)$. Likewise, the distance from A' to side AC and side AB is $x \sin(C + \phi)$ and $x \sin(B + \phi)$ respectively. Therefore,

$$A' = -x \sin(\phi) : x \sin(C + \phi) : x \sin(B + \phi)$$

or equivalently

$$A' = -\sin(\phi) : \sin(C + \phi) : \sin(B + \phi).$$

In the same manner,

$$B' = \sin(C + \phi) : -\sin(\phi) : \sin(A + \phi)$$

and

$$C' = \sin(B + \phi) : \sin(A + \phi) : -\sin(\phi).$$

By the definition of a trilinear line, the line passing through the points A and A' have parameters l , m , and n as follows: $l = 0$, $m = -\sin(B + \phi)$, and $n = \sin(C + \phi)$. Therefore, the equation for line AA' is $-\sin(B + \phi)\beta + \sin(C + \phi)\gamma = 0$. The equations for lines BB' and CC' can be obtained in the same way so that

$$AA' = -\sin(B + \phi)\beta + \sin(C + \phi)\gamma = 0,$$

$$BB' = -\sin(C + \phi)\gamma + \sin(A + \phi)\alpha = 0,$$

$$CC' = -\sin(A + \phi)\alpha + \sin(B + \phi)\beta = 0.$$

The lines AA', BB', and CC' concur if the matrix of their coefficients has a determinant of 0.

$$\begin{vmatrix} 0 & -\sin(B + \phi) & \sin(C + \phi) \\ \sin(A + \phi) & 0 & -\sin(C + \phi) \\ -\sin(A + \phi) & \sin(B + \phi) & 0 \end{vmatrix} =$$

$$0 ((\sin(B + \phi)(\sin(C + \phi)) - (-\sin(B + \phi)((-\sin(A + \phi)(\sin(C + \phi)) + (\sin(C + \phi))((\sin(A + \phi)(\sin(B + \phi))) = 0$$

Therefore the lines AA' , BB' , and CC' concur. The point, P , of concurrence is given by the trilinear coordinates

$$\sin(B + \phi) \sin(C + \phi) : \sin(A + \phi) \sin(C + \phi) : \sin(A + \phi) \sin(B + \phi). \text{(Figure 27)}$$

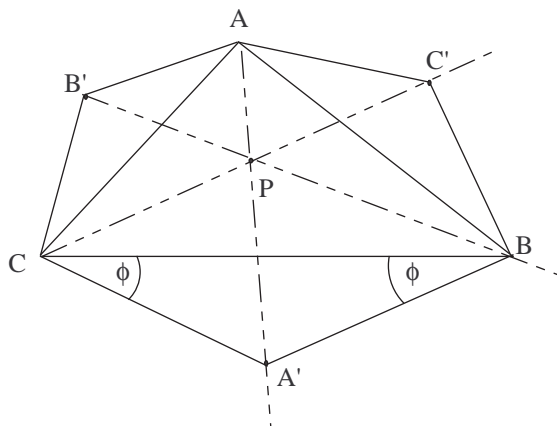


Figure 27: Concurrence

Finally, notice that the same linear algebra computation holds in the case of the given triangle being an obtuse triangle and for the isosceles triangles being constructed on the interior of the given triangle. In these cases, signed distances are used.

As the base angle ϕ varies, the locus of the point P forms a curve with the equation,

$$\frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0,$$

or equivalently,

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - b^2)}{\beta} + \frac{ab(a^2 - b^2)}{\gamma} = 0.$$

To show that the point P of concurrence is a solution to the equation above, we substitute the trilinear coordinates of point

$$P = \sin(B - \phi) \sin(C + \phi) : \sin(A + \phi) \sin(C + \phi) : \sin(A + \phi) \sin(B + \phi)$$

in the formula

$$\frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0$$

for α , β , and γ respectively.

Thus,

$$\begin{aligned} & \frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} =, \\ & \frac{\sin(B - C)}{\sin(B + \phi) \sin(C + \phi)} + \frac{\sin(C - A)}{\sin(C + \phi) \sin(A + \phi)} + \frac{\sin(A - B)}{\sin(A + \phi) \sin(B + \phi)} =, \\ & \frac{\sin(B - C) \sin(A + \phi) + \sin(C - A) \sin(B + \phi) + \sin(A - B) \sin(C + \phi)}{\sin(A + \phi) \sin(B + \phi) \sin(C + \phi)}. \end{aligned}$$

Using trigonometric identities, it can be shown that

$$\frac{\sin(B - C) \sin(A + \phi) + \sin(C - A) \sin(B + \phi) + \sin(A - B) \sin(C + \phi)}{\sin(A + \phi) \sin(B + \phi) \sin(C + \phi)} = 0.$$

Therefore, the locus of the point P as the base angle ϕ varies is given by,

$$\frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0.$$

Next we show that the equation

$$\frac{\sin(B, -C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = 0,$$

is equivalent to the equation

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - b^2)}{\beta} + \frac{ab(a^2 - b^2)}{\gamma} = 0.$$

First, construct the circumcircle of $\triangle ABC$. By moving vertex A along the circumcircle a right triangle, $\triangle ABC$, is constructed with AB having a length of twice the radius of the circumcircle and no change in $\angle A$. (Figure 28)

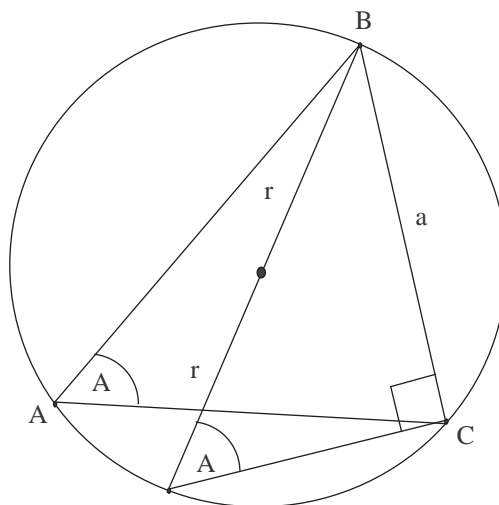


Figure 28: Circumcircle

Therefore, $\sin A = \frac{a}{2r}$. The angles $\sin B$ and $\sin C$ can be found in a similar manner such that

$$\sin A = \frac{a}{2r}, \sin B = \frac{b}{2r}, \text{ and } \sin C = \frac{c}{2r}.$$

By the law of cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

And $\sin(B - C) = \sin B \cos C + \sin C \cos B =$

$$\begin{aligned} & \frac{b}{2r} \frac{a^2 + b^2 - c^2}{2ab} + \frac{c}{2r} \frac{a^2 + c^2 - b^2}{2ac} = \\ & \frac{1}{4r} \left| \frac{a^2 + b^2 - c^2}{a} - \frac{a^2 + c^2 - b^2}{a} \right| \frac{abc}{abc} = \\ & \frac{1}{4abc} (2bc(b^2 - c^2)) = \\ & \frac{1}{2rabc} bc(b^2 - c^2). \end{aligned}$$

$\sin(C - A)$ and $\sin(A - B)$ can be derived in the same way so that,

$$\sin(B - C) = \frac{1}{2rabc}(b^2 - c^2)bc,$$

$$\sin(C - A) = \frac{1}{2rabc}(c^2 - a^2)ca,$$

$$\sin(A - B) = \frac{1}{2rabc}(c^2 - a^2)ab.$$

Therefore,

$$\begin{aligned} & \frac{\sin(B - C)}{\alpha} + \frac{\sin(C - A)}{\beta} + \frac{\sin(A - B)}{\gamma} = \\ & \frac{\frac{1}{2rabc}(b^2 - c^2)bc}{\alpha} + \frac{\frac{1}{2rabc}(c^2 - a^2)ca}{\beta} + \frac{\frac{1}{2rabc}(c^2 - a^2)ab}{\gamma} = \\ & \frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - a^2)}{\beta} + \frac{ab(c^2 - a^2)}{\gamma} = 0. \end{aligned}$$

Now we must show that the equation

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - a^2)}{\beta} + \frac{ab(c^2 - a^2)}{\gamma} = 0$$

is the equation of a conic section.

Using a different notation let,

$$\alpha = d_a,$$

$$\beta = d_b,$$

$$\gamma = d_c,$$

where d_a , d_b , and d_c are the distances to sides a, b, and c respectively. Also let the vertices have the following coordinates,

$$A = (x_1, y_1),$$

$$B = (x_2, y_2),$$

$$C = (x_3, y_3).$$

The vector perpendicular to \overrightarrow{AB} is

$$v = \begin{vmatrix} y_2 - y_1 \\ -(x_2 - x_1) \end{vmatrix}.$$

Let \overrightarrow{PA} be the vector from any point $P = (x, y)$ to A and be given by

$$r = \begin{vmatrix} x - x_1 \\ y - y_1 \end{vmatrix}.$$

Then the distance from P to side AB is given by projecting r onto v which gives

$$d_c = \frac{|(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)|}{c}.$$

In the same manner,

$$d_a = \frac{|(y_3 - y_2)(x - x_2) - (x_3 - x_2)(y - y_2)|}{a},$$

$$d_b = \frac{|(y_1 - y_3)(x - x_3) - (x_1 - x_3)(y - y_3)|}{b}.$$

Therefore,

$$\alpha = \frac{|(y_3 - y_2)(x - x_2) - (x_3 - x_2)(y - y_2)|}{a},$$

$$\beta = \frac{|(y_1 - y_3)(x - x_3) - (x_1 - x_3)(y - y_3)|}{b},$$

$$\gamma = \frac{|(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)|}{c}.$$

Plugging these values in for

$$\frac{bc(b^2 - c^2)}{\alpha} + \frac{ca(c^2 - b^2)}{\beta} + \frac{ab(a^2 - b^2)}{\gamma} = 0.$$

we get

$$\frac{(b^2 - c^2)}{(y_3 - y_2)(x - x_2) - (x_3 - x_2)(y - y_2)} + \frac{(c^2 - b^2)}{(y_1 - y_3)(x - x_3) - (x_1 - x_3)(y - y_3)} + \frac{(a^2 - b^2)}{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)} = 0.$$

By getting a common denominator, the equation becomes a quadratic involving x^2 , y^2 , and xy terms. This quadratic is the equation of a conic section. The equation actually defines a rectangular hyperbola. The proof that this conic section is a rectangular hyperbola is given by Eddy and Fritsch [2].

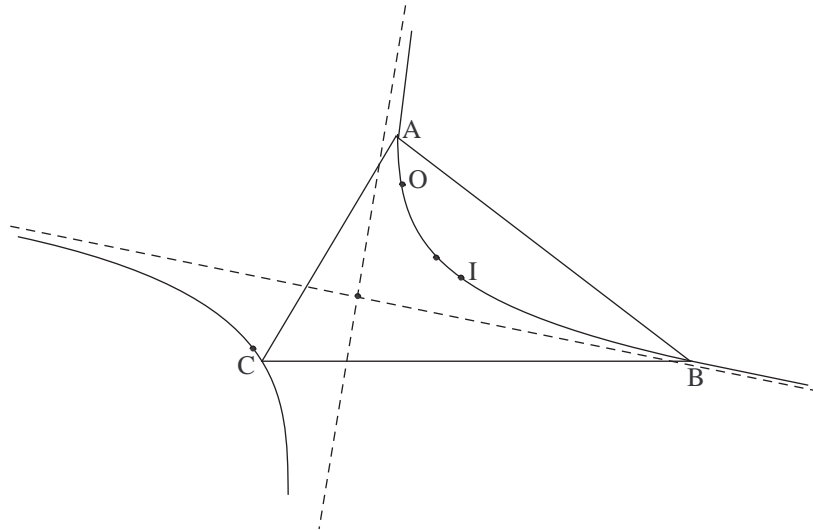


Figure 29: Kiepert's Hyperbola

Kiepert's Hyperbola is a solution to Lemoine's problem. Therefore, the values of ϕ are given which will provide the coordinates of the given triangle. The point of concurrence, P , lies on the vertex A when $\phi = -A$ and $\angle A$ is acute. When $\angle A$ is acute and $\phi = -A$ the lines BB' and CC' pass through vertex A . In Figure 30, BB' , CC' , and AA' intersect at the vertex A .

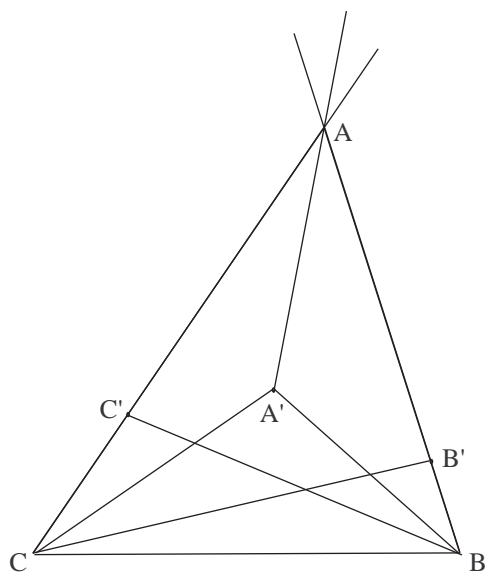


Figure 30: Concurrence at A

If $\angle A$ is obtuse, then $\phi = 180 - A$ when $P = A$. For vertices B and C , $P = B$ and $P = C$ when $\phi = -B$ and $\phi = -C$ respectively. \square

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