On Inverse Limits of Metric Spaces

by

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Abstract

Inverse limit spaces have been a topic studied in various fields of mathematics such as Algebra, Measure Theory, and Topology. Here, we present a theorem that can be summarized as a game in which a given compact metric space X is expressed as an inverse limit built step-by-step by two players. In the i-th step of the game, the first player gives an $\epsilon_i > 0$ and the second player gives a complete space Y_i and two maps, $f_i : X \to Y_i$ and $g_{i-1} : Y_i \to Y_{i-1}$ with the conditions that $\text{dist}(f_{i-1}, g_{i-1} \circ f_i) < \epsilon_i$, and f_i does not mend any two points of X with distance greater than some η_i where $\lim\limits_{i\to\infty} \eta_i = 0$. We prove that the first player can cause the sequence $(g_i \circ \cdots \circ g_{j-1} \circ f_j)_{j=i}^{\infty}$ to converge uniformly to a map $\tilde{f}_i : X \to Y_i$ for each i, and that the map $\tilde{f}$ induced by $\tilde{f}_0, \tilde{f}_1, \ldots$ is a homeomorphism of X onto $\varprojlim\{\tilde{f}_i(X), g_i\}_{i=0}^{\infty}$. Classic theorems by Anderson-Choquet, Mardešić-Segal, and Morton Brown can be reproved by using elements of this game.

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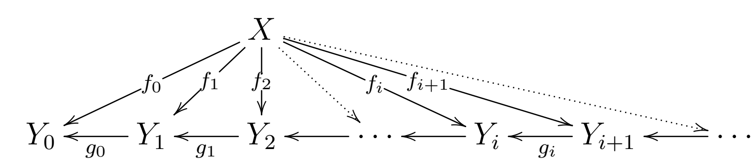
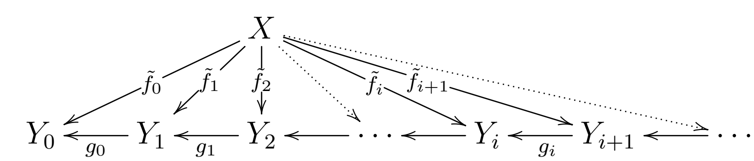
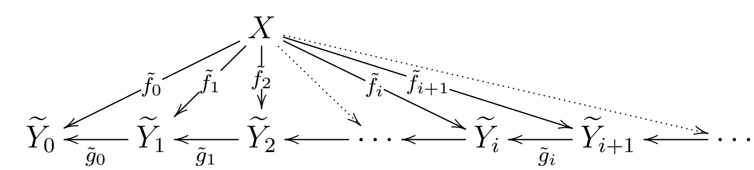
The following are other diagrams that appear throughout the paper.

$$X_0 \xleftarrow[f_0]{} X_1 \xleftarrow[f_1]{} X_2 \xleftarrow[f_2]{} \cdots \xleftarrow[f_{i-1}]{} X_i \xleftarrow[f_i]{} X_{i+1} \xleftarrow[f_{i+1}]{} \cdots$$







Chapter 1

Preliminary Material

In this chapter, we provide the relevant definitions, propositions, theorems, etc. for the topic at hand. The author will assume that any reader of this paper is familiar with basic definitions and concepts related to sets, number systems, relations, orderings, functions, sequences, and basic definitions and theorems from general topology. Any statements given without proof will have references beside them for the curious reader.

We will use $\mathbb{N} = \{1, 2, 3, \ldots\}$ to denote the positive integers and let $\omega = \{0, 1, 2, 3, \ldots\}$, giving respect to the usual linear ordering on both sets. By sequences, we mean infinite sequences. We will often index sequences starting with 0 or 1, depending on which is more appropriate in a given situation.

The first section of this chapter starts with basic definitions and theorems about metric spaces, equivalent metrics, convergent sequences, product spaces, compacta, continua, and more. In the second section, we define inverse sequences and inverse limit spaces, and we provide some examples, important properties, and notable theorems regarding inverse limits.

1.1 Metric Spaces

Definition 1.1. *Let X be a set. We call the function $d : X \times X \to [0, \infty)$ a **metric** on X if for any $x, y, z \in X$, the following hold:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$ (Symmetry)*;*
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)*.*

An ordered pair (X, d) is called a ***metric space*** *whenever X is a set and d is a metric on X.*

The above definition gives three properties that are shared by the notion of distance between points in a set. Thus, if $c \geq 0$ and x and y are two (possibly equal) points in a metric space (X, d) such that $d(x, y) = c$, we may say that x and y are a distance of c away from each other in X. A metric space is a type of topological space, and we may simply say "X is a space" to mean that X is a topological space which may or may not be a metric space. When the use of a metric is not needed in the statement of a theorem, proposition, or lemma, we may simply say "space" rather than "metric space."

Example 1.2. Arguably the most commonly used metric space is $(\mathbb{R}^n, d)$, where $n \in \mathbb{N}$ and where $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, is defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

for every $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, called the *usual metric* or *Euclidean metric.*

That the function given in Example 1.2 is a metric relies on the *Cauchy-Schwarz inequality,*

$$\left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right) \tag{1.1}$$

for every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, which gives us the triangle inequality. For a proof of Equation 1.1 and that d in Example 1.2 satisfies the triangle inequality, see Theorem 1.1.4 and Corollary 1.1.5 in [10]. When $n = 1$, then $\mathbb{R}^n = \mathbb{R}$, and it is common to use the absolute value of the difference between two real numbers, $|x - y|$, to denote the distance between x and y in $\mathbb{R}$.

Example 1.3. Let X be a set, and for all $x, y \in X$, define $d : X \times X \to \{0, 1\}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then one can easily show that d is a metric, called the *discrete metric*, and (X, d) is called a *discrete metric space*.

Definition 1.4. *Let (X, d) be a metric space and let $A \subset X$. The **diameter** of A in X, denoted* $\operatorname{diam}(A)$, *is defined by*

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

*We say A is **bounded** if* $\operatorname{diam}(A) < \infty$.

Definition 1.5. *Let (X, d) be a metric space and let $\mathcal{S}$ be a collection of subsets of X. Then the **mesh** of $\mathcal{S}$ is defined as*

$$\operatorname{mesh}(\mathcal{S}) = \sup\{\operatorname{diam}(S) \mid S \in \mathcal{S}\}.$$

Definition 1.6. *Let X and Y be metric spaces. If a function $f : X \to Y$ has the property that* $\operatorname{diam}\big(f(X)\big) < \infty$ *in Y, then f is called a **bounded function**.*

Definition 1.7. *Let X and Y be metric spaces and let f and g be bounded functions from X to Y. The **distance between** f **and** g, denoted* $\operatorname{dist}(f, g)$, *is given by*

$$\operatorname{dist}(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\},$$

where d is the metric on Y.

In the previous definition, most authors use the notation $\|f - g\|$ to denote the distance between two functions. This is often referred to as the "supremum norm."

Proposition 1.8. *Let X and Y be metric spaces and let $\mathfrak{B}(X,Y)$ be the set of bounded functions from X to Y. Then* dist *is a metric on $\mathfrak{B}(X,Y)$.*

Proof. Let d be the metric on Y. Since $0 \leq d(x,y) < \infty$ for all $x, y \in X$, then $0 \leq d(f(x), g(x)) < \infty$ for all $f, g \in \mathfrak{B}(X,Y)$ and $x \in X$. Thus, $0 \leq \text{dist}(f,g) < \infty$ for all $f, g \in \mathfrak{B}(X,Y)$, meaning dist is a function from $\mathfrak{B}(X,Y) \times \mathfrak{B}(X,Y)$ to $[0,\infty)$.

Let $f, g, h \in \mathfrak{B}(X,Y)$. If $\text{dist}(f,g) = 0$, then $\sup\{d(f(x), g(x)) \mid x \in X\} = 0$, which happens if and only if $f = g$; this satisfies the first condition. Since $d(f(x), g(x)) = d(g(x), f(x))$ for all $x \in X$, we have $\text{dist}(f,g) = \text{dist}(g,f)$, satisfying symmetry. Finally, since $d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x))$ for all $x \in X$, it follows that $\text{dist}(f,h) \leq \text{dist}(f,g) + \text{dist}(g,h)$, giving us the triangle inequality. Therefore, dist is a metric on $\mathfrak{B}(X,Y)$. □

Definition 1.9. *Let (X,d) be a metric space, let $x \in X$, and let $\epsilon > 0$. An **open ball of radius** ϵ **centered at** x, denoted $B_d(x,\epsilon)$, is defined by*

$$B_d(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\},$$

i.e., it is the set of all points of distance less than ϵ away from x.

Lemma 1.10. *Let (X,d) be a metric space, $x \in X$, $\epsilon > 0$, and $y \in B_d(x,\epsilon)$. Then there is a $\delta > 0$ such that $B_d(y,\delta) \subset B_d(x,\epsilon)$.*

Proof. Since $y \in B_d(x,\epsilon)$, $d(x,y) < \epsilon$, implying that $\epsilon - d(x,y) > 0$. Let $\delta = \epsilon - d(x,y)$. If $z \in B_d(y,\delta)$, then $d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta = d(x,y) + \epsilon - d(x,y) = \epsilon$, meaning $z \in B_d(x,\epsilon)$. Therefore, $B_d(y,\delta) \subset B_d(x,\epsilon)$. □

Definition 1.11. *Let (X,d) be a metric space. The **metric topology on** X, or, the **topology on** X **generated by the metric** d (or simply, the **metric topology**), is defined as the collection of all unions of collections of open balls contained in X. We call the members of this topology **open subsets of** X, or we simply say that are open in X. That is, any*

open *subset of* X *is either* X, $\emptyset$, *or* $\bigcup \mathscr{B}$ *where* $\mathscr{B}$ *is any nonempty collection of open balls contained in* X.

Lemma 1.12. *Let* (X, d) *be a metric space. Then the set* $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X,\ \epsilon > 0\}$ *is a base for the metric topology on* X.

Proof. There are two conditions for a collection of sets to be a base for a topology:

1. $\mathcal{B}$ covers X; i.e., for every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

2. If $B_1, B_2 \in \mathcal{B}$ are such that that $B_1 \cap B_2 \neq \emptyset$, then for every $x \in B_1 \cap B_2$, there exists a $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Let $x \in X$. Clearly, there is a $B \in \mathcal{B}$ such that $x \in B$. This satisfies the first condition.

Let $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$. By Lemma 1.10, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $B_d(x, \delta_1) \subset B_1$ and $B_d(x, \delta_2) \subset B_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $B_d(x, \delta) \in \mathcal{B}$ is such that $x \in B_d(x, \delta) \subset B_1 \cap B_2$, satisfying the second condition. $\square$

Lemma 1.13. *Let* (X, d) *be a metric space. A subset* U *of* X *is open in* X *if and only if for every* $x \in U$ *there is an* $\epsilon > 0$ *such that* $B_d(x, \epsilon) \subset U$.

Proof. Let U be open in X. Since $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X\ \epsilon > 0\}$ is a base for the topology on X, then there are sets $\{x_\alpha \mid \alpha \in \Lambda\} \subset U$ and $\{\epsilon_\alpha \mid \epsilon_\alpha > 0\ \forall \alpha \in \Lambda\}$ such that $U = \bigcup_{\alpha \in \Lambda} B_d(x_\alpha, \epsilon_\alpha)$. Let $x \in U$. Then $x \in B_d(x_\alpha, \epsilon_\alpha)$ for some $\alpha \in \Lambda$. By Lemma 1.10, there is some $\epsilon > 0$ such that $B_d(x, \epsilon) \subset B_d(x_\alpha, \epsilon_\alpha) \subset U$.

Let $U \subset X$ be such that for every $x \in U$ there is an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. Then since U is the union of all such open balls, U is also open. $\square$

Definition 1.14. *If* d *and* d' *are two metrics on the same set* X *which generate the same topology on* X, *then they are called* ***equivalent metrics***.

Lemma 1.15 ([32] Theorem 20.1)**.** *Let* (X, d) *be a metric space and let* $d' : X \times X \to [0, \infty)$ *be defined by*

$$d'(x, y) = \min\{d(x, y), 1\}.$$

Then d' is an equivalent metric on X.

Lemma 1.16. *Let (X, d) be a metric space, let $A \subset X$, and let d_A be the metric restricted to $A \times A$. Then (A, d_A) is a metric space.*

Proof. Because $0 \leq d(x, y) < \infty$ for all $x, y \in X$, it is also true that $0 \leq d(x, y) < \infty$ for all $x, y \in A$, whence d_A maps $A \times A$ into $[0, \infty)$. Symmetry and the triangle inequality follow immediately by the fact that d_A inherits these properties from d. □

Given a function f from a set X to a set Y, we say f is *one-to-one*, is an *injection*, or is *injective*, to mean that whenever $x_1, x_2 \in X$ are such that $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. We say that f is *onto*, is a *surjection*, or is *surjective* to mean that for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$. We say that f maps X *into* Y if $f(X) \subset Y$, and we say that f maps *onto* Y if $f(X) = Y$. A function that is both one-to-one and onto is called a *bijection*.

Definition 1.17. *Let X and Y be metric spaces. If the function $f : X \to Y$ is a continuous bijection such that $f^{-1} : Y \to X$ is also continuous, then f is called a* ***homeomorphism****, and X and Y are said the be* ***homeomorphic****.*

Definition 1.18. *Let X and Y be spaces, and let $f : X \to Y$ be a continuous injection. We say that f is an* ***embedding*** *into Y if $f(X)$ is homeomorphic to X, i.e., f is a homeomorphism between X and $f(X)$. If f is also a surjection, we say f is a* ***homeomorphism onto*** *Y, or simply, an* ***onto homeomorphism****.*

From a topological point of view, we are only concerned with equivalent metrics on a given space. In this regard, one may view a metric d on a set X as an equivalence class $[d]$ of metrics on X where $d' \sim d''$ if and only if $d' \in [d]$ and $d'' \in [d]$, but usually (X, d) means that the metric d is fixed. The range of values two equivalent metrics have is not a topological invariant, so it is not important topologically. For example, the interval $(0, 1)$ with the euclidean metric on $\mathbb{R}$ restricted to $(0, 1)$ is homeomorphic to $\mathbb{R}$ with the Euclidean metric, but the two have metrics with different ranges. However, the preservation of the values of a metric is important with respect to geometric properties. Another way to characterize equivalent metrics is that they define the same convergent sequences.

Definition 1.19. *Let (X, d) be a metric space. We say that a sequence of points $(x_n)_{n=0}^{\infty}$ in X* ***converge*** *to a point $x \in X$, denoted $x_n \to x$, if for every $\epsilon > 0$, there exists an $N \in \omega$ such that $d(x_n, x) < \epsilon$ for every $n \geq N$. In this case, we say that our sequence* ***converges*** *or is* ***convergent****.*

Definition 1.20. *Let X be a metric space with metric d. We say that a sequence of points $(x_i)_{i=0}^{\infty}$ in X is a* ***Cauchy sequence*** *(or is* ***Cauchy****) if for every $\epsilon > 0$ there exists an $N \in \omega$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n \geq N$.*

Proposition 1.21. *Let X be a metric space. If a sequence $(x_n)_{n=0}^{\infty}$ converges to a point $x \in X$, then $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.*

Proof. Let $(x_n)_{n=0}^{\infty}$ converge to a point $x \in X$, and let $\epsilon > 0$. Then there exists an $N \in \omega$ such that $d(x_n, x) < \epsilon/2$ for every $n \geq N$. Let $m, n \geq N$. Then

$$d(x_m, x_n) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon,$$

meaning $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. □

Definition 1.22. *A metric space X is called* ***complete*** *if every Cauchy sequence in X converges to a point in X.*

Not all metric spaces are complete. For example, let $\mathbb{Q}$ be the subspace of rational numbers contained in the real numbers endowed the the Euclidean metric restricted to $\mathbb{Q}$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{Q}$ where $x_n = (1 + 1/n)^n$ for every $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$, but it is well-known that this sequence converges to the irrational number, e. Therefore, $\mathbb{Q}$ is not a complete space. In fact, a consequence of the Baire Category Theorem ([10], Theorem 1.6.1) is that $\mathbb{Q}$ cannot be a complete metric space under any metric.

Definition 1.23. *Let X be a set. A collection $\mathcal{U}$ of subsets of X is said to* ***cover*** *X (or $\mathcal{U}$ is a* ***cover of*** *X) if $\bigcup \mathcal{U} = X$. If X is a metric space and $\mathcal{U}$ is a collection of open sets that cover X, then $\mathcal{U}$ is called an* ***open cover of*** *X. A* ***subcover*** *of $\mathcal{U}$ is a set $\mathcal{V} \subset \mathcal{U}$ that covers X. A* ***refinement*** *of $\mathcal{U}$ is a cover $\mathcal{W}$ such that each member of $\mathcal{W}$ is contained in some member of $\mathcal{U}$.*

Definition 1.24. *A metric space X is called* ***compact*** *if every open cover of X has a finite subcover.*

Definition 1.25. *A metric space X is called* ***sequentially compact*** *if every sequence in X has a convergent subsequence.*

Theorem 1.26 ([18], Proposition 3., pg. 84)**.** *A metric space is compact if and only if it is sequentially compact.*

Theorem 1.27. *Let X and Y be spaces where X is compact. If $f : X \to Y$ is continuous, then $f(X)$ is a compact subspace of Y. That is, continuous images of compact spaces are compact.*

Proof. Let $\mathcal{U}$ be an open cover of $f(X)$. Since f is continuous, $f^{-1}(U)$ is open in X for every $U \in \mathcal{U}$. Let $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$. Then $\mathcal{V}$ is a cover of X having a finite subcover $\mathcal{V}' = \{f^{-1}(U_k) \mid 0 \leq k \leq n\}$ since X is compact. Therefore, $\mathcal{U}' = \{U_k \mid 0 \leq k \leq n\}$ is a finite subcover of $\mathcal{U}$, whence $f(X)$ is compact. □

Definition 1.28. *Let (X, d) and (Y, ρ) be metric spaces. We say that the function $f : X \to Y$ is* ***uniformly continuous*** *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$.*

One may clearly see that a uniformly continuous function is also continuous.

Theorem 1.29 ([10], Theorem 1.4.10)**.** *Let X and Y be metric spaces where X is compact. If $f : X \to Y$ is a continuous function, then f is uniformly continuous.*

Theorem 1.30. *A closed subset of a compact space is compact.*

Proof. Let X be compact and let K be a closed subset of X. Let $\mathcal{V}$ be a collection of open subsets of X such that $K \subset \bigcup \mathcal{V}$, and let $\mathcal{U} = \{X \backslash K\} \cup \mathcal{V}$. Then $\mathcal{U}$ is an open cover of X and therefore has a finite subcover $\mathcal{U}'$. It follows that $\mathcal{V}' = \mathcal{U}' \cap \mathcal{V}$ is is a finite collection of open sets such that $K \subset \bigcup \mathcal{V}'$. Let $\mathcal{W} = \{V \cap K \mid V \in \mathcal{V}\}$, and let $\mathcal{W}' = \{V \cap K \mid V \in \mathcal{V}'\}$. Then every member of $\mathcal{W}$ is open in K, making it an open cover of K. Furthermore, $\mathcal{W}'$ is a finite subcover of $\mathcal{W}$, whence K is compact. □

Theorem 1.31. *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space and let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence in X. Since X is compact, Theorem 1.26 says that there is a subsequence, $(x_{n_j})_{j=0}^{\infty}$, of $(x_n)_{n=0}^{\infty}$ which converges to some point, x. Let $\epsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ is Cauchy, there exists an $N \in \omega$ such that $d(x_m, x_n) < \epsilon/2$ whenever $m, n \geq N$. Let $j \in \omega$ be such that $n_j \geq N$ and $d(x_{n_j}, x) < \epsilon/2$. Then for any $n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \epsilon$. Therefore, $(x_n)_{n=0}^{\infty}$ converges to x. □

Theorem 1.32. *If X and Y are metric spaces where X is compact and if $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

The previous theorem is more a corollary that follows from a similar statement that does not require the spaces X and Y be metric but that X is compact and Y is Hausdorff. Since all metric spaces are Hausdorff spaces, Theorem 1.32 follows. We stated this theorem with the added assumption that they are metric since this paper emphasizes metric spaces and not general topological spaces.

Definition 1.33. *Let (X, d) be a metric space and let $A \subset X$. The **ϵ-neighborhood** containing A, denoted $N_\epsilon(A)$, is defined by*

$$N_\epsilon(A) = \{y \in Y \mid \exists a \in A \text{ such that } d(a, y) \leq \epsilon\}.$$

Proposition 1.34. *Given a metric space X and a subset A of X, the following are true.*

1. $\bigcap_{\epsilon>0} N_\epsilon(A) = \overline{A}$, *and*

2. $N_\epsilon(A)$ *is closed if A is compact.*

Proof. To prove the first claim, we observe that if $x \in \bigcap_{\epsilon>0} N_\epsilon(A)$, then $x \in N_\epsilon(A)$ for every $\epsilon > 0$, which implies $x \in \overline{N_\epsilon(A)}$ for every $\epsilon > 0$, giving us $x \in \overline{A}$. If $x \in \overline{A}$, then x is in every closed set containing A, meaning $x \in \overline{N_\epsilon(A)}$ for every $\epsilon > 0$. Therefore, $x \in \overline{N_{1/(n+1)}(A)} \subset N_{1/n}(A)$ for every $n \in \mathbb{N}$, so that $x \in \bigcap_{\epsilon>0} N_\epsilon(A)$.

Suppose A is compact. Let $\epsilon > 0$, and let $x \in \overline{N_\epsilon(A)}$. We want to show $x \in N_\epsilon(A)$. Let $f : A \to \mathbb{R}$ be defined by $f(a) = d(a, x)$ for every $a \in A$. Since $x \in \overline{N_\epsilon(A)}$, we know that for every $n \in \mathbb{N}$, there is a $y_n \in N_\epsilon(A)$ such that $d(y_n, x) \leq 1/n$. Also, there exists $a_n \in A$ such that $d(a_n, y_n) \leq \epsilon$. Since A is compact, the sequence $(a_n)_{n=1}^\infty$ has a convergent subsequence. Without loss of generality, we will assume $a_n \to a$ where $a \in A$. Since f is continuous, we have $f(a_n) \to f(a)$, meaning

$$d(a, x) = f(a) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} d(a_n, x) \leq \lim_{n\to\infty} \big(d(a_n, y_n) + d(y_n, x)\big) \leq \lim_{n\to\infty} (\epsilon + 1/n) = \epsilon,$$

implying $x \in N_\epsilon(A)$, whence $N_\epsilon(A)$ is closed. □

Proposition 1.35 ([10], Proposition 3.1.4 & Exercise 2, pg.78)**.** *Let $(h_j)_{j=0}^\infty$ be a sequence of mappings of a compact metric space (X, ρ) into a complete metric space (Y, d) such that $\sum_{j=0}^\infty \operatorname{dist}(h_j, h_{j+1}) < \infty$. Then, the sequence $(h_j)_{j=0}^\infty$ converges uniformly to a mapping $\tilde{h} : X \to Y$, and*

$$\operatorname{dist}\left(h_k, \tilde{h}\right) \leq \sum_{j=k}^\infty \operatorname{dist}(h_j, h_{j+1})$$

for each positive integer k

Proof. Let $\epsilon > 0$. Since $\sum_{j=0}^\infty \operatorname{dist}(h_j, h_{j+1})$ converges, the sequence of tail ends

$$\Big(\sum_{j=k}^\infty \operatorname{dist}(h_j, h_{j+1})\Big)_{k=0}^\infty$$

converges to 0. Thus, for all $x \in X$ there exists $N_\epsilon \in \omega$ depending only on ϵ such that for every $n \geq N_\epsilon$, $\sum_{j=n}^{\infty} \operatorname{dist}(h_j, h_{j+1}) < \epsilon$. Thus, for all $x \in X$ and for all $m, n \in \mathbb{N}$ such that $m \geq n \geq N_\epsilon$, we have

$$d(h_n(x), h_m(x)) \leq \sum_{j=n}^{m-1} d(h_j(x), h_{j+1}(x)) \leq \sum_{j=n}^{\infty} d(h_j(x), h_{j+1}(x)) < \epsilon.$$

This implies that for all $x \in X$, $(h_j(x))_{j=0}^{\infty}$ is Cauchy. Since Y is complete, there exists a function $\tilde{h} : X \to Y$ such that for all $x \in X$, the sequence $(h_j(x))_{j=0}^{\infty}$ converges to $\tilde{h}(x)$, meaning that $(h_j)_{j=0}^{\infty}$ converges uniformly to $\tilde{h}$. Also, for every positive integer k and for all $x \in X$, we have

$$d(h_k(x), \tilde{h}(x)) \leq \sum_{j=k}^{\infty} d(h_j(x), h_{j+1}(x)),$$

giving us

$$\operatorname{dist}\left(h_k, \tilde{h}\right) \leq \sum_{j=k}^{\infty} \operatorname{dist}\left(h_j, h_{j+1}\right).$$

Since X is compact, each h_j is uniformly continuous by Theorem 1.29. We now show that $\tilde{h}$ is continuous by showing it is uniformly continuous. Let $\epsilon > 0$. Then there is a $j \in \omega$ such that $\operatorname{dist}(h_j, \tilde{h}) < \epsilon/3$. Also, there is a $\delta > 0$ such that whenever $x, y \in X$ are such that $\rho(x, y) < \delta$, we have $d(h_j(x), h_j(y)) < \epsilon/3$. Therefore, whenever $x, y \in X$ are such that $\rho(x, y) < \delta$, we have

$$d\big(\tilde{h}(x), \tilde{h}(y)\big) \leq d\big(\tilde{h}(x), h_j(x)\big) + d\big(h_j(x), h_j(y)\big) + d\big(h_j(y), \tilde{h}(y)\big) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

whence $\tilde{h}$ is continuous. □

Definition 1.36. *Let $\{X_i \mid i \in \omega\}$ be a set of metric spaces. The* ***product space****, $\prod_{i=0}^{\infty} X_i$, is the metric space consisting of sequences $(x_i)_{i=0}^{\infty}$ where $x_i \in X_i$ for every $i \in \omega$, and whose metric is defined in Proposition 1.37.*

The general definition of a product space considers any arbitrary collection $\{X_\alpha \mid \alpha \in \Lambda\}$ of topological spaces where Λ is some indexing set of any cardinality. The product of a countable collection of metric spaces is a metric space, but this is never the case for the product of an uncountable collection of nondegenerate metric spaces. For the purpose of this paper, we focus specifically on countably infinite products of metric spaces with ω as our indexing set.

Proposition 1.37. *Let $\{(X_i, d_i) \mid i \in \omega\}$ be a countably infinite collection of metric spaces and let $X = \prod_{i\in\omega} X_i$. Then the function $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} d'_i(x_i, y_i)$$

for every $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ in X, where d'_i is the equivalent metric on X_i as given by Lemma 1.15. Furthermore, $d(x, y) \leq 1$ for any $x, y \in X$.

Proof. That d maps $X \times X$ into $[0, \infty)$ is clear. Let $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$, and $z = (z_0, z_1, \ldots)$ be points in X. Then $d(x, y) = 0$ if and only if $d'_i(x_i, y_i) = 0$ for all $i \in \omega$ if and only if $x_i = y_i$ for all $i \in \omega$ if and only if $x = y$. Also, since $d'_i(x_i, y_i) = d'_i(y_i, x_i)$ for every $i \in \omega$, we have that $d(x, y) = d(y, x)$, giving us symmetry of d. Since

$$\begin{aligned} d(x, z) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} d'_i(x_i, z_i) &\leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \left(d'_i(x_i, y_i) + d'_i(y_i, z_i)\right) \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} d'_i(x_i, y_i) + \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} d'_i(y_i, z_i) = d(x, y) + d(y, z), \end{aligned}$$

we have that d satisfies the triangle inequality.

Since $\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = 1$ and, for every $i \in \omega$, $d'_i(a, b) \leq 1$ for every $a, b \in X_i$, we have $d(x, y) \leq 1$ for every $x, y \in X$. $\square$

From this point on, whenever we work with a product of a countably infinite collection of metric spaces (X_i, d_i), we may assume that $d_i \leq 1$ for every $i \in \omega$. Also, for convenience, we may use the symbol d to denote a metric in different spaces when there may be no confusion which space is referenced in a given situation.

Given a product space $X = \prod_{i=0}^{\infty} X_i$ of metric spaces X_i with metrics $d_i \leq 1$, the topology on X, called the *product topology*, is defined as the set of all unions of collections of its basic open sets which are of the form $U = \prod_{i=0}^{\infty} U_i$ where U_i is open in X_i for every $i \in \omega$, and $U_i = X_i$ for all by finitely many $i \in \omega$, meaning that the set of all basic open sets in X form a base for the product topology on X. To see that such sets are open, let F be a finite subset of ω and let $U = \prod_{i\in\omega} U_i$ where U_i is an open subset of X_i for every $i \in F$, and where $U_i = X_i$ for every $i \notin F$. Let $x = (x_0, x_1, \ldots) \in U$. Then for each $i \in F$, there is an $\epsilon_i > 0$ such that $B_{d_i}(x_i, \epsilon_i) \subset U_i$. Take $\epsilon = \min\{\frac{\epsilon_i}{2^{i+1}} \mid i \in F\}$. If $y = (y_0, y_1, \ldots)$ is such that $d(x, y) < \epsilon$, then $y_i \in U_i$ for every $i \in F$ and $y_i \in X_i$ for every $i \notin F$, whence $y \in U$, giving us $B_d(x, \epsilon) \subset U$. Likewise, we can show that any open ball B can be expressed as a union of basic open sets, which is equivalent to showing that for any $y \in B$, there is a basic open set U such that $y \in U \subset B$. To do this, let $\epsilon > 0$, let $x \in X$, and let $y \in B_d(x, \epsilon)$. Take $\delta = \epsilon - d(x, y)$. We know there exists some $n \in \omega$ such that $\sum_{i\geq n} \frac{1}{2^{i+1}} = \frac{1}{2^n} < \frac{\delta}{2}$. Thus, for each $i \in \{0, 1, \ldots, n-1\}$, let $U_i = B_{d_i}(y_i, \frac{\delta}{2})$, and let $U = \prod_{i=0}^{n-1} U_i \times \prod_{i=n}^{\infty} X_i$. Then if $z \in U$, we have $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = \epsilon$. Therefore, basic open sets and open balls in $\prod_{i=0}^{\infty} X_i$ generate the same topology.

Definition 1.38. *Given a product space $X = \prod_{i=0}^{\infty} X_i$, we define for each $i \in \omega$ the **projection map**, $\pi_i : X \to X_i$, by $\pi_i(x) = x_i$ for every $x = (x_0, x_1, x_2, \ldots) \in X$.*

Every projection map from a product space to its corresponding factor space is an open and continuous surjection ([10], Proposition 2.6.5). One can easily check that a product space $\prod_{i=0}^{\infty} X_i$ has as a subbase the collection of all subsets of the form $\pi_k^{-1}(U_k)$, where U_k is an open subset of X_k. Thus, basic open sets in the product space are of the form $U = \bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_{i_j})$ where $\{i_1, \ldots, i_n\}$ is some finite subset in ω and for each $j \in \{1, \ldots, n\}$, U_{i_j} is open in X_{i_j}. Clearly, this collection is contained in the collection of basic open sets given in the previous paragraph. However, it is itself not a base for $\prod_{i=0}^{\infty} X_i$.

Theorem 1.39 ([32], Theorem 19.6). *Let $X = \prod_{i=0}^{\infty} X_i$ be a product space, A a space, and let $s : A \to X$ be a function given by the equation, $s(a) = (s_i(a))_{i=0}^{\infty}$ where for each $i \in \omega$, s_i is a function from A into X_i. Then s is continuous if and only if s_i is continuous for every $i \in \omega$.*

Proof. Suppose s is continuous and let $i \in \omega$. Since the projection map π_i is continuous, and because $s_i = \pi_i \circ s$, it follows that s_i is continuous.

Assume now that s_i is continuous for every $i \in \omega$. Chose a member from the subbase of X, $\pi_k^{-1}(U_k)$, where U_k is open in X_k. Then $s^{-1}(\pi^{-1}(U_k)) = (\pi_k \circ s)^{-1}(U_k) = s_k^{-1}(U_k)$ which is open in A since s_k is continuous. Therefore, s is continuous. □

Theorem 1.40 ([10] Theorem 2.6.7). *The product of any set of compact spaces is compact.*

Definition 1.41 ([36], pg. 201). *Let (X, d) be a metric space and let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X. If $\lambda > 0$ is such that for each $x \in X$, there is some $\beta \in A$ such that $B_d(x, \lambda) \subset U_\beta$, then we say that λ is a* ***Lebesgue number*** *for the cover $\{U_\alpha \mid \alpha \in A\}$.*

The next lemma is known as The Lebesgue Covering Lemma, and it says that any open cover a compact metric space will have a Lebesgue number.

Lemma 1.42 ([36], pg. 201-202). *Let (X, d) be a compact metric space and let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of X. Then $\{U_\alpha \mid \alpha \in A\}$ has a Lebesgue number, $\lambda > 0$.*

Proof. Suppose that $\mathcal{U}$ has no Lebesgue number. Then for any $\lambda > 0$, there is an $x \in X$ such that $B_d(x, \lambda)$ is not contained in any member of $\mathcal{U}$. Thus, for every $n \in \mathbb{N}$, there is a point $x_n \in X$, such that $B_d(x_n, 1/n)$ is not contained in any member of $\mathcal{U}$. Since X is compact, it is sequentially compact, meaning $(x_n)_{n=1}^{\infty}$ has a convergent subsequence. Without loss of generality, assume $(x_n)_{n=1}^{\infty}$ converges to some $x \in X$. Since $\mathcal{U}$ is a cover, there is some $\beta \in A$ such that $x \in U_\beta$. Since U_β is open, then there is some $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U_\beta$. Let $N \in \mathbb{N}$ be such that $d(x, x_N) < \epsilon/2$ and $1/N < \epsilon/2$. Then if $y \in B_d(x_N, 1/N)$, then $d(x, y) \leq d(x, x_N) + d(x_N, y) < \epsilon/2 + 1/N < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $B_d(x_N, 1/N) \subset B_d(x, \epsilon) \subset U_\beta$, a contradiction. □

Definition 1.43. *A space X is called* ***connected*** *if it is not the union of two nonempty disjoint open subsets of X. Equivalently, X is connected if it has no nonempty proper subsets which are both closed and open in X.*

Theorem 1.44. *Let X and Y be metric spaces where X is connected. If $f : X \to Y$ is continuous, then $f(X)$ is a connected subspace of Y. That is, continuous images of connected spaces are connected.*

Proof. With out loss of generality, assume f is a continuous surjection. Then $f(X) = Y$, leaving us to show Y is connected. Suppose Y is not connected. Then there exists nonempty, disjoint open subsets U and V of Y such that $Y = U \cup V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty, disjoint open subsets of X such that $X = f^{-1}(U) \cup f^{-1}(V)$, implying X is not connected. □

Theorem 1.45 ([10] Theorem 2.6.10)**.** *The product of any set of connected spaces is connected.*

Definition 1.46. *A* ***compactum*** *(plural: compacta) is a nonempty compact metric space. A* ***continuum*** *(plural: continua) is a connected compactum.*

1.2 Inverse Limits

Definition 1.47. *A sequence $\{X_i, f_i\}_{i=0}^{\infty}$ where for each $i \in \omega$, X_i is a topological space and $f_i : X_{i+1} \to X_i$ is continuous, is called an* ***inverse sequence****. The mappings f_i are called* ***bonding maps****, and the spaces X_i are called* ***factor spaces*** *(or* ***component spaces*** *or* ***coordinate spaces****). The* ***inverse limit*** *of $\{X_i, f_i\}_{i=0}^{\infty}$, denoted $\varprojlim\{X_i, f_i\}_{i=0}^{\infty}$, is defined by*

$$\varprojlim\{X_i, f_i\}_{i=0}^{\infty} = \left\{(x_0, x_1, \ldots) \in \prod_{i=0}^{\infty} X_i \;\middle|\; x_i = f_i(x_{i+1}) \text{ for all } i \in \omega\right\}.$$

The elements of an inverse limit are sequences which we may call ***threads****.*

We can represent an inverse sequence $\{X_i, f_i\}_{i=0}^{\infty}$ by the infinite diagram

$$X_0 \xleftarrow[f_0]{} X_1 \xleftarrow[f_1]{} X_2 \xleftarrow[f_2]{} \cdots \xleftarrow[f_{i-1}]{} X_i \xleftarrow[f_i]{} X_{i+1} \xleftarrow[f_{i+1}]{} \cdots$$

Let $\{X_i, f_i\}_{i=0}^{\infty}$ be any inverse sequence. Given $j, k \in \omega$ such that $j \leq k$, the function $f_{j,k}$ is given by

$$f_{j,k} = \begin{cases} f_j \circ \cdots \circ f_{k-1} & j < k \\ \mathrm{id}_{X_j} & j = k \end{cases}$$

where id_{X_j} is the identity map of X_j onto itself. At times, the comma between the indices j and k will be omitted, yielding f_{jk}, although the comma will often be used to avoid confusion, such as $f_{j,j+1}$ instead of f_{jj+1}. Note also that if $k = j + 1$, then we simply have $f_{j,k} = f_{j,j+1} = f_j$.

Definition 1.48. *A **directed set** is a nonempty set, D, together with an ordering, $\preceq$, such that for every x, y, and z in D,*

1. *$x \preceq x$,*
2. *if $x \preceq y$ and $y \preceq z$, then $x \preceq z$, and*
3. *there exists a $w \in D$ such that $x \preceq w$ and $y \preceq w$.*

That is, $\preceq$ is a preoder on D with the property that any two members x and y of D have an upper bound in D. If, in addition, $\preceq$ has the properties that for every $x, y \in D$,

4. *$x \preceq y$ and $y \preceq x$ implies $x = y$, and*
5. *either $x \preceq y$ or $y \preceq x$,*

*then $\preceq$ is called a **total ordering** (or **linear ordering**), and D together with $\preceq$ is a **totally ordered set** (or **linearly ordered set**).*

A generalization of an inverse sequence, which we will call an *inverse limit system*, is a triple $\{X_\alpha, f_{\alpha\beta}, D\}$ where D is a directed set with ordering $\preceq$, X_α is a topological space for every $\alpha \in D$, and $f_{\alpha\beta} : X_\beta \to X_\alpha$ is a bonding map from for every $\alpha, \beta \in D$ such that $\alpha \preceq \beta$, ([16], pg. 75).

In fact, there is research interest in when $f_{\alpha\beta} : X_\beta \to X_\alpha$ is a set valued function from X_β to the set $2^{X_\alpha} = \{K \subset X_\alpha \mid K \text{ is closed in } X_\alpha \text{ and } K \neq \emptyset\}$ such that for every $x \in X_\beta$ and every open set $V \subset X_\alpha$ containing $f_{\alpha\beta}(x)$, there is an open set U contained in X_β with $x \in U$ such that for any $u \in U$, $f_{\alpha\beta}(u) \subset V$. This kind of function is what is known as an upper semi-continuous set-valued function. More on this notion can be found in [14]. This generalized approach with upper semi-continuous set-valued functions has become a newer branch in the study of inverse systems and is said to originate from William Mahavier in [21]. A recent paper on this topic by Scott Varagona can be found in [37], and an even further generalization can be found in [8]. Though this general definition is not pertinent to the main results in the second chapter of this paper, we will demonstrate some differences between usual inverse limits given in Definition 1.47 and inverse limits in the general setting. In the context of the general definition of inverse sequences, we will always take our directed set to be ω or $\mathbb{N}$ with the usual total ordering, $\leq$, our spaces X_i will be nonempty metric spaces, and our bonding maps will be usual continuous functions.

Example 1.49 ([33], Exercise 2.14)**.** Consider a sequence $(X_i)_{i=0}^{\infty}$ of spaces such that $X_{i+1} \subset X_i$ for every $i \in \omega$ (a nested sequence of spaces). If for each $i \in \omega$ we take $f_i : X_{i+1} \to X_i$ to be the inclusion mapping of X_{i+1} into X_i, then f_i is continuous and $f_i(X_{i+1}) = X_{i+1} = X_i \cap X_{i+1}$. If $x \in \bigcap_{i=0}^{\infty} X_i$, then $x = f_0(x) = f_1(x) = f_2(x) = \ldots$, so that we may identify x with $(x, x, x, \ldots)$. Therefore, $x \in \bigcap_{i=0}^{\infty} X_i \Leftrightarrow x = f_0(x) = f_1(x) = f_2(x) = \ldots \Leftrightarrow x \equiv (x, x, x, \ldots) \Leftrightarrow x \in \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$, giving us

$$\bigcap_{i=0}^{\infty} X_i = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}.$$

That is, the intersection of a nested sequence of spaces can be interpreted as an inverse limit.

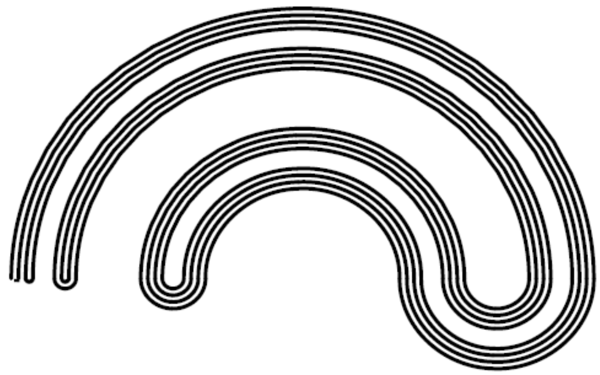


Figure 1.1: A partial geometric representation of the inverse limit defined in Example 1.51, taken from [16], Fig. 1.5.

Example 1.50 ([33], Proposition 2.3)**.** In fact, an inverse limit can be interpreted as a nested intersection of spaces. For example, take any inverse sequence, $\{X_i, f_i\}_{i=0}^{\infty}$, and for every $n \in \omega$, define

$$A_n = \left\{ (x_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty} X_i \;\middle|\; x_i = f_i(x_{i+1}) \text{ whenever } 0 \leq i \leq n \right\}. \tag{1.2}$$

Then it is easy to show that $A_{n+1} \subset A_n$ for all $n \in \omega$ and that $\varprojlim\{X_i, f_i\}_{i=0}^{\infty} = \bigcap_{n=0}^{\infty} A_n$.

Example 1.51. Let $\{X_i, f_i\}_{i=0}^{\infty}$ be the inverse sequence where for each $i \in \omega$, $X_i = [0,1]$ (the unit interval) and $f_i = f$ be defined by

$$f(t) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{2} \\ -2t + 2 & \frac{1}{2} < t \leq 1 \end{cases}$$

called the *tent map*. Then the space $X = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ is called the *Knaster Continuum*, otherwise known as the *Bucket Handle Continuum*. Geometrically, this is a space embeddable in the plane $\mathbb{R}^2$, a partial drawing of which can be found in Figure 1.1.

Example 1.52. Let $S^1 = \{z \in \mathbb{C} \mid z = e^{i\theta},\ 0 \leq \theta < 2\pi\}$ be the unit circle in the complex plane and let the inverse sequence $\{X_i, f_i\}_{i=0}^{\infty}$ be such that for all $i \in \omega$, $X_i = S^1$ and $f_i(z) = z^p$ where $p \in \mathbb{N}\backslash\{1\}$. Then the space $\Sigma_p = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ is called the *p-adic*

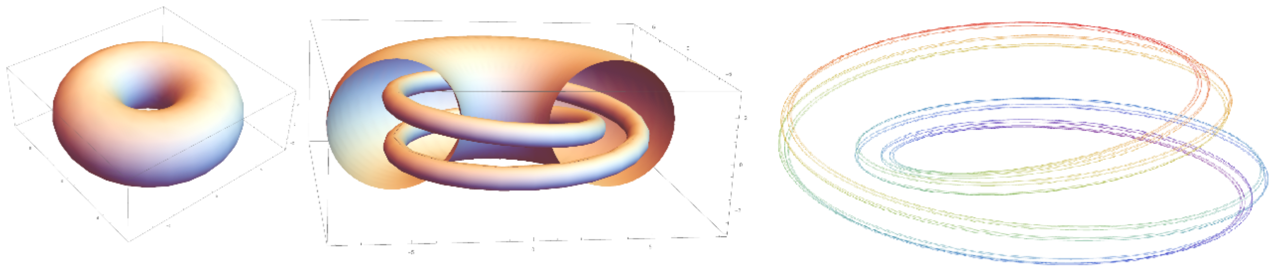


Figure 1.2: T_0, T_1 embedded in the interior of T_0, and Σ_2. (These figures are from James Keesling. [19])

solenoid. Geometrically, this is the space obtained as follows: Start with a solid torus T_0, say with cross diameter equal to 1, and embed in its interior another solid torus T_1 with cross diameter less than 1/2 that wraps around p-many times inside the interior of T_0 without crossing itself. Inductively define T_i for $i > 1$ to be the solid torus with cross diameter less that $\frac{1}{i+1}$ such that it is embedded in the interior of T_{i-1}, wrapping around p-times inside of the interior of T_{i-1} without crossing itself. One can then show that Σ_p is homeomorphic to $\bigcap_{i=0}^{\infty} T_i$. See Figure 1.2 for a geometric representation of the *dyadic solenoid*, Σ_2.

David van Dantzig first introduced solenoids in [11]. Generally, one can consider a solenoid Σ_P, where $P = (p_0, p_1, \ldots)$ is a sequence of positive integers, infinitely many of which are greater than 1 (to avoid triviality), and where for each $i \in \omega$, the bonding map $f_i : S^1 \to S^1$ is defined by $f_i(z) = z^{p_i}$. Even more, we may simply let P be a sequence of prime numbers since if there is a $p_i \in P$ which is the multiple of n prime factors, $q_{i,1}, \ldots, q_{i,n}$, then $z^{p_i} = z^{q_{i,1} \cdots q_{i,n}}$, which is the composition $f_{q_{i,1}} \circ \cdots \circ f_{q_{i,n}} : S^1 \to S^1$ of functions defined by $f_{q_{i,j}} = z^{q_{i,j}}$ for every $j \in \{1, \ldots, n\}$. (See Theorem 1.63.) Again, this is the same space obtained by embedding a solid torus T_{i+1} into the interior of a solid torus T_i by wrapping T_{i+1} around p_i times in the interior of T_i without crossing itself for each $i \in \omega$ with the cross diameters converging to 0 as described previously, and taking the nested intersection of the images of these embeddings. All solenoids are topological groups and are inverse limits of of an inverse sequence of factors spaces, S^1, which is also a topological group. It has been proved by Bing in [4] that P-adic solenoids cannot be embedded in the plane.

The previous two examples yield spaces that have a variety of exotic properties (in particular, indecomposability) that we will not get into here. This demonstrates that inverse limits have the advantage of describing complicated topological spaces in terms of simpler ones.

Theorem 1.53. *The inverse limit of an inverse sequence of nonempty compact metric spaces is a nonempty compact metric space.*

Proof. It is well-known that if $Y = \bigcap_{n=0}^{\infty} A_n$ where $A_{n+1} \subset A_n$ and A_n is a nonempty compact metric space for all $n \in \omega$, then Y is a nonempty compact metric space. Let $\{X_i, f_i\}_{i=0}^{\infty}$ be an inverse sequence where each X_i is a nonempty compact metric space and for each $n \in \omega$, let A_n be as in Equation 1.2. Then for each $n \in \omega$, A_n is a nonempty and also compact since it is the product of compact spaces; also, $A_{n+1} \subset A_n$. Therefore, $\varprojlim\{X_i, f_i\}_{i=0}^{\infty} = \bigcap_{n=0}^{\infty} A_n$ is a nonempty compact metric space. □

The following is an example of an inverse sequence of metric spaces which are all nonempty and non-compact whose inverse limit is empty.

Example 1.54. For each $i \in \mathbb{N}$, let X_i be the open interval $(0, \frac{1}{i})$ which is nonempty and non-compact. Let f_i be the inclusion mapping of $(0, \frac{1}{i+1})$ into $(0, \frac{1}{i})$. Then it is clear that

$$\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \bigcap_{i=1}^{\infty} \left(0, \frac{1}{i}\right) = \emptyset.$$

In the next example, we show how an inverse limit space indexed by a directed set that is not totally ordered can be empty even though each of its factor spaces are nonempty compact metric spaces.

Example 1.55 ([16], Example 106)**.** *Let $D = \mathbb{N} \cup \{a, b\}$ be such that $i \preceq j$ if and only if $i \leq j$, $a \preceq j$ if and only if $j \geq 2$, $b \preceq j$ if and only if $j \geq 3$, $1 \preceq b$, and $a \preceq b$. (That is, a and 1 do not compare while b and 2 do not compare.) For each $\alpha \in D$, let $X_\alpha = \{0, 1\}$.*

Let f_{ij} be the identity function $Id = \mathrm{id}_{\{0,1\}}$ when $3 \le i \le j$. Also let $f_{1,2} = Id$, $f_{a2} = Id$, $f_{ab} = Id$, and $f_{1b} = 1 - Id$. Then $\varprojlim\{X_\alpha, f_{\alpha\beta}, D\} = \emptyset$.

Proof. Suppose there exists some $x \in \varprojlim\{X_\alpha, f_{\alpha\beta}, D\}$. If $x_1 = 0$, then $x_2 = 0$, meaning that $x_a = 0$ and $x_b = 1$ which is a contradiction. Similarly, if $x_1 = 1$, then $x_2 = 1$, meaning $x_a = 1$ and $x_b = 0$ which is also a contradiction. □

Example 1.56. *Let $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ be the metric spaces with the Euclidean metric restricted to X. For every $i \in \mathbb{N}$, let $X_i = \{1, \ldots, i\}$ with the discrete metric, and let $f_i : X_{i+1} \to X_i$ be defined by*

$$f_i(j) = \begin{cases} j & 1 \le j \le i \\ i & j = i+1 \end{cases}$$

Then X is homeomorphic to $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

Proof. Let $Y = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, which is compact since each X_i is compact, and let $h : X \to Y$ be defined by

$$h(x) = \begin{cases} (1, 2, 3, \ldots) & x = 0 \\ (1, \ldots, n-1, n, n, n, \ldots) & x = \frac{1}{n} \end{cases}$$

Since every member of Y is of one of the two forms provided in the definition of h, it follows that h is a surjection. If m and n are positive integers such that $m \ne n$, then $h(1/m) = (1, \ldots, m-1, m, m, m, \ldots) \ne (1, \ldots, n-1, n, n, n, \ldots) = h(1/n)$, whence h is injective. Thus, we are left to show that h is continuous. Let $\epsilon > 0$, and let $N \in \mathbb{N}$ be such that for every $n \ge N$, we have $\sum_{i \ge n+1} \frac{1}{2^{i+1}} = \frac{1}{2^n} < \epsilon$. Let $\delta = \frac{1}{N(N+1)}$. Then if $|x - y| < \delta$, we have

$$d\big(h(x), h(y)\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i\Big(\pi_i\big(h(x)\big), \pi_i\big(h(y)\big)\Big) < \epsilon.$$

□

Proposition 1.57 ([33, Proposition 1.7]). *Let $(A_n)_{n=0}^{\infty}$ be a sequence of compact metric spaces such that $A_{n+1} \subset A_n$ for every $n \in \omega$, and let $X = \bigcap_{n=0}^{\infty} A_n$. If U is an open subset* *of A_0 such that $X \subset U$, then there exists an $N \in \omega$ such that $A_n \subset U$ for all $n \geq N$. In particular, if $A_n \neq \emptyset$ for all $n \in \omega$, then $X \neq \emptyset$.*

The proof of the next theorem applies Proposition 1.57 and follows the proof of [33], Theorem 1.8, which states that the intersection of a nested sequence of continua is a continuum.

Theorem 1.58 ([33], Theorem 2.4)**.** *The inverse limit of an inverse sequence of continua is a continuum.*

Proof. Let $\{X_i, f_i\}_{i=0}^{\infty}$ be an inverse sequence where each space is a continuum and let $X = \varprojlim\{X_i, f_i\}_{i=0}^{\infty} = \bigcap_{n=0}^{\infty} A_n$ where A_n is as in Equation 1.2 for each $n \in \omega$. Note that for each $n \in \omega$, A_n is the product of a collection of connected spaces and is therefore connected. Since continua are nonempty and compact, we know by Theorem 1.53 that X is a nonempty compact metric space. We must only show that X is connected. Suppose X is not connected. Then $X = H \cup K$ where H and K are nonempty, disjoint, and closed subsets of X and therefore of A_0. Since A_0 is also a normal space, there exist nonempty, disjoint open subsets of A_0, O and V, such that $H \subset O$ and $K \subset V$. Let $U = O \cup V$. Then by Proposition 1.57, we have that for some $N \in \omega$, $A_N \subset U$ and $A_N = (A_N \cap O) \cup (A_N \cap V)$, where $A_N \cap O$ and $A_N \cap V$ are nonempty, disjoint open subsets of A_N. Therefore, A_N is not connected, giving us a contradiction. □

Lemma 1.59 ([33], Lemma 2.6)**.** *Let $\{X_i, f_i\}_{i=0}^{\infty}$ be an inverse sequence of metric spaces with inverse limit X, and let A be a compact subset of X. Then $\{\pi_i(A), f_i \restriction \pi_{i+1}(A)\}_{i=0}^{\infty}$ is an inverse sequence with onto bonding maps, and*

$$\varprojlim\{\pi_i(A), f_i \restriction \pi_{i+1}(A)\}_{i=0}^{\infty} = A = \Big[\prod_{i=0}^{\infty} \pi_i(A)\Big] \cap X. \tag{1.3}$$

The subject of this paper is on inverse limits of inverse sequences with factor spaces being metric, usually compact, and sometimes connected as well. That is, we study inverse limits of compacta and continua. When we use the projection mapping π_i, we will always assume it is restricted to the inverse limit. That is, when we say that $\pi_i : \varprojlim\{X_i, f_i\}_{i=0}^{\infty} \to X_i$ is the projection map, we mean that it is the projection map restricted the inverse limit. Although any projection map from an entire product space to its corresponding factor space is surjective, this is not necessarily the case when the projection maps are restricted to the inverse limit. However, given an inverse sequence, all of the projection maps restricted to its inverse limit are surjections if and only if all of the bonding maps are surjections ([20], Remark 2.1.6).

Lemma 1.60. *Given an inverse limit $X = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$, if $i, j \in \omega$ are such that $i \leq j$, then $\pi_i = f_{ij} \circ \pi_j$.*

Proof. Given $x = (x_0, x_1, x_2, \ldots) \in X$, we have $\pi_i(x) = x_i = f_{ij}(x_j) = f_{ij}(\pi_j(x_j)) = f_{ij} \circ \pi_j(x)$. □

Theorem 1.61 ([20], Proposition 2.1.9.)**.** *Given an inverse limit $X = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$, the collection $\mathcal{S} = \{\pi_i^{-1}(U_i) \mid U_i$ is an open subset of $X_i\}$ is a base for X.*

Proof. It is easy to see that $\mathcal{S}$ is a subbase for X. Since X is a subspace of $\prod_{i=0}^{\infty} X_i$, we may take a basic open set in X to be of the form $U = \bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_{i_j})$, where $i_1 < \ldots < i_n$. Since each $f_{i_j i_m}$ is continuous whenever $1 \leq j < m \leq n$, we have that $f_{i_j i_n}^{-1}(U_{i_j})$ is open in X_{i_n}, and thus, so is $V_{i_n} = \bigcap_{j=1}^{n} f_{i_j i_n}^{-1}(U_{i_j})$. Therefore,

$$\pi_{i_n}^{-1}(V_{i_n}) = \pi_{i_n}^{-1}\left(\bigcap_{j=1}^{n} f_{i_j i_n}^{-1}(U_{i_j})\right) = \bigcap_{j=1}^{n} \pi_{i_n}^{-1} f_{i_j i_n}^{-1}(U_{i_j}) = \bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_{i_j}) = U,$$

meaning that a basic open set in X can be expressed as a member of $\mathcal{S}$. Therefore, $\mathcal{S}$ is a base for X. □

There is a considerable body of research regarding inverse limits where the factor spaces are always the unit interval, $I = [0, 1]$, and the same bonding map $f : I \to I$ is always applied. Example 1.51 is such an inverse limit. In this case, we may use $\{I, f\}$ to denote the inverse sequence and $\varprojlim\{I, f\}$ its inverse limit. Another example would be the $\sin(\frac{1}{x})$-continuum, $C = \{(x, \sin(\frac{1}{x})) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$, which can be expressed as the inverse limit of the inverse sequence $\{I, f\}$, where

$$f(t) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{2} \\ -t + \frac{3}{2} & \frac{1}{2} < t \leq 1 \end{cases}$$

The mapping in Example 1.51 is a type of *tent map*. In general, a tent map is a map $T_s : I \to I$ defined by

$$T_s(t) = \begin{cases} st & 0 \leq t \leq \frac{1}{2} \\ s(1-t) & \frac{1}{2} < t \leq 1 \end{cases}$$

where $s \in [1, 2]$ is called the *slope* of the tent map. A famous conjecture dealing with tent maps, known as Ingram's conjecture, states that if $1 \leq s_1 < s_2 \leq 2$, then $\varprojlim\{I, T_{s_1}\}$ is not homeomorphic to $\varprojlim\{I, T_{s_2}\}$. This conjecture was proved by M. Barge, H. Bruin, and S. Štimac in [3].

More examples and theorems regarding inverse limits of the unit interval with the same bonding map can be found in [15] and [16]. In general, if the same factor space X and bonding map $f : X \to X$ is used in an inverse sequence, the inverse sequence can be denoted by $\{X, f\}$ and its inverse limit by $\varprojlim\{X, f\}$. If the factor spaces are a fixed metric space X and the bonding maps f_i can vary for any $i \in \omega$, then we can denote the inverse sequence by $\{X, f_i\}_{i=0}^{\infty}$ and its inverse limit by $\varprojlim\{X, f_i\}_{i=0}^{\infty}$.

Theorem 1.62. *Let $\{X_i, f_i\}$ be an inverse sequence of metric spaces where for each $i \in \omega$, f_i is a homeomorphism of X_{i+1} onto X_i, and let $X = \varprojlim\{X_i, f_i\}_{i=0}^{\infty}$. Then for each $i \in \omega$, the projection map $\pi_i : X \to X_i$ is a homeomorphism.*

Proof. Since f_i is a homeomorphism and thus a bijection for each $i \in \omega$, then for any $i \in \omega$ and any $x_i \in X_i$, there exists a unique $x \in \varprojlim\{X, f_i\}_{i=0}^{\infty}$ such that $\pi_i(x) = x_i$. Therefore, by

Theorem 1.32 and Theorem 1.53, since each π_i is a continuous bijection from the compact space $\varprojlim\{X, f_i\}_{i=0}^{\infty}$ to a metric space X_i, it is a homeomorphism. $\square$

Theorem 1.63 ([20], Theorem 2.1.38.)**.** *Let $\{X_i, f_i\}_{i=0}^{\infty}$ be an inverse sequence, let $(i_n)_{n=0}^{\infty}$ be an increasing subsequence in ω, let $g_n = f_{i_n, i_{n+1}}$, and let $Y_n = X_{i_n}$. Then $\varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ is homeomorphic to $\varprojlim\{Y_n, g_n\}_{n=0}^{\infty}$.*

Proof. Let $h : X \to Y$ be such that for every $x = (x_0, x_1, x_2, \ldots) \in X$, $h(x) = (x_{i_0}, x_{i_1}, x_{i_2}, \ldots)$. Note that for each $j \in \omega$, $x_{i_j} = f_{i_j, i_{j+1}}(x_{i_{j+1}}) = g_j(x_{i_{j+1}})$, meaning that h is well-defined. For each $j \in \omega$, let $k(j)$ be the least integer such that $j \leq i_{k(j)}$. For every $y = (y_0, y_1, y_2, \ldots) \in Y$, let $s_j(y) = f_{j, i_{k(j)}}(y_{k(j)})$ and let $s(y) = (s_0(y), s_1(y), s_2(y), \ldots)$. Let $j \in \omega$ and suppose $k(j) = k(j+1)$. Then $s_j(y) = f_{j, i_{k(j)}}(y_{k(j)}) = f_j \circ f_{j+1, i_{k(j)}}(y_{k(j)}) = f_j \circ f_{j+1, i_{k(j+1)}}(y_{k(j+1)}) = f_j(s_{j+1}(y))$. If $k(j) < k(j+1)$, then $j = i_{k(j)}$, and one can observe that $i_{k(j+1)} = i_{k(j)+1}$. Thus, $s_j(y) = f_{j, i_{k(j)}}(y_{k(j)}) = f_{i_{k(j)}, i_{k(j)}}(y_{k(j)}) = y_{k(j)} = g_{k(j)}(y_{k(j)+1}) = f_{i_{k(j)}, i_{k(j)+1}}(y_{k(j)+1}) = f_{j, i_{k(j+1)}}(y_{k(j+1)}) = f_j \circ f_{j+1, i_{k(j+1)}}(y_{k(j+1)}) = f_j(s_{j+1}(y))$, implying $s(y) \in X$ for every $y \in Y$, meaning s is well-defined. One can see that both h and s are continuous by Theorem 1.39. Also, one can observe that $h \circ s$ is the identity on Y and that $s \circ h$ is the identity on X, making h and s inverses of each other. Therefore, X is homeomorphic to Y. $\square$

Theorem 1.63 is called The Subsequence Theorem by some authors, and it is not always true for generalized inverse limits of inverse systems whose bonding maps are upper semi-continuous set-valued functions.

Corollary 1.64. *If $\{X_i, f_i\}_{i=0}^{\infty}$ is an inverse sequence of compact metrics spaces with surjective bonding maps, then $\varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ is homeomorphic to $\varprojlim\{X_i, f_i\}_{i=n}^{\infty}$ for every $n \in \omega$.*

Corollary 1.65 ([16], Theorem 20)**.** *Let f be a continuous function of a space X into itself, and let $Y = \varprojlim\{X, f\}$. Then the function $h : Y \to Y$ defined by $h(x) = (f(x_0), x_0, x_1, \ldots)$, called the **shift map**, is a homeomorphism of Y onto Y.*

Corollary 1.65 also follows as a result of Corollary 1.64, as one can see it is the result of taking away the first space and bonding map in the sequence.

Constructions of continua (in particular, indecomposable continua) as inverse limits of inverse sequences were first introduced by P.S. Alexandroff, S. Lefschetz, and Hans Freudenthal in the 1920's and 1930's [7]. Inverse limits were discussed more extensively by Eilenberg and Steenrod in the 1950's in Chapter VIII of [12]. An often referenced theorem by Freudenthal is the following.

Theorem 1.66 ([13], Satz 1, pg. 229)**.** *Every compact metric space X is homeomorphic to an inverse limit of an inverse sequence of compact polyhedra, each of whose covering dimension is less than or equal to the covering dimension of X.*

The original proof of this theorem, written in German, can be found in [13]. For a proof written in English, the reader can refer to [34]. Mardešić showed in [23] that Theorem 1.66 for general compact Hausdorff spaces is not always true, with compact Hausdorff spaces of covering dimension equal to 1 and small inductive dimension greater than 1 not having this property. Because of this, Mardešić and Leonard Rubin introduced the notion of approximate inverse systems in [25], in which they show that any compact Hausdorff space with covering dimension less than or equal to n coincides with the limit of some approximate inverse system of compact polyhedra each with dimension less than or equal to n. Watanabe extended this result in [38] by showing that a compact Hausdorff space has covering dimension less than or equal to n if and only if it is the limit of some approximate inverse system of compact polyhedra each with dimension less than or equal to n.

A consequence of Theorem 1.66 worth noting is the following.

Corollary 1.67. *Every 0-dimensional compact metric space is the inverse limit of an inverse sequence of nonempty finite discrete spaces.*

In fact, Example 1.56 follows from Corollary 1.67. The converse of Corollary 1.67 is also true, in that any inverse limit of an inverse sequence of nonempty finite discrete spaces is a 0-dimensional compact metric space. Indeed, since every finite discrete space is compact and metric, any inverse limit of an inverse sequence of finite discrete spaces is also compact and metric. That such an inverse limit would be 0-dimensional follows from the fact that its topology would have a base of closed and open sets inherited from the fact that the set of singletons in each finite discrete spaces is a base of closed and open sets in that space. A nice proof of Corollary 1.67, which can be seen as a theorem of its own, can be found in [9], Theorem 6.C.5. Here, we provide a proof of our own, but we must first state and prove the following proposition and lemma.

Proposition 1.68. *Let X be a space that has a finite covering, $\mathcal{U}$, of closed and open sets. Then there is a refinement $\mathcal{V}$ of $\mathcal{U}$, all of whose members are pairwise disjoint.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$, and for each $j \in \{1, \ldots, n\}$, define $V_j = U_j \setminus \bigcup_{i=j+1}^{n} U_i$, and let $\mathcal{V} = \{V_1, \ldots, V_n\}$. Then $\mathcal{V}$ refines $\mathcal{U}$ and each of its members are pairwise disjoint. □

Lemma 1.69. *Let (X, d) be a 0-dimensional compact metric space. Then there is a sequence $(\mathcal{V}^0, \mathcal{V}^1, \mathcal{V}^2, \ldots)$ of finite closed and open coverings of X such that for every $i \in \omega$, all of the members in $\mathcal{V}^i$ are pairwise disjoint, $\mathcal{V}^{i+1}$ refines $\mathcal{V}^i$, and* $\mathrm{mesh}(\mathcal{V}^i) \to 0$.

Proof. Since X has a base of closed and open sets, we can take a cover $\mathcal{U}^0$ of closed and open subsets of X. Without loss of generality, since X is compact, we may assume assume $\mathcal{U}^0$ is a finite collection, $\{U_1^0, \ldots, U_{n(0)}^0\}$. Let $\mathcal{V}^0 = \{V_1^0, \ldots, V_{n(0)}^0\}$, each of whose members is defined by

$$V_j^0 = U_j^0 \setminus \bigcup_{i=j+1}^{n(0)} U_i^0 \text{ for every } j \in \{1, \ldots, n(0)\}.$$

Since each member of $\mathcal{U}^0$ is closed and open and because $\mathcal{U}^0$ is finite, every member of $\mathcal{V}^0$ is also closed and open. Moreover, $\mathcal{V}^0$ covers X and all of its members are pairwise disjoint. We may now refine $\mathcal{V}^0$; indeed, since it is a cover of X, it has a Lebesgue number, $\lambda_0 > 0$ by Lemma 1.42. That is, for any $A \subset X$ such that $\mathrm{diam}(A) < \lambda_0$, there is a member of $\mathcal{V}^0$ containing A. Let $\gamma_1 = \min\{\lambda_0, 1\}$. Let $\mathcal{U}^1 = \{U_1^1, \ldots, U_{n(1)}^1\}$ be a closed and open cover of X such that,

$$\mathrm{mesh}(\mathcal{U}^1) := \sup\{\mathrm{diam}(S) \mid S \in \mathcal{U}^1\} < \gamma_1.$$

Let $\mathcal{V}^1 = \{V_1^1, \ldots, V_{n(1)}^1\}$, each of whose members is defined by

$$V_j^1 = U_j^1 \setminus \bigcup_{i=j+1}^{n(1)} U_i^1 \text{ for every } j \in \{1, \ldots, n(1)\}.$$

Then $\mathcal{V}^1$ is a closed and open cover of X whose members are pairwise disjoint, and each of which is contained in some member of $\mathcal{V}^0$.

Inductively, for any positive integer i, suppose an open covers, $\mathcal{V}^0, \ldots, \mathcal{V}^{i-1}$, each whose members are pairwise disjoint, have been constructed. Let $\lambda_{i-1} > 0$ be the Lebesgue number for $\mathcal{V}^{i-1}$, let $\gamma_i = \min\{\lambda_{i-1}, \frac{1}{2^{i-1}}\}$, and let $\mathcal{U}^i = \{U_1^i, \ldots U_{n(i)}^i\}$ be an open cover of X such that

$$\text{mesh}(\mathcal{U}^i) := \sup\{\text{diam}(S) \mid S \in \mathcal{U}^i\} < \gamma_i.$$

Let $\mathcal{V}^i = \{V_1^i, \ldots, V_{n(i)}^i\}$, each of whose members is defined by

$$V_j^i = U_j^i \setminus \bigcup_{i=j+1}^{n(i)} U_j^i \text{ for every } j \in \{1, \ldots, n(i)\}.$$

Then $\mathcal{V}^i$ is a closed and open cover of X whose members are pairwise disjoint, each of which is contained in some member of $\mathcal{V}^{i-1}$. $\square$

Proof of Corollary 1.67. Let X be a 0-dimensional compact metric space with metric, d. For every $i \in \omega$, let $\mathcal{V}^i$ be the finite closed and open covering of X with pairwise disjoint members such that $\mathcal{V}^{i+1}$ refines $\mathcal{V}^i$ as guaranteed by Lemma 1.69, and let γ_i be as defined in its proof. We will assume the members of $\mathcal{V}^i$ are nonempty for every $i \in \omega$. Let $g_i : \mathcal{V}^{i+1} \to \mathcal{V}^i$ be such that for every $V \in \mathcal{V}^{i+1}$, $g_i(V)$ is the unique member of $\mathcal{V}^i$ such that $V \subset g_i(V)$.

For each $x \in X$ and each $i \in \omega$, define $f_i(x)$ to be the unique element of $\mathcal{V}^i$ such that $x \in f_i(x) \in \mathcal{V}^i$. We claim that the function $h : X \to \varprojlim\{\mathcal{V}^i, g_i\}_{i=0}^{\infty}$ defined by $h(x) = (f_0(x), f_1(x), f_2(x), \ldots)$ is a homeomorphism. Since X is compact and $\varprojlim\{\mathcal{V}^i, g_i\}_{i=0}^{\infty}$ is a (compact) metric space, we must show only that h is a continuous bijection. To show that h

is injective, let $x, y \in X$ be such that $x \neq y$. Then $d(x, y) > 0$. Thus, there exists some $N \in \omega$ such that $\gamma_j < d(x, y)$ for every $j \geq N$, meaning $f_j(x) \neq f_j(y)$ for every $j \geq N$, implying $h(x) \neq h(y)$, whence h is injective. Take any sequence $(V_0, V_1, V_2, \ldots) \in \varprojlim\{\mathcal{V}^i, g_i\}_{i=0}^{\infty}$. Since $(V_i)_{i=0}^{\infty}$ is a nested sequence of nonempty closed, and thus, compact subsets of X, $\bigcap_{i=0}^{\infty} V_i \neq \emptyset$. Therefore, there is an $x \in X$ such that $h(x) = (V_0, V_1, V_2, \ldots)$, giving us that h is a surjection. We must now show that h is continuous, which we do by showing that it is uniformly continuous. Let $\epsilon > 0$. Then there is an $N \in \omega$ such that $\sum_{i=N}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^N} < \epsilon$. Then if $x, y \in X$ are such that $d(x, y) < \gamma_N$, the distance between $h(x)$ and $h(y)$ is less than ϵ.

□

Notable textbooks in recent decades that provide important theorems from the literature for inverse limits of metric spaces include those of Ingram and Mahavier, Macías, and Nadler in [16], [20], and [33], respectively. Another good discussion on inverse limits comes from Ingram in [15]. Several other authors, including the primary three this paper focuses on, are also mentioned in the next chapter.

Chapter 2

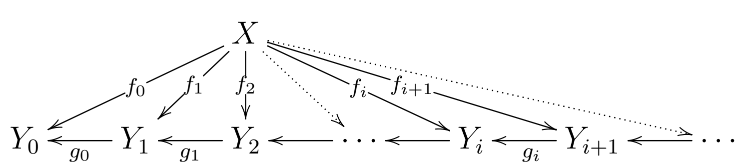
A Common Approach to Anderson-Choquet, Morton Brown, and Mardešić-Segal

Here, we state and prove a theorem that gives a common approach to three other theorems from the literature involving inverse limits of compacta. We begin first by providing several propositions, definitions, and a construction that will lead us to our theorem. After this, we show how we can restate and reprove three classic theorems on inverse limits of compact metric spaces: The Anderson-Choquet Embedding Theorem, Mardešić and Segal's theorem regarding inverse limits of polyhedra, and Morton Brown's Approximation Theorem. Finally, we will show how the construction of the theorem can be restated as a game between two players, allowing us to restate our theorem in the context of a winning strategy for one of these players.

The subsequent material comes from continued joint work done by the author and his advisor, Dr. Piotr Minc. Similar approaches and concepts in this work are present in work by authors such as Isbell, Mioduszewski, McAuley and Robinson, and Marsh and Prajs, which can be found in [17], [31], [29], and [28], respectively. Nonetheless, we believe it sheds new light on these three classic theorems and their proofs.

2.1 The Theorem

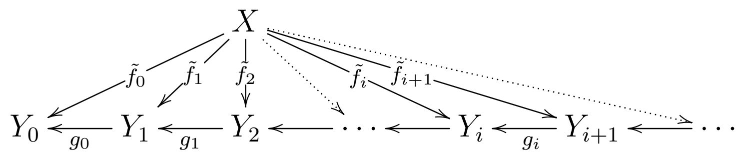
Definition 2.1. *Consider the following (not necessarily commutative) infinite diagram,* (D_∞),



(D_∞)

where X is compact, all Y_i's are complete and all f_i's and g_i's are continuous. We say that ***the diagram*** *(D_∞)* ***converges*** *if, for each $i \in \omega$, the sequence $(g_{ij} \circ f_j)_{j=i}^{\infty}$ converges uniformly to a function $\tilde{f}_i : X \to Y_i$.*

Proposition 2.2. *If the diagram (D_∞) converges, then the diagram*



(2.1)

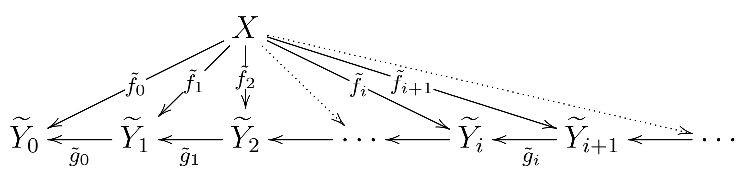
is commutative. In particular, $g_i\left(\tilde{f}_{i+1}(X)\right) = \tilde{f}_i(X)$ for each $i \in \omega$.

Proof. Since (D_∞) converges, we know that for every $i \in \omega$ and every $x \in X$ that

$$g_i \circ \tilde{f}_{i+1}(x) = g_i\Big(\lim_{j\to\infty} g_{i+1,j} \circ f_j(x)\Big) = \lim_{j\to\infty} g_{ij} \circ f_j(x) = \tilde{f}_i(x).$$

Therefore, diagram 2.1 is commutative and $g_i\left(\tilde{f}_{i+1}(X)\right) = \tilde{f}_i(X)$. □

Definition 2.3. *Suppose the diagram (D_∞) converges. In this context, we denote $\tilde{f}_i(X)$ by $\widetilde{Y}_i$, and g_i restricted to $\widetilde{Y}_{i+1} = \tilde{f}_{i+1}(X)$ by $\tilde{g}_i$. By Proposition 2.2, the diagram below, $(\widetilde{D}_\infty)$,*



$(\widetilde{D}_\infty)$

is commutative and all mappings in it are surjective. In this context we say that ***the diagram*** *(D_∞)* ***converges to the diagram*** *$(\widetilde{D}_\infty)$, or $(\widetilde{D}_\infty)$* ***is the limit of*** *(D_∞). Let $\widetilde{Y}$ denote the inverse limit $\varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^{\infty}$. Let $\tilde{f}$ denote the mapping of X onto $\widetilde{Y}$ induced by $(\widetilde{D}_\infty)$. That is, the mapping $\tilde{f}$ such that for every $x \in X$, we have $\tilde{f}(x) = (\tilde{f}_0(x), \tilde{f}_1(x), \tilde{f}_2(x), \ldots)$, is a mapping of X onto $\widetilde{Y}$.*

Definition 2.4. *Suppose that $\eta > 0$ and f is a function defined on a metric space X. We say that* ***the resolution of f is better than*** *η if $f(a) \neq f(b)$ for all $a, b \in X$ such that $d(a, b) \geq \eta$.*

Proposition 2.5. *Suppose $\eta > 0$, X is compact and f is a mapping of X into a metric space Y. If the resolution of f is better than η, then there is $\delta > 0$ such that $d(f(a), f(b)) > \delta$ for all $a, b \in X$ such that $d(a, b) \geq \eta$.*

Proof. Suppose the resolution of f is better than η, but that for every $\delta > 0$ there exists a pair of points $a, b \in X$ such that $d(a, b) \geq \eta$ and $d(f(a), f(b)) \leq \delta$. Then for every $n \in \mathbb{N}$, there exists a pair of points $a_n, b_n \in X$ such that $d(a_n, b_n) \geq \eta$ and $d(f(a_n), f(b_n)) \leq 1/n$. Since X is compact, the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ have convergent subsequences. Without loss of generality, suppose both of these sequences converge to a and b in X, respectively. Since f is continuous, $f(a) = f(b)$. However, this is a contradiction to the assumption that $d(a, b) \geq \eta$ implies $f(a) \neq f(b)$. □

Proposition 2.6. *Suppose $\delta > 0$, $g : Y \to Y'$ is a mapping between metric spaces and Z is a compact subset of Y. Then, there is $\epsilon > 0$ with the property that $d(g(z), g(y)) < \delta$ for each $z \in Z$ and each $y \in Y$ such that $d(z, y) < \epsilon$.*

Proof. Suppose that for each $n \in \mathbb{N}$ there exists a $z_n \in Z$ and a $y_n \in Y$ such that $d(z_n, y_n) < 1/n$ and $d(g(z_n), g(y_n)) \geq \delta$. Since Z is a compact subset of Y, $(z_n)_{n=1}^{\infty}$ has a convergent subsequence. Without loss of generality, we will assume $(z_n)_{n=1}^{\infty}$ converges to some $z \in Z$. Since $d(z_n, y_n) < 1/n$ for every $n \in \mathbb{N}$, $(y_n)_{n=1}^{\infty}$ also converges to z. Thus, since g is continuous, both $g(z_n)$ and $g(y_n)$ converge to $g(z)$. However, this contradicts that $d(g(z_n), g(y_n)) \geq \delta$ for every $n \in \mathbb{N}$. □

Construction 2.7. *Let X be a compact space and let $\eta_0, \eta_1, \ldots$ be a sequence of positive numbers converging to 0. We construct the diagram (D_∞) in the following ω steps:*

Step 0. A complete space Y_0 and a mapping $f_0 : X \to Y_0$ are chosen subject only to the condition that the resolution of f_0 is better than η_0.

Step $i > 0$. First, a positive number ϵ_i is selected depending on the construction in the previous steps. Then, a complete space Y_i and mappings $f_i : X \to Y_i$ and $g_{i-1} : Y_i \to Y_{i-1}$ are chosen subject only to the following conditions:

(a)$_i$ the resolution of f_i is better than η_i, and

(b)$_i$ $\operatorname{dist}(f_{i-1}, g_{i-1} \circ f_i) < \epsilon_i$.

Theorem 2.8. *By choosing a sufficiently small number ϵ_i in each step of Construction 2.7, it is possible to make the diagram (D_∞) converge to the diagram $(\widetilde{D}_\infty)$ such that $\tilde{f}$ described in Definition 2.3 is a homeomorphism of X onto $\widetilde{Y}$.*

Proof. For each $k \in \omega$, $f_k : X \to Y_k$ has resolution better than η_k. It follows from Proposition 2.5 that there is $\delta_k > 0$ such that

$$d\left(f_k\left(a\right), f_k\left(b\right)\right) > \delta_k \quad \text{for all } a, b \in X \text{ such that } d(a, b) \geq \eta_k. \tag{2.2}$$

Let i be an arbitrary positive integer. Observe that $Y_0, \ldots, Y_{i-1}$ and $f_0, \ldots, f_{i-1}$ have been constructed before the i-th step of the construction. If $i > 1$, then $g_0, \ldots, g_{i-2}$ also have been constructed. Thus, $\delta_0, \ldots, \delta_{i-1}$ and $g_{0,i-1}, \ldots, g_{i-1,i-1}$ can be used when selecting ϵ_i. For each $k \in \{0, \ldots, i-1\}$, let $\epsilon_i^{(k)}$ be equal to the ϵ obtained from Proposition 2.6 used with $\delta = 2^{k-i-1}\delta_k$, $Y = Y_{i-1}$, $Y' = Y_k$, $g = g_{k,i-1}$ and $Z = f_{i-1}(X)$. Set $\epsilon_i = \min\left\{\epsilon_i^{(0)}, \ldots, \epsilon_i^{(i-1)}\right\}$.

It follows from (b)$_i$ of Construction 2.7 that

$$\operatorname{dist}\left(g_{k,i-1} \circ f_{i-1}, g_{k,i} \circ f_i\right) < 2^{k-i-1}\delta_k \quad \text{for each } k \in \omega \text{ and each } i > k. \tag{2.3}$$

It follows from Proposition 1.35 that the sequence $(g_{k,j} \circ f_j)_{j=k}^{\infty}$ converges uniformly to a mapping $\tilde{f}_k : X \to Y_k$. Moreover, since $g_{k,k}$ is the identity on Y_k, 2.3 implies that

$$\operatorname{dist}\left(f_k, \tilde{f}_k\right) < \sum_{j=k}^{\infty} 2^{k-j-2}\delta_k = 2^{-1}\delta_k \tag{2.4}$$

By Proposition 2.2, the diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ and $\tilde{f}$, the mapping of X onto $\widetilde{Y}$ induced by $(\widetilde{D}_\infty)$, is a continuous surjection onto $\widetilde{Y} = \varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^\infty$. Thus, to complete the proof, it is enough to observe that $\tilde{f}$ is an injection. For that purpose, take any two points $a \neq b \in X$. Since $\lim_{i\to\infty} \eta_i = 0$, there is a positive integer k such that $d(a,b) > \eta_k$. Since the resolution of f_k is better than η_k (by condition (a)$_k$ of Construction 2.7), it follows from 2.2 that

$$d\left(f_k(a), f_k(b)\right) > \delta_k \tag{2.5}$$

Since $d\left(f_k(a), \tilde{f}_k(a)\right) < 2^{-1}\delta_k$ and $d\left(f_k(b), \tilde{f}_k(b)\right) < 2^{-1}\delta_k$ by 2.4, it follows from 2.5 that $\tilde{f}_k(a) \neq \tilde{f}_k(b)$. Consequently, $\tilde{f}$ is an injection. □

Remark 2.9. *The conditions on the sequence $(\epsilon_i)_{i=0}^\infty$ in the proof of the previous theorem are similar to those established in the theorems by the authors in the following section.*

Remark 2.10. *In context of Construction 2.7, suppose that $S = (s_i)_{i=1}^\infty$ is a sequence of positive functions (understood as procedures producing positive numbers) such that s_1 is a function of Y_0, f_0 and η_1, and for each integer $i > 1$, s_i is a function of $Y_0, \ldots, Y_{i-1}$, $f_0, \ldots, f_{i-1}$, $g_0, \ldots, g_{i-2}$, and η_i. We call S a strategy for Theorem 2.8 if setting $\epsilon_1 = s_1(Y_0, f_0, \eta_1)$ and $\epsilon_i = s_i(Y_0, \ldots, Y_{i-1}, f_0, \ldots, f_{i-1}, g_0, \ldots, g_{i-2}, \eta_i)$ for $i > 1$ produces a construction satisfying Theorem 2.8. Our proof of the theorem describes such a strategy. Observe if S is a strategy for Theorem 2.8, then selecting in the i-th step of the construction any positive number $\epsilon_i \leq s_i$ also leads to a construction satisfying Theorem 2.8. Thus, the theorem can be restated as follows.*

Restatement of Theorem 2.8. *For each positive integer i, it is possible to find before the i-th step of Construction 2.7 a positive number s_i such that*

1. *s_i depends only on the previously defined elements of the construction.*

2. *if any positive $\epsilon_i \leq s_i$ is selected and the construction continues as described in 2.7, then so constructed diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ such that $\tilde{f}$ described in Definition 2.3 is a homeomorphism of X onto $\widetilde{Y}$.*

2.2 Applications of Theorem 2.8

We now demonstrate the usefulness of Theorem 2.8 by showing how it can restate and reprove three others involving inverse limits of metric spaces. To prove these using our theorem in the previous section, we need to interpret each one in terms of Construction 2.7.

2.2.1 The Anderson-Choquet Embedding Theorem

The Anderson-Choquet Embedding Theorem is one providing sufficient conditions for when an inverse limit of compact metric spaces can be embedded in a complete metric space (such as the plane, $\mathbb{R}^2$). We first give the statement of the theorem as it appears Sam B. Nadler Jr.'s book on Continuum Theory, though some notation is changed.

Theorem 2.11 (The Anderson-Choquet Embedding Theorem [33, Theorem 2.10])**.** *Let Y be a compact metric space and let $\{X_i, \varphi_i\}_{i=0}^{\infty}$ be an inverse sequence where each X_i is a nonempty compact subset of Y and each φ_i maps onto X_i. Assume (1) and (2) below:*

(1) For each $\epsilon > 0$, there exists k such that for all $p \in X_k$, $\operatorname{diam}\Big(\bigcup_{j>k} \varphi_{k,j}^{-1}(p)\Big) < \epsilon$;

(2) For each $\delta > 0$, there exists $\delta' > 0$ such that whenever $j > 0$ and $p, q \in X_j$ such that $d_i\big(\varphi_{i,j}(p), \varphi_{i,j}(q)\big) > \delta$, then $d_j(p, q) > \delta'$.

Then $\varprojlim\{X_i, \varphi_i\}_{i=0}^{\infty}$ is homeomorphic to $\bigcap_{n=0}^{\infty}\Big(\overline{\bigcup_{i\geq n} X_i}\Big)$. In particular, if $X_i \subset X_{i+1}$ for each $i \in \omega$, then $\varprojlim\{X_i, \varphi_i\}_{i=0}^{\infty}$ is homeomorphic to $\overline{\bigcup_{i=0}^{\infty} X_i}$.

A version of this theorem in the context of generalized inverse limits with upper semi-continuous set-valued bonding maps has been given by Banič et al. in [2].

We can also state the theorem more in the context of Construction 2.7, with the assumption that Y need not be a compact and that the bonding maps φ_i need not be onto. First, we provide the following statements.

Proposition 2.12. *Let $X = \varprojlim \{X_i, \varphi_i\}_{i=0}^{\infty}$ where X_i is compact. Let π_i denote the projection of X into X_i. Then there is a sequence $\eta_0, \eta_1, \eta_2, \ldots$ of positive numbers converging to 0 such that the resolution of π_i is better than η_i for each $i \in \omega$.*

Proof. Let d denote the metric on X defined by

$$d(a,b) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} d_j(a_j, b_j)$$

for every $a = (a_0, a_1, a_2, \ldots)$ and $b = (b_0, b_1, b_2, \ldots)$ in X where d_j is the metric on X_j and, without loss of generality, $d_j \leq 1$ for every $j \in \omega$. We claim that $\eta_i = 1/2^i$ for every $i \in \omega$. Indeed, suppose $a, b \in X$ are such that $d(a,b) \geq 1/2^i$ and $\pi_i(a) = \pi_i(b)$, that is, $a_i = b_i$. Then $a_k = b_k$ for every $k = 0, \ldots, i-1$ as well. Therefore, $d(a,b) = \sum_{j=i+1}^{\infty} \frac{1}{2^{j+1}} d_j(a_j, b_j) \leq 1/2^{i+1}$, a contradiction to the assumption that $d(a,b) \geq 1/2^i$. □

Proposition 2.13. *Let X be a compact space and let Y be a complete space. Consider a special case of the diagram (D_∞) where each Y_i is a copy of Y and each g_i is the identity on Y. Suppose also $f_0, f_1, f_2, \ldots$ are such that the diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ described in Definition 2.3. Then the sequence $(f_i)_{i=0}^{\infty}$ converges uniformly to a mapping f, $\tilde{f}_i = f$, $\widetilde{Y}_i = f(X)$ and $\tilde{g}_i$ is the identity on $f(X)$ for each $i \in \omega$. Consequently, $\widetilde{Y}$ is homeomorphic to $f(X)$. Moreover, $\bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} f_m(X)} \right) = f(X)$.*

Proof. We will prove only that $\bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} f_m(X)} \right) = f(X)$ and leave the rest of the proof to the reader.

Since $(f_i)_{i=0}^{\infty}$ converges uniformly to f, we have that for each $\epsilon > 0$, there exists an $i \in \omega$ such that $\bigcup_{m \geq i} f_m(X) \subset N_\epsilon(f(X))$. (See Definition 1.33.) Since X is compact and f is continuous, $f(X)$ is compact, implying that $N_\epsilon(f(X))$ is closed by Proposition 1.34. Thus, $\overline{\bigcup_{m \geq i} f_m(X)} \subset N_\epsilon(f(X))$, yielding $\bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} f_m(X)} \right) \subset \bigcap_{\epsilon > 0} N_\epsilon(f(X)) = \overline{f(X)} = f(X)$.

Let $x \in X$. Since $\lim_{m \to \infty} f_m(x) = f(x)$, we have $f(x) \in \overline{\bigcup_{m \geq i} f_m(X)}$ for every $i \in \omega$. Thus, $f(x) \in \bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} f_m(X)} \right)$, whence $f(X) \subset \bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} f_m(X)} \right)$. □

One can summarize this very useful, but a bit technical theorem by the following more intuitive statement.

Theorem 2.14 (A constructive version of Anderson-Choquet Embedding Theorem)**.** *Let* $X = \varprojlim \{X_i, \varphi_i\}_{i=0}^{\infty}$ *where* X_i *is compact and let* h_0 *be an embedding of* X_0 *into a complete space* Y*. Suppose that a sequence of embeddings* $h_1 : X_1 \to Y, h_2 : X_2 \to Y, \dots$ *is constructed one by one such that, for each positive integer* i*, some positive number* ϵ_i *is selected depending on already constructed* $h_0, \dots, h_{i-1}$ *and then an arbitrary embedding* h_i *of* X_i *into* Y *is taken subject only to the condition* $\operatorname{dist}(h_{i-1} \circ \varphi_{i-1}, h_i) < \epsilon_i$*.*

Then by choosing a sufficiently small number ϵ_i *in each step of the above construction, it is possible to make* $\bigcap_{i=0}^{\infty} \left(\overline{\bigcup_{m \geq i} h_m(X_m)} \right)$ *homeomorphic to* X*.*

Proof. Let π_i denote the projection of X into X_i. Let $\eta_0, \eta_1, \eta_2, \dots$ be the sequence promised by Proposition 2.12. We will now convert the construction described in the theorem to a construction of the diagram (D_∞) in the following way. For each $i \in \omega$, set $Y_i = Y$, $f_i = h_i \circ \pi_i$ and observe that the resolution of f_i is better than η_i. If $i > 0$, let $g_i : Y_i \to Y_{i-1}$ be the identity on Y.

Observe that $\pi_{i-1} = \varphi_{i-1} \circ \pi_i$, $f_{i-1} = h_{i-1} \circ \pi_{i-1} = h_{i-1} \circ \varphi_{i-1} \circ \pi_i$ and $g_{i-1} \circ f_i = f_i = h_i \circ \pi_i$. It follows that

$$\operatorname{dist}(f_{i-1}, g_{i-1} \circ f_i) = \operatorname{dist}(h_{i-1} \circ \varphi_{i-1} \circ \pi_i, h_i \circ \pi_i) \leq \operatorname{dist}(h_{i-1} \circ \varphi_{i-1}, h_i) < \epsilon_i.$$

Hence, our construction of (D_∞) is as described in Construction 2.7. By Theorem 2.8, it is possible to choose ϵ_i in each step such that the diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ such that $\tilde{f}$ described in Definition 2.3 is a homeomorphism of X onto $\widetilde{Y}$. Now, the theorem follows from 2.13. ∎

2.2.2 Morton Brown's Approximation Theorem

In his paper, *Some Applications of an Approximation Theorem for Inverse Limits*, Morton Brown provides a theorem ([5], Theorem 3) that gives sufficient conditions for two inverse limits, with the same sequence of factor spaces but different sequences of bonding maps, to be homeomorphic. Though stated differently in his paper, we can state it in the following equivalent way.

Theorem 2.15 (Morton Brown [5], Theorem 3). *Let $X = \varprojlim \{X_i, \varphi_i\}_{i=0}^{\infty}$ where, for each $i \in \omega$, X_i is a compact metric space. For each $i \in \omega$, let $\mathcal{K}_i$ be a collection of maps from X_{i+1} into X_i such that $\varphi_i \in \overline{\mathcal{K}_i}$. Then there is a sequence $g_0, g_1, g_2, \ldots$ such that $g_i \in \mathcal{K}_i$ and $\varprojlim \{X_i, g_i\}_{i=0}^{\infty}$ is homeomorphic to X.*

Proof. Let π_i denote the projection of X into X_i. Let $\eta_0, \eta_1, \ldots$ be a sequence as promised by Proposition 2.12. We will build the diagram (D_∞) prescribing before the construction begins Y_i to be always X_i and f_i to be always π_i. Recall that the resolution of $f_i = \pi_i$ is better than η_i by Proposition 2.12. We will now select a sequence of functions $g_0 \in \mathcal{K}_0, g_1 \in \mathcal{K}_1, \ldots$ one by one. In the i-th step of our construction (where $i \geq 1$) we will select g_{i-1} in the following way. First we select $\epsilon_i > 0$ depending on previously constructed $g_0, \ldots, g_{i-2}$ and take an arbitrary $g_{i-1} \in \mathcal{K}_{i-1}$ subject only to the condition $\operatorname{dist}(g_{i-1}, \varphi_{i-1}) < \epsilon_i$. Since $f_{i-1} = \pi_{i-1} = \varphi_{i-1} \circ \pi_i = \varphi_{i-1} \circ f_i$, it follows that $\operatorname{dist}(f_{i-1}, g_{i-1} \circ f_i) < \epsilon_i$. Thus, our construction is as described in 2.7. Before the i-th step of the construction, for each $i \geq 1$, we find a number s_i satisfying the conclusion of the Restatement of Theorem 2.8. We set $\epsilon_1 = s_1$ and $\epsilon_2 = s_2$. For each $i \geq 3$, let P_i denote the set of those pairs of integers (j, k) such that $0 \leq j < k < i$. For each $(j, k) \in P_i$, we will construct a certain positive number $s_i^{(j,k)}$ and then we will set ϵ_i to be the minimum of s_i and all numbers $s_i^{(j,k)}$ where $(j, k) \in P_i$. The number $s_i^{(j,k)}$ will be constructed in such a way that it depends only on $g_0, \ldots, g_{i-2}$, it does not depend on any g_n with $n \geq i - 1$ and is such that

$$\operatorname{dist}(g_{ji}, g_{jk} \circ \varphi_{ki}) < 2^{-k} \tag{2.6}$$

We construct $s_i^{(j,k)}$ for all $i > k$ by induction with respect to $i - k$. Let $s_{k+1}^{(j,k)}$ be a small positive number with the property $d\left(g_{jk}(a), g_{jk}(b)\right) < 2^{-k}$ for all $a, b \in X_k$ such that $d\left(a, b\right) \leq s_{k+1}^{(j,k)}$. Observe that (2.6) for $i = k+1$ is satisfied since $g_{j,k+1} = g_{jk} \circ g_k$, $\varphi_{k,k+1} = \varphi_k$ and $\text{dist}\left(g_k, \varphi_k\right) \leq s_{k+1}^{(j,k)}$. Now, suppose that $i > k+1$ and $g_{j,i-1}$ have been constructed in such a way that $\text{dist}\left(g_{j,i-1}, g_{jk} \circ \varphi_{k,i-1}\right) < 2^{-k}$. Set $\alpha = 2^{-k} - \text{dist}\left(g_{j,i-1}, g_{jk} \circ \varphi_{k,i-1}\right)$ and let $s_i^{(j,k)}$ be a small positive number with the property $d\left(g_{j,i-1}(a), g_{j,i-1}(b)\right) < \alpha$ for all $a, b \in X_{i-1}$ such that $d\left(a, b\right) \leq s_i^{(j,k)}$. Observe that $\text{dist}\left(g_{ji}, g_{j,i-1} \circ \varphi_{i-1}\right) < \alpha$ since $g_{ji} = g_{j,i-1} \circ g_{i-1}$, X_i is compact, and $\text{dist}\left(g_{i-1}, \varphi_{i-1}\right) \leq s_i^{(j,k)}$. It follows that (2.6) holds because

$$\text{dist}\left(g_{j,i-1} \circ \varphi_{i-1}, g_{jk} \circ \varphi_{ki}\right) = \text{dist}\left(g_{j,i-1} \circ \varphi_{i-1}, g_{jk} \circ \varphi_{k,i-1} \circ \varphi_{i-1}\right)$$

$$\leq \text{dist}\left(g_{j,i-1}, g_{jk} \circ \varphi_{k,i-1}\right) = 2^{-k} - \alpha.$$

Thus, the construction of $s_i^{(j,k)}$ is complete, and, consequently, the construction of $g_0, g_1, g_2, \ldots$ is also complete.

Since $\epsilon_i \leq s_i$ for each positive i, and because the s_i's were selected to satisfy the conclusion of the Restatement of Theorem 2.8, the diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ such that $\tilde{f}$ is a homeomorphism of X onto $\varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^\infty$ where the notation is as described in Definition 2.1. Observe that $\varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^\infty = \varprojlim\{\tilde{f}_i(X), \tilde{g}_i\}_{i=0}^\infty$ is contained in $\varprojlim\{X_i, g_i\}_{i=0}^\infty$.

To complete the proof we must show that $\varprojlim\{\tilde{f}_i(X), \tilde{g}_i\}_{i=0}^\infty = \varprojlim\{X_i, g_i\}_{i=0}^\infty$. To that end, it is enough to prove the following claim.

Claim 2.1. *For each integer $j \geq 0$ and each neighborhood U of $\tilde{f}_j(X)$ in X_j there is an integer $i > j$ such that $g_{ji}(X_i) \subset U$.*

Let $j \geq 0$ and let U be a neighborhood of $\tilde{f}_j(X)$ in X_j. Since $\tilde{f}_j(X)$ is compact, there is a number δ such that the δ-neighborhood of $\tilde{f}_j(X)$ in X_j is contained in U. Since the sequence $(g_{jl} \circ \pi_l)_{l=j}^\infty$ converges uniformly to $\tilde{f}_j$, there is a an integer $L > j$ such that dist $\left(\tilde{f}_j, g_{jl} \circ \pi_l\right) < \delta/3$ for each $l \geq L$. Take an integer $k \geq L$ such that $2^{-k} < \delta/3$. Let V be a neighborhood of $\pi_k(X)$ in X_k such that $g_{jk}(V)$ is contained in $\delta/3$-neighborhood of $g_{jk} \circ \pi_k(X)$ in X_j. Since $X = \varprojlim \{X_l, \varphi_l\}_{l=0}^{\infty}$ and X_l is a compact for each integer $l \geq 0$, there is an integer $i > k$ such that $\varphi_{ki}(X_i) \subset V$.

Take an arbitrary point $x_i \in X_i$. It follows from (2.6) that

$$d\left(g_{ji}(x_i), g_{jk} \circ \varphi_{ki}(x_i)\right) < 2^{-k} < \delta/3 \tag{2.7}$$

By the choice of i, $\varphi_{ki}(x_i) \in V$. It follows from the choice of V that there is $x \in X$ such that

$$d\left(g_{jk} \circ \varphi_{ki}(x_i), g_{jk} \circ \pi_k(x)\right) < \delta/3 \tag{2.8}$$

Since $k \geq L$, we get the result

$$d\left(g_{jk} \circ \pi_k(x), \tilde{f}_j(x)\right) < \delta/3 \tag{2.9}$$

Combining (2.7), (2.8) and (2.9) we infer that for each $x_i \in X_i$ there is $x \in X$ such that

$$d\left(g_{ji}(x_i), \tilde{f}_j(x)\right) < \delta$$

The last inequality and the choice of δ proves the claim and completes the proof of the theorem. □

2.2.3 The Theorem of Mardesic and Segal

Definition 2.16. *Let X and Y be metric spaces, let $f : X \to Y$ be a mapping onto Y, and let $\epsilon > 0$. We say that f is an ϵ-**mapping** (or ϵ-**map**) if for every $y \in Y$,* $\operatorname{diam}\left(f^{-1}(y)\right) < \epsilon$.

Definition 2.17. *Let $\Omega = \{X_\alpha \mid \alpha \in \Lambda\}$ be a class of metric spaces. We say that a metric space X is* ***Ω-like*** *if for each $\epsilon > 0$ there exists a space $X_\alpha \in \Omega$ and an ϵ-mapping $f : X \to X_\alpha$ onto X_α.*

Theorem 2.18 (Mardešić and Segal [27, Lemma 4])**.** *Let X be a continuum, let ψ be a mapping of X onto a polyhedron P, and let $\epsilon > 0$ be an arbitrary positive number. Then there is an $\delta > 0$ such that, for any polyhedron Q and δ-mapping φ of X onto Q, there exists a mapping g of Q onto P, such that the distance* $\operatorname{dist}(\psi, g \circ \varphi) \leq \epsilon$.

For clarity of our subsequent proof, the notation in the above statement differs from the original notation in [27, Lemma 4]. Also, notice that in the original statement of [27, Lemma 4], it is not necessary to assume that $f_1 : X \to P_1$ is an ϵ_1-mapping.

Theorem 2.19 (Mardešić and Segal [27, Theorem 1])**.** *For an arbitrary collection $\mathcal{P}$ of polyhedra, every $\mathcal{P}$-like continuum X is the inverse limit of an inverse sequence $\{Y_i, g_i\}_{i=0}^{\infty}$ with bonding maps g_i onto and with polyhedra $Y_i \in \mathcal{P}$ for every $i \in \omega$.*

Proof of 2.19 from Mardešić - Segal Theorem 2.18 and Theorem 2.8. Set $\eta_i = 2^{-i}$. We will construct the elements of Construction 2.7 in in such a way that all Y_i's are polyhedra from $\mathcal{P}$ and all f_i's and g_i's are surjective. Let f_0 be an η_0-mapping of X onto some polyhedron $Y_0 \in \mathcal{P}$. Now, suppose that steps $0, \ldots, i-1$ of the construction have been completed for some $i > 0$. Chose a positive ϵ_i small enough to satisfy Theorem 2.8. Use 2.18 with $P = Y_{i-1}$, $\psi = f_{i-1}$ and $\epsilon = \epsilon_i$ to get a δ promised by the theorem. Set $\sigma = \min\{\delta, \eta_i\}$. Let f_i be a σ-mapping of X onto a polyhedron $Y_i \in \mathcal{P}$. It follows from the choice of δ that there is a surjection $g_{i-1} : Y_i \to Y_{i-1}$ such that $\operatorname{dist}(f_{i-i}, g_{i-1} \circ f_i) < \epsilon_i$. Observe that the resolution of f_i is better than η_i since f_i is a σ mapping and $\sigma \leq \eta_i$. So, Construction 2.7 is completed, By Theorem 2.8, the diagram (D_∞) converges to the diagram $(\widetilde{D}_\infty)$ such that $\tilde{f}$ described in Definition 2.3 is a homeomorphism of X onto $\varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^{\infty}$. Since all f_i's and g_i's are surjective, it follows that $\widetilde{Y}_i = Y_i$ and $\widetilde{g}_i$ is surjective for each $i \in \omega$. $\square$

Theorem 2.19 has been referenced by numerous authors, thus proving itself to have many applications in Topology. One paper in particular comes from Michael C. McCord in [30] in which he provides theorems stating classes $\mathcal{P}$ of polyhedra for which a universal $\mathcal{P}$-like compactum will exist and classes for when no such universal compactum will exist. The proofs of some of these theorems in the previously mentioned paper use the theorem from Mardešić and Segal as well as Morton Brown's approximation theorem. Mardešić also introduced the notion of approximate resolutions in [22] and generalized results of [27] with Segal in [26] and Matijević in [24].

2.2.4 Another Look at the Proof of Corollary 1.67

We observe here that the proof of Corollary 1.67 contained elements of Construction 2.7 and therefore can be inferred from Theorem 2.8.

Recall that in Corollary 1.67, we expressed a compact 0-dimensional metric space X as the inverse limit of an inverse sequence of finite sets using the following construction. We took a sequence $\mathcal{V}^0, \mathcal{V}^1, \mathcal{V}^2, \ldots$ of coverings of X such that $\mathrm{mesh}(\mathcal{V}^i) \to 0$, and for each $i \in \omega$, $\mathcal{V}^i$ consists of finitely many mutually disjoint, nonempty closed and open sets, and V^{i+1} refines $\mathcal{V}^i$. We then defined $g_i : \mathcal{V}^{i+1} \to \mathcal{V}^i$ by setting, for every $V \in \mathcal{V}^{i+1}$, $g_i(V)$ to be the unique element of $\mathcal{V}^i$ containing V. Observe that the resolution of f_i is better than $2\mathrm{mesh}(\mathcal{V}^i)$. Also observe that f_i and g_i are surjections, and that $f_i = g_i \circ f_{i+1}$. Thus, $\mathrm{dist}(f_{i-1}, g_{i-1} \circ f_i) = 0$ which is less than any positive number ϵ_i. It follows from Theorem 2.8 that $f = (f_0, f_1, f_2, \ldots)$ is homeomorphism of X onto $\varprojlim\{\mathcal{V}^i, g_i\}$.

2.3 The Construction of Theorem 2.8 as a Game

We now demonstrate how Construction 2.7 of Theorem 2.8 can be viewed as a game between two players who we will call Player-ϵ and Player-g. In this game, Player-ϵ provides a sequence $(\epsilon_i)_{i=1}^{\infty}$ of positive numbers that will converge to 0, while Player-g wants to produce a sequence $(g_i)_{i=0}^{\infty}$ of bonding maps between members of a sequence of $(Y_i)_{i=0}^{\infty}$ of complete spaces, where each g_i satisfies certain restrictions depending on the previously constructed ϵ_i's that will become apparent shortly. Each player produces their elements step-by-step, or rather, round-by-round, in this infinite game, all depending on the previously constructed elements of the game. The players are initially provided a "playing field" (which can be thought of as Round 0) before the game begins, which consists of a compactum, X, and a sequence $(\eta_i)_{i=0}^{\infty}$ of positive numbers converging to 0. Player-g will also have to make use of each η_i in the corresponding round, constructing a mapping $f_i : X \to Y_i$ such that $g_{i-1} \circ f_i$ will ϵ_i-commutes with f_{i-1}. In each round, Player-ϵ will go first with their choice with Player-g following depending on the previous choices of Player-ϵ and the aforementioned restrictions. The goal of Player-ϵ is to force Player-g to build an inverse sequence whose inverse limit is homeomorphic to X by providing sufficiently small ϵ_i's in each round while Player-g wants to make choices which prevent such a homeomorphism from occurring.

Here, we exhibit how the game ensues in each round. One may then see how this gives rise to the same as Construction 2.7, albeit in a different and (arguably) more approachable context. We conclude with a restatement of Theorem 2.8 as one in which Player-ϵ has a winning strategy. The interested reader may refer to the paper by Miklós Pintér in [35] for another use of game theory in inverse limits; although, it focuses on inverse limits of measure spaces instead of topological spaces, and the content there does not resemble the methods or ideas here.

The Playing Field: Round 0

The "playing field" in which this game takes place can be thought of the given elements in Step 0 of Construction 2.7, which we would rather call "Round 0." That is, both players are provided a compactum X, a sequence of positive numbers $\eta_0, \eta_1, \ldots$ converging to 0, and a complete space Y_0 and mapping $f_0 : X \to Y_0$ subject only to the condition that the resolution of f_0 is better than η_0.

Round $i > 0$

In this round, ϵ_1 through ϵ_{i-1} have been provided by Player-ϵ, and the complete spaces Y_1 through Y_{i-1}, maps f_1 through f_{i-1}, and bonding maps g_0 through g_{i-2} have been provided by Player-g. Player-ϵ goes first, choosing an $\epsilon_i > 0$ depending on $Y_0, \ldots, Y_{i-1}$, $f_0, \ldots, f_{i-1}$, $g_0, \ldots, g_{i-2}$, and η_i as described in Remark 2.10.

Player-g follows, choosing a complete space Y_i, a mapping $f_i : X \to Y_i$ with resolution better than η_i, and a mapping $g_{i-1} : Y_i \to Y_{i-1}$ subject to the condition that $\text{dist}(f_{i-1}, g_{i-1} \circ f_i) < \epsilon_i$.

After ω-many Rounds

After ω-many rounds have been completed, Player-g will have built the diagram (D_∞) and therefore the diagram $(\widetilde{D}_\infty)$. From here, we provide the following restatement of Theorem 2.8. The proof of this restatement need not be provided since it would require the same proof as that of Theorem 2.8, possibly with some different wording.

Restatement of Theorem 2.8. *With regard the interpretation of Construction 2.7 as a game between Player-ϵ and Player-g, it is possible for Player-ϵ to choose a sufficiently small positive ϵ_i in each i-th round of the game forcing Player-g to build the diagram (D_∞) so that it converges to the diagram $(\widetilde{D}_\infty)$ and making X homeomorphic to $\widetilde{Y} = \varprojlim\{\widetilde{Y}_i, \tilde{g}_i\}_{i=0}^{\infty}$ under the map $\tilde{f}$ induced by $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \ldots$.*

One may argue that this game is rather "unfair" for Player-g, given the numerous choices and restrictions by which this player must abide, while Player-ϵ must simply make a smart enough choice for each positive ϵ_i so that a homeomorphism is forced by the sequence $(\tilde{f}_i)_{i=0}^{\infty}$, where again, $\tilde{f}_i$ is the uniform limit of $(g_i \circ \cdots \circ g_{j-1} \circ f_j)_{j=i}^{\infty}$. Regardless, viewing the construction of Theorem 2.8 and its proof as a game sheds light on a the common characteristics of these three three main theorems from the literature and their proofs. Though more inquiry would be needed, this construction may provide sufficient conditions for when a given compactum and an inverse limit space are homeomorphic.

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