Global existence of classical solutions, spreading speeds, and traveling waves of chemotaxis models with logistic source on $\mathbb{R}^N$

by

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Abstract

This dissertation is devoted to the study of the classical Keller-Segel chemotaxis systems with space-time heterogeneous logistic source function on $\mathbb{R}^N$. Chemotaxis systems are mathematical models describing aggregation phenomena of cells due to chemotaxis. That is, phenomena of directed movement of cells in response to the gradient of a chemical attractant, which may be produced by the cells themselves.

We first study the fundamental problems such as local existence and global existence of nonnegative classical solutions for given nonnegative initial function in various spaces. Among our results, we prove that it is enough for the self-limitation coefficient of the logistic source function to be greater than or equal to the chemotaxis sensitivity coefficient to guarantee the existence of time-global classical solutions.

Next, we discuss the pointwise and uniform persistence of classical solutions, the existence of positive entire solutions, the existence of time-periodic solution if the logistic function is time-periodic, and, the existence of steady state solutions if the logistic function is time homogeneous. In particular, we show that any classical solution with a positive initial function enjoys pointwise persistence under the same assumption of the existence of time-global classical solution. Moreover, we study the stability of positive entire solutions, and the spreading feature of solutions with compactly supported or front like initials. In this direction, our results recover as a special case the stability and spreading speeds for the classical Fisher-KPP equations.

Finally, we establish the existence and non-existence of traveling wave solutions. When the logistic function is homogeneous and the chemotaxis sensitivity coefficient is sufficiently small, we show that there are traveling wave solutions with arbitrarily large speeds and there is no traveling wave solution of arbitrarily small speeds. That is there are positive constant $0 < c_-^* \leq c_+^* < \infty$ such that for any $c \geq c_+^*$, there is a traveling wave solution with speed c connecting the two trivial constant solutions and no such solutions exist with speed $c < c_-^*$.

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Chapter 1

Introduction

Chemotaxis is a biological process through which living organisms orient their movement along a chemical concentration gradient. Such movement may be towards or away from a higher concentration of a chemical substance. The process is present in different types of biological phenomena such as bacteria aggregation, immune system response or angiogenesis in the embryo formation and in tumour development. Recent studies describe macroscopic processes such as population dynamics or gravitational collapses, in terms of chemotaxis. Because of its crucial role in the aforementioned processes, chemotaxis has attracted significant interest and has been investigated not only from a biological point of view but also from a mathematical perspective.

Many mathematical models to describe chemotaxis have been proposed since the pioneering work of Keller and Segel during the 1970s [24, 25]. Let $u(x,t)$ denotes the population density function at time t and location $x \in D$, $D \subset \mathbb{R}^N$, of some mobile species moving toward the concentration gradient of some chemical substance with density function $v(x,t)$, which is produced by the mobile species themselves. Then the time evolution of the population density function $u(x,t)$ can be described by the following differential equation

$$u_t = \underbrace{\Delta u}_{\text{Diffusion term}} - \underbrace{\chi \nabla \cdot (u \nabla v)}_{\text{Chemotaxis term}} + \underbrace{(a(x,t) - b(x,t)u)u}_{\text{Reaction term}}, \quad x \in D, \tag{1.1}$$

where $u_t(x,t)$ and $\Delta u(x,t)$ stand for the partial derivatives with respect to time and the space of $u(x,t)$, respectively. In the reaction term $(a(x,t) - b(x,t)u)u$, the functions $a(x,t)$ and $b(x,t)$, which will be assumed to be positive, bounded, and uniformly Hölder continuous, measure the production rate and self-limitation rate of the mobile species at time t and location $x \in D$, respectively. The positive constant χ in the chemotaxis term $\chi \nabla \cdot (u \nabla v)$ measures the sensitivity of the mobile species with respect to the effect of the chemical substance. We consider the case that the chemical substance is produced by the mobile species, and hence suppose that the concentration function $v(x,t)$ is given by the partial differential equations

$$\tau v_t = \Delta v - \lambda v + \mu u, \tag{1.2}$$

where $\tau \geq 0$, λ and μ are positive constant. The term $+\mu u$ in (1.2) indicates that the mobile species produce the chemical substance themselves while the positive constant λ measures the self-degradation rate of the chemical substance. The nonnegative constant τ is related to the diffusion rate of the chemical substance with respect to the mobile species. Combining (1.1) and (1.2) and taking $\tau = 0$, we obtain the following coupled system of parabolic-elliptic partial differential equations

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + (a(x,t) - b(x,t)u)u, \quad x \in D, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in D, \end{cases} \tag{1.3}$$

complemented with certain boundary conditions on ∂D when D is bounded. Thus, the chemotaxis system (1.3) models the situation where the chemical substance diffuses very fast compared to the mobile species.

This dissertation is devoted to the study of several dynamic aspects of the deterministic chemotaxis model (1.3) with space-time dependent logistic source on the unbounded domain $D = \mathbb{R}^N$. The chemotaxis system (1.3) is a time and space logistic source dependent variant of the classical parabolic-elliptic Keller and Segel models [24], and describes the situations in which the chemical substance diffuses very fast. In the last two decades, considerable progress has been made in the analysis of (1.3) on both bounded and unbounded domains.

When $a \equiv b \equiv 0$ and $N = 1$, it is well known that classical/weak solutions of (1.3) with given smooth initial functions always exist globally when D is either a bounded domain with smooth boundary or is the whole space $\mathbb{R}^N$. However, when $a \equiv b \equiv 0$, $N = 2$ and D is a ball centered at the origin, Herrero and Velázquez [17, 18, 19] constructed a radial solution to

(1.3) which blows up in finite time and forms a $\delta-$function singularity at the origin. J.I. Diaz et al. [8] and T. Nagai [34] also proved the existence of solutions which blow up in a finite time when $N \geq 2$.

It is believed that, if the self-limitation rate function $b(x,t)$ is positive and large enough, in the sense that $\inf_{x,t} b(x,t) \gg 0$, then solutions to (1.3) will always exist for all time. This was in fact proved in 2007 by Tello and Winkler [55] when (1.3) is considered on a bounded domain D complemented with Neumann boundary condition and with the choice $\lambda = \mu = 1$, and $a(x,t) = b(x,t) \equiv b$ is space and time independent. With these choices, it was shown in [55] that if either $N \leq 2$ or $\chi < \frac{Nb}{(N-2)_+}$, (1.3) has a globally bounded classical solution for any nonnegative and uniformly continuous initial data. Furthermore, if $b > 2\chi$, then for any $u_0 \in C^{0,\alpha}(\bar{D})$ with $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$,

$$\lim_{t\to\infty} \left[\|u(\cdot;t;u_0) - 1\|_{L^\infty(D)} + \|v(\cdot,t;u_0) - 1\|_{L^\infty(D)} \right] = 0,$$

where $(u(x,t;u_0), v(x,t;u_0))$ is the solution of (1.3) complemented with Newmann boundary condition and with $u(x,0;u_0) = u_0(x)$. It should be pointed out that when $N \geq 3$ and $b \leq \frac{N-2}{N}\chi$, it remains open whether for any given initial data $u_0 \in C^{0,\alpha}(\bar{D})$, (1.3) supplemented by Newmann boundary condition possesses a global classical solution $(u(x,t;u_0), v(x,t;u_0))$ with $u(x,0;u_0) = u_0(x)$, or whether finite-time blow-up occurs for some initial data. The works [28], [63], [66] should be mentioned along this direction. It is shown in [28], [66] that in the presence of suitably weak logistic dampening (that is, small b), certain transient growth phenomena do occur for some initial data. It is shown in [63] that, with the reaction term $f(u) = au - bu^\kappa$ with suitable $\kappa < 2$ (for instance, $\kappa = 3/2$) and the second equation of (1.3) replaced by $0 = \Delta v(x,t) - \frac{1}{|D|}\int_D u(x,t)dx + u(x,t)$, then finite-time blow-up is possible. The reader is referred to [2], [7], [15], [57], [60], [62], [63], [64], [66], [67], [69], and references therein for other studies of (1.3) on bounded domain with Neumann or Dirichlet boundary conditions and various kinds of source functions.

It is worth mentioning that most of the existing results are established for the space-time homogeneous logistic function $f(u) = (a - bu)u$. Furthermore, in contrast to bounded domains, there is not much study of (1.3) on unbounded domains. Besides the difficulties induced from the lack of comparison principle for solutions of (1.3), the unboundedness of the spatial domain induces many additional difficulties in the study of (1.3) on unbounded domains.

There are also several studies of (1.3) when D is the whole space $\mathbb{R}^N$ and $a(x,t) \equiv b(x,t) \equiv 0$ (see [8, 23, 35, 53, 52]). For example, in the case of $a(x,t) = b(x,t) \equiv 0$, it is known that blow-up occurs if either N=2 and the total initial population mass is large enough, or $N \geq 3$ (see [2, 8, 35] and references therein). However, there is little study of (1.3) when $D = \mathbb{R}^N$ and $a(x,t) \neq 0$ and $b(x,t) \neq 0$.

In reality, the environments of many living organisms are spatially and temporally heterogeneous. It is of both biological and mathematical interests to study chemotaxis models with certain time and space dependence. In the case that the chemotaxis is absent, i.e., $\chi = 0$ in (1.3), the population density $u(x,t)$ of the mobile species satisfies the following scalar reaction diffusion equation,

$$\partial_t u = \Delta u + u(a(x,t) - b(x,t)u), \ x \in D \tag{1.4}$$

complemented with certain boundary conditions if $D \subset \mathbb{R}^N$ is a bounded domain. Equation (1.4) is called Fisher or KPP type equation in literature because of the pioneering works by Fisher ([10]) and Kolmogorov, Petrowsky, Piskunov [26] in the special case $a(t,x) = b(t,x) = 1$. A huge amount of research has been carried out toward the asymptotic dynamics of (1.4), see, for example, [3, 4, 5, 6, 11, 12, 31, 32, 33, 36, 37, 50, 51, 58, 59, 70], etc. for the asymptotic dynamics of (1.4) on bounded and unbounded domains.

In this dissertation, we study several dynamical features of nonnegative solutions of (1.3) when $D = \mathbb{R}^N$, $a(x,t)$ and $b(x,t)$ are Hölder's continuous functions and satisfy

$$0 < \inf_{x,t} \min\{a(x,t), b(x,t)\} \leq \sup_{x,t} \max\{a(x,t), b(x,t)\} < \infty.$$

We first investigate the local existence and global existence of solution $u(t,x)$ of (1.3) with given initial condition $u(x,t_0) = u_0(x)$ for various initial functions $u_0(x)$. Note that, due to biological interpretations, only nonnegative initial functions will be of interest. Let $t_0 \in \mathbb{R}$ and $T > 0$ be given. We call $(u(x,t), v(x,t))$ a *classical solution* of (1.3) on $[t_0, t_0 + T)$ if $(u, v) \in$

$C(\mathbb{R}^N \times [t_0, t_0+T)) \cap C^{2,1}(\mathbb{R}^N \times (t_0, t_0+T))$ and satisfies (1.3) for $(x,t) \in \mathbb{R}^N \times (t_0, t_0+T)$ in the classical sense. A classical solution $(u(x,t), v(x,t))$ of (1.3) on $[t_0, t_0+T)$ is called *nonnegative* if $u(x,t) \geq 0$ and $v(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times [t_0, t_0+T)$. A *global classical solution* of (1.3) is a classical solution on $\mathbb{R}^N \times [t_0, \infty)$ for some $t_0 \in \mathbb{R}$. In this dissertation, among others, we prove the following results.

• *For any given $t_0 \in \mathbb{R}$ and given nonnegative uniformly continuous and bounded initial function $u_0(x)$,* (1.3) *has a unique local nonegative classical solution with $u(x,t_0) = u_0(x)$ (see Theorem 2.1 for detail).*

• *If $\chi\mu < \inf_{x,t} b(x,t)$, then for any given $t_0 \in \mathbb{R}$ and given nonnegative uniformly continuous and bounded initial function $u_0(x)$,* (1.3) *has a unique bounded global nonegative classical solution with $u(x,t_0) = u_0(x)$ on $[t_0, \infty)$ (see Theorem 2.2 for detail).*

Therefore, as already mentioned above, it is enough for the self-limitation function $b(x,t)$ to be large enough to rule out any possible finite time blow-up phenomena. When a classical solution is globally defined in time with a strictly positive initial function, it is important to know whether this solution will remain uniformly strictly positive over time or it will eventually die out. Likewise, it is also of great biological interest to know how globally defined solutions with compactly supported initial functions spread over time. These questions are related to persistence and asymptotic spreading, which are well studied in the absence of chemotaxis but are hardly studied in the presence of chemotaxis. In this dissertation, we prove that

• *If $\chi\mu < \inf_{x,t} b(x,t)$, the pointwise persistence phenomena occurs in* (1.3). *Furthermore if $\chi\mu < \left(1 + \frac{\sup_{x,t} a(x,t)}{\inf_{x,t} a(x,t)}\right)^{-1} \inf_{x,t} b(x,t)$ then uniform persistence phenomena occurs in* (1.3) *(see Theorem 2.3 for details).*

These results guarantee that persistence phenomena occurs in (1). Hence, it is natural to study the existence, uniqueness, and stability of strictly positive entire solutions of (1.3). These are very basic problems in the heterogeneous case, but are very nontrivial problems in chemotaxis models. In this dissertation, we prove that

• *If $\chi\mu < \inf_{x,t} b(x,t)$ then* (1.3) *has a strictly positive entire solution, which is time-periodic if the logistic source function is time-periodic, and time-homogeneous if the logistic source function is time homogeneous. Moreover, if $0 < \chi \ll 1$, then* (1.3) *has a unique strictly* *positive entire solution which is uniformly and exponentially stable with respect to strictly positive perturbation (see Theorems 2.4 and 2.5 for details).*

When a positive entire solution is stable, it is natural to know the asymptotic behavior of solutions with front-like or compactly supported initial functions. This question is strongly related to the spreading speeds of solutions, the existence, uniqueness and stability of transition front solutions of (1.3) connecting the unique positive entire solution and the trivial solution $u(x,t) \equiv 0$. Transition waves are very important as they describe how mobile species transit between two entire solutions. Transition waves are also used to characterize the spreading speeds, however, transition wave solutions of (1.3) are hardly studied. In this dissertation, we prove the following.

• *Suppose that the functions $a(x,t)$ and $b(x,t)$ are both constant. If $0 < \chi\mu \ll b$, then for every $\varepsilon > 0$ and every classical solution $u(x,t)$ of* (1.3) *associated with a nonnegative and nonempty compactly supported initial function $u_0(x) = u(x,0)$, it holds that*

$$\limsup_{t\to\infty} \sup_{|x|\geq (c^*_{\text{upper}}+\varepsilon)t} u(x,t) = 0,$$

where $c^*_{\text{upper}} = 2\sqrt{a} + \frac{\mu\chi a\sqrt{N}}{2(b-\chi\mu)\sqrt{\lambda}}$. Furthermore, if $0 < \chi\mu \ll b$ then there is a positive constant c^*_{lower} such that for every $0 < \varepsilon \ll 1$ and every classical solution $u(x,t)$ of (1.3) *associated with a nonnegative and nonempty compactly supported initial $u_0(x) = u(x,0)$, it holds that*

$$\liminf_{t\to\infty} \inf_{|x|\leq (c^*_{\text{lower}}-\varepsilon)t} u(x,t) > 0$$

(see Theorem 2.6 for details and also for the general case).

• *Suppose that $a(x,t) \equiv a$ and $b(x,t) \equiv b$ are both constant functions. Then for every positive constants χ and μ satisfying $0 < \mu\chi < \frac{b}{2}$, there is a constant $c^*(\chi) > 2\sqrt{a}$ such that for every $c \geq c^*(\chi)$,* (1.3) *has a traveling wave solution $(u(x,t), v(x,t)) = (\phi(x-ct), \psi(x-ct))$ with speed c connecting the constant equilibrium solutions $(\frac{a}{b}, \frac{\mu a}{\lambda b})$ and $(0,0)$. There is no such traveling wave solution of speed less than $2\sqrt{a}$ (see Theorem 2.7 for more details).*

Chapter 2

Notations, definitions, and main results

In this chapter, we start with the notations that will be used throughout the rest of this work. Also, we introduce the relevant definitions of the concepts discussed. The last part this chapter is concerned with the statements of the main results of this dissertation. The proofs of these results will be given in the subsequent chapters.

2.1 Notations and standing assumptions

Let N be a positive integer. For every $x \in \mathbb{R}^N$ and $r > 0$, let $|x|_\infty = \max\{|x_i| \mid i = 1, \cdots, N\}$, $|x| = \sqrt{\sum_{i=1}^N |x_i|^2}$ and $B(x, r) = \{y \in \mathbb{R}^N \mid |x - y| < r\}$. For every function $w : \mathbb{R}^N \times I \to \mathbb{R}$, where $I \subset \mathbb{R}$, we set $w_{\inf}(t) = \inf_{x \in \mathbb{R}^N} w(x,t)$, $w_{\sup}(t) = \sup_{x \in \mathbb{R}^N} w(x,t)$, $w_{\inf} = \inf_{x,t} w(x,t)$ and $w_{\sup} = \sup_{x,t} w(x,t)$. Let

$$C^b_{\rm unif}(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) \mid u(x) \text{ is uniformly continuous in } x \in \mathbb{R}^N \text{ and } \sup_{x \in \mathbb{R}^N} |u(x)| < \infty\}$$

equipped with the norm $\|u\|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|$. For any $0 \le \nu < 1$, let

$$C^{b,\nu}_{\rm unif}(\mathbb{R}^N) = \{u \in C^b_{\rm unif}(\mathbb{R}^N) \mid \sup_{x,y \in \mathbb{R}^N, x \ne y} \frac{|u(x) - u(y)|}{|x - y|^\nu} < \infty\}$$

with norm $\|u\|_{C^{b,\nu}_{\rm unif}(\mathbb{R}^N)} = \|u\|_\infty + \sup_{x,y \in \mathbb{R}^N, x \ne y} \frac{|u(x)-u(y)|}{|x-y|^\nu}$ and

$$\begin{aligned} &C^\theta((t_1,t_2), C^\nu_{\rm unif}(\mathbb{R}^N)) \\ &= \{u(\cdot) \in C((t_1,t_2), C^\nu_{\rm unif}(\mathbb{R}^N)) \mid u(t) \text{ is locally Hölder continuous in t with exponent } \theta\}. \end{aligned}$$

In particular for every $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, we set $u_{0\inf} = \inf_x u_0(x)$ and $u_{0\sup} = \sup_x u_0(x)$.

Throughout this work, we shall always suppose there is some $0 < \nu \ll 1$ such that the following hypothesis holds:

(H) $a(\cdot,\cdot), b(\cdot,\cdot) \in C^{\nu,\nu}_{\rm unif}(\mathbb{R}^N \times \mathbb{R})$, $\min\{a_{\inf}, b_{\inf}\} > 0$, *and* $\max\{a_{\sup}, b_{\sup}\} < \infty$.

We will be concerned with the Banach space $X = C^b_{\mathrm{unif}}(\mathbb{R}^N)$ and the analytic semigroup $\{T(t)\}_{t\geq 0}$ generated by $A = \Delta - I$ on $X = C^b_{\mathrm{unif}}(\mathbb{R}^N)$. Explicitly, it holds that

$$(T(t)u)(x) = e^{-t}(G(\cdot,t) * u)(x) = \int_{\mathbb{R}^N} e^{-t}G(x-y,t)u(y)dy \tag{2.1}$$

for every $u \in X$, $t \geq 0$, and $x \in \mathbb{R}^N$, where $X = C^b_{\mathrm{unif}}(\mathbb{R}^N)$ and $G(x,t)$ is the heat kernel defined by

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}. \tag{2.2}$$

Let $X = C^b_{\mathrm{unif}}(\mathbb{R}^N)$ and $X^\alpha = \mathrm{Dom}((I-\Delta)^\alpha)$ be the fractional power spaces of $I - \Delta$ on X ($\alpha \in [0,1]$). Note that $X^0 = X$ and $X^1 = \mathrm{Dom}(I-\Delta)$. It is well known that Δ generates a contraction C_0-semigroup defined by the heat kernel, $\{G(t)\}_{t\geq 0}$, on X with spectrum $\sigma(\Delta) = (-\infty, 0]$ (see [16]). Thus, the Hille-Yosida theorem implies that the resolvent operator $R(\lambda)$ associated with Δ is the Laplace transform of $\{G(\cdot,t)\}_t$. Thus the operator $\Delta - \lambda I$ is invertible with

$$(\lambda I - \Delta)^{-1}u = \int_0^\infty e^{-\lambda t}G(\cdot,t) * u dt \tag{2.3}$$

for all $u \in X$ and $\lambda > 0$. Furthermore the restriction operator $(\Delta - \lambda I)^{-1}|_{X^\alpha} : X^\alpha \to X^\alpha$ is a bounded linear map. For every $\alpha \geq 0$ there is a positive constant $C_\alpha > 0$ such that

$$\|T(t)u\|_{X^\alpha} \leq C_\alpha t^{-\alpha} e^{-t}\|u\|_\infty, \quad t > 0,\ u \in C^b_{\mathrm{unif}}(\mathbb{R}^N), \tag{2.4}$$

with $C_0 = 1$. Furthermore, it holds that

$$\|(T(t) - I)u\|_{X^0} \leq C_\alpha t^\alpha e^{-t}\|u\|_{X^\alpha}, \quad t > 0,\ u \in X^\alpha, 0 < \alpha \leq 1. \tag{2.5}$$

We refer to [16] for the proofs of the inequalities (2.4) and (2.5).

We end this section by stating an important result that will be needed to complete the proof of the main theorem on the uniqueness of solutions.

Lemma 2.1. *([16, Exercise 4*, page 190]) Assume that* a_1, a_2, α, β *are nonnegative constants, with* $0 \le \alpha,\ \beta < 1$, *and* $0 < T < \infty$. *There exists a constant* $M(a_2, \beta, T) < \infty$ *so that for any integrable function* $u : [0,\ T] \to \mathbb{R}$ *satisfying*

$$0 \le u(t) \le a_1 t^{-\alpha} + a_2 \int_0^t (t-s)^{-\beta} u(s) ds$$

for a.e. t in $[0,\ T]$, *we have*

$$0 \le u(t) \ \le \ \frac{a_1 M}{1-\alpha} t^{-\alpha}, \ \ \text{a.e. on} \ \ 0 < t < T.$$

2.2 Statements of the main results

As stated in the previous chapter, this dissertation is concerned with the study of the following partial differential equations (PDE) :

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v) + (a(x,t) - b(x,t)u)u, \quad x \in \mathbb{R}^N, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N, \end{cases} \tag{2.6}$$

where the functions $a(x,t)$ and $b(x,t)$ satisfy the standing assumption **(H)**. The objective of the current work is to investigate the global existence and persistence of nonnegative bounded classical solutions of (2.6), existence and stability of positive entire bounded classical solutions of (2.6), spreading properties of classical solutions of (2.6) with compactly supported initial functions, and traveling wave solutions for (2.6). For the sake of clarity, we introduce some definitions.

Definition 2.1. *For given* $u_0 \in X := C^b_{\rm unif}(\mathbb{R}^N)$ *and* $t_0 \in \mathbb{R}$, $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ *is said to be a* *<u>classical solution</u>* *of* (2.6) *on* $[t_0, T)$ *with* $u(x,t_0;t_0,u_0) = u_0(x)$ *for every* $x \in \mathbb{R}^N$ *if* $u(\cdot,\cdot;t_0,u_0), v(\cdot,\cdot;t_0,u_0) \in C([t_0,T) : X) \cap C^{2,1}(\mathbb{R}^N \times (t_0,T))$ *and satisfies* (2.6) *for* $(x,t) \in \mathbb{R}^N \times (t_0,T)$ *in the classical sense with* $\lim_{t\to 0^+} u(\cdot, t_0+t) = u_0$ *in* X. *When a classical solution* $(u(x,t), v(x,t))$ *of* (2.6) *on* $[t_0,T)$ *satisfies* $u(x,t) \ge 0$ *and* $v(x,t) \ge 0$ *for every* $(x,t) \in \mathbb{R}^N \times [t_0,T)$, *we say that it is* *<u>nonnegative</u>.* *A* *<u>global classical solution</u>* *of* (2.6) *on* $[t_0,\infty)$ *is a* *classical solution on* $[t_0, T)$ *for every* $T > 0$. *We say that* $(u(x,t), v(x,t))$ *is an* *entire solution* *of* (2.6) *if* $(u(x,t), v(x,t))$ *is a global classical solution of* (2.6) *on* $[t_0, \infty)$ *for every* $t_0 \in \mathbb{R}$. *For given uniformly continuous function* u_0 *and* $t_0, T \in \mathbb{R}$ *with* $T > t_0$, *if* $(u(x,t), v(x,t))$ *is a* *classical solution* *of* (2.6) *on* $\mathbb{R}^N \times (t_0, T)$ *with* $u(x, t_0) = u_0(x)$ *for all* $x \in \mathbb{R}$, *we denote it as* $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ *and call it the* *solution of* (2.6) *with initial function* $u_0(x)$ *at time* t_0.

Note that, due to biological interpretations, only nonnegative initial functions will be of interest. It is of great interest to determine under which circumstance that (2.6) has a unique nonnegative solution for a given initial function. The main results stated below are selected from our works [43, 44, 46, 47, 48, 49].

We have the following result on the local existence and uniqueness of solution of (2.6) for initial data belonging to $C^b_{\text{unif}}(\mathbb{R}^N)$.

Theorem 2.1. *For any* $t_0 \in \mathbb{R}$ *and* $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ *with* $u_0 \geq 0$, *there exists* $T_{max} \in (0, \infty]$ *such that* (2.6) *has a unique non-negative classical solution* $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ *on* $[t_0, t_0 + T_{\max})$ *satisfying* $\lim_{t\to 0+} u(\cdot, t_0 + t; t_0, u_0) = u_0$ *in the* $C^b_{\text{unif}}(\mathbb{R}^N)$*-norm,*

$$u(\cdot,\cdot;t_0,u_0) \in C([t_0, t_0 + T_{\max}), C^b_{\text{unif}}(\mathbb{R}^N)) \cap C^1((t_0, t_0 + T_{\max}), C^b_{\text{unif}}(\mathbb{R}^N)) \tag{2.7}$$

and

$$u,\ u_{x_i},\ u_{x_i x_j},\ u_t \in C^{\theta}((t_0, t_0 + T_{\max}), C^{\nu}_{\text{unif}}(\mathbb{R}^N)) \tag{2.8}$$

for all $i, j = 1, 2, \cdots, N$, $0 < \theta \ll 1$, *and* $0 < \nu \ll 1$. *Moreover, if* $T_{\max} < \infty$, *then* $\limsup_{t\to T_{\max}} \|u(\cdot, t_0 + t; u_0)\|_\infty = \infty$.

Note that for $u_0 \equiv 0$, $(u(x,t;t_0,u_0), v(x,t;t_0,u_0)) \equiv (0,0)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. From both the mathematical and biological points of view, it is important to find conditions which guarantee the global existence of $(u(x,t;t_0;u_0), v(x,t;t_0,u_0))$ for every $t_0 \in \mathbb{R}$ and $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N) \setminus \{0\}$ with $u_0 \geq 0$. The following is our main result on the global existence.

Theorem 2.2. *Suppose that* $\chi\mu \leq b_{\inf}$, *then for every* $t_0 \in \mathbb{R}$ *and nonnegative function* $u_0 \in C^b_{\text{unif}}(\mathbb{R}^n) \setminus \{0\}$, (2.6) *has a unique nonnegative global classical solution* $(u(x,t;t_0,u_0),$

$v(x,t;t_0,u_0))$ *satisfying* $\lim_{t\searrow 0}\|u(\cdot,t_0+t;t_0,u_0)-u_0\|_\infty=0$. *Moreover, it holds that*

$$\|u(\cdot,t+t_0;t_0,u_0)\|_\infty\le\|u_0\|_\infty e^{a_{\sup}t}. \tag{2.9}$$

Furthermore, if

$(\mathbf{H1})$ $\qquad b_{\inf}>\chi\mu$

holds, then the following hold.

(i) *For every nonnegative initial function* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)\setminus\{0\}$ *and* $t_0\in\mathbb{R}$, *there holds*

$$\|u(\cdot,t+t_0;t_0,u_0)\|_\infty\le\max\{\|u_0\|_\infty,\frac{a_{\sup}}{b_{\inf}-\chi\mu}\}\ \forall\ t>0, \tag{2.10}$$

and

$$\limsup_{t\to\infty}\|u(\cdot,t+t_0;t_0,u_0)\|_\infty\le\frac{a_{\sup}}{b_{\inf}-\chi\mu}. \tag{2.11}$$

(ii) *For every* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $\inf_{x\in\mathbb{R}^N}u_0(x)>0$ *and* $t_0\in\mathbb{R}$, *there holds*

$$\frac{a_{\inf}}{b_{\sup}}\le\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N}u(x,t+t_0;t_0,u_0),\quad \liminf_{t\to\infty}\inf_{x\in\mathbb{R}^N}u(x,t+t_0;t_0,u_0)\le\frac{a_{\sup}}{b_{\inf}}. \tag{2.12}$$

(iii) *For every positive real number* $M>0$, *there is a constant* $K_1=K_1(\nu,M,a,b)$ *such that for every* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $0\le u_0\le M$, *we have*

$$\|v(\cdot,t+t_0;t_0,u_0)\|_{C^{1,\nu}_{\rm unif}(\mathbb{R}^N)}\le K_1,\quad\forall\ t_0\in\mathbb{R},\ \forall\ t\ge 0. \tag{2.13}$$

The so called *persistence* is an important concept in population models.

Definition 2.2. *Assume* **(H1)**. *We say that* *pointwise persistence* *occurs in* (2.6) *if for any* $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $\inf_{x\in\mathbb{R}^N}u_0(x)>0$, *there is* $m(u_0)>0$ *such that*

$$m(u_0)\le u(x,t+t_0;t_0,u_0)\le\max\{\|u_0\|_\infty,\frac{a_{\sup}}{b_{\inf}-\chi\mu}\}\ \forall\ t_0\in\mathbb{R}\text{ and }t>0. \tag{2.14}$$

We say that <u>*uniform persistence*</u> *occurs in* (2.6) *if there are* $0 < m < M < \infty$ *such that for any* $t_0 \in \mathbb{R}$ *and any positive initial function* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $\inf_{x\in\mathbb{R}} u_0(x) > 0$*, there exists* $T(t_0, u_0) > 0$ *such that*

$$m \le u(x, t + t_0; t_0, u_0) \le M \quad \forall\ t \ge T(t_0, u_0).$$

By Theorem 2.2, for any $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_{0\inf} > 0$, $\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} u(x, t + t_0; t_0, u_0)$ has a positive lower bound and $\liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x, t + t_0; t_0, u_0)$ has a positive upper bound. But it is not clear whether there is a positive lower bound (respectively, a positive lower bound independent of u_0) for $\liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x, t+t_0; t_0, u_0)$ with $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$ under hypothesis **(H1)**, which would imply pointwise persistence (respectively, uniform persistence) in (2.6). We have the following results on the pointwise persistence and uniform persistence of solutions of (2.6) with positive initial functions.

Theorem 2.3. *(i) (Pointwsie persistence) Suppose that* **(H1)** *holds. Then pointwise persistence occurs in* (2.6)*.*

(ii) (Uniform persistence) Suppose that **(H2)** *holds. If, furthermore,*

$$\textbf{(H2)} \quad b_{\inf} > (1 + \frac{a_{\sup}}{a_{\inf}})\chi\mu$$

holds, then uniform persistence occurs in (2.6)*. In particular, for every strictly positive initial function* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *and* $\varepsilon > 0$*, there is* $T_\varepsilon(u_0) > 0$ *such that the unique classical solution* $(u(x, t + t_0; t_0, u_0), v(x, t + t_0; t_0, u_0))$ *of* (2.6) *with* $u(\cdot, t_0; t_0, u_0) = u_0(\cdot)$ *satisfies*

$$\underline{M} - \varepsilon \le u(x, t + t_0; t_0, u_0) \le \overline{M} + \varepsilon, \quad \forall t \ge T_\varepsilon(u_0),\ x \in \mathbb{R}^N,\ \ t_0 \in \mathbb{R}, \tag{2.15}$$

and

$$\frac{\mu\underline{M}}{\lambda} - \varepsilon \le v(x, t + t_0; t_0, u_0) \le \frac{\mu\overline{M}}{\lambda} + \varepsilon, \quad \forall t \ge T_\varepsilon(u_0),\ x \in \mathbb{R}^N,\ \ t_0 \in \mathbb{R}, \tag{2.16}$$

where

$$\underline{M} := \frac{(b_{\inf} - \chi\mu)a_{\inf} - \chi\mu a_{\sup}}{(b_{\sup} - \chi\mu)(b_{\inf} - \chi\mu) - (\chi\mu)^2} > \frac{a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu}}{b_{\sup} - \chi\mu}, \tag{2.17}$$

and

$$\overline{M} := \frac{(b_{\sup} - \chi\mu)a_{\sup} - \chi\mu a_{\inf}}{(b_{\sup} - \chi\mu)(b_{\inf} - \chi\mu) - (\chi\mu)^2} < \frac{a_{\sup}}{b_{\inf} - \chi\mu}. \tag{2.18}$$

Furthermore, the set

$$I_{inv} := \{u \in C^b_{\rm unif}(\mathbb{R}^N) \mid \underline{M} \le u_0(x) \le \overline{M},\ \forall\, x \in \mathbb{R}^N\} \tag{2.19}$$

is a positively invariant set for solutions of (2.6) *in the sense that for every* $t_0 \in \mathbb{R}$ *and* $u_0 \in I_{inv}$*, we have that* $u(\cdot, t + t_0; t_0, u_0) \in I_{inv}$ *for every* $t \ge 0$.

Remark 2.1. *(1) Assume* **(H1)**. *By Theorem 2.2 (ii), and Theorem 2.3 (i), it holds that* $m(u_0) \le \frac{a_{\sup}}{b_{\inf}}$ *and* $\limsup_{t\to\infty} \|u(\cdot, t + t_0; t_0, u_0)\|_\infty \le \frac{a_{\sup} - \chi\mu m(u_0)}{b_{\inf}-\chi\mu}$ *uniformly in* $t_0 \in \mathbb{R}$ *for every* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $u_{0\inf} > 0$. *It remains open whether uniform persistence occurs under* **(H1)**.

(2) The proof of Theorem 2.3 (i) is highly nontrivial and is based on a key and fundamental result proved in Lemma 4.5. Roughly speaking, Lemma 4.5 shows that for any given time $T > 0$*, the concentration* $u(x, t; t_0, u_0)$ *of the mobile species at time* $t_0 + T$ *is bounded below by* $u_{0\inf}$ *provided that* $u_{0\inf}$ *is sufficiently small. This result will also play a crucial role in the study of existence of strictly positive entire solutions stated below.*

(3) When the functions $a(x, t)$ *and* $b(x, t)$ *are both space and time homogeneous,* $\underline{M} = \overline{M} = \frac{a}{b}$*, where* $\underline{M}$ *and* $\overline{M}$ *are as in Theorem 2.3 (ii).*

Assume **(H1)**. By Theorems 2.2 and 2.3, for any $t_0 \in \mathbb{R}$ and strictly positive $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$,

$$0 < \liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x, t + t_0; t_0, u_0) \le \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} u(x, t + t_0; t_0, u_0) < \infty.$$

Naturally, it is important to know whether there is a *strictly positive entire solution*, that is, an entire solution $(u^+(x, t), v^+(x, t))$ of (2.6) with $\inf_{t\in\mathbb{R},x\in\mathbb{R}^N} u^+(x, t) > 0$. It is also important to know the stability of strictly positive entire solutions of (2.6) (if such exist) and to investigate the asymptotic behavior of globally defined classical solutions with nonnegative initial functions. We have the following result on the existence of strictly positive entire solutions.

Theorem 2.4 (Existence of strictly positive entire solutions)**.** *Suppose that* **(H1)** *holds. Then* (2.6) *has a strictly positive entire solution* $(u^+(x,t), v^+(t,x))$*. Moreover, the following hold.*

(i) *Any strictly positive entire solution* $(u^+(x,t), v^+(x,t))$ *of* (2.6) *satisfies*

$$(a_{\inf} - \chi\mu u^+_{\sup})_+ \le (b_{\sup} - \chi\mu)u^+_{\inf} \quad \textit{and} \quad (b_{\inf} - \chi\mu)u^+_{\sup} \le (a_{\sup} - \chi\mu u^+_{\inf})_+.$$

In particular, we have that

$$\frac{a_{\inf}}{b_{\sup}} \le u^+_{\sup} \le \frac{a_{\sup} - \chi\mu u^+_{\inf}}{b_{\inf} - \chi\mu}, \tag{2.20}$$

for every positive entire solution $(u^+(x,t), v^+(t,x))$ *of* (2.6).

(ii) *If* **(H2)** *holds, then any strictly positive entire solution* $(u^+(x,t), v^+(x,t))$ *of* (2.6) *satisfies*

$$\underline{M} \le u^+(x,t) \le \overline{M}, \quad \forall x \in \mathbb{R}^N, \ \forall t \in \mathbb{R}, \tag{2.21}$$

where $\underline{M}$ *and* $\overline{M}$ *are given by* (2.17) *and* (2.18)*, respectively.*

(iii) *If there is* $T > 0$ *such that* $a(x, t+T) = a(x,t)$ *and* $b(x, t+T) = b(x,t)$ *for very* $x \in \mathbb{R}^N$*,* $t \in \mathbb{R}$*, then* (2.6) *has a strictly positive entire solution* $(u^+(x,t), v^+(x,t))$ *satisfying* $(u^+(x, t+T), v^+(x, t+T)) = (u^+(x,t), v^+(x,t))$ *for every* $x \in \mathbb{R}^N$*,* $t \in \mathbb{R}$*.*

(iv) *If* $a(x,t) = a(x)$ *and* $b(x,t) = b(x)$*, then* (2.6) *has a strictly positive steady state solution.*

Remark 2.2. **(i)** *Theorem 2.4 (i) provides an explicit a priori lower and upper bounds for the supremum of all positive entire solutions. This lower bound is in fact achieved in the case when the functions* $a(x,t)$ *and* $b(x,t)$ *are constant.*

(ii) *Theorem 2.4 (ii) shows that if* **(H2)** *holds, then the explicit lower bound and upper bound for all positive entire solutions which coincide with the lower bound and upper bound of the attraction region given by Theorem 2.3 (ii).*

We have the following result on the uniqueness and stability of positive entire solutions of (2.6).

Theorem 2.5 (Uniqueness and stability of strictly positive entire solutions)**.** *There exists* $\chi_0 > 0$ *such that when* $0 \leq \chi < \chi_0$*, there is* $\alpha_\chi > 0$ *such that* (2.6) *has a unique strictly positive entire solution* $(u_\chi^+(x,t), v_\chi^+(x,t))$ *which is uniformly and exponentially stable with respect to strictly positive perturbations in the sense that for any* $u_0 \in C^b_{\rm unif}(\mathbb{R})$ *with* $u_{0\inf} > 0$*, there is* $M > 0$ *such that*

$$\|u(\cdot, t+t_0; t_0, u_0) - u_\chi^+(\cdot, t+t_0)\|_\infty \leq Me^{-\alpha_\chi t},\ \forall t \geq 0,\ \forall t_0 \in \mathbb{R}, \tag{2.22}$$

and

$$\|v(\cdot, t+t_0; t_0, u_0) - v_\chi^+(\cdot, t+t_0)\|_\infty \leq \frac{\mu}{\lambda} Me^{-\alpha_\chi t},\ \forall t \geq 0,\ \forall t_0 \in \mathbb{R}.,\ \forall t_0 \in \mathbb{R}. \tag{2.23}$$

Furthermore, if the logistic function $f(x,t,u) = (a(x,t) - ub(x,t))u$ *is either space homogeneous or is of form* $f(x,t,u) = b(x,t)(\kappa - u)u$*,* $\kappa > 0$*, then* χ_0 *can be taken to be* $\chi_0 = \frac{b_{\inf}}{2\mu}$*, and* $u_\chi^+(x,t) = u_0^+(t)$*,* $0 < \chi < \chi_0$*, is the only stable positive entire solution of the Fisher-KPP equation* (1.4).

Remark 2.3. (i) *If we suppose that the logistic function is space homogeneous (resp. the function* $\mathbb{R}^N \times \mathbb{R} \ni (x,t) \mapsto \frac{a(x,t)}{b(x,t)}$ *is constant), Theorem 2.5 establishes the stability of the unique space homogeneous (resp. space-time homogeneous) positive entire solution of* (2.6) *when the chemotaxis sensitivity satisfies* $0 < \chi < \frac{b_{\inf}}{2\mu}$*. Furthermore, this result goes beyond the stability of the constant equilibrium given by Theorem 2.3 (ii) when the logistic source is constant, and show that all strictly positive solutions of* (2.6) *converge exponentially to* $(\frac{a}{b}, \frac{\mu a}{\lambda b})$ *when* $0 < \chi < \frac{b_{\inf}}{2\mu}$*. It should be noted that the hypothesis* $0 < \chi < \frac{b_{\inf}}{2\mu}$ *is weaker than hypothesis* **(H2)**.

(ii) *It is worth mentioning that the techniques developed to prove Theorem 2.5 can be adopted to study the uniqueness and stability of positive entire solution of* (2.6) *studied on bounded domains with Neuwmann boundary conditions. Hence the same result is true in this latter case.*

We have the following results on the asymptotic behavior or spreading properties of solutions to (2.6) with compactly supported initial functions.

Theorem 2.6 (Asymptotic spreading). *(1) Suppose that* **(H1)** *holds. Then for every* $t_0 \in \mathbb{R}$ *and every nonnegative initial function* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with nonempty compact support* $supp(u_0)$*, we have that*

$$\lim_{t\to\infty} \sup_{|x|\geq ct} u(x, t+t_0; t_0, u_0) = 0, \quad \forall c > c_+^*, \tag{2.24}$$

where

$$c_+^*(a, b, \chi, \lambda, \mu) := 2\sqrt{a_{\rm sup}} + \frac{\chi\mu\sqrt{N}a_{\rm sup}}{2(b_{\rm inf} - \chi\mu)\sqrt{\lambda}}. \tag{2.25}$$

(2) Suppose that

$$(\mathbf{H3}) \qquad b_{\rm inf} > \left(1 + \frac{\left(1 + \sqrt{1 + \frac{N a_{\rm inf}}{4\lambda}}\right) a_{\rm sup}}{2a_{\rm inf}}\right)\chi\mu. \tag{2.26}$$

Then for every $t_0 \in \mathbb{R}$ *and nonnegative initial function* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with*

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(x, t+t_0; t_0, u_0) > 0, \quad \forall 0 \leq c < c_-^*(a, b, \chi, \lambda, \mu), \tag{2.27}$$

where

$$c_-^*(a, b, \chi, \lambda, \mu) := 2\sqrt{a_{\rm inf} - \frac{\chi\mu a_{\rm sup}}{b_{\rm inf} - \chi\mu}} - \chi\frac{\mu\sqrt{N}a_{\rm sup}}{2\sqrt{\lambda}(b_{\rm inf} - \chi\mu)} \tag{2.28}$$

Remark 2.4. *Let χ_0 be given by Theorem 2.5. One can prove that for every,*

$$0<\chi<\min\left\{\chi_0, \frac{b_{\inf}}{\mu}\left(1+\frac{\left(1+\sqrt{1+\frac{Na_{\inf}}{4\lambda}}\right)a_{\sup}}{2a_{\inf}}\right)^{-1}\right\},$$

it holds that

$$\lim_{t\to\infty}\sup_{|x|\le ct}|u(x,t+t_0;t_0,u_0)-u_\chi^+(x,t)|=0,\quad \forall 0\le c<c_-^*(a,b,\chi,\lambda,\mu), \forall\ t_0\in\mathbb{R},$$

whenever $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ is nonnegative with nonempty compact support $supp(u_0)$, where the constant $c_-^*(a,b,\chi,\lambda,\mu)$ is given by Theorem 2.6.

We studied the existence and non-existence of transition wave solution of (2.6) when the functions $a(x,t)$ and $b(x,t)$ are both constant.

Definition 2.3. *Suppose that the functions $a(x,t)$ and $b(x,t)$ are both constant. Given a vector $\xi\in\mathbb{R}^N$ with $\|\xi\|=1$, an entire solution $(u(x,t),v(x,t))$ is called a traveling wave solution of* (2.6) *in the direction of ξ with speed c, connecting $(\frac{a}{b},\frac{a\mu}{b\lambda})$ and $(0,0)$ if it can be written as $(u(x,t),v(x,t))=(U(x\cdot\xi-ct),V(x\cdot\xi-ct))$ for some nonnegative functions $U,V\in C^2(\mathbb{R})$ satisfying $\lim_{x\to-\infty}(U(x),V(x))=(\frac{a}{b},\frac{a\mu}{b\lambda})$ and $\lim_{x\to\infty}(U(x),V(x))=(0,0)$. The function $(U(x),V(x))$ is called the profile of the traveling wave.*

Among others, we proved the following results.

Theorem 2.7 (Existence and non-existence of Traveling wave solutions)**.** *Suppose that $N=1$.*

(i) *(Existence of planar traveling wave solutions) Suppose that the functions $a(x,t)$ and $b(x,t)$ are constant. There is a function $c^*_{up}:(0,\frac{b}{\mu})\ni\chi\mapsto c^*_{up}(\chi)\in(2\sqrt{a},\infty)$ satisfying*

$$\lim_{\chi\to 0^+}c^*_{up}(\chi)=\begin{cases}2\sqrt{a} & \text{if } a\le\lambda,\\ \frac{a+\lambda}{\sqrt{\lambda}} & \text{if } a\ge\sqrt{\lambda},\end{cases}$$

such that for every $0 < \chi < \frac{b}{2\mu}$*, and* $c \geq c^*_{up}(\chi)$*,* (2.6) *has a traveling wave solution* $(u(x,t), v(x,t)) = (U(x-ct), V(x-ct))$ *with speed* c *and connecting* $(\frac{a}{b}, \frac{a\mu}{b\lambda})$ *and* $(0,0)$ *(i.e.* $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ *and* $(U(\infty), U(\infty)) = (0,0)$*). Moreover,*

$$\lim_{x\to\infty} \frac{U(x)}{e^{-\mu x}} = 1,$$

where μ *is the only solution of the equation* $c = \mu + \frac{a}{\mu}$ *in* $(0, \sqrt{a})$*.*

(ii) *(Non-existence of planar traveling wave solutions) Suppose that the functions* $a(x,t)$ *and* $b(x,t)$ *are both constant. Then* (2.6) *has no traveling wave solution* $(u(x,t), v(x,t)) = (U(x-ct), V(x-ct))$ *with a speed* $c < 2\sqrt{a}$ *and connecting* $(\frac{a}{b}, \frac{a\mu}{b\lambda})$ *and* $(0,0)$*.*

Remark 2.5. *We note that supposing* $N = 1$ *in Theorem 2.7 is not a restriction. Indeed, if* $(u(x,t), v(x,t)) = (U(x-ct), V(x-ct))$*,* $x \in \mathbb{R}$*, is a traveling wave solution of* (2.6) *with speed c connecting* $(\frac{a}{b}, \frac{a\mu}{b\lambda})$ *and* $(0,0)$ *in* $\mathbb{R}$ *then for any given unit vector* $\xi \in \mathbb{R}^N$ *and* $N \geq 1$ *the function* $(u(x,t), v(x,t)) = (U(x\cdot\xi - ct), V(x\cdot\xi - ct))$*,* $x \in \mathbb{R}^N$*, is also a traveling wave in the direction of* ξ *with speed c connecting* $(\frac{a}{b}, \frac{a\mu}{b\lambda})$ *and* $(0,0)$*. Hence Theorem 2.7 applies to* $N \geq 1$*.*

Chapter 3

Local and global existence of nonnegative classical solutions

In this chapter, which contains two sections, we study the local and global existence of nonnegative classical solutions of (2.6) and prove Theorems 2.1 and 2.2. Section 1 is devoted to the study of local existence of nonnegative classical solution. In section 2, we provide explicit conditions on the parameters which ensure that time-local classical solutions are defined globally in time.

3.1 Local existence of classical solutions

In this section, we investigate the local existence and uniqueness of classical solutions of (2.6) with given initial functions in $C^b_{\rm uinf}(\mathbb{R}^N)$ and prove Theorem 2.1. Our approach to prove Theorem 2.1 is first to prove the existence of a mild solution (see Definition 3.1 below) and then to prove the mild solution is a classical solution.

Definition 3.1. *For given $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $t_0 \in \mathbb{R}$ and $T > 0$, a function $(u(x,t), v(x,t)) \in [C([t_0, t_0+T] \,:\, C^b_{\rm unif}(\mathbb{R}^N))]^2$, with $v = \mu(\lambda I - \Delta)^{-1}u$, is called a* <u>mild solution</u> *of* (2.6) *if it satisfies the integral equation*

$$\begin{aligned} u(\cdot, t+t_0) =& T(t)u_0 - \chi \int_0^t T(t-s)\nabla\cdot(u(s+t_0)\nabla v(s+t_0))ds \\ &+ \int_0^t T(t-s)(1+a(s+t_0)-b(s+t_0)u(s+t_0))u(s+t_0)ds, \ \forall\, t\in[0,T], \end{aligned}$$

where $\{T(t)\}_{t\geq 0}$ is the analytic C_0-semigroup in (2.1).

It is clear that any classical solution of (2.6) is also a mild solution in the sense of Definition 3.1. Next, we establish some important lemmas.

Lemma 3.1. *Let $T(t)_{t\geq 0}$ be the semigroup in* (2.1) *generated by $\Delta - I$ on $C^b_{\rm unif}(\mathbb{R}^N)$. For every $t > 0$, the operator $T(t)\nabla\cdot$ has a unique bounded extension on $\left(C^b_{\rm unif}(\mathbb{R}^N)\right)^N$ satisfying*

$$\|T(t)\nabla\cdot u\|_\infty \leq \frac{N}{\sqrt{\pi}} t^{-\frac{1}{2}} e^{-t}\|u\|_\infty \quad \forall\, u \in \left(C^b_{\rm unif}(\mathbb{R}^N)\right)^N,\ \forall\, t > 0. \tag{3.1}$$

Proof. Let $C^{1,b}_{\rm unif}(\mathbb{R}^N) = \{u \in C^1(\mathbb{R}^N)\,|\, u(\cdot), \partial_{x_i}u(\cdot) \in C^b_{\rm unif}(\mathbb{R}^N),\ i = 1, 2, \cdots, N\}$. Since $C^{1,b}_{\rm unif}(\mathbb{R}^N)$ is a dense subspace of $C^b_{\rm unif}(\mathbb{R}^N)$, it is enough to prove that inequality (3.1) hold on $\left(C^{1,b}_{\rm unif}(\mathbb{R}^N)\right)^N$. For every $u = (u_1, u_2, \cdots, u_N) \in \left(C^{1,b}_{\rm unif}(\mathbb{R}^N)\right)^N$ and $t > 0$, we have

$$T(t)\partial_{x_i}u_i = \frac{e^{-t}}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}}\partial_{x_i}u_i(x-z)dz = \lim_{R\to\infty}\left[\frac{e^{-t}}{(4\pi t)^{\frac{N}{2}}}\int_{B(0,R)} e^{-\frac{|z|^2}{4t}}\partial_{x_i}u_i(x-z)dz\right]. \tag{3.2}$$

Next, for every $R > 0$ using integration by parts, we have

$$\begin{aligned}
&\int_{B(0,R)} e^{-\frac{|z|^2}{4t}}\partial_{x_i}u_i(x-z)dz && (3.3)\\
=&\frac{1}{2t}\int_{B(0,R)} z_i e^{-\frac{|z|^2}{4t}}u_i(x-z)dz - \int_{\partial B(0,R)} e^{-\frac{|z|^2}{4t}}u_i(x-z)\nu_i(z)ds(z)\\
=&\frac{1}{2t}\int_{B(0,R)} z_i e^{-\frac{|z|^2}{4t}}u_i(x-z)dz - e^{-\frac{R^2}{4t}}\int_{\partial B(0,R)} u_i(x-z)\frac{z_i}{R}ds(z). && (3.4)
\end{aligned}$$

Since u is uniformly bounded and the function $z \in \mathbb{R}^N \mapsto z_i e^{-\frac{|z|^2}{4t}}$ belongs to $L^1(\mathbb{R}^N)$, then

$$\lim_{R\to\infty}\frac{1}{2t}\int_{B(0,R)} z_i e^{-\frac{|z|^2}{4t}}u_i(x-z)dz = \frac{1}{2t}\int_{\mathbb{R}^N} z_i e^{-\frac{|z|^2}{4t}}u_i(x-z)dz. \tag{3.5}$$

On the other hand, we have

$$\left|e^{-\frac{R^2}{4t}}\int_{\partial B(0,R)} u_i(x-z)\frac{z_i}{R}ds(z)\right| \leq \|u_i\|_\infty e^{-\frac{R^2}{4t}}\int_{\partial B(0,R)} ds(z) \to 0 \text{ as } R\to\infty. \tag{3.6}$$

Combining (3.2), (3.3),(3.5) and (3.6), we obtain that

$$T(t)\partial_{x_i}u_i = \frac{e^{-t}}{2t\,(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} z_i e^{-\frac{|z|^2}{4t}}u_i(x-z)dz = \frac{e^{-t}}{2t\,(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} H_i(z,t)u_i(x-z)dz, \tag{3.7}$$

where the function $H_i(z,t) = z_i e^{-\frac{|z|^2}{4t}}$. Observe that, taking $y = \frac{1}{2\sqrt{t}} z$, then

$$\|H_i(\cdot,t)\|_{L^1(\mathbb{R}^N)} = 2\sqrt{t}\,(4t)^{\frac{N}{2}} \int_{\mathbb{R}^N} |y_i| e^{-|y|^2} dy = 2\sqrt{t}\,(4t)^{\frac{N}{2}}\, \|H_i(\cdot, \frac{1}{4})\|_{L^1(\mathbb{R}^N)}.$$

This, combined with Hölder's inequality and (3.7) yield that

$$\|T(t)\partial_{x_i} u_i\|_\infty \leq \frac{t^{-\frac{1}{2}} e^{-t}}{\pi^{\frac{N}{2}}} \|H_i(\cdot, \frac{1}{4})\|_{L^1(\mathbb{R}^N)} \|u_i\|_\infty.$$

Direct computations yield that $\|H_i(\cdot, \frac{1}{4})\|_{L^1(\mathbb{R}^N)} = \pi^{\frac{N-1}{2}}$. Hence

$$\|T(t)\partial_{x_i} u_i\|_\infty \leq \frac{t^{-\frac{1}{2}} e^{-t}}{\sqrt{\pi}} \|u_i\|_\infty. \tag{3.8}$$

Inequality (3.1) easily follows from (3.8). □

The next Lemma provides an explicit a priori estimate of the gradient of the solution $v(\cdot,\cdot)$ in the second equation of (2.6). This a priori estimate will be useful in the proof of existence theorem and the discussion on the asymptotic behavior of the solution.

Lemma 3.2. *For every $u \in C^b_{\rm unif}(\mathbb{R}^N)$, with $u(x) \geq 0$, and $\lambda > 0$, we have that*

$$\|(\Delta - \lambda I)^{-1} u\|_\infty \leq \frac{1}{\lambda} \|u\|_\infty \quad \textit{and} \quad \|\partial_{x_i}(\Delta - \lambda I)^{-1} u\|_\infty \leq \frac{1}{2\sqrt{\lambda}} \|u\|_\infty \tag{3.9}$$

for each $i = 1, \cdots, N$. Therefore we have

$$\|\nabla(\Delta - \lambda I)^{-1} u\|_\infty \leq \frac{\sqrt{N}}{2\sqrt{\lambda}} \|u\|_\infty,\ \forall\, u \in C^b_{\rm unif}(\mathbb{R}^N). \tag{3.10}$$

Proof. Let $u \in C^b_{\rm unif}(\mathbb{R}^N)$ and set $v = (I - \Delta)^{-1} u$. According to (2.3) it follows that

$$\begin{aligned} |v(x)| \leq & \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda s}}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x-z|^2}{4s}} |u(z)| dz ds \\ \leq & \|u\|_\infty \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda s}}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x-z|^2}{4s}} dz ds = \frac{1}{\lambda} \|u\|_\infty, \quad \forall\, x \in \mathbb{R}^N. \end{aligned} \tag{3.11}$$

On the other hand, observe that from (2.3) that

$$\begin{aligned}
\partial_{x_i} v(x) =& \int_0^\infty \int_{\mathbb{R}^N} \frac{(z_i - x_i)e^{-\lambda s}}{2s(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x-z|^2}{4s}} u(z) dz ds \\
=& \int_0^\infty \int_{\mathbb{R}^N} \frac{z_i e^{-\lambda s} e^{-|z|^2}}{\pi^{\frac{N}{2}} \sqrt{s}} u(x + 2\sqrt{s} z) dz ds \\
=& \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda s} e^{-|\pi_i(z)|^2}}{\pi^{\frac{N-1}{1}} \sqrt{s}} \left[\int_0^\infty z_i e^{-z_i^2} u(x + 2\sqrt{s}(\pi_i(z) + z_i e_i)) dz_i \right] d\pi_i(z) ds \\
& - \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda s} e^{-|\pi_i(z)|^2}}{\pi^{\frac{N}{1}} \sqrt{s}} \left[\int_0^\infty z_i e^{-z_i^2} u(x + 2\sqrt{s}(\pi_i(z) - z_i e_i)) dz_i \right] d\pi_i(z) ds
\end{aligned} \tag{3.12}$$

where $\pi_i(z) = (z_1, \cdots, z_{i-1}, 0, z_{i+1}, \cdots, z_N)$ for every $z \in \mathbb{R}^N$. Since $u(z) \geq 0$, for every $\tau \in \{-1, 1\}$ and $i \in \{1, \cdots, N\}$, using the facts that $\int_0^\infty z_i e^{-z_i^2} dz_i = \frac{1}{2}$, $\int_{\mathbb{R}^{N-1}} \frac{e^{-|\pi_i(z)|^2}}{\pi^{\frac{N-1}{1}} \sqrt{s}} d\pi_i(z) = \pi^{\frac{N-1}{2}}$, and $\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{\lambda}} ds = \frac{\sqrt{\pi}}{\sqrt{\lambda}}$, we have

$$\begin{aligned}
0 \leq & \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda s} e^{-|\pi_i(z)|^2}}{\pi^{\frac{N-1}{1}} \sqrt{s}} \left[\int_0^\infty z_i e^{-z_i^2} u(x + 2\sqrt{s}(\pi_i(z) + \tau z_i e_i)) dz_i \right] d\pi_i(z) ds \\
\leq & \|u\|_\infty \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda s} e^{-|\pi_i(z)|^2}}{\pi^{\frac{N-1}{1}} \sqrt{s}} \left[\int_0^\infty z_i e^{-z_i^2} dz_i \right] d\pi_i(z) ds \\
= & \|u\|_\infty \left[\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{\lambda}} ds \right] \left[\int_{\mathbb{R}^{N-1}} \frac{e^{-|\pi_i(z)|^2}}{\pi^{\frac{N-1}{1}} \sqrt{s}} d\pi_i(z) \right] \left[\int_0^\infty z_i e^{-z_i^2} dz_i \right] = \frac{\|u\|_\infty}{2\sqrt{\lambda}}.
\end{aligned} \tag{3.13}$$

Therefore, (3.9) follows from (3.11), (3.12) and (3.13). The lemma thus follows. □

Next, we prove Theorems 2.1. The main tools for the proof of this theorem are based on the contraction mapping theorem and the existence of classical solutions for linear parabolic equations with Hölder continuous coefficients. Throughout the rest of this subsection, C denotes a constant independent of the initial functions and the solutions under consideration, unless specified otherwise. We let $X = C^b_{\rm unif}(\mathbb{R}^N)$ and X^β is the fractional power space of $I - \Delta$ on X ($\beta \in (0, 1)$).

Proof of Theorem 2.1. **(i) Existence of mild solution**. We first prove the existence of a mild solution of (2.6) with given initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, which will be done by proving five claims.

Fix $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$. For every $t_0, T > 0$ and $R > 0$, let

$$\mathcal{S}_{R,T}(t_0) := \left\{ u \in C([t_0, t_0 + T], C^b_{\rm unif}(\mathbb{R}^N)) \mid \|u(\cdot, t)\|_X \le R \right\}.$$

Note that $\mathcal{S}_{R,T}(t_0)$ is a closed subset of the Banach space $C([t_0, t_0 + T], C^b_{\rm unif}(\mathbb{R}^N))$ with the sup-norm.

Claim 1. For any $u \in \mathcal{S}_{R,T}(t_0)$ and $t \in [0, T]$, $(Gu)(t + t_0)$ is well defined, where

$$\begin{aligned}(Gu)(t+t_0) =& T(t)u_0 + \chi \int_0^t T(t-s)\nabla \cdot (u(s+t_0)\nabla(\Delta - \lambda I)^{-1}u(s+t_0))ds \\ &+ \int_0^t T(t-s)\left((1 + a(\cdot, s+t_0))u(s+t_0) - b(\cdot, s+t_0)u^2(s+t_0)\right) ds,\end{aligned}$$

with $(Gu)(t_0) = u_0$, and the integrals are taken in $C^b_{\rm unif}(\mathbb{R}^N)$. Indeed, let $u \in \mathcal{S}_{R,T}(t_0)$ and $0 < t \le T$ be fixed. Since the function

$$[0, t] \ni s \mapsto (a(\cdot, s+t_0) + 1)u(s+t_0) - b(\cdot, s+t_0)u^2(s+t_0) \in C^b_{\rm unif}(\mathbb{R}^N)$$

is continuous, then the function $F_1 : [0, t] \to C^b_{\rm unif}(\mathbb{R}^N)$ defined by

$$F_1(s) := T(t-s)\left((1 + a(\cdot, s+t_0))u(s+t_0) - b(\cdot, s+t_0)u^2(s+t_0)\right)$$

is continuous. Hence the Riemann integral $\int_0^t F_1(s)ds$ in $C^b_{\rm unif}(\mathbb{R}^N)$ exists. Observe that for every $0 < \varepsilon < t$ and $s \in [0, t-\varepsilon]$, we have

$$\begin{aligned}F_{2,\varepsilon}(s) :=& T(t-s)\nabla \cdot (u(s+t_0)\nabla(\Delta - \lambda I)^{-1}u(s+t_0)) \\ =& T(t-\varepsilon-s)T(\varepsilon)\nabla \cdot (u(s+t_0)\nabla(\Delta - \lambda I)^{-1}u(s+t_0)),\end{aligned}$$

and the function $[0, t-\varepsilon] \ni s \mapsto T(\varepsilon)\nabla \cdot (u(s+t_0)\nabla(\Delta - \lambda I)^{-1}u(s+t_0)) \in C^b_{\rm unif}(\mathbb{R}^N)$ is continuous. Thus the function $F_{2,\varepsilon} : [0, t-\varepsilon] \to C^b_{\rm uinf}(\mathbb{R}^N)$ is continuous for every $0 < \varepsilon < t$. Thus, the function $F_2 : [0, t) \to C^b_{\rm uinf}(\mathbb{R}^N)$ defined by

$$F_2(s) := T(t-s)\nabla \cdot (u(s+t_0)\nabla(\Delta - \lambda I)^{-1}u(s+t_0))$$

is continuous. Moreover, using Lemma 3.1 and inequality (3.10), we have that

$$\begin{aligned}\int_0^t \|F_2(s)\|_\infty ds &\leq \chi \int_0^t \|T(t-s)\nabla\cdot\| \|u(s+t_0)\|_\infty \|\nabla(\Delta-\lambda I)^{-1}u(s+t_0)\|_\infty ds \\ &\leq \frac{\mu N\sqrt{N}}{2\sqrt{\lambda\pi}}\chi \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|u(s+t_0)\|_\infty^2 ds \\ &\leq \frac{\mu N R^2\sqrt{N}}{2\sqrt{\lambda\pi}}\chi\Gamma(\frac{1}{2}) = \frac{\mu N R^2\sqrt{N}}{2\sqrt{\lambda}}\chi.\end{aligned}$$

Hence, the Riemann integral $\int_0^t F_2(s)ds$ in $C^b_{\rm unif}(\mathbb{R}^N)$ exists. Note that $(Gu)(t+t_0) = T(t)u_0 + \int_0^t F_2(s)ds + \int_0^t F_1(s)ds$. Whence, Claim 1 follows.

Claim 2. For every $u \in \mathcal{S}_{R,T}(t_0)$ and $0<\beta<\frac{1}{2}$, the function $(0,T] \ni t \to (Gu)(t+t_0) \in X^\beta$ is locally Hölder continuous, and G maps $\mathcal{S}_{R,T}(t_0)$ into $C([t_0, t_0+T], C^b_{\rm unif}(\mathbb{R}^N))$.

First, observe that

$$\begin{aligned}(Gu)(t+t_0) = \underbrace{T(t)u_0}_{I_0(t)} + \chi \underbrace{\int_0^t T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))ds}_{I_1(t)} \\ + \underbrace{\int_0^t T(t-s)((a(\cdot,s+t_0)+1)u(s+t_0) - b(\cdot,s+t_0)u^2(s+t_0))ds}_{I_2(t)}. \end{aligned} \tag{3.14}$$

For every $t>0$, it is clear that $T(t)u_0 \in X^\beta$ because the semigroup $\{T(t)\}_t$ is analytic. Furthermore, the divergence operator $T(t)\nabla\cdot$ satisfies

$$T(t)\nabla\cdot w = T(\frac{t}{2})(T(\frac{t}{2})\nabla\cdot w) \in \text{Dom}(\Delta) \subset X^\beta$$

for all $t > 0,\ w \in \left(C^b_{\rm unif}(\mathbb{R}^N)\right)^N$. Using Lemma 3.1 and inequalities (2.4) and (3.10), we obtain

$$\begin{aligned}
&\int_0^t \|T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_{X^\beta}ds \\
=&\int_0^t \|(\Delta-I)^\beta T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds \\
\le& C\int_0^t (t-s)^{-\beta-\frac{1}{2}}e^{-(t-s)}\|u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0)\|_\infty ds \\
\le& CR^2\int_0^t (t-s)^{-\beta-\frac{1}{2}}e^{-(t-s)}ds \le CR^2\Gamma(\frac{1}{2}-\beta).
\end{aligned} \tag{3.15}$$

Since the operator $(I-\Delta)^\beta$ is closed, we have that

$$I_1(t) = \int_0^t T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))ds \in X^\beta$$

for every $t > 0$. Similar arguments show that $I_2(t) \in X^\beta$ for every $0 < t \le T$. Hence $u(t) \in X^\beta$ for every $t > 0$.

Next, let $t \in (0,T)$ and $h > 0$ such that $t+h \le T$. Using (2.4) and (2.5), we have

$$\begin{aligned}
\|I_0(t+h)-I_0(t)\|_{X^\beta} &\le \|(T(h)-I)T(t)u_0\|_{X^\beta} \le Ch^\beta\|T(t)u_0\|_{X^{2\beta}} \\
&\le Ch^\beta t^{-2\beta}e^{-t}\|u_0\|_\infty \le Ch^\beta t^{-2\beta}\|u_0\|_\infty,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\|I_1(t+h)-I_1(t)\|_{X^\beta} \\
\le&\int_0^t \|(T(h)-I)T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_{X^\beta}ds \\
&+\int_t^{t+h} \|T(t+h-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_{X^\beta} \\
\le& Ch^\beta\int_0^t (t-s)^{-\beta-\frac{1}{2}}e^{-(t+h-s)}\|(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds \\
&+C\int_t^{t+h}(t+h-s)^{-\beta-\frac{1}{2}}e^{-(t+h-s)}\|u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds \\
\le& CR^2h^\beta\int_0^t \frac{e^{-(t+h-s)}}{(t-s)^{\beta+\frac{1}{2}}}ds + CR^2\int_t^{t+h}\frac{e^{-(t+h-s)}}{(t+h-s)^{\beta+\frac{1}{2}}}ds \le CR^2(h^\beta+h^{\frac{1}{2}-\beta}),
\end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
&\|I_2(t+h)-I_2(t)\|_{X^\beta}\\
\leq &\int_0^t \|(T(h)-I)T(t-s)((a(\cdot,s+t_0)+1)u(s+t_0)-b(\cdot,s+t_0)u^2(s+t_0))\|_{X^\beta}ds\\
&+\int_t^{t+h} \|T(t+h-s)((a(\cdot,s+t_0)+1)u(s+t_0)-b(\cdot,s+t_0)u^2(s+t_0))\|_{X^\beta}ds\\
\leq &Ch^\beta \int_0^t (t-s)^{-\beta}e^{-(t+h-s)}\|(a(\cdot,s)+1)u(s+t_0)-b(\cdot,s)u^2(s+t_0)\|_\infty ds\\
&+C\int_t^{t+h} \frac{e^{-(t+h-s)}}{(t+h-s)^\beta}\|(a(\cdot,s+t_0)+1)u(s+t_0)-b(\cdot,s+t_0)u^2(s+t_0)\|_\infty ds\\
\leq &CR^2(h^\beta+h^{1-\beta}).
\end{aligned} \tag{3.18}
$$

Combining (3.14),(3.16),(3.17) and (3.18), we deduce that the function $(0,T]\ni t\to (Gu(t+t_0))\in X^\beta$ is locally Hölder continuous.

Now it is clear that $t\to (Gu)(t+t_0)\in C^b_{\rm unif}(\mathbb{R}^N)$ is continuous in t at $t=0$. The claim then follows.

Claim 3. For every $R>\|u_0\|_\infty$, there exists $T:=T(R)$ such that $G(\mathcal{S}_{R,T}(t_0))\subset \mathcal{S}_{R,T}(t_0)$.

First, observe that for any $u\in\mathcal{S}_{R,T}(t_0)$, we have

$$
\begin{aligned}
&\|G(u)(t+t_0)\|_\infty\\
\leq\ & \|T(t)u_0\|_\infty+\chi\int_0^t \|T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds\\
&+(1+a_{\rm sup})\int_0^t\|T(t-s)u(s+t_0)\|_\infty ds+b_{\rm sup}\int_0^t\|T(t-s)u^2(s+t_0)\|_\infty ds\\
\leq\ & e^{-t}\|u_0\|_\infty+\chi\int_0^t \|T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds\\
&+(1+a_{\rm sup})R\int_0^t e^{-(t-s)}ds+b_{\rm sup}R^2\int_0^t e^{-(t-s)}ds\\
=\ & e^{-t}\|u_0\|_\infty+R\left((1+a_{\rm sup})+b_{\rm sup}R\right)(1-e^{-t})\\
&+\chi\int_0^t \|T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds.
\end{aligned} \tag{3.19}
$$

Using Lemma 3.1 and inequality (3.10), the last inequality can be improved to

$$
\begin{aligned}
\|G(u)(t+t_0)\|_\infty \le & e^{-t}\|u_0\|_\infty + R\left((1+a_{\sup})+bR\right)(1-e^{-t}) \\
& + C\chi\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))\|_\infty ds \\
\le & \frac{\|u_0\|_\infty}{e^t} + R\left((1+a_{\sup})+bR\right)(1-e^{-t}) + \chi C R^2\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}ds \\
\le & \frac{\|u_0\|_\infty}{e^t} + R\left((1+a_{\sup})+b_{\sup}R\right)(1-e^{-t}) + 2C\chi R^2 t^{\frac{1}{2}}. \qquad (3.20)
\end{aligned}
$$

Now, by (3.20), we can now chose $T>0$ such that

$$
\|G(u)(t+t_0)\|_\infty \le \frac{\|u_0\|_\infty}{e^{-t}} + R\left((1+a_{\sup})+b_{\sup}R\right)(1-e^{-t}) + 2C\chi R^2 t^{\frac{1}{2}} < R \;\forall\, t\in[0,T].
$$

This together with Claim 2 implies Claim 3.

Claim 4. G is a contraction map for T small and hence has a fixed point $u(\cdot)\in\mathcal{S}_{R,T}(t_0)$. Moreover, for every $0<\beta<\frac{1}{2}$, $(0,T]\ni t\to u(t+t_0)\in X^\beta$ is locally Holder continuous.

For every $u,w\in\mathcal{S}_{R,T}$, using again Lemma 3.1 and (2.4), we have

$$
\begin{aligned}
& \|(G(u)-G(w))(t+t_0)\|_\infty \\
\le & \chi\int_0^t \|T(t-s)\nabla\cdot(u\nabla(\Delta-\lambda I)^{-1}u - w\nabla(\Delta-\lambda I)^{-1}w)(s+t_0)\|_\infty ds \\
& + (1+a_{\sup})\int_0^t \|T(t-s)(u(s+t_0)-w(s+t_0))\|_\infty ds \\
& + b_{\sup}\int_0^t \|T(t-s)(u^2(s+t_0)-w^2(s+t_0)\|_\infty ds \\
\le & C\chi\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|(u\nabla(\Delta-\lambda I)^{-1}u - w\nabla(\Delta-\lambda I)^{-1}w)((s+t_0))\|_\infty ds \\
& + ((1+a_{\sup})+2Rb_{\sup})\int_0^t e^{-(t-s)}\|(u(s)-w(s))\|_\infty ds \\
\le & C\chi\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|(u(s+t_0)-w(s+t_0))\|_\infty\|\nabla(\Delta-\lambda I)^{-1}u(s)\|_\infty ds \\
& + C\chi\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|w(s+t_0)\|_\infty\|\nabla(\Delta-\lambda I)^{-1}(w(s+t_0)-u(s+t_0))\|_\infty ds \\
& + (1+a_{\sup}+2Rb_{\sup})\|u-w\|_{\mathcal{S}_{R,T}(t_0)}\int_0^t e^{-(t-s)}ds \\
\le & \left(\frac{CR\chi\mu\sqrt{N}}{\sqrt{\lambda}}\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}ds + (1+a_{\sup}+2Rb_{\sup})\int_0^t e^{-(t-s)}ds\right)\|u-w\|_{\mathcal{S}_{R,T}(t_0)} \\
\le & \left(2\frac{CR\chi\mu\sqrt{N}}{\sqrt{\lambda}}t^{\frac{1}{2}} + (1+a_{\sup}+2Rb_{\sup})t\right)\|u-w\|_{\mathcal{S}_{R,T}(t_0)}.
\end{aligned}
$$

Hence, choose T small satisfying

$$2\frac{CR\chi\mu\sqrt{N}}{\sqrt{\lambda}}t^{\frac{1}{2}} + (1 + a_{\sup} + 2Rb_{\sup})t < 1 \quad \forall\, t \in [0, T],$$

we have that G is a contraction map. Thus there is $T > 0$ and a unique function $u \in \mathcal{S}_{R,T}(t_0)$ such that

$$u(t+t_0) = T(t)u_0 + \chi\int_0^t T(t-s)\nabla\cdot(u(s+t_0)\nabla(\Delta-\lambda I)^{-1}u(s+t_0))ds$$
$$+ \int_0^t T(t-s)((a(\cdot, s+t_0)+1)u(s+t_0) - b(\cdot, s+t_0)u^2(s+t_0))ds.$$

Moreover, by Claim 2, for every $0 < \beta < \frac{1}{2}$, the function $t \in (0, T] \to u(t+t_0) \in X^\beta$ is locally Holder continuous. Clearly, $u(t)$ is a mild solution of (2.6) on $[t_0, T+t_0)$ with $\alpha = 0$ and $X^0 = C^b_{\rm unif}(\mathbb{R}^N)$.

Claim 5. There is $T_{\max} \in (0, \infty]$ such that (2.6) has a mild solution $u(\cdot)$ on $[t_0, t_0 + T_{\max})$ with $\alpha = 0$ and $X^0 = C^b_{\rm unif}(\mathbb{R}^N)$. Moreover, for every $0 < \beta < \frac{1}{2}$, $(0, T_{\max}) \ni t \mapsto u(\cdot) \in X^\beta$ is locally Hölder continuous. If $T_{\max} < \infty$, then $\limsup_{t\to T_{\max}} \|u(t+t_0)\|_\infty = \infty$.

This claim follows the regular extension arguments.

(ii) Regularity and non-negativity. We next show that the mild solution $u(\cdot)$ of (2.6) on $[t_0, t_0 + T_{\max})$ obtained in (i) is a nonnegative classical solution of (2.6) on $[t_0, t_0 + T_{\max})$ and satisfies (2.7), (2.8).

Without loss of generality, we may suppose that $t_0 = 0$. Let $0 < t_1 < T_{\max}$ be fixed. It follows from claim 2 that for $0 < \nu \ll 1$, $u_1 := u(t_1) \in C^\nu_{\rm unif}(\mathbb{R}^N)$, and the mappings

$$t \to u(\cdot, t+t_1) := u(t+t_1)(\cdot) \in C^\nu_{\rm unif}(\mathbb{R}^N), \;\; t \mapsto v(\cdot, t+t_1) \in C^\nu_{\rm unif}(\mathbb{R}^N)$$

$$t \mapsto \frac{\partial v(\cdot, t+t_1)}{\partial x_i} \in C^\nu_{\rm unif}(\mathbb{R}^N), \quad t \mapsto \frac{\partial^2 v(\cdot, t+t_1)}{\partial x_i \partial x_j} \in C^\nu_{\rm unif}(\mathbb{R}^N)$$

are locally Hölder continuous in $t \in (-t_1, T_{\max} - t_1)$, where $v(\cdot, t+t_1) := \mu(\lambda I - \Delta)^{-1}u(\cdot, t+t_1)$ and $i, j = 1, 2, \cdots, N$. Consider the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u} = (\Delta - 1)\tilde{u} + \tilde{F}(t, \tilde{u}), \quad x \in \mathbb{R}^N, \; t > 0 \\ \tilde{u}(x, 0) = u_1(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{3.21}$$

where $\tilde{F}(t, \tilde{u}) = -\chi\nabla v(\cdot, t+t_1)\nabla\tilde{u} + (a(\cdot, t+t_1) + 1 - \chi v(\cdot, t+t_1) - (b(\cdot, t+t_1) - \chi)u(\cdot, t+t_1))\tilde{u}$.
Then by [13, Theorem 11 and Theorem 16 in Chapter 1], (3.21) has a unique classical solution

$\tilde u(x,t)$ on $[0,T_{\max}-t_1)$ with $\lim_{t\to 0+}\|\tilde u(\cdot,t)-u_1\|_\infty=0$. In fact $\tilde u$ has the representation

$$\tilde u(x,t)=\int_{\mathbb{R}^N}\Gamma(x,t,y,t_1)u_1(y)dy$$

with the function Γ satisfying the inequalities

$$|\Gamma(x,t,y,\tau)|\le C\frac{e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{N}{2}}}\quad\text{and}\quad |\partial_{x_i}\Gamma(x,t,y,\tau)|\le C\frac{e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{(N+1)}{2}}}$$

for every $0<\lambda_0<1$. By a priori interior estimates for parabolic equations (see [13, Theorem 5]), we have that

$$\tilde u(\cdot,\cdot)\in C^1((0,T_{\max}-t_1),C^b_{\text{unif}}(\mathbb{R}^N),$$

and the mappings

$$t\mapsto \tilde u(\cdot,t)\in C^\nu_{\text{unif}}(\mathbb{R}^N),\quad t\mapsto \frac{\partial\tilde u}{\partial x_i}(\cdot,t)\in C^\nu_{\text{unif}}(\mathbb{R}^N),$$
$$t\mapsto \frac{\partial^2\tilde u}{\partial x_i\partial x_j}(\cdot,t)\in C^\nu_{\text{unif}}(\mathbb{R}^N),\quad t\mapsto \frac{\partial\tilde u}{\partial t}(\cdot,t)\in C^\nu_{\text{unif}}(\mathbb{R}^N)$$

are locally Hölder continuous in $t\in(0,T_{\max}-t_1)$ for $i,j=1,2,\cdots,N$ and $0<\nu\ll 1$. Hence, by [16, Lemma 3.3.2], $\tilde u(t)(\cdot)=\tilde u(\cdot,t)$ is also a mild solution of (3.21) and then satisfies the following integral equation,

$$\tilde u(t)=T(t)u_1-\chi\int_0^t T(t-s)(\nabla v(s+t_1)\nabla\tilde u(s))ds$$
$$+\int_0^t T(t-s)(a(\cdot,s+t_1)+1-\chi v(s+t_1)-(b(\cdot,s+t_1)-\chi)u(s+t_1))\tilde u(s))ds$$

for $t\in[0,T_{\max}-t_1)$. Now, using the fact that $\nabla\tilde u\cdot\nabla v(\cdot+t_1)=\nabla\cdot(\tilde u\nabla v(\cdot+t_1))-(v(\cdot+t_1)-u(\cdot+t_1))\tilde u$, we have

$$\begin{aligned}\tilde u(t)=&T(t)u_1-\chi\int_0^t T(t-s)(\nabla\cdot(\tilde u(s)\nabla v(s+t_1))ds\\
&+\chi\int_0^t T(t-s)(v(s+t_1)-u(s+t_1))\tilde u(s)ds\\
&+\int_0^t T(t-s)(a+1-\chi v-(b-\chi)u(s+t_1))\tilde u(s)ds\\
=&T(t)u_1-\chi\int_0^t T(t-s)\nabla\cdot(\tilde u(s)\nabla v(s+t_1))ds\\
&+\int_0^t T(t-s)(a+1-bu(s+t_1))\tilde u(s)ds.\end{aligned}\tag{3.22}$$

On the other hand from equation (3.14), we have that

$$
\begin{aligned}
u(t+t_1) &= T(t)u_1 - \chi \int_0^t T(t-s)\nabla \cdot (u(s+t_1)\nabla v(s+t_1))ds \\
&\quad + \int_0^t T(t-s)(a+1-bu(s+t_1))u(s+t_1)ds. \qquad (3.23)
\end{aligned}
$$

Taking the difference side by side of (3.22) and (3.23) and using Lemma 3.1 and (2.4), we obtain for any $\epsilon > 0$ and $0 < t < T_\epsilon < T_{\max} - t_1 - \epsilon$ that

$$
\begin{aligned}
&\|\tilde{u}(t) - u(t+t_1)\|_\infty \\
\leq\ & \chi \int_0^t \|T(t-s)\nabla \cdot ((u(s+t_1) - \tilde{u}(s))\nabla v(s+t_1))\|_\infty ds \\
&+ \int_0^t \|T(t-s)(a+1-bu(s+t_1))(u(s+t_1) - \tilde{u}(s))\|_\infty ds \\
\leq\ & C\chi \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|(u(s+t_1) - \tilde{u}(s))\nabla v(s+t_1))\|_\infty ds \\
&+ \int_0^t e^{-(t-s)} \|(a+1-bu(s+t_1))(u(s+t_1) - \tilde{u}(s))\|_\infty ds \\
\leq\ & C\chi \sup_{s\in[0,T_\epsilon]} \|\nabla v(s+t_1)\|_\infty \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|u(s+t_1) - \tilde{u}(s)\|_\infty ds \\
&+ C(a_{\sup} + 1 + b_{\sup} \sup_{s\in[0,T_\epsilon]} \|u(s+t_1)\|_\infty) \int_0^t e^{-(t-s)} \|u(s+t_1) - \tilde{u}(s)\|_\infty ds.
\end{aligned}
$$

Combining this last inequality with Lemma 2.1, we conclude that $\tilde{u}(t) = u(t+t_1)$ for every $t \in [0, T_\epsilon]$. We then have that u is a classical solution of (2.6) on $[0, T_{\max})$ and satisfies (2.7) and (2.8). Since $u_0 \geq 0$, by comparison principle for parabolic equations, we get $u(x,t) \geq 0$.

Let $u(\cdot, t; u_0) = u(t)(\cdot)$ and $v(\cdot, t; u_0) = \mu(\lambda I - \Delta)^{-1} u(\cdot, t; u_0)$. We have that the function $(u(\cdot, \cdot; u_0), v(\cdot, \cdot; u_0))$ is a nonnegative classical solution of (2.6) on $[0, T_{\max})$ with initial function u_0 and $u(\cdot, t; u_0)$ satisfies (2.7) and (2.8).

(iii) Uniqueness. We now prove that for given $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, (2.6) has a unique classical solution $(u(\cdot, \cdot; u_0), v(\cdot, \cdot; u_0))$ satisfying (2.7) and (2.8).

Any classical solution of (2.6) satisfying the properties of Theorem 2.1 clearly satisfies the integral equation (3.23). Suppose that for given $u_0 \in C^b_{\rm unif}(\mathbb{R}^1)$ with $u_0 \geq 0$, $(u_1(t,x), v_1(t,x))$ and $(u_2(t,x), v_2(t,x))$ are two classical solutions of (2.6) on $\mathbb{R}^N \times [0,T)$ satisfying the properties of Theorem 2.1. Let $0 < t_1 < T' < T$ be fixed. Thus $\sup_{0\leq t\leq T'}(\|u_1(\cdot,t)\|_\infty + \|u_2(\cdot,t)\|_\infty) < \infty$. Let $u_i(t) = u_i(\cdot, t)$ and $v_i(t) = (I - \Delta)^{-1} u_i(t)$ for every $i = 1, 2$ and

$0 \le t < T$. For every $t \in [t_1, T']$, and $i = 1, 2$ we have that

$$u_i(t) = T(t-t_1)u_i(t_1)+\chi\int_{t_1}^{t} T(t-s)\nabla\cdot(u_i(s)\nabla v_i(s))ds+\int_{t_1}^{t} T(t-s)(a+1-bu_i(s))u_i(s)ds.$$

Hence for $t_1 \le t \le T^{'}$, using Lemma 3.2 and inequality (2.4), we obtain

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_\infty \\
\le&\|(u_1(t_1) - u_2(t_1))\|_\infty + C\chi\int_{t_1}^{t} \frac{e^{-(t-s)}}{\sqrt{t-s}}\|u_1(s)\nabla v_1(s) - u_2(s)\nabla v_2(s)\|_\infty ds \\
&+ \int_0^t e^{-(t-s)}\|u_1(s) - u_2(s)\|_\infty(a + 1 + b(\|u_1(s)\|_\infty + \|u_2(s)\|_\infty))ds \\
\le&\|(u_1(t_1) - u_2(t_1))\|_\infty + C\chi\int_{t_1}^{t} \frac{e^{-(t-s)}}{\sqrt{t-s}}\|u_1(s) - u_2(s)\|_\infty\|\nabla v_1(s)\|_\infty \\
&+ C\chi\int_{t_1}^{t} \frac{e^{-(t-s)}}{\sqrt{t-s}}\|u_2(s)\|\|\nabla(v_2(s) - v_1(s)\|_\infty ds \\
&+ (a_{\sup} + 1 + b_{\sup}\sup_{0\le\tau\le T'}(\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty))\int_0^t e^{-(t-s)}\|u_1(s) - u_2(s)\|_\infty ds \\
\le&\|(u_1(t_1) - u_2(t_1))\|_\infty + \frac{C\mu\sqrt{N}\chi}{2\sqrt{\lambda}}\int_{t_1}^{t} \frac{e^{-(t-s)}}{\sqrt{t-s}}\|u_1(s) - u_2(s)\|_\infty(\|u_1(s)\|_\infty + \|u_2(s)\|_\infty) \\
&+ (a_{\sup} + 1 + b_{\sup}\sup_{0\le\tau\le T'}(\|u_1(\tau)\|_\infty + \|u_2(\tau)\|_\infty))\int_{t_1}^t e^{-(t-s)}\|u_1(s) - u_2(s)\|_\infty ds \\
\le&\|(u_1(t_1) - u_2(t_1))\|_\infty + M\int_{t_1}^{t} \frac{e^{-(t-s)}}{\sqrt{t-s}}\|u_2(s) - u_1(s)\|_\infty ds,
\end{aligned}$$

where $M = a_{\sup}+1+(\frac{C\sqrt{N}\chi}{2\sqrt{\lambda}}+b_{\sup}\sqrt{T'})\sup_{0\le t\le T'}(\|u_1(\tau)\|_\infty+\|u_2(\tau)\|_\infty) < \infty$. Let $t_1 \to 0$, we have

$$\|u_1(t) - u_2(t)\|_\infty \le M\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\|u_2(s)\|\|u_2(s) - u_1(s)\|_\infty ds.$$

By Lemma 2.1 again, we get $u_1(t) \equiv u_2(t)$ for all $0 \le t \le T'$. Since $T' < T$ was arbitrary chosen, then $u_1(t) \equiv u_2(t)$ for all $0 \le t < T$. The theorem is thus proved. □

3.2 Global existence of classical solutions

This section is devoted to the study of the global existence of classical solutions to (2.6). For every $t_0 \in \mathbb{R}$ and nonnegative function $u_0 \in C^b_{\text{unif}}(\mathbb{R}^n)$, it follows from Lemma 3.2 that,

$$\|v(\cdot, t + t_0; t_0, u_0)\|_\infty \le \frac{\mu}{\lambda}\|u(\cdot, t + t_0; t_0, u_0)\|_\infty,\ \forall\ 0 \le t < T_{\max}(t_0, u_0), \tag{3.24}$$

and

$$\|\nabla v(\cdot,t+t_0;t_0,u_0)\|_\infty \le \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\|u(\cdot,t+t_0;t_0,u_0)\|_\infty,\ \forall\ 0\le t<T_{\max}(t_0,u_0). \tag{3.25}$$

Now we present the proof of Theorem 2.2. Note that Theorem 2.2 provides a sufficient condition on the parameters χ and $b_{\inf}$ to guarantee the existence of time globally defined classical solutions.

Proof of Theorem 2.2. Let $t_0 \in \mathbb{R}$ and $u_0 \in C^b_{\rm unif}(\mathbb{R}^n)$, $u_0 \ge 0$, be given. According to Theorem 2.1, there is $T_{\max} = T_{\max}(t_0,u_0) \in (0,\infty]$ such that (2.6) has a unique nonnegative classical solution $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ on $[t_0, t_0+T_{\max})$. Since $b_{\inf} \ge \chi\mu$, we have that $(u(x,t;t_0,u_0), v(x,t;t_0,u_0))$ satisfies

$$\begin{aligned} u_t &= \Delta u - \chi\nabla v\cdot\nabla u + u(a(x,t) - u(b(x,t)-\chi\mu) - \chi\lambda v)\\ &\le \Delta u - \chi\nabla v\cdot\nabla u + u(a(x,t) - u(b(x,t)-\chi\mu))\\ &\le \Delta u - \chi\nabla v\cdot\nabla u + u(a_{\sup} - (b_{\inf}-\chi\mu)u) \end{aligned} \tag{3.26}$$

for $t \in (t_0, t_0+T_{\max})$. Thus, by comparison principles for parabolic equations, it follows from (3.26) that

$$u(x,t+t_0;t_0,u_0) \le \overline{u}(t;\|u_0\|_\infty), \quad \forall\, 0\le t<T_{\max}(t_0,u_0),\ \forall\, x\in\mathbb{R}^N, \tag{3.27}$$

where $\overline{u}(t;\|u_0\|_\infty)$ solves the ODE

$$\begin{cases} \frac{d}{dt}\overline{u} = \overline{u}(a_{\sup} - (b_{\inf}-\chi\mu)\overline{u})\\ \overline{u}(0) = \|u_0\|_\infty. \end{cases} \tag{3.28}$$

Since $b_{\inf} \ge \chi\mu$, then $\overline{u}(t;\|u_0\|_\infty)$ is defined for all $t \ge 0$. This implies that $T_{\max}(t_0,u_0) = \infty$. Moreover, $\overline{u}(t;\|u_0\|_\infty) \le |u_0\|_\infty e^{ta_{\sup}}$ for all $t > 0$. Hence (2.9) holds.

(i) If $b_{\inf} > \chi\mu$, we have that $\overline{u}(t;\|u_0\|_\infty) \le \max\{\|u_0\|_\infty, \frac{a_{\sup}}{b_{\inf}-\chi\mu}\}$ for all $t > 0$, and $\lim_{t\to\infty}\overline{u}(t;\|u_0\|_\infty) = \frac{a_{\sup}}{b_{\inf}-\chi\mu}$ provided that $\|u_0\|_\infty > 0$. Hence (2.10) and (2.11) hold.

(ii) First, by (3.24),

$$
\begin{aligned}
u_t &= \Delta u - \chi \nabla v \cdot \nabla u + u(a(x,t) - u(b(x,t) - \chi\mu) - \chi\lambda v) \\
&\geq \Delta u - \chi \nabla v \cdot \nabla u + u(a_{\inf} - \|u(\cdot,t;t_0,u_0)\|_\infty (b_{\sup} - \chi\mu) - \chi\lambda \frac{\mu}{\lambda} \|u(\cdot,t;t_0,u_0)\|_\infty) \\
&= \Delta u - \chi \nabla v \cdot \nabla u + u(a_{\inf} - \|u(\cdot,t;t_0,u_0)\|_\infty b_{\sup}) \qquad (3.29)
\end{aligned}
$$

for $t > t_0$. By comparison principle for parabolic equations, we have

$$
u(x,t+t_0;t_0,u_0) \geq e^{\int_{t_0}^{t+t_0}(a_{\inf} - \|u(\cdot,s+t_0;t_0,u_0)\|_\infty b_{\sup})ds} u_{0\inf} \quad \forall t \geq t_0.
$$

This together with $u_{0\inf} > 0$ implies that

$$
\inf_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0) > 0 \quad \forall\, t \geq t_0.
$$

Next, for any $\epsilon > 0$, there is $T^\epsilon > 0$ such that

$$
u(x,t+t_0;t_0,u_0) \leq u^\infty + \epsilon \ \text{ and } \ v(x,t+t_0;t_0,u_0) \leq \frac{\mu}{\lambda}(u^\infty + \epsilon) \quad \forall\, t \geq T^\epsilon,
$$

where $u^\infty = \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0)$. This combined with (3.29) imply that

$$
u_t \geq \Delta u - \chi \nabla v \cdot \nabla u + u(a_{\inf} - (u^\infty + \epsilon) b_{\sup})
$$

for $t \geq T^\epsilon$. By comparison principle for parabolic equations again, we have

$$
u(x,t+t_0;t_0,u_0) \geq e^{(a_{\inf} - (u^\infty+\epsilon)b_{\sup})(t-T^\epsilon)} \inf_{x\in\mathbb{R}^N} u(x,T^\epsilon + t_0;t_0,u_0) \quad \forall\, t \geq T^\epsilon.
$$

By the boundedness of $u(x,t+t_0,t_0,u_0)$ for $t \geq 0$, we must have

$$
a_{\inf} - (u^\infty + \epsilon) b_{\sup} \leq 0 \quad \forall\, \epsilon > 0.
$$

The first inequality in (2.12) then follows.

Now, if $\liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x,,t+t_0;t_0,u_0) = 0$, then the second inequality in (2.12) holds trivially. Assume $u_\infty := \liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x,,t+t_0;t_0,u_0) > 0$. Then for any

$0 < \epsilon < u_\infty$, there is $T_\epsilon > 0$ such that

$$u(x, t + t_0; t_0, u_0) \geq u_\infty - \epsilon \text{ and } v(x, t + t_0; t_0, u_0) \geq \frac{\mu}{\lambda}(u_\infty - \epsilon) \quad \forall\, t \geq T_\epsilon.$$

This combined with (3.26) yields that

$$u_t \leq \Delta u - \chi \nabla v \cdot \nabla u + u(a_{\sup} - (u_\infty - \epsilon) b_{\inf})$$

for $t \geq T_\epsilon$. By comparison principle for parabolic equations, we have

$$u(x, t + t_0; t_0, u_0) \leq e^{(a_{\sup} - (u_\infty - \epsilon) b_{\inf})(t - T_\epsilon)} \|u(\cdot, T_\epsilon + t_0; t_0, u_0)\| \quad \forall\, t \geq T_\epsilon.$$

This together with the first inequality in (2.12) implies that

$$a_{\sup} - (u_\infty - \epsilon) b_{\inf} \geq 0 \quad \forall\, 0 < \epsilon < u_\infty.$$

The second inequality in (2.12) then follows.

(iii) Let $x \in \mathbb{R}^N$ and $t \geq 0$ be fixed. Define

$$f(y) = \lambda v(x + y, t + t_0; t_0, u_0) - \mu u(x + y, t + t_0; t_0, u_0), \quad \forall y \in B(0, 3)$$

and

$$\phi(y) = v(x + y, t + t_0; t_0, u_0), \quad \forall y \in \bar{B}(0, 3)$$

Let G_1 be the solution of

$$\begin{cases} \Delta G_1 = f, & y \in B(0, 3) \\ G_1 = 0, & \text{on } \partial B(0, 3). \end{cases}$$

Choose $p \gg N$ such that $W^{2,p}(B(0,3)) \subset C^{1+\nu}_{\text{unif}}(B(0,3))$ (with continuous embedding). Thus, by regularity for elliptic equations, there is $c_{1,\nu} > 0$ (depending only on ν, N and the Lebesgue measure $|B(0,3)|$ of $B(0,3)$) such that

$$\|G_1\|_{C^{1+\nu}_{\text{unif}}(B(0,3))} \leq c_{1,\nu} \|f\|_{L^p(B(0,3))}. \tag{3.30}$$

Next, define

$$G_2(y) = v(x + y, t + t_0; t_0, u_0) - G_1(y), \quad \forall y \in \bar{B}(0, 3).$$

Hence G_2 solves

$$\begin{cases} \Delta G_2 = 0, \quad y \in B(0,3) \\ G_2(y) = \phi(y), \quad y \in \partial B(0,3). \end{cases}$$

Thus, (see [9, page 41]),

$$G_2(y) = \frac{2 - \|y\|^2}{2N\omega_N} \int_{\partial B(0,3)} \frac{\phi(z)}{|y-z|^N} dS(z), \quad \forall y \in B(0,3),$$

where $\omega_N = |B(0,1)|$ is the Lebesgue measure of $B(0,1)$, and

$$\partial_{y_i} G_2(y) = -\frac{y_i}{N\omega_N} \int_{\partial B(0,3)} \frac{\phi(z)}{|y-z|^N} dS(z) + \frac{2 - \|y\|^2}{2\omega_N} \int_{\partial B(0,3)} \frac{(y_i - z_i)\phi(z)}{|y-z|^{N+2}} dS(z), \forall y \in B(0,3). \tag{3.31}$$

But

$$|y+h-z| \geq |z| - |y+h| \geq 1, \quad \forall z \in \partial B(0,3),\ y,h \in B(0,1)$$

and

$$||y+h-z| - |y-z|| \leq |h|, \forall y,h,z \in \mathbb{R}^N.$$

It follows from (3.31), that there is $c_{2,\nu} > 0$ (depending only on ν, N and $|B(0,3)|$) such that

$$|\partial_{y_i} G_2(y+h) - \partial_{y_i} G_2(y)| \leq c_{2,\nu} |h|^\nu \|\phi\|_\infty, \quad \forall y,h \in B(0,1).$$

Combining the last inequality with (3.30), there is $c_\nu(N,P)$(depending only on ν, N and $|B(0,3)|$) such that

$$\|G_1 + G_2\|_{C^{1+\nu}_{\rm unif}(B(0,1))} \leq c_\nu [\|f\|_\infty + \|\phi\|_\infty]. \tag{3.32}$$

Note that $v(x+h, t+t_0; t_0, u_0) = (G_1 + G_2)(h)$, thus (iii) follows from (2.10), (3.24), (3.25), and (3.32). □

Chapter 4

Pointwise and uniform persistence phenomena

In this chapter we explore the pointwise and unifrom persistence of positive classical solutions and prove Theorem 2.3. In order to do so, we first prove some important lemmas.

4.1 Important lemmas

This section is devoted to establishing the tools that will be needed to prove our main result on pointwise and uniform persistence of solutions of (2.6). The next Lemma provides a finite time pointwise persistence for solutions $(u(x,t;t_0,u_0),v(x,t;t_0,u_0))$ of (2.6) with strictly positive function u_0.

Lemma 4.1. *Suppose that* **(H1)** *holds. Then for every $T>0$, $t_0\in\mathbb{R}$, and for every nonnegative initial function $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$, it holds that*

$$\inf_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0)\geq u_{0\,\inf}e^{t(a_{\inf}-b_{sup}\|u_0\|_\infty e^{Ta_{\sup}})},\quad \forall 0\leq t\leq T. \tag{4.1}$$

In particular, for every $T>0$ and for every nonnegative initial function $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ satisfying $\|u_0\|_\infty\leq M_T:=\frac{a_{\inf}e^{-a_{\sup}T}}{b_{\sup}}$, we have that

$$\inf_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0)\geq \inf_{x\in\mathbb{R}^N}u_0(x),\quad \forall 0\leq t\leq T,\ \forall t_0\in\mathbb{R}. \tag{4.2}$$

Proof. Let $t_0\in\mathbb{R}$ and $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$, $u_0\geq 0$, be given. Since **(H1)** holds, it follows from Theorem 2.2 that $(u(\cdot,t+t_0;t_0,u_0),v(\cdot,t+t_0;t_0,u_0))$ is defined for all $t\geq 0$. By (2.9) and (3.24),

$$\chi\lambda\|v(\cdot,t+t_0;t_0;u_0)\|_\infty\leq\chi\mu\|u(\cdot,t+t_0;t_0;u_0)\|_\infty\leq\chi\mu\|u_0\|_\infty e^{a_{\sup}t},\quad\forall\, t\geq 0.$$

Hence, for every $t_0<t\leq t_0+T$, it follows from the previous inequality and (3.29) that

$$u_t \geq \Delta u - \chi \nabla v \cdot \nabla u + u(a_{\inf} - b_{\sup}\|u_0\|_\infty e^{a_{\sup}T}). \tag{4.3}$$

Thus, by comparison principle for parabolic equations, it follows from (4.3) that

$$\inf_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0) \geq u_{0\inf} e^{t(a_{\inf} - b_{sup}\|u_0\|_\infty e^{a_{\sup}T})}, \quad \forall\, 0 \leq t \leq T,\ T>0,\ t_0 \in \mathbb{R}. \tag{4.4}$$

Observe that $\|u_0\|_\infty \leq M_T := \frac{a_{\inf} e^{-a_{\sup}T}}{b_{\sup}}$ implies that $a_{\inf} - b_{\sup}\|u_0\|_\infty e^{a_{\sup}T} \geq 0$. This combined with (4.4) yields (4.2). □

Remark 4.1. *We note that a slight modification of the proof of Lemma 4.1 yields that if* **(H1)** *does not hold then*

$$\inf_{x\in\mathbb{R}^N} u(x,t+t_0;t_0,u_0) \geq u_{0\inf} e^{t(a_{\inf} - (b_{\sup}+\chi\mu)\|u_0\|_\infty e^{a_{\sup}T})}, \quad \forall\ 0 \leq t \leq T < T_{\max}(u_0), \tag{4.5}$$

for every nonnegative initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$. Hence for every initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$, it always holds that

$$\inf_{x\in\mathbb{R}^N, 0\leq t\leq T} u(x,t+t_0;t_0,u_0) > 0, \quad \forall\, 0 \leq T < T_{\max}(u_0), \forall t_0 \in \mathbb{R}.$$

It should be noted (4.5) *and* (4.1) *do not implies the pointwise persistence of $u(x,t+t_0;t_0,u_0)$.*

Lemma 4.2. *Assume that (H1) holds. Let $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $\{u_{0n}\}_{n\geq 1}$ be a sequence of non-negative functions in $C^b_{\rm unif}(\mathbb{R}^N)$, and let $\{t_{0n}\}_{n\geq1}$ be a sequence of real numbers. Suppose that $0 \leq u_{0n}(x) \leq M := \frac{a_{\sup}}{b_{\inf}-\chi\mu}$ and $\{u_{0n}\}_{n\geq1}$ converges locally uniformly to u_0. Then there exist a subsequence $\{t_{0n'}\}$ of $\{t_{0n}\}$, functions $a^*(x,t), b^*(x,t)$ such that $(a(x,t+t_{0n'}), b(x,t+t_{0n'})) \to (a^*(x,t), b^*(x,t))$ locally uniformly as $n' \to \infty$, and $u(x,t+t_{0n'};t_{0n'},u_{0n'}) \to u^*(x,t;0,u_0)$ locally uniformly in $C^{2,1}(\mathbb{R}^N \times (0,\infty))$ as $n' \to \infty$, where $(u^*(x,t;0,u_0), v^*(x,t;0,u_0)$ is the classical solution of*

$$\begin{cases} u_t(x,t) = \Delta u(x,t) - \chi\nabla\cdot(u(x,t)\nabla v(x,t)) + (a^*(x,t) - b^*(x,t)u(x,t))u(x,t), & x \in \mathbb{R}^N \\ 0 = (\Delta - \lambda I)v^*(x,t) + \mu u^*(x,t), \quad x \in \mathbb{R}^N \\ u^*(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases}$$

Proof. Without loss of generality, by Arzela-Ascoli's Theorem, we may suppose that $(a(x,t+t_{0n}), b(x,t+t_{0n})) \to (a^*(x,t), b^*(x,t))$ locally uniformly in $\mathbb{R}^N \times \mathbb{R}$ as $n \to \infty$. Recall that

$(u(x,t+t_{0n};t_{0n},u_{0n}), v(x,t+t_{0n};t_{0n},u_{0n}))$ satisfies for $x\in\mathbb{R}^N,\ t>0$,

$$u_t(\cdot,\cdot+t_{0n};t_{0n},u_{0n}) = \Delta u(\cdot,\cdot+t_{0n};t_{0n},u_{0n}) - \chi\nabla\cdot(u(\cdot,\cdot+t_{0n};t_{0n},u_{0n})\nabla v(\cdot,\cdot+t_{0n};t_{0n},u_{0n}))$$
$$+ (a(\cdot,\cdot+t_{0n}) - b(\cdot,\cdot+t_{0n})u(\cdot,\cdot+t_{0n};t_{0n},u_{0n}))u(\cdot,\cdot+t_{0n};t_{0n},u_{0n}).$$

So, by variation of constant formula, we have that

$$\begin{aligned}
&u(\cdot,t+t_{0n};t_{0n},u_{0n}) - \underbrace{e^{t(\Delta-I)}u_{0n}}_{I_{0n}}(t)\\
&=\underbrace{\int_0^t e^{(t-s)(\Delta-I)}(a(\cdot,s+t_{0n})+1)u(\cdot,s+t_{0n};t_{0n},u_{0n})ds}_{I_{1n}(t)}\\
&\quad-\chi\underbrace{\int_0^t e^{(t-s)(\Delta-I)}\nabla\cdot(u(\cdot,s+t_{0n};t_{0n},u_{0n})\nabla v(\cdot,t+t_{0n};t_{0n},u_{0n}))ds}_{I_{2n}(t)}\\
&\quad-\underbrace{\int_0^t e^{(t-s)(\Delta-I)}b(\cdot,s+t_{0n})u^2(\cdot,s+t_{0n};t_{0n},u_{0n})ds}_{I_{3n}(t)},\ \forall\, t>0, \qquad (4.6)
\end{aligned}$$

where $\{e^{t(\Delta-I)}\}_{t\ge 0}$ denotes the analytic semigroup generated on $X^0 := C^b_{\rm unif}(\mathbb{R}^N)$ by $\Delta - I$. Let X^β, $\beta>0$, denote the fractional power spaces associated with $\Delta - I$. Let $0<\beta<\frac{1}{2}$ be fixed.

Using inequalities (2.4), (2.5), and (3.16), (3.18), there is a constant $C_\beta>0$, such that

$$\|I_{0n}(t+h)-I_{0n}(t)\|_{X^\beta} \le C_\beta h^\beta t^{-\beta}\|u_{0n}\|_\infty \le C_\beta h^\beta t^{-\beta}M,$$

$$\begin{aligned}
\|I_{1n}(t+h)-I_{1n}(t)\|_{X^\beta} &\le C_\beta(a_{\sup}+1)M\left[h^\beta\int_0^t\frac{e^{-(t-s)}}{(t-s)^\beta}ds + \int_t^{t+h}\frac{e^{-(t+h-s)}}{(t+h-s)^\beta}ds\right]\\
&\le C_\beta(a_{\sup}+1)M\left[h^\beta\Gamma(1-\beta)+\frac{h^{1-\beta}}{1-\beta}\right],
\end{aligned}$$

and

$$\|I_{2n}(t+h)-I_{2n}(t)\|_{X^\beta} \le C_\beta b_{\sup}M^2\left[h^\beta\Gamma(1-\beta)+\frac{h^{1-\beta}}{1-\beta}\right].$$

It follows from Lemma 3.2 and inequalities (2.4), (2.5), and (3.18) that

$$\begin{aligned}\|I_{3n}(t+h)-I_{3n}(t)\|_{X^\beta} &\le \frac{\mu\sqrt{N}C_\beta M^2}{2\sqrt{\lambda}}\left[h^\beta\int_0^t \frac{e^{-(t-s)}}{(t-s)^{\beta+\frac{1}{2}}}ds+\int_t^{t+h}\frac{e^{-(t+h-s)}}{(t+h-s)^{\beta+\frac{1}{2}}}ds\right]\\ &\le \frac{\mu\sqrt{N}C_\beta M^2}{2\sqrt{\lambda}}\left[h^\beta\Gamma(1-\beta)+\frac{h^{1-\beta}}{1-\beta}\right].\end{aligned}$$

Hence the function $(0,\infty)\ni t\mapsto u(\cdot,t+t_{0n};t_{0n},u_{0n})\in X^\beta$ is locally uniformly Hölder continuous. It then follows from the Arzela-Ascoli Theorem and [13, Theorem 15] that there is a subsequence $\{t_{0n'}\}$ of $\{t_{0n}\}$ and a function $u\in C^{2,1}(\mathbb{R}^N\times(0,\infty))$ such that $u(x,t+t_{0n'};t_{0n'},u_{0n'})$ converges to $u(x,t)$ locally uniformly in $C^{2,1}(\mathbb{R}^N\times(0,\infty))$ as $n'\to\infty$. Furthermore, taking $v(x,t)=\mu(\lambda I-\Delta)^{-1}u(x,t)$, we have that

$$u_t(x,t)=\Delta u(x,t)-\chi\nabla\cdot(u(x,t)\nabla v(x,t))+(a^*(x,t)-b^*(x,t)u(x,t))u(x,t),\quad x\in\mathbb{R}^N$$

for $t>0$. Since $u_{0n'}(x)\to u_0(x)$ locally uniformly as $n\to\infty$, it is not hard to show from (4.6) that $u(x,t)$ satisfies

$$\begin{aligned}u(x,t)=&e^{t(\Delta-I)}u_0-\chi\int_0^t e^{(t-s)(\Delta-I)}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))ds\\ &+\int_0^t e^{(t-s)(\Delta-I)}((1+a^*(\cdot,s))u-b^*(\cdot,s)u^2(\cdot,s)))ds.\end{aligned}\tag{4.7}$$

Note that $(u^*(x,t;0,u_0),v^*(x,t;0,u_0))$ also satisfies the integral equation (4.7). It thus follows from Lemma 2.1, that $u(x,t)=u^*(x,t;0,u_0)$. □

Lemma 4.3. *Assume that (H1) holds. For every $M>0$, $\varepsilon>0$, and $T>0$, there exist $L_0=L(M,T,\varepsilon)\gg 1$ and $\delta_0=\delta_0(M,\varepsilon)$ such that for every initial function $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ with $0\le u_0\le M$ and every $L\ge L_0$,*

$$u(x,t+t_0;t_0,u_0)\le\varepsilon,\quad \forall\, 0\le t\le T,\ t_0\in\mathbb{R},\ \forall\ |x|_\infty<2L\tag{4.8}$$

whenever $0\le u_0(x)\le\delta_0$ for $|x|_\infty<3L$.

Proof. It follows from (3.26) and comparison principle for parabolic equations that

$$0\le u(x,t+t_0;t_0,u_0)\le U(x,t+t_0;t_0,u_0),\quad \forall\,x\in\mathbb{R}^N,\ t\ge 0,\tag{4.9}$$

where U solves

$$
\begin{cases} U_t = \Delta U - \chi \nabla v(\cdot,\cdot;t_0,u_0)\cdot \nabla U + a_{\sup} U, \quad t > t_0 \\ U(\cdot,t_0) = u_0 \end{cases} \tag{4.10}
$$

It follows from Theorem 2.2 (iii) and [13, Theorem 12] that $U((x,t_0+t;t_0,u_0))$ can be written in the form

$$
U(x,t_0+t;t_0,u_0) = \int_{\mathbb{R}^N} \Gamma(x,t,y,0)u_0(y)dy. \tag{4.11}
$$

Moreover, for every $0 < \lambda_0 < 1$, there is a constant $K_2 = K_2(\lambda_0, N, \nu, K_1, T)$, where K_1 is given by Theorem 2.2 (iii), such that for $x \in \mathbb{R}^N,\ \tau \le t \le \tau + T$,

$$
|\Gamma(x,t,y,\tau)| \le K_2 \frac{e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{N}{2}}} \text{ and } |\partial_{x_i}\Gamma(x,t,y,\tau)| \le K_2 \frac{e^{-\frac{\lambda_0|x-y|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{N+1}{2}}}. \tag{4.12}
$$

We then have

$$
\begin{aligned}
U(x,t_0+t;t_0,u_0) &\le K_2 \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0|x-y|^2}{4t}}}{t^{\frac{N}{2}}} u_0(y)dy \\
&= K_2 \int_{R^N} e^{-\frac{\lambda_0}{4}|z|^2} u_0(x+t^{\frac{1}{2}}z)dz \\
&\le K_2 \Big[\int_{|z|_\infty \le \frac{L}{\sqrt{T}}} e^{-\frac{\lambda_0}{4}|z|^2} u_0(x+t^{\frac{1}{2}}z)dz + \int_{|z|_\infty \ge \frac{L}{\sqrt{T}}} e^{-\frac{\lambda_0}{4}|z|^2} u_0(x+t^{\frac{1}{2}}z)dz \Big].
\end{aligned}
$$

This implies that for $|x| \le 2L$,

$$
\begin{aligned}
U(x,t_0+t;t_0,u_0) &\le K_2\delta_0 \int_{\mathbb{R}^N} e^{-\frac{\lambda_0}{4}|z|^2}dz + K_2\|u_0\|_\infty \int_{|z|_\infty \ge \frac{L}{\sqrt{T}}} e^{-\frac{\lambda_0}{4}|z|^2}dz \\
&\le K_2\delta_0 \left(\frac{4\pi}{\lambda_0}\right)^{\frac{N}{2}} + K_2 M \int_{|z|_\infty \ge \frac{L}{\sqrt{T}}} e^{-\frac{\lambda_0}{4}|z|^2}dz.
\end{aligned} \tag{4.13}
$$

Take $\delta_0 = \frac{\varepsilon}{2K_2}\left(\frac{4\pi}{\lambda_0}\right)^{-\frac{N}{2}}$ and choose $L_0 \gg 1$ such that $\int_{|z|_\infty \ge \frac{L_0}{\sqrt{T}}} e^{-\frac{\lambda_0}{4}|z|^2}dz < \frac{\varepsilon}{2K_2M}$, it follows from (4.13) that for every $L \ge L_0$, there holds that $U(x,t+t_0;t_0,u_0) \le \varepsilon$ for every $|x|_\infty \le 2L$ whenever $u_0(x) \le \delta_0$ for all $|x|_\infty \le 3L$. This combined with (4.9) yields the lemma. □

Lemma 4.4. *Suppose that* **(H2)** *holds. Consider the sequence* $(\underline{M}_n, \overline{M}_n)_{n\geq 0}$ *defined inductively by* $\underline{M}_0 = 0$ *and*

$$\overline{M}_n = \frac{a_{\sup} - \chi\mu\underline{M}_n}{b_{\inf} - \chi\mu}, \quad \textit{and} \quad \underline{M}_{n+1} = \frac{a_{\inf} - \chi\mu\overline{M}_n}{b_{\sup} - \chi\mu}, \quad \forall\, n \geq 0. \tag{4.14}$$

Then for every $n \geq 0$*, it holds that*

$$\underline{M}_{n+1} > \underline{M}_n \geq 0 \quad \textit{and} \quad \overline{M}_n > \overline{M}_{n+1} > 0.$$

Moreover, we have that
$$\lim_{n\to\infty} (\underline{M}_n, \overline{M}_n) = (\underline{M}, \overline{M}),$$
where $\underline{M}$ *and* $\overline{M}$ *are given by* (2.17) *and* (2.18)*, respectively.*

Proof. For every $n \geq 0$, it holds that

$$\underline{M}_{n+1} = \frac{(b_{\inf} - \chi\mu)a_{\inf} - \chi\mu a_{\sup} + (\chi\mu)^2\underline{M}_n}{(b_{\inf} - \chi\mu)(b_{\sup} - \chi\mu)} \tag{4.15}$$

and

$$\overline{M}_{n+1} = \frac{(b_{\sup} - \chi\mu)a_{\sup} - \chi\mu a_{\inf} + (\chi\mu)^2\overline{M}_n}{(b_{\inf} - \chi\mu)(b_{\sup} - \chi\mu)}. \tag{4.16}$$

Thus, since $\underline{M}_0 = 0$, $\overline{M}_0 = \frac{a_{\sup}}{b_{\inf}-\chi\mu} > 0$, and **(H2)** holds, it follows by mathematical induction that $\underline{M}_n \geq 0$ and $\overline{M}_n \geq 0$ for every $n \geq 0$. Therefore, it follows from (4.14) that

$$0 \leq \underline{M}_n \leq \frac{a_{\inf}}{b_{\sup} - \chi\mu} \quad \text{and} \quad 0 \leq \overline{M}_n \leq \frac{a_{\sup}}{b_{\inf} - \chi\mu}, \quad \forall\, n \geq 0.$$

Observe that $\underline{M}_0 < \underline{M}_1$. Hence, (4.15) implies that $\underline{M}_n < \underline{M}_{n+1}$ for every $n \geq 0$. Similarly, we have that $\overline{M}_0 > \overline{M}_1$. Hence (4.16) implies that $\overline{M}_{n+1} < \overline{M}_n$ for every $n \geq 0$. Thus the sequence $(\underline{M}_n, \overline{M}_n)$ is convergent. By passing to the limit in (4.15) and (4.16), it is easily seen that $\lim_{n\to\infty}(\underline{M}_n, \overline{M}_n) = (\underline{M}, \overline{M})$, where $\underline{M}$ and $\overline{M}$ are given by (2.18) and (2.17) respectively. □

Lemma 4.5. *For fixed* $T > 0$*, there is* $0 < \delta_0^*(T) < M^+ = \frac{a_{\sup}}{b_{\inf}-\chi\mu} + 1$ *such that for any* $0 < \delta \leq \delta_0^*(T)$ *and for any* u_0 *with* $\delta \leq u_0 \leq M^+$*,*

$$\delta \leq u(x, t_0 + T; t_0, 0, u_0) \leq M^+ \quad \forall\, x \in \mathbb{R}^N,\ \forall\, t_0 \in \mathbb{R}. \tag{4.17}$$

Proof. We divide the proof into four steps.

First of all, let $a_0 = \frac{a_{\inf}}{3}$ and

$$D_L = \{x \in \mathbb{R}^N \,|\, |x_i| < L \quad \text{for } \ i = 1, 2, \cdots, N\}.$$

Consider

$$\begin{cases} u_t = \Delta u + a_0 u, \quad x \in D_L \\ u = 0, \quad x \in \partial D_L, \end{cases} \tag{4.18}$$

and its associated eigenvalue problem

$$\begin{cases} \Delta u + a_0 u = \sigma u, \quad x \in D_L \\ u = 0, \quad x \in \partial D_L. \end{cases} \tag{4.19}$$

Let $\sigma_{_L}$ be the principal eigenvalue of (4.19) and $\phi_L(x)$ be its principal eigenfunction with $\phi_L(0) = 1$. Note that

$$\phi_L(x) = \Pi_{i=1}^N \cos\big(\frac{\pi}{2L} x_i\big) \quad \text{and} \quad 0 < \phi_L(x) \le \phi_L(0), \quad \forall x \in D_L.$$

Note also that $u(x,t) = e^{\sigma_L t}\phi_L(x)$ is a solution of (4.18). Let $u(x,t;u_0)$ be the solution of (4.18) with $u_0 \in C(\bar{D}_L)$. Then

$$u(x,t;\kappa\phi_L) = \kappa e^{\sigma_L t}\phi_L(x) \tag{4.20}$$

for all $\kappa \in \mathbb{R}$.

In the following, let $L_0 \gg 0$ be such that $\sigma_{_L} > 0 \quad \forall L \ge L_0$.

Step 1. Let $T > 0$ be fixed. Consider

$$\begin{cases} u_t = \Delta u + b_\epsilon(x,t)\cdot\nabla u + a_0 u, \quad x \in D_L \\ u = 0, \quad x \in \partial D_L, \end{cases} \tag{4.21}$$

where $|b_\epsilon(x,t)| < \epsilon$ for $x \in \bar{D}_L$ and $t_0 \le t \le t_0 + T$. Let $u_{b_\epsilon,L}(x,t;t_0,u_0)$ be the solution of (4.21) with $u_{b_\epsilon,L}(x,t_0;t_0,u_0) = u_0(x)$.

We claim that *there is* $\epsilon_0(T) > 0$ *such that for any* $L \ge L_0$*,* $\kappa > 0$*, and* $0 \le \epsilon \le \epsilon_0(T)$*,*

$$u_{b_\epsilon,L}(0,t_0+T;t_0,\kappa\phi_L) \ge e^{\frac{T\sigma_{L_0}}{2}}\kappa > \kappa \tag{4.22}$$

provided that $|b_\epsilon(x,t)| < \epsilon$ *for* $x \in D_L$*; and for any* $L \ge L_0$ *and* $0 \le \epsilon \le \epsilon_0(T)$*,*

$$0 \le u_{b_\epsilon,L}(x,t+t_0;t_0,\kappa\phi_L) \le e^{a_0 t}\kappa \quad \forall\ 0 \le t \le T,\ x \in D_L. \tag{4.23}$$

In fact, by (4.20), there is $\epsilon_0(T) > 0$ such that for any $0 \le \epsilon \le \epsilon_0(T)$,

$$u_{b_\epsilon,L_0}(0,t_0+T;t_0,\kappa\phi_{L_0}) > e^{\frac{T\sigma_{L_0}}{2}}\kappa \tag{4.24}$$

provided that $|b_\epsilon(x,t)| < \epsilon$ for $x \in D_{L_0}$. Note that for $L \ge L_0$,

$$\phi_L(x) \ge \phi_{L_0}(x) \quad \forall\ x \in D_{L_0},$$

and

$$\begin{cases} \partial_t u_{b_\epsilon,L}(\cdot,t_0+t;t_0,\kappa\phi_L) = \Delta u_{b_\epsilon,L}(x,t_0+t;t_0,\kappa\phi_L) + b_\epsilon(x,t)\cdot\nabla u_{b_\epsilon,L}(x,t_0+t;t_0,\kappa\phi_L) \\ \qquad\qquad +a_0 u_{b_\epsilon,L}(x,t_0+t;t_0,\kappa\phi_L), \quad x \in D_{L_0}, \\ u_{b_\epsilon,L}(x,t_0+t;t_0,\kappa\phi_L) > 0, \quad x \in \partial D_{L_0}. \end{cases}$$

Then by comparison principle for parabolic equations,

$$u_{b_\epsilon,L}(x,t_0+t;t_0,\kappa\phi_L) \ge u_{b_\epsilon,L_0}(x,t_0+t;t_0,\kappa\phi_{L_0})$$

for $x \in D_{L_0}$, which together with (4.24) implies (4.22). (4.23) follows directly from comparison principle for parabolic equations.

Step 2. Consider

$$\begin{cases} u_t = \Delta u + b_\epsilon(x,t)\cdot\nabla u + u(2a_0 - c(x,t)u), \quad x \in D_L \\ u = 0, \quad x \in \partial D_L, \end{cases} \tag{4.25}$$

where $0 \le c(x,t) \le b_{\sup}$. Let $u(x,t;t_0,u_0)$ be the solution of (4.25) with $u_\varepsilon(x,t_0;t_0,u_0) = u_0(x)$. Assume $L \ge L_0$ and $0 \le \epsilon \le \epsilon_0(T)$.

We claim that

$$u_\varepsilon(0,t_0+T;t_0,\kappa\phi_L) \ge e^{\frac{T\sigma_{L_0}}{2}}\kappa \tag{4.26}$$

provided that $0 < \kappa \le \kappa_0(T) := \frac{a_0 e^{-a_0 T}}{b_{\sup}}$.

Note that (4.23) yields,

$$
\begin{aligned}
&\partial_t u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L) - \Delta u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L) - b_\epsilon(x,t)\cdot\nabla u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L)\\
&- u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L)\big(2a_0 - c(x,t)u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L)\big)\\
&= -u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L)\big(a_0 - c(x,t)u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L)\big)\\
&\le 0 \quad \text{for} \quad t_0 \le t \le t_0+T, \quad x\in D_L
\end{aligned}
$$

when $0 < \kappa \le \frac{a_0 e^{-a_0 T}}{b_{\sup}}$. Then by comparison principal for parabolic equations,

$$
u_\varepsilon(x,t;t_0,\kappa\phi_L) \ge u_{b_\epsilon,L}(x,t;t_0,\kappa\phi_L) \quad \text{for} \quad t_0\le t\le t_0+T, \quad x\in D_L.
$$

This together with (4.22) implies (4.26).

Step 3. For any given $x_0 \in \mathbb{R}^N$, consider

$$
u_t = \Delta u - \chi\nabla v\cdot\nabla u + u(a(x+x_0,t) - \chi\lambda v(x,t;t_0,x_0,u_0) - (b(x+x_0,t)-\chi\mu)u), \quad x\in\mathbb{R}^N, \tag{4.27}
$$

where $v(x,t;t_0,x_0,u_0)$ is the solution of

$$
0 = \Delta v - \lambda v + \mu u, \quad x\in\mathbb{R}^N.
$$

Let $u(x,t;t_0,x_0,u_0)$ be the solution of (4.27) with $u(x,t_0;t_0,x_0,u_0) = u_0(x)$. Let $\epsilon_0(T) > 0$ and $\kappa_0(T) > 0$ be as in Steps 1 and 2, respectively.

We claim that *there is* $0 < \delta_0(T) \le \kappa_0(T)$ *such that for any* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $0 \le u_0 \le M^+$ *and* $u_0(x) < \delta_0(T)$ *for* $|x_i| \le 3L$, $i = 1,2,\cdots,N$, $x_0\in\mathbb{R}^N$

$$
0 \le \lambda v(x,t;t_0,x_0,u_0) \le \frac{a_0}{2\chi}, \;\; |\nabla v(x,t;t_0,x_0,u_0)| < \frac{\epsilon_0(T)}{2\chi} \;\; \text{for} \;\; t_0\le t\le t_0+T, \; x\in D_L, \tag{4.28}
$$

provided that $L \gg 1$.

Indeed, let $0 < \varepsilon \le \varepsilon_0(T)$ be fixed. Lemma 4.3 implies that there is $\delta_1 = \delta_1(M^+,\varepsilon)$ and $L_1 = L_1(M^+,T,\varepsilon) > L_0$ such that for every $L \ge L_1$, there holds

$$
u(x,t+t_0;t_0,x_0,u_0) \le \varepsilon, \quad \forall\, 0\le t\le T, \; t_0\in\mathbb{R}, \; \forall x_0\in\mathbb{R}^N, \; \forall |x| \in D_{2L} \tag{4.29}
$$

whenever $0 \le u_0(x) \le \delta_1$, $\forall\, x\in D_{3L}$.

Next, note that

$$v(x,t_0+t;t_0,x_0,u_0)=\mu\int_0^\infty\int_{\mathbb{R}^N}\frac{e^{-\lambda s}}{(4\pi s)^{\frac{N}{2}}}e^{-\frac{|x-z|^2}{4s}}u(z,t;t_0,x_0,u_0)dzds$$

and

$$\partial_{x_i}v(x,t_0+t;t_0,x_0,u_0)=\mu\int_0^\infty\int_{\mathbb{R}^N}\frac{(z_i-x_i)e^{-\lambda s}}{2s(4\pi s)^{\frac{N}{2}}}e^{-\frac{|x-z|^2}{4s}}u(z,t;t_0,x_0,u_0)dzds.$$

Hence, by (4.29), for $L\geq L_1$, $0\leq t\leq T$, and $|x|_\infty<L$, we have

$$\begin{aligned}v(x,t_0+t;t_0,x_0,u_0)\leq&\frac{\mu}{\pi^{\frac{N}{2}}}\left[\int_0^L\int_{|z|_\infty\leq\frac{L}{2\sqrt{T}}}e^{-\lambda s}e^{-|z|^2}dzds\right]\sup_{0\leq t\leq T,|z|_\infty\leq 2L}u(z,t+t_0;t_0,x_0,u_0)\\&+\frac{\mu a_{\sup}}{(b_{\inf}-\chi\mu)\pi^{\frac{N}{2}}}\int\int_{s\geq L\,\text{or}\,|z|_\infty\geq\frac{L}{2\sqrt{T}}}e^{-\lambda s}e^{-|z|^2}dzds\\\leq&\frac{\mu}{\lambda}\varepsilon+\frac{\mu a_{\sup}}{(b_{\inf}-\chi\mu)\pi^{\frac{N}{2}}}\int\int_{s\geq L\,\text{or}\,|z|_\infty\geq\frac{L}{2\sqrt{T}}}e^{-\lambda s}e^{-|z|^2}dzds\end{aligned}\tag{4.30}$$

and

$$\begin{aligned}|\partial_{x_i}v(x,t_0+t;t_0,0,u_0)|\leq&\frac{\varepsilon\mu}{\pi^{\frac{N}{2}}}\int_0^L\int_{|z|_\infty\leq\frac{L}{2\sqrt{T}}}\frac{|z_i|e^{-\lambda s}e^{-|z|^2}}{\sqrt{s}}dzds\\&+\frac{\mu a_{\sup}}{(b_{\inf}-\chi\mu)\pi^{\frac{N}{2}}}\int\int_{s\geq L\,\text{or}\,|z|_\infty\geq\frac{L}{2\sqrt{T}}}\frac{|z_i|e^{-\lambda s}e^{-|z|^2}}{\sqrt{s}}dzds\\\leq&\frac{\varepsilon\mu}{\sqrt{\lambda}}+\frac{\mu a_{\sup}}{(b_{\inf}-\chi\mu)\pi^{\frac{N}{2}}}\int\int_{s\geq L\,\text{or}\,|z|_\infty\geq\frac{L}{2\sqrt{T}}}\frac{|z_i|e^{-\lambda s}e^{-|z|^2}}{\sqrt{s}}dzds,\end{aligned}\tag{4.31}$$

whenever $0\leq u_0(x)\leq\delta_1$ for every $|x|_\infty\leq 3L$. These together with (4.29) implies (4.28).

Note that

$$\begin{cases}u_t\geq\Delta u-\chi\nabla v\cdot\nabla u+u(2a_0-(b(x+x_0,t)-\chi\mu)u),\quad x\in D_L\\u(x,t;t_0,x_0,u_0)>0,\quad x\in\partial D_L.\end{cases}$$

Let $\kappa=\inf_{x\in D_L}u_0(x)$. Then $\kappa\leq\delta_0(T)\leq\kappa_0(T)$. By comparison principle for parabolic equations,

$$u(x,t;t_0,x_0,u_0)\ge u_\epsilon(t,x;t_0,\kappa\phi_L)\quad \text{for}\quad x\in D_L,\quad t_0\le t\le t_0+T.$$

This together with the conclusion in Step 2

$$u(0,T+t_0;t_0,x_0,u_0)\ge e^{\frac{T\sigma_{L_0}}{2}}\kappa=e^{\frac{T\sigma_{L_0}}{2}}\inf_{x\in D_L}u_0(x). \tag{4.32}$$

Step 4. In this step we claim that *there is* $0<\delta_0^*(T)<\min\{\delta_0(T),M^+\}$*, where* $M^+=\frac{a_{\sup}}{b_{\inf}-\chi\mu}$*, such that for any* $0<\delta\le\delta_0^*(T)$ *and for any* u_0 *with* $\delta\le u_0\le M^+$*,*

$$\delta\le u(x,t_0+T;t_0,0,u_0)\le M^+\quad\forall\ x\in\mathbb{R}^N. \tag{4.33}$$

Assume that the claim does not hold. Then there are $\delta_n\to 0$, $t_{0n}\in\mathbb{R}$, u_{0n} with $\delta_n\le u_{0n}\le M^+$, and $x_n\in\mathbb{R}^N$ such that

$$u(x_n,t_{0n}+T;t_{0n},0,u_{0n})<\delta_n. \tag{4.34}$$

Note that

$$u(x+x_n,t;t_{0n},0,u_{0n})=u(x,t;t_{0n},x_n,u_{0n}(\cdot+x_n)).$$

Let $\epsilon_0:=\epsilon(T)>0$, $\delta_0:=\delta_0(T)>0$, and $\kappa_0:=\kappa_0(T)>0$ be fixed and be such that the conclusions in Steps 2 and 3 hold. Let

$$D_{0n}=\{x\in\mathbb{R}^N\,|\,|x_i|<3L,\ u_{0n}(x+x_n)>\frac{\delta_0}{2}\}.$$

Without loss of generality, we may assume that $\lim_{n\to\infty}|D_{0n}|$ exists.

Case 1. $\lim_{n\to\infty}|D_{0n}|=0$. We claim that *in this case,* $|\chi\nabla v(x+x_n,t+t_{0n};t_{0n},0,u_{0n})|<\epsilon_0$ *and* $0\le v(x+x_n,t+t_{0n};t_{0n},0,u_{0n})\le a_0$ *for* $|x_i|\le L$*,* $i=1,2,\cdots,N$*,* $L\gg 1$ *and* $n\gg 1$.

Indeed, let $\{\tilde u_{0n}\}_{n\ge 1}$ be sequence of elements of $C^b_{\rm unif}(\mathbb{R}^N)$ satisfying

$$\begin{cases}\delta_n\le\tilde u_{0n}(x)\le\frac{\delta_0}{2},\quad x\in D_{3L}\ \text{ and}\\ \|\tilde u_{0n}(\cdot)-u_{0n}(\cdot+x_n)\|_{L^p(\mathbb{R}^N)}\to 0,\quad\forall p>1.\end{cases}$$

Let $w_n(x,t) := u(t+t_{0n}, x; t_{0n}, x_n, u_{0n}(\cdot + x_n)) - u(t+t_{0n}, x; t_{0n}, x_n, \tilde{u}_{0n})$ and $v_n(x,t) := v(t+t_{0n}, x; t_{0n}, x_n, u_{0n}(\cdot + x_n)) - v(t+t_{0n}, x; t_{0n}, x_n, \tilde{u}_{0n})$. Hence $\{(w_n, v_n)\}_{n\geq}$ satisfies

$$\begin{cases} \partial_t w_n = \Delta w_n + b_n(t,x)\cdot \nabla w_n + f_n(t,x)w_n + g_n(t,x)v_n + h_n \cdot \nabla v_n, \quad x\in\mathbb{R}^N, t>0 \\ 0 = \Delta v_n - \lambda v_n + \mu w_n, \qquad x\in\mathbb{R}^N, t>0 \\ w_n(0,x) = u_{0n}(x+x_n) - \tilde{u}_{0n}(x), \quad x\in\mathbb{R}^N, \end{cases} \tag{4.35}$$

where $b_n(t,x) = -\chi\nabla v(t+t_{0n}, x+x_n; t_{0n}, x_n, u_{0n}(\cdot+x_n))$, $g_n(t,x) := -\chi\lambda u(t+t_{0n}, x+x_n; t_{0n}, x_n, \tilde{u}_{0n})$, $h_n(t,x) := -\chi\nabla u(t+t_{0n}, x+x_n; t_{0n}, x_n, \tilde{u}_{0n})$, and

$$\begin{aligned} f_n(t-t_{0n}, x-x_n) :=& a(t,x) - \chi\lambda v(t,x;t_0,x_n,u_{0n}(\cdot+x_n)) \\ & - (b(t,x)-\chi\mu)(u(t,x;t_0,x_n,u_{0n}(\cdot+x_n)) + u(t,x;t_0,x_n,\tilde{u}_{0n})). \end{aligned}$$

For $w_{0n}(0,\cdot)\in L^p(\mathbb{R}^N)$, (4.35) has a unique solution $w(t,x;w_{0n})$ with $w(0,x;w_{0n}) = w_{0n}(x)$ in $L^p(\mathbb{R}^N)$. Note that $\nabla\cdot(w_nb_n) = b_n\cdot\nabla w_n + w_n\nabla\cdot b_n$ and $\nabla\cdot b_n = -\chi(\lambda v - \mu u)(t + t_{0n}, x; t_{0n}, x_n, u_{0n}(\cdot+x_n))$. Hence

$$\partial_t w_n = \Delta w_n + \nabla\cdot(w_nb_n) + (f_n(t,x) - \nabla\cdot b_n)w_n + g_n(t,x)v_n + h_n\cdot\nabla v_n, \quad x\in\mathbb{R}^N, t>0.$$

Thus, the variation of constant formula yields that

$$\begin{aligned} w_n(t,\cdot) =& e^{t(\Delta-I)}w_n(0) + \underbrace{\int_0^t e^{(t-s)(\Delta-I)}\nabla\cdot(w_n(s,\cdot)b_n(s,\cdot))ds}_{I_1} \\ & + \underbrace{\int_0^t e^{(t-s)(\Delta-I)}((1+f_n(s,\cdot)-\nabla\cdot b_n(s,\cdot))w_n(s,\cdot) + g_n(s,\cdot)v_n(s,\cdot) + h_n\cdot\nabla v_n)ds}_{I_2}, \end{aligned}$$

where $\{e^{t(\Delta-I)}\}_{t\geq 0}$ denotes the C_0-semigroup on $L^p(\mathbb{R}^N)$ generated by $\Delta - I$.

Observe that $\|b_n(t,\cdot)\|_\infty \leq \mu\|u(t+t_{0n}, x+x_n; t_{0n}, u_{0n}(\cdot+x_n))\|_\infty \leq \mu\frac{a_{\sup}}{b_{\inf}-\chi\mu}$. Hence, as shown in [49, Lemma 3.1], we have

$$\begin{aligned} \|I_1\|_{L^p(\mathbb{R}^N)} &\leq C\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|\nabla b_n(s,\cdot)\|_\infty\|w_n(s,\cdot)\|_{L^p(\mathbb{R}^N)}ds \\ &\leq \frac{C\mu a_{\sup}}{b_{\inf}-\chi\mu}\int_0^t \frac{e^{-(t-s)}}{\sqrt{t-s}}\|w_n(s,\cdot)\|_{L^p(\mathbb{R}^N)}ds. \end{aligned}$$

We also observe that $\sup_{0\le t\le T, n\ge 1}\|1+f_n(t,\cdot)-\nabla\cdot b_n(t,\cdot)\|_\infty<\infty$, $\sup_{0\le t\le T,n\ge 1}\|g_n(t,\cdot)\|_\infty<\infty$, and $\sup_{0\le t\le T,n\ge 1}\|h_n(t,\cdot)\|_\infty<\infty$, thus we have

$$\|I_2\|_{L^p(\mathbb{R}^N)}\le C\int_0^t e^{-(t-s)}\left\{\|w_n(s,\cdot)\|_{L^p(\mathbb{R}^N)}+\|v_n(s,\cdot)\|_{W^{1,p}(\mathbb{R}^N)}\right\}ds.$$

Since $(\Delta-\lambda I)v_n=-\mu w_n$, then by elliptic regularity, we have that

$$\|v_n(t,\cdot)\|_{W^{2,p}(\mathbb{R}^N)}\le C\|w_n(t,\cdot)\|_{L^p(\mathbb{R}^N)}.$$

Hence, since $\|e^{t(\Delta-I)}w_n(0,\cdot)\|_{L^p(\mathbb{R}^N)}\le e^{-t}\|w_n(0,\cdot)\|_{L^p(\mathbb{R}^N)}$, we obtain

$$\|w_n(t,\cdot)\|_{L^p(\mathbb{R}^N)}\le\|w_n(0)\|_{L^p(\mathbb{R}^N)}+C\int_0^t(t-s)^{-\frac{1}{2}}\|w_n(s,\cdot)\|_{L^p(\mathbb{R}^N)}ds$$

for some constant $C>0$. Therefore it follows from Lemma 2.1 that

$$\|w_n(t,\cdot)\|_{L^p(\mathbb{R}^N)}\le C_T\|w_n(0,\cdot)\|_{L^p(\mathbb{R}^N)},\quad\forall\, 0\le t\le T,\ \forall\, n\ge 1,$$

where $C_T>0$ is a constant. Thus

$$\lim_{n\to\infty}\sup_{0\le t\le T}\|w_n(t,\cdot)\|_{L^p(\mathbb{R}^N)}=0. \tag{4.36}$$

For $p>N$, by regularity and a priori estimates for elliptic operators, there is a constant $C>0$ such that

$$\|(\Delta-\lambda I)^{-1}w\|_{C^{1,b}_{\rm unif}(\mathbb{R}^N)}\le C\|w\|_{L^p(\mathbb{R}^N)},\quad\forall w\in L^p(\mathbb{R}^N).$$

Combining this with (4.36) we have that

$$\lim_{n\to\infty}\sup_{0\le t\le T}\|v_n(t,\cdot)\|_{C^{1,b}_{\rm unif}(\mathbb{R}^N)}=0. \tag{4.37}$$

It follows from the claim in Step 3 that for every $n\ge 1$,

$$0\le\lambda v(t+t_{0n},x;t_{0n},x_n,\tilde{u}_{0n})\le\frac{a_0}{2\chi},\ \ |\chi\nabla v(t+t_{0n},x;t_{0n},x_n,\tilde{u}_{0n})|\le\frac{\varepsilon_0}{2},\quad\forall 0\le t\le T,\ x\in D_L.$$

Thus (4.37) implies that, for $n\gg 1$, $\forall 0\le t\le T$, $x\in D_L$, there holds

$$0\le\chi\lambda v(t+t_{0n},x;t_{0n},x_n,u_{0n}(\cdot+x_n))\le a_0,\ \ |\chi\nabla vt+t_{0n},x;t_{0n},x_n,u_{0n}(\cdot+x_n))|\le\varepsilon_0.$$

Hence, it follows from the arguments of (4.32) that

$$u(T+t_{0n},0;t_{0n},x_n,u_{0n}(\cdot+x_n))>\delta_n,$$

which is a contradictions. Hence **case 1** does not hold.

Case 2. $\liminf_{n\to\infty} |D_{0n}| > 0$.

In this case, without loss of generality, we might suppose that $\inf_{n\geq 1} |D_{0n}| > 0$, and there a suitable $N-$cube, $D \subset\subset D_{3L}$ with $\inf_{n\geq 1} |D \cap D_{0n}| > 0$. Let $\Psi_n(x,t)$ denotes the solution of

$$\begin{cases} u_t = \Delta u, \quad x \in D_{3L} \\ u = 0, \quad \text{on } (0,T) \times \partial D_{3L} \\ u(\cdot,0) = \frac{\delta_0}{2}\chi_{D\cap D_{0n}}. \end{cases} \tag{4.38}$$

Thus, by comparison principle for parabolic equations, we have

$$e^{t\Delta}u_{0n}(x+x_n) \geq \Psi_n(x,t), \quad \forall x \in D_{3L}, 0 \leq t \leq T, \ n \geq 1.$$

From this, it follows that

$$\|e^{t\Delta}u_{0n}(\cdot + x_n)\|^2_{C^\infty(D_{3L})} \geq \frac{1}{|D_{3L}|}\int_{D_{3L}} \Psi_n^2(x,t)dx, \quad \forall 0 \leq t \leq T, \ n \geq 1. \tag{4.39}$$

Note that for every $n \geq 1$, $\Psi_n(x,t)$ can be written as

$$\Psi_n(x,t) = \frac{\delta_0}{2}\sum_{k=1}^{\infty} e^{-t\tilde{\lambda}_k}\phi_k(x)\left[\int_{D_{3L}} \phi_k(y)\chi_{D\cap D_{0n}}(y)dy\right],$$

where $\{\phi_k\}_{k\geq 1}$ denotes the orthonormal basis of $L^2(D_{3L})$ consisting of eigenfunctions with corresponding eigenvalues $\{\tilde{\lambda}_k\}$ of $-\Delta$ with Dirichlet boundary conditions on D_{3L}. Since $\tilde{\lambda}_1$ is principal, then we might suppose that $\phi_1(x) > 0$ for every $x \in D_{3L}$. Thus

$$\begin{aligned} \|\Psi_n(\cdot,t)\|^2_{L^2(D_{3L})} &= \sum_{k=1}^{\infty} e^{-2t\tilde{\lambda}_k}\left[\frac{\delta_0}{2}\int_{D_{3L}} \phi_k(y)\chi_{D\cap D_{0n}}(y)dy\right]^2 \\ &\geq e^{-2t\tilde{\lambda}_1}\left[\int_{D_{3L}} \phi_1(y)\chi_{D\cap D_{0n}}(y)dy\right]^2 \\ &\geq e^{-2t\tilde{\lambda}_1}\left[\frac{\delta_0}{2}|D\cap D_{0n}|\min_{y\in D}\phi_1(y)\right]^2. \end{aligned} \tag{4.40}$$

Since $\inf_{n\geq 1} |D_{0n}| > 0$ and $\min_{y\in D}\phi_1(y) > 0$, it follows from (4.39) and (4.40) that

$$\inf_{0\leq t\leq T, n\geq 1} \|e^{t(\Delta - I)}u_{0n}(\cdot + x_n)\|_{C(D_{3L})} > 0.$$

Thus there is $0 < T_0 \ll 1$ such that

$$\inf_{n\geq 1} \|u(\cdot, T_0 + t_{0n}; t_{0n}, x_n, u_{0n}(\cdot + x_n))\|_{C^0(D_{3L})} > 0.$$

Hence, we might suppose that $u(\cdot, T_0 + t_{0n}; t_{0n}, x_n, u_{0n}(\cdot + x_n)) \to u_0^*$ locally uniformly and $\|u_0^*\|_{C(D_{3L})} > 0$. Moreover, by Lemma 4.2, we might assume that $(u(\cdot, T + t_{0n}; t_{0n}, x_n, u_{0n}(\cdot + x_n)), v(\cdot, T + t_{0n}; t_{0n}, x_n, u_{0n}(\cdot + x_n))) \to (u^*(x,t), v^*(x,t))$, $a(x + x_n, t) \to a^*(x,t)$, and $b(x + x_n, t) \to b^*(x,t)$, where (u^*, v^*) satisfies

$$\begin{cases} u_t^* = \Delta u^* - \chi\nabla \cdot (u^*\nabla v^*) + (a^* - b^*u^*)u^* \\ 0 = (\Delta - \lambda I)v^* + \mu u^* \\ u^*(\cdot, 0) = u_0^*. \end{cases}$$

Sine $\|u_0^*\|_\infty > 0$ and $u^*(x,t) \geq 0$, it follows from comparison principle for parabolic equations that $u^*(x,t) > 0$ for every $x \in R^N$ and $t \in (0, T]$. In particular $u^*(0,T) > 0$. Note by (4.34) that we must have $u^*(0,T) = 0$, which is a contradiction. Hence the result holds. □

4.2 Proof of Theorem 2.3

In this section, using the preliminary results established in the previous section, we present the proof of Theorem 2.3.

Proof of Theorem 2.3. (i) Let $u_0 \in C^b_{\text{uinf}}(\mathbb{R}^N)$, with $u_{0\inf} > 0$ be given. It follows from (2.11) that there is $T_1 > 0$ such that

$$u(x, t + t_0; t_0, u_0) \leq M^+ := \frac{a_{\sup}}{b_{\inf} - \chi\mu} + 1, \quad \forall\, t \geq T_1, \quad ,\forall\, t_0 \in \mathbb{R}.$$

Note that T_1 is independent of t_0. We claim that

$$m(u_0) := \inf_{t_0\in\mathbb{R},(x,t)\in\mathbb{R}^N\times[0,\infty)} u(x, t + t_0; t_0, u_0) > 0. \tag{4.41}$$

In fact, since $u_{0\inf} > 0$, by Lemma 4.1, we have that

$$\delta_1 := \inf_{t_0\in\mathbb{R},(x,t)\in\mathbb{R}^N\times[t_0,t_0+T_1]} u(x, t + t_0; t_0, u_0) \geq u_{0\inf} e^{-T_1(a_{\inf}+b_{\sup}\|u_0\|_\infty e^{T_1 a_{\sup}})} > 0. \tag{4.42}$$

Let

$$\delta_2 = \min\{\delta_1, \delta_0(T_1)\},$$

where $\delta_0(T_1)$ is given by Lemma 4.5. Then $\delta_2 > 0$. By induction, it follows from Lemma 4.5 that

$$\delta_2 \leq \inf_x u(x, t_0 + nT_1; t_0, u_0)) \leq M^+, \quad \forall\, t_0 \in \mathbb{R} \text{ and } \forall\, n \in \mathbb{N}. \tag{4.43}$$

Lemma 4.1 implies that for every $t_0 \in \mathbb{R}$, $x \in \mathbb{R}^N,\ t \in [0, T_1]$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} u(x, t_0 + nT_1 + t; t_0, u_0) &= u(x, t_0 + nT_1 + t; t_0 + nT_1, u(x, t_0 + nT_1, t_0, u_0)) \\ &\geq \delta_2 e^{t(a_{\inf} - b_{\sup} M^+ e^{T_1 a_{\sup}})} \\ &\geq \delta_2 e^{-T_1(a_{\inf} + b_{\sup} M^+ e^{T_1 a_{\sup}})} \end{aligned} \tag{4.44}$$

By (4.44), we obtain that

$$\inf_{t_0 \in \mathbb{R}, (x,t) \in \mathbb{R}^N \times [0,\infty)} u(x, t_0 + t; t_0, u_0)) \geq \delta_2 e^{-T_1(a_{\inf} + b_{\sup} M^+ T_1)}$$

The last inequality yields that $m(u_0) > 0$. Hence (4.41) holds.

(ii) Let $(\underline{M}_n, \overline{M}_n)_{n \geq 0}$ be the sequence define by (4.14). Let $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_{0\inf} > 0$ be fixed.

We first claim that for every $n \geq 0$, and $\varepsilon > 0$ there is $T^n_\varepsilon(u_0)$ such that

$$\underline{M}_n - \varepsilon \leq u(x, t + t_0; t_0, u_0) \leq \overline{M}_n + \varepsilon \quad \forall\, x \in \mathbb{R}^N, \forall\, t \geq T^n_\varepsilon(u_0), \forall\, t_0 \in \mathbb{R}, \tag{4.45}$$

which implies that for any $\varepsilon > 0$ there is $T_\varepsilon(u_0)$ such that (2.15) holds.

In fact, for $n = 0$, it is clear that $\underline{M}_0 = 0 \leq u(x, t + t_0; t_0, u_0)$ for every $x \in \mathbb{R}^N,\ t \geq 0$, and $t_0 \in \mathbb{R}$. It follows (3.27) that there is $T^0_\varepsilon(u_0)$ such that

$$u(x, t + t_0; t_0, u_0) \leq \overline{M}_1 + \varepsilon, \quad \forall\, x \in \mathbb{R}^N,\, t \geq T^0_\varepsilon(u_0), \forall\, t_0 \in \mathbb{R}.$$

Hence (4.45) holds for $n = 0$. Suppose that (4.45) holds for $n - 1$, ($n \geq 1$). We show that (4.45) also holds for n. Indeed, let $\varepsilon > 0$. It follows from the induction hypothesis that there is

$\tilde{T}_\varepsilon^{n-1}(u_0) \gg 1$ that

$$\underline{M}_{n-1} - \frac{\varepsilon}{4} \le u(x, t+t_0; t_0, u_0) \le \overline{M}_{n-1} + \frac{\varepsilon}{4} \quad \forall\, x \in \mathbb{R}^N,\, \forall\, t \ge T_\varepsilon^{n-1}(u_0),\, \forall\, t_0 \in \mathbb{R}. \tag{4.46}$$

This implies that

$$\begin{cases} u_t(\cdot, \cdot + t_0; t_0, u_0) \\ \ge \Delta u(\cdot, \cdot + t_0; t_0, u_0) - \chi(\nabla v \cdot \nabla u)(\cdot, \cdot + t_0; t_0, u_0) \\ +(a_{\inf} - \chi\mu(\overline{M}_{n-1} + \frac{\varepsilon}{4}) - (b_{\sup} - \chi\mu) u(\cdot, \cdot + t_0; t_0, u_0)) u(\cdot, \cdot + t_0; t_0, u_0),\ t > \tilde{T}_\varepsilon^{n-1}(u_0), \\ u(\cdot, \tilde{T}_\varepsilon^{n-1}(u_0) + t_0; t_0, u_0) \ge m(u_0), \end{cases} \tag{4.47}$$

where $m(u_0) := \inf\{u(x, t+t_0; t_0, u_0) \mid x \in \mathbb{R}^N,\ t \in [0, \infty),\ t_0 \in \mathbb{R}\} > 0$. Hence, it follows from comparison principle for parabolic equations that there is $\tilde{T}_\varepsilon^{n}(u_0) \ge \tilde{T}_\varepsilon^{n-1}(u_0)$ such that

$$u(x, t+t_0; t_0, u_0) \ge \frac{(a_{\inf} - \chi\mu(\overline{M}_{n-1} + \frac{\varepsilon}{4}))_+}{b_{\sup} - \chi\mu} - \frac{\varepsilon}{4}, \quad \forall\, t \ge \tilde{T}_\varepsilon^{n}(u_0),\, x \in \mathbb{R}^N,\, \forall\, t_0 \in \mathbb{R}. \tag{4.48}$$

Note that, since **(H2)** holds, then it follows from Lemma 4.4 that

$$\frac{(a_{\inf} - \chi\mu(\overline{M}_{n-1} + \frac{\varepsilon}{4}))_+}{b_{\sup} - \chi\mu} \ge \underline{M}_n - \frac{\chi\mu\varepsilon}{4(b_{\sup} - \chi\mu)} \ge \underline{M}_n - \frac{\varepsilon}{4}. \tag{4.49}$$

It follows from (4.48) and (4.49) that

$$\begin{cases} u_t(\cdot, \cdot + t_0; t_0, u_0) \\ \le \Delta u(\cdot, \cdot + t_0; t_0, u_0) - \chi(\nabla v \cdot \nabla u)(\cdot, \cdot + t_0; t_0, u_0) \\ \quad +(a_{\sup} - \chi\mu(\underline{M}_n - \frac{\varepsilon}{2}) - (b_{\inf} - \chi\mu) u(\cdot, \cdot + t_0; t_0, u_0)) u(\cdot, \cdot + t_0; t_0, u_0),\ t > \tilde{T}_\varepsilon^{n}(u_0), \\ u(\cdot, \tilde{T}_\varepsilon^{n}(u_0) + t_0; t_0, u_0) \le \overline{M}_{n-1} + \frac{\varepsilon}{4}. \end{cases} \tag{4.50}$$

Hence, it follows from comparison principle for parabolic equations that there is $T_\varepsilon^{n}(u_0) \ge \tilde{T}_\varepsilon^{n}(u_0)$ such that

$$u(x,t+t_0;t_0,u_0) \le \frac{(a_{\inf} - \chi\mu(\underline{M}_n - \frac{\varepsilon}{2}))_+}{b_{\sup} - \chi\mu} + \frac{\varepsilon}{2}, \quad \forall\, t \ge T_\varepsilon^n(u_0),\ x \in \mathbb{R}^N,\ \forall\, t_0 \in \mathbb{R}. \tag{4.51}$$

Observe that, sine **(H2)** holds, Lemma 4.4 implies that

$$\frac{(a_{\inf} - \chi\mu(\underline{M}_n - \frac{\varepsilon}{2}))_+}{b_{\inf} - \chi\mu} \le \overline{M}_n + \frac{\varepsilon}{2}. \tag{4.52}$$

Hence, it follows from (4.48)-(4.52) that (4.45) also holds for n. Thus we conclude that (4.45) holds for every $n \ge 0$.

Next, we show that the set I_{inv} given by (2.19) is an invariant set for solutions of (2.6).

By Lemma 4.4 we have that $\underline{M}_n \nearrow \underline{M}$ and $\overline{M}_n \searrow \overline{M}$. It suffices to show that the set $I_{inv}^n := \{u_0 \in C^b_{\rm unif}(\mathbb{R}^N) \mid \underline{M}_n \le u_0(x) \le \overline{M}_n\}$, $n \ge 0$, is positively invariant for (2.6). This is also done by induction on $n \ge 0$. The case $n = 0$ is guaranteed by Theorem 2.2 (i). Suppose that I_{inv}^n is a positive invariant set for (2.6). Let $u_0 \in I_{inv}^{n+1}$. Since, by Lemma 4.4, $\overline{M}_n > \overline{M}_{n+1}$, it follows from (4.47) and comparison principle for parabolic equations that

$$u(x,t+t_0;t_0,u_0) \ge \min\Big\{\underline{M}_{n+1}\ ,\ \underbrace{\frac{a_{\inf} - \chi\mu\overline{M}_n}{b_{\sup} - \chi\mu}}_{=\underline{M}_{n+1}}\Big\} = \underline{M}_{n+1},\ \forall\, x \in \mathbb{R}^N,\ \forall\, t \ge 0, \forall\, t_0 \in \mathbb{R}.$$

Using this last inequality, by (4.50), it follows from comparison principle for parabolic equations that

$$u(x,t+t_0;t_0,u_0) \le \max\Big\{\overline{M}_{n+1}\ ,\ \underbrace{\frac{a_{\sup} - \chi\mu\underline{M}_{n+1}}{b_{\inf} - \chi\mu}}_{=\overline{M}_{n+1}}\Big\} = \overline{M}_{n+1},\ \forall\, x \in \mathbb{R}^N,\ \forall\, t \ge 0, \forall\, t_0 \in \mathbb{R}.$$

Thus, I_{inv}^{n+1} is also a positive invariant set for (2.6). The result thus follows. □

Chapter 5

Existence, uniqueness and stability of positive entire solutions.

This chapter is concerned with the existence, uniqueness, and stability of strictly positive entire solutions of (2.6). In the first section, we study the existence of strictly positive entire solutions and prove Theorem 2.4. The uniqueness and stability of these strictly positive entire solutions are studied in Section 2, where we prove Theorem 2.5.

5.1 Existence of strictly positive entire solutions

While the proof of Lemma 4.5 is presented in the previous chapter, for the sake of clarity in the arguments in the proof of our main result in this section, it is convenient to point out some fundamental results developed in its proof. Letting $a_0 = \frac{a_{\inf}}{3}$, $D_L := \{x \in \mathbb{R}^N \ : \ |x_i| < L\, \forall\, i = 1, \cdots, N\}$, consider the PDE

$$\begin{cases} u_t - \Delta u - a_0 u = 0, & x \in D_L \\ u = 0 & x \in \partial D_L \end{cases} \tag{5.1}$$

and its corresponding eigenvalue problem

$$\begin{cases} -\Delta u - a_0 u = \sigma u, & x \in D_L \\ u = 0 & x \in \partial D_L. \end{cases} \tag{5.2}$$

There exists $L_0 > 1$ such that the principal eigenvalue of (4.19), denoted by σ_L, is negative for every $L \geq L_0$. Moreover, a principal eigenfunction, ϕ_L, associated to the principal eigenvalue σ_L can be chosen in such away that $0 < \phi_L(x) < \phi(0) = 1$ for all $x \in D_L \setminus \{0\}$. Moreover, for every $0 < \varepsilon_0 \ll 1$, *there is* $0 < \delta_0 \ll 1$ *such that for any* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with* $0 \le u_0 \le M^+$ *and* $u_0(x) < \delta_0$ *for* $|x_i| \le 3L$, $i = 1, 2, \cdots, N$, $x_0 \in \mathbb{R}^N$,

$$0 \le \lambda v(x,t;t_0,x_0,u_0) \le \frac{a_0}{2\chi},\ \ |\nabla v(x,t;t_0,x_0,u_0)| < \frac{\epsilon_0}{2\chi}\ \text{ for }\ t_0 \le t \le t_0{+}1,\ x \in D_L, \tag{5.3}$$

provided that $L \gg 1$.

Next we consider the following related periodic-perturbation of (4.18),

$$\begin{cases} u_t - \Delta u - b_\varepsilon(x,t)\nabla u - a_0 u = 0, & x \in D_L \\ u = 0 & x \in \partial D_L. \end{cases} \tag{5.4}$$

with $|b_\varepsilon(x,t)| \le \varepsilon$, $b_\varepsilon(x,t+1) = b_\varepsilon(x,t)$, and its corresponding periodic eigenvalue problem

$$\begin{cases} u_t - \Delta u - b_\varepsilon(x,t)\nabla u - a_0 u = \sigma u, & x \in D_L,\ 0 < t < 1, \\ u(x,t) = 0, & x \in \partial D_L, 0 < t < 1, \\ u(x,0) = u(x,1), & x \in D_L. \end{cases} \tag{5.5}$$

We suppose that $b_\varepsilon(x,t)$ is 1−periodic in $t \in \mathbb{R}$, that is, $b_\varepsilon(x,t+1) = b_\varepsilon(x,t)$ for all $x \in D_L$, and $t \in \mathbb{R}$ and we let $U_{L,b_\varepsilon}(t,\tau)$, $\tau < t$, denotes the solution operator of (5.4) on $L^p(D_L)$, $N \ll p < \infty$. For, $\tau < t$, the evolution operator $U_{L,\varepsilon}(t,\tau)$ is a compact and strongly positive operator on $W^{2,p}_0(D_L) := \{u \in W^{2,p}(D_L)\ :\ u = 0 \text{ on } \partial D_L\}$. Letting $K_{L,b_\varepsilon} := U_{L,b_\varepsilon}(1,0)$, which is compact and strongly positive, thus its spectrum radius $r_{L,\varepsilon}$, is positive. By Krein-Rutman Theorem, $r_{L,\varepsilon}$ is an eigenvalue of $K_{L,\varepsilon}$ with a corresponding positive eigenfunction $u_{L,\varepsilon}$. It is well known that $\sigma^\varepsilon_L := -\ln(r_{L,\varepsilon})$ is the principal eigenvalue of (5.5) with positive 1-periodic eigenfunction $\phi_{L,\varepsilon}(t) = e^{t\sigma_{L,\varepsilon}} U_{L,b_\varepsilon}(1,0)u_{L,\varepsilon}$, (see [20]). Note that $U_L(t)(\phi_L) = e^{-t\sigma_L}\phi_L$, where $U_L(t)$ denotes the solution operator of (4.18). It follows that $K_L(\phi_L) = U_L(1)(\phi_L) = e^{-\sigma_L}\phi_L$, which implies that $r_L \ge e^{-\sigma_L}$. By perturbation theory for parabolic equations, we have that $U_{L,b_\varepsilon}(1,0) \to U_L(1)$ as $\|b_\varepsilon\|_{C(\bar{D}_L\times[0,1])} \to 0$. Thus, there is $0 < \varepsilon_0(L) << 1$ such that $r_{L,\varepsilon} \geq e^{-\frac{\sigma_L}{2}}$ whenever $\|b_\varepsilon\|_{C(\bar{D}_L\times[0,1])} \leq \varepsilon_0(L)$. Hence

$$\sigma_{L,\varepsilon} = -\ln(r_{L,\varepsilon}) \leq \frac{\sigma_L}{2} < 0, \quad 0 < \varepsilon < \varepsilon_0(L).$$

Note that $U_{L,b_\varepsilon}(t,\tau)\phi_{L,\varepsilon}(\tau) = e^{-(t-\tau)\sigma_{L,\varepsilon}}\phi_{L,\varepsilon}(t)$. Thus for every nonnegative initial function $u_0 \in C(\overline{D}_L)$ with $\|u_0\|_\infty > 0$, we have that

$$\sup_{x\in D_L,\tau<t} |(U_{L,b_\varepsilon}(t,\tau)u_0)(x)| = \infty, \quad \forall \|b_\varepsilon\|_{C(\bar{D}_L\times[0,1])} < \varepsilon_0(L). \tag{5.6}$$

Proof of Theorem 2.4. Let $T > 0$ be fixed and $\delta_0 := \delta_0^*(T)$ and $M^+ = \frac{a_{\sup}}{b_{\inf}-\chi\mu} + 1$ be given in Lemma 4.5. It follows from Lemma 4.5 that

$$\delta_0 \leq u(x, T-kT; -kT, u_0) \leq M^+, \quad x \in \mathbb{R}^n,\ k \geq 1,\ \delta_0 \leq u_0 \leq M^+. \tag{5.7}$$

Thus, it follows by induction and uniqueness of solution that

$$\delta_0 \leq u(x, nT-kT; -kT, u_0) \leq M^+, \quad x \in \mathbb{R}^n,\ k \geq 1,\ n \geq 1,\ \delta_0 \leq u_0 \leq M^+. \tag{5.8}$$

Let $u_n^k(x) := u(x, -nT; -kT, \delta_0)$ for all $x \in \mathbb{R}^n$, and $k \geq n \geq 0$. Then by a priori estimates for parabolic equations (see [13]), the sequence $\{u_0^k\}_{k\geq 1}$ has a locally uniformly convergent subsequence $\{u_0^{k'}\}_{k\geq 1}$ to some u^* with $u^* \in C^\nu_{\rm unif}(\mathbb{R}^n)$ for $0 < \nu < 1$. Let $u^+(x,t) = u(x,t;0,u^*)$ for every $x \in \mathbb{R}^n$ and $t \geq 0$. We claim that $u^+(\cdot,\cdot)$ has a backward extension. Indeed, by uniqueness of solution of (2.6), for every $1 \leq n \leq k'$, we have that

$$u_0^{k'}(\cdot) = u(\cdot, 0; -nT, u(\cdot, -nT; -k'T, \delta_0)) = u(\cdot, 0; -nT, u_n^{k'}). \tag{5.9}$$

Similarly as above, for every $n \geq 1$, there is a function $u_n^* \in C^b_{\rm unif}(\mathbb{R}^n)$ and a subsequence $\{u_n^{k'_n}\}_{k\geq 1}$ of $\{u_n^{k'}\}$ with $u_n^{k'_n} \to u_n^*$ locally uniformly as $k_n^{'} \to \infty$.

Since $u_0^{k'_n} \to u^*$ for each $n \geq 1$ locally uniformly, it follows from (5.9) and Lemma 4.2 that

$$u^*(\cdot) = u(\cdot, 0; -nT, u_n^*).$$

Therefore

$$u^+(x,t) = u(x,t;0,u^*) = u(x,t,0,u(\cdot,0;-nT,u_n^*)) = u(x,t;-nT,u_n^*) \tag{5.10}$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$. Since $u(\cdot,t;-nT,u_n^*)$ is defined for all $t \geq -nT$, then it follows from (5.10) that $u^+(x,t)$ has an extension to $\mathbb{R}^N \times [-nT,\infty)$ for every $n \in \mathbb{N}$. Therefore, $u^+(x,t)$ has a backward extension on $\mathbb{R}^N \times \mathbb{R}$. Note that (5.8) implies that $\delta_0 \leq u^* \leq M^+$. Thus, by Theorem 2.3 and Lemma 4.1, we obtain that $0 < \inf_{x,t} u^+(x,t) \leq \sup_{x,t} u^+(x,t) \leq M^+$. Hence $(u^+(x,t), v^+(x,t))$ is a positive entire solution of (2.6).

(i) Suppose that $(u^+(x,t), v^+(x,t))$ is a strictly positive entire solution of (2.6). Then,

$$\underline{u}(t-t_0;u^+_{\inf}) \leq u^+(x,t) \leq \overline{u}(t-t_0;u^+_{\sup}), \forall\, t_0 \in \mathbb{R}, t \geq t_0,\ x \in \mathbb{R}^N, \tag{5.11}$$

where $\underline{u}(t;u^+_{\inf})$ solves

$$\begin{cases} \frac{d}{dt}\underline{u} = \underline{u}(a_{\inf} - \chi\mu u^+_{\sup} - (b_{\sup} - \chi\mu)\underline{u}), & t > 0 \\ \underline{u}(0) = u^+_{\inf}, \end{cases}$$

and $\overline{u}(t;u^+_{\sup})$ solves

$$\begin{cases} \frac{d}{dt}\overline{u} = \overline{u}(a_{\sup} - \chi\mu u^+_{\inf} - (b_{\inf} - \chi\mu)\overline{u}), & t > 0 \\ \underline{u}(0) = u^+_{\sup}. \end{cases}$$

Note that

$$\lim_{t\to\infty} \underline{u}(t;u^+_{\inf}) = \frac{(a_{\inf} - \chi\mu u^+_{\sup})_+}{b_{\sup} - \chi\mu}, \quad \text{and} \quad \lim_{t\to\infty} \overline{u}(t;u^+_{\sup}) = \frac{(a_{\sup} - \chi\mu u^+_{\inf})_+}{b_{\inf} - \chi\mu}. \tag{5.12}$$

Hence, it follows from (5.11) and (5.12) that

$$(a_{\inf} - \chi\mu u^+_{\sup})_+ \leq (b_{\sup} - \chi\mu)u^+_{\inf} \quad \text{and} \quad (b_{\inf} - \chi\mu)u^+_{\sup} \leq (a_{\sup} - \chi\mu u^+_{\inf})_+. \tag{5.13}$$

Thus, (i) follows.

(ii) Since $0 < u^+_{\inf} \le u^+_{\sup}$, (5.13) implies that

$$a_{\inf} - \chi\mu u^+_{\sup} \le (b_{\sup} - \chi\mu)u^+_{\inf} \quad \text{and} \quad (b_{\inf} - \chi\mu)u^+_{\sup} \le a_{\sup} - \chi\mu u^+_{\inf}.$$

This together with (5.13) implies that

$$(b_{\inf} - \chi\mu)a_{\inf} - \chi\mu a_{\sup} \le \big((b_{\inf} - \chi\mu)(b_{\sup} - \chi\mu) - (\chi\mu)^2\big)u^+_{\inf} \tag{5.14}$$

and

$$\big((b_{\inf} - \chi\mu)(b_{\sup} - \chi\mu) - (\chi\mu)^2\big)u^+_{\sup} \le (b_{\sup} - \chi\mu)a_{\sup} - \chi\mu a_{\inf}. \tag{5.15}$$

Since **(H2)** holds and $u^+_{\inf} > 0$, it follows from (5.14) that $(b_{\inf} - \chi\mu)(b_{\sup} - \chi\mu) - (\chi\mu)^2 > 0$. Thus (2.21) follows from (5.14) and (5.15).

(iii) Let $\delta^*_0(T)$ be given by Lemma 4.5 and $E(T) := \{u \in C^b_{\rm unif}(\mathbb{R}^N) \mid \delta^*_0(T) \le u_{\inf} \le u_{\sup} \le \frac{a_{\inf}}{b_{\sup} - \chi\mu}\}$ endowed with the open compact topology. Lemma 4.5 implies that the map $\mathbb{P}_T : E(T) \ni u_0 \mapsto u(\cdot, T; 0, u_0) \in E(T)$ is well defined. Note that $E(T)$ is a closed bounded convex subset of $C^b_{\rm uinf}(\mathbb{R}^N)$ endowed with the open compact topology. Let $\{u_{0n}\}_{n\ge 1} \subset E(T)$ and $u_0 \in E(T)$ such that $u_{0n} \to u_0$ uniformly on every compact subset of $\mathbb{R}^N$. For every $n \ge 1$, we have

$$u_t(\cdot, \cdot; 0, u_{0n}) = \Delta u - \chi\nabla v(\cdot, \cdot; u_{0n}) \cdot \nabla u + (a - \chi\lambda v(\cdot, \cdot; 0, u_{0n}) - (b - \chi\mu)u)u, \quad t > 0$$

and Theorem 2.2 (ii) gives

$$\sup_{0 \le t \le T, n \ge 1} \|v(\cdot, t; 0, u_{0n})\|_{C^{1,\nu}_{\rm unif}(\mathbb{R}^N)} < \infty. \tag{5.16}$$

Since $u_{0n} \to u_0$ locally uniformly, it follows from Lemma 4.2 that there is a subsequence $\{(u(\cdot, \cdot; 0, u_{0n'}), v(\cdot, \cdot; 0, u_{0n'}))\}_{n\ge 1}$ of $\{(u(\cdot, \cdot; 0, u_{0n}), v(\cdot, \cdot; 0, u_{0n}))\}_{n\ge 1}$ and a function $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, \infty))$ such that $(u(\cdot, \cdot; 0, u_{0n'}), v(\cdot, \cdot; 0, u_{0n'})) \to (u, v)$ locally uniformly in

$C^{2,1}(\mathbb{R}^N \times (0,\infty))$. Moreover, (u,v) satisfies $\Delta v - \lambda v + \mu u = 0$ and

$$\begin{cases} u_t = \Delta u - \chi \nabla v \cdot \nabla u + (a - \chi\lambda v - (b-\chi\mu)u)u, \quad 0 < t \le T \\ u(0) = u_0. \end{cases}$$

Thus $(u(x,t), v(x,t)) = (u(x,t;0,u_0), v(x,t;0,u_0))$ for every $x \in \mathbb{R}^N$, $t \in [0,T]$. This implies that $u(\cdot,T;0,u_{0n'}) \to u(\cdot,T;0,u_0)$ locally uniformly. Hence $\mathbb{P}_T$ is continuous.

Next let $\{u_{0n}\}_{n\ge} \in E(T)$ be given. It follows from (5.16) and a priori estimate for parabolic equations that

$$\sup_n \|u(\cdot,T;0,u_{0n})\|_{C^\nu(\mathbb{R}^N)} < \infty.$$

Thus $\{u(\cdot,T;0,u_{0n})\}_{n\ge1}$ has a convergent subsequence in the open compact topology in $E(T)$. Hence $\mathbb{P}_T$ is a compact map. Therefore, Shauder's fixed theorem implies that there is $u^* \in E(T)$ such that $u(\cdot,T;0,u^*) = u^*$. Clearly $(u(\cdot,\cdot;0,u^*), v(\cdot,\cdot;0,u^*))$ is a $T-$periodic solution of (2.6) and can extended uniquely to a positive entire solution.

(iv) For every $n \ge 1$, let $t_n = \frac{1}{n}$ and $u_{0n} \in C^b_{\rm unif}(\mathbb{R}^N)$, such that $(u(x,t;u_{0n}), v(x,t;u_{0n}))$ is positive t_n- periodic solution of (2.6) with $\frac{a_{\inf}}{b_{\sup}} \le \sup_{(x,t)} u(x,t;u_{0n}) \le \frac{a_{\sup}}{b_{\inf}-\chi\mu}$.

Claim 1 :*There exists $L \gg 1$ large enough, such that*

$$\inf_{n\ge1, x_0\in\mathbb{R}^N} \sup_{|x|<L} u_{0n}(x+x_0) > 0. \tag{5.17}$$

Let $a_0 = \frac{a_{\inf}}{3}$ and $L_0 \gg 1$ be fixed such that the principal eigenvalue λ_L of (5.2) is negative for every $L \ge L_0$. Note that for every nonnegative uniformly continuous function $u_0(x)$ in D_L, $L \ge L_0$, with $\|u_0\|_{L^\infty(D_L)} > 0$, we have that $\|u(\cdot,t;u_0)\|_\infty \to \infty,$ as $t \to \infty$, where $u(x,t;u_0)$ solves the initial-boundary problem (5.1). Hence, by (5.6), for every $L \ge L_0$, there is $\varepsilon_0(L) > 0$ such that if $\sup_{x\in D_L, 0\le t\le 1} |b_{\varepsilon_0}(x,t)| \le \varepsilon_0$, $b_{\varepsilon_0}(x,t+1) = b_{\varepsilon_0}(x,t)$ for every $x \in D_L,\ t \ge 0$, then for every nonnegative continuous function $u_0(x)$ on D_L with $\|u_0\|_{L^\infty(D_L)} > 0$, we have that

$$\sup_{x\in D_L, t>0} (U_{L,b_{\varepsilon_0(L)}}(t;0)u_0)(x) = \infty, \tag{5.18}$$

where $U_{b_{\varepsilon_0(L)},L}(x,t;0)u_0$ solves the initial boundary value problem (5.4) (with $T=1$).

Taking $T=1$, $M=\frac{a_{\sup}}{b_{\inf}-\chi\mu}$, and $\varepsilon=\min\{\frac{a_{\inf}}{3(\chi\lambda+b_{\sup}-\chi\mu)},\varepsilon_0(L_0)\}$, it follows from Lemma 4.3 and inequalities (4.8) and (4.28) that there is $L_1>L_0$ and $\delta_0>0$ such that for every $L\geq L_1$, $x\in D_L$, and $0\leq t\leq 1$, we have

$$u(x,t+t_0;u_0)<\varepsilon,\;\; v(x,t+t_0;t_0,u_0)<\varepsilon,\;\; \text{and}\;\; |\nabla v(x,t+t_0;t_0,u_0)|\leq\varepsilon, \tag{5.19}$$

whenever $0\leq u_0(x)\leq\delta_0,\ \forall\,|x|\leq 3L,\ \ i=1,\cdots,N$. Suppose that there is some $n\geq 1$ and $x_0\in\mathbb{R}^N$, such that

$$\sup_{|x|_\infty<3L_1} u_{0n}(x+x_0)<\delta_0. \tag{5.20}$$

Thus, since $(u(x,t;0,u_{0n}),v(x,t;0,u_{0n}))$ is t_n-periodic with $t_n\leq 1$, it follows from (5.19) that, for $|x-x_0|<L_1,\ t\geq 0$,

$$\begin{aligned}u_t(,\cdot;0,u_{0n}) &= \Delta u(\cdot,\cdot;0,u_{0n})-\chi(\nabla v\nabla u)(\cdot,\cdot;0,u_{0n})+u(a-(b-\chi\mu)u-\chi\lambda v)\\ &\geq \Delta u(\cdot,\cdot;0,u_{0n})-\chi(\nabla v\nabla u)(\cdot,\cdot;0,u_{0n})+\frac{a_{\inf}}{3}u(\cdot,\cdot,0,u_{0n}).\end{aligned}$$

Therefore, by comparison principle for parabolic equations, since $L_1\geq L_0$, we have that

$$u(x+x_0,t;0,u_{0n})\geq U_{b_{\varepsilon_0},L_0}(x,t;0)u_{0n|_{D_{L_0}}},\quad \forall\,|x|_\infty<L_1,\ \forall t\geq 0 \tag{5.21}$$

where $u_{0n|_{D_{L_0}}}$ denotes the restriction of u_{0n} on D_{L_0} and $b_{\varepsilon_0}(x,t)=\nabla v(x+x_0,t;0,u_{0n})$ for every $x\in D_{L_0}, t\geq 0$. It follows from (5.18) and (5.21) that $\sup_{x,t}u(x,t;0,u_{0n})=\infty$, which is a contradiction. Hence Claim 1 follows.

By a priori estimate for parabolic equations, we may suppose that $u_{0n}\to u^*\in C^b_{\rm unif}(\mathbb{R}^N)$ in the open compact topology. Let $u^+(x,t)=u(x,t;0,u^*)$.

Claim 2: $u^+(x,t)=u^*(x)$ *for every* $x\in\mathbb{R}^N$, *and* $t\geq 0$.

Without loss of generality, let us suppose that $u_{0n} \to u^*$ in the open compact topology. Let $x \in \mathbb{R}^N$ and $t > 0$ be fixed. For every $n \geq 1$, we have that

$$u^+(x,t) - u^*(x) = \underbrace{u(x,t;0,u^*) - u(x,t;0,u_{0n})}_{I_{1,n}(x,t)} + \underbrace{u(x,t;0,u_{0n}) - u(x,[nt]T_n;0,u_{0n})}_{I_{2,n}(x,t)} + \underbrace{u(x,[nt]t_n;0,u_{0n}) - u^*}_{I_{3,n}(x,t)}. \tag{5.22}$$

Since $u(x,t;0,u_{0n})$ is t_n-periodic, then

$$I_{3,n}(x,t) = u_{0n}(x) - u^*(x) \to 0, \quad \text{as } n \to \infty$$

in the open compact topology. It follows from the variation of constant formula that

$$I_{2,n}(x,t) = \underbrace{-\chi \int_0^{t-[nt]T_n} T(t-[nt]t_n - s)\nabla \cdot ((u\nabla v)(x, s+[nt]t_n;0,u_{0n})))ds}_{I^1_{2,n}(x,t)} + \underbrace{\int_0^{t-[nt]t_n} T(t-[nt]t_n - s)(((a+1-bu)u)(x, s+[nt]t_n;0,u_{0n}))ds}_{I^2_{2,n}(x,t)}$$

where $\{T(t)\}_{t\geq 0}$ denotes the analytic semigroup in (2.1). Since $\|u_{0n}\|_\infty \leq M$, there is a constant C depending only on M such that

$$|I^1_{2,n}(x,t)| \leq C\int_0^{t-[nt]t_n} \frac{e^{-(t-[nt]t_n-s)}}{\sqrt{t-[nt]t_n - s}}ds \leq C(t-[nt]t_n)^{\frac{1}{2}} \to 0, \text{ as } n \to \infty$$

and

$$|I^2_{2,n}(x,t)| \leq C\int_0^{t-[nt]t_n} e^{-(t-[nt]t_n-s)}ds = C(1-e^{-(t-[nt]t_n)}) \to 0, \quad \text{as } n \to \infty.$$

Hence $I_{2,n}(x,t) \to 0$ as $n \to \infty$ in the open compact topology . Since $u_{0n} \to u^*$ in the open compact topology, by Lemma 4.2, we have that $I_{1,n}(x,t) \to 0$ as $n \to \infty$ in the open compact topology. Therefore, we conclude from (5.22) that $u^+(x,t) = u^*(x)$, which completes the proof of Claim 2.

Next, it follows from Claim 1 that there exists $L \gg 1$ such that

$$\inf_{x_0 \in \mathbb{R}^N} \sup_{|x|_\infty \le L} u^*(x+x_0) > 0. \tag{5.23}$$

Suppose by contradiction that $u^*_{\inf} = 0$. Then there is a sequence $\{x_n\}_{n\ge 1}$ such that $u^*(x_n) \to 0$ as $n \to \infty$. Let $u_n(x) = u^*(x+x_n)$ for every $n \ge 1$. By a prior estimate for parabolic equations, as above, we may suppose that $u_n \to \tilde{u}$ in the open compact topology and $\tilde{u}$ is a steady state solution of (2.6). Furthermore, (5.23) implies that $\|\tilde{u}\|_\infty > 0$. Hence by comparison principle for parabolic equations, we that that $\tilde{u}(0) > 0$. But $\tilde{u}(0) = \lim_{n\to\infty} u^*(x_n) = 0$, which impossible. Thus $u^*_{\inf} > 0$. Therefore $u^*(x)$ is a positive steady state solution of (2.6). □

5.2 Uniqueness and stability of strictly positive entire solutions

In this section, we study the uniqueness and stability of strictly positive entire solutions of (2.6) and prove Theorem 2.5. First, we study these questions for general logistic type source function $f(x,t,u) = u(a(x,t) - ub(x,t))$, and prove that there is a positive constant χ_0 such that for every $0 \le \chi < \chi_0$, (2.6) has a unique exponentially stable positive entire solution. Next, we examine two frequently encountered cases of logistic source in the literature, namely space independent logistic source function $f_0(x,t,u) = u(a(t) - ub(t))$ and a logistic source function of the form $f_1(x,t,u) = b(x,t)(\kappa - u)u$, $\kappa > 0$, and derive explicit lower bound for χ_0. In this section, we shall always assume that **(H1)** holds, so that pointwise persistence phenomena occurs in (2.6) (see Theorem 2.3 (i)). Furthermore, for every initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $\inf_x u_0(x) > 0$, every initial time $t_0 \in \mathbb{R}$, it follows from Remark 2.1 that there exists $T_1(u_0) \gg 1$ such that the unique nonnegative global classical solution $(u(x,t+t_0;t_0,u_0), v(x,t+t_0;t_0,u_0))$ of (2.6) with $(u(x,t_0;t_0,u_0) = u_0(x)$, satisfies

$$0 < m(u_0) \le u(x,t+t_0;t_0,u_0) \le \frac{a_{\sup}}{b_{\inf} - \chi\mu}, \forall\, t \ge T_1(u_0),\ \forall\, x \in \mathbb{R}^N,\ \forall t_0 \in \mathbb{R}. \tag{5.24}$$

Henceforth, we shall always suppose that $0 < u_{0\,\inf} \le u_{0\,\sup} \le \frac{a}{b_{\inf}-\chi\mu}$. Note that, by a variation of constant formula, we have that

$$u(\cdot,t+t_1+t_0;t_0,u_0) = T(t)u(\cdot,t_1+t_0;t_0,u_0) - \chi\int_0^t T(t-s)\nabla(u\nabla v)(\cdot,s+t_1+t_0;t_0,u_0)ds$$
$$+\int_0^t T(t-s)((a+1-ub)u)(\cdot,s+t_1+t_0;t_0,u_0))ds, \tag{5.25}$$

where $\{T(t)\}_{t\ge 0}$ denotes the analytic semigroup in (2.1). We let X^β, $0<\beta\le 1$, stand for the fractional power space associated with $I-\Delta$.

Thus, it holds that (see [16]) $X^{\frac{1}{2}+\beta}$ is continuously embedding in $C^b_{\text{unif}}(\mathbb{R}^N)$ with

$$\|\nabla u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \le \frac{\sqrt{N}\Gamma(\beta)}{\sqrt{\pi}\Gamma(\frac{1}{2}+\beta)}\|u\|_{X^{\frac{1}{2}+\beta}}, \quad \forall\, u\in X^{\beta+\frac{1}{2}}, \quad \forall\, 0<\beta<\frac{1}{2}, \tag{5.26}$$

$$\|u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \le \|u\|_{X^\beta}, \quad \forall\, u\in X^\beta, \quad \forall\, 0<\beta<1, \tag{5.27}$$

and

$$\|T(t)u\|_{X^\beta} \le C_\beta t^{-\beta}e^{-t}\|u\|_\infty, \quad \forall\, t>0,\ \forall\, u\in\ X^\beta,\ \forall\, 0<\beta<1. \tag{5.28}$$

The next lemma provides an a priori bound on the sup-norm of the gradient of positive entire solutions to (2.6).

Lemma 5.1. *There is a positive constant C independent of χ, a, b, λ and μ such that for any positive entire solution $(u^+_\chi(x,t), v^+_\chi(x,t))$ of* (2.6)*, it holds that*

$$\|\nabla u^+_\chi(\cdot,t+t_0)\|_\infty \le C_{\frac{3}{4}}\frac{\sqrt{N}\Gamma(\frac{1}{4})}{\sqrt{\pi}\Gamma(\frac{3}{4})}M_0e^{-t}t^{-\frac{3}{4}}\left(1+CM_2t^{\frac{1}{4}}\right)e^{2t\left(\Gamma(\frac{1}{4})M_2\right)^4}, \quad \forall t_0\in\mathbb{R},\ \forall t>0, \tag{5.29}$$

where $M_0=\frac{a_{\sup}}{b_{\inf}-\chi\mu}$ and $M_1=2a_{\sup}+1+\chi\mu M_0$ and $M_2:=C_{\frac{3}{4}}\left(\frac{\chi\mu N\Gamma(\frac{1}{4})}{\sqrt{\pi}\lambda\Gamma(\frac{3}{4})}M_0+M_1\right)$ and $C_{\frac{3}{4}}$ is given by (5.28)*.*

Proof. Observe from (5.25) that for every $t > 0$, $t_0 \in \mathbb{R}$, $u_\chi^+(\cdot, t+t_0)$ can be written as

$$
\begin{aligned}
u_\chi^+(t+t_0) =& T(t)u_\chi^+(t_0) - \chi \int_0^t T(t-s)(\nabla u_\chi^+ \cdot \nabla v_\chi^+)(s+t_0)ds \\
&+ \int_0^t T(t-s)\left((a+1-\chi\lambda v_\chi^+ - (b-\chi\mu)u_\chi^+)u_\chi^+\right)(s+t_0)ds. \quad (5.30)
\end{aligned}
$$

Note from Lemma 3.2 and (2.20) that $\|\nabla v_\chi^+(\cdot, t+t_0)|_\infty \leq \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\|u_\chi^+(\cdot, t+t_0)\|_\infty \leq \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}M_0$. Thus, it follows from (2.4), (5.28) and (5.26) and Lemma 3.2 that

$$
\begin{aligned}
&\|\chi \int_0^t T(t-s)(\nabla u_\chi^+ \cdot \nabla v_\chi^+)(s+t_0)ds\|_{X^{\frac{3}{4}}} \\
\leq& \frac{\chi\mu\sqrt{N}C_{\frac{3}{4}}M_0}{2\sqrt{\lambda}} \int_0^t \frac{e^{-(t-s)}}{(t-s)^{\frac{3}{4}}}\|\nabla u_\chi^+(s+t_0)\|_\infty ds \\
\leq& \frac{\chi\mu N\Gamma(\frac{1}{4})C_{\frac{3}{4}}M_0}{2\sqrt{\pi\lambda}\Gamma(\frac{3}{4})} \int_0^t \frac{e^{-(t-s)}}{(t-s)^{\frac{3}{4}}}\|\nabla u_\chi^+(s+t_0)\|_\infty ds.
\end{aligned}
$$

Similarly since $\lambda\|v_\chi^+(t+t_\tau)\|_\infty \leq \chi\mu\|u_\chi^+(t+t_\tau)\|_\infty \leq \chi\mu M_0$, using (5.27) and (5.28), we obtain

$$
\begin{aligned}
&\|\int_0^t T(t-s)\left((a+1-\chi\lambda v_\chi^+ - (b-\chi\mu)u_\chi^+)u_\chi^+\right)(s+t_0)ds\|_{X^{\frac{3}{4}}} \\
\leq& C_{\frac{3}{4}}\left(a_{\sup} + 1 + \chi\lambda\sup_\tau\|v_\chi^+(\tau)\|_\infty + (b_{\sup} - \chi\mu)\sup_\tau\|u_\chi^+(\tau)\|_\infty\right)\int_0^t \frac{e^{-(t-s)}}{(t-s)^{\frac{3}{4}}}\|u_\chi^+(s+t_0)\|_\infty ds \\
\leq& C_{\frac{3}{4}}\left(2a_{\sup} + 1 + \chi\mu M_0\right)\int_0^t \frac{e^{-(t-s)}}{(t-s)^{\frac{3}{4}}}\|u_\chi^+(s+t_0)\|_\infty ds.
\end{aligned}
$$

Therefore, we have from (5.30) that

$$
\|e^t u_\chi^+(t+t_0)\|_{X^{\frac{3}{4}}} \leq C_{\frac{3}{4}}M_0 t^{-\frac{3}{4}} + \underbrace{C_{\frac{3}{4}}\left(\frac{\chi\mu N\Gamma(\frac{1}{4})}{2\sqrt{\pi\lambda}\Gamma(\frac{3}{4})}M_0 + M_1\right)}_{:=M_2}\int_0^t \frac{e^s\|u_\chi^+(s+t_0)\|_\infty}{(t-s)^{\frac{3}{4}}}ds.
$$

Therefore, it follows from [1, Theorem 3.1.1] that there is $C > 0$ such that

$$
\|e^t u_\chi^+(t+t_0)\|_{X^{\frac{3}{4}}} \leq C_{\frac{3}{4}}M_0 t^{-\frac{3}{4}}\left(1 + CM_2 t^{\frac{1}{4}}\right)e^{2t\left(\Gamma(\frac{1}{4})M_2\right)^4}.
$$

Combining this with (5.26), we obtain (5.29). The Lemma is thus proved. □

Remark 5.1. *It follows from Lemma 5.1 that*

$$\|\nabla u_\chi^+(\cdot,t)\|_\infty = \|\nabla u_\chi^+(1+(t-1))\|_\infty \le C_{\frac{3}{4}}\frac{\sqrt{N}\Gamma(\frac{1}{4})}{\sqrt{\pi}\Gamma(\frac{3}{4})}M_0e^{-1}\left(1+CM_2\right)e^{2\left(\Gamma(\frac{1}{4})M_2\right)^4}, \quad \forall t\in\mathbb{R},$$

where C, M_0, M_1 and M_2 are given by Lemma 5.1 whenever $(u_\chi^+(x,t), v_\chi^+(x,t))$ is a positive entire solution of (2.6). *Therefore, by setting*

$$C_0(\chi) := \sup\{\|\nabla u_\chi^+(\cdot,t)\|_\infty, t\in\mathbb{R},\ (u_\chi^+(x,t), v_\chi^+(x,t)) \textit{ is a positive entire solution of } (2.6)\}, \tag{5.31}$$

we have that $C_0(\chi) < \infty$ for every $0<\chi<\frac{b_{\inf}}{\mu}$. Moreover taking $C_1(\chi) = 1+\frac{\mu C_0(\chi)\sqrt{N}}{2u^+_{\chi\inf}\sqrt{\lambda}}$, it follows from (2.21) *that*

$$\lim_{\chi\to0^+}\frac{\chi\mu C_1(\chi)u^+_{\chi\sup}}{(b_{inf}-\chi\mu)u^+_{\chi\inf}} = 0,$$

for any positive entire solution $(u_\chi^+(x,t), v_\chi^+(x,t))$ of (2.6). *Thus, we introduce the following definition*

$$\chi_0 := \sup\{\chi\in(0,\frac{b_{\inf}}{\mu})\ :\ \forall\, 0<\tilde\chi<\chi,\ \exists\,(u^+_{\tilde\chi}, v^+_{\tilde\chi}) \textit{ satisfying } \frac{\tilde\chi\mu C_1(\chi)u^+_{\tilde\chi\sup}}{(b_{\inf}-\tilde\chi\mu)u^+_{\tilde\chi\inf}} < 1\,\}. \tag{5.32}$$

Lemma 5.2. *For given $u_0\in C^b_{\rm unif}(\mathbb{R}^N)$ and positive entire solution $(u_\chi^+(x,t), v_\chi^+(x,t))$ of* (2.6) *we let*

$$U(x,t+t_0;t_0,u_0) := \frac{u(x,t+t_0;t_0,u_0)}{u_\chi^+(x,t+t_0)} \quad \textit{and} \quad V(x,t+t_0;t_0,u_0) := \frac{v(x,t+t_0;t_0,u_0)}{v_\chi^+(x,t+t_0)}.$$

Then $U(x,t+t_0;t_0,u_0)$ satisfies

$$U_t = \Delta U+\nabla U\nabla(2\ln(u_\chi^+)-\chi v)+\chi\left(\lambda(v_\chi^+-v)+\nabla\ln(u_\chi^+)\nabla(v_\chi^+-v)\right)U+(b-\chi\mu)u_\chi^+U(1-U). \tag{5.33}$$

In particular, if $u_\chi^+(x,t)=u_\chi^+(t)$, that is u_χ^+ is space independent, we have that

$$U_t=\Delta U-\chi\nabla U\nabla v+\left(\chi\lambda(1-V)U+(b-\chi\mu)U(1-U)\right)u_\chi^+(t). \tag{5.34}$$

Proof. We have that

$$\begin{aligned}U_t=&\frac{1}{(u_\chi^+)^2}\left(u_\chi^+\left(\Delta u-\chi\nabla\cdot(u\nabla v)+(a-bu)u\right)-u\left(\Delta u_\chi^+-\chi\nabla\cdot(u_\chi^+\nabla v_\chi^+)+(a-bu_\chi^+)u_\chi^+\right)\right)\\=&\frac{1}{u_\chi^+}\left(\Delta u-U\Delta u_\chi^+-\chi\left(\nabla\cdot(u\nabla v)-U\nabla\cdot(u_\chi^+\nabla u_\chi^+)\right)\right)+bu_\chi^+U(1-U)\\=&\Delta U+2\nabla U\cdot\nabla\ln(u_\chi^+)-\frac{\chi}{u_\chi^+}\left(\nabla\cdot(u\nabla v)-U\nabla\cdot(u_\chi^+\nabla u_\chi^+)\right)+bu_\chi^+U(1-U).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\nabla\cdot(u\nabla v)-U\nabla\cdot(u_\chi^+\nabla v_\chi^+)\\=&\nabla u\cdot\nabla v+Uu_\chi^+\Delta v-U\nabla u_\chi^+\cdot\nabla v_\chi^+-Uu_\chi^+\Delta v^+\\=&Uu_\chi^+\Delta(v-v_\chi^+)+U\nabla u_\chi^+\cdot\nabla(v-v_\chi^+)+u_\chi^+\nabla U\cdot\nabla v\\=&\lambda Uu_\chi^+(v-v_\chi^+)+\mu(u_\chi^+)^2U(1-U)+U\nabla u_\chi^+\cdot\nabla(v-v_\chi^+)+u_\chi^+\nabla U\cdot\nabla v.\end{aligned}$$

Hence, we have that

$$U_t=\Delta U+\nabla U\cdot\nabla(2\ln(u_\chi^+)-\chi v)-\chi\left(\lambda(v-v_\chi^+)+\nabla ln(u_\chi^+)\nabla(v-v_\chi^+)\right)U+(b-\chi\mu)u_\chi^+U(1-U).$$

□

We note that to show the stability of the positive entire solution $u_\chi^+(x,t)$ it is enough to show that $\|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty\to 0$ as $t\to\infty$. We first prove the following theorem, which will be used in for the proof of our main result in this section.

Theorem 5.1. *For every $\varepsilon > 0$, and for every $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ satisfying $0 < u_{0\inf} \le u_{0\sup} \le \frac{a_{\sup}}{b_{\inf}-\chi\mu}$, and $n \ge 1$ there is $T_{\varepsilon,n} > 0$ such that*

$$\|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty \le \left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n \frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\sup}}+\varepsilon, \quad \forall\, t \ge T_{\varepsilon,n},\ t_0 \in \mathbb{R}, \tag{5.35}$$

where $C_1(\chi) := 1 + \frac{\mu C_0(\chi)\sqrt{N}}{2u^+_{\chi\inf}\sqrt{\lambda}}$ and C_0 is given by (5.31)*. Furthermore, if $u^+_\chi(x,t) = u^+_\chi(t)$, is space homogeneous, then $T_{\varepsilon,n}$ can be chosen so that*

$$\|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty \le \left(\frac{\chi\mu}{b_{\inf}-\chi\mu}\right)^n \frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}+\varepsilon, \quad \forall\, t \ge T_{\varepsilon,n},\ t_0 \in \mathbb{R}, \tag{5.36}$$

Proof. The proof of this theorem is divided in two parts. In the first part, we shall give the proof of the general case. Next, in the second part, we consider the proof of the particular cases.

Let $\varepsilon > 0$ be given. Since, by (3.25), $\|\nabla(v-v^+_\chi)(\cdot,t+t_0;t_0,u_0)\|_\infty \le \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\|(u-u^+_\chi)(\cdot,t+t_0;t_0,u_0)\|_\infty$ and $\|\lambda(v-v^+_\chi)(\cdot,t+t_0;t_0,u_0)\|_\infty \le \mu\|(u-u^+_\chi)(\cdot,t+t_0;t_0,u_0)\|_\infty$ for every $t \ge 0$, we have from Remark 5.1 that

$$\begin{aligned}
&\|(\lambda(v-v^+_\chi)+\nabla\ln(u^+_\chi)\cdot\nabla(v-v^+_\chi))(\cdot,t+t_0;t_0,u_0)\|_\infty \\
\le&\underbrace{\left(1+\frac{C_0(\chi)\sqrt{N}}{2u^+_{\chi\inf}\sqrt{\lambda}}\right)}_{=C_1(\chi)}\mu\|(u-u^+_\chi)(\cdot,t+t_0;t_0,u_0)\|_\infty \\
\le&\mu C_1(\chi)u^+_{\chi\sup}(t+t_0)\|(U-1)(\cdot,t+t_0;t_0,u_0)\|_\infty, \forall\, t \ge 0,
\end{aligned} \tag{5.37}$$

where $C_0(\chi)$ is given by (5.31). Observe from Theorem 2.2 (i) and Theorem 2.4 (i) that

$$\|u(\cdot,t+t_0;t_0,u_0)-u^+_\chi\|_\infty \le \frac{a_{\sup}}{b_{\inf}-\chi\mu}, \quad \forall t \ge 0,\ \forall\, t_0 \in \mathbb{R}$$

Thus it follows from the first inequality in (5.37) that

$$\|(\lambda(v-v^+_\chi)+\nabla\ln(u^+_\chi)\cdot\nabla(v-v^+_\chi))(\cdot,t+t_0;t_0,u_0)\|_\infty \le \frac{\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}, \quad \forall t \ge 0,\ \forall\, t_0 \in \mathbb{R}.$$

This combined with (5.33) yields that

$$U_t \le \Delta U + \nabla U \cdot \nabla(2\ln(u_\chi^+) - \chi v) + \frac{\chi\mu C_1(\chi) a_{\sup}}{b_{\inf} - \chi\mu} U + (b - \chi\mu) u_\chi^+ U(1-U), \tag{5.38}$$

and

$$U_t \ge \Delta U + \nabla U \cdot \nabla(2\ln(u_\chi^+) - \chi v) - \frac{\chi\mu C_1(\chi) a_{\sup}}{b_{\inf} - \chi\mu} U + (b - \chi\mu) u_\chi^+ U(1-U). \tag{5.39}$$

Let $\underline{U}_1(t)$ denote the solutions of the ODE

$$\begin{cases} \frac{d\underline{U}}{dt} = -\frac{\chi\mu C_1(\chi) a_{\sup}}{b_{\inf} - \chi\mu} \underline{U} + (b_{\inf} - \chi\mu) u^+_{\chi\inf} \underline{U}(1-\underline{U}), \\ \underline{U}(0) = \min\{\frac{u_{0\inf}}{u^+_{\chi\sup}}, 1\} \end{cases}$$

and $\overline{U}_1(t)$ denote the solutions of the ODE

$$\begin{cases} \frac{d\overline{U}}{dt} = \frac{\chi\mu C_1(\chi) a_{\sup}}{b_{\inf} - \chi\mu} \overline{U} + (b_{\inf} - \chi\mu) u^+_{\chi\inf} \overline{U}(1-\overline{U}) \\ \overline{U}(0) = \max\{\frac{u_{0\sup}}{u^+_{\chi\inf}}, \frac{(b_{\inf}-\chi\mu)u^+_{\chi\inf} + \frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\} \end{cases}$$

Thus, it follows from comparison principle for ODE's that

$$\overline{U}_1(t) \ge 1 + \frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf} - \chi\mu)u^+_{\chi\inf}} \quad \text{and} \quad 0 < \underline{U}_1(t) \le 1 \quad \forall\, t \ge 0. \tag{5.40}$$

Furthermore, it holds that

$$\lim_{t\to\infty} \underline{U}_1(t) = \left(1 - \frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf} - \chi\mu)u^+_{\chi\inf}}\right)_+ \quad \text{and} \quad \lim_{t\to\infty} \overline{U}_1(t) = 1 + \frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf} - \chi\mu)u^+_{\chi\inf}}. \tag{5.41}$$

We claim that

$$\underline{U}_1(t) \le U(x, t+t_0; t_0, u_0) \le \overline{U}_1(t), \quad \forall\, x \in \mathbb{R}, \forall\, t \ge 0,\ \forall\, t_0 \in \mathbb{R}. \tag{5.42}$$

Indeed, by setting

$$\mathcal{L}_1^+(U) := \Delta U + \frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}U + (b(x,t+t_0)-\chi\mu)u_\chi^+ U(1-U) \quad \text{and}$$

$$\mathcal{L}_2^-(U) := \Delta U - \frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}U + (b(x,t+t_0)-\chi\mu)u_\chi^+ U(1-U),$$

it follows from (5.40) that

$$\frac{d\overline{U}_1}{dt} - \mathcal{L}_1^+(\overline{U}_1) = ((b_{\inf}-\chi)u_{\chi\inf}^+ - (b(x,t+t_0)-\chi\mu)u_\chi^+)\overline{U}_1(1-\overline{U}_1) \geq 0 \tag{5.43}$$

and

$$\frac{d\underline{U}_1}{dt} - \mathcal{L}_1^-(\underline{U}_1) = ((b_{\inf}-\chi)u_{\chi\inf}^+ - (b(x,t+t_0)-\chi\mu)u_\chi^+)\underline{U}_1(1-\underline{U}_1) \leq 0. \tag{5.44}$$

Therefore, using (5.38), (5.39), (5.43), (5.44), and comparison principle for parabolic equations, we deduce that (5.42) holds. Thus, it follows from (5.41) and (5.42) that there is $T_{1,\varepsilon} \gg 1$ such that for every t_0,

$$1-\frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf}-\chi\mu)u_{\chi\inf}^+}-\varepsilon \leq U(x,t+t_0;t_0,u_0) \leq 1+\frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf}-\chi\mu)u_{\chi\inf}^+}+\varepsilon, \quad \forall\, t \geq T_{1,\varepsilon}, \forall\, x \in \mathbb{R}^N,$$

which is equivalent to

$$\|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty \leq \frac{\frac{\chi\mu C_1(\chi)a_{\sup}}{b_{\inf}-\chi\mu}}{(b_{\inf}-\chi\mu)u_{\chi\inf}^+} + \varepsilon, \quad \forall\, t \geq T_{1,\varepsilon},\ \forall\, t_0 \in \mathbb{R}.$$

This complete the proof of (5.35) for the case $n = 1$.

Next, proceed by induction and suppose that (5.35) holds for some $n \geq 1$. We show that (5.35) holds for $n+1$. Indeed, using the last inequality in (5.37), we may suppose that for

$0<\tilde\varepsilon\ll 1$, we have for every $t_0\in\mathbb{R}$

$$
\begin{aligned}
&\|(\lambda(v-v_\chi^+)+\nabla\ln(u_\chi^+)\cdot\nabla(v-v_\chi^+))(\cdot,t+t_0;t_0,u_0)\|_\infty\\
\le& C_1(\chi)\mu u^+_{\chi\sup}(t+t_0)\|(U-1)(\cdot,t+t_0;t_0,u_0)\|_\infty\\
\le& C_1(\chi)\mu u^+_{\chi\sup}(t+t_0)\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\sup}}+\tilde\varepsilon,\quad \forall\, t\ge T_{n,\tilde\varepsilon},\ x\in\mathbb{R}^N,
\end{aligned}
\tag{5.45}
$$

for some $T_{n,\tilde\varepsilon}\gg 1$. Therefore, similar arguments as in the case of $n=1$ from (5.38) to (5.44) yield for every $t_0\in\mathbb{R}$,

$$
\begin{aligned}
\|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty \le& \frac{\chi C_1(\chi)\mu u^+_{\chi\sup}\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\sup}}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}+\varepsilon,\\
=&\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^{n+1}\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\sup}}+\varepsilon,\quad \forall\, t\ge T_{n+1,\varepsilon},
\end{aligned}
$$

for some $T_{n+1,\varepsilon}\gg 1$.

If $u_\chi^+(x,t)=u_\chi^+(t)$, then using (5.34) instead of (5.33) in the proof of the general case given above, (5.38) and (5.39) become

$$
U_t\le\Delta U-\chi\nabla U\cdot\nabla v+(\chi\mu\|V-1\|_\infty U+(b-\chi\mu)U(1-U))\,u_\chi^+(t), \tag{5.46}
$$

and

$$
U_t\ge\Delta U-\chi\nabla U\cdot\nabla v+(-\chi\mu\|V-1\|_\infty U+(b-\chi\mu)U(1-U))\,u_\chi^+(t). \tag{5.47}
$$

Observe that $\|V(\cdot,t+t_0;t_0,u_0)-1\|_\infty\le\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}$ for every $t\ge 0$, $t_0\in\mathbb{R}$. Hence, by considering $\underline{U}_1(t)$ and $\overline{U}_1(t)$ solutions of the ODE

$$
\begin{cases}
\frac{d\underline{U}}{dt}=\left(-\chi\mu\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\underline{U}+(b_{\inf}-\chi\mu)\underline{U}(1-\underline{U})\right)u_\chi^+(t+t_0)\\
\underline{U}(0)=\min\{\frac{u_{0\inf}}{u^+_{\chi\sup}},1\}
\end{cases}
$$

and

$$\begin{cases} \frac{d\overline{U}}{dt} = \left(\chi\mu\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\overline{U} + (b_{\inf} - \chi\mu)\overline{U}(1-\overline{U})\right) u^+_\chi(t+t_0) \\ \overline{U}(0) = \max\{\frac{u_{0\sup}}{u^+_{\chi\inf}}, 1 + \left(\frac{\chi\mu}{b_{\inf}-\chi\mu}\right)\frac{a_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\}. \end{cases}$$

Hence, following similar arguments as in the general case, (5.40)-(5.44), we deduce that (5.36) also holds. This completes the proof of the theorem. □

We now present the proof of Theorem 2.5, which is based on the previous result.

Let $\tilde{U}(x,t;t_0,u_0) = U(x,t;t_0;u_0) - 1$ and $\tilde{V}(x,t;t_0,u_0) = V(x,t;t_0,u_0) - 1$. Then it follows from (5.33) that $\tilde{U}(x,t;t_0,u_0)$ satisfies

$$\begin{aligned} \tilde{U}_t =& \Delta\tilde{U} + \nabla\tilde{U}\cdot\nabla(2\ln(u^+_\chi) - \chi v) - (b(x,t)-\chi\mu)u^+_\chi(t)\tilde{U} \\ &+ U\left(\lambda(v - v^+_\chi) + \nabla ln(u^+_\chi)\nabla(v - v^+_\chi)\right) - (b(x,t)-\chi\mu)u^+_\chi\tilde{U}^2. \end{aligned} \tag{5.48}$$

Let $\Phi_\chi(t,s)$ be the solution operator in $C^b_{\text{unif}}(\mathbb{R}^N)$ of

$$u_t = \Delta u + \nabla u\cdot\nabla(2\ln(u^+_\chi) - \chi v) - (b(x,t)-\chi\mu)u^+_\chi u. \tag{5.49}$$

Then, by the comparison principle for parabolic equations, we have

$$\|\Phi_\chi(t,s)\| \le e^{-(t-s)(b_{\inf}-\chi\mu)u^+_{\chi\inf}}, \quad \forall\, t-s\ge 0. \tag{5.50}$$

Proof of Theorem 2.5. We shall give the proof of the general case. The proof of the particular case follows similar arguments. We suppose that $0<\chi<\chi_0$, where χ_0 is given by (5.32). Hence, by definition of χ_0, there is a positive entire solution of (2.6) $(u^+_\chi(x,t), v^+_\chi(x,t))$ satisfying

$$(\tilde{H}): \quad \frac{\chi\mu C_1(\chi)u^+_{\chi\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}} < 1.$$

Exponential Stability of $(u_\chi^+(x,t), v_\chi^+(x,t))$**:** By Theorem 5.1 we may suppose that there is $t_n \gg 1$, $t_n < t_{n+1}$, such that

$$\|\tilde{U}(\cdot,t+t_0;t_0,u_0)\|_\infty = \|U(\cdot,t+t_0;t_0,u_0)-1\|_\infty \le 2\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n, \ \forall\, t\ge t_n,\ t_0\in\mathbb{R}. \tag{5.51}$$

By the variation of constant formula, it follows from (5.48) that for every $t \ge 0$,

$$\tilde{U}(\cdot,t+t_n+t_0;t_0,u_0) = I_{1,n}(t;t_0) + \chi I_{2,n}(t;t_0) - I_{3,n}(t,t_0), \tag{5.52}$$

where

$$I_{1,n}(t,t_0) := \Phi_\chi(t+t_n+t_0;t_n+t_0)\tilde{U}(\cdot,t_n+t_0;t_0,u_0), \forall t\ge 0, \forall\, n\ge 1,$$

$$I_{2,n}(t,t_0) := \int_0^t \Phi_\chi(t+t_n+t_0, s+t_n+t_0)\Big(U\big(\lambda(v-v_\chi^+)+\nabla\ln(u_\chi^+)\cdot\nabla(v-v_\chi^+)\big)\Big)(\cdot,s+t_n+t_0)ds,$$

and

$$I_{3,n}(t,t_0) := \int_0^t \Phi_\chi(t+t_n+t_0, s+t_n+t_0)(b-\chi\mu)u_\chi^+\tilde{U}^2(\cdot,s+t_n+t_0)ds.$$

Next, it follows from (5.51) and (5.50) that for every $n\ge 1$, $t_0\in\mathbb{R}$, and $t \ge 0$,

$$\begin{aligned}\|I_{1,n}(t,t_0)\|_\infty \le& e^{-t(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\|\tilde{U}(\cdot,t_n+t_0;t_0,u_0)\|_\infty \\ \le& \underbrace{2\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n}_{:=K_{1,n}} e^{-t(b_{\inf}-\chi\mu)u^+_{\chi\inf}}.\end{aligned} \tag{5.53}$$

Next, for every $0 \le s \le t$, $n \ge 1$, and $t_0 \in \mathbb{R}$, we have

$$\begin{aligned}
&\|\Phi_\chi(t+t_n+t_0, s+t_n+t_0)((U\left(\lambda(v-v_\chi^+)+\nabla\ln(u_\chi^+)\cdot\nabla(v-v_\chi^+))(s+t_n+t_0)\right)\|_\infty \\
\le& e^{-(t-s)(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\|\left(U\left(\lambda(v-v_\chi^+)+\nabla\ln(u_\chi^+)\cdot\nabla(v-v_\chi^+)\right)\right)(s+t_n+t_0)\|_\infty \\
\le& \left(1+2\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n\right)\frac{\|\left(\lambda(v-v_\chi^+)+\nabla\ln(u_\chi^+)\cdot\nabla(v-v_\chi^+)\right)(s+t_n+t_0)\|_\infty}{e^{(t-s)(b_{\inf}-\chi\mu)u^+_{\chi\inf}}} \\
\le& \left(1+2\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n\right)\left(1+\frac{C_0(\chi)\sqrt{N}}{2u^+_{\chi\inf}\sqrt{\lambda}}\right)\frac{\mu\|(u-u_\chi^+)(s+t_n+t_0)\|_\infty}{e^{(b_{\inf}-\chi\mu)u^+_{\chi\inf}(t-s)}} \\
\le& \underbrace{\left(1+2\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n\right)\left(1+\frac{C_0(\chi)\sqrt{N}}{2u^+_{\chi\inf}\sqrt{\lambda}}\right)\mu u^+_{\chi\sup}}_{:=K_{2,n}}\frac{\|\tilde{U}(\cdot,s+t_n+t_0)\|_\infty}{e^{(b_{\inf}-\chi\mu)u^+_{\chi\inf}(t-s)}}
\end{aligned} \tag{5.54}$$

We also have

$$\begin{aligned}
&\|\Phi_\chi(t+t_n+t_0, s+t_n+t_0)(b-\chi\mu)u_\chi^+\tilde{U}^2(\cdot,s+t_n+t_0)\|_\infty \\
\le& 2\underbrace{(b_{\sup}-\chi\mu)u^+_{\chi\sup}\left(\frac{\chi\mu C_1(\chi)u^+_{\sup}}{(b_{\inf}-\chi\mu)u^+_{\chi\inf}}\right)^n}_{:=K_{3,n}}\|\tilde{U}(\cdot,s+t_n+t_0)\|_\infty e^{-(t-s)(b_{\inf}-\chi\mu)u^+_{\inf}}.
\end{aligned} \tag{5.55}$$

Thus, it follows from (5.52), (5.53), (5.54), and (5.55) that

$$\begin{aligned}
&\|\tilde{U}(\cdot,t+t_n+t_0;t_0,u_0)\|_\infty \\
\le& K_{1,n}e^{-(b_{\inf}-\chi\mu)t}+(\chi K_{2,n}+K_{3,n})\int_0^t e^{-(t-s)(b_{\inf}-\chi\mu)u^+_{\chi\sup}}\|\tilde{U}(\cdot,s+t_n+t_0;t_0,u_0)\|_\infty ds,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&e^{t(b_{\inf}-\chi\mu)u^+_{\chi\sup}}\|\tilde{U}(\cdot,t+t_n+t_0;t_0,u_0)\|_\infty \\
\le& K_{1,n}+(\chi K_{2,n}+K_{3,n})\int_0^t e^{s(b_{\inf}-\chi\mu)u^+_{\chi\sup}}\|\tilde{U}(\cdot,s+t_n+t_0;t_0,u_0)\|_\infty ds, \forall\, t\ge 0.
\end{aligned}$$

Therefore, by Gronwall's inequality, we obtain that

$$e^{t(b_{\inf}-\chi\mu)u^+_{\chi\sup}}\|\tilde{U}(\cdot,t+t_n+t_0;t_0,u_0)\|_\infty \le K_{1,n}e^{(\chi K_{2,n}+K_{3,n})t}, \quad \forall\, t\ge 0.$$

That is

$$|\tilde{U}(\cdot, t+t_n+t_0; t_0, u_0)\|_\infty \le K_{1,n} e^{-\left((b_{\inf}-\chi\mu)u^+_{\chi\sup}-\chi K_{2,n}-K_{3,n}\right)t}, \quad \forall\, t \ge 0. \tag{5.56}$$

By $(\tilde{H})$, we have

$$\lim_{n\to\infty} K_{1,n} = \lim_{n\to\infty} K_{3,n} = 0 \quad \text{and} \quad \lim_{n\to\infty} K_{2,n} = \Big(1 + \frac{C_0(\chi)\sqrt{N}}{2\sqrt{\lambda} u^+_{\chi\inf}}\Big)\mu u^+_{\chi\sup} = \mu C_1(\chi) u^+_{\chi\sup}.$$

Since $(\tilde{H})$ holds, then there is $n_0 \gg 1$ such that

$$\alpha_\chi := \sup_{n\ge n_0}((b_{\inf}-\chi\mu)u^+_{\chi\sup} - \chi K_{2,n} - K_{3,n}) > 0.$$

This combined with (5.56) yield that

$$\|u(\cdot, t+t_{n_0}+t_0; t_0, u_0) - u^+_\chi(t+t_{n_0}+t_0)\|_\infty \le u^+_{\chi\sup} K_{1,n_0} e^{-t\alpha_\chi} \quad \forall\, t \ge 0,$$

whch implies that $(u^+_\chi(x,t), v^+_\chi(x,t))$ is exponentially stable.

Uniqueness of $(u^+_\chi(x,t), v^+_\chi(x,t))$**:** Let $(\tilde{u}^+_\chi(x,t), \tilde{v}^+_\chi(x,t))$ be a positive entire solution of (2.6). Then, since $0 < \tilde{u}^+_{\chi\inf} \le \tilde{u}^+_{\chi\sup} < \infty$, it follows from the exponential stability of $(u^+_\chi(x,t), v^+_\chi(x,t))$ that there is a positive constant K depending only of $\tilde{u}^+_{\chi\inf}, \tilde{u}^+_{\chi\sup}, u^+_{\chi\inf}$, and $u^+_{\chi\sup}$, such that

$$\begin{aligned}
&\|\tilde{u}^+_\chi(\cdot,t) - u^+_\chi(\cdot,t)\|_\infty \\
=&\|\tilde{u}^+_\chi(\cdot, n+(t-n); t-n, u^+_\chi(\cdot,t-n)) - u^+_\chi(\cdot, n+(t-n); t-n, u^+_\chi(\cdot,t-n))\|_\infty \\
\le& K e^{-n\alpha_\chi}, \quad \forall\, n \ge 1.
\end{aligned}$$

Letting $n \to \infty$ in the last inequality yields that $u^+_\chi(x,t) \equiv \tilde{u}^+_\chi(x,t)$. This completes the proof of Theorem 2.5. □

Chapter 6

Asymptotic spreading and traveling wave solutions

In this chapter we study spreading speeds of solutions with compactly supported initial functions, as well as the existence and non-existence of traveling wave solutions of (2.6). Section 1 is devoted to the proof of our main result on the spreading speeds. Section 2 contains results which will be used later in Section 3 for the proof of the existence of traveling wave solutions.

6.1 Asymptotic spreading

In this section, we study the spreading properties of positive solutions and prove Theorem 2.6. We first present two lemmas.

Lemma 6.1. *Consider*

$$u_t = \Delta u + q_0(x,t) \cdot \nabla u + u(a_0 - b_0 u), \quad x \in \mathbb{R}^N, \tag{6.1}$$

where $q_0 \in \mathbb{R}^N$ *is a continuous vector function and* a_0, b_0 *are positive constants. Let* $u(x,t;u_0)$ *be the solution of* (6.1) *with* $u(\cdot,0;u_0) = u_0(\cdot) \in C^b_{\rm unif}(\mathbb{R}^N)$ *(*$u_0(x) \geq 0$*). If*

$$\liminf_{|x|\to\infty} \inf_{t\geq 0} (4a_0 - |q_0(x,t)|^2) > 0, \tag{6.2}$$

then for any nonnegative initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *with nonempty support,*

$$\liminf_{t\to\infty, |x|\leq ct} u(x,t;u_0) > 0 \quad \forall\ 0 < c < c_0^*,$$

where $c_0^* = \liminf_{|x|\to\infty} \inf_{t\geq 0}(2\sqrt{a_0} - |q_0(x,t)|)$.

Proof. It follows from Theorem 1.5 in [3]. □

Next, we prove Theorem 2.6.

Proof of Theorem 2.6. (1) Let $t_0 \in \mathbb{R}$ and $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_0 \geq 0$ such that there is $R \gg 1$ with $u_0(x) = 0$ for all $\|x\| \geq R$. By (2.11), for every $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that

$$\|u(\cdot, t+t_0; t_0, u_0)\| \leq \frac{a_{\sup}}{b_{\inf} - \chi\mu} + \varepsilon, \quad \forall t \geq T_\varepsilon. \tag{6.3}$$

By (3.25) and (6.3), we have that

$$\|\nabla v(\cdot, t+t_0; t_0, u_0)\|_\infty \leq \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\big(\frac{a_{\sup}}{b_{\inf} - \chi\mu} + \varepsilon\big), \quad \forall t \geq T_\varepsilon. \tag{6.4}$$

Choose $C > 0$ such that

$$u_0(x) \leq Ce^{-\sqrt{a_{\sup}}|x|}, \quad \forall\, x \in \mathbb{R}^N,$$

and let

$$K_\varepsilon := \sup_{0 \leq t \leq T_\varepsilon} \|\nabla v(\cdot, t+t_0; t_0, u_0)\|_\infty.$$

Let $\xi \in \mathbb{S}^{N-1}$ be given and consider

$$\overline{U}(x, t; \xi) := Ce^{-\sqrt{a_{\sup}}(x\cdot\xi - (2\sqrt{a_{\sup}} + \chi K_\varepsilon)t)}.$$

Recall from inequality (3.26) that $u_t(\cdot, \cdot + t_0; t_0, u_0) \leq \mathcal{L}_0 u(\cdot, \cdot + t_0; t_0, u_0)$, where

$$\mathcal{L}_s(w) := \Delta w - \chi\nabla v(\cdot, \cdot + s + t_0; t_0, u_0)\cdot\nabla w + (a_{\sup} - (b_{\inf} - \chi\mu)w)w, \forall\, w \in C^{2,1}(\mathbb{R}^N \times (0, \infty)), \forall\, s \geq 0.$$

We have that

$$
\begin{aligned}
&\overline{U}_t - \mathcal{L}_0\overline{U}\\
&= \big((2a_{\sup} + \chi\sqrt{a_{\sup}}K_\varepsilon) - a_{\sup} + \chi\sqrt{a_{\sup}}\xi\cdot\nabla v(\cdot,\cdot+t_0;t_0,u_0) - (a_{\sup} - (b_{\inf}-\chi\mu)\overline{U})\big)\,\overline{U}\\
&= \big(\chi\sqrt{a_{\sup}}(K_\varepsilon - \xi\cdot\nabla v(\cdot,\cdot+t_0;t_0,u_0)) + (b_{\inf}-\chi\mu)\overline{U}\big)\,\overline{U}\\
&\geq \big(\chi\sqrt{a_{\sup}}(K_\varepsilon - \|\xi\cdot\nabla v(\cdot,\cdot+t_0;t_0,u_0)\|_\infty) + (b_{\inf}-\chi\mu)\overline{U}\big)\,\overline{U} \geq 0.
\end{aligned}
$$

Since $u_0(x) \leq \overline{U}(x,0;\xi)$, then it follows from the comparison principle for parabolic equations that

$$
u(x,t+t_0;t_0,u_0) \leq \overline{U}(x,t;\xi), \quad \forall\, x\in\mathbb{R}^N,\ \forall\, t\in[0,T_\varepsilon],\ \forall\, \xi\in\mathbb{S}^{N-1}. \tag{6.5}
$$

Next, let

$$
L_\varepsilon := \frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\big(\frac{a_{\sup}}{b_\infty-\chi\mu}+\varepsilon\big)
$$

and

$$
\overline{W}(x,t;\xi) := e^{-\sqrt{a_{\sup}}(\xi\cdot x-(2\sqrt{a_{\sup}}+\chi L_\varepsilon)t)}\overline{U}(0,T_\varepsilon;\xi), \quad \forall t\geq 0,\ \forall\, x\in\mathbb{R}^N,\ \forall\, \xi\in\mathbb{S}^{N-1}.
$$

Similarly, using inequality (6.4), we have that

$$
\begin{aligned}
\overline{W}_t - \mathcal{L}_{T_\varepsilon}\overline{W} &= \big(\chi\sqrt{a_{\sup}}(L_\varepsilon - \xi\cdot\nabla v(\cdot,\cdot+T_\varepsilon+t_0;t_0,u_0) + (b_{\inf}-\chi\mu)\overline{W}\big)\,\overline{W}\\
&\geq \big(\chi\sqrt{a_{\sup}}(L_\varepsilon - \|\xi\cdot\nabla v(\cdot,\cdot+T_\varepsilon+t_0;t_0,u_0)\|_\infty) + (b_{\inf}-\chi\mu)\overline{W}\big)\,\overline{W}\\
&\geq 0.
\end{aligned}
$$

But by (6.5), we have that $\overline{W}(x,0;\xi) = \overline{U}(x,T_\varepsilon;\xi) \geq u(\cdot,T_\varepsilon+t_0;t_0,u_0)$. Hence by the comparison principle for parabolic equations we obtain that

$$
u(x,t+t_0;t_0,u_0) \leq \overline{W}(x,t;\xi), \quad \forall\, x\in\mathbb{R}^N,\ \forall\, t\geq T_\varepsilon,\ \forall\, \xi\in\mathbb{S}^{N-1}. \tag{6.6}
$$

Observe that

$$\lim_{\varepsilon\to 0^+}(2\sqrt{a_{\sup}}+\chi L_\varepsilon)=2\sqrt{a_{\sup}}+\frac{\chi\mu\sqrt{N}a_{\sup}}{2(b_{\inf}-\chi\mu)\sqrt{\lambda}}=c_+^*(a,b,\chi,\lambda,\mu).$$

Thus, it follows from (6.6) and the definition of $\overline{W}$ that

$$\lim_{t\to\infty}\sup_{|x|\geq ct}u(x,t+t_0;t_0,u_0)=0,$$

whenever $c>c_+^*(a,b,\chi,\lambda,\mu)$. This completes the proof of (1).

(2) We first claim that

$$4(a_{\inf}-\frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu})-\frac{N\mu^2\chi^2a_{\sup}^2}{4\lambda(b_{\inf}-\chi\mu)^2}>0. \tag{6.7}$$

Indeed, let $\tilde{\mu}=\frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu}$. (6.7) is equivalent to $4(a_{\inf}-\tilde{\mu})-\frac{N}{4\lambda}\tilde{\mu}^2>0$. This implies that

$$0<\tilde{\mu}=\frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu}<\frac{2a_{\inf}}{1+\sqrt{1+\frac{Na_{\inf}}{4\lambda}}}$$

and then

$$\frac{b_{\inf}-\chi\mu}{\chi\mu}>\frac{\left(1+\sqrt{1+\frac{Na_{\inf}}{4\lambda}}\right)a_{\sup}}{2a_{\inf}}.$$

This proves the claim.

Next, by (2.11), (3.24), and (3.25), for every $\varepsilon>0$, we can choose T_ε with $T_\varepsilon\to\infty$ as $\varepsilon\to 0$ such that

$$\|v(\cdot,t+t_0;t_0,u_0)\|_\infty<\frac{\mu a_{\sup}}{\lambda(b_{\inf}-\chi\mu)}+\varepsilon \text{ and } \|\nabla v(\cdot,t+t_0;t_0,u_0)\|_\infty<\frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\Big(\frac{a_{\sup}}{b_{\inf}-\chi\mu}+\varepsilon\Big) \tag{6.8}$$

for all $t\geq T_\varepsilon$.

Note that for every $\varepsilon > 0$ and $t \geq T_\varepsilon + t_0$, we have

$$\begin{aligned} u_t(\cdot,\cdot;t_0,u_0) \geq{}& \Delta u(\cdot,\cdot;t_0,u_0) - \chi \nabla v(\cdot,\cdot;t_0,u_0)\cdot \nabla u(\cdot,\cdot;t_0,u_0) \\ &+ (a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu} - \chi\mu\varepsilon - (b_{\sup}-\chi\mu)u(\cdot,\cdot;t_0,u_0))u(\cdot,\cdot;t_0,u_0). \end{aligned} \tag{6.9}$$

For every $\varepsilon > 0$, let $U(\cdot,\cdot;\varepsilon)$ denotes the solution of the initial value problem

$$\begin{cases} U_t(\cdot,\cdot;\varepsilon) = \mathcal{A}_\varepsilon(U)(\cdot,\cdot;\varepsilon),\ t>0,\ x\in\mathbb{R}^N \\ U(\cdot,0;\varepsilon) = u(\cdot,T_\varepsilon+t_0;t_0), \end{cases} \tag{6.10}$$

where

$$\mathcal{A}_\varepsilon(U)(\cdot,\cdot;\varepsilon) = \Delta U(\cdot,\cdot;\varepsilon) + q(\cdot,\cdot;\varepsilon)\cdot\nabla U(\cdot,\cdot;\varepsilon) + U(\cdot,\cdot;\varepsilon)F_\varepsilon(U(\cdot,\cdot;\varepsilon)),$$

$$F_\varepsilon(s) = a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu} - \chi\mu\varepsilon - (b_{\sup}-\chi\mu)s, \quad \forall s\in\mathbb{R}.$$

and

$$q(x,t;\varepsilon) = \begin{cases} -\chi\nabla v(\cdot,t+T_\varepsilon+t_0;t_0,u_0),\ t\geq 0 \\ -\chi\nabla v(\cdot,T_\varepsilon+t_0;t_0,u_0),\ t<0. \end{cases}$$

Hence, by the comparison principle for parabolic equations, it follows from (6.9) and (6.10) that

$$u(x,t+T_\varepsilon+t_0;u_0) \geq U(x,t;\varepsilon), \quad \varepsilon>0,\ t\geq 0,\ x\in\mathbb{R}^N. \tag{6.11}$$

Observe that for $0<\varepsilon\ll 1$, since **(H3)** holds, it follows form (6.7) and (6.8) that

$$\lim_{R\to\infty}\inf_{t\geq 0,|x|\geq R}(4F_\varepsilon(0) - \|q(x,t;\varepsilon)\|^2) \geq 4F_\varepsilon(0) - \chi^2\frac{\chi^2\mu^2 N}{4\lambda}\Big(\frac{a_{\sup}}{b_{\inf}-\chi\mu}+\varepsilon\Big)^2 > 0. \tag{6.12}$$

By Lemma 6.1, it holds that

$$\liminf_{t\to\infty}\inf_{|x|\leq ct} U(x,t;\varepsilon) > 0 \tag{6.13}$$

for every $0 < \varepsilon \ll 1$ and $0 \le c < c_\varepsilon^*$ where

$$c_\varepsilon^* := \liminf_{|x|\to\infty} \inf_{t\ge T_\varepsilon} \left(2\sqrt{a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu} - \chi\mu\varepsilon} - \chi\|\nabla v(x,t+t_0;t_0,u_0)\| \right).$$

Combining inequalities (6.11) and (6.13), we obtain that

$$\liminf_{t\to\infty} \inf_{|x|\le ct} u(x,t+T_\varepsilon+t_0;t_0,u_0) > 0 \quad \forall\ 0<\varepsilon\ll 1,\ \forall\, 0\le c< c_\varepsilon^*. \tag{6.14}$$

Using (6.8), we have that

$$c_\varepsilon^* \ge 2\sqrt{a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu} - \chi\mu\varepsilon} - \chi\frac{\mu\sqrt{N}}{2\sqrt{\lambda}}\Big(\frac{a_{\sup}}{b_{\inf}-\chi\mu}+\varepsilon\Big)$$

Hence

$$\liminf_{\varepsilon\to 0^+} c_\varepsilon^* \ge 2\sqrt{a_{\inf} - \frac{\chi\mu a_{\sup}}{b_{\inf}-\chi\mu}} - \frac{\chi\mu\sqrt{N}a_{\sup}}{2\sqrt{\lambda}(b_{\inf}-\chi\mu)} := c_-^*(a,b,\chi,\lambda,\mu) \tag{6.15}$$

This together with (6.14) implies (2.27). □

6.2 Super- and sub-solutions

In this section and the next one, we take $N = 1$ and suppose that the functions $b(x,t)$ and $a(x,t)$ are both constant. We construct super- and sub-solutions of some equations related to (2.6). They will be used to prove the existence of traveling wave solutions in next the section.

We first note that $(u(x,t), v(x,t))$ is solution of (2.6) if and only if the function $(\tilde u(x,t), \tilde v(x,t)) = (\frac{a}{b}u(\frac{\sqrt{a}}{a}x, \frac{1}{a}t), \frac{\mu}{b}v(\frac{\sqrt{a}}{a}x, \frac{1}{a}t))$ solves

$$\begin{cases} \tilde u_t = \tilde u_{xx} - \tilde\chi(\tilde u\tilde v_x)_x + (1-\tilde u)\tilde u, \\ 0 = \tilde v_{xx} - \tilde\lambda\tilde v + \tilde u \end{cases}$$

where $\tilde{\chi} = \frac{\chi\mu}{b}$ and $\tilde{\lambda} = \frac{\lambda}{a}$. Hence, it is enough to prove Theorem 2.7 under the assumption $a = b = \mu = 1$. So, without loss of generality, we shall suppose that $a = b = \mu = 1$ in this section and the next section.

Observe that, under the assumption $a(x,t) \equiv b(x,t) = \mu = 1$, if $(u(x,t), v(x,t)) = (U(x - ct), V(x - ct))$ is a traveling wave solution of (2.6) connecting $(1, \frac{1}{\lambda})$ and $(0,0)$ with speed c, then $(u,v) = (U(x), V(x))$ is a stationary solution of

$$\begin{cases} u_t = u_{xx} + cu_x - \chi u_x v_x + u(1 - \chi\lambda v - (1-\chi)u), \quad x \in \mathbb{R}, \\ 0 = v_{xx} - \lambda v + u, \quad x \in \mathbb{R}, \end{cases} \tag{6.16}$$

connecting $(1, \frac{1}{\lambda})$ and $(0,0)$. For a given c, showing the existence of a traveling wave solution of (2.6) connecting $(1, \frac{1}{\lambda})$ and $(0,0)$ is then equivalent to showing the existence of a stationary solution connecting $(1, \frac{1}{\lambda})$ and $(0,0)$. Throughout this section, we assume that $0 < \chi < 1$, unless specified otherwise.

For every $0 < \tau < \min\{1, \sqrt{\lambda}\}$ and $x \in \mathbb{R}$ define

$$\varphi_\tau(x) = e^{-\tau x} \quad \text{and} \quad c_\tau = \tau + \frac{1}{\tau}.$$

Note that for every fixed $0 < \tau < \min\{1, \sqrt{\lambda}\}$, the function φ_τ is decreasing, infinitely differentiable, and it satisfies

$$\varphi_\tau''(x) + c_\tau \varphi_\tau'(x) + \varphi(x) = 0, \quad \forall\, x \in \mathbb{R}, \tag{6.17}$$

and

$$\frac{1}{\lambda - \tau^2}\varphi_\tau''(x) - \frac{\lambda}{\lambda - \tau^2}\varphi_\tau(x) = -\varphi_\tau(x) \quad \forall\, x \in \mathbb{R}. \tag{6.18}$$

For every $\tau \in (0, \min\{1, \sqrt{\lambda}\})$ define

$$U_\tau^+(x) = \min\{\frac{1}{1-\chi}, \varphi_\tau(x)\} = \begin{cases} \frac{1}{1-\chi} & \text{if } x \le \frac{\ln(1-\chi)}{\tau}, \\ e^{-\tau x} & \text{if } x \ge \frac{\ln(1-\chi)}{\tau}. \end{cases} \tag{6.19}$$

and

$$V_\tau^+(x) = \min\{\frac{1}{\lambda(1-\chi)},\ \frac{1}{\lambda-\tau^2}\varphi_\tau(x)\}. \tag{6.20}$$

Since φ_τ is decreasing, then the functions U_τ^+ and V_τ^+ are both non-increasing. Furthermore, the functions U_τ^+ and V_τ^+ belong to $C^\delta_{\rm unif}(\mathbb{R})$ for every $0\le\delta<1$ and $0<\tau<1$.

Let $0<\tau<1$ be fixed. Next, let $\tau<\tilde\tau<\min\{1,2\tau\}$ and $d>1$. The function $\varphi_\tau - d\varphi_{\tilde\tau}$ achieves its maximum value at $\bar a_{\tau,\tilde\tau,d} := \frac{\ln(d\tilde\tau)-\ln(\tau)}{\tilde\tau-\tau}$ and takes the value zero at $\underline{a}_{\tau,\tilde\tau,d} := \frac{\ln(d)}{\tilde\tau-\tau}$.

Define

$$U_\tau^-(x) := \max\{0,\varphi_\tau(x)-d\varphi_{\tilde\tau}(x)\} = \begin{cases} 0 & \text{if } x\le \underline{a}_{\tau,\tilde\tau,d} \\ \varphi_\tau(x)-d\varphi_{\tilde\tau}(x) & \text{if } x\ge \underline{a}_{\tau,\tilde\tau,d}. \end{cases} \tag{6.21}$$

Clearly, $0\le U_\tau^-\le U_\tau^+\le\frac{1}{1-\chi}$ and $U_\tau^-\in C^\delta_{\rm unif}(\mathbb{R})$ for every $0\le\delta<1$.

Let us consider the set $\mathcal{E}_\tau$ defined by

$$\mathcal{E}_\tau = \{u\in C^b_{\rm unif}(\mathbb{R})\,|\,U_\tau^-\le u\le U_\tau^+\} \tag{6.22}$$

for every $0<\tau<1$. It should be noted that U_τ^- and $\mathcal{E}_\tau$ all depend on $\tilde\tau$ and d. Later on, we shall provide more information on how to choose d and $\tilde\tau$ whenever τ is given.

For every $u\in C^b_{\rm unif}(\mathbb{R})$, consider

$$U_t = U_{xx} + (c_\tau - \chi V'(x;u))U_x + (1-\chi\lambda V(x;u)-(1-\chi)U)U,\quad x\in\mathbb{R},\ t>0, \tag{6.23}$$

where

$$V(x;u) = \int_0^\infty\int_{\mathbb{R}} \frac{e^{-\lambda s}}{\sqrt{4\pi s}} e^{-\frac{|x-z|^2}{4s}} u(z)dzds. \tag{6.24}$$

It is well known that the function $V(x;u)$ is the solution of the second equation of (2.6) in $C^b_{\rm unif}(\mathbb{R})$ with given $u\in C^b_{\rm unif}(\mathbb{R})$.

For given open intervals $D \subset \mathbb{R}$ and $I \subset \mathbb{R}$, a function $U(\cdot,\cdot) \in C^{2,1}(D \times I, \mathbb{R})$ is called a *super-solution* or *sub-solution* of (6.23) on $D \times I$

$$U_t \geq U_{xx} + (c_\tau - \chi V'(x;u))U_x + (1 - \chi\lambda V(x;u) - (1-\chi)U)U \quad \text{for } x \in D, \;\; t \in I$$

or

$$U_t \leq U_{xx} + (c_\tau - \chi V'(x;u))U_x + (1 - \chi\lambda V(x;u) - (1-\chi)U)U \quad \text{for } x \in D, \;\; t \in I,$$

respectively.

Theorem 6.1. *Suppose that* $0 < \chi < \frac{1}{2}$ *and* $0 < \tau < \min\{1, \sqrt{\lambda}\}$ *satisfy*

$$\frac{\tau(\tau + \sqrt{\lambda - \tau^2})}{\lambda - \tau^2} \leq \frac{1-\chi}{\chi}. \tag{6.25}$$

Then for every $u \in \mathcal{E}_\tau$*, the following hold.*

(1) $U(x,t) = \frac{1}{1-\chi}$ *and* $U(x,t) = \varphi_\tau(x)$ *are super-solutions of* (6.23) *on* $\mathbb{R} \times \mathbb{R}$.

(2) There is $d_0 > 0$ *such that* $U(x,t) = U_\tau^-(x)$ *is a sub-solution of* (6.23) *on* $(\underline{a}_{\tau,\tilde{\tau},d}, \infty) \times \mathbb{R}$ *for all* $d \geq d_0$ *and* $\tau < \tilde{\tau} < \min\{1, 2\tau, \tau + \frac{\lambda}{\tau + \sqrt{\lambda - \tau^2}}\}$*. Moreover,* $U(x,t) = U_\tau^-(x_\delta)$ *is a sub-solution of* (6.23) *on* $\mathbb{R} \times \mathbb{R}$ *for* $0 < \delta \ll 1$*, where* $x_\delta = \underline{a}_{\tau,\tilde{\tau},d} + \delta$.

We recall from Lemma 3.2 that

$$\max\{\|V(\cdot;u)\|_\infty, \; \|V'(\cdot;u)\|_\infty, \|V''(\cdot;u)\|_\infty\} \leq \max\{1, \frac{1}{\lambda}\}\|u\|_\infty \quad \forall \, u \in \mathcal{E}_\tau. \tag{6.26}$$

The next lemma provides a pointwise estimate for $|V(\cdot;u)|$ with $u \in \mathcal{E}_\tau$.

Lemma 6.2. *For every* $0 < \tau < \min\{1, \sqrt{\lambda}\}$ *and* $u \in \mathcal{E}_\tau$*, let* $V(\cdot;u)$ *be defined as in* (6.24)*, then*

$$0 \leq V(\cdot;u) \leq V_\tau^+(\cdot). \tag{6.27}$$

Proof. For every $u \in \mathcal{E}_\tau$, since $0 \le U_\tau^- \le u \le U_\tau^+$ then

$$0 \le V(\cdot; U_\tau^-) \le V(\cdot; u) \le V(\cdot; U_\tau^+).$$

Hence it is enough to prove that $V(\cdot; U_\tau^+) \le V_\tau^+(\cdot)$. For every $x \in \mathbb{R}$, $0 < \tau < 1$, we have that

$$\begin{aligned}
&\int_0^\infty \Big(\int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|x-z|^2}{4s}} \varphi_\tau(z)}{\sqrt{4\pi s}} dz\Big) ds \\
=&\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\lambda s}\Big(\int_{\mathbb{R}} e^{-z^2} e^{-\tau(x-2\sqrt{s}z)} dz\Big) ds = \frac{e^{-\tau x}}{\sqrt{\pi}} \int_0^\infty e^{-\lambda s}\Big(\int_{\mathbb{R}} e^{-|z-\tau\sqrt{s}|^2} e^{\tau^2 s} dz\Big) ds \\
=&\frac{e^{-\tau x}}{\sqrt{\pi}} \int_0^\infty e^{-(\lambda-\tau^2)s}\Big(\int_{\mathbb{R}} e^{-|z-\tau\sqrt{s}|^2} dz\Big) ds = e^{-\tau x} \int_0^\infty e^{-(\lambda-\tau^2)s} ds = \frac{\varphi_\tau(x)}{\lambda - \tau^2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&V(x; U_\mu^+) \\
=&\int_0^\infty \Big(\int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|x-z|^2}{4s}} U_\tau^+(z)}{\sqrt{4\pi s}} dz\Big) ds = \int_0^\infty \Big(\int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|x-z|^2}{4s}}}{\sqrt{4\pi s}} \min\{\frac{1}{1-\chi}\ ,\ \varphi_\tau(z)\} dz\Big) ds \\
\le&\min\Big\{\frac{1}{1-\chi} \underbrace{\int_0^\infty \int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|x-z|^2}{4s}}}{\sqrt{4\pi s}} dz ds}_{=\frac{1}{\lambda}}\ ,\ \int_0^\infty \Big(\int_{\mathbb{R}} \frac{e^{-\lambda s} e^{-\frac{|x-z|^2}{4s}}}{\sqrt{4\pi s}} \varphi_\tau(z) dz\Big) ds\Big\} = V_\tau^+(x).
\end{aligned}$$

□

Next, we present a pointwise estimate for $|V'(\cdot; u)|$ with $u \in \mathcal{E}_\tau$.

Lemma 6.3. *Let* $u \in C^b_{\text{unif}}(\mathbb{R})$ *and* $V(\cdot; u) \in C^{2,b}_{\text{unif}}(\mathbb{R})$ *be the corresponding function satisfying the second equation in* (2.6)*. Then*

$$|V'(x; u)| \le \frac{\tau + \sqrt{\lambda - \tau^2}}{\lambda - \tau^2} \varphi_\tau(x) \tag{6.28}$$

for every $x \in \mathbb{R}$ *and every* $u \in \mathcal{E}_\tau$ *and* $0 < \tau < \min\{1, \sqrt{\lambda}\}$*.*

Proof. Let $u \in \mathcal{E}_\tau$ and fix any $x \in \mathbb{R}$.

$$V'(x;u) = \int_0^\infty \int_{\mathbb{R}} \frac{(z-x)e^{-\lambda s}}{2s\sqrt{4\pi s}} e^{-\frac{|z-x|^2}{4s}} u(z)dzds = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} \frac{ze^{-\lambda s}}{\sqrt{s}} e^{-z^2} u(x+2\sqrt{s}z)dzds. \tag{6.29}$$

Observe that

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} \frac{|z|}{\sqrt{s}} e^{-\lambda s} e^{-|z|^2} \varphi_\tau(x+2\sqrt{s}z)dzds \\
\le& \frac{\varphi_\tau(x)}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(\lambda-\tau^2)s}}{\sqrt{s}} \Big(\int_{\mathbb{R}} \frac{|z|}{e^{|z-\tau\sqrt{s}|^2}} dz \Big) ds = \frac{\varphi_\tau(x)}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(\lambda-\tau^2)s}}{\sqrt{s}} \Big(\int_{\mathbb{R}} \frac{|z+\tau\sqrt{s}|}{e^{|z|^2}} dz \Big) ds \\
\le& \frac{\varphi_\tau(x)}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(\lambda-\tau^2)s}}{\sqrt{s}} \Big(\int_{\mathbb{R}} (|z| + \tau\sqrt{s}) e^{-|z|^2} dz \Big) ds \\
=& \frac{\varphi_\mu(x)}{\sqrt{\pi}} \int_0^\infty \frac{(1+\mu\sqrt{\pi s})e^{-(1-\mu^2)s}}{\sqrt{s}} ds = \Big(\frac{1}{\sqrt{\lambda-\tau^2}} + \frac{\tau}{\lambda-\mu^2}\Big)\varphi_\tau(x).
\end{aligned} \tag{6.30}$$

Since $u \le \varphi_\tau$, (6.28) follows from (6.29) and (6.30). The lemma is thus proved. □

Proof of Theorem 6.1. For every $U \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+)$, let

$$\mathcal{L}U = U_{xx} + (c_\mu - \chi V'(\cdot;u))U_x + (1 - \chi V(\cdot;u) - (1-\chi)U)U. \tag{6.31}$$

(1) First, we have that

$$\mathcal{L}\Big(\frac{1}{1-\chi}\Big) = (1 - \chi\lambda V(\cdot;u) - 1)\frac{1}{1-\chi} = -\frac{\chi}{1-\chi}\lambda V(\cdot;u) \le 0.$$

Hence $U(x,t) = \frac{1}{1-\chi}$ is a super-solution of (6.23) on $\mathbb{R} \times \mathbb{R}$.

Next, it follows from Lemma 6.3 and (6.25) that

$$\begin{aligned}\mathcal{L}(\varphi_\tau) =&\varphi_\tau''(x) + (c_\tau - \chi V'(\cdot;u))\varphi_\tau'(x) + (1 - \chi\lambda V(\cdot;u) - (1-\chi)\varphi_\tau)\varphi_\tau \\ =&\underbrace{(\varphi_\tau'' + c_\tau\varphi_\tau' + \varphi_\tau)}_{=0} + (\tau\chi V'(\cdot;u) - \chi\lambda V(\cdot;u) - (1-\chi)\varphi_\tau)\varphi_\tau \\ =&(\tau\chi V'(\cdot;u) - \chi V(\cdot;u) - (1-\chi)\varphi_\tau)\varphi_\tau \\ \le&\chi\Big(\frac{\tau(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2} - \frac{(1-\chi)}{\chi}\Big)\varphi_\tau^2 \le 0.\end{aligned} \tag{6.32}$$

Hence $U(x,t) = \varphi_\tau(x)$ is also a super-solution of (6.23) on $\mathbb{R}\times\mathbb{R}$.

(2) Let $O = (\underline{a}_{\tau,\tilde{\tau},d}, \infty)$. Then for $x \in O$, $U_\mu^-(x) > 0$. For $x \in O$, it follows from inequality (6.28) that

$$\begin{aligned}&\mathcal{L}U_\tau^- \\ =&\mu^2\varphi_\tau - \tilde{\tau}^2 d\varphi_{\tilde{\mu}} + (c_\tau - \chi V'(\cdot;u))(-\tau\varphi_\tau + d\tilde{\tau}\varphi_{\tilde{\tau}}) + (1-\chi\lambda V(\cdot;u) - (1-\chi)U_\tau^-)U_\tau^- \\ =&\underbrace{(\tau^2 - \tau c_\tau + 1)}_{=0}\varphi_\tau + d\underbrace{(\tilde{\tau}c_\tau - \tilde{\tau}^2 - 1)}_{=A_0}\varphi_{\tilde{\tau}} - \chi V'(\cdot;u)(-\tau\varphi_\tau + d\tilde{\tau}\varphi_{\tilde{\tau}}) \\ &-(\chi\lambda V + (1-\chi)U_\tau^-)U_\tau^- \\ \ge& dA_0\varphi_{\tilde{\tau}} - \chi|V'(\cdot;u)|(\tau\varphi_\tau + d\tilde{\tau}\varphi_{\tilde{\tau}}) - \chi\lambda V_\tau^+ U_\tau^- - (1-\chi)[U_\tau^-]^2 \\ \ge& dA_0\varphi_{\tilde{\tau}} - \chi\frac{(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2}\Big(\tau\varphi_\tau + d\tilde{\tau}\varphi_{\tilde{\tau}}\Big)\varphi_\tau - \chi\lambda V_\tau^+ U_\tau^- - (1-\chi)[U_\tau^-]^2 \\ \ge& dA_0\varphi_{\tilde{\tau}} - \chi\frac{(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2}\Big(\tau\varphi_\tau + d\tilde{\tau}\varphi_{\tilde{\tau}}\Big)\varphi_\tau - \frac{\chi\lambda}{\lambda-\tau^2}\varphi_\tau U_\tau^- - (1-\chi)[U_\tau^-]^2 \\ =& dA_0\varphi_{\tilde{\tau}} - \underbrace{(\chi\frac{\tau(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2} + \frac{\chi\lambda}{\lambda-\tau^2} + 1 - \chi)}_{=A_1}\varphi_\tau^2 \\ &+ d\Big(2(1-\chi) - \chi\frac{\tilde{\tau}(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2} + \frac{\chi\lambda}{\lambda-\tau^2}\Big)\varphi_\tau\varphi_{\tilde{\tau}} - d^2(1-\chi)\varphi_{\tilde{\tau}}^2.\end{aligned}$$

Note that $U_\tau^-(x) > 0$ is equivalent to $\varphi_\tau(x) > d\varphi_{\tilde{\tau}}(x)$, which is also equivalent to

$$d(1-\chi)\varphi_\tau(x)\varphi_{\tilde{\tau}}(x) > d^2(1-\chi)\varphi_{\tilde{\tau}}^2(x).$$

Since $A_1 > 0$, thus for $x \in O$, we have

$$
\begin{aligned}
&\mathcal{L}U_\tau^-(x)\\
\geq& dA_0\varphi_{\tilde{\tau}}(x) - A_1\varphi_\tau^2(x) + d\underbrace{\Big((1-\chi) - \chi\frac{\tilde{\tau}(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2} + \frac{\chi\lambda}{\lambda-\tau^2}\Big)}_{A_2}\varphi_\tau(x)\varphi_{\tilde{\tau}}(x)\\
=& A_1\Big(\frac{dA_0}{A_1}e^{(2\tau-\tilde{\tau})x} - 1\Big)\varphi_\tau^2(x) + dA_2\varphi_\tau(x)\varphi_{\tilde{\tau}}(x).
\end{aligned}
$$

Note also that, by (6.25),

$$
\begin{aligned}
A_2 =& \chi\Big(\frac{1-\chi}{\chi} - \frac{\tau(\tau+\sqrt{\lambda-\tau^2})}{\lambda-\tau^2}\Big) + \frac{\chi}{\lambda-\tau^2}\Big(\lambda - (\tilde{\tau}-\tau)(\tau+\sqrt{\lambda-\tau^2})\Big)\\
\geq& \frac{\chi}{\lambda-\tau^2}\Big(\lambda - (\tilde{\tau}-\tau)(\tau+\sqrt{\lambda-\tau^2})\Big) \geq 0,
\end{aligned}
\tag{6.33}
$$

whenever $\tilde{\tau} \leq \tau + \frac{\lambda}{\tau+\sqrt{\lambda-\tau^2}}$. Observe that

$$
A_0 = \frac{(\tilde{\mu}-\mu)(1-\mu\tilde{\mu})}{\mu} > 0.
$$

Furthermore, we have that $U_\mu^-(x) > 0$ implies that $x > 0$ for $d \geq 1$. Thus, for every $d \geq d_0 := \max\{1, \frac{A_1}{A_0}\}$, we have that

$$
\mathcal{L}U_\tau^-(x) > 0 \tag{6.34}
$$

whenever $x \in O$ and $\tilde{\tau} \leq \min\{2\tau, \tau + \frac{\lambda}{\tau+\sqrt{\lambda-\tau^2}}\}$. Hence $U(x,t) = U_\tau^-(x)$ is a sub-solution of (6.23) on $(\underline{a}_{\tau,\tilde{\tau},d}, \infty) \times \mathbb{R}$.

Note that for $0 < \delta \ll 1$,

$$
(1 - \chi\lambda V(x_\delta; u) - (1-\chi)U_\tau^-(x_\delta))U_\tau^-(x_\delta) \geq (1 - \frac{\chi}{1-\chi} - (1-\chi)U_\tau^-(x_\delta))U_\tau^-(x_\delta) > 0 \quad \forall\, x \in \mathbb{R},
$$

whenever $0 < \chi < \frac{1}{2}$, where $x_\delta = \underline{a}_{\tau,\tilde{\tau},d} + \delta$. This implies that $U(x,t) = U_\tau^-(x_\delta)$ is a sub-solution of (6.23) on $\mathbb{R} \times \mathbb{R}$. □

6.3 Traveling wave solutions

In this section, we investigate the existence of traveling wave solutions of (2.6) connecting $(1, \frac{1}{\lambda})$ and $(0, 0)$ and prove Theorem 2.7 (i). We first prove the following theorem and then prove Theorem 2.7 (i).

Theorem 6.2. *Suppose that* $0 < \tau < \min\{1, \sqrt{\lambda}\}$ *and* $0 < \chi < \frac{1}{2}$ *satisfy* (6.25). *Let* $c_\tau = \tau + \frac{1}{\tau}$. *Then* (2.6) *has a traveling wave solution* $(u(x, t), v(x, t)) = (U(x - c_\tau t), V(x - c_\tau t))$ *satisfying*

$$\lim_{x \to -\infty} U(x) = 1, \quad \lim_{x \to \infty} \frac{U(x)}{e^{-\tau x}} = 1.$$

Our key idea to the proof of the above theorem is to prove that, for any $\tau > 0$ and $0 < \chi < \frac{1}{2}$ satisfying (6.25), there is $u^*(\cdot) \in \mathcal{E}_\tau$ such that $U = u^*(\cdot)$ is a stationary solution of (6.23) and $u^*(-\infty) = 1$ and $u^*(\infty) = 0$, which implies that $(u(x, t), v(x, t)) = (u^*(x - c_\tau t), V(x - c_\tau t; u^*))$ is a traveling wave solution of (2.6) connecting $(1, \frac{1}{\lambda})$ and $(0, 0)$.

In order to prove Theorem 6.2, we first prove some lemmas. Fix $u \in \mathcal{E}_\tau$. For given $u_0 \in C^b_{\rm unif}(\mathbb{R})$, let $U(x, t; u_0)$ be the solution of (6.23) with $U(x, 0; u_0) = u_0(x)$. By the arguments in the proof of Theorem 2.2, we have $U(x, t; U^+_\tau)$ exists for all $t > 0$ and $U(\cdot, \cdot; U^+_\tau) \in C([0, \infty), C^b_{\rm unif}(\mathbb{R})) \cap C^1((0\ ,\ \infty), C^b_{\rm unif}(\mathbb{R})) \cap C^{2,1}(\mathbb{R} \times (0,\ \infty))$ satisfying

$$U(\cdot, \cdot; U^+_\tau), U_x(\cdot, \cdot; U^+_\tau), U_{xx}(\cdot, t; U^+_\tau), U_t(\cdot, \cdot; U^+_\tau) \in C^\theta((0, \infty), C^\nu_{\rm unif}(\mathbb{R})) \qquad (6.35)$$

for $0 < \theta, \nu \ll 1$.

Lemma 6.4. *Assume that* $0 < \tau < \min\{1, \sqrt{\lambda}\}$, *and* $0 < \chi < 1$ *satisfy* (6.25). *Then for every* $u \in \mathcal{E}_\tau$, *the following hold.*

(i) $0 \le U(\cdot, t; U^+_\tau) \le U^+_\tau(\cdot)$ *for every* $t \ge 0$.

(ii) $U(\cdot, t_2; U^+_\tau) \le U(\cdot, t_1; U^+_\tau)$ *for every* $0 \le t_1 \le t_2$

Proof. (i) Note that $U_\tau^+(\cdot) \leq \frac{1}{1-\chi}$. Then by the comparison principle for parabolic equations and Theorem 6.1(1), we have

$$U(x,t;U_\tau^+) \leq \frac{1}{1-\chi} \quad \forall\, x \in \mathbb{R},\ t \geq 0.$$

Similarly, note that $U_\tau^+(x) \leq \varphi_\tau(x)$. Then by the comparison principle for parabolic equations and Theorem 6.1(1) again, we have

$$U(x,t;U_\tau^+) \leq \varphi_\tau(x) \quad \forall\, x \in \mathbb{R}\ t \geq 0.$$

Thus $U(\cdot,t;U_\tau^+) \leq U_\tau^+$. This completes of (i).

(ii) For $0 \leq t_1 \leq t_2$, since

$$U(\cdot,t_2;U_\tau^+) = U(\cdot,t_1,U(\cdot,t_2-t_1;U_\tau^+))$$

and by (i), $U(\cdot,t_2-t_1;U_\tau^+) \leq U_\tau^+$, (ii) follows from comparison principle for parabolic equations. □

Let us define $U(x;u)$ to be

$$U(x;u) = \lim_{t\to\infty} U(x,t;U_\tau^+) = \inf_{t>0} U(x,t;U_\tau^+). \tag{6.36}$$

By the a priori estimates for parabolic equations, the limit in (6.36) is uniform in x on compact subsets of $\mathbb{R}$ and $U(\cdot;u) \in C^b_{\rm unif}(\mathbb{R})$. We shall provide sufficient hypothesis on the choice of d to guarantee that the function $U(\cdot;u)$ constructed above is not identically zero for each $u \in \mathcal{E}_\tau$. Now, we are ready to prove that the function $u \in \mathcal{E}_\tau \to U(\cdot;u) \in \mathcal{E}_\tau$ for d large enough.

Lemma 6.5. *For every* $0 < \chi < \frac{1}{2}$, $0 < \tau < \tilde{\tau} < \min\{1, 2\tau, \tau + \frac{\lambda}{\tau+\sqrt{\lambda-\tau^2}}\}$, *there is* $d_0 > 1$ *such that*

$$U(x;u) \geq \begin{cases} U_\tau^-(x), & x \geq \underline{a}_{\tau,\tilde{\tau},d} \\ U_\tau^-(x_\delta), & x \leq x_\delta = \underline{a}_{\tau,\tilde{\tau},d} + \delta \end{cases} \tag{6.37}$$

for every $u \in \mathcal{E}_\tau$, $t \geq 0$, and $0 < \delta \ll 1$, whenever $d \geq d_0$.

Proof. Let $u \in \mathcal{E}_\tau$ be fixed. Let $O = (\underline{a}_{\tau,\tilde{\tau},d}, \infty)$. Note that $U_\tau^-(\underline{a}_{\tau,\tilde{\tau},d}) = 0$. By Theorem 6.1(2), $U_\tau^-(x)$ is a sub-solution of (6.23) on $O \times (0, \infty)$ for $d \geq d_0$. Note also that $U_\tau^+(x) \geq U_\tau^-(x)$ for $x \geq \underline{a}_{\tau,\tilde{\tau},d}$ and $U(\underline{a}_{\tau,\tilde{\tau},d}, t; U_\tau^+)) > 0$ for all $t \geq 0$. Then by the comparison principle for parabolic equations, we have that

$$U(x, t; U_\tau^+) \geq U_\tau^-(x) \quad \forall\, x \geq \underline{a}_{\tau,\tilde{\tau},d},\ t \geq 0$$

for $d \geq d_0$.

Now for any $0 < \delta \ll 1$, by Theorem 6.1(2), $U(x,t) = U_\tau^-(x_\delta)$ is a sub-solution of (6.23) on $\mathbb{R} \times R$. Note that $U_\tau^+(x) \geq U_\tau^-(x_\delta)$ for $x \leq x_\delta$ and $U(x_\delta, t; U_\tau^+) \geq U_\tau^-(x_\delta)$ for $t \geq 0$. Then by the comparison principle for parabolic equations again,

$$U(x, t; U_\tau^+) \geq U_\tau^-(x_\delta) \quad \forall\, x \leq x_\delta,\ t > 0.$$

The lemma then follows. $\square$

Remark 6.1. *It follows that under the assumptions of Lemmas 6.4 and 6.5*

$$U_{\tau,\delta}^-(\cdot) \leq U(\cdot, t; U_\tau^+) \leq U_\tau^+(\cdot)$$

for every $u \in \mathcal{E}_\tau$, $t \geq 0$ and $0 \leq \delta \ll 1$, where

$$U_{\tau,\delta}^-(x) = \begin{cases} U_\tau^-(x), & x \geq \underline{a}_{\tau,\tilde{\tau},d} + \delta \\ U_\tau^-(x_\delta), & x \leq x_\delta = \underline{a}_{\tau,\tilde{\tau},d} + \delta. \end{cases}$$

This implies that

$$U_{\tau,\delta}^-(\cdot) \leq U(\cdot; u) \leq U_\tau^+(\cdot)$$

for every $u \in \mathcal{E}_\tau$. Hence $u \in \mathcal{E}_\tau \mapsto U(\cdot; u) \in \mathcal{E}_\tau$.

From now on, we suppose that $0 < \tau < \min\{1, \sqrt{\lambda}\}$, and $0 < \chi < 1$ are fixed and satisfy inequality (6.25). Next choose $\tilde{\tau}$ such that

$$\tau < \tilde{\tau} < \min\{1, 2\tau, \tau + \frac{1}{\tau + \sqrt{\lambda - \tau^2}}\},$$

and take $d \geq d_0$, where d_0 is given by Lemma 6.5. We have the following important result.

Lemma 6.6. *Assume that* $0 < \mu, \chi < 1$ *satisfy* (6.25). *Then for every* $u \in \mathcal{E}_\tau$ *the associated function* $U(\cdot; u)$ *satisfies the elliptic equation,*

$$0 = U_{xx} + (c_\tau - \chi V'(x;u))U_x + (1 - \chi\lambda V(x;u) - (1-\chi)U)U \quad \forall\, x \in \mathbb{R}. \tag{6.38}$$

Proof. Let $\{t_n\}_{n\geq 1}$ be an increasing sequence of positive real numbers converging to ∞. For every $n \geq 1$, define $U_n(x,t) = U(x, t+t_n; u)$ for every $x \in \mathbb{R},\ t \geq 0$. For every n, U_n solves the PDE

$$\begin{cases} \partial_t U_n = \partial_{xx} U_n + (c_\tau - \chi V'(x;u))\partial_x U_n + (1 - \chi\lambda V(x;u) - (1-\chi)U_n)U_n, \ x \in \mathbb{R},\ t > 0, \\ U_n(\cdot, 0) = U(\cdot, t_n; u). \end{cases}$$

Let $\{T(t)\}_{t\geq 0}$ be the analytic semigroup on $C^b_{\rm unif}(\mathbb{R})$ generated by $\Delta - I$ and let $X^\beta = {\rm Dom}((I-\Delta)^\beta)$ be the fractional power spaces of $I - \Delta$ on $C^b_{\rm unif}(\mathbb{R})$ ($\beta \in [0,1]$).

The variation of constant formula and the fact that $V''(x;u) - \lambda V(x;u) = -u(x)$ yield that

$$\begin{aligned} U(\cdot, t; u) = &\underbrace{T(t)U^+_\tau}_{I_1(t)} + \underbrace{\int_0^t T(t-s)(((c_\tau - \chi V'(\cdot;u))U)_x) ds}_{I_2(t)} \\ &+ \underbrace{\int_0^t T(t-s)(2 - \chi u)U(\cdot, s; u) ds}_{I_3(t)} - (1-\chi) \underbrace{\int_0^t T(t-s)U^2(\cdot, s; u) ds}_{I_4(t)}. \end{aligned}$$

Let $0 < \beta < \frac{1}{2}$ be fixed. We have that

$$\|I_1(t)\|_{X^\beta} \leq C_\beta t^{-\beta} e^{-t} \|U^+_\mu\|_\infty = \frac{C}{1-\chi} t^{-\beta} e^{-t}.$$

Next, using inequality (3.1) , we have that

$$\begin{aligned}\|I_2(t)\|_{X^\beta} \leq & C_\beta\int_0^t (t-s)^{-\frac{1}{2}-\beta}e^{-(t-s)}\|(c_\tau-\chi V'(\cdot;u))U(\cdot,s;u)\|_\infty \\ \leq & \frac{C_\beta}{1-\chi}(c_\tau+\frac{\chi}{\lambda(1-\chi)})\int_0^t \frac{e^{-(t-s)}}{(t-s)^{\beta+\frac{1}{2}}}ds \leq \frac{C_\beta}{1-\chi}(c_\tau+\frac{\chi}{\lambda(1-\chi)})\Gamma(\frac{1}{2}-\beta).\end{aligned}$$

And

$$\begin{aligned}\|I_3(t)\|_{X^\beta} \leq & C_\beta\int_0^t (t-s)^{-\beta}e^{-(t-s)}\|(2-\chi u)U(\cdot,s;u)\|_\infty ds \\ \leq & \frac{C_\beta}{1-\chi}(2+\frac{\chi}{1-\chi})\int_0^t (t-s)^{-\beta}e^{-(t-s)}ds \leq \frac{C_\beta}{1-\chi}(2+\frac{\chi}{1-\chi})\Gamma(1-\beta).\end{aligned}$$

Similar arguments yield that

$$\|I_4(t)\|_{X^\beta} \leq \frac{C_\beta}{(1-\chi)^2}\Gamma(1-\beta).$$

Therefore, for every $T>0$ we have that

$$\sup_{t\geq T}\|U(\cdot,t;u)\|_{X^\beta} \leq M_T < \infty, \tag{6.39}$$

where

$$M_T = \frac{C_\beta}{1-\chi}\Big[T^{-\beta}e^{-T} + (c_\tau+\frac{1}{\lambda(1-\chi)})(2\Gamma(1-\beta)+\Gamma(\frac{1}{2}-\beta))\Big]. \tag{6.40}$$

Hence it follows that

$$\sup_{n\geq 1, t\geq 0}\|U_n(\cdot,t)\|_{X^\beta} \leq M_{t_1} < \infty. \tag{6.41}$$

Next, for every $t, h \geq 0$ and $n\geq 1$, we have that

$$\|I_1(t+h+t_n) - I_1(t+t_n)\|_{X^\beta} \leq \frac{C_\beta h^\beta e^{-(t+t_n)}}{(t+t_n)^\beta}\|U_\tau^+\|_\infty \leq C_\beta h^\beta t_1^{-\beta}e^{-t_1}\|U_\tau^+\|_\infty, \tag{6.42}$$

$$
\begin{aligned}
&\|I_2(t+h+t_n)-I_2(t+t_n)\|_{X^\beta}\\
&\le \int_0^{t+t_n}\|(T(h)-I)T(t+t_n-s)(((c_\tau-\chi V'(\cdot,s;u))U(\cdot,s;u))_x)\|_{X^\beta}ds\\
&\ +\int_{t+t_n}^{t+t_n+h}\|T(t+t_n+h-s)(((c_\tau-\chi V'(\cdot,s;u))U(\cdot,s;u))_x)\|_{X^\beta}ds\\
&\le C_\beta h^\beta\int_0^{t+t_n}(t+t_n-s)^{-\beta-\frac{1}{2}}e^{-(t+t_n-s)}\|(c_\tau-\chi V'(\cdot,s;u))U(\cdot,s;u)\|_\infty ds\\
&\ +C_\beta\int_{t+t_n}^{t+t_n+h}\frac{e^{-(t+t_n+h-s)}\|(c_\tau-\chi V'(\cdot,s;u))U(\cdot,s;u)\|_\infty}{(t+t_n+h-s)^{\beta+\frac{1}{2}}}ds\\
&\le\frac{C_\beta}{1-\chi}(c_\tau+\frac{\chi}{\lambda(1-\chi)})\Big[h^\beta\Gamma(\frac{1}{2}-\beta)+\int_{t+t_n}^{t+t_n+h}\frac{e^{-(t+t_n+h-s)}}{(t+t_n+h-s)^{\beta+\frac{1}{2}}}ds\Big]\\
&\le\frac{C_\beta}{1-\chi}(c_\tau+\frac{\chi}{\lambda(1-\chi)})\Big[h^\beta\Gamma(\frac{1}{2}-\beta)+\frac{h^{\frac{1}{2}-\beta}}{\frac{1}{2}-\beta}\Big],
\end{aligned}
\tag{6.43}
$$

and

$$
\begin{aligned}
&\|I_3(t+t_n+h)-I_3(t+t_n)\|_{X^\beta}\\
&\le\int_0^{t+t_n}\|(T(h)-I)T(t+t_n-s)((2-\chi u)U(\cdot,s;u))\|_{X^\beta}ds\\
&\ +\int_{t+t_n}^{t+t_n+h}\|T(t+t_n+h-s)((2-\chi u)U(\cdot,s;u))\|_{X^\beta}ds\\
&\le\frac{C_\beta}{1-\chi}(2+\frac{\chi}{1-\chi})\Big[h^\beta\Gamma(1-\beta)+\frac{h^{1-\beta}}{1-\beta}\Big],
\end{aligned}
\tag{6.44}
$$

and

$$
\begin{aligned}
&\|I_4(t+t_n+h)-I_4(t+t_n)\|_{X^\beta}\\
&\le\int_0^{t+t_n}\|(T(h)-I)T(t+t_n-s)U^2(\cdot,s;u)\|_{X^\beta}ds\\
&\ +\int_{t+t_n}^{t+t_n+h}\|T(t+t_n+h-s)U^2(\cdot,s;u)\|_{X^\beta}ds\\
&\le\frac{C_\beta}{(1-\chi)^2}\Big[h^\beta\Gamma(1-\beta)+\frac{h^{1-\beta}}{1-\beta}\Big].
\end{aligned}
\tag{6.45}
$$

It follows from inequalities (6.41), (6.42), (6.43), (6.44) and (6.45), the functions $U_n:[0,\infty)\to X^\beta$ are uniformly bounded and equicontinuous. Since X^β is continuously imbedded in $C^\nu(\mathbb{R})$ for every $0\le\nu<2\beta$ (see [16]), therefore, the Arzelá-Ascoli Theorem and Theorem 3.15 in [13], imply that there is a function $\tilde{U}(\cdot,\cdot;u)\in C^{2,1}(\mathbb{R}\times(0,\infty))$ and a subsequence $\{U_{n'}\}_{n\ge1}$

of $\{U_n\}_{n\geq 1}$ such that $U_{n'} \to \tilde{U}$ in $C^{2,1}_{loc}(\mathbb{R}\times(0,\infty))$ as $n\to\infty$ and $\tilde{U}(\cdot,\cdot;u)$ solves the PDE

$$\begin{cases} \partial_t\tilde{U} = \partial_{xx}\tilde{U} + (c_\tau - \chi V'(x;u))\partial_x\tilde{U} + (1-\chi\lambda V(x;u) - (1-\chi)\tilde{U})\tilde{U} \;\; x\in\mathbb{R}\,,\; t>0 \\ \tilde{U}(x,0) = \lim_{n\to\infty} U(x,t_{n'};u). \end{cases}$$

But $U(x;u) = \lim_{t\to\infty} U(x,t;u)$ and $t_{n'} \to \infty$ as $n\to\infty$, hence $\tilde{U}(x,t;u) = U(x;u)$ for every $x\in\mathbb{R},\ t\geq 0$. Hence $U(\cdot;u)$ solves (6.38). □

Lemma 6.7. *Assume that* $0<\mu<1$ *and* $0<\chi<\frac{1}{2}$ *satisfy* (6.25). *Then, for any given* $u\in\mathcal{E}_\tau$*,* (6.38) *has a unique bounded non-negative solution satisfying*

$$\liminf_{x\to-\infty} U(x) > 0 \quad \text{and} \quad \lim_{x\to\infty}\frac{U(x)}{e^{-\tau x}} = 1. \tag{6.46}$$

Proof. First, note that for any two $U_1, U_1 \in C^b_{\text{unif}}(\mathbb{R})$ satisfying (6.46) with $U_i(x) > 0$ for $x\in\mathbb{R}$, we can define the so called part metric $\rho(U_1,U_2)$ as follows:

$$\rho(U_1,U_2) = \inf\{\ln\alpha \,|\, \alpha\geq 1,\ \frac{1}{\alpha}U_1(x)\leq U_2(x)\leq \alpha U_1(x),\quad \forall\, x\in\mathbb{R}\}.$$

Moreover, there is $\alpha\geq 1$ such that

$$\rho(U_1,U_2) = \ln\alpha \quad \text{and} \quad \frac{1}{\alpha}U_1(x)\leq U_2(x)\leq\alpha U_1(x),\quad \forall\, x\in\mathbb{R}.$$

Next, fix $u\in\mathcal{E}_\tau$. Suppose that $U_1(x)$ and $U_2(x)$ are two solutions of (6.38) satisfying (6.46). Let $\alpha\geq 1$ be such that $\rho(U_1,U_2) = \ln\alpha$. Note that $U(x,t;U_i) = U_i$ for all $t\geq 0$ and every $i = 1,2$. Hence

$$\rho(U(\cdot,t;U_1),U(\cdot,t;U_2)) = \ln\alpha,\quad \forall\, t\geq 0.$$

Assume that $\alpha > 1$. Note that

$$\frac{1}{\alpha}U_1(x)\leq U_2(x)\leq \alpha U_1(x),\quad \forall\, x\in\mathbb{R},$$

and

$$(\alpha U_i)_t > (\alpha U_i)_{xx} + (c_\tau - \chi V'(\cdot; u))(\alpha U_i)_x + (1 - \chi\lambda V(\cdot; u) - (1-\chi)(\alpha U_i))(\alpha U_i)$$

for $i = 1, 2$. Thus the comparison principle for parabolic equations implies that

$$\begin{cases} U_2(x) \le U(x, t, \alpha U_1) < \alpha U_1(x) & \forall\, x \in \mathbb{R},\ t > 0 \\ U_1(x) \le U(x, t, \alpha U_2) < \alpha U_2(x) & \forall\, x \in \mathbb{R},\ t > 0. \end{cases} \tag{6.47}$$

Since $U_i(x) > 0$ for every $x \in \mathbb{R}$ and $\lim_{x\to\infty} \frac{U_i(x)}{e^{-\tau x}} = 1$ for each $i = 1, 2$, then for every $1 < \alpha' < \alpha$, there is $R_{\alpha'} \gg 1$ such that

$$U_2(x) < \alpha' U_1(x), \quad U_1(x) < \alpha' U_2(x) \quad \forall\, x \ge R_{\alpha'}. \tag{6.48}$$

Since $U_i(x) > 0$ for every $x \in \mathbb{R}$ and $\liminf_{x\to-\infty} U_i(x) > 0$ for each $i = 1, 2$, then

$$l_{\alpha'} := \min\{\inf_{x \le R_{\alpha'}} U_1(x), \inf_{x \le R_{\alpha'}} U_2(x)\} > 0, \qquad \forall\, 1 < \alpha' < \alpha. \tag{6.49}$$

For every $1 < \alpha' < \alpha$, $i = 1, 2$ and $x \le R_{\alpha'}$, we have

$$\begin{aligned} (\alpha U_i)_t =& (\alpha U_i)_{xx} + (c_\tau - \chi V'(x; u))(\alpha U_i)_x \\ &+ (1 - \chi\lambda V(x; u) - (1-\chi)(\alpha U_i))(\alpha U_i) + (1-\chi)(\alpha - 1)U_i(\alpha U_i) \\ \ge& (\alpha U_i)_{xx} + (c_\tau - \chi V'(x; u))(\alpha U_i)_x \\ &+ (1 - \chi\lambda V(x; u) - (1-\chi)(\alpha U_i))(\alpha U_i) + (1-\chi)(\alpha - 1)l_{\alpha'}(\alpha U_i). \end{aligned} \tag{6.50}$$

On the other hand, if we set $W^i(x,t) = e^{\varepsilon t}U(x,t;\alpha U_i)$, it follows from (6.47) that

$$
\begin{aligned}
W_t^i =& \varepsilon W^i + e^{\varepsilon t}U_t(x,t;\alpha U_i)\\
=& \varepsilon W^i + W_{xx}^i + (c_\tau - \chi V'(x;u))W_x^i + (1 - \chi\lambda V(x;u) - (1-\chi)W^i)W^i\\
&+ (1-\chi)(e^{\varepsilon t} - 1)U(x,t;\alpha U_i)W^i\\
\leq& W_{xx}^i + (c_\tau - \chi V'(x;u))W_x^i + (1 - \chi\lambda V(x;u) - (1-\chi)W^i)W^i + \varepsilon W^i\\
&+ \alpha(1-\chi)(e^{\varepsilon t} - 1)U_i W^i\\
\leq& W_{xx}^i + (c_\tau - \chi V'(x;u))W_x^i + (1 - \chi\lambda V(x;u) - (1-\chi)W^i)W^i\\
&+ \Big(\varepsilon + \alpha(1-\chi)(e^{\varepsilon t} - 1)L_{\alpha'}\Big)W^i,
\end{aligned}
$$

where

$$L_{\alpha'} = \max\{\sup_{x \leq R_{\alpha'}} U_1(x), \sup_{x \leq R_{\alpha'}} U_2(x)\}.$$

Choose $0 < \varepsilon \ll 1$ such that

$$\varepsilon + \alpha(1-\chi)(e^{\varepsilon t} - 1)L_{\alpha'} < (1-\chi)(\alpha - 1)l_{\alpha'} \quad 0 \leq t \leq 1.$$

Then, for $x \leq R_{\alpha'}$ and $0 \leq t \leq 1$ we have

$$W_t^i \leq W_{xx}^i + (c_\tau - \chi V'(x;u))W_x^i + (1 - \chi\lambda V(x;u) - (1-\chi)W^i)W^i + (1-\chi)(\alpha - 1)l_{\alpha'}W^i. \tag{6.51}$$

But inequality (6.47) implies that $U(R_{\alpha'},t;\alpha U_i) < \alpha U_i(R_{\alpha'})$ for every $t > 0$ and $i = 1,2$. So, choose $0 < \varepsilon \ll 1$ such that

$$W^i(R_{\alpha'},t) = e^{\varepsilon t}U(R_{\alpha'},t;\alpha U_i) \leq \alpha U_i(R_{\alpha'}) \quad \frac{1}{2} \leq t \leq 1, \ \ i = 1,2. \tag{6.52}$$

Therefore, using the comparison principle for parabolic equations, it follows from inequalities (6.50), (6.51) and (6.52) that

$$W^i(x,t) = e^{\varepsilon t}U(x,t;\alpha U_i) \leq \alpha U_i(x) \quad \forall\, x \leq R_{\alpha'},\ \frac{1}{2} \leq t \leq 1,\ i = 1,2.$$

for $0 < \varepsilon \ll 1$. Hence there is $0 < \varepsilon_0 \ll 1$ such that

$$U(x, 1; \alpha U_i) \leq e^{-\varepsilon} \alpha U_i(x) \quad \forall\, x \leq R_{\alpha'},\ i = 1, 2.$$

Combining this with (6.47), we obtain that

$$\begin{cases} U_2(x) \leq e^{-\varepsilon_0} \alpha U_1(x) & x \leq R_{\alpha'} \\ U_1(x) \leq e^{-\varepsilon_0} \alpha U_2(x) & x \leq R_{\alpha'}. \end{cases} \tag{6.53}$$

Combining inequalities (6.49) and (6.53) we have that

$$\frac{1}{\max\{\alpha', e^{-\varepsilon_0}\alpha\}} U_1(x) \leq U_2(x) \leq \max\{\alpha', e^{-\varepsilon_0}\alpha\} U_1(x) \quad \forall x \in \mathbb{R}.$$

Which implies that

$$\alpha \leq \max\{\alpha', e^{-\varepsilon_0}\alpha\} < \alpha,$$

which is a contradiction. Hence $\alpha = 1$ and then $U_1 = U_2$. The lemma is thus proved. □

We now prove Theorem 6.2.

Proof of Theorem 6.2. First of all, let us consider the normed linear space $\mathcal{E} = C^b_{\text{unif}}(\mathbb{R})$ endowed with the norm

$$\|u\|_* = \sum_{n=1}^{\infty} \frac{1}{2^n} \|u\|_{L^\infty([-n,\, n])}.$$

For every $u \in \mathcal{E}_\tau$ we have that

$$\|u\|_* \leq \frac{1}{1-\chi}.$$

Hence $\mathcal{E}_\tau$ is a bounded convex subset of $\mathcal{E}$. Furthermore, since the convergence in $\mathcal{E}$ implies the pointwise convergence, then $\mathcal{E}_\tau$ is a closed, bounded, and convex subset of $\mathcal{E}$. Furthermore, a sequence of functions in $\mathcal{E}_\tau$ converges with respect to norm $\|\cdot\|_*$ if and only if it converges locally uniformly on $\mathbb{R}$.

We prove that the mapping $\mathcal{E}_\tau \ni u \mapsto U(\cdot; u)$ has a fixed point. We divide the proof in two steps.

Step 1. In this step, we prove that the mapping $\mathcal{E}_\tau \ni u \mapsto U(\cdot; u)$ is compact.

Let $\{u_n\}_{n\geq 1}$ be a sequence of elements of $\mathcal{E}_\tau$. Since $U(\cdot; u_n) \in \mathcal{E}_\tau$ for every $n \geq 1$ then $\{U(\cdot; u_n)\}_{n\geq 1}$ is clearly uniformly bounded by $\frac{1}{1-\chi}$. Using inequality (6.39), we have that

$$\sup_{t\geq 1} \|U(\cdot, t; u_n)\|_{X^\beta} \leq M_1$$

for all $n \geq 1$ where M_1 is given by (6.40). Therefore, there is $0 < \nu \ll 1$ such that

$$\sup_{t\geq 1} \|U(\cdot, t; u_n)\|_{C^\nu_{\mathrm{unif}}(\mathbb{R})} \leq \tilde{M}_1 \tag{6.54}$$

for every $n \geq 1$ where $\tilde{M}_1$ is a constant depending only on M_1. Since for every $n \geq 1$ and every $x \in \mathbb{R}$, we have that $U(x, t; u_n) \to U(x; u_n)$ as $t \to \infty$, it follows from (6.54) that

$$\|U(\cdot; u_n)\|_{C^\nu_{\mathrm{unif}}} \leq \tilde{M}_1 \tag{6.55}$$

for every $n \geq 1$. This implies that the sequence $\{U(\cdot; u_n)\}_{n\geq 1}$ is equicontinuous. The Arzelá-Ascoli's Theorem implies that there is a subsequence $\{U(\cdot; u_{n'})\}_{n\geq 1}$ of the sequence $\{U(\cdot; u_n)\}_{n\geq 1}$ and a function $U \in C(\mathbb{R})$ such that $\{U(\cdot; u_{n'})\}_{n\geq 1}$ converges to U locally uniformly on $\mathbb{R}$. Furthermore, the function U satisfies inequality (6.55). Combining this with the fact $U^-_\tau(x) \leq U(x; u_{n'}) \leq U^+_\tau(x)$ for every $x \in \mathbb{R}$ and $n \geq 1$, by letting $n \to \infty$, we obtain that $U \in \mathcal{E}_\tau$.

Step 2. In this step, we prove that the mapping $\mathcal{E}_\tau \ni u \mapsto U(\cdot; u)$ is continuous.

Let $u \in \mathcal{E}_\tau$ and $\{u_n\}_{n\geq 1} \in \mathcal{E}^{\mathbb{N}}_\tau$ be such that $\|u_n - u\|_* \to 0$ as $n \to \infty$. Suppose by contradiction that $\|U(\cdot; u_n) - U(\cdot; u)\|_*$ does not converge to zero. Hence there is $\delta > 0$ and a subsequence $\{u_{n_1}\}_{n\geq 1}$ such that

$$\|U(\cdot; u_{n_1}) - U(\cdot; u)\|_* \geq \delta \quad \forall\, n \geq 1. \tag{6.56}$$

For every $n \geq 1$, we have that $U(\cdot, u_{n_1})$ satisfies

$$\begin{aligned}0 =&U_{xx}(x;u_{n_1})+(c_\tau-\chi V(x;u_{n_1}))U_x(x;u_{n_1})\\&+(1-\chi\lambda V(x;u_{n_1})-(1-\chi)U(x;u_{n_1}))U(x;u_{n_1})\quad \forall\ x\in\mathbb{R}.\end{aligned}\tag{6.57}$$

Claim 1. $\|V(\cdot;u_n)-V(\cdot;u)\|_* \to 0$ as $n\to\infty$. Indeed, for every $R>0$, it follows from (6.24) that

$$\begin{aligned}|V(x;u_n)-V(x;u)| &\leq \frac{1}{\sqrt{\pi}}\int_0^\infty\int_{\mathbb{R}} e^{-\lambda s}e^{-z^2}|u_n(x-2\sqrt{t}z)-u(x-2\sqrt{s}z)|dzds\\&\leq \frac{1}{\sqrt{\pi}}\int_0^R\int_{B(0,R)} e^{-\lambda s}e^{-z^2}|u_n(x-2\sqrt{t}z)-u(x-2\sqrt{s}z)|dzds\\&+\frac{2}{(1-\chi)\sqrt{\pi}}\int_{\{s\geq R \text{ or } |z|\geq R\}} e^{-\lambda s}e^{-z^2}dzds.\end{aligned}\tag{6.58}$$

Thus for every $k\in\mathbb{N}$ and every $R>1$, we have that

$$\begin{aligned}&\|V(\cdot;u_n)-V(\cdot;u)\|_{L^\infty([-k\ ,k])}\\\leq&\frac{1}{\sqrt{\pi}}\Big[\int_0^R\int_{B(0,R)} e^{-\lambda s}e^{-z^2}dzds\Big]\|u_n-u\|_{L^\infty([-(k+2R^{\frac{3}{2}})\ ,\ (k+2R^{\frac{3}{2}})])}\\&+\frac{2}{(1-\chi)\sqrt{\pi}}\int_{\{s\geq R \text{ or } |z|\geq R\}} e^{-\lambda s}e^{-z^2}dzds\\\leq&\frac{1}{\sqrt{\pi}}\underbrace{\Big[\int_0^\infty\int_{\mathbb{R}} e^{-\lambda s}e^{-z^2}dzds\Big]}_{\sqrt{\pi}/\lambda}\|u_n-u\|_{L^\infty([-(k+2R^{\frac{3}{2}})\ ,\ (k+2R^{\frac{3}{2}})])}\\&+\frac{2}{(1-\chi)\sqrt{\pi}}\int_{\{s\geq R \text{ or } |z|\geq R\}} e^{-\lambda s}e^{-z^2}dzds\\\leq&\frac{2^{k+2R^2}}{\lambda}\|u_n-u\|_*+\frac{2}{(1-\chi)\sqrt{\pi}}\int_{\{s\geq R \text{ or } |z|\geq R\}} e^{-\lambda s}e^{-z^2}dzds.\end{aligned}\tag{6.59}$$

Now, let $\varepsilon>0$ be given. Choose $R\gg 1$ and $k\gg 1$ such that

$$\frac{2}{(1-\chi)\sqrt{\pi}}\int_{\{s\geq R \text{ or } |z|\geq R\}} e^{-\lambda s}e^{-z^2}dzds<\frac{\varepsilon}{3}\qquad\text{and}\qquad\sum_{i\geq k}\frac{2}{(1-\chi)2^i}<\frac{\varepsilon}{3}.\tag{6.60}$$

Next, choose $N \gg 1$ such that

$$\frac{2^{k+2R^2}}{\lambda}\|u_n - u\|_* < \frac{\varepsilon}{3} \quad \forall\, n \geq N. \tag{6.61}$$

It follows from inequalities (6.59), (6.60) and (6.61) that for every $n \geq N$, we have

$$\begin{aligned} &\|V(\cdot;u_n) - V(\cdot;u)\|_* \\ \leq &\sum_{i\geq k} \frac{1}{2^i}\|V(\cdot;u_n) - V(\cdot;u)\|_{L^\infty([-i\ ,\ i])} + \|V(\cdot;u_n) - V(\cdot;u)\|_{L^\infty([-k\ ,\ k])} \\ \leq &\sum_{i\geq k} \frac{2}{(1-\chi)2^i} + \|V(\cdot;u_n) - V(\cdot;u)\|_{L^\infty([-k\ ,\ k])} < \varepsilon. \end{aligned} \tag{6.62}$$

Thus, the claim follows.

Claim 2. $\|V'(\cdot;u_n) - V'(\cdot;u)\|_* \to 0$ as $n \to \infty$. Indeed, it follows from (6.24) that

$$\begin{aligned} V'(x;w) = &\int_0^\infty \int_{\mathbb{R}} \frac{(z-x)e^{-\lambda s}}{2s\sqrt{4\pi s}} e^{-\frac{|x-z|^2}{4s}} w(z)dzds \\ = &\frac{-1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} y e^{-\lambda s} e^{-y^2} w(x - 2\sqrt{s}y)dzds \ \ \forall\, x \in \mathbb{R}\,, w \in C^b_{\text{unif}}(\mathbb{R}). \end{aligned} \tag{6.63}$$

Since

$$\lim_{R\to\infty} \int_{\{s\geq R \text{ or } |y|\geq R\}} |y|e^{-\lambda s}e^{-y^2}dzds = 0,$$

same arguments as in the proof of Claim 1 yield Claim 2.

Now, since $V''(\cdot;u_n) - V''(\cdot;u) = (V(\cdot;u_n) - V(\cdot;u)) - (u_n - u)$, it follows from Claim 1 that

$$\|V''(\cdot;u_n) - V''(\cdot;u)\|_* \to 0 \quad \text{as} \quad n \to \infty. \tag{6.64}$$

Combining inequality (6.55), Claim 1, Claim 2, (6.64), Theorem 3.15 of [13], and the Arzelá-Ascoli's Theorem, there is a subsequence $\{U(\cdot;u_{n_2})\}_{\geq 1}$ of $\{U(\cdot;u_{n_1})\}_{n\geq 1}$ and a function $U \in C^2(\mathbb{R})$ such that $\{U(\cdot;u_{n_2})\}_{\geq 1}$ converges to U in $C^2_{loc}(\mathbb{R}^N)$ and U satisfies

$$0 = U_{xx} + (c_\tau - \chi V'(x;u))U_x + (1 - \chi\lambda V(x;u) - (1-\chi)U)U. \tag{6.65}$$

Hence $U \in \mathcal{E}_\tau$ and

$$\|U(\cdot; u_{n_2}) - U\|_* \to 0 \quad \text{as} \quad n \to \infty. \tag{6.66}$$

But

$$\begin{aligned} 0 = U_{xx}(x; u) + (c_\tau - \chi V'(x; u))U_x(x; u) + (1 - \chi\lambda V(x; u) \\ - (1 - \chi)U(x; u))U(x; u) \quad \forall\, x \in \mathbb{R}. \end{aligned} \tag{6.67}$$

By Lemma 6.7, $U(\cdot) = U(\cdot; u)$. By (6.56),

$$\|U(\cdot) - U(\cdot; u)\| \geq \delta,$$

which is a contradiction. Hence the mapping $\mathcal{E}_\tau \ni u \mapsto U(\cdot; u)$ is continuous.

Now by Schauder's Fixed Point Theorem, there is $U \in \mathcal{E}_\tau$ such that $U(\cdot; U) = U(\cdot)$. Then $(U(x), V(x; U))$ is a stationary solution of (6.16) with $c = c_\tau$. It is clear that

$$\lim_{x\to\infty} \frac{U(x)}{e^{-\tau x}} = 1.$$

We claim that if $\chi < \frac{1}{2}$, then

$$\lim_{x\to-\infty} U(x) = 1.$$

For otherwise, we may assume that there is $x_n \to -\infty$ such that $U(x_n) \to a \neq 1$ as $n \to \infty$. Define $U_n(x) = U(x + x_n)$ for every $x \in \mathbb{R}$ and $n \geq 1$. By observing that $U_n = U(\cdot; U_n)$ for every n, hence it follows from the Step 1, that there is a subsequence $\{U_{n'}\}_{n\geq 1}$ of $\{U_n\}_{n\geq}$ and a function $U^* \in \mathcal{E}_\mu$ such that $\|U_{n'} - U^*\|_* \to 0$ as $n \to \infty$. Next, it follows from Step 2 that $(U^*, V(\cdot; U^*))$ is also a stationary solution of (6.16).

Claim 3. $\inf_{x\in\mathbb{R}} U^*(x) > 0$. Indeed, let $0 < \delta \ll 1$ be fixed. For every $x \in \mathbb{R}$, there $N_x \gg 1$ such that $x + x_{n'} < x_\delta$ for all $n \geq N_x$. Hence, it follows from Remark 6.1 that

$$0 < U_\mu^-(x_\delta) \leq U(x + x_{n'})\; \forall\, n \geq N_n.$$

Letting n go to infinity in the last inequality, we obtain that $U_{\mu}^{-}(x_{\delta}) \leq U^{*}(x)$ for every $x \in \mathbb{R}$. The claim thus follows.

Since $\chi < \frac{1}{2}$, it follows from Theorem 2.5 that $U^{*}(x) = V(x; U^{*}) = 1$ for every $x \in \mathbb{R}$. In particular, $a = U^{*}(0) = 1$, which is a contradiction. This implies that $U^{*}(0) = 1 = a$, which is a contradiction. Hence $\lim_{x\to-\infty} U(x) = 1$. $\square$

As a direct consequence of Theorem 6.2 we present the proof of Theorem 2.7.

Proof of Theorem 2.7 (i). Let $0 < \chi < \frac{1}{2}$ be fixed. According to Theorem 6.2, it is enough to show that for every $c \geq c^{*}(\chi)$ there is $0 < \tau(c) < 1$ with $c_{\tau(c)} = c$ and $\tau(c)$ satisfying (6.25). To this end, let $\tau^{*}(\chi) \in (0, \min\{1, \sqrt{\lambda}\}]$ be given by

$$\tau^{*}(\chi) = \sup\left\{ 0 < \tau < \min\{1, \sqrt{\lambda}\} \ : \ \frac{\tau\Big(\tau + \sqrt{\lambda - \tau^2}\Big)}{\lambda - \tau^2} \leq \frac{1-\chi}{\chi} \right\}.$$

Recall that $c^{*}(\chi) := c_{\tau^{*}(\chi)} = \tau^{*}(\chi) + \frac{1}{\tau^{*}(\chi)}$. Since the function $(0,1) \ni \mu \mapsto c_{\tau} = \tau + \frac{1}{\tau}$ is continuous and decreasing with $\lim_{\tau\to0^{+}} c_{\tau} = \infty$, then for every $c > c^{*}(\chi)$, there is a unique $\tau(c) \in (0, \tau^{*}(\chi))$ such that $c = c_{\tau(c)}$. Observe that

$$\frac{\tau\Big(\tau + \sqrt{\lambda - \tau^2}\Big)}{\lambda - \tau^2} = \frac{1}{\frac{\lambda}{\tau^2} - 1} + \frac{1}{\sqrt{\frac{\lambda}{\tau^2} - 1}}, \quad \forall\, 0 < \tau < \min\{1, \sqrt{\lambda}\}. \tag{6.68}$$

Hence the function $(0, \min\{1, \sqrt{\lambda}\}) \ni \tau \mapsto \frac{\tau\left(\tau + \sqrt{\lambda - \tau^2}\right)}{\lambda - \tau^2}$ is also strictly increasing. Thus, since $\tau(c) < \tau^{*}(\chi)$, we have that

$$\frac{\tau(c)\Big(\tau(c) + \sqrt{\lambda - \tau(c)^2}\Big)}{\lambda - \tau(c)^2} \leq \frac{\tau^{*}(\chi)\Big(\tau^{*}(\chi) + \sqrt{\lambda - (\tau^{*}(\chi))^2}\Big)}{\lambda - (\tau^{*}(\chi))^2} = \frac{1-\chi}{\chi}.$$

Hence, applying Theorem 6.2 the result follows. Observe from the definition of τ^{*} that

$$\lim_{\chi\to0^{+}} \tau^{*}(\chi) = \min\{1, \sqrt{\lambda}\}. \tag{6.69}$$

Indeed, by a change of variable $\beta = \frac{1}{\sqrt{\frac{\lambda}{\tau^2}-1}}$, the algebraic equation

$$\frac{\tau\left(\tau+\sqrt{\lambda-\tau^2}\right)}{\lambda-\tau^2} = \frac{1-\chi}{\chi},\ \tau\in(0,\sqrt{\lambda}) \tag{6.70}$$

is transformed into a quadratic equation

$$\beta^2+\beta = \frac{1-\chi}{\chi}, \quad \beta>0. \tag{6.71}$$

Note that $\beta = \frac{\sqrt{1+\frac{4(1-\chi)}{\chi}}-1}{2}$ is the only positive solution of (6.71). Thus solution of (6.70) is given by

$$\tilde{\tau}_\chi = \sqrt{\lambda}\left[1+\frac{4}{\left[\sqrt{1+\frac{4(1-\chi)}{\chi}}-1\right]^2}\right]^{-\frac{1}{2}}$$

Thus, we have that

$$\tau^*(\chi) = \min\{1,\tilde{\tau}_\chi\},$$

which implies (6.69). Thus,

$$\lim_{\chi\to 0^+} c^*(\chi) = c_{\min\{1,\sqrt{\lambda}\}} = \begin{cases} 2 & \text{if } 1\le\lambda, \\ \frac{1+\lambda}{\sqrt{\lambda}} & \text{if } 1\ge\sqrt{\lambda}. \end{cases}$$

□

In order to prove Theorem 2.7(ii) we first prove some lemmas.

Lemma 6.8. *(1) Let $0\le c<2\sqrt{a}$ be fixed and $\lambda_0\ge 0$ be such that $c^2-4a+4\lambda_0<0$. Let $\lambda_D(L)$ be the principal eigenvalue of*

$$\begin{cases} \phi_{xx}+c\phi_x+a\phi = \lambda\phi, & 0<x<L \\ \phi(0)=\phi(L)=0. \end{cases} \tag{6.72}$$

Then there is $L>0$ such that $\lambda_D(L)=\lambda_0$.

(2) Let c and L be as in (1). Let $\lambda_D(L; b_1, b_2)$ be the principal eigenvalue of

$$\begin{cases} \phi_{xx} + (c + b_1(x))\phi_x + (a + b_2(x))\phi = \lambda\phi, \quad 0 < x < L \\ \phi(0) = \phi(L) = 0, \end{cases} \tag{6.73}$$

where $b_1(x)$ and $b_2(x)$ are continuous functions. If there is a C^2 function $\phi(x)$ with $\phi(x) > 0$ for $0 < x < L$ such that

$$\begin{cases} \phi_{xx} + (c + b_1(x))\phi_x + (a + b_2(x))\phi \le 0, \quad 0 < x < L \\ \phi(0) \ge 0, \quad \phi(L) \ge 0 \end{cases} \tag{6.74}$$

Then $\lambda_D(L, b_1, b_2) \le 0$.

Proof. (1) Let $L > 0$ be such that

$$\frac{\sqrt{4a - 4\lambda_0 - c^2}}{2} L = \pi.$$

Then $\lambda = \lambda_0$ is the principal eigenvalue of (6.72) and $\phi(x) = e^{-\frac{c}{2}x} \sin\left(\frac{\sqrt{4a-4\lambda_0-c^2}}{2}x\right)$ is a corresponding positive eigenfunction. Hence $\lambda_D(L) = \lambda_0$ and (1) follows.

(2) Consider

$$\begin{cases} u_t = u_{xx} + (c + b_1(x))u_x + (a + b_2(x))u, \quad 0 < x < L \\ u(x, 0) = u(x, L) = 0. \end{cases} \tag{6.75}$$

Let $u(x, t; u_0, b_1, b_2)$ be the solution of (6.75) with $u(x, 0; u_0, b_1, b_2) = u_0(x)$ for $u_0 \in L^2(0, L)$. Then we have

$$\lambda_D(L; b_1, b_2) = \lim_{t\to\infty} \frac{\ln \|u(\cdot, t; u_0, b_1, b_2)\|_{L^2}}{t}$$

for any $u_0 \in L^2(0, L)$ with $u_0 \ge 0$ and $u_0 \ne 0$. By the comparison principle for parabolic equations, $u(x, t; \phi, b_1, b_2) \le \phi(x)$ for all $t \ge 0$ and $0 < x < L$. It then follows that $\lambda_D(L; b_1, b_2) \le 0$. □

Lemma 6.9. *(1) Let $c < 0$ be fixed and let $\lambda_0 > 0$ be such that $0 < \lambda_0 < a$. Let $\lambda_{N,D}(L)$ be the principal eigenvalue of*

$$\begin{cases} \phi_{xx} + c\phi_x + a\phi = \lambda\phi, \quad 0 < x < L \\ \phi_x(0) = \phi(L) = 0. \end{cases} \tag{6.76}$$

Then there is $L > 0$ such that $\lambda_{N,D}(L) = \lambda_0$.

(2) Let c and L be as in (1). Let $\lambda_{N,D}(L; b_1, b_2)$ be the principal eigenvalue of

$$\begin{cases} \phi_{xx} + (c + b_1(x))\phi_x + (a + b_2(x))\phi = \lambda\phi, \quad 0 < x < L \\ \phi_x(0) = \phi(L) = 0, \end{cases} \tag{6.77}$$

where $b_1(x)$ and $b_2(x)$ are continuous functions. If there is a C^2 function $\phi(x)$ with $\phi(x) > 0$ for $0 < x < L$ such that

$$\begin{cases} \phi_{xx} + (c + b_1(x))\phi_x + (a + b_2(x))\phi \le 0, \quad 0 < x < L \\ \phi_x(0) \le 0, \quad \phi(L) \ge 0 \end{cases} \tag{6.78}$$

Then $\lambda_{N,D}(L, b_1, b_2) \le 0$.

Proof. (1) Fix $c < 0$ and $0 < \lambda_0 < a$ with $4a - 4\lambda_0 < c^2$. Let

$$L = \frac{1}{\sqrt{c^2 - 4a + 4\lambda_0}} \ln \frac{-c + \sqrt{c^2 - 4a + 4\lambda_0}}{-c - \sqrt{c^2 - 4a + 4\lambda_0}}.$$

Then $L > 0$, $\lambda_{N,D}(L) = \lambda_0$ is the principal eigenvalue of (6.76), and $\phi(x)$ is a corresponding positive eigenfunction, where

$$\phi(x) = -e^{\frac{-c+\sqrt{c^2-4a+4\lambda_0}}{2}x} + \frac{-c + \sqrt{c^2 - 4a + 4\lambda_0}}{-c - \sqrt{c^2 - 4a + 4\lambda_0}} e^{\frac{-c-\sqrt{c^2-4a+4\lambda_0}}{2}x}.$$

(1) then follows.

(2) can be proved by the similar arguments as those in Lemma 6.8 (2). □

Proof of Theorem 2.7. We first consider the case that $0 \le c < 2\sqrt{a}$. Then there is $\lambda_0 > 0$ such that

$$c^2 - 4a + 4\lambda_0 < 0.$$

By Lemma 6.8 (1), there is $L > 0$ such that $\lambda_D(L) = \lambda_0 > 0$.

Fix $0 \le c < 2\sqrt{a}$ and choose L as above. Assume that (2.6) has a traveling wave solution $(u, v) = (U(x - ct), V(x - ct))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$. Then (6.16) has a stationary solution $(u, v) = (U(x), V(x))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$. Moreover, for any $\epsilon > 0$, this is $x_\epsilon > 0$ such that

$$0 < U(x) < \epsilon, \quad 0 < V(x) < \epsilon, \quad |V_x(x)| < \epsilon \quad \forall\, x \ge x_\epsilon.$$

Consider the eigenvalue problem,

$$\begin{cases} \phi_{xx} + (c - \chi V_x)\phi_x + (a - \chi(\lambda V - \tau c V_x) - (b - \chi\mu)U)\phi = \lambda\phi, & x_\epsilon < x < x_\epsilon + L \\ \phi(x_\epsilon) = \phi(x_\epsilon + L) = 0. \end{cases} \tag{6.79}$$

Let $\lambda_D^\epsilon(L)$ be the principal eigenvalue of (6.79). By Lemma 6.8 (1) and perturbation theory for principal eigenvalues of elliptic operators, $\lambda_D^\epsilon(L) > 0$ for $0 < \epsilon \ll 1$.

Note that

$$U_{xx} + (c - \chi V_x)U_x + (a - \chi(\lambda V(x) - \tau c V_x) - (b - \chi)U(x))U = 0 \quad \forall\, x_\epsilon \le x \le x_\epsilon + L$$

and $U(x_\epsilon) > 0$, $U(x_\epsilon + L) > 0$. Then, by Lemma 6.8 (2), $\lambda_D^\epsilon(L) \le 0$. We get a contradiction. Therefore, (2.6) has no traveling wave solution $(u, v) = (U(x - ct), V(x - ct))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$ and $0 \le c < 2\sqrt{a}$.

Next, we consider the case that $c < 0$. Let λ_0 and L be as in Lemma 6.9 (1). Then $\lambda_{N,D}(L) = \lambda_0 > 0$.

Fix $c < 0$ and the above L. Assume that (2.6) has a traveling wave solution $(u, v) = (U(x - ct), V(x - ct))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$. Then

(6.16) has a stationary solution $(u, v) = (U(x), V(x))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$. Similarly, for any $\epsilon > 0$, this is $x_\epsilon > 0$ such that

$$0 < U(x) < \epsilon, \quad 0 < V(x) < \epsilon, \quad |V_x(x)| < \epsilon \quad \forall\, x \geq x_\epsilon.$$

Moreover, since $U(\infty) = 0$, there is $\tilde{x}_\epsilon > x_\epsilon$ such that

$$U_x(\tilde{x}_\epsilon) < 0.$$

Consider the eigenvalue problem,

$$\begin{cases} \phi_{xx} + (c - \chi V_x)\phi_x + (a - \chi(\lambda V - \tau c V_x) - (b - \chi\mu)U)\phi = \lambda\phi, \quad \tilde{x}_\epsilon < x < \tilde{x}_\epsilon + L \\ \phi_x(\tilde{x}_\epsilon) = \phi(\tilde{x}_\epsilon + L) = 0. \end{cases} \tag{6.80}$$

Let $\lambda^\epsilon_{N,D}(L)$ be the principal eigenvalue of (6.80). By Lemma 6.9 (1) and using perturbation theory for principal eigenvalues of elliptic operators, $\lambda^\epsilon_{N,D}(L) > 0$ for $0 < \epsilon \ll 1$.

Note that

$$U_{xx} + (c - \chi V_x)U_x + (a - \chi(\lambda V(x) - \tau c V_x) - (b - \chi\mu)U(x))U = 0 \quad \forall\, \tilde{x}_\epsilon \leq x \leq \tilde{x}_\epsilon + L$$

and $U_x(\tilde{x}_\epsilon) < 0$, $U(\tilde{x}_\epsilon + L) > 0$. Then, by Lemma 6.9 (2), $\lambda^\epsilon_{N,D}(L) \leq 0$, contradiction. Therefore, (2.6) has no traveling wave solution $(u, v) = (U(x - ct), V(x - ct))$ with $(U(-\infty), V(-\infty)) = (\frac{a}{b}, \frac{a\mu}{b\lambda})$ and $(U(\infty), V(\infty)) = (0, 0)$ and $c < 0$.

Theorem 2.7 (ii) is thus proved. □

Chapter 7

Remarks and further works

When $\chi = 0$ and the functions $a(x,t)$ and $b(x,t)$ are constants, it is known that $c^*_{up} = 2\sqrt{a}$ and that the traveling wave solution associated with any speed $c \geq 2\sqrt{a}$ is unique up to a translation which is also stable. In which case the constant $2\sqrt{a}$ is called the minimal wave speed and coincide with the spreading speeds. There is also a huge amount of research on the transition wave solutions in the case $\chi = 0$ and $a(x,t)$ and $b(x,t)$ depend on x and/or t. The following questions associated to (2.6) arise from the results obtained in the above chapters.

P1. Suppose that the logistic source function is constant. Does a minimal wave speed exist in (2.6)? That is, is there a positive constant $c^*(a,b,\chi,\mu,\lambda)$ such that (2.6) has traveling wave solutions connecting the two constant equilibria solutions for every $c > c^*(a,b,\chi,\mu,\lambda)$ and no such solution exists of speed $c < c^*(a,b,\chi,\mu,\lambda)$?

P2. Uniqueness and stability of transition waves in (2.6).

P3. For space and/or time dependent logistic source, does (2.6) admit transition wave solutions?

P4. Finite time blow-up of solution in chemotaxis models with logistic type source on both bounded and unbounded domains.

The results on traveling wave solutions of (2.6) presented in this dissertation have been extended to the following full parabolic chemotaxis system,

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + (a - bu)u, \\ \tau v_t = v_{xx} - \lambda v + \mu u. \end{cases}$$

Indeed, in [46], we considered this problem and developed new techniques which are nontrivial generalizations of the ones presented in this dissertation. In general, the study of dynamics of solutions of the full parabolic chemotaxis system is more complex. In fact, most of the results established in this dissertation on the parabolic-elliptic case, such as global existence of classical solutions, stability of strictly positive entire solutions, spreading speeds, are still open, that is, the following problem remains to be studied.

P5. Dynamics in full parabolic-parabolic chemotaxis systems on $\mathbb{R}^N$,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + (a(x,t) - b(x,t)u)u, \quad x \in D, \\ \tau v_t = \Delta v - \lambda v + \mu u, \quad x \in D. \end{cases} \tag{1}$$

It is well known that micro-organisms usually have mixed directed movement toward the gradient of chemical substances, in the sense that the mobile species move toward higher concentration of the chemical substances or away from it. These phenomena are describe by the attraction-repulsion chemotaxis models. In [45, 47], joint works with Dr.W. Shen, we studied the existence of global classical solutions, stability of constant equilibria, spreading speeds, and existence and non-existence of traveling wave solutions in the following attraction-repulsion chemotaxis systems

$$\begin{cases} u_t = \Delta u - \chi_1\nabla(u\nabla v_1) + \chi_2\nabla(u\nabla v_2) + u(a - bu), \qquad x \in \mathbb{R}^N \\ \tau\partial_t v_1 = (\Delta - \lambda_1 I)v_1 + \mu_1 u, \qquad x \in \mathbb{R}^N \\ \tau\partial_t v_2 = (\Delta - \lambda_2 I)v_2 + \mu_2 u, \qquad x \in \mathbb{R}^N. \end{cases}$$

In particular, taking $\tau = 0$ and $\chi_2 = 0$ in the last system of partial differential equations, we recover as a special case system (2.6) when the functions $a(x,t)$ and $b(x,t)$ are both constant.

In a joint work with Issa Bachar Tahir [22], we considered the following extended attraction chemotaxis system with two speciesof parabolic-parabolic-elliptic type with nonlocal terms

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla(u\cdot\nabla w) + u\left(a_0 - a_1 u - a_2 v - a_3\int_\Omega u - a_4\int_\Omega v\right), \quad x\in\Omega \\ v_t = d_2\Delta v - \chi_2\nabla(v\cdot\nabla w) + v\left(b_0 - b_1 u - b_2 v - b_3\int_\Omega u - b_4\int_\Omega v\right), \quad x\in\Omega \\ 0 = d_3\Delta w + ku + lv - \lambda w, \quad x\in\Omega \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with smooth boundary, where $a_0, b_0,\ a_1$, and b_2 are positive and $a_2,\ a_3,\ a_4,\ b_1,\ b_3$, and b_4 are real numbers. Under some explicit conditions on these paramaters, we proved the global existence of non-negative classical solutions, coexistence of the two species in a sense that the system has a unique positive constant steady state solution which is globally asymptotically stable. We also found some conditions on the coefficients $a_i,\ b_i$, and on the chemotaxis sensitivities χ_i for which the phenomenon of competitive exclusion occurred, i.e. one of the species dies out asymptotically, whereas the other reaches its carrying capacity in the large time limit. Meanwhile, the following problem has been rarely studied.

P6. Traveling wave solutions in competitive/cooperative chemotaxis systems of two species.

In the future, I plan to continue working on various dynamic aspects of chemotaxis models with logistic source functions $f(t,x,u,v) = u(a(t,x) - b(t,x)u)$, including the problems described above. I also plan to study chemotaxis models with bistable source functions.

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