Decomposing Graphs With Two Associate Classes Into Paths Of Length 3 And The Intersection Problem Of Latin Rectangles

by

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Abstract

In this thesis, the decomposition problem of graphs with two associate classes into paths of length 3 is completely settled. The intersection problem for latin rectangles is completely solved as well. In addition, an Euler circuit of $K(n,p)$ with diameter at least $(n-3)p/2+1$ is constructed and the intersection problem of latin squares of order n and $n+1$ is discussed.

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Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Basics

A graph G consists of a set $V(G)$ of vertices together with a set $E(G)$ of edges, and a mapping associating to each edge e an unordered pair x, y of vertices called the endpoints of e. There may be multiple edges associated to the same pair of vertices. Two vertices are called adjacent if they are distinct and joined by an edge. A path of length n is a sequence of $n+1$ distinct vertices $(v_1, v_2, ..., v_{n+1})$ such that v_i and v_{i+1} are adjacent for $1 \leq i \leq n$. A decomposition of a graph G is a partition of its edge set $E(G)$. An H-decomposition of G is a decomposition D of G in which each element of D induces a copy of graph H. For nonnegative integers n, p, λ_1 and λ_2, the equipartite graph with two associate classes $G(n, p, \lambda_1, \lambda_2)$ is defined to be the graph with np vertices, partitioned into p parts $V_1, ..., V_p$, each of size n, in which two vertices are joined by λ_1 edges if they are in the same part, and by λ_2 edges if they are in different parts.

A walk is a sequence $(v_0, e_1, v_1, e_1, ..., v_k)$ of vertices v_i and edges e_i in a graph such that for $1 \leq i \leq k$, e_i has endpoints v_{i-1} and v_i. A walk is called closed if it starts and ends at the same vertex. A trail is a walk without repeated edges. An Euler circuit of graph G is a closed trail which includes every edge of G exactly once. The distance of two appearances of the same vertex v in a walk W is the number of edges between the two appearances of v along W. The distance of vertex v in a walk W, denoted by $d_W(v)$, is the least distance among all pairs of appearances of v along W. The diameter $d(W)$ of a walk W is defined by $d(W) = \min\{d_W(v) \mid v \in V(W)\}$ (i.e. the minimum distance of all vertices in W).

For positive integers r, n with $r \leq n$, a latin rectangle is an $r \times n$ array of n symbols in which each symbol occurs exactly once in each row and at most once in each column, and each cell contains exactly one symbol. A latin square of order n is an $n \times n$ latin rectangle. If L is a latin rectangle then let $L_{i,j}$ denote the symbol in cell (i, j) of L. For $n \leq m$, let L and S be latin squares of order n and m, respectively. The intersection number of L and S is defined to be $I(L, S) = |\{(i, j) \mid 1 \leq i, j \leq n, L_{i,j} = S_{i,j}\}|$. Let R and Q be $r \times n$ latin rectangles. The intersection number of R and Q is defined to be $I(R, Q) = |\{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq n, R_{i,j} = Q_{i,j}\}|$. The problem of determining the set of all the possible intersection numbers is referred as the intersection problem.

1.2 Outline

This thesis contains four topics that are described in their own chapters. The first two topics are highly related to each other, as are the last two as well.

In Chapter 2, a complete solution to the decomposition problem for equipartite graphs with two associate classes into paths of length 3 is presented. Necessary conditions for the existence of such decomposition is determined, and it is shown that these necessary conditions are also sufficient by constructing a decomposition of equipartite graphs with two associate classes into paths of length 3 whenever the necessary conditions are satisfied.

In Chapter 3, for odd n and p we construct an Euler circuit E of $K(n, p)$ with the property that the diameter of $E \geq (n-3)p/2 + 1$, where $K(n, p) = G(n, p, 0, 1)$ is the complete multipartite graph of p parts with equal part sizes n. Then E is used to obtain some results on the decomposition problem for equipartite graphs with two associate classes into paths of various lengths.

In Chapter 4, two latin squares of order n and $n + 1$ are constructed, which partially answers the intersection problem for latin squares of order n and $n + 1$.

In Chapter 5, the intersection problem for latin rectangles of same order is completely settled by finding necessary and sufficient condition for the existence of two $r \times n$ latin rectangles with specified intersection numbers for all integers r and n with $1 \leq r \leq n$.

Chapter 2

Decomposing Graphs With Two Associate Classes Into Paths Of Length 3

2.1 Basics

It is common to refer the path with k vertices as P_k, which has $k-1$ edges. We however focus on the number of edges in a path, therefore we will call the path with k edges L_k. Thus $P_{k+1} = L_k$.

Let $G = (V, E)$ be a graph, and A, B be subsets of V. We use $G[A]$to denote the subgraph of G induced by A. Furthermore, we use $G[A, B]$ to denote the subgraph of G whose vertex set is $A \cup B$ and whose edge set consists of all of the edges in E that have exactly one endpoint in A and one endpoint in B.

A decomposition of a graph G is a partition of its edge set $E(G)$. An H-decomposition of G is a decomposition D of G in which each element of D induces a copy of H. G is said to be H-decomposable if there exists an H-decomposition of G. It causes no confusion to denote an H-decomposition D of G by the subgraph induced by the elements of D instead of the actual partition of $E(G)$.

If G is a graph then let λG denote the graph with vertex set $V(G)$ in which for each $\{u, v\} \in V(G)$, u and v are joined by λx edges in λG if and only if they are joined by x edges in G.

2.2 History

Decomposing general graphs into paths has been considered over the last 50 years. L_1-decompositions are trivial. For L_2, Kotzig [39] showed a connected simple graph is L_2-decomposable if and only if it has even number of edges. According to [9], the following elegent short proof is due to Dr. Dean G. Hoffman: assign an arbitrary orientation to the graph. Since there are even number of edges, there must be even number of vertices with odd out-degree. Pick two of those vertices. Since the graph is connected, there must be a path in the underlying graph between these two vertices. Reverse the orientation on the edges of the path. The out-degree remains the same for any vertices on the path except for the two end vertices, and they have even out-degree now. Repeat until there is no vertex with odd out-degree. For every vertex, pair its outgoing edges and use the vertex as center to form paths of length 2.

Tarsi [58] solved the path decomposition problem for complete multigraphs.

Theorem 2.1 (Tarsi, 1981 [58]). *λK_n can be decomposed into L_k's if and only if $\lambda n(n-1) \equiv 0$ (mod $2k$) and $n \geq k+1$.*

Parker [51] completely solved the case when it comes to simple complete bipartite graphs.

Theorem 2.2 (Parker, 1998 [51]). *Let k, m, n be positive integers. $K_{m,n}$ has an L_k decomposition if and only k divides mn and the parity conditions in Table 2.1 are satisfied.*

Table 2.1: Parity Conditions for $K_{m,n}$ to have an L_k-decomposition

Case	k	m	n	Parity Conditions
1	even	even	even	$k \leq 2m$, $k \leq 2n$, not both equalities
2	even	even	odd	$k \leq 2m-2$, $k \leq 2n$
3	even	odd	even	$k \leq 2m$, $k \leq 2n-2$
4	even	odd	odd	not possible
5	odd	even	even	$k \leq 2m-1$, $k \leq 2n-1$
6	odd	even	odd	$k \leq 2m-1$, $k \leq n$
7	odd	odd	even	$k \leq m$, $k \leq 2n-1$
8	odd	odd	odd	$k \leq m$, $k \leq n$

Truszczyński [60] found L_k-decompositions of $\lambda K_{m,n}$ in many cases. In particular when it comes to $\lambda K_{n,n}$, Shyu [57] extended the result by settling the existence in all but one case, namely when $n = 15$, $\lambda = 3$ and $k = 27$. (See Theorem 2.3)

Theorem 2.3 (Shyu, 2007 [57]). *Suppose $(n, \lambda, k) \neq (15, 3, 27)$. $\lambda K_{n,n}$ has a decomposition into L_k's if and only*

(1) $k \mid \lambda n^2$.

(2) $k \leq n$ *if* $\lambda = 1$ *and* n *is odd*

(3) $k + 1 \leq 2n$ *if* $\lambda \geq 2$ *or* n *is even*

Billington, Cavenagh and Smith [8, 9] solved the problem of decomposing the simple complete equipartite graphs with 3, 4 and 5 parts into copies of L_k for all $k \geq 1$. Lee, Lee and Lin [42] solved the existence problem for L_k-decompositions of $\lambda K_{n,n,n}$.

As is the case in this paper, several results have been found that restricted attention to L_3-decompositions. Kumar [40] and Billington and Hoffman [10] independently settled existence problem of L_3-decompositions when G is a simple complete multipartite graph. Billington and Hoffman [10] also solved the problem for L_4 in the same paper.

Theorem 2.4 (Kumar, 2003 [40])**.** *Suppose* $r \geq 3$*,* $n_i > 0$ *for all* $1 \leq i \leq r$*. Then the complete multipartite graph* $G = K_{n_1,n_2,\ldots,n_r}$ *is* L_3*-decomposable if and only if* 3 *divides* $|E(G)|$ *and* $G \neq K_{1,1,1}$*.*

Heinrich, Liu and Yu [34] proved a simple graph G is L_3-decomposable if G is $3k$-regular and G has no cut-edge when $3k$ is odd. They also showed that a simple connected 4-regular graph G is L_3-decomposable if and only if 3 divides $|E(G)|$. Diwan, Dion, Mendell, Plantholt and Tipnis [17] showed that each connected 4-regular multigraph G with maximum edge-multiplicity at most 2 is L_3-decomposable if and only if no 3 vertices of G induce a subgraph with more than 4 edges and 3 divides $|E(G)|$.

A special kind of path decomposition is the balanced path decomposition where balanced means each vertex appears in same number of elements of the decomposition as each other vertex. Balanced path decomposition for λK_n was settled by Huang [36] and Hung and Mendelsohn [37], independently. Lee and Lin [43] found necessary and sufficient conditions for $\lambda K_{n,n}$ to have a balanced L_k-decomposition for all $k \geq 1$.

A special kind of balanced path decomposition is the path factorization, or known as resolvable path designs, meaning that the paths in the decomposition can be partitioned into vertex-disjoint spanning subgraphs. Horton [35] settled the L_2-factorization problem for λK_n. Bermond, Heinrich and Yu [7] extended the result to all L_k, $k \geq 3$. Yu [62] settled the L_k-factorization problem for $\lambda K_{n,n}$ and $\lambda K_{n,n,n}$. Yu also settled L_k-factorization problem for $\lambda K_{n,\ldots,n}$ when $k-1$ is prime. Muthusamy and Paulraja [47] extended Yu's result to when k is prime.

Barát and Thomassen conjectured in [4] that for any fixed tree T, any simple graph G with sufficiently large edge-connectivity for which $|E(T)|$ divides $|E(G)|$ is T-decomposable. Several attempts to settle this conjecture focused on the case where T is a path [59, 12, 38]. The full conjecture was finally proved in [5].

For more on path decomposition, see survey by Heinrich [33].

In this thesis we consider L_3-decompositions of another family of graphs that arises in the literature [28, 30, 29, 2, 13, 48, 41, 54, 55]. Motivated by statistical applications [52, 11, 15], these graphs are known as complete graphs with two associate classes. While in general these graphs need not have the same number of vertices in each part, the focus of this thesis is the equipartite family, defined as follows.

Definition 2.1. *Let $n, p, \lambda_1, \lambda_2$ be nonnegative integers. Define $G(n, p, \lambda_1, \lambda_2)$ to be the graph with np vertices, partitioned into p parts $V_1, ..., V_p$, each of size n, in which two vertices are joined by λ_1 edges if they are in the same part, and by λ_2 edges if they are in different parts. We say an edge is pure if both of its endpoints belong to the same part, and mixed otherwise.*

Bose and Shimamoto [11] classified partially balanced designs with two association classes into five types: group divisible, simple, triangular, latin square type and cyclic. In graph thoery terms, the group divisible designs can be described as the decomposing of graphs of two associate classes into complete graphs. For a wealth of information of group divisible designs, see Raghavarao [52, p. 121].

H-decompositions of $G = G(n, p, \lambda_1, \lambda_2)$ have been studied for a few choices of H. Fu, Rodger and Sarvate [28, 30] settled the decomposition problem of G into 3-cycles. Fu and Rodger [29] also decomposed G into 4-cycles, finding necessary and sufficient conditions for their existence. Bahmanian and Rodger [2] decomposed G into Hamilton cycles whenever it is possible. Ndungo and Sarvate [48] showed $G(n, 2, 3, 4)$ can be decomposed into K_4's if and only if 3 divides n except possibly when $n = 18$; they also showed the obvious necessary conditions for a K_4-decomposition is also sufficient for: $G(7m, 2, 5m, 7m - 1)$ for all $m \geq 2$; $G(5m + 1, 2, 5m + 1, 7m)$ whenever m is even; and $G(5m + 1, 2, 2(5m + 1), 14m)$ for all m. A generalization of G allows the parts to have different sizes: in such a case where this generalized graph has exactly 2 parts, if either one of the two parts has size 2 or $\lambda_1 \geq \lambda_2$, Chaffee and Rodger [13] settled the K_3-decomposition problem.

Note that when $p = 1$ or $\lambda_2 = 0$, each component of the graph $G(n, p, \lambda_1, \lambda_2)$ is $\lambda_1 K_n$. On the other hand, when $n = 1$ the graph is $\lambda_2 K_p$. In both cases Tarsi's theorem suffices to solve the path decomposition problem. Therefore throughout the rest of this chapter, we will assume that $n \geq 2$, $p \geq 2$ and $\lambda_2 \geq 1$.

2.3 Lemmas

This lemma will be needed:

Lemma 2.1. *Suppose $n \geq 5$ and $n \equiv 2$ (mod 3). There exists an L_3-decomposition of $K_n - e$ for any $e \in E(K_n)$.*

Proof. Note if $n \equiv 0$ or 1 (mod 3), $n \geq 4$, then K_n can be completely decomposed into L_3's by Theorem 2.1.

It is not hard to decompose K_5 into L_3's and a single edge. Suppose $n > 5$ and $n \equiv 2$ (mod 3). Let $\{x, y\} \subset V(K_n)$ and let $V' = V(K_n) \setminus \{x, y\}$. Then $|V'| = 3k$ for some integer $k > 1$. Using Theorem 2.1, let (V', B_1) be an L_3-decomposition of K_{n-2}. By Theorem 2.2, let (V, B_2) be an L_3-decomposition of $K_{2,3k}$ with bipartition $\{\{x, y\}, V'\}$ of the vertex set. Then $(V, B_1 \cup B_2)$ is an L_3-decomposition of $K_n - e$ with $e = \{x, y\}$. □

With Lemma 2.1 in mind, let $(V_i, D_i(\{x, y\}))$ denote an L_3-decomposition of $G = K_n - e$ with vertex set V_i and e being the edge $\{x, y\} \subset V_i$. Refer back to Theorem 2.2, if $n \equiv 0 \pmod 3$, then $K_{n,n}$ can be completely decomposed into L_3's.

Lemma 2.2. *Suppose $n \geq 2$ and $n \equiv 1$ or $2 \pmod 3$. There exists an L_3-decomposition of $K_{n,n} - e$ for any $e \in E(K_{n,n})$.*

Proof. When $n = 2$, it is easy to see $K_{2,2}$ composes of a L_3 and a single edge. It is also not hard to decompose $K_{4,4}$ into L_3's and a single edge. Thus now suppose $n \geq 5$. Let V_i, $i = 1, 2$ be the vertex set of part i.

If $n \equiv 2 \pmod 3$, pick two vertices x_i, y_i from V_i for $i = 1, 2$. Then $|V_1 \setminus \{x_1, y_1\}| = |V_2 \setminus \{x_2, y_2\}| = 3k$ for some positive integer k, and the graph induced by $(V_1 \setminus \{x_1, y_1\}) \cup (V_2 \setminus \{x_2, y_2\})$ is a $K_{3k,3k}$ which can be decomposed into L_3's by Theorem 2.2. The graph induced by $(V_1 \setminus \{x_1, y_1\}) \cup \{x_2, y_2\}$ is a $K_{2,3k}$, so is the one induced by $(V_2 \setminus \{x_2, y_2\}) \cup \{x_1, y_1\}$, and $K_{2,3k}$ is decomposable by Theorem 2.2. The only edges left now are the edges between the vertices $\{x_1, y_1, x_2, y_2\}$, which is a $K_{2,2}$ and therefore a L_3 with a edge left.

If $n \equiv 1 \pmod 3$, pick four vertices w_i, x_i, y_i, z_i from each V_i. Then $|V_i \setminus \{w_i, x_i, y_i, z_i\}| = 3k$ for some positive integer k for all i, and the graph induced by $\bigcup_{i=1,2}(V_i \setminus \{w_i, x_i, y_i, z_i\})$ is a $K_{3k,3k}$. $(V_1 \setminus \{w_1, x_1, y_1, z_1\}) \cup \{w_2, x_2, y_2, z_2\}$ is a $K_{4,3k}$, so is $(V_2 \setminus \{w_2, x_2, y_2, z_2\}) \cup \{w_1, x_1, y_1, z_1\}$. Finally, $\bigcup_{i=1,2}\{w_i, x_i, y_i, z_i\}$ induces a $K_{4,4}$. All graph above can be decomposed into L_3's except $K_{4,4}$ has leave being a single edge. Therefore the lemma is proved.

□

Let $(V_i, V_j, D_{i,j}(\{x_i, x_j\}))$ denote an L_3-decomposition of $G = K_{n,n} - e$ with bipartition $\{V_i, V_j\}$ of the vertex set and with $e = \{x_i, x_j\}$, $x_i \in V_i$ and $x_j \in V_j$.

Proving the main result Theorem 2.6 when $n \in \{2, 3\}$ is most difficult because then no 3-path can be completely within one part. In these cases the proof technique makes use of the method of differences. Given graph $G = G(2, p, \lambda_1, \lambda_2)$, consider the multiset $M = \lambda_2\{i \mid 1 \leq i \leq p-1\} \cup \lambda_1\{p\}$, which is the multiset of the differences; a difference is called pure if it is equal to p, and mixed otherwise. This set of differences is a well-known useful way to partition the edge set of G: $\{\{i, i+d \mid i \in Z_{2p}\} \mid d$ is a mixed difference in $M\}$ partitions the mixed edges of G and $\{\{i, i+d \mid i \in Z_p\} \mid d$ is a pure difference in $M\}$ partitions the pure edges of G. It will be useful to let $E(d) = \{\{i, i+d\} \mid i \in Z_{2p}$ if d is mixed$\}$, and $\{\{i, i+d\} \mid i \in Z_p$ if d is pure$\}$. So to find the L_3-decomposition of G, we first form a partition Π of M into multisets (possibly Π contains repetitions) such that for each $S \in \Pi$, $\cup_{d \in S} E(d)$ induces a graph $G(S)$, which has a L_3-decomposition.

This approach is also used when $n = 3$, and $M = \lambda_2\{i \mid 1 \leq i \leq \lfloor \frac{3p}{2} \rfloor, i \neq p\} \cup \lambda_1\{p\}$ where again p is the pure difference and all other differences are mixed. The partition of $E(G(3, p, \lambda_1, \lambda_2))$ is $\{\{i, i+d \mid i \in Z_{3p}\} \mid d$ is a mixed difference in $M, d \neq \frac{3p}{2}\} \cup \{\{i, i+\frac{3p}{2} \mid i \in Z_{\frac{3p}{2}}\} \mid$ if $\frac{3p}{2} \in M\} \cup \{\{i, i+d \mid i \in Z_{3p}\} \mid d$ is a pure difference in $M\}$.

For any multiset D, each element being in $\{1, 2, ..., \lfloor \frac{v}{2} \rfloor\}$, define $G_v(D)$ to be the graph with vertex set Z_v, and with edges in the multiset $\bigcup_{d \in D} \{\{i, i+d\} \mid d \in D, i \in Z_v$ if $d < \frac{v}{2}, i \in Z_{\frac{v}{2}}$ if $d = \frac{v}{2}\}$. Notice that if d occurs x times in the multiset D then the edge $\{i, i+d\}$ appears x times in $G_v(D)$.

Bermond, Favaron and Maheo [6] proved a much more general result than the following that shows when the edges of two differences can be used to form two edge-disjoint hamilton cycles.

Theorem 2.5 ([6])**.** *Let s, t, n be positive integers with $s \leq t < \frac{n}{2}$. If the greatest common divisor among s, t, n is 1, then the graph $G_n(\{s, t\})$ has a hamilton cycle decomposition.*

The next two lemmas provide graphs that have L_3-decompositions in the cases where n is 2 or 3 respectively.

Lemma 2.3. *Let $p \geq 2$ be an integer and let d, d', d'' be distinct elements in $\{1, 2, \ldots, p-1\}$. Then $G_{2p}(D)$ has L_3-decomposition, (Z_{2p}, P), if D is one of the following sets:*

1. *$k\{1\}$ if 3 divides kp*

2. *$\{p, d\}$*

3. $\{1, 2\}$ *if* 3 *divides* p

4. $\{d, d', d''\}$

5. $\{1, p, p\}$ *if* 3 *divides* p

6. $\{2, 3, 3\}$ *if* $p = 3$

7. $\{\frac{p}{3}, p, p, p\}$ *if* 3 *divides* p

8. $\{d, p, p, p, p\}$

Proof. We consider each case in turn. It is not hard to check that each graph defined is a path, and that the edges are covered by the paths as required.

1. $G_{2p}(k\{1\})$ is isomorphic to kC_{2p}, which is clearly L_3-decomposable when 3 divides kp.

2. Let $P = \{(0+i, d+i, d+p+i, p+i) \mid i \in Z_p\}$.

3. Since 3 divides p, Theorem 2.5 implies that $G_{2p}(D)$ has a decomposition into two C_{2p}'s, each being L_3-decomposable.

4. Let $P = \{(0+i, d'+i, d'+d+i, d'+d-d''+i) \mid i \in Z_{2p}\}$, where $d > d' > d''$.

5. Let $P = \{(i, i+p, i+p+1, i+1) \mid i \in \mathbb{Z}_p\} \cup \{(3i, 3i+1, 3i+2, 3i+3) \mid i \in \mathbb{Z}_{p/3}$.

6. Let $P = \{(0+i, 2+i, 5+i, 1+i), (1+i, 4+i, 0+i, 3+i) \mid i \in \{0, 3\}\}$.

7. Each component of $G_{2p}(D)$ is isomorphic to $G_6(\{1, 3, 3, 3\})$, which has the following L_3-decomposition: $\{(0+2i, 3+2i, 4+i, 1+2i) \mid i \in \mathbb{Z}_3\} \cup \{(0, 1, 4, 5), (5, 2, 3, 0)\}$.

8. Let $P = \{(0+i, p+i, p+d+i, p+d+p+i) \mid i \in Z_{2p}\}$. $\square$

Lemma 2.4. *Let* $p \geq 2$ *be an integer and let* $\{d, d', d''\} \subset \{1, 2, ..., \lfloor \frac{3p-1}{2} \rfloor\} \setminus \{p\}$. *Then* $G_{3p}(D)$ *has* L_3*-decomposition,* (Z_{3p}, P)*, if* D *is one of the following sets:*

1. $\{1\}$

2. $\{\frac{3p}{2}, p\}$ *if* 2 *divides* p

3. $\{\frac{3p}{2}, d\}$ *if* 2 *divides* p

4. $\{d, p, p\}$

5. $\{d, d', d''\}$

6. $\{g_1, g_2\}$ *with* $gcd(\{g_1, g_2, 3p\}) = 1$, $g_1, g_2 \in \{1, ..., \lfloor \frac{3p}{2} \rfloor\} \setminus \{\frac{3p}{2}\}$

Proof. Following the approach in Lemma 2.3, we consider each case in turn.

1. $G_{3p}(\{1\})$ is isomorphic to C_{3p}, which is clearly L_3-decomposable as $p \geq 2$.
2. Let $P = \{(0+i, p+i, p+\frac{3p}{2}+i, \frac{3p}{2}+i) \mid i \in \mathbb{Z}_{\frac{3p}{2}}\}$.
3. Let $P = \{(0+i, d+i, d+\frac{3p}{2}+i, \frac{3p}{2}+i) \mid i \in \mathbb{Z}_{\frac{3p}{2}}\}$.
4. Let $P = \{(0+i, d+i, d+p+i, d+2p+i) \mid i \in \mathbb{Z}_{3p}\}$.
5. If $d = d' = d''$ then, since $d \neq p$, let $P = \{(0+i, d+i, 2d+i, 3d+i) \mid i \in \mathbb{Z}_{3p}\}$. If $d \geq d' \geq d''$ with $d > d''$ then let $P = \{(0+i, d'+i, d'+d+i, d+d'-d''+i) \mid i \in \mathbb{Z}_{3p}\}$.
6. Since $p \geq 2$, Theorem 2.5 implies that $G_{3p}(D)$ has a decomposition into two C_{3p}'s, each being L_3-decomposable.

□

Let $S(p, \lambda, l)$ be the graph formed from λK_p by adding p vertex disjoint paths of length l, each path intersecting $V(\lambda K_p)$ in one of the path's end vertices.

Lemma 2.5. *Let* $p \geq 2$ *be an integer. There exists a* L_3*-decomposition of the following graphs:*

1. $S(p, 1, 1)$ *when* $p \equiv 0$ *or* 2 $(mod\ 3)$ *and* $p \neq 3$

2. $S(p, 1, 2)$ *when* $p \equiv 0$ $(mod\ 3)$

Proof. First we prove the $S(p, 1, 1)$ case. Let $S = S(p, 1, 1)$ have vertex set $V(S) = Z_p \times Z_2 = \{(i, j) \mid i \in Z_p, j \in Z_2\}$, where $S[\{(i, 0) \mid i \in Z_p\}] = K_p$ and $(i, 1)$ has degree 1 being adjacent to $(i, 0)$ for each $i \in Z_p$.

Suppose $p \equiv 0 \pmod 3$. The proof is by induction on p. Suppose $p = 6$. Then $\{((i, 1), (i, 0), (i+$

$3,0),(i+3,1) \mid i \in Z_3\}$ together with an L_3-decomposition of $G_6[\{1,2\}]$ (see Lemma 2.3 case (3)) provides the decomposition. Assume $p = 9$. Similarly, $\{((i,1),(i,0),(i+3,0),(i-1,0)) \mid i \in Z_9\}$ and an L_3-decomposition of $G_9[\{1,2\}]$ by Lemma 2.4 case (7) provides the decomposition.

Assume $S(p,1,1)$ exists for $p \leq k$. When $p = k+6$, let $p = 3q+6$ with $3q \geq 6$. Let $S = S(3q+6,1,1)$ have vertex set $V(S) = A \cup B$ where $A = \{(i,j) \mid 1 \leq i \leq 3q, j \in Z_2\}$ and $B = \{(i,j) \mid 3q+1 \leq i \leq 3q+6, j \in Z_2\}$. Then $S[A] = S(3q,1,1)$ and $S[B] = S(6,1,1)$ both are L_3-decomposable by induction hypothesis, furthermore $S[A,B] = K_{3q,6}$ is also L_3-decomposable by Theorem 2.2. Therefore S is L_3-decomposable.

Suppose $p \equiv 2 \pmod 3$. When $p = 2$, $S(2,1,1)$ is isomorphic to a L_3. When $p = 5$, $\{((1,1),(1,0),(3,0),(3,1)),((2,1),(2,0),(4,0),(4,1)),((5,1),(5,0),(3,0),(4,0)),((1,0),(4,0),(5,0),(2,0))$ $((5,0),(1,0),(2,0),(3,0))\}$ is a L_3-decomposition for S. When $p \geq 8$, express $p = 3q+2$ for some $q \geq 2$ since $p \equiv 2 \pmod 3$ and $p \geq 8$. Let $S = S(3q+2,1,1)$ with vertex set $V(S) = A \cup B$ where $A = \{(i,j) \mid 1 \leq i \leq 3q, j \in Z_2\}$ and $B = \{(i,j) \mid 3q+1 \leq i \leq 3q+2, j \in Z_2\}$. Then $S[A] = S(3q,1,1)$ and $S[B] = S(2,1,1)$ both are L_3-decomposable by the previous cases, furthermore $S[A,B] = K_{3q,2}$ is also L_3-decomposable by Theorem 2.2. Therefore S is L_3-decomposable.

Second we prove the $S(p,1,2)$ case. Let $S = S(p,1,2)$ has vertex set $V(S) = Z_p \times Z_3 = \{(i,j) \mid i \in Z_p, j \in Z_3\}$, where $S[\{(i,0) \mid i \in Z_p\}] = K_p$ and $((i,0),(i,1),(i,2))$ is a path of length 3 for all $i \in Z_p$. Suppose $p \equiv 0 \pmod 3$. We will prove by induction on p. Suppose $p = 3$. Clearly $S(3,1,2)$ is L_3-decomposiable, namely $\{((i,0),(i+1,0),(i+1,1),(i+1,2)) \mid i \in Z_p\}$. Assume $S(p,1,2)$ exists for $p \leq k$. When $p = k+3$, express $p = 3q+3$ with $q \geq 1$. Let $S = S(3q+3,1,2)$ with vertex set $V(S) = A \cup B$ where $A = \{(i,j) \mid i \in Z_{3q}, j \in Z_3\}$ and $B = \{(i,j) \mid 3q \leq i \leq 3q+2, j \in Z_3\}$. Then $S[A] = S(3q,1,2)$ and $S[B] = S(3,1,2)$ both are L_3-decomposable by induction hypothesis, furthermore $S[A,B] = K_{3q,3}$ is also L_3-decomposable by Theorem 2.2. Therefore S is L_3-decomposable. □

2.4 Main Result

We now prove the main theorem.

Theorem 2.6. *Suppose $n \geq 2$, $p \geq 2$ and $\lambda_2 \geq 1$. $G(n,p,\lambda_1,\lambda_2)$ has a decomposition into L_3's if and only if the following conditions hold:*

(1) 3 *divides* $|E(G)| = \frac{1}{2}\lambda_1 pn(n-1) + \frac{1}{2}\lambda_2 p(p-1)n^2$.

(2) If $n = 2$, *then* $\lambda_1 \leq 4(p-1)\lambda_2$.

(3) If $n = 3$, *then* $\lambda_1 \leq 3(p-1)\lambda_2$.

Proof. We first prove the necessity of conditions (1-3). The necessity of (1) follows from the total number of edges in $G = G(n,p,\lambda_1,\lambda_2)$ must be a multiple of 3, as the edge set can be partitioned into L_3's. Now suppose $n = 2$. First note that in any L_3-decomposition of $G(2,p,\lambda_1,\lambda_2)$, each copy of L_3 has at most 2 pure edges and therefore must have at least one mixed edge. Thus the number of pure edges is at most twice the number of mixed edges. Since there are $p\lambda_1$ pure edges and $4\frac{p(p-1)}{2}\lambda_2$ mixed edges, it follows that

$$\frac{1}{2}p\lambda_1 \leq 4\frac{p(p-1)}{2}\lambda_2$$

is a necessary condition for the existence of a L_3-decomposition of G. Finally suppose $n = 3$. Similarly as in $n = 2$ case, each copy of L_3 has at most 2 pure edges in any L_3-decomposition of $G(3,p,\lambda_1,\lambda_2)$. There are $3p\lambda_1$ pure edges and $9\frac{p(p-1)}{2}\lambda_2$ mixed edges and the inequality follows. So (2) and (3) are necessary.

We now turn to the sufficiency, considering five cases in turn. The first two cases use Lemmas 2.3 and 2.4, finding a suitable partition of M. The last three cases produce an L_3-decomposition $(V(G),B)$ of G by using Lemmas 2.1 and 2.2. The cases are:

1. $n = 2$.

2. $n = 3$.

3. $n \geq 4, n \equiv 1 \pmod 3$.

4. $n \geq 5, n \equiv 2 \pmod 3$.

5. $n \geq 6, n \equiv 0 \pmod 3$.

Case 1: $n = 2$. First suppose $p = 2$, so the set of difference is $M = \lambda_2\{1\} \cup \lambda_1\{2\}$. Express the number of pure differences as $\lambda_1 = 4t + r$ for some integers t, r with $0 \leq r \leq 3$, then let $\lambda_2 - t = s$ for some integer s; so by necessary condition (2), $s \geq 0$. Furthermore, by (2) it follows that if $r \geq 1$ then $s \geq 1$. Moreover, since $|E(G)| = 2\lambda_1 + 4\lambda_2 = 12t + 2r + 4s$, which by (1) is divisible by 3: if $r = 2$ then $s \geq 2$ and if $r = 3$ then $s \geq 3$. In particular, $s \geq r$.

We begin by forming a partition Π of $M = (s + t)\{1\} \cup (4t + r)\{2\}$. Let Π contain t copies of $\{1, 2, 2, 2, 2\}$, $s - r$ copies $\{1\}$ and r copies of $\{1, 2\}$; it was just shown that $s \geq r$, so this is possible. By condition (1), 3 divides $|E(G)| = 2\lambda_1 + 4\lambda_2 = 12t + 2r + 4s$, thus 3 divides $(2r + 4s)$. Therefore, writing $s - r = (2r + 4s) - (3r + 3s)$, it follows that the right hand side of the equation is divisible by 3, so 3 divides $s - r$. Then, by Lemma 2.3 case (8), (1) and (2) respectively, Π induces an L_3-decomposition of G.

Next suppose $p = 3$, so $M = \lambda_2\{1, 2\} \cup \lambda_1\{3\}$. Express the number of pure differences as $\lambda_1 = 4t + r$ for some integers t, r with $0 \leq r \leq 3$. Then let $2\lambda_2 - t = 2u + s$ for some integers u, s where $u \geq 0$, if $r = 0$ then $s \in \{0, 1\}$ and if $r \geq 1$ then $s \in \{1, 2\}$. Such u, s always exist by necessary condition (2). Let $n_1 = 0$ if $(r, s) \in \{(0, 0), (1, 1), (2, 1)\}$ and 1 otherwise. Let $n_2 = 0$ if $(r, s) \in \{(0, 0), (0, 1), (3, 1)\}$ and 1 otherwise (It will be useful later to note that in all 8 cases, $n_1 + n_2 = s$.) We now form the partition Π of $M = \frac{1}{2}(t + 2u + s)\{1, 2\} \cup (4t + r)\{3\}$. Let Π contain: $\min\{t, \lambda_2 - n_2\}$ copies of $\{2, 3, 3, 3, 3\}$; $\max\{t - (\lambda_2 - n_2), 0\}$ copies of $\{1, 3, 3, 3, 3\}$; $(\lambda_2 - n_2 - \min\{t, \lambda_2 - n_2\})$ copies of $\{1, 2\}$; $(\lambda_2 - n_1 - \max\{t - (\lambda_2 - n_2), 0\} - (\lambda_2 - n_2 - \min\{t, \lambda_2 - n_2\}))$ copies of $\{1\}$; and also 0, 1 or 2 more sets depending on the values of r and s as described in Table 2.2.

Table 2.2: $n = 2, p = 3$

(r,s)	Π also contains	(r,s)	Π also contains
(0,0)	None	(2,1)	$\{2,3,3\}$
(0,1)	$\{1\}$	(2,2)	$\{1,3\},\{2,3\}$
(1,1)	$\{2,3\}$	(3,1)	$\{1,3,3,3\}$
(1,2)	$\{1\},\{2,3\}$	(3,2)	$\{1,3\},\{2,3,3\}$

We now prove this partition Π of M is always possible. We have two cases. Recall that $\lambda_2 \geq 1$ and $n_2 \leq 1$.

First, if $t \leq \lambda_2 - n_2$ then Π has t copies of $\{2,3,3,3,3\}$, 0 copies of $\{1,3,3,3,3\}$, $(\lambda_2 - n_2 - t)$ copies of $\{1,2\}$, $(t + n_2 - n_1)$ copies of $\{1\}$ and up to two sets from Table 2.2. Being in the first case implies that $\lambda_2 - n_2 - t \geq 0$. Clearly $t + n_2 - n_1 \geq 0$ unless possibly when $t = 0$, $n_2 = 0$ and $n_1 = 1$. This exceptional case can not happen since if $t = 0$, $n_2 = 0$ and $n_1 = 1$, then $(r,s) \in \{(0,1),(3,1)\}$, so $s = 1$. Thus $2\lambda_2 = 2\lambda_2 - t = 2u + s = 2u + 1$, a contradiction.

Second, suppose $t > \lambda_2 - n_2$. Then Π has $\lambda_2 - n_2$ copies of $\{2,3,3,3,3\}$, $t - (\lambda_2 - n_2)$ copies of $\{1,3,3,3,3\}$, 0 copies of $\{1,2\}$, $(\lambda_2 - n_1 - t + \lambda_2 - n_2)$ copies of $\{1\}$ and up to two sets from Table 2.2. Being in the second case, $t - (\lambda_2 - n_2) > 0$. Also $(\lambda_2 - n_1 - t + \lambda_2 - n_2) = 2\lambda_2 - t - n_1 - n_2 = t + 2u + s - t - n_1 - n_2 = 2u + s - n_1 - n_2 = 2u \geq 0$.

In both cases, it is easy to check that Π contains exactly λ_2 copies of differences 1 and 2 and λ_1 copies of differences 3.

Finally assume that $p \geq 4$. We construct the decomposition on the vertex set Z_{2p} with parts $P_i = \{i, i+p\}$ for each $i \in Z_p$.

Again we consider a few cases. Write the number of pure differences as $\lambda_1 = 4t + r$ for some integers t, r with $0 \leq r \leq 3$, then let $(p-1)\lambda_2 - t = 3u + s$ for some integers u, s with $u \geq 0$ and $0 \leq s \leq 3$ where $s \geq 1$ whenever $r \geq 1$. Such u, s always exist by necessary condition (2). We begin by placing r pure differences and s mixed differences into multisets in Π, as defined in Table 2.3. (Whenever the difference $\frac{p}{3}$ appears in the table, it will be shown that $p \equiv 0 \pmod 3$.)

Table 2.3: $n = 2, p \geq 4$

(r, s)	Π also contains	(r, s)	Π also contains
(0,0)	None		
(0,1)	$\{1\}$	(2,1)	$\{1, p, p\}$
(0,2)	$\{1, 2\}$	(2,2)	$\{p, d\}, \{p, d'\}$
(0,3)	$\{d, d', d''\}$	(2,3)	$\{1\}, \{p, d\}, \{p, d'\}$
(1,1)	$\{p, d\}$	(3,1)	$\{\frac{p}{3}, p, p, p\}$
(1,2)	$\{1\}, \{p, d\}$	(3,2)	$\{1\}, \{\frac{p}{3}, p, p, p\}$
(1,3)	$\{1, 2\}, \{p, d\}$	(3,3)	$\{p, d\}, \{p, d'\}, \{p, d''\}$
Where d, d', d'' are arbitrary distinct mixed differences.			

Next place into Π, u sets containing three distinct mixed differences in M. This can be done greedily with the proviso that for $1 \leq x < \lambda_2$ each difference is placed in x elements of Π before it occurs in the $(x + 1)^{th}$ element of Π; note that since $p \geq 4$ there are at least 3 different mixed differences.

There now remain t mixed differences and $4t$ copies of p in M which do not occur in a set currently in Π; partition these into t sets of size 5, each of which contains exactly one of the remaining mixed differences and place these sets Π.

We now prove that 3 divides p whenever the difference $\frac{p}{3}$ appears in the Table 2.3. Note $\frac{p}{3}$ only appears in Table 2.3 when $r = 3$ and $s = 1$ or 2. By necessary condition (1), $|E(G)| = \lambda_1 p + (p-1)\lambda_2 2p = (4t + r)p + (t + 3u + s)2p = 6tp + 6up + rp + 2sp$ is divisible by 3 . Therefore $3 \mid rp + 2sp$. Thus $3 \mid 5p$ and $3 \mid 7p$ when $(r, s) = (3, 1)$ and $(3, 2)$, respectively. In either case, it follows that 3 divides p.

Case 2: $n = 3$**.** Following the approach of Case 1, a partition Π of $M = \lambda_2\{i \mid 1 \leq i \leq \lfloor \frac{3p}{2} \rfloor, i \neq p\} \cup \lambda_1\{p\}$ is defined below, such that by Lemma 2.4 there exists an L_3-decomposition of $G_{3p}(D)$ for each $D \in \Pi$.

First suppose 2 divides p. Then the set of difference is $M = \lambda_2\{i \mid 1 \leq i \leq \frac{3p}{2}, i \neq p\} \cup \lambda_1\{p\}$. We now form the partition Π of M.

Let Π contain: (a) $\min\{\lambda_1, \lambda_2\}$ copies of $\{\frac{3p}{2}, p\}$; (b) $\{d_i, p, p\}$ for $1 \leq i \leq \max\{\lfloor \frac{\lambda_1 - \lambda_2}{2} \rfloor, 0\}$ with $d_i \notin \{p, \frac{3p}{2}\}$; (c) $(\lambda_1 - \min\{\lambda_1, \lambda_2\} - 2\max\{\lfloor \frac{\lambda_1 - \lambda_2}{2} \rfloor, 0\})$ copies of $\{p - 1, p\}$; and (d) $\{d_i, \frac{3p}{2}\}$ for $1 \leq i \leq \max\{\lambda_2 - \lambda_1, 0\}$ with $d_i \notin \{p, \frac{3p}{2}\}$. The particular assignment of the mixed differences to these sets can be done greedily in a way that ensures that if U is the set of unused differences then $\Delta = \{\delta_1, \delta_2\} \subseteq U$ if $|U| \equiv 2 \pmod 3$ and $\Delta = \{\delta_1\} \subseteq U$ if $|U| \equiv 1 \pmod 3$, where $\delta_1 = 1$ and $\delta_2 \in \{1, 2\}$. Note that clearly $\lambda_1 - \min\{\lambda_1, \lambda_2\} - 2\max\{\lfloor \frac{\lambda_1 - \lambda_2}{2} \rfloor, 0\} \geq 0$. Also notice that in the elements of Π just identified there occur λ_1 copies of p and λ_2 copies of $\frac{3p}{2}$.

Complete the formation of Π as follows. If $|U| \equiv 0$, 1 or 2 (mod 3) then let P be a partition of U, $U \setminus \{\delta_1\}$ or $U \setminus \{\delta_1, \delta_2\}$ into sets of size 3 respectively. Let Π contain: (e) the elements of P, and (f) the set Δ.

Use Lemma 2.4 case (2), (4), (6), (3), (5), and (1) (or (6) if $\Delta = \{1, 2\}$) to obtain the required L_3-decomposition in the cases (a-f) respectively.

Second, suppose 2 does not divide p; then $p \geq 3$ (since $p \geq 2$ is assumed). Then the set of differences is $M = \lambda_2\{i \mid 1 \leq i \leq \frac{3p-1}{2}, i \neq p\} \cup \lambda_1\{p\}$. Let $\lambda_1 = 2t + r$ where t, r are nonnegative integers with $0 \leq r \leq 1$. Then define nonnegative integers u and s with $0 \leq s \leq 2$ by letting $(\frac{3p-1}{2} - 1)\lambda_2 - t - r = 3u + s$. This is always possible by necessary condition (3).

We begin by placing into Π: (a) $\{1\}$ if $s = 1$ and $\{1, 2\}$ if $s = 2$, and put (b) r copies of $\{4, p\}$ into Π. This is always possible since $p \geq 3$ in this case. Next we put into Π (c) $3u$ mixed differences partitioned into sets of size three. There now remain t mixed differences and $2t$ pure differences in M to be placed in sets in Π: (d) partition them into t sets of size 3, each of which contains exactly one of the mixed differences.

Use Lemma 2.4 cases (1) (or (6) if $s = 2$), (6), (5) and (4) to obtain required the L_3-decomposition in the cases (a-d) respectively.

Case 3: $n \equiv 1$ **(mod** 3**), and** $n \geq 4$. The pure edges induce p copies of $\lambda_1 K_n$, each of which is L_3-decomposable by Theorem 2.1; so let (V, B') be an L_3-decomposition of the graph induced by all the pure edges. Thus it remains to consider the mixed edges in G. Consider three cases, forming an L_3-decomposition (V, B) of G in each case.

If $p = 2$ then $\lambda_2 \equiv 0 \pmod 3$ by (1). Let $B = \frac{\lambda_2}{3} D_{1,2}(x_1, x_2) \cup \frac{\lambda_2}{3} D_{1,2}(y_1, x_2) \cup \frac{\lambda_2}{3} D_{1,2}(y_1, y_2) \cup \lambda_2\{(x_1, x_2, y_1, y_2)\} \cup B'$, where $y_1 \neq x_1$ and $y_2 \neq x_2$.

If $p = 3$ then let $B = \lambda_2 D_{1,2}(x_1, x_2) \cup \lambda_2 D_{2,3}(x_2, x_3) \cup \lambda_2 D_{1,3}(y_1, x_3) \cup \lambda_2\{(x_1, x_2, x_3, y_1)\} \cup B'$, where $y_1 \neq x_1$.

If $p \geq 4$ then, by (1), either $\lambda_2 \equiv 0 \pmod 3$ or $p \equiv 0$ or $1 \pmod 3$. In both cases, by Theorem 2.1, there exists an L_3-decomposition $(\{x_i \mid x_i \in V_i, 1 \leq i \leq p\}, B_1)$ of $\lambda_2 K_p$. Let $B = B' \cup B_1 \cup (\bigcup_{1 \leq i < j \leq p} \lambda_2 D_{i,j}(x_i, x_j))$.

Case 4: $n \equiv 2$ **(mod** 3**), and** $n \geq 5$. We begin by considering a special case when $\lambda_1 \equiv \lambda_2 \equiv 0 \pmod 3$. Note $\lambda_1 K_n$ and $\lambda_2 K_{n,n}$ are both L_3-decomposable by Theorem 2.1 and Theorem 2.3 respectively, thus so is $G(n, p, \lambda_1, \lambda_2)$. So suppose either $\lambda_1 \not\equiv 0 \pmod 3$ or $\lambda_2 \not\equiv 0 \pmod 3$. For $1 \leq i \leq p$, let x_i, y_i, z_i, w_i be 4 distinct vertices in V_i (recall $n \geq 5$).

Suppose $p = 3$.

For $1 \leq i \leq 3$ and $1 \leq k \leq \lambda_1$, let

$$\{x_{i,k}, y_{i,k}\} = \begin{cases} \{x_i, y_i\}, & \text{if } k \equiv 1 \pmod 3 \\ \{y_i, z_i\}, & \text{if } k \equiv 2 \pmod 3 \\ \{z_i, w_i\}, & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

and place $(V_i, D_i(n, \{x_{i,k}, y_{i,k}\}))$ into B. For $1 \leq i \leq 3$ let B contain $\lfloor \frac{\lambda_1}{3} \rfloor$ copies of (x_i, y_i, z_i, w_i). Then for $1 \leq i \leq 3$, there are at most two pure edges in $G[V_i]$ remaining to place in 3-paths: none if $\lambda_1 \equiv 0 \pmod 3$, $\{x_i, y_i\}$ if $\lambda_1 \equiv 1 \pmod 3$ and $\{x_i, y_i\}$ and $\{y_i, z_i\}$ if $\lambda_1 \equiv 2 \pmod 3$.

For $1 \leq i < j \leq 3$, let

$$\{x_{i,j}, y_{i,j}\} = \begin{cases} \{x_i, x_j\}, & \text{if } (i,j) \neq (1,3) \\ \{y_1, x_3\}, & \text{if } (i,j) = (1,3) \end{cases}$$

and place into B λ_2 copies of $(V_i, V_j, D_{i,j}(n, n, \{x_{i,j}, y_{i,j}\}))$. Let B contain $\lambda_2 - 1$ copies of (x_1, x_2, x_3, y_1). Then the mixed edges remaining to be place in 3-paths are $\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, y_1\}\}$.

If $\lambda_1 \equiv 0 \pmod 3$, then place (x_1, x_2, x_3, y_1) into B to complete the decomposition.

If $\lambda_1 \equiv 1 \pmod 3$, then the leaves remaining are $\{\{x_i, y_i\} \mid 1 \le i \le 3\} \cup \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, y_1\}\}$. Place (x_2, x_1, y_1, x_3) and (y_2, x_2, x_3, y_3) into B.

If $\lambda_1 \equiv 2 \pmod 3$, then the leaves remaining are $\{\{x_i, y_i\}, \{y_i, z_i\} \mid 1 \le i \le 3\} \cup \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, y_1\}\}$. Place (z_1, y_1, x_1, x_2), (z_2, y_2, x_2, x_3) and (y_1, x_3, y_3, z_3) into B.

So it remains to consider the case where $p = 2$ or $p \ge 4$. For $1 \le i \le p$ and $1 \le k \le \lambda_1$, let

$$\{x_{i,k}, y_{i,k}\} = \begin{cases} \{x_i, y_i\}, & \text{if } k \equiv 1 \pmod 3 \\ \{y_i, z_i\}, & \text{if } k \equiv 2 \pmod 3 \\ \{z_i, w_i\}, & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

where $\{x_i, y_i, z_i, w_i\} \subseteq V_i$, and place $(V_i, D_i(n, \{x_{i,k}, y_{i,k}\}))$ into B. For $1 \le i \le p$ let B contain $\lfloor \frac{\lambda_1}{3} \rfloor$ copies of (x_i, y_i, z_i, w_i). Then for $1 \le i \le p$, there are at most two pure edges in $G[V_i]$ remaining to place in 3-paths: none if $\lambda_1 \equiv 0 \pmod 3$, $\{x_i, y_i\}$ if $\lambda_1 \equiv 1 \pmod 3$ and $\{x_i, y_i\}$ and $\{y_i, z_i\}$ if $\lambda_1 \equiv 2 \pmod 3$.

For $1 \le i < j \le p$ and $1 \le k \le \lambda_2$, let

$$\{x_{i,j,k}, y_{i,j,k}\} = \begin{cases} \{x_i, x_j\}, & \text{if } k \equiv 1 \pmod 3 \text{ or } k \ge 3\lfloor \frac{\lambda_2 - 1}{3} \rfloor + 1 \\ \{y_i, x_j\}, & \text{if } k \equiv 2 \pmod 3, k \le 3\lfloor \frac{\lambda_2 - 1}{3} \rfloor \\ \{y_i, y_j\}, & \text{if } k \equiv 0 \pmod 3, k \le 3\lfloor \frac{\lambda_2 - 1}{3} \rfloor \end{cases}$$

and place $(V_i, V_j, D_{i,j}(n, n, \{x_{i,j,k}, y_{i,j,k}\}))$ into B. For $1 \le i < j \le p$ let B contain $\lfloor \frac{\lambda_2 - 1}{3} \rfloor$ copies of (x_i, x_j, y_i, y_j). Then for $1 \le i < j \le p$, there are at most three mixed edges in $G[Vi, V_j]$ remaining to place in 3-paths: $3\{x_i, x_j\}$ if $\lambda_2 \equiv 0 \pmod 3$, $\{x_i, x_j\}$ if $\lambda_2 \equiv 1 \pmod 3$ and $2\{x_i, x_j\}$ if $\lambda_2 \equiv 2 \pmod 3$.

Now we consider 8 cases in turn.

If $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 1 \pmod 3$, then $p \equiv 0$ or $1 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, x_j\} \mid 1 \le i < j \le p\}$ is isomorphic to K_p, which is L_3-decomposable by Theorem 2.1.

If $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 2 \pmod 3$, then $p \equiv 0$ or $1 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $2\{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $2K_p$, which is L_3-decomposable by Theorem 2.1.

If $\lambda_1 \equiv 1 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$, then $p \equiv 0 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, y_i\} \mid 1 \leq i \leq p\} \cup 3\{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $2K_p \cup S(p, 1, 1)$, which is L_3-decomposable by Theorem 2.1 and Lemma 2.5 respectively.

If $\lambda_1 \equiv \lambda_2 \equiv 1 \pmod 3$, then $p \equiv 0$ or $2 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, y_i\} \mid 1 \leq i \leq p\} \cup \{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $S(p, 1, 1)$, which is L_3-decomposable by Lemma 2.5.

If $\lambda_1 \equiv 1 \pmod 3$ and $\lambda_2 \equiv 2 \pmod 3$, then $p \equiv 0 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, y_i\} \mid 1 \leq i \leq p\} \cup 2\{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $K_p \cup S(p, 1, 1)$, which is L_3-decomposable by Theorem 2.1 and Lemma 2.5 respectively.

If $\lambda_1 \equiv 2 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$, then $p \equiv 0 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, y_i\}, \{y_i, z_i\} \mid 1 \leq i \leq p\} \cup 3\{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $2K_p \cup S(p, 1, 2)$, which is L_3-decomposable by Theorem 2.1 and Lemma 2.5 respectively.

If $\lambda_1 \equiv 2 \pmod 3$ and $\lambda_2 \equiv 1 \pmod 3$, then $p \equiv 0 \pmod 3$ by necessary condition (1). The graph induced by the set of leaves $\{\{x_i, y_i\}, \{y_i, z_i\} \mid 1 \leq i \leq p\} \cup \{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $S(p, 1, 2)$, which is L_3-decomposable by Lemma 2.5.

If $\lambda_1 \equiv \lambda_2 \equiv 2 \pmod 3$, then $p \equiv 0$ or $2 \pmod 3$ by necessary condition (1). If $p = 2$ then the leaves induce $S(2, 2, 2)$, which has L_3-decomposition $(x_2, x_1, y_1, z_1) \cup (x_1, x_2, y_2, z_2)$. If $p \geq 4$ and $p \equiv 0 \pmod 3$, then graph induced by the set of leaves $\{\{x_i, y_i\}, \{y_i, z_i\} \mid 1 \leq i \leq p\} \cup 2\{\{x_i, x_j\} \mid 1 \leq i < j \leq p\}$ is isomorphic to $K_p \cup S(p, 1, 2)$, which is L_3-decomposable by Theorem 2.1 and Lemma 2.5 respectively. If $p \geq 4$ and $p \equiv 2 \pmod 3$, then let $p = 3q + 2$ where q is an integer with $q \geq 1$. The graph induced by the leaves is isomorphic to $S(2,2,2) \cup K_{2,3q} \cup K_{3q} \cup S(3q,1,2)$, which is L_3-decomposable by above case, Theorem 2.2, Theorem 2.1 and Lemma 2.5 respectively.

Case 5: $n \equiv 0$ (**mod** 3)**, and** $n \geq 6$. Since 3 divides $|E(K_n)| = \frac{1}{2}n(n-1)$, by Theorem 2.1 there exists an L_3-decomposition (V_i, B_i) of $\lambda_1 K_n$ for $1 \leq i \leq p$. By Theorem 2.2 there exists an L_3-decomposition $(V_i, V_j, B_{i,j})$ of $\lambda_2 K_{n,n}$ for $1 \leq i < j \leq p$. Then $(V(G), (\bigcup_{1 \leq i \leq p} B_i) \cup (\bigcup_{1 \leq i < j \leq p} B_{i,j}))$ is the required decomposition.

□

Chapter 3

Euler Circuits With Large Minimum Distance in Graphs With Two Associate Classes

3.1 Basics

A latin square of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and column. A transversal of a latin square of order n is a set of n entries with no pair of entries that share the same row, column or symbol.

For the rest of the chapter, we will assume that n is an odd positive integer with $n = 2k+1$ for some integer k.

We define L to be the $n \times n$ array with $(i,j)^{\text{th}}$ entry $L_{i,j} = (i+j)(k+1) \pmod{n}$ for $i, j \in \mathbb{Z}_n$. The following is well-known but proof is included for completeness.

Lemma 3.1. *L is an idempotent latin square. Moreover, entries of L can be partitioned into n transversals.*

Proof. To see L is a latin square, note that L is obtained by renaming the addition table of Z_n by multipling each entry with $(k+1)$. L is idempotent since for $0 \leq i \leq n-1$, $L_{i,i} \equiv 2i(k+1) \equiv 2i(2^{-1}) \equiv i \pmod{2k+1}$.

We now proceed to prove L can be partitoned into n transversals. For $0 \leq i \leq n-1$, let $T_i = \{L_{i+j,j} \mid 0 \leq j \leq n-1\}$ where the subindices are calculated modulo n. We claim that $\{T_i \mid 0 \leq i \leq n-1\}$ is a set of transversals that partition the entries of latin square L. Suppose for some $0 \leq j \neq j' \leq n-1$, $L_{i+j,j} = L_{i+j',j'}$. Thus $((i+j)+j)(k+1) \equiv ((i+j')+j')(k+1) \pmod{2k+1}$. Clearly this implies $j = j'$ since $k+1 = 2^{-1}$ and so T_i is indeed a transversal. It is easy to see $\{T_i \mid 0 \leq i \leq n-1\}$ partitions the cells of L by definition of the T_i's. □

The graph $G(n, p, 0, 1)$ is more commonly denoted by $K(n, p)$, since it is the complete multipartite graph of p parts with equal part sizes n. Let $v_{i,j}$ denote the ith vertex of jth part of $K(n, p)$ for $0 \leq i \leq n-1$ and $0 \leq j \leq p-1$.

A walk is a sequence $(v_0, e_1, v_1, e_1, ..., v_k)$ of vertices v_i and edges e_i in a graph such that for $1 \leq i \leq k$, e_i has endpoints v_{i-1} and v_i. We often omit the edges while writing down a walk. A walk is called closed if it starts and ends at the same vertex. A trail is a walk without repeated edges. An Euler circuit of graph G is a closed trail which includes every edge of G exactly once.

If $W_1, W_2, ..., W_x$ are walks in G where, for $1 \leq i \leq x-1$, W_i ends at the same vertex where W_{i+1} starts, then define the walk $W = (W_1, W_2, ..., W_x)$ to be the concatenation of these x walks. The distance of two appearances of the same vertex v in a walk W is the number of edges between the two appearances of v along W. The distance of vertex v in a walk W, denoted by $d_W(v)$, is the least distance among all pairs of appearances of v along W. In other words, it is the length of the shortest closed walk in W starting at v. The diameter $d(W)$ of a walk W is defined by $d(W) = \min\{d_W(v) \mid v \in V(W)\}$ (i.e. the minimum distance of all vertices in W).

For the rest of the chapter, let p be an odd positive integer. Define the hamilton cycle $H = (0, 1, 2, p-1, 3, p-2, ..., \frac{p-1}{2}, \frac{p+3}{2}, \frac{p+1}{2})$, and σ be the permutation $(0)(123...(p-2)(p-1))$. Let $H_i = \sigma^i(H)$. Then $\{H_i \mid 1 \leq i \leq \frac{p-1}{2}\}$ is a hamilton cycle decomposition of K_p whose vertices are labelled $\{0, 1, 2, ..., p-1\}$. This is known as the Walecki construction. By Tarsi's result [58], the diameter of the walk $(H_1, H_2, ..., H_{(p-1)/2})$ is $p-2$.

For a graph G, define $spread(G) = \max\{d(E) \mid E$ is an Euler circuit of $G\}$. Ramirez-Alfonsin [53] showed the spread of $K_{4m+1} \geq 2m-1$. In [49], Oksimets showed $p - 4 \leq spread(K_p) \leq p-2$ for $p \geq 5$ and $d(K_{2n,2n}) = 4n-4$ for $n \geq 2$.

Given hamilton cycle H_i of K_p, we now construct a family of hamilton cycles of $K(n, p)$. Let $\pi_i(j)$ be the jth vertex in the hamilton cycle H_i for $1 \leq j \leq p$ and $1 \leq i \leq (p-1)/2$.

For $0 \le a, b \le n-1$, define $P_{i,a,b} = (v_{\pi_i(1),x(j)}, v_{\pi_i(2),x(j)}, ..., v_{\pi_i(j),x(j)}, ..., v_{\pi_i(p),x(j)}, v_{\pi_i(1),x(j)+1})$ (addition modulo n) where

$$x(j) = \begin{cases} a & \text{if } j \text{ is odd, } j \ne p, \\ b & \text{if } j \text{ is even, and} \\ L_{a,b} & \text{if } j = p, \end{cases}$$

and L is the latin square in Lemma 3.1. Let $C_{i,a,b}$ be $(P_{i,a,b}, P_{i,a+1,b+1}, ..., P_{i,a+n-1,b+n-1})$ and $E_i = (C_{i,0,0}, C_{i,0,1}, ..., C_{i,0,n-1})$. We show next that $C_{i,a,b}$ is a hamilton cycle of $K(n,p)$ and $E = (E_1, E_2, ..., E_{\frac{p-1}{2}})$ is an Euler circuit of $K(n,p)$.

Throughout the chapter, all the second and third subindex of $P_{i,a,b}$ and $C_{i,a,b}$ are module n.

3.2 Lemmas

Lemma 3.2. *For* $1 \le i \le \frac{p-1}{2}$ *and* $0 \le a, b \le n-1$, $C_{i,a,b}$ *is a hamilton cycle of* $K(n,p)$*. Moreover, for* $0 \le x \le p-1$ *and* $0 \le y \le n-1$, $v_{x,y}$ *is the* $(zp+w)^{th}$ *vertex of* $C_{i,a,b}$ *where* $w = \pi_i^{-1}(x)$ *and* z *is given by*

$$z = \begin{cases} y - a \pmod n & \text{if } w \text{ is odd, } w \ne p, \\ y - b \pmod n & \text{if } w \text{ is even, and} \\ y + k(a+b) \pmod n & \text{if } w = p. \end{cases} \tag{3.1}$$

Proof. We first prove that $C_{i,a,b}$ is a closed walk of $K(n,p)$. Note the last vertex of $P_{i,a,b}$ and the first vertex of $P_{i,a+1,b+1}$ is the same vertex, namely $v_{\pi_i(1),a+1}$. Furthermore, the first and last vertex of $C_{i,a,b}$ is the same one, namely $v_{\pi_i(1),a}$. This shows $C_{i,a,b}$ is a closed walk.

We now show property (1), in particular this shows that each vertex appears at least once in $C_{i,a,b}$. For any vertex $v_{x,y}$, $0 \le x \le p-1$ and $0 \le y \le n-1$, let $w = \pi_i^{-1}(x)$. There are three cases.

Case 1: w is odd, $w \neq p$. Then $v_{x,y}$ appears in $P_{i,a+j,b+j}$, $0 \leq j \leq n-1$ only when $y = a+j$. Thus $j = y - a \pmod n$ and $v_{x,y}$ is the wth vertex in $P_{i,y,b+y-a}$. Moreover, prior to $P_{i,a+j,b+j}$ there are $j = y-a$ paths in $C_{i,a,b}$, namely $P_{i,a,b}$, $P_{i,a+1,b+1}, ..., P_{i,a+j-1,b+j-1}$. Therefore $v_{x,y}$ is the $((y-a)p+w)$th vertex in $C_{i,a,b}$.

Case 2: w is even. Clearly $w \neq p$. Then $v_{x,y}$ appears in $P_{i,a+j,b+j}$, $0 \leq j \leq n-1$ only when $y = b+j$. Thus $j = y - b \pmod n$ and $v_{x,y}$ is the wth vertex in $P_{i,a+y-b,y}$. Similarly as in case 1, there are $j = y - b$ paths in $C_{i,a,b}$ prior to $P_{i,a+y-b,y}$. Therefore $v_{x,y}$ is the $((y-b)p+w)$th vertex in $C_{i,a,b}$.

Case 3: $w = p$. Then $v_{x,y}$ is the pth vertex in $P_{i,a+j,b+j}$, for some unique j with $L_{a+j,b+j} = y$ by Lemma 3.1. $L_{a+j,b+j} = (a+j+b+j)(k+1) \equiv (a+b+2j)(k+1) \equiv (a+b)(k+1)+j(2k+2) \equiv (a+b)(k+1)+j \pmod{2k+1}$. Thus $j \equiv y-(a+b)(k+1) \equiv y-(a+b)(k+1)+(2k+1)(a+b) \equiv y+k(a+b) \pmod{2k+1}$, and $v_{x,y}$ is the wth vertex in $P_{i,a+y+k(a+b),b+y+k(a+b)}$. Again there are $j = y+k(a+b)$ paths in $C_{i,a,b}$ prior to $P_{i,a+y+k(a+b),b+y+k(a+b)}$. Therefore $v_{x,y}$ is the $((y+k(a+b))p+w)$th vertex in $C_{i,a,b}$.

Since $C_{i,a,b}$ has length np and hence by cases 1-3 each vertex appear at least once in $C_{i,a,b}$, each vertex appears exactly once in $C_{i,a,b}$. So $C_{i,a,b}$ is a hamilton cycle.

□

Lemma 3.3. *E is an Euler circuit of $K(n,p)$ and the diameter of E is at least $\frac{n-3}{2}p+1$.*

Proof. We begin by proving $E = (E_1, E_2, ..., E_n)$ is an Euler circuit of $K(n,p)$. First of all, for any $C \in \{C_{i,1,j} \mid 1 \leq i \leq \frac{p-1}{2}, 0 \leq j \leq n-1\}$, the first and last vertex is $v_{\pi_i(1),0} = v_{0,0}$ since $\pi_i(1) = 0$ for all i. Therefore, E is indeed a closed walk.

We now show that every edge appears in E exactly once. Let $e = \{v_{u,v}, v_{u',v'}\}$, $u \neq u'$ be an edge of $K(n,p)$. Since $\{H_i \mid 1 \leq i \leq \frac{p-1}{2}\}$ is a hamilton cycle decomposition of K_p, there is a unique i such that the edge $\{u,u'\}$ is in H_i. Thus e can only possibly appear in E_i. Without loss of generality, assume that $\pi_i(w) = u$ and $\pi_i(w+1) = u'$ for some w, $1 \leq w \leq p$. We have four cases.

Case 1: $w \neq p$ and w is odd. Clearly, $w+1 \neq p$ since $w+1$ is even. By definition of E, e only appears in $P_{i,v,v'}$, which is a part of $C_{i,0,v-v'}$.

Case 2: w is even and $w \neq p-1$. By definition of E, e only appears in $P_{i,v',v}$, which is a part of $C_{i,0,v'-v}$.

Case 3: w is even and $w = p-1$. By definition of E, e only appears in $P_{i,x,v'}$, where $0 \leq x \leq n-1$ with $L_{x,v'} = v$. By Lemma 3.1, x is unique, namely $2v - v'$.

Case 4: $w = p$. By definition of E, e only appears in $P_{i,x,v'-1}$, where $0 \leq x \leq n-1$ with $L_{x,v'-1} = v$. By Lemma 3.1, x is unique, namely $2v - v' + 1$.

This concludes the proof that E is an Euler circuit of $K(n,p)$. We now move to determine the diameter of E.

For any i, j, $C_{i,0,j}$ is a hamilton cycle and no vertex appears twice except the start/end vertex $v_{0,0}$. Since E is obtained by concatenating these cycles, $d_E(v_{0,0}) = np$. For vertex $v_{x,y} \neq v_{0,0}$, any two appearances of $v_{x,y}$ must be in different hamilton cycles. By definition of E, there are two cases: (1) $v_{x,y}$ is in $C_{i,0,a}$ and $C_{i,0,a+1}$, for some a, $0 \leq a \leq n-2$ and (2) $v_{x,y}$ is in $C_{i,0,n-1}$ and $C_{i+1,0,0}$, for some i, $1 \leq i \leq \frac{p-3}{2}$.

Case 1: the consecutive appearances of $v_{x,y}$ are in $C_{i,0,a}$ and $C_{i,0,a+1}$, for some a, $0 \leq a \leq n-2$. The distance of $v_{x,y}$ in $(C_{i,0,a}, C_{i,0,a+1})$ is determined by finding its location in $C_{i,0,a}$ and $C_{i,0,a+1}$, respectively. Let $w = \pi_i^{-1}(x)$.

i) Suppose w is odd and $w \neq p$.

By Lemma 3.2, $v_{x,y}$ is the $((y-1)p+w)$th vertex in both $C_{i,0,a}$ and $C_{i,0,a+1}$. Clearly, the distance of two appearances of $v_{x,y}$ is np.

ii) Suppose w is even. Clearly $w \neq p$.

By Lemma 3.2, $v_{x,y}$ is the $((y-a)p+w)$th and $((y-(a+1))p+w)$th vertex in $C_{i,0,a}$ and $C_{i,0,a+1}$, respectively. The distance is $np - ((y-a)p+w) + ((y-(a+1))p+w) = (n-1)p$.

iii) Suppose $w = p$.

By Lemma 3.2, $v_{x,y}$ is the $((y+k(1+a))p+w)$th and the $((y+k(1+a+1))p+w)$th vertex in $C_{i,0,a}$ and $C_{i,0,a+1}$, respectively. The distance is $np-((y+k(1+a))p+w)+((y+k(1+a+1))p+w)=(n+k)p=\frac{3n-1}{2}p$.

Case 2: the consecutive appearances of $v_{x,y}$ are in $C_{i,0,n-1}$ and $C_{i+1,0,0}$, for some i, $1 \le i \le \frac{p-3}{2}$. Let $w=\pi_i^{-1}(x)$ and $w'=\pi_{i+1}^{-1}(x)$. We have six cases, as showed in the table below:

	w' is odd, $w \neq p$	w' is even	$w'=p$
w is odd, $w \neq p$	Case 1	Case 1	Case 3
w is even	Case 2	Case 2	Case 4
$w=p$	Case 5	Case 5	Case 6

The following facts from Lemma 3.2 are used repeatedly in this case.

		location of the vertex
1*	w is odd, $w \neq p$	$yp+w$
2*	w is even	$(y+1)p+w$ if $y \neq n-1$, w if $y=n-1$
3*	$w=p$	$(y-k+1)p$ if $y \geq k$, $(y+k+2)p$ if $y<k$
4*	w' is odd, $w \neq p$	$yp+w'$
5*	w' is even	$yp+w'$
6*	$w'=p$	$(y+1)p$

Subcase 1: w is odd, $w \neq p$ and $w' \neq p$.

Using 1* and 4*/5*, the distance is $(np-(yp+w))+(yp+w')=np-w+w' \geq np-p+0=(n-1)p$.

Subcase 2: Both w and w' are even.

Using 2* and 5*, if $y=n-1$ then the distance is $(np-w)+((n-1)p+w')=(2n-1)p-w+w' \geq (2n-2)p$. If $y \neq n-1$ then the distance is $(np-((y+1)p+w))+(yp+w')=(n-1)p-w+w' \geq (n-2)p$.

Subcase 3: w is odd, $w \neq p$ and $w'=p$.

Using 1* and 6*, the distance is $(np-(yp+w))+(y+1)p=(n+1)p-w \geq np$.

Subcase 4: w is even and $w' = p$.

Using 2* and 6*, if $y = n-1$ then the distance is $(np - (0p + w)) + np = (2n)p - w' \geq (2n-1)p$. If $y \neq n-1$ then the distance is $(np-((y+1)p+w))+(y+1)p = np-w \geq (n-1)p$.

Subcase 5: $w = p$ and $w' \neq p$.

Using 3* and 4*/5*, if $y \geq k$ then distance is $(np - (y - k + 1)p) + (yp + w') = (n+k-1)p + w' \geq (n+k-1)p$. If $y < k$ then distance is $(np - (y+k+2)p) + (yp+w') = (n-k-2)p + w' \geq (n-k-2)p + 1 = (\frac{n-3}{2})p + 1$.

Subcase 6: $w = p$ and $w' = p$.

This is not possible as this implies $\pi_i(p) = x = \pi_{i+1}(p)$, which means the edge $\{x, 0\} \in E(K_p)$ is in both hamilton cycle H_i and H_{i+1}, but $\{E(H_i) \mid 1 \leq i \leq \frac{p-1}{2}\}$ is a partition of the edges of K_p. □

3.3 Main Result

Theorem 3.1. *Let n, p be odd positive integers, and $\{k_1, k_2, ..., k_m\}$ be a multiset of integers which satisfies $1 \leq k_i \leq \frac{1}{2}(n-3)p$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} k_i = \frac{1}{2}n^2p(p-1)$. Then $K(n,p)$ can be decomposed into paths $P_1, P_2, ..., P_m$, where the length of P_i is k_i for $1 \leq i \leq m$.*

Proof. By Lemma 3.3, E is an Euler circuit of $K(n,p)$ with diameter at least $\frac{1}{2}(n-3)p + 1$. Then $P_1, P_2, ..., P_m$ are obtained by cutting E into paths of lengths $k_1, k_2, ..., k_m$. Since the diameter of E is greater than the length of any of $P_1, P_2, ..., P_m$, they are indeed paths. □

The path-arboreal question asks, given graph G with $|V(G)| = n$, suppose that $\{k_i \mid 1 \leq i \leq m\}$ is a multiset of m positive integers satisfying $1 \leq k_i \leq n-1$ and $\sum_{i=1}^{m} k_i = |E(G)|$, can G be decomposed into paths of lengths $k_1, k_2, ..., k_m$? The above Theorem partially answers the question for $K(n,p)$.

Corollary 3.1. *Let n, p be odd positive integers and k be an integer with $1 \leq k \leq \frac{1}{2}(n-3)p$. If $\frac{1}{2}n^2p(p-1)$ is divisible by k, then there exists an L_k-decomposition of $K(n,p)$.*

Theorem 3.2. *Suppose n, p are odd positive integers; λ_1, λ_2 are positive integers with $\lambda_1 \leq \lambda_2$; and $k \leq \frac{1}{2}(n-3)p$. Then graph $G(n, p, \lambda_1, \lambda_2)$ is L_k-decomposable if and only if k divides $|E(G)| = \frac{1}{2}n(n-1)p\lambda_1 + \frac{1}{2}p(p-1)n^2\lambda_2$.*

Proof. Since $\lambda_1 \leq \lambda_2$, $G(n, p, \lambda_1, \lambda_2) = G(n, p, \lambda_1, \lambda_1) \cup G(n, p, 0, \lambda_2 - \lambda_1) = \lambda_1 K_{np} \cup (\lambda_2 - \lambda_1)K(n, p)$.

For $(\lambda_2 - \lambda_1)K(n, p)$, by Lemma 3.3 E is an Euler circuit of $K(n, p)$ with diameter at least $\frac{1}{2}(n-3)p + 1$. Since $k \leq (n-3)p/2$, E can be used to form a L_k-decomposition D_1 of $(\lambda_2 - \lambda_1)K(n, p)$ with possibly some edge(s) left, which induce a path of length less than k. These edges are called the leaves of D_1.

For $\lambda_1 K_{np}$, by Tarsi's result [58] the Walecki construction produces an Euler circuit W with diameter $d(W) = np - 2$. Since $k \leq (n-3)p/2 < np - 2$, W can be used to form a L_k-decomposition D_2 of $\lambda_1 K_{np}$ into copies of L_k with possibly some edge(s) left, which induce a path of length less than k. These edges are called the leaves of D_2. Since the $\lambda_1 K_{np}$ is edge-transitive, we can assume that none of the vertices in the leaves of D_2 appears in the leaves of D_1. Thus leaves of D_1 and D_2 together form a path P.

Since k divides $E(G) = D_1 \cup D_2 \cup P$, k divides $|E(P)|$ and path P has a trivial L_k-decomposition D_3. Then $D_1 \cup D_2 \cup D_3$ is a L_k-decomposition of $G(n, p, \lambda_1, \lambda_2)$.

□

3.4 Future Directions

A pair of latin squares are orthogonal if the n^2 ordered pairs of symbols formed by juxtaposing the pairs of symbols appearing in the same cell of the two arrays are all distinct. A latin square of order n which can be partitoned into n transversals is equivalent to a pair of orthogonal latin squares of order n. To see this, given such a latin square L we can make a new latin square L' from L by assigning $L'_{x,y} = i$ if $L_{x,y}$ is part of ith transversal. Then clearly, L and L' are orthogonal. The reverse can be proved similarly. Given orthogonal latin squares L and L' with symbols $\{0, 1, 2, ..., n-1\}$, define $T_i = \{L_{x,y} \mid L'_{x,y} = i\}$ for

$0 \le i \le n-1$. It's easy to see that every T_i is a transversal and the T_i's partition L. This means to find such a latin square of order $4n+2$ satisfying the additional useful structure found in L would be diffcult in general, as we know from the eventual disproof of the famous Euler's conjecture. One approach to try to generalize what follows might be to try to use symmetric idempotent latin square of even order with holes in place of L.

Chapter 4

The Intersection Problem for Two Latin Squares of size difference one

4.1 Basics

Let r, s and n be positive integers with $n \geq r, s$. A partial latin rectangle is an $r \times s$ array of n symbols (we usually use $\{1, 2, ..., n\}$) in which each symbol occurs at most once in each row and column and each cell contains at most one symbol. An incomplete latin rectangle is a partial latin rectangle in which every cell contains a symbol. A latin rectangle is an incomplete latin rectangle in which each symbol appears exactly once in each row. A latin square of order n is an $n \times n$ latin rectangle. If L is a (partial or incomplete) latin rectangle then let $L_{i,j}$ denote the symbol in cell (i, j) of L.

Latin squares satisfying additional properties are also of interest. Let L be a latin square of order n. L is said to be *idempotent* if $L_{i,i} = i$ for $1 \leq i \leq n$. L is *unipotent* if $L_{i,i} = c$ for $1 \leq i \leq n$ and a fixed symbol c. If n is even then L is said to be *half-idempotent* if $L_{i,i} = i$ for $1 \leq i \leq \frac{n}{2}$ and $L_{i,i} = i - \frac{n}{2}$ for $\frac{n}{2} + 1 \leq i \leq n$. If $L_{i,j} = L_{j,i}$ for $1 \leq i, j \leq n$, then L is said to be *symmetric* (or *commutative*). L is *semi-symmetric* if for all i, j, the entry in cell $(i, L_{j,i})$ is j. L is *totally symmetric* if for any i, j, the entries in cell $(i, L_{i,j})$ and $(L_{i,j}, j)$ are j and i, respectively. It is well-known that an idempotent totally symmetric latin square of order n is equivalent to a Steiner triple system (STS) of order n (see [16], Remark III.2.12).

Finally, L is said to have holes of size k if (1) $H = \{h_1, h_2, ..., h_{\frac{n}{k}}\}$ partitions the set $\{1, 2, ..., n\}$ with $|h_i| = k$ for $1 \leq i \leq n/k$, and (2) the cells in $h_i \times h_i$ are filled with symbols from h_i for $1 \leq i \leq n/k$ (so are latin subsquares).

Throughout this chapter, assume that if L has order n then the cells of L are (i, j) for $1 \leq i, j \leq n$. Given two latin squares L and S, possibly of different orders, a cell (i, j) is said to be (L, S)-different if (i, j) is a cell in both L and S, and these two cells contain different symbols; if it is clear to which latin squares are being referred to, then (i, j) is simply called a different cell. In this paper, of particular interest is the possible number of (L, S)-different cells two latin squares L and S can have. To this end, for any two latin squares L and S, let $D(L, S)$ denote the number of (L, S)-different cells where L and S are latin squares of orders x and y respectively with $x \leq y$. The intersection number $I(L, S)$ is defined to be the number of cells (i, j) for which cell (i, j) in L and S contains the same symbol; so clearly $I(L, S) = x^2 - D(L, S)$. More formally we have the following definition.

Definition 4.1. *Suppose $x \leq y$ and let L and S be latin squares of order x and y respectively. The number of (L, S)-different cells is defined to be $|\{(i, j) \mid 1 \leq i, j \leq x, L_{i,j} \neq S_{i,j}\}|$; that is, the number of cells of L and the top left partial square of S of order x that contain different symbols. The intersection number of L and S, denoted by $I(L, S)$, is defined to be $x^2 - D(L, S)$. Define $I(n) = \{I(L, S) \mid \ L$ and S are latin squares of order n $\}$.*

4.2 History

The problem of determining $I(n)$ is referred as the intersection problem for latin squares of the same order. This was settled by Fu [26] who proved the following result.

Theorem 4.1 (Fu, 1980 [26])**.** *Let L, S be latin squares of order n. Then*

$$I(n) = \begin{cases} \{1\} & \text{if } n = 1; \\ \{0, 4\} & \text{if } n = 2; \\ \{0, 3, 9\} & \text{if } n = 3; \\ \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\} & \text{if } n = 4; \\ \{0, 1, 2, ..., n^2\} \setminus \{n^2 - 1, n^2 - 2, n^2 - 3, n^2 - 5\} & \text{if } n \geq 5. \end{cases}$$

A natural extension to finding $I(n)$ is to add the requirement that L and S both satisfy an additional property. In [26], Fu also solved the intersection problem for idempotent latin squares and for unipotent latin squares. Webb [61] settled the symmetric idempotent case. The half-idempotent case was solved in [21] by Fu and Fu. The symmetric case was solved by Fu, Fu and Guo [23]. Fu [27] and Lindner and Wallis [46] solved the symmetric unipotent case. Fu, Gwo and Wu [24] settled the semi-symmetric case. Fu, Huang, Shih and Yaon [25] solved the totally symmetric case.

Since an idempotent totally symmetric latin square is equivalent to a Steiner triple system, the intersection problem was solved by Lindner and Rosa [44, 45] and DiPaola and Nemeth [50].

Fu and Fu [21] settled the intersection problem for latin squares with holes of size 2 and commutative latin squares with holes of size 2. Baker [3] settled the intersection problem for latin squares with holes of size 2 and 3. There is some doubt about whether there exist two latin squares with holes of size 2 such that they have exactly 35 cells in common, but it appears that no such pair exists. Chang and Faro [14] solved the intersection problem for latin squares which have orthogonal mates.

For a survey of results of intersection problem for many kinds of pairs of latin squares of the same order, see [22] by Fu and Fu.

In [20], Fu and Fu extended the intersection problem to considering three latin squares, all of same order, n. They found all the possible numbers k such that there exist three distinct latin squares of order n in which there are exactly k cells in which all three latin squares contain the same symbol. Note that for the remaining $n^2 - k$ cells, it was only required that at least one pair of the three latin squares had different entries. In [1], Adams, Billington, Bryant and Mahmoodian completely solved a stronger version of the problem where they required that all three latin squares contain different symbols in each of the remaining $n^2 - k$ cells.

In [19], Dukes and Mendelsohn introduced a generalization of the intersection problem, namely that of finding intersection numbers for latin squares of orders n and $n + k$. They settled the problem for most values of n and k. Dukes and Howell [18] later completely solved the problem. Unknown to the development in [18] at the time, we were interested by the problem solving the case $k = 1$; this is one of the unsolved cases in [19]. The proof when

$k = 1$ presented in Chapter 5 is basically self-contained, thus differing from the proof in [18] which relies on the deep theorem of Heinrich ([32], Theorem 4.4 - 4.12).

4.3 Main Result

Let $L(2)$ denote the latin square of order $2x$ formed from the latin square L of order x by the particular direct product defined as follows: for $1 \le i, j \le x$, if cell (i, j) of L contains symbol k, then cells $(2i-1, 2j-1)$, $(2i-1, 2j)$, $(2i, 2j-1)$ and $(2i, 2j)$ of $L(2)$ form a 2×2 subsquare $c(i, j)$, the cells containing symbols $2k-1$, $2k$, $2k$ and $2k-1$ respectively. Throughout the following construction, many of the 2×2 subsquares in $L(2)$ remain intact. It will be convenient to refer them as the special 2×2 subsquares.

Theorem 4.2. *Let $n \ge 12$, and let $c \in [n^2 - (n-3)(n-1), n^2 - n + 1]$. There exist two latin squares K and K^+ of order n and $n+1$ respectively, which have exactly $n^2 - c$ different cells.*

Proof. Assume that $n \ge 12$ and that $c \in [n^2 - (n-3)(n-1), n^2 - n + 1]$. Let ∞ be the only symbol occurring in K^+ that does not occur in K. Since ∞ can occur at most once in each of row $n+1$ and column $n+1$ of K^+, necessary there are at least $n-1$ cells (i, j) that are (K, K^+)-different because in K^+ cell (i, j) contains ∞. With this in mind, let $d = n^2 - c - (n-1)$. So d counts the number of (K, K^+)-different cells that exclude $n-1$ mandatory (K, K^+)-different cells that in K^+ contain symbol ∞. Possibly there is one more such cell, which could occur when $K^+_{i,j} = \infty$; so exactly 0 or 1 of these d cells contain ∞ in K^+.

Our construction depends on the value of n modulo 4, and the parity of $m = \lfloor \frac{n}{2} \rfloor$ is also relevant. Therefore define $m = 2x + \epsilon_1$ and $n = 4x + 2\epsilon_1 + \epsilon_2$, where ϵ_1, $\epsilon_2 \in \{0, 1\}$. By assumption since $n \ge 12$ it follows that $x \ge 3$. Furthermore, let $d = p(n-1) + 4q + r$ where $p, q, r \in \mathbb{N} \cup \{0\}$, $0 \le r \le 3$, $4q + r \le n - 1$, and $p \le n - 3$. In the following construction, K is formed from $L(2)$ for some careful choice of L. so K is constructed to contain many

2×2 subsquares. Then K^+ is formed from K. In so doing, p of the rows of K are deranged, and in some of the remaining rows (in particular the first 4 rows) q of the 2×2 subsquares are selected and the symbols within them are switched, thus providing 4 different cells for each 2×2 subsquare. There are 3 further cells that we can control to ensure that in the end exactly r cells are (K, K^+)-different (for any $r \in \{0, 1, 2, 3\}$).

First we construct an $m \times m$ partial latin square T on symbol set $\{1, 2, ..., n\}$, from which K will then be constructed, using the following seven steps,

1. Let L be any idempotent latin square of order $x \geq 2$; this exists because we know that $x \geq 3$.

2. Consider $L(2)$. Let S_3 be the 2×2 subsquare in the first 2 rows of $L(2)$ that contains symbols 3 and 4. Let S_2 be the 2×2 subsquare in the 3rd and 4th rows of $L(2)$ that contains symbols 1 and 2. Replace each symbol α in the 8 cells in S_2 and S_3 with $m + \alpha$ to form an incomplete latin square, L_2 of order $2x$.

3. For $1 \leq i \leq x$, remove the symbol $2i$ in cells $(2i-1, 2i)$, $(2i, 2i-1)$ from L_2, and for $1 \leq i \leq 2x$, replace the symbol $2\lceil \frac{i}{2} \rceil - 1$ in cell (i, i) with symbol i to form the partial idempotent latin square L_3 of order $2x$.

4. If $r \in \{0, 1\}$, then let $L_4 = L_3$. If $r \in \{2, 3\}$ then form L_3 from the partial idempotent latin square L_4 as follows:

   (a) Remove the symbol from each of the cells $(3, 1)$, $(3, 2)$, $(4, 1)$ and $(4, 2)$,

   (b) Remove the occurence of symbol 3 from column 1, and the occurence of symbol 2 from column 1, and

   (c) Fill cells $(2, 1)$ and $(3, 1)$ with 3 and 2 respectively.

   (Eventually, once K and K^+ are formed, exactly r of the cells $(1, 1)$, $(2, 1)$ and $(3, 1)$ will be (K, K^+)-different.)

5. If $\epsilon_1 = 1$ then L_4 has order $2x = m - 1$, so to L_4 add a new row and column (i.e. the mth row and column) in which all cells are empty except that cell (m, m) contains symbol m. So the result is the partial idempotent latin square L_5. If $\epsilon_1 = 0$, then let $L_5 = L_4$. In both cases L_5 has order m.

6. If $\epsilon_2 = 0$, then change all symbols in last two columns of L_5 by replacing i with $m + i$ for $1 \leq i \leq m$. Let the result be L_6. If $\epsilon_2 = 1$ then let $L_6 = L_5$. This ensures that eachl symbol in $\{m+1, m+2, ..., n\}$ appears at least once* in L_6. So L_6 has order m.

7. Fill the empty cells in L_6 greedily with symbols in $\{1, 2, ..., n\}$ to form the incomplete idempotent latin square T; this is possible since $n \geq 2m$.

By Step 6, if $\epsilon_2 = 0$ then each of the symbols $m+1, ..., n$ appears in L_6 and thus in T. Therefore by theorem embed T in an idempotent latin square K of order $2m + \epsilon_2 = n$.

We now turn our attention to forming K^+. To do so we modify and expand K into a latin square of order $n + 1$ in such a way that $I(K, K^+) = c$. This can be accomplished as follows. As K^+ is being formed, it will be helpful to identify the number of (K, K_i)-different cells formed in Step i.

Step 1. From K remove the symbols in cells (i, i) and fill them with ∞ for $2 \leq i \leq n$ to form the incomplete latin square K_1. (These are the $n - 1$ mandatory occurrences of the symbol ∞ in K^+, producing $n - 1$ (K, K^+)-different cells.)

Step 2. Note that since $n \geq 12$, $4q + r \leq n - 1$, and $x = \lfloor \frac{n}{4} \rfloor$, it follows that $q \leq 2x - 3$. Also K contains at least $2x - 3$ 2×2 subsquares. Therefore we can pick q of the $2x - 3$ special 2×2 subsquares in the first 4 rows of K_1 and switch the symbols in each such subsquare to form the imcomplete latin square K_2. This provides $4q$ (K, K_2)-different cells (in addition to the $n - 1$ (K, K_2)-different cells formed in Step 1).

Step 3. If $r = 1$ or 3, then replace the symbol 1 in cell $(1, 1)$ of K_3 with ∞. If $r = 2$ or 3, then replace the symbol in cell $(2, 1)$ with 2 and the symbol in cell $(3, 1)$ with 3. Let the resulting incomplete latin square be named K_3. This step introduces r (K, K_3)-different cells (so there are exactly $(n-1)+4q+r$ (K, K_3)-different cells in total).

Step 4. Derange the bottom p rows of K_3 to create K_4, using some permutation π on $\{1, ..., n\}$, which has $\pi(i) = i$ if $1 \leq i \leq n-p$, and is a bijective derangement on $[n-p+1, n]$. This step creates $p(n-1)$ further (K, K_4)-different cells since each such row already has a cell containing ∞ which was used in Step 1 to identify a (K, K_1)-different cell (and so has already been counted).

So in the incomplete $n \times n$ latin square K_4 we have $(n-1)+4q+r+p(n-1) = n^2 - c$ (K, K_4)-different cells.

We now expand K_4 to a latin square K^+ of order $n+1$ by adding a new column and row as the $(n+1)^{th}$ column and row as follows:

1. If the symbol in cell $(1,1) = 1$ or ∞, then fill cells $(n+1, 1)$ and $(1, n+1)$ with ∞ or fill them with 1 respectively. Fill cell $(n+1, n+1)$ with the symbol in cell $(1,1)$.

2. By step 1, for $2 \leq i \leq n$ column i is missing symbol $c(i) = i$. By step 3, column 1 is missing symbol $c(i) = 1$ if $r \in \{1, 3\}$ and symbol $c(i) = \infty$ otherwise. For $1 \leq i \leq n$ fill cell $(n+1, i)$ with symbol $c(i)$.

3. By step 4, for $2 \leq i \leq n$ row i is missing symbol $c(i) = \pi(i)$. For $2 \leq i \leq n$ fill cell $(i, n+1)$ with symbol $c(i)$.

□

Chapter 5

The Intersection Problem for Latin Rectangles

5.1 Basics

For definitions and history of intersection problems, see Chapter 4.

5.2 Lemmas

We now turn to solving the intersection problem of latin rectangles. In the rest of the chapter, assume that $1 \leq r < n$. Let R and S be $r \times n$ latin rectangles. Define the intersection number of R and S to be $I(R,S) = |\{(i,j) \mid 1 \leq i \leq r, 1 \leq j \leq n, R_{i,j} = S_{i,j}\}|$. Similarly to defining $I(n)$, define $I(r,n) = \{I(R,S) \mid R, S \text{ are } r \times n \text{ latin rectangles}\}$.

The following well-known facts will be useful.

Theorem 5.1 (Hall [31], 1945)**.** *Any $r \times n$ latin rectangle can be embedded in an $n \times n$ latin square.*

Theorem 5.2 (Ryser [56], 1951)**.** *Let T be an $r \times s$ latin rectangle on the symbols in $\{1, 2, ..., n\}$. Let $N(i)$ denote the number of times that the symbol i occurs in T. Then T can be embedded in an $r \times n$ latin rectangle if and only if $N(i) \geq r + s - n$ for $1 \leq i \leq n$.*

Lemma 5.1. *Let R, S be $r \times n$ latin rectangles on the symbols in $\{1, 2, ..., n\}$. Let d_i be the number of cells in row i in which R and S differ. Then $d_i \neq 1$ for $1 \leq i \leq r$.*

Proof. Observe that if two rows from two latin rectangles agree in $n-1$ cells, then they must agree in the last cell as well since each symbol appears exactly once in each row. Therefore, two rows can not differ by exactly one cell. □

Corollary 5.1. *$rn - 1 \notin I(r,n)$ for all $r < n$.*

Proof. If $I(R,S) = rn - 1$, then R and S differ by exactly one cell. Then there is a row in which where R and S differ in exactly one cell, which contradicts Lemma 5.1. $\square$

Lemma 5.2. $I(2,3) = \{0, 2, 3, 6\}$.

Proof. It is trivial that 0 and 6 are in $I(2,3)$. By Corollary 5.1, $5 \notin I(2,3)$.

For 2 and 3, there are three latin rectangles listed below. The first two have intersection number 2 and the the last two have intersection number 3.

$$\begin{matrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{matrix} \quad \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$$

Now we show 4 and 1 are not in $I(2,3)$.

If two 2×3 latin rectangles R and S have intersection number 4, then they differ by exactly 2 cells. By Lemma 5.1, these two cells must be in the same row. Without the loss of generality, assume they are cell $(1,1)$ and $(1,2)$, and $R_{1,1} = 1$ and $R_{1,2} = 2$. Then $R_{1,3} = 3 = S_{1,3}$, and $S_{1,1}, S_{1,2}$ must be 2 and 1, respeactively. Consider $R_{2,1}$. Since $R_{2,1} = S_{2,1}$, it can not be 1 ($R_{1,1} = 1$) or 2 ($S_{1,1} = 2$). Thus $R_{2,1} = 3$. Same argument on $R_{2,2}$ shows $R_{2,2} = 3$, a contradiction.

If two 2×3 latin rectangles R and S have intersection number 1, then they agree on exactly 1 cells. Without the loss of generality, assume it is the cell $(1,1)$ and $R_{1,1} = S_{1,1} = 1$. We can further assume that $R_{1,2} = 2$ and $R_{1,3} = 3$. Since S has to differ with R at rest of the cells, we have $S_{1,2} = 3$ and $S_{1,3} = 2$. There are only two possiblities for the second row of R: 231 or 312. Similarly for S, the second row can only be 321 or 213. In any of the four cases, R and S must agree on one cell in the second row, therefore completing the proof. $\square$

Lemma 5.3. *For* $1 \leq r < n$ *and* $4 \leq n$, $\{rn-2, rn-3, ..., (r-1)n+2, (r-1)n\} \subseteq I(r,n)$.

Proof. Let d be an integer with $2 \leq d \leq n-2$ or $d = n$. We will first construct an $n \times n$ latin square, and then obtain two $r \times n$ latin rectangles with d different cells from it.

We start by constructing the first and second row as follows. If $2 \le d \le n-2$, then consider

$$\begin{array}{cccccccccc} 1 & 2 & \ldots & d-1 & d & d+1 & \ldots & n-1 & n \\ 2 & 3 & \ldots & d & 1 & d+2 & \ldots & n & d+1 \end{array}$$

which is possible as long as $d \ge 2$ and $n - d \ge 2$, which is true since $2 \le d \le n-2$.

If $d = n$, then we simply use

$$\begin{array}{ccccc} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & 1 \end{array}$$

In either case, by Theorem 5.1 there is an $n \times n$ latin square L with such first and second rows. We obtain L' by switching the first d entries of the first row and second row. Note L' is still a $n \times n$ latin square as every symbol appears exactly once in every row and column. We then obtain two $r \times n$ latin rectangles R and S by deleting the second row and other arbitrarily chosen $n - r - 1$ rows, excluding the first, from L and L', respectively. Clearly, R and S agree on every cell that is not in the first row, where they agree on exactly $n - d$ cells. So $I(R, S) = rn - d$. □

Lemma 5.4. *Let $n \ge 3$. For $1 \le r \le n-2$, $(r-1)n + 1 \in I(r, n)$.*

Proof. Consider the latin rectangle with first three rows being $12...n$, $23...n1$, $3...n12$, respectively. This is possible since $n \ge 3$. By Theorem 5.1, there is a $n \times n$ latin square L with such first three rows. Let R be the latin rectangle obtained by deleting the second, the third and arbitrary other $n - r - 2$ rows (but not the first row) from L. Form S by replacing the first $n - 1$ entries in the first row with $234...(n-1)1$.

Since the first $n - 2$ entries of second row of L is $234...(n-1)$, and $L_{3,n-1} = 1$, no symbol appears more than once in any column of S and S is still a latin rectangle. Clearly, $I(R, S) = rn - (n-1) = (r-1)n + 1$. □

Lemma 5.5. *For $4 \le r$ and $n = r + 1$, $(r-1)n + 1 \in I(r, n)$.*

Proof. We will prove by constructing two $r \times n$ latin rectangles R, S with $I(R,S) = (r-1)n+1$. For $5 \leq n \leq 8$, R and S are presented below. The (R,S)-different cells in S are in bold font.

For $n = 5$:

1	2	3	4	5		**2**	**1**	3	4	5
2	1	4	5	3		**1**	**2**	4	5	3
3	4	5	1	2		3	4	5	1	2
5	3	1	2	4		5	3	1	2	4

For $n = 6$:

1	2	3	4	5	6		**2**	**3**	**1**	4	5	6
2	1	4	5	6	3		**1**	**2**	4	5	6	3
3	4	6	2	1	5		3	4	6	2	1	5
4	6	5	1	3	2		4	6	5	1	3	2
6	5	2	3	4	1		6	5	2	3	4	1

For $n = 7$:

1	2	3	4	5	6	7		**2**	**3**	**1**	4	5	6	7
2	3	1	5	6	7	4		**1**	**2**	**3**	5	6	7	4
3	1	2	6	7	4	5		3	1	2	6	7	4	5
4	5	6	7	1	3	2		4	5	6	7	1	3	2
5	6	7	1	4	2	3		5	6	7	1	4	2	3
6	7	4	2	3	5	1		6	7	4	2	3	5	1

For $n = 8$:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 6 & 7 & 8 & 3 \\ 3 & 4 & 6 & 1 & 8 & 2 & 5 & 7 \\ 4 & 5 & 7 & 8 & 1 & 3 & 6 & 2 \\ 5 & 3 & 8 & 2 & 7 & 1 & 4 & 6 \\ 7 & 8 & 2 & 6 & 3 & 4 & 1 & 5 \\ 8 & 6 & 1 & 7 & 2 & 5 & 3 & 4 \end{array} \quad \begin{array}{cccccccc} \mathbf{2} & \mathbf{1} & \mathbf{5} & \mathbf{3} & \mathbf{4} & 6 & 7 & 8 \\ \mathbf{1} & \mathbf{2} & 4 & 5 & 6 & 7 & 8 & 3 \\ 3 & 4 & 6 & 1 & 8 & 2 & 5 & 7 \\ 4 & 5 & 7 & 8 & 1 & 3 & 6 & 2 \\ 5 & 3 & 8 & 2 & 7 & 1 & 4 & 6 \\ 7 & 8 & 2 & 6 & 3 & 4 & 1 & 5 \\ 8 & 6 & 1 & 7 & 2 & 5 & 3 & 4 \end{array}$$

For $n \geq 9$, consider the $3 \times n$ latin rectangle:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \ldots & n-4 & n-3 & n-2 & n-1 & n \\ 2 & 3 & 1 & 5 & \ldots & 4 & n-2 & n-1 & n & n-3 \\ 3 & 1 & 2 & 6 & \ldots & n-2 & n-1 & n & 4 & 5 \end{array}$$

This exists since $(n-4)-4+1 \geq 2$, because $n \geq 9$. By Theorem 5.1 there is an $n \times n$ latin square L with such first three rows. We obtain a new latin square L' from L by switching the first $n-4$ cells in the first and second rows, then switch the first three cells in the second and third rows. So the first three rows of L' are:

$$\begin{array}{ccccccccc} 2 & 3 & 1 & 5 & \ldots & 4 & n-3 & n-2 & n-1 & n \\ 3 & 1 & 2 & 4 & \ldots & n-4 & n-2 & n-1 & n & n-3 \\ 1 & 2 & 3 & 6 & \ldots & n-2 & n-1 & n & 4 & 5 \end{array}$$

We then obtain two $r \times n$ latin rectangles R and S by deleting the second row and another arbitrarily chosen $n-r-1$ rows, excluding the first and third rows, from L and L', respectively. Clearly, R and S differ on first $n-4$ cells in the first row and the first 3 cells in the second. Hence, $I(R, S) = rn - (n-4) - 3 = (r-1)n + 1$.

We note that all pairs (R, S) in this lemma only have different cells in the first two rows, which will be useful in Lemma 5.9. □

Lemma 5.6. *For* $2 \leq r < n$, $(r-1)n - 1 \in I(r, n)$.

Proof. If $n = 3$ then $(r, n) = (2, 3)$, and this case is shown in Lemma 5.2.

For $4 \leq n \leq 6$ and $r = n-1$, R and S are presented below. If $r < n-1$, then delete the bottom $n-1-r$ rows to obtain R and S. The (R, S)-different cells in S are in bold font.

For $n = 4$,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{array} \qquad \begin{array}{cccc} \mathbf{2} & \mathbf{3} & \mathbf{1} & 4 \\ \mathbf{1} & \mathbf{2} & 4 & 3 \\ 3 & 4 & 2 & 1 \end{array}$$

For $n = 5$,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 2 & 3 & 1 \end{array} \qquad \begin{array}{ccccc} \mathbf{2} & \mathbf{3} & \mathbf{1} & 4 & 5 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 5 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 2 & 3 & 1 \end{array}$$

For $n = 6$,

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 2 & 3 & 1 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{array} \qquad \begin{array}{cccccc} \mathbf{2} & \mathbf{1} & \mathbf{4} & \mathbf{3} & 5 & 6 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 5 & 6 & 4 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 2 & 3 & 1 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{array}$$

For $n \geq 7$, consider the $3 \times n$ latin rectangle:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \dots & n-3 & n-2 & n-1 & n \\ 2 & 3 & 1 & 5 & \dots & n-2 & 4 & n & n-1 \\ 3 & 1 & 2 & 6 & \dots & n-1 & n & 4 & 5 \end{array}$$

This exists since $(n-2) - 4 + 1 \geq 2$, because $n \geq 7$. By Theorem 5.1 there is a $n \times n$ latin square L with these first three rows. We obtain a new latin square L' by switching the first $n-2$ cells in the first and second rows, then switch the first three cells in the second and third rows. So the first three rows of L' are:

$$\begin{array}{cccccccc} 2 & 3 & 1 & 5 & \ldots & n-2 & 4 & n-1 & n \\ 3 & 1 & 2 & 4 & \ldots & n-3 & n-2 & n & n-1 \\ 1 & 2 & 3 & 6 & \ldots & n-1 & n & 4 & 5 \end{array}$$

We then obtain two $r \times n$ latin rectangles R and S by deleting their second row and another arbitrarily chosen $n-r-1$ rows excluding the first and third rows, from L and L', respectively. Clearly, R and S differ in the first $n-2$ cells in the first row and the first three cells in the second row. Hence, $I(R, S) = rn - (n-2) - 3 = (r-1)n - 1$.

We note that all pairs (R, S) in this lemma only have different cells in the first two rows, which will be useful in Lemma 5.9. □

Corollary 5.2. *For* $3 \le r < n$, $(r-2)n - 1 \in I(r, n)$.

Proof. For $4 \le n \le 6$ and $r = n - 1$, the following are the required R and S, which are obtained by modifying the latin rectangles from Lemma 5.6. If $r < n - 1$, then delete the bottom $n - 1 - r$ rows to obtain R and S. The (R, S)-different cells in S are in bold font.

For $n = 4$,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 1 & 2 \end{array} \qquad \begin{array}{cccc} \mathbf{2} & \mathbf{3} & \mathbf{1} & 4 \\ \mathbf{1} & \mathbf{2} & 4 & 3 \\ \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{1} \end{array}$$

For $n = 5$,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 2 & 3 & 1 \end{array} \qquad \begin{array}{ccccc} \mathbf{2} & \mathbf{3} & \mathbf{1} & 4 & 5 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 5 & 4 \\ \mathbf{5} & \mathbf{1} & \mathbf{4} & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 2 & 3 & 1 \end{array}$$

For $n = 6$,

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 6 & 1 & 4 & 3 & 2 & 5 \\ 4 & 5 & 6 & 2 & 3 & 1 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{array} \qquad \begin{array}{cccccc} \mathbf{2} & \mathbf{1} & \mathbf{4} & \mathbf{3} & 5 & 6 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 5 & 6 & 4 \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{1} & \mathbf{2} \\ 4 & 5 & 6 & 2 & 3 & 1 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{array}$$

For $n \geq 7$, consider the latin squares L and L' constructed in Lemma 5.6. Let K be the latin square obtained by switching the second and fourth rows of L'. We then obtain two $r \times n$ latin rectangles R and S by deleting the second row and arbitrary other $n - r - 1$ rows (but not the first, third or fourth row) from L and K, respectively. Then R and S differ on first $n - 2$ cells in the first row, the first 3 cells in the seconds row and the entire third row. Hence, $I(R, S) = rn - (n - 2) - 3 - n = (r - 2)n - 1$.

□

Lemma 5.7. *For* $2 \leq r < n$, $\{(r-2)n+2, ..., (r-1)n-2\} \subset I(r, n)$.

Proof. Let $k \in \{(r-2)n+2, ..., (r-1)n-2\}$. We will prove the result by constructing two $r \times n$ latin rectangles R, S with $I(R, S) = k$. Let $d = rn - k$. Then $n + 2 \leq d \leq 2n - 2$ and $2 \leq d - n \leq n - 2$. Let L and L' be latin squares constructed in Lemma 5.3 that have $d - n$ different cells in each of the first and second rows. Switch the second and third rows of L'. The two $r \times n$ latin rectangles R and S formed by deleting the third row and arbitrary $n - r - 1$ rows (other than the first and second row) from both L and L'. Clearly, R and S differ in exactly $d - n$ cells in the first row and n cells at the second row. Hence $I(R, S) = rn - ((d - n) + n) = rn - d = k$.

□

Lemma 5.8. *For* $r \geq 3$ *and* $n = r + 1$, $\{(r-2)n+1, (r-2)n+2\} \subseteq I(r, n)$.

Proof. We begin with the $(r-2)n+2$ case. We start with the 3×3 incomplete latin rectangle

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 4 \\ * & 3 & 2 \end{array}$$

where $* = 4$ if $n = 4$, and $* = 5$ if $n \geq 4$. Embed this into a $3 \times n$ latin square L by Theorems 5.1 and 5.2. Let R be the $(n-1) \times n$ latin rectangle obtained by deleting the third row of L. Let S be the latin rectangle by switching the symbols 1 and 2 in R. Since R and S disagree in exactly two cells per row, $I(R, S) = rn - (2n-2) = (r-2)n + 2$.

For the $(r-2)n+1$ case, modify S as follows. Form S' by switching symbols in $S_{1,2}$ and $S_{1,3}$. Note $S_{1,2} = 1$ and $S_{1,3} = 3$. S' is still an latin rectangle, as no symbol in the second and third column is 3 and 1, respectively, since $L_{3,2} = 3$ and $L_{3,3} = 2$. Since R and S' differ at cells $(1,1)$ $(1,2)$ and $(1,3)$ and two cells per row (except the first row), $I(R, S') = rn - (2n-1) = (r-2)n + 1$. □

Lemma 5.9. *For* $3 \leq r < n$, $\{2, ..., (r-2)n-2\} \cup \{(r-2)n\} \subseteq I(r, n)$.

Proof. Let $d = qn + p$ with $0 \leq p \leq n-1$ and $2 \leq q \leq r-1$.

Suppose $p = 0$ or $2 \leq p \leq n-2$. Let R, S be $r \times n$ latin rectangles with p different cells as constructed in Lemma 5.3. Note R and S agree on all cells outside the first row. Derange the bottom q rows of S to create S', using some permutation π on $\{1, ..., r\}$, which has $\pi(i) = i$ if $1 \leq i \leq n-q$, and is a bijective derangement on $[r-q+1, r]$. Clearly, R and S' differ in $p + qn$ cells.

Suppose $p = n-1$. Let R, S be $r \times n$ latin rectangles with $n-1$ different cells as constructed in Lemma 5.4 if $r \leq n-2$ and Lemma 5.5 if $r = n-1$, respectively. Note R and S agree on all cells outside the first and second rows. Derange the bottom q rows of S to create S', using some permutation π on $\{1, ..., r\}$, which has $\pi(i) = i$ if $1 \leq i \leq n-q$, and is a bijective derangement on $[r-q+1, r]$. We note that since $d \neq rn-1$, $(q, p) \neq (r-1, n-1)$.

Thus $p = n-1$ implies $q \leq r-2$, so $r-q+1 \geq 3$ and thus the first two rows of S' are the same as in S. Therefore, R and S' differ at $p+qn$ cells.

Suppose $p = 1$. Let R, S be $r \times n$ latin rectangles with $n+1$ different cells as constructed in Lemma 5.6. Then $d = qn+1 = (q-1)n+(n+1)$. We note that since $d \neq 2n+1$, $q \geq 3$. Moreover, $q \leq r-1$. Therefore $2 \leq q-1 \leq r-2$. Derange the bottom $q-1$ rows of S to create S', using some permutation π on $\{1, ..., r\}$, which has $\pi(i) = i$ if $1 \leq i \leq n-q$, and is a bijective derangement on $[r-q+1, r]$. Since $q \leq r-2$, so $r-q+1 \geq 3$ and thus the first two rows of S' are the same as in S. Therefore, R and S' differ at $(q-1)n+(n+1) = qn+1$ cells. □

Lemma 5.10. *For $1 \leq r < n$ and $4 \leq n$, $1 \in I(r, n)$.*

Proof. By Theorem 4.1, there exist two $n \times n$ latin squares L and L' with $I(L, L') = 1$. Without loss of generality, we may assume they agree at cell $(1, 1)$. Delete the bottom $n-r$ rows from both L and L'. This results two $r \times n$ latin rectangles which only agree at cell $(1, 1)$. □

Lemma 5.11. *For $2 \leq r \leq n-2$, $(r-2)n+1 \in I(r, n)$.*

Proof. Consider the $4 \times n$ partial latin rectangle:

$$\begin{array}{ccccc}
2 & 3 & ... & n & 1 \\
1 & 2 & ... & n-1 & n \\
n & 1 & ... & n-2 & n-1 \\
3 & 4 & ... & 1 & 2
\end{array}$$

This exists since $n \geq 4$. By Theorem 5.1 there is a $n \times n$ latin square L with such first four rows. Let L' be the latin square obtained by switching the first and third rows in L. We then obtain two $r \times n$ latin rectangles R and S by deleting the third, fourth and another arbitrary arbitrarily chosen $n-r-2$ rows excluding the first and second rows from L and L', respectively.

Note the second, third and fourth rows of L are exactly the same as the first three rows of L defined in Lemma 5.4. Therefore by the exactly same argument as in Lemma 5.4, let S' be the $r \times n$ latin rectangle formed by replacing the first $n-1$ cells of the second row of S with $234...(n-1)1$. Then $I(R, S') = rn - n - (n-1) = (r-2)n + 1$.

□

5.3 Main Result

Theorem 5.3. *Let $1 \leq r < n$ be positive integers. Let $J(r,n) = \{0, 1, 2, ..., rn-2, rn\}$. Then*

$$I(r,n) = \begin{cases} \{0,2,3,6\} & \textit{if } (r,n) = (2,3); \\ J(3,4) \setminus \{9\} & \textit{if } (r,n) = (3,4); \\ J(r,n) & \textit{otherwise.} \end{cases}$$

Proof. We first show $9 \notin I(3,4)$. Suppose $9 \in I(3,4)$. Then there exists two 3×4 latin rectangles R and S, such that $I(R,S) = 9$. By Lemma 5.1, all the three cells in which R and S differ must be in the same row. Without loss of generality, we may assume the first row of R and S are 1234 and 2314, respectively. Since R and S agree in the last 2 rows, $R_{2,1} \cup R_{3,1} = \{3,4\}$, $R_{2,2} \cup R_{3,2} = \{1,4\}$ and $R_{2,3} \cup R_{3,3} = \{2,4\}$. This is impossible since symbol 4 cannot appear three times in two rows.

The case 0 and rn are trivial to show.

In the case $r = 1$ and $n \in \{2,3\}$, it is easy to create examples with desired intersection numbers. If $n \geq 4$, use Lemma 5.3, 5.10 and Corollary 5.1 for intersection numbers $\{2, ..., n-2\}$, 1 and $n-1$, respectively.

If $(r,n) = (2,3)$, then Lemma 5.2 solves the case.

For all other cases, the results are arranged into the following table.

<table>
<tr><th rowspan="2">Intersection number $k =$</th><th rowspan="2">$r = 2$, $n \geq 4$</th><th colspan="2">$r \geq 3$</th></tr>
<tr><th>$n \geq r + 2$</th><th>$n = r + 1$</th></tr>
<tr><td>rn</td><td colspan="3">trivial</td></tr>
<tr><td>$rn - 1$</td><td colspan="3">Corollary 5.1</td></tr>
<tr><td>$rn-2 \geq k \geq (r-1)n+2$ or $k = (r-1)n$</td><td colspan="3">Lemma 5.3</td></tr>
<tr><td>$(r-1)n + 1$</td><td colspan="2">Lemma 5.4</td><td>Lemma 5.5</td></tr>
<tr><td>$(r-1)n - 1$</td><td colspan="3">Lemma 5.6</td></tr>
<tr><td>$(r-1)n-2 \geq k \geq (r-2)n + 2$</td><td colspan="3">Lemma 5.7</td></tr>
<tr><td>$(r-2)n + 1$</td><td colspan="2">Lemma 5.11</td><td>Lemma 5.8</td></tr>
<tr><td>$(r-2)n$</td><td>See $k = 0$</td><td colspan="2">Lemma 5.9</td></tr>
<tr><td>$(r-2)n - 1$</td><td>n/a</td><td colspan="2">Corollary 5.2</td></tr>
<tr><td>$(r-2)n - 2 \geq k \geq 2$</td><td>n/a</td><td colspan="2">Lemma 5.9</td></tr>
<tr><td>1</td><td colspan="3">Lemma 5.10</td></tr>
<tr><td>0</td><td colspan="3">trivial</td></tr>
</table>

$\square$

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