On Chain and Duo Rings

by

Francisco Javier Santillán-Covarrubias

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 4, 2018

Keywords: Chain ring, Duo ring, Maximal ring, Self-injective ring, *FGC*-ring,
Hopfian module

Approved by

Ulrich Albrecht, Professor of Mathematics and Interim Chair
Wenxian Shen, Professor of Mathematics
Luke Oeding, Assistant Professor of Mathematics
Ziqin Feng, Assistant Professor of Mathematics

Abstract

The aim of this work is to study a variety of concepts in a *non-commutative* setting, which originally arose in Commutative Ring Theory. The discussion focuses particularly on properties like hopficity and co-hopficity, maximality and almost maximality, self-injectivity and FGC-rings. We investigate these in the context of chain and duo rings and extend some fundamental results.

Acknowledgments

I am grateful to all of those with whom I have had the pleasure to work during my studies. I would especially like to thank Dr. Ulrich Albrecht, professor, and advisor, who has taught me and support me more than I could ever give him credit for here. Particularly, I am very grateful to him for help me to bring the *Ring Theory* back to my mathematics and for the piece of Algebra that generously shared with me. As well, I want to thank the rest of the members of my dissertation committee: Dr. Wenxian Shen, Dr. Ziqin Feng, and Dr. Luke Oeding who have provided me extensive academical guidance and taught me a great deal about their fields of study. Also, I want to thank Dr. Edward Loewenstein for his time and support as an outsider reader.

I would not have been able to complete this degree without the support of the Department of Mathematics and Statistics at Auburn University and the Instituto de Educación Media Superior de la Ciudad de México. In particular, I want to thank Dr. Tin-Yau Tam, Dr. Andras Bezdek, and Mrs. Joyce Griffin for their constant and kindly support during the last two years. Besides, I would like to thank the department staff members Ms. Gwen Kirk and Ms. Mallory Lipscomb, along with the personnel at the Graduate School and the Office of International Programs for their courteous assistance during my stay at AU.

Nobody has been more important to me than the members of my family in México and USA; who (in some way) have always been with me. My affection and gratitude to all of you. To my friends from México, and my new friends in Auburn: Thank y'all for being there!

Finally, I wish to thank my best friend ever Gisela for all the laughs, all the music, and all the *cats*. Thanks for bringing me to Auburn inside your suitcase! *Gracias... totales!*

D.N.E.

To my parents

Refugio Covarrubias and Francisco Santillán

and the memory of my second mothers

Guadalupe Santillán and Yolanda Santillán

Hay hombres que luchan un día
Y son buenos
Hay otros que luchan un año
Y son mejores
Hay quienes luchan muchos años
Y son muy buenos
Pero hay los que luchan toda la vida
Esos son los imprescindibles

There are men who fight one day
And they are good
There are others who fight a year
And they are better
There are those who fight for many years
And they are very good
But there are those who fight their whole lives
Those are the indispensable ones

Bertolt Brecht

Auburn, Alabama, August 2018

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Chapter 1

Introduction

Throughout this work, by a *ring* R we mean an *associative ring* with identity and by an R-*module* we mean a *unitary* R-*module*. Precise definitions and concrete examples of all the concepts mentioned in the present introduction will be given in the corresponding chapters.

A ring R with unit is a *right (left) chain ring* R if its right (left) ideals are linearly ordered by inclusion. Commutative chain rings are called *valuation rings*. Furthermore, a *right (left) duo ring* R is a ring in which every right (left) ideal is two-sided. In particular, $Rx \subseteq xR$ ($xR \subseteq Rx$) for all $x \in R$, i.e. in a right (left) duo ring, every right (left) principal ideal *absorbs* its left (right) counterpart. Duo rings are also called *invariant rings* [4].

Chain rings arise as a natural generalization of valuation rings. The latter have been studied extensively by several authors, and a summary of the rich structure theory available for these rings can be found in Fuchs' and Salce's book ([13]). Chain rings themselves also arise in Geometry, for instance in the investigation of Helmslev planes, and have been discussed in detail by Bessenrodt, Brungs, and Törner ([4], [5] and [9]). Naturally, the question arises which properties of modules over valuation rings carry over to the non-commutative setting. For instance, Albrecht and Scible addressed this question in [1] by considering the structure theory of finitely generated modules over valuation domains. They showed that this structure theory extends to chain rings if and only if the chain ring R also is a duo ring. The results in [4] further emphasize the importance of the duo condition in the investigation of chain rings.

Concepts like hopficity and co-hopficity, maximality and almost maximality, and FGC-rings have been the focus of research in commutative ring theory for several decades, and can be found in [29], [14] and [6].

The main goal of this dissertation is to examine non-commutative versions of these notions in the the framework of one-sided chain and duo rings.

Section 2 introduces the notions of chain and duo ring, gives several examples and describes their basic properties which are used in the remainder of our discussion. Section 3 considers *repetitive elements* and *repetitive rings*, which were introduced by Goodearl in [15] to characterize the rings R for which all surjective endomorphisms of finitely generated R-modules are automorphisms. We continue the discussion of [15] by describing how to construct repetitive elements, and investigate repetitive elements in chain rings. Our main result shows that one-sided duo rings are repetitive, and a chain ring is a duo ring if all its units are repetitive (Theorem 3.2.2).

A right R-module M is *Hopfian* if every surjective endomorphism of M is an automorphism. Dually, M is *co-Hopfian* if injective endomorphisms are automorphisms. The notions of hopficity and co-hopficity were introduced by Hiremath in [16], and have been studied by Goodearl, Varadarajan and other, see, e.g. [28]. Section 4 presents some examples and results about rings R that are Hopfian as right R-modules. In particular, we show that modules with a local endomorphisms ring satisfy a cancellation property (Proposition 4.0.16). We use this result to show that a free R-module is Hopfian if and only if it is finitely generated whenever R is a right (left) chain ring or right (left) duo ring (Theorem 4.0.18). In addition, we observe that cyclic R-modules are Hopfian if R is a right duo ring. This allows us to show that finite direct sums of cyclic R-modules are Hopfian (Theorem 4.0.21) if R is a right chain and right duo ring. As a corollary, we obtain that finite direct sums of cyclic modules *cancel* in direct sums for these rings.

Section 5 studies one-sided *annihilators*. Theorem 5.1.2 shows that every left principal ideal of a chain ring R is a left annihilator if R_R is co-Hopfian. Using the right-left symmetry of this result, we obtain that these conditions on ${}_RR$ imply that the nil-radical $N(R)$ of R, it is contained in the left singular ideal $Z_l(R)$ of R. Moreover, the Jacobson radical $J(R)$, and the right singular ideal $Z_r(R)$ are both nilpotent in this case.

Also, Section 5 considers maximality, almost maximality and self-injectivity. Gill's had shown that these concepts are closely related in the commutative case [14]. We extend several of his results to the non-commutative setting. For instance, we establish that right almost maximality of a ring R is equivalent to its right maximality if R is a right duo and right chain ring which is not a domain. This allows us to characterize right self-injective rings, as the left maximal rings R such that R_R is co-Hopfian whenever R is a chain and right duo ring. This also extends Klatt's and Levy's characterization of valuation rings in [21].

In Section 6, we show that the *right-left localizations* of a duo ring R at a completely prime (prime) ideal P are equivalent. In particular, we obtain that, if R is an almost maximal chain and duo ring, then the localization R_P also is an almost maximal chain ring. Our discussions in Section 4 naturally give rise to the question of which rings R have the property that all finitely generated right R-modules are direct sums of cyclic modules. Such rings are called *right FGC-rings*. Kaplansky has shown that an almost maximal valuation ring is an FGC-ring [19, Theorem 14]. However, a combination of results by Kaplansky, Matlis, Gill, Lafon and Warfield shows that a local commutative ring R is an FGC-ring if only if R is an almost maximal ring ([12, p.133-135]). Behboodi extended this result in [3] to a non-commutative setting by showing that a local duo ring R is an FGC-ring if and only if it is a chain-ring for which R/I is a linearly compact left R-module for every non-zero ideal I of R. We use this result to extend Kaplansky's characterization to right chain and right duo rings (Theorem 6.2.4). We also show that the FGC-property is *closed under localizations*. Additionally, we address a question asked by Fuchs and Salce in [13], namely to find the integral domains R, over which the finitely generated (finitely presented) modules cancel in direct sums. Our last theorem, collects many of our findings about chain and duo rings, and links all the main concepts of this work as follows:

Theorem 6.2.8. *Let R be a chain and duo ring, and P a prime ideal of R such that $R \setminus P$ does not contains zero divisors. If R is a FGC-ring, the following hold:*

a) Any finitely generated module cancels in direct sums.

b) R is an almost maximal ring. If R is not a domain, then R is a maximal ring.

c) If ${}_RR$ or R_R is co-Hopfian and if R is not a domain, then R is self-injective.

d) Every finitely generated R-module M is Hopfian. If M is also injective, then M is both Hopfian and co-Hopfian.

e) All matrix rings $M_n(R)$ are repetitive.

f) The localization R_P is an almost maximal chain ring.

g) If $E = E(R/J(R))$ is Noetherian, then E has a finite number of submodules.

h) All finitely generated injective R-modules satisfy Fitting's Lemma.

i) An R-module M is distributive if only if all cyclic submodules of M are distributive.

Although we intended to make this document as self-contained as possible, the reader has to be aware that most of the topics studied in this dissertation are related to well-known problems in Ring Theory. While we introduce many of the basic definitions and facts related to the concepts studied, the reader is referred to standard texts like [2], [22], [26] or [30] for additional background details.

1.1 Basic Notions

As we mentioned before, all rings R are associative rings with identity and all the R-modules are unitary. The word *ideal* refers a two-sided ideal; and adjectives like co-Hopfian or Noetherian, likewise means both right and left co-Hopfian or Noetherian. Let R be a ring and S a subset of R, the *right annihilator of* S is the set $ann_r(S) = \{a \in R \mid sa = 0 \ \ for\ any \ \ s \in S\}$. In a similar way, we define *the left annihilator* $ann_l(S)$. Also, an element $0 \neq x \in R$ is a *right (left) zero divisor* if $ann_l(x) \neq 0$ $(ann_r(x) \neq 0)$.

A *domain* is a ring with no right or left zero divisors, and a *right (left) unit* u in R is an element of R such that there is $a \in R$ with $au = 1$ $(ua = 1)$. Additionally, $M \neq 0$ is a *simple* module if M has no proper submodules different from zero.

An object satisfies the *Ascending Chain Condition (ACC)* with respect to a property P, if any non-trivial chain of subobjects satisfying P eventually terminates. A *Noetherian right (left) module* is a module that satisfies the *ACC* for right (left) submodules, while a *right (left) Noetherian ring* satisfies this condition for right (left) ideals. Similarly, we define the *Descending Chain Condition (DCC)*, *Artinian right (left) module* and *Artinian right (left) ring*. On the other hand, an object satisfies the *Maximum Condition (MC)* for some type of subobjects, if every non-empty family of these subobjects has a maximal element. It is a well-known fact that a module satisfies the *MC* with respect to P if and only if satisfies the *ACC* with respect to P.

The intersection of all maximal right (or left) ideals of R is called the *Jacobson radical*, $J(R)$, of R. A useful characterization of this ideal is

$$J(R) = \{y \in R \mid 1 - xyz \ is\ a\ unit\ for\ every\ x, z \in R\}$$

A submodule N of M is an *essential submodule* if every non-zero submodule of M has a non-zero intersection with N, and this is denoted by $N \subseteq_e M$. The *right singular ideal* of R is $Z_r(R) = \{x \in R \mid ann_r(x) \subseteq_e R\}$. Similarly, we define the *left singular ideal* $Z_l(R)$.

1.2 Notation

$\mathbb{Z}$	ring of integers
$\mathbb{N}$	set of natural numbers
$\mathbb{Q}$	field of rational numbers
$\mathcal{M}_R$ $({}_R\mathcal{M})$	category of right (left) R-modules
$\mathbb{Z}_n$	ring of residues $\mathbb{Z}/n\mathbb{Z}$
$\subset$	strict inclusion
ω	first infinite cardinal
$A \setminus B$	set-theoretic difference
$M_n(R)$	ring of $n \times n$ matrices with entries from R
$C(R)$	the center of the ring R
$M^{(I)}$, $\bigoplus_{i\in I} M$	direct sum of I copies if M
$End_R(M)$	ring of R-endomorphisms of M
xR, Rx	right, left cyclic R-module generated by x
$Rad(M)$	radical of M
$J(R)$	Jacobson radical of R
$N(R)$	nilradical of R
$E(M)$	injective hull (or envelope) of M
$Z_r(R)$, $Z_l(R)$	right, left singular ideal of R
$dim(M)$	Goldie-dimension of M
$U(R)$	group of units of the ring R
$(R_P)_r$, $(R_P)_l$	right, left localization of R at the completely prime ideal P

Chapter 2

One-sided Chain Rings and Duo Rings

This chapter discusses some of the basic properties of chain and duo rings which will be used throughout our discussion. A more extensive study of right chain rings can be found in [4].

A ring is a *right chain ring* if $aR \subseteq bR$ or $bR \subseteq aR$ for all $a, b \in R$. Similarly, we define *left chain rings.* If R is both a right and a left chain ring, then R is *a chain ring.* A commutative chain ring R is called *valuation ring.* An example of a *finite* valuation ring is the ring $\mathbb{Z}_{p^k}$ with p a prime and $k \in \mathbb{N}$. For a non-commutative example, consider the field $\mathbb{F}_4$ with four elements and define the standard vector-space addition on $V = \mathbb{F}_4 \bigoplus \mathbb{F}_4$. To make V into a ring R define a multiplication by $(a, b) \cdot (c, d) = (ac, ad + bc^2)$. According to [17], R is smallest non-commutative chain ring. Observe that finite one-sided chain rings are automatically two-sided chain rings ([17, Theorem 2.1]). A further example is obtained by considering a division ring D which admits a one-to-one D-endomorphism f which is not an automorphism. The ring $R^* = D \times D$ which is obtained by using the standard addition and by defining the multiplication by $(a, b) \cdot (c, d) = (ac, f(a)d + bc)$, is a right chain ring that is not left chain ring (nor a domain) [5, Example 6.16].

Recall that a non-zero R-module M is called *local* if it has a largest proper submodule. In general, $Rad(M)$ is the intersection of all maximal submodules of M. Examples of local $\mathbb{Z}$-modules are simple modules and the modules $\mathbb{Z}_{p^k}$, where p is prime and $k < \omega$. The radical of the right R_R is the Jacobson radical $J(R)$ of R. It is a two-sided ideal of R, and coincides with the radical of ${}_RR$. In a local ring R, $J(R)$ is the unique maximal right ideal of R. Examples of local rings include fields, rings in which every element is either a unit or nilpotent, the endomorphism rings of *indecomposable modules* (non-zero modules with no non-trivial direct summands) of finite length and of course, any chain ring.

Although the next results are well-known properties of chain rings, we include some of the proofs for the convenience of the reader.

Lemma 2.0.1. [4, Lemma 1.2 & Lemma 1.4] *Let R be a right chain ring.*

a) R is a local ring such that all right ideals are linearly ordered by the inclusion.

b) Every finitely generated right ideal is principal.

c) If $M \in {}_R\mathcal{M}$ and $x \in M$, then xR is indecomposable.

d) For every $a \in R$, $Ra \subseteq UaR$

e) A right ideal of R is a two-sided ideal exactly if $uI \subseteq I$ for every $u \in U(R)$.

Proof. a) If I and J are right ideals of R such that $I \not\subset J$, then there exists $a \in I \setminus J$. For any $b \in J$, we have $bR \subseteq aR$, since $a \notin J$. Therefore $J = \sum_{b \in J} bR \subseteq aR \subseteq I$. In particular, any two maximal right ideals must be equal. b) follows immediately from the definition. Finally, to see c) observe that the submodules of xR are linearly ordered since $xR \cong R/ann_r(x)$. In particular, xR is indecomposable.

d) Let $x \in R$. If x is either a unit or $xa \in aR$, then we are done. Suppose $x \in J$ and $xa \notin aR$. Since R is a right chain ring, we have that $aR \subseteq xaR$, and $a = xas$ for some $s \in R$. Note that s must be in J, for otherwise s is a unit and $xa \in aR$. Then $1 + s \in U$ and

$$xa(1 + s) = xa + xas = xa + a = (1 + x)a.$$

Since $x \in J$ by assumption, $(1 + x) \in U$ and we obtain

$$xa = (1 + x)a(1 + s)^{-1} \in UaR.$$

e) Clearly the result holds if I is a two-sided ideal. Conversely, suppose I is a right ideal such that $uI \subseteq I$ for every $u \in U$. If $a \in I$, then $Ua \subseteq I$ by assumption. Then $ra \in Ra \subseteq UaR \subseteq I$ for every $r \in R$ by d). Therefore I is an ideal. □

A ring R is a *right (left) duo ring* if every right (left) ideal is a two-sided ideal, or equivalently, a ring such that $Ra \subseteq aR$ ($aR \subseteq Ra$) for any $a \in R$. We will say that R is *duo ring* if R is a right and left duo ring, i.e. if $aR = Ra$ for all $a \in R$. Trivial examples of duo rings are, of course, commutative rings and division rings. Furthermore, the right chain ring $R^* = D \times D$ defined previously is also a right duo ring that is not a left duo ring.

Another example is given in [8]. Let F be the splitting field of the polynomial $p(x) = x^3 - 2$ over the rational numbers $\mathbb{Q}$. Let α be the real root of $p(x)$ and β be one of the complex roots. Then there exists an automorphism σ of F with $\sigma(\alpha) = \beta$. We consider the twisted power series ring $K = F[[x, \sigma]]$ in one variable x over F. It consists of all formal series $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in F$ with the addition $\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$ and the multiplication defined using the distributive law and the rule $xa = \sigma(a)x$. The subring $R = \mathbb{Q} + xK$ of K, consisting of all those series whose constant term is in $\mathbb{Q}$ is a non-commutative duo domain.

For one-sided annihilators in a right duo ring, the following is well-known:

Lemma 2.0.2. *Let M be a right module over a right duo ring R.*

a) If M is finitely generated, say $M = \sum_{i=1}^{n} x_i R$, then $ann_r(M) = \cap_{i=1}^{n} ann_r(x_i)$.

b) $ann_r(x) = ann_r(xR)$ for all $x \in M$.

Proof. a). If $a \in \bigcap_{i=1}^{n} ann_r(x_i)$ and $m \in M$, then, $ma = \sum_{i=1}^{n} x_i r_i a$. Since R is a right duo ring there is a family $\{t_i\}_{i=1}^{n} \in R$ such that $r_i a = a t_i$ for any i. Hence, $ma = \sum_{i=1}^{n} (x_i a) t_i = 0$. Therefore $\bigcap_{i=1}^{n} ann_r(x_i) \subseteq ann_r(M)$. Since $ann_r(M) \subseteq \bigcap_{i=1}^{n} ann_r(x_i)$, the result follows. Note that b) is an immediate consequence of a). □

Chapter 3

Repetitive Rings

An element a of a ring R is *right repetitive* if, for each finitely generated right ideal I of R, the right ideal $\sum_{n=0}^{\infty} a^n I$ is also finitely generated. Rings in which every element is right repetitive are called *right repetitive rings*. Repetitiveness is not a *left-right* symmetric condition as it is shown in [15] (Examples 9, 10 and 13).

Proposition 3.0.1. [15, p. 591] *Let R be any ring and $a \in R$, then, the following conditions are equivalent:*

a) $\sum_{n=0}^{\infty} a^n I$ is a finitely generated right ideal of R for any finitely generated right ideal I of R.

b) $\sum_{n=0}^{\infty} a^n I = \sum_{n=0}^{k} a^n I$ for some positive integer k for any finitely generated right ideal I of R.

c) for any $x \in R$, there exist $k_x < \omega$ and $r_0, r_1, ..., r_{k_x-1} \in R$ such that

$$a^{k_x} x = xr_0 + axr_1 + a^2 xr_2 + ...a^{k_x-1} xr_{k_x-1} \qquad (1)$$

.

Proof. *a)* $\implies$ *b)* Let $\{y_1, y_2, ..., y_m\}$ be a set of the generators for $\sum_{n=0}^{\infty} a^n I$. Then $y_i = a^{n_{i_1}} b_{i_1} + a^{n_{i_2}} b_{i_2} + ... + a^{n_{i_s}} b_{i_s}$ for some $s < \omega$. If $k_i = max\{n_{i_j}\}_{j=1}^{s}$, it is clear that $y_i \in \sum_{n=0}^{k_i} a^n I$. Therefore, if $k = max\{k_i\}_{i=1}^{m}$, we have $\sum_{n=0}^{\infty} a^n I \subset \sum_{n=0}^{k} a^n I$.

b) $\implies$ *c)*. Let $x \in R$, consider the right principal ideal xR. By b) $\sum_{n=0}^{\infty} a^n xR = \sum_{n=0}^{k} a^n xR$ for some positive integer k. Then, $a^{k+1}x \in \sum_{n=0}^{k} a^n xR$, i.e. $a^{k+1}x = \sum_{n=0}^{k} a^n x r_{x_n}$.

c) $\implies$ *a)*. Let $y \in \sum_{n=0}^{\infty} a^n I$. Thus $y = a^{n_1}b_1 + a^{n_2}b_2 + ... + a^{n_s}b_s$ for some $s < \omega$, where $b_i \in I$ for any $i = 1, 2, ..., s$. Let $\{x_1, x_2, ..., x_m\}$ be a set of the generators for I, then $b_i = \sum_{j=1}^{m} x_j c_{b_{i_j}}$ with $c_{b_{i_j}} \in R$, and therefore $a^{n_i}b_i = \sum_{j=1}^{m} a^{n_i} x_j c_{b_{i_j}}$. By c), for each x_j, there exist k_j and $x_{j_0}, x_{j_1}, ..., x_{j_{k_j-1}} \in R$ such that

$$a^{k_j}x_j = x_j x_{j_0} + a x_j x_{j_1} + a^2 x_j x_{j_2} + a^{k_j-1} x_j x_{j_{k_j-1}} \in \sum_{n=0}^{k_j-1} a^n I \qquad (2)$$

If $n_i \leq k_j$, clearly $a^{n_i}x_j \in \sum_{n=0}^{k_j-1} a^n I$. Suppose that $k_j < n_i$. Observe that equation (2) implies $a^{k_j+t}x_j \in \sum_{n=0}^{k_j} a^n I$ for any $t < \omega$. This means that $a^{n_i}x_j = a^{k_j+(n_i-k_j)}x_j$ belongs to $\sum_{n=0}^{k_j} a^n I$. Thus, in both cases $a^{n_i}x_j c_{b_{i_j}} \in \sum_{n=0}^{k_j} a^n I$ for any $j = 1, 2, ..., m$. Let $k = max\{k_j\}_{j=1}^{m}$. Then, $a^{n_i}b_i = \sum_{j=1}^{m} a^{n_i} x_j c_{b_{i_j}} \in \sum_{n=0}^{k} a^n I$, and therefore $y \in \sum_{n=0}^{k} a^n I$. $\square$

Note that c) establishes that it suffices to check repetitiveness only for right principal ideals I.

3.1 Right Repetitive Elements

Let $C(R)$ be the center of R, which consist of all the elements in R that commute with arbitrary elements in R. If $a \in R$ is integral over $C(R)$, then there exists a monic polynomial $p(x) \in C(R)[x]$ such that $p(a) = 0$, and consequently $p(a)r = 0$ for all $r \in R$. Therefore, a satisfies an equation of the type (1) (in Proposition 3.0.1) since the coefficients of $p(x)$ and r commute. Thus, a is a repetitive element in R. Repetitiveness, in this sense, is a kind of integrality condition. Trivially, nilpotent elements in a ring are repetitive. In general, rings that are integrally closed over their center, commutative rings and right Noetherian rings are examples of right repetitive rings.

We now discuss how to construct additional repetitive elements from given ones.

Proposition 3.1.1. *Let R be a ring, and $x \in R$ right repetitive.*

a) If $u \in U(R)$, then $u^{-1}xu$ is right repetitive.

b) If $c \in C(R)$, then xc is right repetitive.

c) $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$ *are right repetitive in the matrix ring $M_2(R)$.*

Proof. a) If a is any element in R, by Proposition 3.0.1 x repetitive implies that there exist $k < \omega$ and $r_0, r_1, ..., r_{k-1} \in R$ such that

$$x^k(ua) = \sum_{n=0}^{k-1} x^n(ua)r_n$$

Hence $u^{-1}(x^k ua) = \sum_{n=0}^{k-1} u^{-1}x^n(ua)r_n$. Since $(u^{-1}xu)^m = (u^{-1}x^m u)$ for any m, we conclude $(u^{-1}xu)^k a = \sum_{n=0}^{k-1}(u^{-1}xu)^n ar_n$. Thus, $u^{-1}xu$ is repetitive.

b) As in a), there exist $k < \omega$ and $r_0, r_1, ..., r_{k-1} \in R$ such that $x^k a = \sum_{n=0}^{k-1} x^n ar_n$ for any $a \in R$. Hence,

$$(xc)^k a = x^k c^k a = c^k(x^k a) = c^k(\sum_{n=0}^{k-1} x^n ar_n) = \sum_{n=0}^{k-1}(xc)^n a(c^{k-n}r_n)$$

as required.

c) Let x be a right repetitive element in R, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element in $M_2(R)$. We only show that $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ is right repetitive since the other cases can be treated in a similar way. Since x is right repetitive, there are $k < \omega$ and $r_0, r_1, ..., r_{k-1}, s_0, s_1, ..., s_{k-1} \in R$, such that $x^k a = \sum_{i=0}^{k-1} x^i ar_i$ and $x^k b = \sum_{j=0}^{k-1} x^j bs_j$.

Therefore,

$$\begin{aligned}
\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}^k \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} x^k a & x^k b \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=0}^{k-1} x^n a r_n & \sum_{n=0}^{k-1} x^n b s_n \\ 0 & 0 \end{bmatrix} \\
&= \sum_{n=0}^{k-1} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r_n & 0 \\ 0 & s_n \end{bmatrix}
\end{aligned}$$

which implies that $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ is right repetitive in $M_2(R)$. □

Additionally, we have:

Corollary 3.1.2. *If R is a right repetitive ring, then $M_2(R)$ is right repetitive if and only if the finite sum of right repetitive elements in this ring is also right repetitive.*

Proof. The direct implication is it clear. It remains to show that $M_2(R)$ is right repetitive if the sum of repetitive elements is repetitive. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element in $M_2(R)$. Since R is right repetitive, a, b, c and d are right repetitive elements in R. By c) in the previous proposition, $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ are right repetitive elements. By hypothesis, their sum must be also right repetitive. □

Note that the arguments in the proof of Proposition 3.1.1 c) hold for matrices in the ring $M_n(R)$ for all $1 < n < \omega$. Therefore, to verify that the ring $M_n(R)$ is repetitive (whenever R is repetitive), it is enough to show that the sum of repetitive elements is also repetitive. The repetitiveness of the ring $M_n(R)$ will play an important role in the next section.

3.2 Right Repetitive Chain Rings

We now investigate repetitive elements in chain rings.

Theorem 3.2.1. *Let R be a right chain ring. If x is a right repetitive element of R, then $1+x$ is right repetitive.*

Proof. Let $a \in R$. We need to show that $\Sigma_{n=0}^{\infty}(1+x)^n aR$ is a finitely generated right ideal of R. Since R is a right chain ring, the right ideals $(1+x)R$, xaR and aR have to satisfy one of the following cases:

If $(1+x)aR \subseteq xaR, aR$, then $(1+x)a = as$ for some $s \in R$. For $0 < n < \omega$, we obtain

$$(1+x)^n a = (1+x)^{n-1}(1+x)a = (1+x)^{n-1}as \in (1+x)^{n-1}aR.$$

Thus, $(1+x)^n aR \subseteq (1+x)^{n-1}aR$, and

$$aR \subseteq \sum_{n=0}^{\infty}(1+x)^n aR \subseteq (1+x)aR \subseteq aR$$

Then $\sum_{n=0}^{\infty}(1+x)^n aR = aR$ is a finitely generated right ideal of R.

On the other hand, if $aR \subseteq (1+x)aR$, then $xa = (-a) + (1+x)a \in (1+x)aR$, and $xaR \subseteq (1+x)aR$. Since $(1+x)aR \subseteq aR + xaR$, we obtain $(1+x)aR = aR + xaR$. Also, if $xaR \subseteq (1+x)aR$, then $a = (-xa) + (1+x)a \in (1+x)aR$, and $(1+x)aR = aR + xaR$ as before. We now show

$$\Sigma_{k=0}^{n}(1+x)^k aR = \Sigma_{k=0}^{n} x^k aR$$

for all $n < \omega$ in both of these cases. By what has been shown, $aR + (1+x)aR = aR + xaR$. Clearly,

$$\Sigma_{k=0}^{n+1}(1+x)^k aR \subseteq \Sigma_{k=0}^{n+1} x^k aR.$$

We can find $\ell_0, \ldots, \ell_n < \omega$ such that

$$x^{n+1}a = (1+x)^{n+1}a - [\Sigma_{k=0}^{n} x^k \ell_k]a \in (1+x)^{n+1}aR + \Sigma_{k=0}^{n} x^k aR.$$

By the induction hypothesis, $\Sigma_{k=0}^{n} x^k aR = \Sigma_{k=0}^{n}(1+x)^k aR$. Hence, $x^{n+1}a \in \Sigma_{k=0}^{n+1}(1+x)^k aR$.

Since x is right repetitive, there are $k, \ell < \omega$ such that

$$\Sigma_{n=0}^{\infty} x^n aR = \Sigma_{n=0}^{k} x^n aR = x^\ell aR$$

using the fact that R is a right chain ring. But

$$\Sigma_{n=0}^{\infty}(1+x)^n aR = \Sigma_{n=0}^{\infty} x^n aR = x^\ell aR$$

by what was shown in the last paragraph. □

We conclude this section by looking at repetitiveness in chain and duo rings:

Theorem 3.2.2. *Let R be a ring.*

a) If R is a right duo ring, then R is right repetitive.

b) If R is a right chain ring and all the units are right repetitive, then R is right duo.

Proof. a) Let R be a right duo ring. If $x, a \in R$, then $xa = as$ for some $s \in R$. Thus, $x^n a = x^{n-1}(xa) = x^{n-1}(as)$ for all $0 < n < \omega$. Hence, $x^n aR \subseteq aR$ if $0 < n < \omega$, and

$$aR \subseteq \sum_{n=0}^{\infty} x^n aR = aR + \sum_{n=1}^{\infty} x^{n-1} aR \subseteq aR$$

which implies that x is right repetitive. Hence, R is a right repetitive ring.

b) Let R be a right chain ring such that every $u \in U(R)$ is right repetitive. There are $k, \ell < \omega$ such that

$$\Sigma_{n=0}^{\infty} u^n aR = \Sigma_{n=0}^{k} u^n aR = u^\ell aR$$

For all $r \in R$, we can find $s \in R$ such that $u^{\ell+1}ar = u^{\ell}as$. Since u is a unit, this implies $uaR \subseteq aR$. By part e) of Lemma 2.0.1, aR is a two-sided ideal of R, for all $a \in R$, and R is a right duo ring. $\square$

Chapter 4

R_R Hopfian and Hopfian Modules

A right R-module M is *Hopfian* if every epimorphic endomorphism $f : M \to M$ is an automorphism. Hopficity is a categorical notion that has been studied for groups, rings, modules, and topological spaces. Hiremath introduced the concept of Hopfian rings and modules in [16]. This section presents some examples and results on Hopfian modules. We are interested exclusively in the hopficity of rings considered as modules.

According to Varadarajan ([28, Theorem 1.3]), a ring is Hopfian as a left module if only if is Hopfian as a right module. Thus, we just say that the ring is Hopfian as a module. Clearly, every simple module is Hopfian. Some other examples of these modules are given by the following results:

Proposition 4.0.1 ([16], Proposition 1.2)**.** *Any commutative ring R is Hopfian as an R-module.*

Proof. Let $f : R_R \to R_R$ be any epimorphism with $f(1_R) = a$. Then, $f(r) = f(1)r = ar$ for any $r \in R$. Since exist $b \in R$ such that $f(b) = 1_R$, we have $1_R = f(b) = f(1_R)b = ab = ba$. Thus, a is a unit, and $Ker\ f = Ann_R(a) = 0$. □

Moreover, finite-dimensional vector spaces are Hopfian by a simple dimension argument. Our next result extends this to a more general setting. Recall that the *Goldie-dimension, $dim(M)$ of a module M* is the maximal number of non-zero summands in any direct sum of submodules of M.

Proposition 4.0.2. *Let R be a ring.*

a) Any module which satisfies the ACC for kernels of its endomorphism ring is Hopfian.

b) If R has finite right Goldie-dimension, then finitely generated projective and finite dimensional nonsingular right R-modules are Hopfian.

Proof. a) Let $f : M \to M$ be an epimorphism with $Ker\ f = K$. Since M satisfies the ACC, the chain

$$K = Ker\ f \subseteq Ker\ f^2 \subseteq Kerf^3 \subseteq ...$$

becomes stationary. Select $k < \omega$ such that $Ker\ f^k = Ker\ f^\ell$ for $\ell > k$. Pick $y \in Kerf^k$, and note that f^k is also an epimorphism. Therefore, there exists $x \in M$ such that $f^k(x) = y$. Hence, $f^{2k}(x) = f^k(y) = 0$. Thus, $x \in Ker\ f^{2k} = Ker\ f^k$, and $y = f^k(x) = 0$. Therefore, $K \subseteq Ker\ f^k = 0$.

b) Let P be a finitely generated projective right R-module. If $f : P \to P$ is an epimorphism, then the projectivity of P projective yields $P \cong P \oplus Ker\ f$. Since there exist a finitely generated free module F such that $F \cong P \oplus N$ for some submodule N of F, we obtain that P has finite right Goldie dimension. Since $dim(P)+dim(Ker\ f) = dim(P) < \infty$, we conclude $Ker\ f = 0$. The case of a finite dimensional nonsingular module is discussed in the same way. □

In particular, note that every finitely generated right module over a right Noetherian ring is Hopfian. Also, if $f : R_R \to R_R$ is an epimorphism, then $Kerf = ann_r(f(1_R))$ and $ann_r(f^n(1_R)) \subseteq ann_r(f^{n+1}(1_R))$ for any $n < \omega$, then by Part a),

Corollary 4.0.3. *A ring which satisfies the ACC on right or left annihilators is Hopfian.*

Furthermore, Vasconcelos' Theorem [29, Proposition 1.2] reduces the discussion to non-commutative rings:

Theorem 4.0.4. [29] *A finitely generated module over a commutative ring is Hopfian.*

Proof. (Vasconcelos) Let f be the given endomorphism of M. One can consider M as a module over $R[x]$, where the action of the indeterminate is given by $xm = f(m)$ for $m \in M$.

Then M is a finitely generated $R[x]$-module with $M = xM$. Consequently, $(1 + sx)M = 0$ for some $s \in R[x]$ using determinants. It is then clear that f is a monomorphism. $\square$

Unfortunately, arguments like this using determinants fail to carry over to a non-commutative setting in most cases.

The following theorem which is due to K. Goodearl characterizes the rings R for whose every finitely generated R-module is Hopfian.

Theorem 4.0.5. [15, Theorem 7] *For a ring R, the following conditions are equivalent:*

a) All finitely generated right $R-$modules are Hopfian.

b) All matrix rings $M_n(R)$ are right repetitive.

c) In every matrix ring $M_n(R)$, all units are right repetitive.

Corollary 4.0.6. [15, Proposition 4] *If R is a right repetitive ring, then all cyclic right R-modules are Hopfian.*

The next result provides an example of a ring R for which all finitely generated R-modules are Hopfian.

Proposition 4.0.7. [15, Example 10] *Let* $R = \begin{bmatrix} K & 0 \\ K[x] & K[x] \end{bmatrix}$ *where K is a field and x is an indeterminate. The matrix ring $M_n(R)$ is left repetitive for any $n < \omega$ since R is left Noetherian.*

It is important to note that according to Propositions 4.0.1 and 4.0.6, commutative and repetitive rings are Hopfian as modules, but also is true, that this is not the case for every ring, as we will observe in Proposition 4.0.19.

We now investigate the Hopfian property in the context of right duo and right chain rings. Combining the last corollary with Theorem 3.2.2 yields

Corollary 4.0.8. *If R is a right duo ring, then all cyclic right R-modules are Hopfian.*

In the following, we make use of a classical result called Shur's Lemma:

Lemma 4.0.9. [2, Shur's Lemma] *If M is a simple R-module, then $End_R(M)$ is a division ring.*

Proposition 4.0.10. *Any local ring is Hopfian as a module over itself.*

Proof. Let R be a local ring and $J = J(R)$ be the Jacobson radical of R. It suffices to consider R as a right module by what has been said previously. It is clear that R/J is a simple R-module. By the Schur's Lemma, $End_R(R/J)$ is a division ring. Let $\phi : End_R(R) \to End_R(R/J)$ be the ring homomorphism defined as $\phi(f) = \bar{f}$, where $\bar{f}(a + J) = f(a) + J$. Since R_R is free, ϕ is an epimorphism. Note that, for any epimorphism $f \in End_R(R)$, we have that $R = f(R) \supset J$. Then $f \notin Hom_R(R, J) = Ker\ \phi$. This implies that $\bar{f}$ has an inverse $\bar{g} = \phi(g)$ for some $g \in End_R(R)$. Note that $\phi(gf) = \phi(g)\phi(f) = 1_{End_R(R/J)} = \phi(1_{End_R(R)})$ where $End_R(R) = R$. Thus, $1_R - gf \in Ker\ \phi$, and therefore $(1_R - gf)(a) \in J$ for any $a \in R$. Since R local, we have that $1_R - (1_R - g(f(1_R)))$ is a unit. Since $g(f(1_R)) = g(1_R)f(1_R)$, we obtain that $f(1_R)$ has a left inverse. This means that, if $0 = f(x) = f(1_R)x$, then $x = 0$. Therefore, f is an isomorphism. □

A ring R is *Dedekind-finite* (DF) if $ab = 1_R$ implies $ba = 1_R$ for all $a, b \in R$. For example, if M is a Hopfian R-module, and $f, g \in E = End_R(M)$ satisfy $fg = 1_E$, then f must be epic. The hopficity of M gives $gf = 1_E$. Thus, the endomorphism rings of Hopfian (or co-Hopfian) modules are example of DF-ring. Modules with the latter property are called *DF-Modules*. Several properties of these modules are summarized in [7]. For example, it is easy to see that:

Proposition 4.0.11. *The class of rings R such that R_R is Hopfian coincides with the class of DF-rings.*

Proof. As it is observed above, if R_R is Hopfian then $End_R(R_R) = R$ is a DF-ring. On the other hand, every endomorphism f of R_R has the form $f(r) = ar$ for some $a \in R$. Thus, if

f is onto, then $aR = R$, and $ab = 1_R$ for some $b \in R$. Since R is a DF-ring, $ba = 1_R$. Thus, $g(r) = br$ is the inverse of f. □

Proposition 4.0.12. [23, Exercise 20.8] *For any ring R and any integer $n \geq 1$, the following statements are equivalent:*

a) $M_n(R)$ is DF.

b) For any right $R-$module M, $R_R^n \cong R_R^n \oplus K$ implies that $K = 0$.

c) R_R^n is Hopfian.

Lemma 4.0.13. [23, Exercise 20.9] *$M_n(R)$ is DF for all $n < \omega$ if R is a right duo ring R.*

Proof. (Goodearl, [23, p.284]). Let $\{J_i\}_{i \in I}$ be the set of maximal right ideals of R. Since J_i is an ideal, R/J_i is a division ring for any $i \in I$. Thus, $M_n(R)/M_n(J_i) \cong M_n(R/J_i)$ is a simple Artinian ring for all $n < \omega$, which we know to be Dedekind-finite by using arguments from Linear Algebra. The natural ring homomorphism $M_n(R) \to \prod_i M_n(R/J_i)$ has $M_n(Rad(R)) = Rad(M_n(R))$ as its kernel. Thus, $M_n(R)/Rad(M_n(R))$ is a DF-ring, and this implies the same for $M_n(R)$. □

A ring R has *right stable range* 1 if, whenever $aR + bR = R$ for some $a, b \in R$, then there exists $e \in R$ such that $a + be \in U(R)$. Observe that if $b = 0$, then this condition is equivalent to R being a DF-ring. Due to a result of Vaserstein left and right stable range 1 are equivalent properties [22, p.300]. For this type of rings we have,

Proposition 4.0.14. *Any local ring R has stable range 1.*

Proof. Let $a, b \in R$ such that $aR + bR = R$. Then, $aR = R$ or $bR = R$ since otherwise $aR, bR \subseteq J$ yields $R = J(R)$. Without lost of generality assume $bR = R$. Then $b \notin J$ yields that b is a unit. If a also is a unit, then $(1_R - a) + (-(1_R - b)) = b - a$ cannot be a non-unit since R is local. Thence, $1_R - (b - a) = a + b(b^{-1} - 1_R)$ is invertible as is required. Similarly, if a is not a unit, then $(1_R - b) + (-a) = 1_R - (a + b)$ cannot be invertible, which implies that $a + b(1_R)$ is a unit. □

Theorem 4.0.15. [22, Cancellation Theorem of Evans] *Let R be a ring, and M, N, K be right R-modules. Suppose $End_R(M)$ has stable range 1. Then $M \oplus N \cong M \oplus K$ implies $N \cong K$.*

Combining the last two results, we obtain:

Proposition 4.0.16. *Any module with a local endomorphism ring cancels in direct sums.*

Observe that the cancellation property does not hold for modules in general; given two non-isomorphic R-modules N and K, if we set $M = N \oplus K \oplus N \oplus K \oplus ...$, then $M \oplus N \cong M \oplus K \cong M$, but M cannot be cancelled. This construction, known as *Eilenberg's trick*, suggests restricting the study of the cancellation property to finitely generated modules ([11],p.192). As a first step, we investigate free Hopfian modules. We have,

Corollary 4.0.17. *If R is a local ring, then any finitely generated free right R-module is Hopfian.*

Proof. Let F be a finite free R-module, and consider an epimorphism $g : F \to F$ with kernel K. Since F is projective, then $F \cong F \oplus K$, or equivalently, $R_R^n \cong R_R^n \oplus K$ for some $n < \omega$. By the previous proposition, we can cancel R_R one by one since R is local. Thus, $K = 0$. □

According to Hiremath [16], it is not known whether hopficity is closed under taking (finite) direct sums. However, Hiremath showed that *Hopfian free modules are finitely generated* [16, Proposition 12]. Thus, combining Hiremath's proposition with Proposition 4.0.12, Lemma 4.0.13, and Corollary 4.0.17, we obtain,

Theorem 4.0.18. *Let R be a local ring or a right duo ring. Then a free right R-module is Hopfian if only if it is finitely generated.*

Interestingly, a free module of finite rank over a non-commutative ring need not to be Hopfian. In fact, as we mentioned before, not every ring is Hopfian as a module as Hiremath pointed out in the following example.

Proposition 4.0.19. [16, p.899] *Let K be a field, and $V = \oplus_{n=1}^{\infty} K$ be a countably infinite dimensional vector space over K. Then the ring R of all linear operators of V is not Hopfian as an R-module.*

Proof. To show this, define two K-endomorphisms of V by $f((k_i)_{i=1}^{\infty}) = (0, k_1, k_2, ...)$, and $g((k_i)_{i=1}^{\infty}) = (k_2, k_3, k_4, ...)$ for $(k_i)_{i=1}^{\infty} \in V$. Let $\pi_1 : V \to K$ be the projection onto the first coordinate with kernel $\oplus_{n=2}^{\infty} K$. Then $gf = 1_R$ and $\pi_1 f = 0$. If R were Hopfian, then R would be a DF-ring by Proposition 4.0.11. Therefore, $fg = 1_R$. But this means that $\pi_1 = \pi_1 1_R = \pi_1(fg) = (\pi_1 f)g = 0$, a contradiction. □

Recall that a module is *indecomposable* in case it is non-zero and has no non-trivial direct summands. A direct decomposition $M = \oplus_A M_\alpha$ of a module M as a direct sum of idecomposables submodules $\{M_\alpha\}_{\alpha \in A}$ is an *indecomposable decomposition.*

To prove the main result of this chapter, we need the following result:

Theorem 4.0.20. [2, Azumaya's Theorem] *If a module has a direct decomposition $M = \oplus_A M_\alpha$, where each endomorphism ring $End(M_\alpha)$ is local, then this is an indecomposable decomposition and,*

a) Every non-zero direct summand of M has an indecomposable direct summand;

b) The decomposition $M = \oplus_A M_\alpha$ complements maximal direct summands and thus is equivalent to every indecomposable decomposition of M.

Theorem 4.0.21. *Let R be a right duo and right chain ring. A right R-module of the form $M = \oplus_{i=1}^{n} x_i R$ is Hopfian.*

Proof. Let be $\phi : M \to M$ an epimorphism with $Ker\ \phi = K$. We induct on the number of generators of M. For $n = 1$, we have that M is a cyclic R-module, and then by Corollary 4.0.8, M is Hopfian. For the general case, suppose that the result is true for any module with less than n generators. Consider the family of right ideals $\{ann_r(x_i)\}_{i=1}^{n}$. Since R is a right chain ring we may assume that this family of ideals satisfies $ann_r(x_i) \subseteq ann_r(x_{i+1})$ for any i. Observe, that R a right duo ring implies that $ann_r(x_i)$ is a two-sided ideal for any $i = 1, ..., n$. Therefore, we have that $I = ann_r(M) = ann_r(x_1)$.

Moreover, each module x_iR has a local endomorphism ring. To see this, let $\alpha \in End(x_iR)$, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ann_r(x_i) & \longrightarrow & R & \longrightarrow & x_iR & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & ann_r(x_i) & \longrightarrow & R & \longrightarrow & x_iR & \longrightarrow & 0 \end{array}$$

where α_0 and α_1 are obtained by the projectivity of R. Hence, α_1 is left multiplication by some $r_1 \in R$. Furthermore, left multiplication by $s \in R$ induces an endomorphism of x_iR if and only if $s\ ann_r(x_i) \subseteq ann_r(x_i)$. Since $ann_r(x_i)$ is a two-sided ideal, $End(x_iR)$ is an epimorphic image of R, and hence local.

Since I is a two-sided ideal, M can be viewed as an (R/I)-module. Note that $x_1R \cong R/I$, and let k be the largest index such that $I = ann_r(x_1R) = ann_r(x_2R) = \ldots = ann_r(x_kR)$. Then, $F = \oplus_{i=1}^{k} x_iR$ is a free (R/I)-module, and $M = F \oplus (\oplus_{i=k+1}^{n} x_iR)$. Since $M \cong M/K$, there is an (R/I)-submodule N of M containing K such that $N/K \cong \oplus_{i=k+1}^{n} x_iR$. Then,

$$M/N \cong (M/K)/(N/K) \cong F.$$

But F projective as an $R/I-$module implies $M/K \cong F \oplus (N/K)$, and $M = F \oplus N$ as an R/I-module since $F \cong M/N$. Now, observe that x_iR is indecomposable for any i since R is a right chain ring.

Since the rings $End(x_iR)$ are local for all $i = 1, \ldots, n$, we obtain that $N \cong N/K \cong \oplus_{i=k+1}^{n} x_iR$ by Azumaya's Theorem. Hence, there is an epimorphism $N \to N$ with kernel K. Since $N \cong \oplus_{i=k+1}^{n} x_iR$, this epimorphism has to be an isomorphism, i.e. $K = 0$ by induction hypothesis. $\square$

Corollary 4.0.22. *Let R be a right chain and right duo ring such that every finitely generated right R-module is a direct sum of cyclic modules. Then*

a) all the finitely generated right modules are Hopfian.

b) the endomorphism ring of every finitely generated right module is DF.

c) any finite direct sum of cyclic modules cancel in direct sums.

Proof. a) Is immediate from previous theorem. By Proposition 4.0.11, b) is equivalent to a).

c) As in the proof of the theorem above, the endomorphism ring of each cyclic module in the direct sum is a local ring. By Corollary 4.0.16, each cyclic module has the cancellation property. Therefore the direct sum of these modules also cancel in direct sums. □

It is important to note that these last results raise the problem to determine when a chain and duo ring is one of the so called FGC-rings. We revisit this question in Chapter 6.

Chapter 5

Annihilators, Co-Hopficity and Self-Injectivity

5.1 Annihilator Conditions

In this section we discuss basic properties of the right and left annihilators. We begin our discussion with a well-know result on annihilators.

Theorem 5.1.1. [26, Ikeda-Nakayama's Theorem] *The following properties of R are equivalent:*

a) Every homomorphism $f : I \to R$, where I is finitely generated right ideal, has the form $f(x) = rx$ for some $r \in R$.

b) R satisfies:

i) $ann_l(A_1 \cap A_2) = ann_l(A_1) + ann_l(A_2)$ for all finitely generated right ideals A_1 and A_2.

ii) $ann_l(ann_r(a)) = Ra$ for all $a \in R$.

Then we have,

Theorem 5.1.2. *Let R be a chain ring. Then,*

a) $ann_l(A_1 \cap A_2) = ann_l(A_1) + ann_l(A_2)$ for all right ideals A_1 and A_2.

b) If every $a \in R$ with $ann_r(a) = 0$ is a unit, then $ann_l(ann_r(B)) = B$ for every principal left ideal B. If additionally R is a right duo ring, then

i) $ann_l(ann_r(a)) = Ra$, and

ii) for any principal right ideal I of R, every homomorphism $f : I \to R$ is defined by a multiplication (by the left) by a fixed element of R.

Proof. a) is immediate since R is a two-sided chain ring and $A_1 \subseteq A_2$ implies that $ann_l(A_2) \subseteq ann_l(A_1)$. We now prove b). Let B a principal left ideal of R with $B = Rx$ for some $x \in R$. Clearly, $Rx \subseteq ann_l(ann_r(Rx))$. If $b \in ann_l(ann_r(Rx))$, then $Rb \subseteq Rx$ or $Rx \subseteq Rb$ since R a left chain ring. Clearly, the first inclusion implies the result. If the second inclusion holds, then $ann_r(Rb) \subseteq ann_r(Rx)$. Note that $Rb \subseteq ann_l(ann_r(Rx))$ implies

$$ann_r(Rb) \supseteq ann_r(ann_l(ann_r(Rx))) = ann_r(Rx)$$

Then, $ann_r(Rb) = ann_r(Rx)$. This implies $xa = 0$ if and only if $ba = 0$ for all $a \in R$. Recall that $x = sb$ for some $s \in R$. If $bt \in bR \cap ann_r(s)$, then $0 = s(bt) = (sb)t = xt$. Thus, by the equality of the annihilators above, $bt = 0$. Therefore, $bR \cap ann_r(s) = 0$. Since $bR \neq 0$ and R is a right chain ring, $ann_r(s) = 0$. By hypothesis s is a unit, and therefore, $b = s^{-1}x \in Rx$ which implies the desired equality. For b-i), note that Lemma 2.0.2 b) implies $ann_r(a) = ann_r(aR) \subseteq ann_r(Ra)$ for every $a \in R$ since $Ra \subseteq aR$. Thus, $ann_r(a) = ann_r(Ra)$. By what has already been shown in Part b), $ann_l(ann_r(a)) = ann_l(ann_r(Ra)) = Ra$. Finally, observe that b-i) is equivalent to b-ii) according to the Ikeda-Nakayama's Theorem. □

Note the left-right symmetry in the preceding theorem. The rings that satisfy a) are called left *Ikeda-Nakayama rings* (IN-rings) and are studied in [10]. Also note that b) implies that every principal left ideal is a left annihilator.

5.2 R_R co-Hopfian

Let R be a ring satisfying condition b) in Theorem 5.1.2, i.e., a ring in which every element a with $ann_r(a) = 0$ is a unit. If f is a non-zero injective endomorphism of R_R. Then $f(r) = f(1_R)r$. Since $0 = Kerf = ann_r(f(1_R))$, we have that $f(1_R)$ is a unit. Therefore, f is onto. This means that rings with the latter property are co-Hopfian as right modules. On the other hand, if R_R is co-Hopfian and $a \in R$ satisfies $ann_r(a) = 0$, then the assignment $1_R \mapsto a$ defines an injective endomorphism g of R_R. Because of the co-hopficity of R_R, it is an isomorphism. Thus, $aR = R$. This last condition implies that a has a right inverse. Observe that the ring of endomorphisms of a co-Hopfian module is a DF-ring. Since $End_R(R_R) = R$, a must be a unit. Consequently,

Proposition 5.2.1. *R_R is co-Hopfian if only if every $a \in R$ with $ann_r(a) = 0$ is a unit.*

Co-Hopfian modules were studied in [16] and [28]. In the latter, Varadarajan observed ([28, p.294]) that co-Hopficity is not a right-left symmetric condition. Therefore, we say that R_R (or ${}_RR$) is co-Hopfian to indicate that R is co-Hopfian as a right (or as a left) module. We can show as in 4.0.2 a) that Artinian modules are co-Hopfian. Although, $\mathbb{Z}$ is Hopfian as a $\mathbb{Z}$-module, it is not co-Hopfian. On the other hand, $\mathbb{Q}$ is both Hopfian and co-Hopfian as both a $\mathbb{Q}$-module and as a $\mathbb{Z}$-module [28]. For further examples of co-Hopfian modules, see [15], [16] and [28]. Observe that if R is co-Hopfian as right or left module over itself, then R is Hopfian as a module since it is easy to see that projective co-Hopfian objects are Hopfian. It is also true that injective Hopfian objects are co-Hopfian. The next proposition is an easy application of this.

A module is called *semi-simple* if it is a sum of simple modules. A ring R is called semi-simple if R_R is a semi-simple module. It is a well known fact that semi-simple rings are those for which all the modules are projective and injective [26, Proposition 7.7, p.24]. Thus,

Proposition 5.2.2. *Over a semi-simple ring a module is Hopfian if only if is co-Hopfian.*

Examples of co-Hopfian modules are given in the following proposition. First, a well-known result,

Lemma 5.2.3. *Let M be a R-module with finite Goldie dimension and let $f \in End_R(M)$ be a monomorphism. Then $f(M)$ is an essential submodule of M.*

Also, a module M is *right quasi-injective* if and only if for every right R-module N, every R-monomorphism $j : N \to M$, and every R-homomorphism $f : N \to M$, there is an $\bar{f} \in End_R(M)$ such that $\bar{f}j = f$. It is clear that injective modules are quasi-injective.

Proposition 5.2.4. *Let R be any ring. Then, any injective or quasi-injective R-module of finite Goldie dimension is co-Hopfian.*

Proof. Let M be a quasi-injective R-module, and let $f : M \to M$ be any R-monomorphism. Then, $M = f(M) \oplus N$ for some submodule N of M. Observe that $dim(f(M)) = dim(M)$ since $f(M) \subseteq_e M$ by the previous lemma. Therefore $N = 0$, and then f is an automorphism. □

On a different topic, a ring R has the *maximum condition* on (right or left) annihilators if and only if every non-empty set of (right or left) annihilators has a maximal element, or equivalently, if any non-trivial chain of (right or left) annihilators eventually terminates (ACC).

Proposition 5.2.5. [18, Lemma 1.8 & Theorem 1.9]

a) Let R be a ring in which every principal right ideal is a right annihilator. Then $N(R) \subseteq Z_l(R)$.

b) If R satisfies the maximum condition on left annihilators and if every principal right ideal is a right annihilator, then $J(R)$ and $Z_r(R)$ are both nilpotent.

Combining the above result with Part b) of Theorem 5.1.2, and using the right-left symmetry in this theorem, we obtain,

Corollary 5.2.6. *If R be a right chain ring such that ${}_RR$ is co-Hopfian, $N(R) \subseteq Z_l(R)$. If R also satisfies the ACC condition on left annihilators, then $J(R)$ and $Z_r(R)$ are both nilpotent.*

We conclude this section with a technical result which is a generalization of the commutative case in [21] and that we will use later.

Proposition 5.2.7. *Let R be a chain ring such R_R is co-Hopfian, I and A left ideals of R, and let $J = J(R)$ be the Jacobson radical of R.*

a) If I is not an annihilator ideal, then there is a smallest principal ideal Rx containing I, such that $Rx = ann_l(ann_r(I))$ and $I = Jx$.

b) If $I \subset A$, then $ann_l(ann_r(I)) \subseteq A$.

Proof. a) Since I is left ideal that is not an annihilator, $I \subset ann_l(ann_r(I))$. If $x \in ann_l(ann_r(I)) \setminus I$, then $I \subset Rx$ since R is a left chain ring. Hence, $I \subset Rx \subseteq ann_l(ann_r(I))$. Taking right annihilators, we obtain $ann_r(Rx) = ann_r(I)$, which yields $ann_l(ann_r(Rx)) = ann_l(ann_r(I))$. By Proposition 5.1.2 b), $Rx = ann_l(ann_r(I))$. We now show $Jx = I$. If $Jx \subset K \subset Rx$ for some left ideal $K \subset R$, then $K = Lx$ for some left ideal L such that $J \subset L \subset R$, which contradicts the maximality of J. This means that there are no left ideals between Jx and Rx. On the other hand, the fact that R is a left chain ring implies $Jx \subseteq I$ or $I \subseteq Jx$. Note that the first inclusion implies $I = Jx$ by the observation above. If the second inclusion were strict, then we would have $I \subset Rb \subset Rx$ for some $b \in Jx$ since R is a left chain ring. Taking double annihilators would yield $Rx = Rb$. Therefore, $Jx = Rx$ which clearly is a contradiction. This also shows that Rx is the smallest left ideal with the mentioned property.

b) Note that $I \subseteq ann_l(ann_r(I))$. If equality holds, then the result is obvious. If the inclusion is strict, then I cannot be an annihilator ideal. By a), $ann_l(ann_r(I)) = Rx$ for some $x \in R \setminus I$. For any $a \in A \setminus I$, we have $Rx \subseteq Ra$ or $Ra \subset Rx$. If the first inclusion holds, we are done. So, assume $Ra \subset Rx$. Then $I \subset Ra \subseteq Rx$, which implies $ann_l(ann_r(I)) = Ra \subseteq A$. □

5.3 One-Sided Maximal, Almost Maximal and Self-injective Rings

In this section, we discuss the relationship between right Maximal and right Almost Maximal rings for right chain and right duo rings. Our results extend Gill's work on the commutative case [14]. We characterize right self-injective rings in terms of their right co-Hopficity and left maximality.

A ring R is right (left) maximal if any system of pairwise solvable congruences of the form $x_\alpha \equiv x_\beta(I_\beta)$ with $\alpha, \beta \in \Lambda$, $x_\alpha, x_\beta \in R$ and I_β a right (left) ideal of R, has a simultaneous solution in R. We will say that R is right (left) almost maximal if the above congruences have a simultaneous solution whenever $\bigcap_\Lambda I_\alpha \neq 0$. Examples of maximal rings are fields, the ring of p-adic integers $\mathbb{Z}_p$, and the ring $\mathbb{Z}/p^n\mathbb{Z}$. Also, $\mathbb{Z}$ is an almost maximal ring that is not a maximal ring.

The next theorem, establishes conditions for the equivalence of the concepts *right almost maximal* and *right maximal* in right chain rings:

Theorem 5.3.1. *If R is a right (left) chain ring which contains an element $b \neq 0$ such that $b^2 = 0$, then, R is right (left) almost maximal if only if R is right (left) maximal.*

Proof. It is clear that R right maximal implies R right almost maximal. Suppose that R is right almost maximal. Let Λ be any set, and consider a system $x_\gamma \equiv x_\alpha(I_\alpha)$ of pairwise solvable congruences with right ideals with $\gamma, \alpha \in \Lambda$. We want to show that this system has a simultaneous solution. If $\bigcap_{\alpha\in\Lambda} I_\alpha \neq 0$, then the fact that R right almost maximal yields that this system has a simultaneous solution. Therefore, we may assume that $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$. There is $0 \neq b \in R$ such that $b^2 = 0$. Since R is a right chain ring there exists $\alpha_0 \in \Lambda$ such that $I_{\alpha_0} \subseteq bR$, otherwise $0 \neq bR \subseteq \bigcap_{\alpha\in\Lambda} I_\alpha$, which is not possible. Define $\Lambda' = \{\alpha \in \Lambda \mid I_\alpha \subseteq I_{\alpha_0}\}$. By hypothesis, $x_\alpha \equiv x_{\alpha_0}(I_{\alpha_0})$ has solution for any $\alpha \in \Lambda$. In particular, for $\alpha \in \Lambda'$, we have that $x_\alpha - x_{\alpha_0} \in I_{\alpha_0} \subseteq bR$ and therefore, $x_\alpha - x_{\alpha_0} = br_\alpha$ for some $r_\alpha \in R$. Now, consider the right ideal $K_\alpha = \{r \in R \mid br \in I_\alpha\}$ of R. Note that $b^2 = 0 \in I_\alpha$, implies $b \in K_\alpha$ for all $\alpha \in \Lambda$. Therefore $\bigcap_{\alpha\in\Lambda'} K_\alpha \supseteq \bigcap_{\alpha\in\Lambda} K_\alpha \neq 0$. For $\gamma, \alpha \in \Lambda'$ with

$I\gamma \subseteq I_\alpha$, consider the system of congruences $r_\gamma \equiv r_\alpha(K_\alpha)$. Note that $x_\gamma - x_\alpha = b(r_\gamma - r_\alpha)$. Since $x_\gamma - x_\alpha \in I_\alpha$, we have $r_\gamma - r_\alpha \in K_\alpha$. By what has been shown, this last system of congruences admits a simultaneous solution r_0. Finally, note that, since $r_0 - r_\alpha \in K_\alpha$, we have $(x_{\alpha_0} + br_0) - x_\alpha = b(r_0 - r_\alpha) \in I_\alpha$. Therefore $x = x_{\alpha_0} + br_0$ is the required simultaneous solution. □

Proposition 5.3.2. *If R is a right duo and right chain ring which is not a domain, then there is $0 \neq a \in R$ such that $a^2 = 0$.*

Proof. Since R is not a domain, there exist $a \in R$ such that $ann_r(a) \neq 0$. Since R is a right chain ring, then $ann_r(a) \subseteq aR$ or $aR \subseteq ann_r(a)$. The first inclusion implies that, for any $0 \neq s \in ann_r(a)$, $s = ar$ for some $r \in R$. Then, $s^2 = (ar)s = a(rs)$. Since R is a is a right duo ring, we have there is $r' \in R$ such that $rs = sr'$ and therefore $s^2 = (as)r' = 0$. Note that $aR \subseteq ann_r(a)$, which clearly implies $a^2 = 0$. □

Corollary 5.3.3. *A right duo and right chain ring that is not a domain is right (left) almost maximal if and only if is right (left) maximal.*

Recall that R is *right self-injective* if R is injective as a right module over itself, or equivalently, if any R-homomorphism $f : I \to R$ from a right ideal I of R can be extended to all of R. Examples of self-injective rings are all the rings $\mathbb{Z}_n$. Also, if V is an infinite-dimensional right vector space over a division ring D, then consider the full ring R of linear transformations of V, where linear transformations act on the left. The ring R is right self-injective, but is not left self-injective [24].

We now establish the relationship between right self-injective, right co-Hopfian and left maximal rings for chain and right duo rings.

Theorem 5.3.4. *Let R be a chain ring and a right duo ring. Then R is a right self-injective ring if and only if the following conditions hold:*

a) R_R is co-Hopfian.

b) R is a left maximal ring.

Proof. Assume that a) and b) holds. Let I be a right ideal of R and let $f : I \to R$ be a right R-homomorphism. Take $x \in I$, and note that $0 = f(x(ann_r(x))) = f(x)(ann_r(x))$. Then $f(x) \in ann_l(ann_r(x))$. By applying a), Propositions 5.1.2 b-i) and 5.2.1, $f(x) \in Rx = ann_l(ann_r(x))$ since R is also a right duo ring. Thus, $f(x) = r_x x$ for some $r_x \in R$. We want to prove that f can be extended to all of R, or equivalently, that f is defined by left multiplication with a fixed element of R. For this, let $a, b \in R$. Since R is a right chain ring, we may assume $aR \subseteq bR$. Consider the system of congruences $r_a \equiv r_b(ann_l(a))$, where r_a and r_b are defined by f. We want to show that this system is pairwise solvable. We have that $a = bs$ for some $s \in R$. Then, $(r_a - r_b)a = r_a a - r_b(bs) = f(a) - f(b)s = f(a) - f(bs) = 0$. Therefore $r_a - r_b \in ann_l(a)$ for any $a, b \in R$. By b), the system has a simultaneous solution $\hat{r} \in R$. Then, $\hat{r} - r_x \in ann_l(x)$ for all $x \in I$, and consequently $(\hat{r} - r_x)x = 0$. This implies $f(x) = r_x x = \hat{r}x$ for all $x \in I$. Therefore, $f(x) = \hat{r}x$ is the required extension, and R is right self-injective.

Conversely, let $f : R \to R$ be any injective right R-homomorphism. Note that R right self-injective implies that there exist an R-endomorphism g such that $gf = 1_R$. Thus g is an epimorphism. Since R is a right duo ring, R_R is Hopfian by Corollary 4.0.8. Therefore, g is an injection. Hence f is also an automorphism and R_R is co-Hopfian.

We now show b). Let Λ be a set, and consider a family $\{I_\alpha \mid \alpha \in \Lambda\}$ of left ideals of R. Suppose that $x_\gamma \equiv x_\alpha(I_\alpha)$ with $\gamma, \alpha \in \Lambda$ is a system of pairwise solvable congruences with $x_\alpha \in R$ for all $\alpha \in \Lambda$. We want to show that this system admits a simultaneous solution. Since R is a left chain ring, we may assume $I_\gamma \subseteq I_\alpha$ and therefore $x_\alpha - x_\gamma \in I_\alpha$. Observe that if there is a non-zero left ideal I_{α_0} such that $I_{\alpha_0} \subseteq I_\alpha$ for all $\alpha \in \Lambda$, then $x_{\alpha_0} - x_\alpha \in I_\alpha$

for all $\alpha \in \Lambda$, and this implies that the system has a simultaneous solution. Assume that there is no such left ideal. Define $J_\alpha = ann_r(I_\alpha)$. Then, if $I_\gamma \subseteq I_\alpha$ we have that $J_\alpha \subseteq J_\gamma$. Consider the multiplication maps $f_{x_\gamma} : J_\gamma \to R$ and $f_{x_\alpha} : J_\alpha \to R$ (left multiplication by x_γ and x_α, respectively) on their common domain J_α. Therefore, we can define the map $f : \bigcup_{\alpha\in\Lambda} J_\alpha \to R$ as the union of the these maps. Since R is a right chain ring, f can be viewed as an R-homomorphism from a right ideal of R to R. By hypothesis, R is right self-injective, and therefore, f can be extended to all of R. This implies that f is a multiplication from the left by some element $x \in R$. Our claim, is that x is a solution for the system of congruences. To show this, let $a_\alpha \in J_\alpha$. Then, $(x - x_\alpha)a_\alpha = xa_\alpha - x_\alpha a_\alpha = f(a_\alpha) - f_{x_\alpha}(a_\alpha) = 0$. Therefore $x - x_\alpha \in ann_l(J_\alpha) = ann_l(ann_r(I_\alpha))$. We know $I_\alpha \subseteq ann_l(ann_r(I_\alpha))$. If the equality holds, then x is the desired solution. Suppose $I_\alpha \subset ann_l(ann_r(I_\alpha))$. Recall that we are assuming that there is no a smallest left ideal in the family $\{I_\alpha\}_\alpha \in \Lambda$. Thus, $I_\gamma \subset I_\alpha$ for some $I_\gamma \in \Lambda$. By Proposition 5.2.7 b), $ann_l(J_\gamma) = ann_l(ann_r(I_\gamma)) \subseteq I_\alpha$. Then $x - x_\gamma \in I_\alpha$ since $x - x_\gamma \in ann_l(J_\gamma)$ (as we showed for α). By the pairwise solvability, $x_\gamma - x_\alpha \in I_\alpha$. Thus, $(x - x_\gamma) + (x_\gamma - x_\alpha) = x - x_\alpha \in I_\alpha$ for all $\alpha \in \Lambda$. Hence, x is again a simultaneous solution. □

As a corollary of this theorem, we obtain a different characterization of commutative self-injective rings, than the one given by Klatt and Levy [21, Theorem 2.3]:

Corollary 5.3.5. *A valuation ring R is self-injective if only if:*

(a) R_R or ${}_RR$ is co-Hopfian.

(b) R is a maximal ring.

Combining this last result with the Klatt' and Levy' theorem we have:

Corollary 5.3.6. *Let R be a valuation ring that is also a maximal ring. Then, R_R or ${}_RR$ is co-Hopfian if only if $Ann(Ann(B)) = B$ for all principal ideals B of R.*

Chapter 6

Localizations and FGC-Rings

6.1 Localizations in Chain and Duo rings

Let R be any ring. A completely prime ideal P of R is an ideal such that $ab \in P$ implies $a \in P$ or $b \in P$. For such an ideal, we define the right localization of R at P, by $(R_P)_r = \{ab^{-1} \mid a \in R, b \in R \setminus P\}$ whenever $R \setminus P$ does not contain zero divisors. As in the commutative case, we have that $(R_P)_r$ is a local ring with right maximal ideal $P(R_P)_r = \{ab^{-1} \mid a \in P, b \in R \setminus P\}$. To see this, just note that if $P(R_P)_r$ is contained strictly in some right maximal ideal $\bar{I}$ of $(R_P)_r$, then there is an element $ab^{-1} \in \bar{I}$ such that $ab^{-1} \notin P(R_P)_r$. This implies that $ba^{-1} \in R_P \setminus P(R_P)_r$. Therefore, $\bar{I}$ contains a unit. In a similar fashion, we can verify that $P(R_P)_r$ is the unique right maximal ideal of $(R_P)_r$. Symmetrically, we have that $(R_P)_l$ is a local ring with left maximal ideal $(R_P)_lP$. It is important to note that in a duo ring localizations at prime ideals always exist [25].

Lemma 6.1.1. *Let R be a left (right) chain ring, and let P be a completely prime ideal of R such that $R \setminus P$ does not contain zero divisors. Then, the right (left) localization $(R_P)_r$ ($(R_P)_l$) is a left (right) chain ring.*

Proof. Let $\bar{a}, \bar{b} \in (R_P)_r$. Suppose that $\bar{a} = rt^{-1}$ and $\bar{b} = st^{-1}$. Since R is a left chain ring and $r, s \in R$, we may assume that $r = us$ for some $u \in R$. Therefore, $\bar{a} = rt^{-1} = u(st^{-1}) \in (R_P)_r\bar{b}$, thus $(R_P)_r\bar{a} \subseteq (R_P)_r\bar{b}$. □

It is important to mention, that, in general, the duo property is not closed under localizations. However, if additionally the ring is Noetherian, the localization results a duo ring ([8]).

Recall that an ideal P of R is a prime ideal if for any two A, B ideals of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. Observe that if R is a two-sided duo ring, then the notions of completely prime and prime ideal, are equivalent. In that case, we have:

Lemma 6.1.2. *Let R be a duo ring and P a prime ideal of R such that $R \setminus P$ does not contain zero divisors. Then $(R_P)_r = (R_P)_l$.*

Proof. Let $\bar{a} \in (R_P)_r$. Then $\bar{a} = st^{-1}$ for some $s \in R$ and $t \in R - P$. Since R is a duo ring, we have $tR = Rt$. Then $ts = bt$ for some $b \in R$. Thus, $\bar{a} = st^{-1} = t^{-1}b \in (R_P)_l$. Symmetrically, we have the second containment and therefore the equality. □

In view of the above, if R is a duo ring and there is no confusion we will denote the two-sided localization just as R_P. For the next result, recall that given a ring homomorphism $f : R \to S$ and a right ideal J of S, $f^{-1}(J)$ is a right ideal of R.

Theorem 6.1.3. *Let R be a duo and chain ring, and let P be a prime ideal of R such that $R \setminus P$ does not contain zero divisors. Then, R right maximal (almost maximal) ring implies then R_P is right maximal (almost maximal) ring.*

Proof. Assume that R is a right maximal (almost maximal) ring. Consider the system $y_\gamma \equiv y_\alpha(J_\alpha)$ of pairwise solvable congruences with J_α right ideals of R_P, $y_\gamma, y_\alpha \in R_P$ and $\alpha, \gamma \in \Lambda$ for some set Λ. We want to show that this system has a simultaneous solution in R_P. By the two previous lemmas, R_P is a right chain ring ($(R_P)_r = (R_P)_l$). Without lost of generality assume $R_P = (R_P)_r$. Then $J_\alpha \subset PR_P$ for any $\alpha \in \Lambda$. Consider the localization map $\phi : R \to R_P$ given by $\phi(r) = r/1$. Suppose first that $y_\alpha \in \phi(R)$ for any $\alpha \in \Lambda$. Then, for each $\alpha \in \Lambda$, there exists $x_\alpha \in R$ such that $\phi(x_\alpha) = y_\alpha$. Define $I_\alpha = \phi^{-1}(J_\alpha)$, and consider the system of congruences $x_\gamma \equiv x_\alpha(I_\alpha)$ in R. This system is pairwise solvable in R since $\phi(x_\gamma) - \phi(x_\alpha) \in J_\alpha$. Also, R is right maximal implies that the system has a simultaneous solution $x_0 \in R$. It follows that $\phi(x_0)$ is a common solution for the system in R_P (if R is right almost maximal, observe that $\bigcap_{\alpha\in\Lambda} J_\alpha \neq 0$ implies $\bigcap_{\alpha\in\Lambda} I_\alpha \neq 0$). Now, assume that $y_{\alpha_0} \notin \phi(R)$ for some $\alpha_0 \in \Lambda$. The pairwise solvability of the system implies that $y_{\alpha_0} - y_\alpha \in J_{\alpha_0} \bigcup J_\alpha \subseteq PR_P$. Moreover, for any element y in the localization, $y = ab^{-1} = (a/1)(1/b) = \phi(a)(\phi(b))^{-1}$. Then, if $y \in PR_P$, we have $y \in \phi(R)$. Otherwise, $y = (\phi(b))^{-1}$ contradicting the maximality of PR_P. Then, $PR_P \subseteq \phi(R)$, and therefore, $y_{\alpha_0} - y_\alpha \in \phi(R)$. Observe that $y_\alpha \notin \phi(R)$ for all $\alpha \in \Lambda$, since $y_{\alpha_0} - y_\alpha + y_\alpha \in \phi(R)$ contradicts our hypothesis. Thus, if $y_\alpha \notin \phi(R)$ for some α, then $y_\beta \notin \phi(R)$ for any $\beta \in \Lambda$. Let $y_\gamma \equiv y_\alpha(J_\alpha)$ be a pairwise solvable system such that $y_\alpha \notin \phi(R)$ for any $\alpha \in \Lambda$. Since $y = \phi(a)(\phi(b))^{-1} \in \phi(a)R \subseteq \phi(R)$ for all $y \in R_P$, we conclude that $y_\alpha = x_\alpha^{-1} = (\phi(x_\alpha))^{-1}$ for some $x_\alpha \in R \backslash P$. Then $y_\alpha^{-1} = \phi(x_\alpha)$. Since R is a right chain and left duo ring, we may assume that $x_\gamma = x_\alpha s = t x_\alpha$ for some $s, t \in R$. Thus, $x_\gamma^{-1} - x_\alpha^{-1} = (1-t)(tx_\alpha)^{-1} = y_\gamma - y_\alpha \in J_\alpha$. Therefore $(1-t)(tx_\alpha)^{-1}(tx_\alpha)x_\alpha = x_\alpha - x_\gamma \in J_\alpha$. This means that $y_\gamma^{-1} - y_\alpha^{-1} \in J_\alpha$. Hence, $y_\gamma^{-1} \equiv y_\alpha^{-1}(J_\alpha)$ is a pairwise solvable system with $y_\alpha^{-1} \in \phi(R)$ for all $\alpha \in \Lambda$. By the first part of the proof, this system has a simultaneous solution of the form $\phi(x_0)$ with $x_0 \in R \setminus P$. Thus, $(\phi(x_0))^{-1}$ is a common solution for the system $y_\gamma \equiv y_\alpha(J_\alpha)$, and therefore R_P is a right maximal (almost maximal) ring. $\square$

6.2 Right FGC-Rings

Classic examples of FGC-rings are principal ideal domains. The characterization of FGC-rings is a problem that has been of interest for many years. For 20 years, the only known FGC-domains were the principal ideal domains and the almost maximal valuations domains. In this section we establish some results for these rings for the non-commutative case. For this, consider the next theorem due to Kaplansky, Matlis, Gill, Lafond, and Warfield ([12, p.134]). Recall that a *uniserial module* is a module whose submodules are totally ordered by inclusion.

Theorem 6.2.1. [14, Main Theorem] *Let R be a local ring with maximal ideal $J(R)$. Then following are equivalent:*

a) The unique simple module $R/J(R)$ has uniserial injective hull $E(R/J(R))$.

b) R is an almost maximal valuation ring.

c) Every idecomposable injective R-module is uniserial.

d) Every finitely generated R-module is a direct sum of cyclic modules R-modules.

For the non-commutative case, Behboodi recently obtained a similar characterization for local duo rings [3]:

Theorem 6.2.2. [3, Theorem 3.5] *Let R be a local duo ring. Then following statements are equivalent:*

a) R is a left FGC-ring.

b) Every 2-generated left R-module is a direct sum of cyclic modules.

c) Every factor module of the free left R-module $R \oplus R$ is a direct sum of cyclic modules.

d) Every factor module of the free left R-module $R \oplus R$ is serial.

e) R is uniserial and for every non-zero ideal I of R, R/I is a linearly compact left R-module.

f) R is uniserial and every idecomposable injective left R-module is left uniserial.

g) R is a right FGC-ring.

Our next theorem considers the latter characterization in the framework of right chain and duo rings. The proof it is based on the techniques developed by Gill in [14], and on an application of the previous theorem. First, a technical result,

Lemma 6.2.3. [2, Nakayama's Lemma] *For a left ideal I of a ring, the following are equivalent:*

a) $I \leq J(R)$.

b) For every finitely generated left R-module M, if $IM = M$, then $M = 0$.

c) For every finitely generated left R-module M, IM is superfluous in M.

Theorem 6.2.4. *The following conditions are equivalent for a right duo and right chain ring R:*

a) R is a right FGC ring.

b) R is a right almost maximal ring.

c) $E(R/I)$ is a uniserial right R-module for any right ideal I of R.

Proof. It remains to show $c) \implies a)$: Let M a finitely generated right $R-$module. We are going to induct on the number of generators of M. If $n = 1$, the result is obvious. Let $M = \sum_{i=1}^{n} x_i R$ with $\{x_i\}_{1=1}^{n} \subseteq M$, for some $n > 1$. Assume that the result is true for any module with less that n generators. Since R is a right duo and right chain ring, we may also assume that $ann_r(x_1) = ann_r(M)$. Let $I = ann_r(x_1)$. Since R is right duo, I is a two-sided ideal and then $x_1 R \cong R/I$. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & x_1R & \hookrightarrow & M \\ & & \downarrow^{\cong} & & \\ & & R/I & \swarrow_f & \\ & & \downarrow^{j} & & \\ & & E(R/I) & & \end{array}$$

Note that, since $E(R/I)$ is injective, there exists $M \xrightarrow{f} E(R/I)$ which makes the diagram commutative. Furthermore, M finitely generated implies that $f(M)$ is a finitely generated submodule of $E(R/I)$. By hypothesis $E(R/I)$ is uniserial, thus $f(M)$ is a cyclic R-module. Then there exists $y \in E(R/I)$ such that $f(M) = yR$. This implies that we can find $m \in M$ such that $y = f(m)$. We claim $mR \cong yR$. To prove this, suppose $r \in ann_r(m)$. Then $mr = 0$ yields $0 = f(mr) = f(m)r = yr$. This means $ann_r(m) \subseteq ann_r(y)$. On the other hand, for $\alpha : x_1R \xrightarrow{\cong} R/I \xrightarrow{j} E(R/I)$, we have $\alpha(x_1) = ys$ for some $s \in R$. For any $a \in R$ we obtain $\alpha(x_1 a) = y(sa)$. Since R is a right duo ring, $Ra \subseteq aR$, there is $t \in R$

such that $sa = at$. Then $\alpha(x_1a) = y(sa) = y(at) = (ya)t$. Hence, if $a \in ann_r(y)$, then $\alpha(x_1a) = 0$. Since α is a monomorphism $x_1a = 0$. Thus, $ann_r(y) \subseteq ann_r(x_1) = I$. Also, $I \subseteq ann_r(m)$, and $I = ann_r(m) = ann_r(y)$. Then, $R/I = R/ann_r(m) = R/ann_r(y)$, and therefore, $R/I \cong mR \cong yR$. Now, consider the diagram

$$\begin{array}{ccc} mR & \overset{i}{\to} M \overset{f}{\to} & yR \\ \downarrow^{\cong} & & \downarrow^{\cong} \\ R/I & \overset{\alpha}{\to} & R/I \end{array}$$

We have that $\alpha : x_1R \cong R/I \overset{j}{\to} R/I$ is an isomorphism. Thus, fi is also an isomorphism, from wich we get that $mR \overset{i}{\to} M$ is splits, say $M = mR \oplus (M/mR)$. Finally, our claim is that $U = M/mR$ is a finitely generated R-module with less than n generators. To show this, let $J = J(R)$. By the Nakayama's Lemma, $mR/(mR)J$ is a non-zero cyclic (R/J)-module and therefore $mR \neq 0$. Thus, U is finitely generated module with $n-1$, generators. By the induction hypothesis, U is direct sum of cyclic R-modules, and the same holds for M.

Observe that by Lemma 2.0.1 a right chain rings are local rings. According to Brandal [6], *b)* is equivalent to *e)* in Theorem 6.2.2. Thus, for the remaining implications, we can apply this theorem since $E(R/I)$ is an indecomposable injective R-module whenever R is a right chain ring. □

Is important to note the right-left symmetry in $c) \implies a)$. This means that if R is a right maximal chain duo ring, then R is a two-sided FGC-ring.

For a ring that is not necessarily chain-duo we also have:

Proposition 6.2.5. *Let R be an FGC-ring and P a prime ideal of R such that $R \setminus P$ does not contains zero divisors. Then the localization R_P is also an FGC-ring.*

Proof. Let M be a finitely generated right R_P-module, i.e. $M = y_1R_P + y_2R_P + ... + y_nR_P$. Suppose ϕ is the localization map defined by $r \mapsto r/1$. It is clear that M and R_P are both right R-modules with the scalar multiplication induced by ϕ. Then the R-submodule

$N = y_1R + y_2R + ... + y_nR$ of M is a direct sum of cyclic modules since R is an FGC-ring, say $N = x_1R \oplus x_2R \oplus ... \oplus x_kR$ for some $x_1, x_2, ..., x_k \in N$. Consider the canonical projection $M \xrightarrow{\pi} M/N$ which induces the epimorphism $M \otimes_R R_P \xrightarrow{\pi \otimes Id_{R_P}} M/N \otimes_R R_P$. If $z \in M/N \otimes_R R_P$, then $z = (\sum_{i=1}^n y_i r_i t_i^{-1} + N) \otimes_R rt^{-1}$ with rt^{-1} and $r_i t_i^{-1}$ elements in R_P for all $i = 1, 2, ..., n$. Observe that there exists $0 \neq d \in R \setminus P$ and $s_1, s_2, ..., s_n \in R$ such that $\sum_{i=1}^n y_i r_i t_i^{-1} = \sum_{i=1}^n y_i s_i d^{-1}$. Thus, $z = (\sum_{i=1}^n y_i s_i d^{-1} + N) \otimes_R dd^{-1} rt^{-1} = \bar{0} \otimes_R d^{-1} rt^{-1} = 0$, which implies $0 = (\pi \otimes Id_{R_P})(m \otimes 1_{R_P}) = \pi(m) \otimes 1_{R_P}$ for any $m \in M$. Therefore $\pi(M) = \bar{0}$, and then $M = N \otimes_R R_P = \oplus_{i=1}^k x_i R_P$. □

The next result partially addresses a question by Fuchs and Salce ([13, Problem 12]): Which are the domains R, over which the finitely generated (finitely presented) modules cancel in direct sums?

Corollary 6.2.6. *Let R be a local right duo ring. If R is a right almost maximal right chain ring (a right FGC-ring), then finitely generated modules cancel in direct sums.*

Proof. If M is a finitely generated right R-module, then M is a finite direct sum of cyclic modules. By Corollary 4.0.22. M cancels in direct sums. □

For the next result, recall that a module M is called *distributive module* if for any submodules K, L, N,

$$K \cap (L + N) = K \cap L + K \cap N.$$

Observe that the inclusion $K \cap L + K \cap N \subseteq K \cap (L + N)$ always holds.

For the next theorem, we will need the following lemma:

Lemma 6.2.7. [20, Proposition 1.3] *Let M be an R-module. Then M is a distributive module if and only if all submodules of M with two generators are distributive modules.*

Finally, the next theorem collects many of our findings about chain and duo rings and links all the main concepts in this work through the FGC-rings:

Theorem 6.2.8. *Let R be a chain and duo ring, and P a prime ideal of R such that $R \setminus P$ does not contains zero divisors. The following hold in case R is an FGC-ring:*

a) *Any finitely generated module cancels in direct sums.*

b) *R is an almost maximal ring. If R is not a domain, then R is a maximal ring.*

c) *If ${}_RR$ or R_R is co-Hopfian and if R is not a domain, then R is self-injective.*

d) *Every finitely generated R-module M is Hopfian. If M is also injective, then M is both Hopfian and co-Hopfian.*

e) *All matrix rings $M_n(R)$ are repetitive.*

f) *The localization R_P is an almost maximal chain ring.* [1]

g) *If $E = E(R/J(R))$ is Noetherian, then E has a finite number of submodules.*

h) *All finitely generated injective R-modules satisfy Fitting's Lemma.* [2]

i) *An R-module M is distributive if only if all cyclic submodules of M are distributive.*

Proof. a) is shown in Corollary 6.2.6. For b), Theorem 6.2.4 shows that R is an almost maximal ring. Then, by the hypothesis and Proposition 5.3.2, R is a maximal ring. By Theorem 5.3.4, c) holds. In d), Corollary 4.0.22 implies that M is Hopfian. Now, let $f : M \to M$ be an injective R-homomorphism. The fact that M is injective implies that there exists $g : M \to M$ such that $gf = Id_M$. Thus, g is an epimorphism, and since M is Hopfian, g is an automorphism. Therefore f is an isomorphism too. Note that e) is a direct consequence of d) and Goodearl's Theorem (Theorem 4.0.5). For f), Lemma 6.1.1, Theorem

[1] Although we know that R_P is an FGC-ring (Proposition 6.2.5), we cannot use any the characterizations for FGC-rings since R_P is not necessarily a duo ring, unless R is a commutative or a Noetherian ring [8].

[2] Fitting's Lemma: $M \in \mathcal{M}_R$ of finite length n, $f \in End_R(M)$,then $M = Imf^n \bigoplus Kerf^n$.

6.1.3, and b) imply that R_P is a an almost maximal chain ring. To show g), observe that Theorem 4 in [25] establishes that with the conditions in g), E is right Noetherian if only if E has finite length. Thus, by Theorem 6.2.4 c), E is uniserial, and the result follows. Observe that by Theorem 8 in [15], h) and the first part of d) are equivalent. Finally, to prove i), let M be an R-module such that any cyclic module is distributive. Let S be a submodule of M with two generators and K, L, N submodules of S. We want to show that $K \cap (L + N) \subseteq K \cap L + K \cap N$. Note that $S = s_1R \oplus s_2R$ for some $s_1, s_2 \in M$ since R is an FGC-ring. Suppose $K \subseteq s_iR$, $L \subseteq s_jR$ and $N \subseteq s_qR$ with $i, j, q = 1$ or 2. Observe that the result is clear if $i = 1$ and $j = q = 2$, or $i = j = q$. Assume $i = j = 1$ and $q = 2$. Let $x \in K \cap (L + N)$. Then $x = l + n \in K$, and therefore, $x - l \in s_1R \cap N = 0$. But, this means $x \in K \cap L = K \cap L + K \cap N$ since $K \cap N = 0$. Therefore S is a distributive module, and by Lemma 6.2.7 the result follows. □

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