On The Number of Cylinders Touching a Sphere

by

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Abstract

The kissing number problem is a packing problem in geometry where one has to find the maximum number of congruent non-overlapping copies of a given body so that they can be arranged each touching a common copy.

The most studied version of this problem is about the kissing number of the unit ball.

A similar question was proposed by Wlodzimierz Kuperberg in 1990. Kuperberg asked for the maximum number of non-overlapping infinitely long unit cylinders touching a unit ball. He conjectured that at most six disjoint infinitely long unit cylinders can touch a unit sphere. W. Kuperberg's so called *six cylinder problem* [WK90] is a well known, 28 year old problem in discrete geometry and it is still an open problem.

In 2015, Moritz Firsching showed an arrangement of 6 disjoint cylinders with radii 1.0496594, where each cylinder touched a given unit ball.

In this dissertation several variants of W. Kuperberg's problem are considered and solved. For example new bounds will be proved concerning the number of tangent cylinders with various radii. Some already known bounds will be improved by elaborating on the method introduced by Brass and Wenk [BW00]. Application of a deep theorem on circle packing by Musin [OM03] also provides some non-trivial bounds. The major part of the dissertation is about proving theorems concerning the size and the number of discs which one can place on a concentric sphere avoiding the cylinders. This way new lower bounds are proved for the total area between cylinders on a concentric sphere. Such lower bounds can improve the existing results concerning Kuperberg's type cylinder problems. Most of the lemmas will be proved with pure geometric arguments, but in some cases the final answer uses Maple computations. We give several different lower bounds for the total area of gaps. Even our best lower bound does not solve Kuperberg's 6 cylinder problem. The last section

contains an application of our lower bound (joint work with Andras Bezdek) where it is proved that seven infinitely long cylinders of radii 1.04965 (Firsching's radius) cannot touch a unit sphere. In view of Firsching's construction this settles the Kuperberg question for radius 1.04965.

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Table of Contents

List of Figures

List of Tables

On The Number of Cylinders Touching a Sphere

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Chapter 1

Introduction - Kuperberg's Conjecture

The kissing number problem is a packing problem in geometry asking for the maximum number of congruent non-overlapping bodies touching a congruent central body. The most studied version of this problem is finding maximum number of non-overlapping unit balls tangent to a central unit ball. In two dimensional space the answer is obvious: the maximum is 6. However in three dimensional space it took a long time to prove that the answer is 12, because there was a promising arrangement of 13 spheres to contact a unit sphere and it was impossible to rule out the existence of such arrangement by naked eye. The disagreement between Isaac Newton and David Gregory [GS03] is an interesting historical detail: At first Newton gave an incomplete proof for 12 to be the maximum, while Gregory believed that 13 spheres could fit. K. Schütte and B. Waerden [SW53] proved that 13 unit spheres cannot contact a unit sphere. Later a simpler proof was given by J. Leech [LS71]. In 4D, the analogous question was answered by O. Musin [OM03] who proved that at most 24 spheres can touch a unit sphere.

A similar question was proposed by Wlodzimierz Kuperberg in 1990. Kuperberg asked for the maximum number of non-overlapping infinitely long unit cylinders touching a unit ball.

Conjecture 1. [W. Kuperberg (1990)] *At most six disjoint infinitely long unit cylinders can touch a unit ball.*

Several simple arrangements were given by W. Kuperberg (Figure 1.1), where six cylinders touched a sphere. The question of whether seven infinitely long cylinders could contact one sphere, remained open and was recognised to be very difficult.

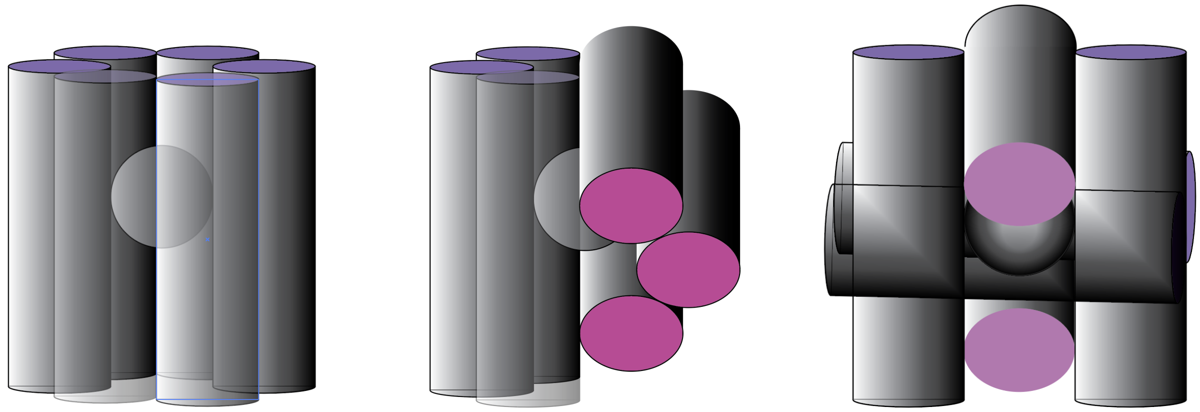


Figure 1.1: Six disjoint, unit radius cylinders touching a unit sphere

In 1991, Aladár Heppes [HSz91], using the area of the "shadows" - orthogonal projections on the surface of the unit sphere, showed that nine is not possible. In the same paper, László Szabó by using Blichfeldt's Density Lemma provided the same result. Aladár Heppes also used the "shadow" area method to provide specific radii for cylinders where 6, 7, and 8 such cylinders could not contact a unit ball.

Theorem 1.1. [A. Heppes and L. Szabó (1991), [HSz91]] *Nine disjoint infinitely long unit cylinders cannot touch a unit ball.*

Theorem 1.2. [A. Heppes and L. Szabó (1991), [HSz91]] *Eight disjoint infinitely long cylinders of radius* $r > 0.96$ *cannot touch a unit ball; similarly seven disjoint infinitely long cylinders of radius* $r > 1.075$ *cannot touch a unit ball; and six disjoint infinitely long cylinders of radius* $r > 1.275$ *cannot touch a unit ball.*

Later on Peter Brass and Carola Wenk [BW00] used a very elegant and short spherical surface area argument to prove that 8 unit cylinders cannot contact a unit ball. The method is as following; consider a sphere of radius $\sqrt{4.7}$ which is concentric with a unit ball. They computed the surface area of the concentric sphere which is enclosed in a unit infinitely long cylinder touching the unit ball . They finished their proof by noticing that the total area of 8 such disjoint congruent patches exceeds the total surface area of the sphere.

Theorem 1.3. **[P. Brass and C. Wenk (2000), [BW00]]** *Eight disjoint infinitely long unit cylinders cannot touch a unit ball.*

In 2015, Moritz Firsching showed an arrangement of six disjoint cylinders with radius 1.0496594, where each cylinder touched a given unit ball [MF15]. Firsching using Brass and Wenk's method also gave some upper bounds in terms of n for the radii r, for which n cylinders of radius r cannot touch a unit ball.

Theorem 1.4. **[M. Firsching (2015), [MF15]]** *Eleven disjoint infinitely long cylinders with radius* $r > 0.592$ *cannot touch a unit ball; ten disjoint infinitely long cylinders with radius* $r > 0.663$ *cannot touch a unit ball; nine disjoint infinitely long cylinders with radius* $r > 0.756$ *cannot touch a unit ball; eight disjoint infinitely long cylinders with radius* $r > 0.884$ *cannot touch a unit ball; seven disjoint infinitely long cylinders with radius* $r > 1.069$ *cannot touch a unit ball; six disjoint infinitely long cylinders with radius* $r > 1.362$ *cannot touch a unit ball; five disjoint infinitely long cylinders with radius* $r > 1.893$ *cannot touch a unit ball; four disjoint infinitely long cylinders with radius* $r > 3.119$ *cannot touch a unit ball; three disjoint infinitely long cylinders with radius* $r > 8.123$ *cannot touch a unit ball.*

Theorem 1.5. **[O. Ogievetsky and S. Shlosman (2018), [OS18]]** *[OS18] 6 infinitely long cylinders with radius* $r = \dfrac{1}{8}(3+\sqrt{33}) \approx 1.093070331$ *can touch a unit ball.*

The given radius r in Theorem 1.5 is conjectured to be the largest possible. However 7 disjoint infinitely long cylinders with radius $r = 1.093070331$ cannot touch a unit ball as seen from Theorem 3.2 and Theorem 1.4.

In this dissertation, we study several variants of W. Kuperberg's problem. For example, we wanted to know how many cylinders with a specific radius can touch simultaneously two unit balls. For specific numbers of cylinders, we used new methods to find upper bounds for their radii. Some of the upper bounds improved previous ones, but some of them turned out to be weaker. Since the weaker bounds were obtained with a simpler new method, we will present them also. As an example of new method, we will show that seven infinitely long cylinders of radii 1.04965 (Firsching's radius) cannot touch a unit sphere. In view of Firsching's construction, this settles the Kuperberg type question for cylinders of radius 1.04965.

Chapter 2

Informal Summary of New Results of This Dissertation

W. Kuperberg's so called *six cylinder problem* [WK90] is a well known, 28 years old problem in discrete geometry. For readers who are already familiar with the problem, with the terminology and with partial results, this section gives a glimpse of what will be proved in this dissertation. Other readers might want to return to this section after reading the formal introduction and the first few sections of the thesis.

The following is a list of our new statements, estimates and solutions of various Kuperberg type questions. For the sake of brevity, we avoid many details, definitions; our purpose is to give an intuitive description only.

Remark 1: Heppes was very close to discovering Brass and Wenk's elegant proof. This remark will explain why.

Remark 3.3: Musin's result on the densest packing of 14 discs almost solves the 6 cylinder problem.

Remark 3: describes our approach to improve Brass and Wenk's method.

Theorem 4.1: minimizes the sum of distances from a point to three non-overlapping cylinders.

Theorem 3.1: proves that the gaps around spherical patches contain two specific discs.

Remark 5: For small radii, parallel cylinders do not maximize the number of contacting cylinders.

Theorem 3.2: An area-formula is redone, since Brass and Wenk's paper omitted its details.

Theorem 3.3: For $n = 3, 4, \ldots, 11$, we give upper bounds for the radii of n contacting cylinders.

Theorem 3.7: How many cylinders can simultaneously touch two unit spheres? First proof.

Theorem 3.4: How many cylinders can touch a sphere and avoid a given tangent sphere?

Corollary 3.5: At most six unit cylinders can touch two tangent unit balls.

Theorem 3.6: At most six unit cylinders can be simultaneously tangent to two unit spheres. Second proof.

Lemmas and theorems of section 4: A sequence of lemmas are proved to provide different lower bounds for the total area of gaps among the contacting cylinders on a concentric sphere. Computing area on a sphere can be very difficult. The main idea is to place small discs in the gaps, assign their area to cylinders and estimate how much area each cylinder gets. In this way we avoid area computations; what we do is more like combinatorial geometric study.

Theorem 5.1(Joint work with Andras Bezdek): At most sxi disjoint infinitely long cylinders of radii $1.0496594\ldots$ can touch a unit ball such cylinders will be called Firsching cylinders.

Chapter 3

Congruent Cylinders Touching a Unit Ball

In this section we start listing our results in the form of theorems and remarks. If a theorem or remark is not followed by a reference of origin, then it is our work. At the beginning, the theorems and remarks have short proofs so we prove them right after stating them.

The best known method to obtain upper bounds for the number of disjoint infinitely long cylinders of radii $r \geq 1$ touching a unit sphere is due to Brass and Wenk. First they considered a larger sphere of radius $R > 1$, concentric with the unit sphere. They noticed that the tangent cylinders intersect the surface area of the larger sphere in disjoint congruent spherical patches. Thus, if k cylinders touch the unit sphere then k times the area of a single patch must not exceed $4R^2\pi$, the surface area the larger sphere. It turned out that with a good choice of R, if $r = 1$ the inequality $k < 8$ follows. Kuperberg's problem is intriguing, because proving the stronger inequality $k < 7$ seems to be a very difficult step to make.

Heppes [HSz91] assigned for each cylinder a 'cone' by connecting the center of the unit sphere to each point of the cylinder which is not further than a $\sqrt{3}$ unit distance. He used an elementary argument (depicted on Figure 3.1) to show that the cones assigned to different cylinders are disjoint (Figure 3.2). Then Heppes computed the angular measure of a cone at its vertex and verified that 9 times this angular measure is greater than the total surface area of the sphere. Thus, he proved that 9 cylinders cannot touch the sphere.

3.1 Prior Results of Heppes, Szabo, Brass, Wenk, Musin and Firsching

Remark 1. *Heppes failed to notice that one can get the same cones by intersecting the cylinders with a concentric sphere of radius* $\sqrt{3}$ *and thus they are automatically disjoint. In*

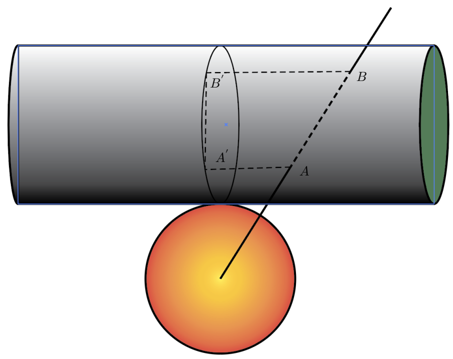


Figure 3.1: Heppes's proof

our view, Heppes was very close to discovering Brass and Wenk's solution by changing the radius of $\sqrt{3}$ *to a more suitable one* ($\sqrt{4.7}$).

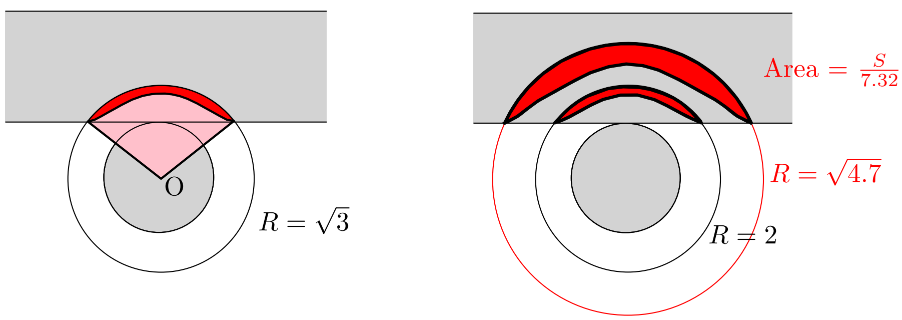


Figure 3.2: Heppes's proof compared to Brass' and Wenk's proof

Remark 2. *We noticed a connection between Kuperberg's problem and the celebrated result of Oleg Musin, who proved that if 14 congruent discs are packed on a sphere then the radius of the discs is at most a* $27.56°$ *central angle. Let us intersect a tangent unit cylinder with a concentric sphere of a specific radius (say of radius 2) (Figure 3.3). Then inscribe in the trace of each cylinder two circles in symmetrical position. If the radii of these circles has larger than* $27.56°$ *central angle, then Musin's result disproves the existence of an arrangement of 7 disjoint touching cylinders. Unfortunately, this was not the case, so all we can conclude with some elementary calculations is that 7 cylinders of radii* 1.119 *cannot touch a unit ball.*

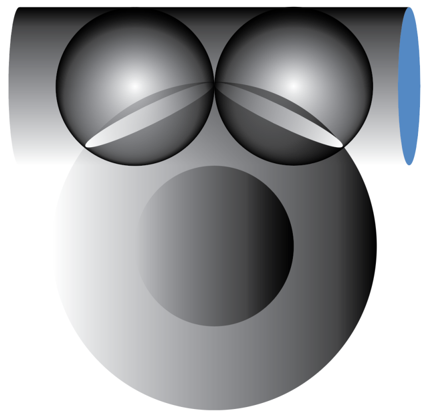


Figure 3.3: Application of Musin's theorem

Remark 3. *Brass and Wenk's argument ignores the fact that the patches do not tile the sphere, while it is obvious that there must be gaps between the patches. Our goal is to give a lower bound, say G, for the total area of the gaps. Then, in the spirit of Brass and Wenk's argument, we hoped for better upper bounds for the number of cylinders: If k cylinders touch the unit sphere then k times the area of a single patch must not exceed $4R^2\pi - G$, the surface area of the larger sphere minus the guaranteed gap area. Right at the beginning, we had to make a decision on how to approach computing the area of the gaps.*

Our main strategy is based on the following observation: In sections 3 and 4 of this thesis, we will give lower a bound for the total area of the gaps by presenting various lemmas concerning the number of disjoint discs of radii ρ which can be placed around the patches.

Definition 1. *We will call a cylinder a* **Firsching Cylinder***, if it is tangent to a unit ball and has radius* 1.0496594*. We will refer to the common part of a Firsching cylinder and the surface of a sphere concentric with the given unit sphere, as* **Firsching patch***. Consider a cylinder a radius greater than 1.0496594 and coaxial with the Firsching cylinder. Similarly to Firsching patches we can define patches of such cylinders as well. This larger patch contains the Firsching patch. We define* **Firsching patch's complement** *as the set theoretical difference of the two patches. A spherical cap on the surface of a concentric sphere whose*

Euclidean radius is $k = 3(1.0496594)\left(\frac{2}{\sqrt{3}} - 1\right) = 0.1623828741$ *will be called a* **Firsching cap**. *The area of a Firsching cap will be denoted by* P.

3.2 Archimedes' Hat-Box Theorem

Remark 4. [Archimedes' work], [HS98] *Archimedes knew that the formula of the surface area of a spherical cap is the same as that of a planar disc, i.e. it is* πr^2*, where* r *is the Euclidean radius of the cap.*

This formula easily follows from the more general Hat-Box theorem of Archimedes: by slicing twice perpendicularly to the cylinder's axis one cuts out a portion of the surface area from both the sphere and the cylinder; these two pieces have the same surface area (Figure 3.4).

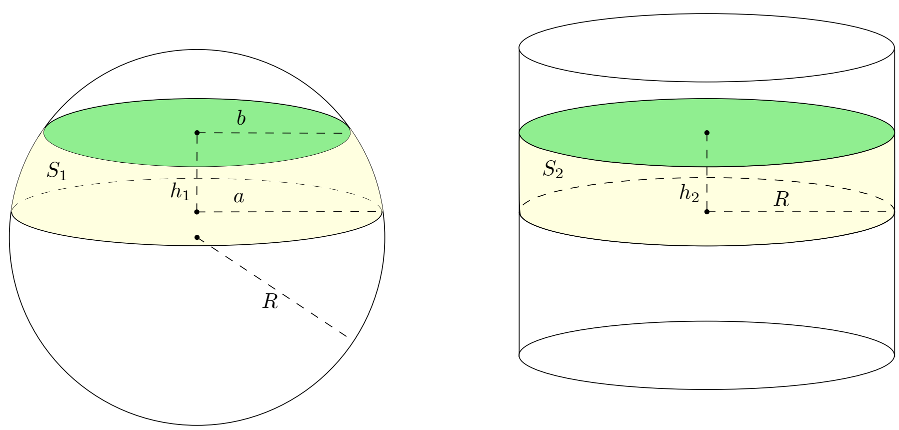


Figure 3.4: If $h_1 = h_2$ then $S_1 = S_2$

We also include a variation of the same result of Archimedes using the Figure 3.5:
Let's consider the sphere S_1 centered at the origin and the sphere S_2 centered at the point $(0, 0, R)$ where R is the radius of S_1. By a simple calculation we can find the coordinates of center of the circle common to both spheres. $x^2 + y^2 + z^2 = R^2$ is the equation of S_1 and $x^2 + y^2 + (z - R)^2 = k^2$ is the equation of S_2.

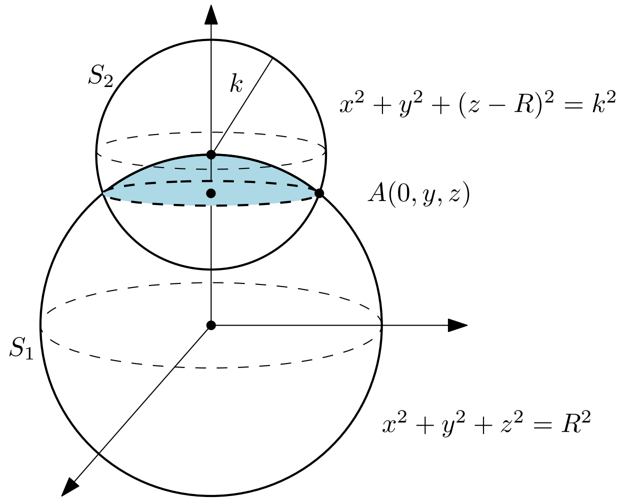


Figure 3.5: Areas of spherical caps

Then the intersection of the equations gives us

$$z = \frac{2R^2 - k^2}{2R}$$

Since the area of the cap is equal to $A = 2\pi R(R - z)$ we have that

$$A = 2\pi R \left(R - \frac{2R^2 - k^2}{2R} \right) = \pi k^2$$

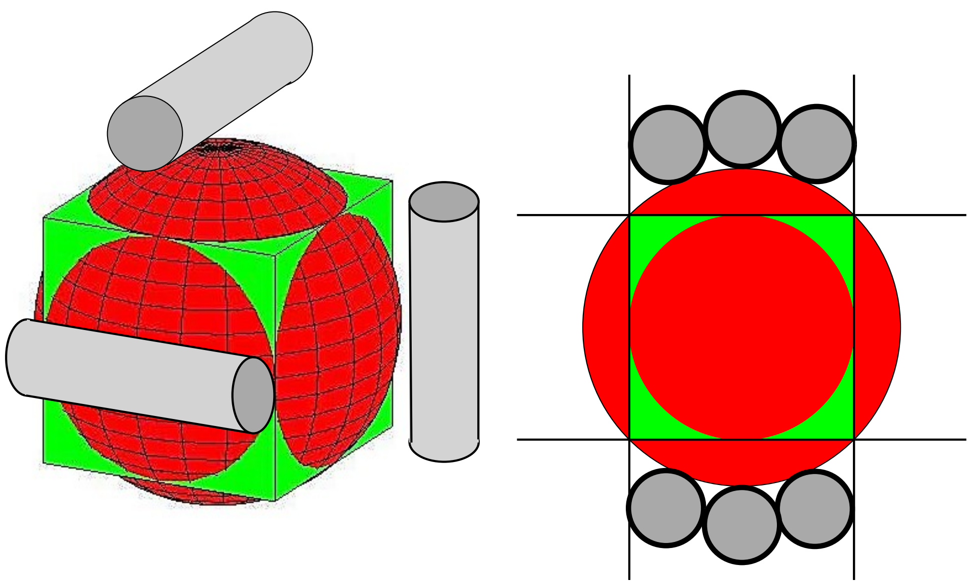
$\square$

3.3 Touching a Unit Sphere with Congruent Cylinders of Radii r

Kuperberg's question, from another view, is the following. He arranged 6 parallel cylinders around a unit sphere, so that they formed an annulus and each of them touched the unit sphere. Then, Kuperberg raised the question if the cylinders could be arranged to have space for one more tangent cylinder. We then ask the following,

Question 1. *For a given $n \geq 3$, let r_n be the radius so that n parallel cylinders, each tangent to a unit sphere, form an annulus. For which n can the cylinders be rearranged so that with an additional tangent cylinder, the $n+1$ cylinders can touch the unit sphere?*

Remark 5. *Certainly the answer for the previous question is negative for $n = 3$. Brass and Wenk's area argument gives negative answer for n=5. The question is open for $n = 6, 7$. Figure 3.6 shows that the answer is positive for $n = 17$.*



a) The axes-parallel cube helps to visualize the position of the 3×6 axes-parallel cylinders.

b) Front view of the unit sphere and the six cylinders perpendicular to the dirrection of view.

Figure 3.6: An arrangement of $3 \times 6 = 18$ cylinders of radii $> r_{17}$

The case of $n = 17$: The axes parallel cube whose edges are tangent to the unit sphere is used to describe the arrangement of 18 tangent cylinders. In style, the arrangement is similar to

Kuperberg's 3rd construction of 6 tangent cylinders. Here 6 cylinders are parallel to each of the three axis. Figure 3.6b is the front view (say view from the direction of the x-axis), thus 6 of the tangent cylinders are depicted with 6 congruent circles. Since three of the circles span the space between opposite faces, we have that diameters of these three circles add up to more than the edge length ($\sqrt{2}$) of the cube, thus $r > \frac{\sqrt{2}}{6}$. If a circle of radius $\frac{\sqrt{2}}{6}$ is tangent to a unit circle, then from the center it spans an angle $\alpha(r) = \arcsin \frac{\frac{\sqrt{2}}{6}}{1+\frac{\sqrt{2}}{6}} = 10.99°$. Since, $17 \times 10.99° = 186°$ we have that putting cylinders of radii r_{17} in parallel position is not maximizing the number of contacting cylinders of radii r_{17}.

$\square$

3.4 Bezdek and Kuperberg's Lemma on a Sphere Touching Three Cylinders

Remark 6. **[Bezdek and Kuperberg (1991)]** *Bezdek and Kuperberg, while studying the densest packing of cylinders, used a lemma stating that the smallest sphere that can touch three disjoint infinitely long cylinders of radius* r *has a radius* $k = r\left(\frac{2\sqrt{3}-3}{3}\right)$. *Moreover, the cylinders that touch a sphere of radius* k *must be mutually parallel.*

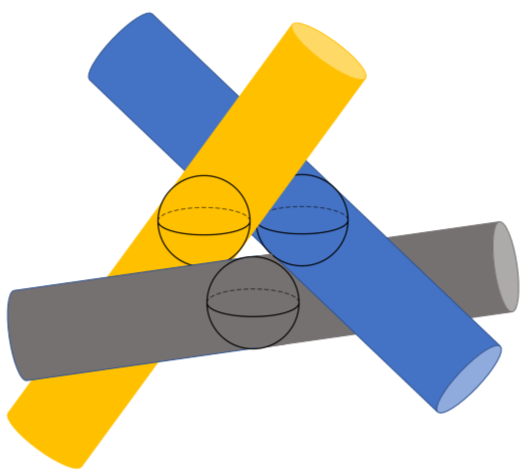
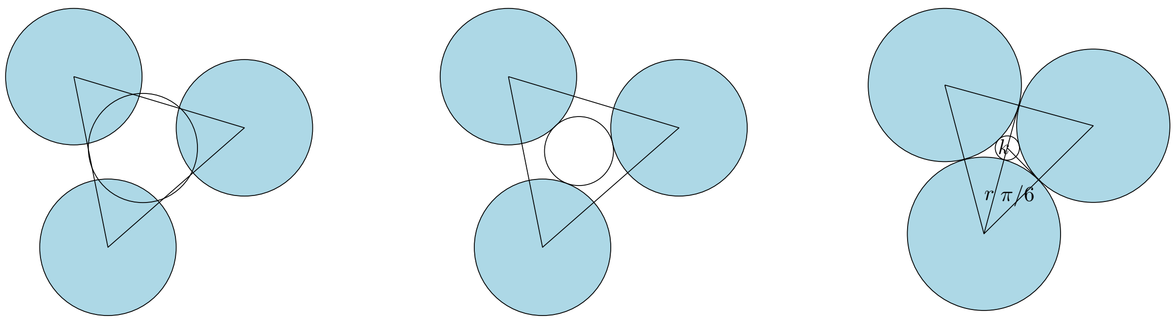


Figure 3.7: Three cylinders touching a unit ball

Proof of Remark 6: Let us start with three infinitely long cylinders of radii r, touching a unit ball. Inside each cylinder, there is a sphere of radius r touching the unit ball (Figure 3.7).

Let's consider the plane P that contains the centers of these three spheres. Then, consider the projection of these three spheres and the ball to a plane parallel to the plane P. Figure 3.8a shows the projected circles. Continuously shrink the small circle until it touches all three of the circles of radii r. (Figure 3.8b).



a) General projection b) Smaller circle c) Smallest circle

Figure 3.8: Proof of the lemma of Bezdek and Kuperberg

It is easy to see that the circle in the center is smallest when the circles of radius r mutually touch each other (Figure 3.8c).

Let k denote the radius of the central circle. It turns out that

$$\cos\left(\frac{\pi}{6}\right)(r+k) = r \quad \Rightarrow \quad k = r\left(\frac{2\sqrt{3}-3}{3}\right)$$

Finally we need to point out that a small circle of radius k can indeed touch three cylinders of radius r

$\square$

3.5 Placing Two Discs in the Gaps Around Cylinders on the Surface of a Concentric Sphere

Theorem 3.1. *Consider a packing of infinitely long cylinders of radius $r \leq 1$, touching a unit ball. Let again $k = r\left(\frac{2\sqrt{3}-3}{3}\right)$. For each $R \geq 1+k$, there exist two disjoint spheres of radius of k outside the cylinders, whose centers are on the sphere of radius R concentric with the given unit ball. Moreover, if we restrict the number of cylinders to 3, then the same holds without requiring $r \leq 1$.*

Proof of 3.1: First consider the case when $r \leq 1$. Let us consider a packing of cylinders C_i $(i = 1, 2, \dots)$ of radii r, which are touching a unit ball and let us consider the cylinders C_i^+ $(i = 1, 2, \dots)$, which are coaxial with cylinders C_i and have radii $r + k$. Then, because of Theorem 3.1, no three of the cylinders C_i^+ have common points, therefore the double covered regions $C_i^+ \cap C_j^+$ are mutually disjoint.

For the rest of the proof, assume that *i*) the orientation is such that the center of the unit ball is at the origin O and *ii*) C_1 is horizontal, i.e. the axis of C_1 lies in the yz coordinate plane and is parallel to the y coordinate axis (Figure 3.9).

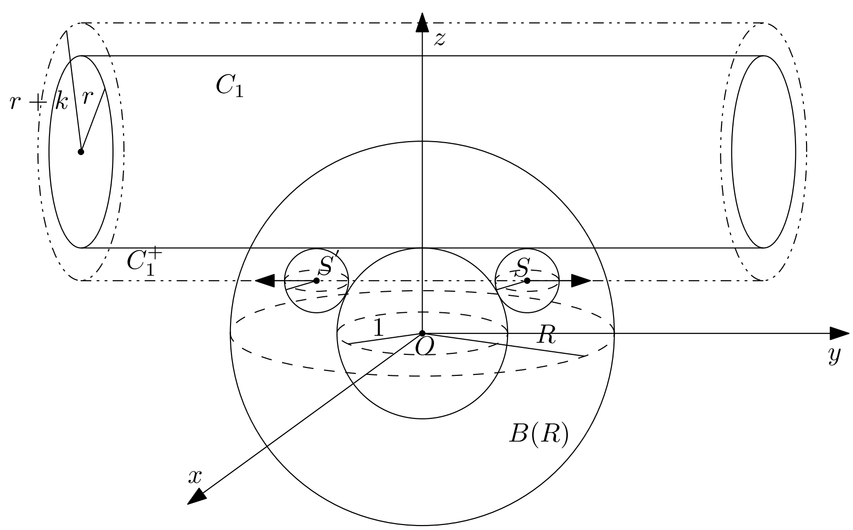


Figure 3.9: Two spheres of radius k touch the unit ball and the cylinder.

Consider two spheres S and S' of radius k and centers in the yz plane so that they touch both the unit ball and cylinder the C_1 (Figure 3.9). The spheres S and S' are disjoint and assume S is to the right of S'. The center of sphere S is inside the sphere $B(R)$ of radius R and with center O and it is on the surface of the cylinder C_1^+, in fact it lies on the line $z = 1 - k$. Start moving the center of S to the right on the line $z = 1 - k$. When the center hits another cylinder C_i^+, then it can be moved around this cylinder so that the center of S stays on the surface of the cylinder C_1^+ and returns to the line $z = 1 - k$. There are two ways to go around the cylinder C_i^+. We choose the one, where the y-coordinate of the center of S never gets smaller than before the collision. That means the center of S moves continuously on a path from inside the sphere S_R to outside the sphere S_R. Thus, at some point the center must be on the sphere $B(R)$ and must have a y-coordinate greater than that of the center of S. Now we move S' similarly but to the left to obtain a second sphere whose center is on the sphere $B(R)$ and has a y-coordinate smaller than that of the center of S'. The two newly selected spheres are disjoint, which proves what we wanted.

Next consider the case when exactly three congruent cylinders of radius r touch the unit ball. The above argument takes care of the case when $r \le 1$. In view of Theorem 1, we know that $1 < r \le 6.4641...$ We also have $k < 1$. We have three contact points on the unit sphere, hence there is a hemisphere containing the three contact points. Let the great circle C be the boundary of this hemisphere. Let H and $H^{'}$ be the half-spaces determined by the plane of C. Specifically let $H^{'}$ be the half-space containing the three contact points. We will explicitly tell where the centers of spheres S and S' of radius k are. They are on the concentric sphere $B(R)$ of radius $R > 1 + k$. Let S be the sphere of radius k whose center is the most distant from O in H. Let S' be the sphere of radius k touching the three given cylinders and the unit ball. It is easy to see that S' lies in H' and is disjoint from S. $\square$

When W. Kuperberg stated his 6 cylinder problem, he also noticed that it is very natural to ask for each integer n, how large r can be so that n congruent cylinders of radii r can touch a unit ball. In the context of this problem, finding upper bound means, the following for each integer n, we want to give a radius R so that if $r > R$ then n congruent cylinders of radii r cannot touch a unit ball. The first upper bound is given by Brass and Wenk in their paper where they proved that 8 unit cylinders cannot touch a unit ball. In short, Brass and Wenk computed the area of the trace of a single touchinr on a concentric sphere of radius r. They gave a formula for the area omitting details of the computation. Since we will use the their formulas, for completeness, we recreate here all the details of this calculus type argument.

3.6 New Upper Bounds for the Radii of n Contacting Cylinders

Theorem 3.2. [P. Brass and C. Wenk (2000)] *Assume a cylinder of radius r is tangent to a unit sphere. The surface area of the trace of this cylinder on a concentric sphere of radius R is*

$$2\sqrt{R}\int_{-\sqrt{R-1}}^{\sqrt{R-1}} \arcsin\left(\frac{\sqrt{4(R-x^2)(r+1)^2-(x^2-2r-R-1)^2}}{2(r+1)\sqrt{R-x^2}}\right)dx.$$

Let's consider a unit ball centered at the origin and let $S_{\sqrt{R}}$ be a sphere with radius $\sqrt{R}$ and centered at the origin and let C_r be a infinitely long cylinder with radius r which is touching the unit ball and is parallel to y-axis. For a given R ($1 \leq R \leq (2r+1)^2$), we will compute the surface area of the intersection of the surface of the sphere $S_{\sqrt{R}}$ and that of the solid cylinder C_r. Such intersections will be called *spherical patches*, or *patches* in short.

The sphere $S_{\sqrt{R}}$ and the cylinder C_r have the following set of points:

$$\begin{aligned} S_{\sqrt{R}} \quad &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = R\} \\ C_r \quad &= \{(x,y,z) \in \mathbb{R}^3 \mid y^2+(z-(1+r))^2 \leq r^2\} \end{aligned}$$

The boundary of the region of intersection (boundary of a patch) contains the following set of points:

$$D_{(R,r)} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) \in S_{\sqrt{R}} \cap C_r \text{ where } z = \sqrt{R-x^2-y^2}\}$$

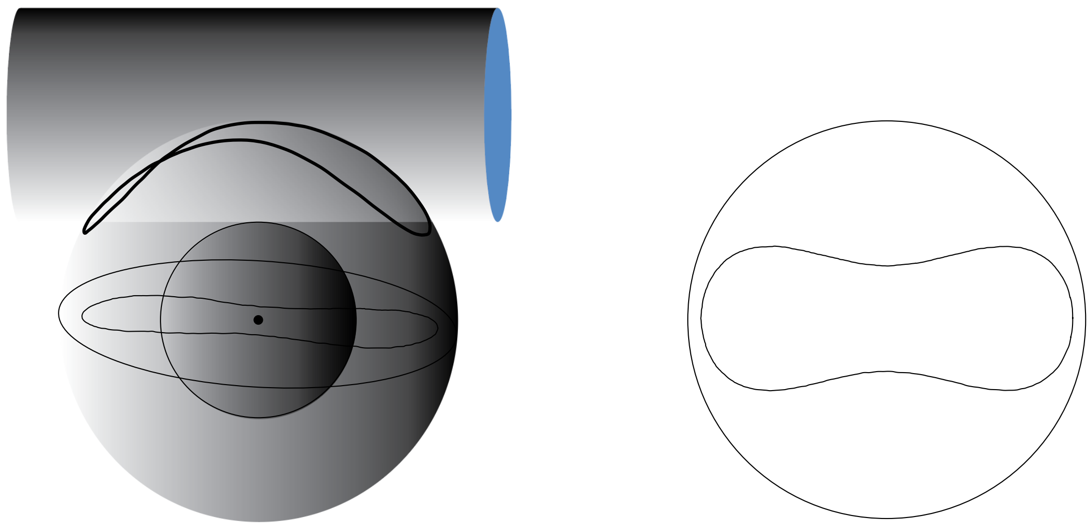


Figure 3.10: Firsching patches

Since $1 \leq R \leq (2r+1)^2$, the spherical patch is connected. Its projection $D_{(R,r)}$ on the xy-coordinate plane is not necessarily convex (Figure 3.10). Formally $D_{(R,r)}$ contains the following points:

$$\begin{aligned} D_{(R,r)} &= \{(x,y) \in \mathbb{R}^2 \mid y^2 + (\sqrt{R - x^2 - y^2} - (1+r))^2 \leq r^2\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \mid |x| \leq \sqrt{R-1} \text{ and } |y| \leq \frac{\sqrt{4(R-x^2)(r+1)^2 - (x^2 - 2r - R - 1)^2}}{2(r+1)}\right\} \end{aligned}$$

The enclosed spherical surface area of the patch is calculated by the following integral,

$$A(S_{\sqrt{R}} \cap C_r) = \int_{D^1_{(R,r)}} \frac{\sqrt{R}}{\sqrt{R - x^2 - y^2}} dydx \tag{3.1}$$

$$= \int_{-\sqrt{R-1}}^{\sqrt{R-1}} \int_{-\frac{\sqrt{4(R-x^2)(r+1)^2-(x^2-2r-R-1)^2}}{2(r+1)}}^{\frac{\sqrt{4(R-x^2)(r+1)^2-(x^2-2r-R-1)^2}}{2(r+1)}} \frac{\sqrt{R}}{\sqrt{R-x^2-y^2}} dydx \tag{3.2}$$

$$= 2\sqrt{R} \int_{-\sqrt{R-1}}^{\sqrt{R-1}} \arcsin\left(\frac{\sqrt{4(R-x^2)(r+1)^2 - (x^2-2r-R-1)^2}}{2(r+1)\sqrt{R-x^2}}\right) dx \tag{3.3}$$

$\square$

According to Theorem 3.1, for any arrangement of tangent cylinders with radius $r \leq 1$, there exist at least two disjoint spheres with radius $k = r\left(\frac{2\sqrt{3}-3}{3}\right)$ whose centers are on the surface of a concentric sphere with unit ball and has radius R.

Let's assume that the number of the contacting cylinders is n. Consider a concentric sphere of radius $\sqrt{R}$. (One could work with radius R, but we want to follow closely the computation of Bras and Wenk, and seems they found more convenient to work with radius $\sqrt{R}$). Then, consider the total area of their patches and the corresponding two spherical caps of radii k. Let us denote this total area by A. For each r there exists a concentric sphere of radius $\sqrt{R}$ such that the area A over the surface area of the sphere $\sqrt{R}$ is maximized. Let's call that maximum ratio by $S(r)$.

$$S(r) = \max_{1 \leq R \leq (2r+1)^2} \frac{A}{4\pi R}$$

We will derive upper bounds considering the patches on the concentric sphere of radius $\sqrt{R}$, where the maximum $S(r)$ is attained. The following table gives our new upper bounds. Note that in the first column n is the number of tangent cylinders. The second column contains the actual bounds. The third column contains the specific values of R which are telling us that the upper bound was obtained by computing the patch areas on the concentric sphere of radius $\sqrt{R}$. For example for $n = 9$, the maximum radii of 9 contacting cylinders cannot be larger than 0.75494110976.

Theorem 3.3.

n	upper bound for r (i.e. for any radius r greater then the numbers bellow, n cylinders cannot touch the unit ball)	R where the maximum S(r) attained
3	7.3319289	24.7713
4*	3.04984027	12.4084
5**	1.874707795	8.0509
6	1.354550711	6.0614771
7	1.06513747	4.9497859
8	0.881589897	4.245643
9	0.75494110976	3.7613642505
10	0.66227158213	3.4088017
11	0.591478922	3.14049555

Table 3.1: New upper bounds in case for various radii
Note: (*) and (**) are conjectured.

Remark 7.

n	Heppes	Firsching	Yardimci
3	NA	8.123015726697261129873583	7.3319289
4*	NA	3.119690083242860621653928	3.04984027
5**	NA	1.893940144132469649262296	1.874707795
6	1.275	1.362728791829127036542209	1.354550711
7	1.075	1.069484644843172117577150	1.06513747
8	0.960	0.8842320082596736518347155	0.881589897
9	NA	0.7566957511004313621344271	0.75494110976
10	NA	0.6635122014473178737120738	0.66227158213
11	NA	0.5923978489139096427663741	0.591478922

Table 3.2: Comparisons of upper bounds of radii

3.7 On the Number of Cylinders Touching Two Unit Balls

Next in order to test the power of Brass and Wenk's area argument, we came up with the following variant. This time, we want to touch with cylinders two disjoint unit spheres simultaneously (Figure 3.11). The questions are i) at most how many disjoint unit cylinders can do this, ii) what can we say about the maximum, if we change the radii of the cylinders? This section contains our results.

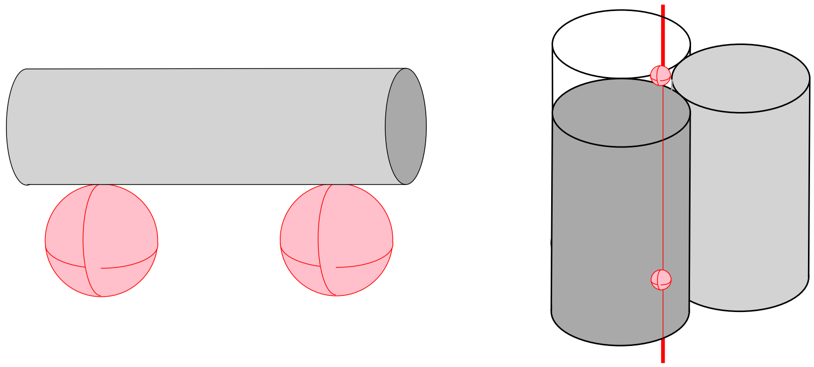


Figure 3.11: Maximizing the radii of n cylinders tangent to two spheres (Illustration for $n = 3$)

Theorem 3.4. *Consider a ball with radius $r > 0.9001799674$ which is tangent to a unit ball. In addition to this ball, at most six disjoint infinitely long unit cylinders can touch the unit ball.*

Proof of Theorem 3.4: Suppose S and S_1 are two spheres centered at the origin with radius $r_1 = 1$ and $R = \sqrt{4.698271645}$, respectively. Consider a unit cylinder C that touches S at the point $(0, 0, 1)$ and parallel to y-axis. We calculate the patch area of the sphere S_1 which is inside the cylinder C by Theorem 3.2 using Maple.

So the area is

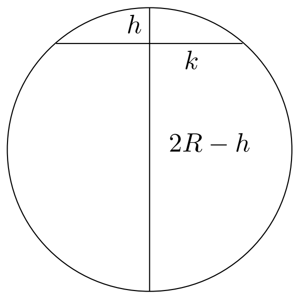
$$2\sqrt{R}\int_{-\sqrt{R-1}}^{\sqrt{R-1}} \arcsin\left(\frac{\sqrt{4(R-x^2)(r_1+1)^2-(x^2-2r_1-R-1)^2}}{2(r_1+1)\sqrt{R-x^2}}\right) dx \approx 8.056070935$$

Now suppose 7 unit sphere touch a unit ball. Then the total area of the surface of sphere S_1 which is covered by these seven cylinders is $7P$. So the uncovered surface area of the sphere is $A = 4\pi R^2 - 7P$

This time consider a sphere S_2 of radius r ($r \geq (R-1)/2$) which is tangent to the sphere S at the point $(0, 0, 1)$. Consider the spherical cap which is part of sphere S_1 and inside the sphere S_2. The area of the cylindrical cap is equal to $2\pi hR$ where h is the height of the cap. The area of the spherical cap could be at most an area A when 7 unit cylinders touching the unit ball. Consider that the area of the spherical cap is A so that

$$2\pi hR = A \;\Rightarrow\; h = \frac{A}{2\pi R}$$

Let the point $(0, k, R-h)$ be one of the intersection points of the spheres S_1 and S_2 in the yz-coordinate plane. By using the cord property we have



$$k^2 = h(2R-h) \;\Rightarrow\; k = \sqrt{h(2R-h)}$$

Figure 3.12: Finding the value of k by the cord property

The equation of the sphere S_2 is $x^2+y^2+(z-1-r)^2 = r^2$ so by plugging the coordinates of the point $(0, k, R-h)$ in this equation, we have

$$(k)^2 + (R-h-1-r)^2 = r^2 \;\Rightarrow\; r = \frac{k^2 + (R-h-1)^2}{2(R-h-1)} \approx 0.9001799674$$

So if $r > 0.9001799674$, then at most six unit infinitely long cylinders can touch the unit ball in addition to a tangent ball of radius r.

□

Corollary 3.5. *Consider a unit ball B which touching another given unit ball C then at most six infinitely long unit radius cylinders can touch the unit ball B, and avoid the ball C.*

Theorem 3.6. *Consider two non-overlapping unit balls. Then at most six disjoint infinitely long unit cylinders can simultaneously touch both balls.*

Proof of 3.6: Consider a cylinder which is tangent to two given unit balls. The two contact points can be connected by a line segment which is completely inside the cylinder. Notice

Figure 3.13, that points of the surface of the left hemisphere of the left sphere and points of the surface of the right hemisphere of the right sphere cannot be a contact point.

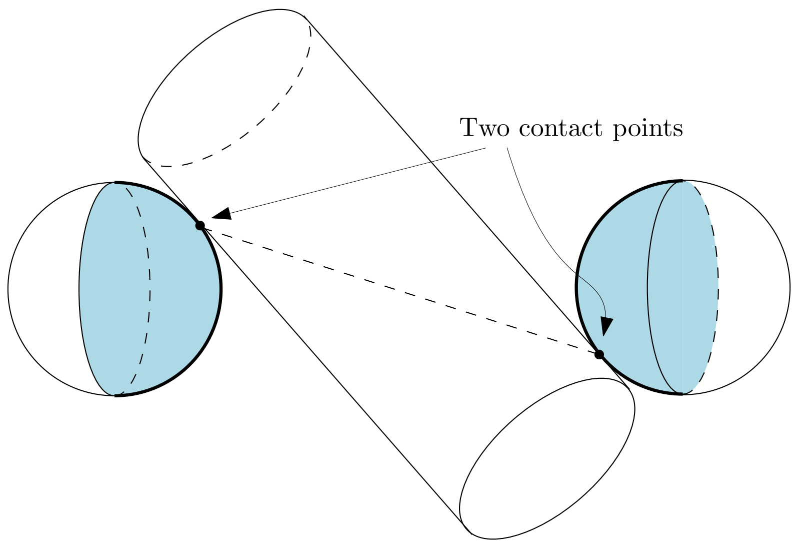


Figure 3.13: Two non-overlapping spheres in general position

Consider a unit ball which is tangent to one of the unit balls, does not separate the balls and has its center on the line which passes through the centers of balls as in Figure 3.14.

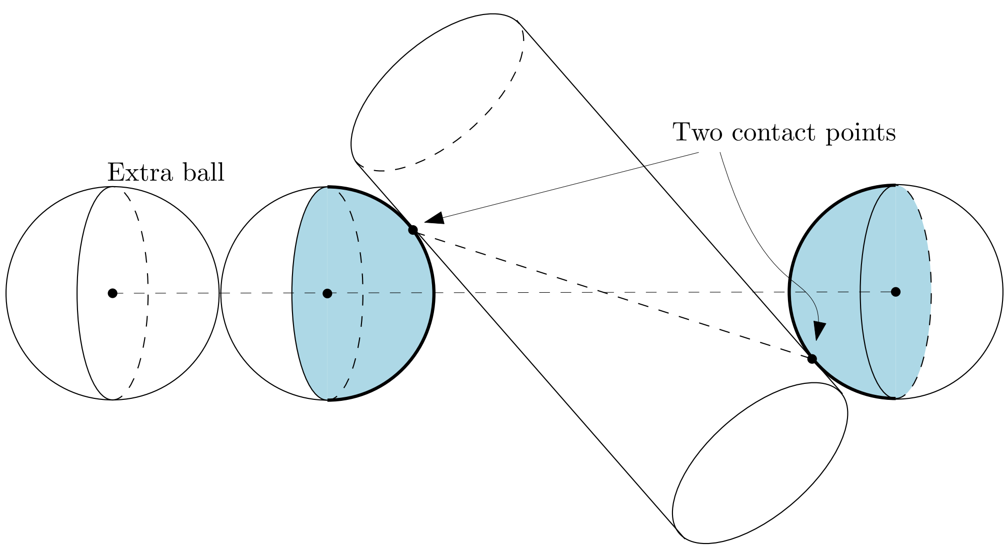


Figure 3.14: Two non-overlapping spheres in general position with an imaginary extra ball.

This extra ball is disjoint from any segment connecting possible contact points. Thus, this extra ball is disjoint from the cylinders tangent to both given balls. Corollary 3.5 says that at most six unit cylinders can touch a unit ball which touches another unit ball.

□

The rest of this section contains second proof for Theorem 3.6 under the condition $O_1O_2 \geq 4$. We start with recalling

Theorem 3.7. *For $n = 3, 4, 5, 6$ let r_n be equal to that radius, which allows a unit circle to be touched by a ring of n equal circles of radii r_n. r_n is the largest radius r so that n disjoint congruent cylinders of radii r can touch simultaneously two given unit balls whose centers are at a sufficient large distance (in case of $n = 6$, the distance 4 is sufficeint). In the optimal arrangement the cylinders must be parallel to the line connecting the centers of the unit balls.*

Proof of Theorem 3.7: Our short proof will be based on a new lemma and will apply a known theorem on densest circle packings. We start by giving these details,

Lemma 3.8. *Let e be a line in three dimensional space so that it is not perpendicular to the x-axis. Let $P(x)$ be the point of line e, whose first coordinate is x. Let $f(x)$ be the distance between the points $(x, 0, 0)$ and $P(x)$ (Figure 3.15a). It turns out that $f(x)$ is a convex function.*

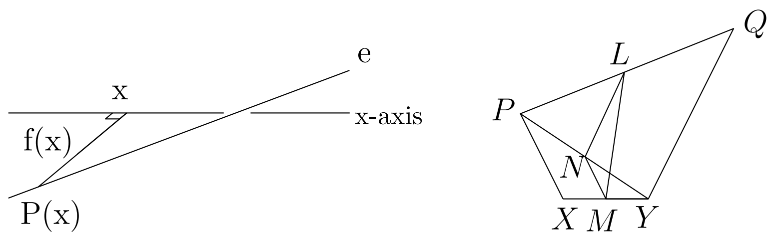


Figure 3.15: Convexity of a distance function

Proof of Lemma 3.15: The midpoint convexity (a sufficient condition for convexity) follows immediately from an elementary geometric property of quadrilaterals. Take the $3D$ quadrilateral with vertices $X = (x, 0, 0)$, $Y = (y, 0, 0)$, $P = P(x)$, $Q = P(y)$ (Figure 3.15b). Let M be the midpoint of XY, N be the midpoint of PY, L be the midpoint of PQ. By similar triangles $\frac{1}{2}XP = MN$, $\frac{1}{2}YQ = NL$. By triangle inequality $MN + NL \leq ML$. Since $f(x) = XP$ and $f(y) = YQ$, we have $\frac{(f(x)+f(y)}{2} \geq ML$. Since both XP and YQ are perpendicular to the x axis so is ML. Thus the last inequality is the midpoint convexity of the function $f(x)$.

□

Next, we give a brief account of the history of circle packings in a circle. The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was to find the smallest circle in which we can pack n congruent unit circles. The densest packing of n congruent circles in a circle are known for $n \leq 13$ and $n = 19$. The densest packing of n congruent circles in a circle were discussed by Kravitz [K67] for $n = 2, \ldots, 16$. It turns out that we can use/apply the solutions for $n \leq 6$ (the answers for $n > 6$ will not help as the optimal arrangements do not have the ring structure what we see for $n = 3, 4, 5, 6$).

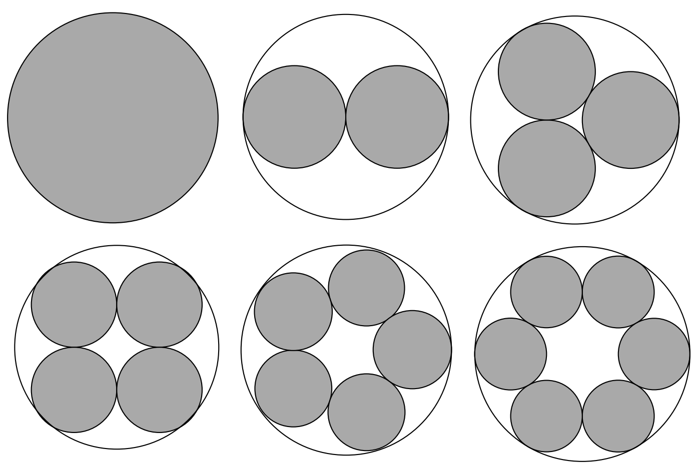


Figure 3.16: Densest packings of $n = 1, 2, \ldots, 6$ circles in a circle

Theorem 3.9. [Pirl (1969) and Graham (1968)] *The densest packings of* $n = 1, 2, \ldots, 6$ *circles in a circle are exactly those which are given in Figure 3.16.*

Proof of Theorem 3.9: Concerning the proof of these intuitively correct optimal arrangements, we refer to Pirl [P69] who proved that these arrangements are optimal for $n \leq 9$ and he also found that the optimal configuration for $n = 10$. For $n \leq 6$ proofs were given independently by Graham [CG68]. A proof for $n = 6$ and 7 was also given by Crilly and Suen [CS87]. $\square$

Now, we are ready to give the second proof of Theorem 3.6. Let O_1 and O_2 be the centers of the two given unit balls with $O_1O_2 \geq 4$. We explain our argument for $n = 6$ and note the similar argument holds for $n = 3, 4$ and 5. The line O_1O_2 will play the role of x-axis with the origin at the midpoint of O_1O_2. Let e_i, $i = 1, \ldots, 6$ be the axis of the cylinders. Introduce the functions f_i , $i = 1, \ldots, 6$ as above. Let H be the plane passing through the origin so that it is perpendicular to the x-axis. The plane H cuts each cylinder in an ellipse, whose center has a distance $f_i(0)$ from the origin. Once we establish that $f_i(0) \leq r_n - 1$ we are done (for example, in case of six circles $f_i(0) \leq 2$). Indeed, then n disjoint circles are contained in a circle of radius r_n (for example, in case of six circles radius 3), which in view of the above theorem implies uniqueness of the circles and that implies the parallel positions of the cylinders.

Drop perpendicular from O_1 to e_i and then from the foot of this back to O_1O_2. Let Q_1 be the foot. Of course $O_1Q_1 \leq 2$. Do the same with O_2 to get Q_2. Q_1Q_2 contains M. Using the convexity function argument we can finish the proof.

$\square$

Similar argument proves Theorem 3.7 for $n = 3, 4, 5$.

Chapter 4

Estimating the Total Area of Gaps Around Cylinders on the Surface of a Concentric Sphere

In this section we develop an area approach which can be used to solve weaker versions of Kuperberg's 6 cylinder problem or in the future could help solving Kuperberg's conjecture itself. Kuperberg had several arrangements where 6 unit cylinders touched a unit sphere and he asked for showing that 7 unit cylinders cannot do the same.

It is quite natural to check if one can prove this with Brass and Wenk's original idea. Assume we consider a sphere of radius $R > 1$ concentric to the unit sphere and compute on its surface, the area of the trace/patch of a tangent cylinder of radius 1.0496594. Unfortunately, it turns out that there is no $R > 1$ for which 7 times the area of this trace is more than $4R^2\pi$. This argument ignores the fact that the patches do not tile the sphere, while it is obvious that there must be gaps between the patches. Our goal is to give a lower bound say $G(R)$ for the total area of the gaps on the concentric sphere of radius R. All what we want is that 7 times the area of the trace of a cylinder exceed $4R^2\pi - G(R)$, the surface area of the larger sphere minus the area of the guaranteed gap area.

In order to refer more pictorially to certain numbers we will refer to them by names or by special visually simple terms:

Definition 2. *We will call a cylinder* **Firsching Cylinder***, if it is tangent to a given unit sphere and has radius* $1.0496594\ldots$. *Letter* r *will refer to* $1.0496594\ldots$ *(or in short to* $1.049\ldots$*), which is the numerical value of the radii in the example of Firsching.*

Definition 3. *As before, we call/refer to the common part of a Firsching cylinder and the surface of a sphere concentric to the given unit ball, as* **Firsching patch***, or in short* FP*. Both the shape and the area of the Firsching patches depend on the radius* R *of the concentric sphere, so sometimes we will use the notation* $FP(R)$*.*

We will use the following generalization of the lemma of Bezdek and Kuperberg [BK91]:

4.1 The Minimum Sum of Distances From a Point to Three Cylinders

Theorem 4.1. *Consider three congruent cylinders of unit radii, then the sum of distances from a point to the surfaces of the cylinders is at least* $3\left(\frac{2}{\sqrt{3}}-1\right)$. *Equality holds if the cylinders are parallel and mutually touch each other and the point is at equal distances from the cylinders.*

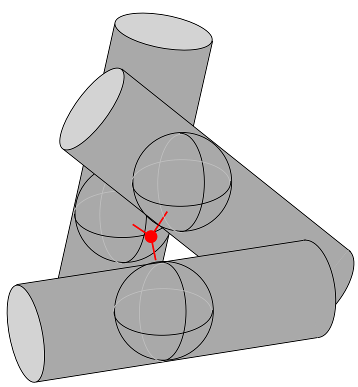


Figure 4.1: The minimum of the sum of distances from a point to three cylinders.

Proof of Theorem 4.1: Consider three infinitely long cylinders with unit radius in general position. Let P be a point and consider the shortest distances from each of the cylinders to the point P (Figure 4.2a). Each of the cylinders contains a ball with unit radius which is closest to the given point (Figure 4.2b). Project these three balls and the point P to a plane which is parallel to the plane that passes through the centers of the three balls (Figure 4.2b). In general, the three projected circles mutually may not touch each other (Figure 4.2c). The mutually touching circles will provide the smallest sum of the distances (Figure 4.2d). The mutually touching circles case gives us an equilateral triangle whose vertices are centers of the three circles and thus have an edge length 2. By Fermat Points [WE18], the smallest sum of distances from the vertices of a triangular to an inside point is provided by the center point of an equilateral triangular. So that

by basic calculation, we get that the sum of the three distances is at least $3\left(\frac{2}{\sqrt{3}}-1\right)$.
$\square$

Let us continue with defining neighbourhoods of Firsching patches:

4.2 Terminology and Some Important Numerical Constants

Definition 4. *Consider a cylinder which has radius $\rho + 1.049\ldots$ ($\rho \geq 0$) and is coaxial with the Firsching cylinder. Similarly to Firsching patches, we can define patches of such cylinders as well. On any concentric sphere this larger patch includes the Firsching patch. We define* ρ-**neighbourhood of the Firsching patch**, *as the set theoretical difference of the two patches. The area of this neighbourhood will be denoted by ρ-FP (or ρ-$FP(R)$, if the radius of the concentric sphere is to be emphasised).*

Definition 5. *It was mentioned in the previous section that, independently from the radius of a sphere, the spherical cap of Euclidean radius ρ has area $\rho^2\pi$ (Remark 4). Remember that k is a radius so that a sphere of radius k can touch three mutually tangent parallel cylinders of radius r. Since r is the radius of a Firsching cylinder, it is natural to refer to a spherical disc (or cap) of Euclidean radius k as* **Firsching cap**.

Our proof of Theorem 5.1 will be driven by the numerical values of certain distances, radii, disc area, and gap area. For example when we estimate the area of gaps we will do it by comparing it to P, and prove lemmas like the gap has area at least $5P$ or $7P$ etc. Let us summarize in a list some of the frequently used constants and their references, which will play a key role in the lemmas,

- The radius of the Firsching cylinders is $\mathbf{r = 1.0496594}\ldots$ (or in short $\mathbf{r = 1.049}\ldots$).

- For any point outside of three disjoint unit cylinders the sum of the distances to three unit cylinders is at least $3\left(\frac{2}{\sqrt{3}}-1\right) = 3 \times 0.1547\ldots$. This also means that $\mathbf{0.1547}\ldots$

is the radius of the largest sphere which can have common points with each of three disjoint unit cylinders.

- By simple scaling, we have that for any point outside of three disjoint Firsching cylinders the sum of the distances to these cylinders is at least $3r\left(\frac{2}{\sqrt{3}}-1\right) = 3 \times 0.1623828741$. Letter k will refer to the constant $\mathbf{k} = \mathbf{0.1623828741}$. This also means that k is the radius of the largest sphere which can have common points with each of three disjoint Firsching cylinders of radii r.

- The area of a Firsching cap is $\mathbf{P} = \mathbf{k^2}\pi = \mathbf{0.8283813644}\ldots$.

4.3 On a Lower Bound for the Area of Gaps, Which Once Proved, Would Imply New Results

In this section we show;

Lemma 4.2. *For $R = \sqrt{4.836}$, the Firsching patch FP(R), and the area P of the Firsching disc, we have the following inequality,*

$$4\pi R^2 \leq 7FP(R) + 9.28P.$$

Proof of Lemma 4.2: This lemma needs only numerical verification. Appendix B shows that three numerical values satisfy the stated inequality.

Lemma 4.2 is very important because it says that in order to prove Theorem 5.1 all we need to do is to place disjoint discs in the gaps with a total area $9.28P$.

- Throughout the section patches, neighbourhoods, gaps will be considered on a concentric sphere of radius $\mathbf{R} = \sqrt{\mathbf{4.836}}$. This radius is different than that in the paper of Brass and Wenk, and Firsching. We checked other radii, but apparently this choice was working for us, because by this radius we have the smallest needed gap area to be filled on the surface of a sphere.

Throughout the thesis, we will give lower bound for the total area of the gaps by presenting various lemmas concerning the number of disjoint discs of radii ρ which can be placed around the patches.

In order to get a better understanding of various area elements, we state the following:

4.4 Area of the ρ-neighbourhood of a Firsching Patch

Lemma 4.3. *(i) On the surface of the concentric sphere of radius $R = \sqrt{4.836}$ the area of the $\frac{k}{2}$-neighbourhood of a Firsching patch has an area greater than $11.34P$ (Figure 4.2). In notation*

$$area\left(\frac{k}{2} - FP(R)\right) > 11.34P$$

.

(ii) On the surface of the concentric sphere of radius $R = \sqrt{4.836}$ *the area of the* 0.6*-neighbourhood of a Firsching patch has an area greater than* $13.65P$*. In notation*

$$area\left(\frac{3k}{5} - FP(R)\right) > 13.65P$$

.

(iii) On the surface of the concentric sphere of radius $R = \sqrt{4.836}$ *the area of the* $0.724k$*-neighbourhood of a Firsching patch has an area greater than* $16.54P$*. In notation*

$$area\left(0.724k - FP(R)\right) > 16.54P$$

.

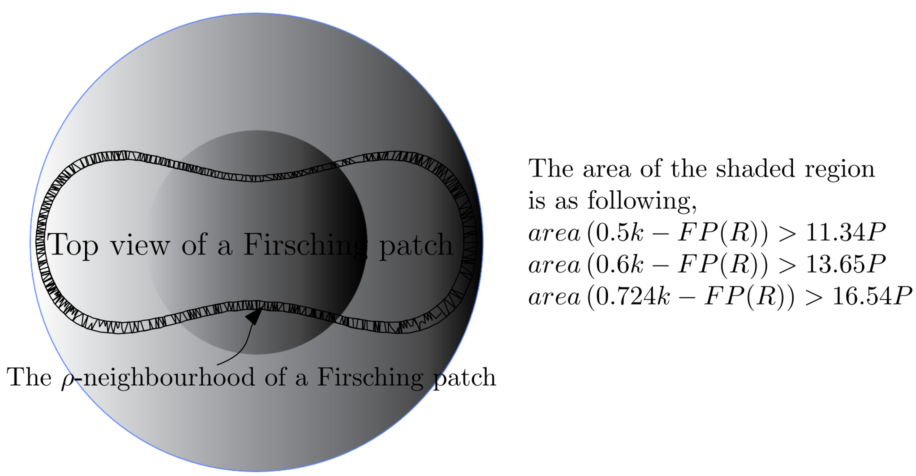


Figure 4.2: The shaded spherical region is the ρ-neighbourhood of a Firsching patch.

The proof of Lemma 4.3 is a matter of plugging in the numerical values of R and ρ. $area(\rho - FP(R))$ is evaluated by a Maple code (Appendix B).

4.5 Placing Discs in the Gaps, a Way to Estimate the Area of Gaps

In terms of application Lemma 4.3, if there is a Firsching patch whose ρ-neighbourhood does not overlap any other Firsching patches, then the gap is already large enough to prove that 7 Firsching cylinders cannot touch a unit sphere. In this rear situation, we do not need to place discs in order to estimate the area of gaps.

According to Lemma 4.2 we will place disjoint discs with a total area $9.28P$, in the gaps left by Firsching patches on the surface of a concentric sphere of radius R. We know that no point in the gap on the R-sphere can be closer than distance k to three of the Firsching cylinders. Thus, no point of the gap on the R-sphere can be closer than distance k to three of the Firsching patches. This is an information which suggests that we should place discs whose Euclidean radius is in the range of k. Specifically we will study three possibilities:

a) the Euclidean radii of the inserted discs are $\rho = 0.5k$,

b) the Euclidean radii of the inserted discs are $\rho = 0.6k$,

c) the Euclidean radii of the inserted discs are $\rho = 0.724k$,

If we decide on placing smaller discs in the gaps, then we have to place many of them, preferably disjoint ones, to reach the desired total area. If we decide on placing somewhat larger ones, then although we do not need as many, but it will be more difficult to show that there is room for larger discs.

It turns out that cases a and b do not provide large enough total gap area to prove Lemma 4.2 but in case c we will prove sufficiently large gap area and thus complete the proof of theorem 5.1.

Let us start describing how we place discs in the gaps in case a).

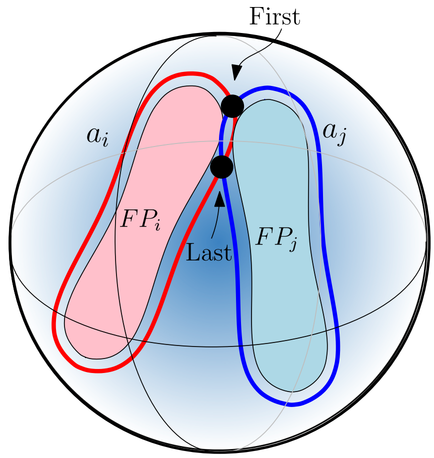


Figure 4.3: Where and how to place discs in the gap

We will give instructions for choosing the centers of discs to be placed in the gaps. For $\rho \in \{0.5k, 0.6k, 0.724k\}$, the boundary curves of the ρ-neighbourhoods of the Firsching patches will guide us in the selection. Let FP_i for $i = 1, 2, \ldots, 7$ be the seven Firsching patches. The Firsching patches of course are disjoint, but the ρ-neighbourhoods of them are not necessarily disjoint. Let a_i for $i = 1, 2, \ldots, 7$ be the boundary curves of the seven Firsching ρ-neighbourhoods. For each pair of the Firsching patches FP_i and FP_j, we consider the intersection of the boundary curves a_i and a_j. If the ρ-neighbourhoods do not overlap, then we do not select any center point. If the ρ-neighbourhoods overlap, then so do the boundary curves a_i and a_j. Intuitively speaking,

i) we choose the two furthest pair of intersection points for the centers, or we could also say that

ii) we choose the first and the last crossings of the curves a_i and a_j.

Just by looking at Figure 4.4, it seems that the above selection is well defined. However, since the patches are not convex, the boundary curves could intersect each other several times, so we quickly would run into problems seeing that the two ways of selection give the same points or whether they are well defined at all. The problem is similar to a situation where you place n chairs around a round table and want to tell someone which chair is the first, which is the last, and which pair of chairs are at the largest distance. We will take a closer look at the situation and then restate how to choose the center points.

4.6 On the Overlap of Two Neighbourhoods of Firsching Patches

Lemma 4.4. *Let a_i and a_j be the boundary curves of the ρ-neighbourhoods of the Firsching patches FP_i and FP_j. Assume a_i and a_j intersect each other.*

For $\rho \in \{0.5k, 0.6k, 0.724k\}$, both a_i and a_j can be split in two halves of equal lengths so that one of the halves does not contain any intersection point of a_i and a_j. Thus, on the other halves of the boundary arcs, the first and the last points of crossings of the curves a_i and a_j become well defined.

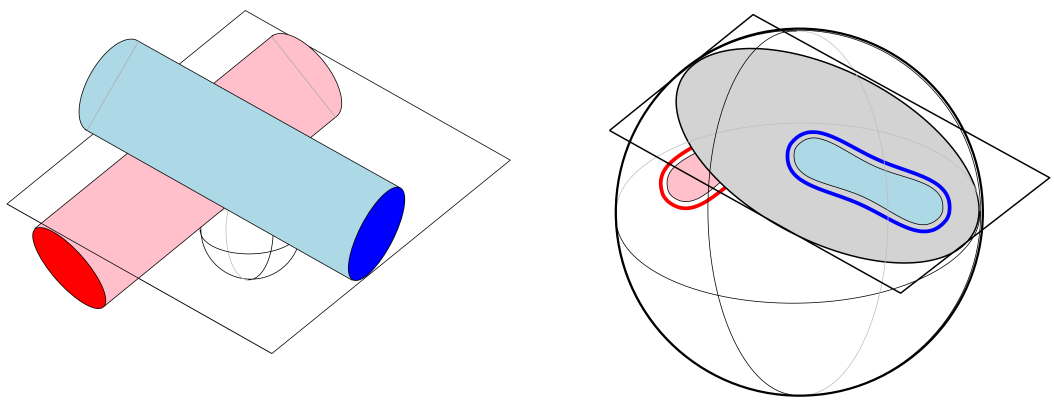
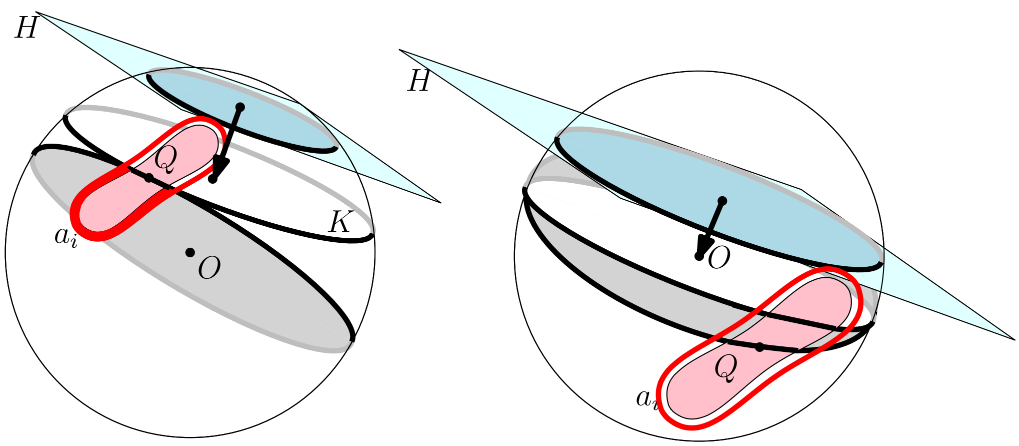


Figure 4.4: Separating Firsching patches by a plane

Proof of Lemma 4.4: Two disjoint cylinders are always separable by a plane. Let H be a plane which separates two Firsching cylinders c_i and c_j so two Firsching patches FP_i and FP_j. Since the cylinders are tangent to the unit sphere, H cuts the unit sphere in a circle. Under our assumption, at least one of a_i and a_j intersect this circle. Without loss of generality assume a_i and the center of the unit sphere belongs to the same half space bounded by H.

First we prove Lemma 4.4 for a_i. Both the patches and their neighbourhoods are central symmetrical on the concentric sphere with center O of radius R. Let Q be the center of the region bounded by a_i. At this point we will use the numerical fact that ρ is smaller than the Euclidean radius of the largest disc centered at Q and contained inside of a_i. In fact, this is the reason why we can say that both O and Q are on the same side of the plane H. Let us move the plane H parallel towards the center O of the concentric sphere, and stop when it passes through Q or O. In the first case, denote by K the circle into which the shifted plane cuts the concentric sphere. It is easy to see (Figure 4.5a) that the great circle which contains Q and which is tangent to K lies on the same side of H where a_i lies. Because of the central symmetry of a_i this completes the proof of Lemma 4.4. In the second case (Figure 4.5b), the great circle whose farthest point from H is Q does the same as the great circle in case a). Now we turn to the other boundary curve a_j. We argue similarly, as above. This time we move the plane H parallel away from the center O of the concentric sphere, and stop when it passes through Q. Denote again by K the circle into which the shifted plane cuts the concentric sphere. It is easy to see (Figure 4.5a) that the great circle which contains Q and which is tangent to K does the needed partition.



a) b)

Figure 4.5: Half of a_i does not contain intersection points

□

4.7 Estimating the Number of Special Discs Placed around a Firsching Patch

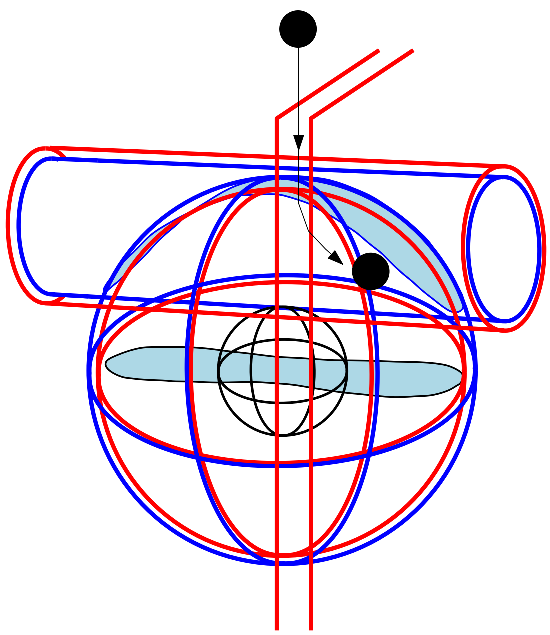
Lemma 4.5. *(i) $6P$ is a lower bound of the area of $\frac{k}{2}$-neighbourhood of a Firsching patch.*

(ii) $7.2P$ is a lower bound of the area of $0.6k$-neighbourhood of a Firsching patch.

(iii) $8.6P$ is a lower bound of the area of $0.724k$-neighbourhood of a Firsching patch.

Proof of Lemma 4.5:

(i) Consider spheres of radius $\frac{k}{4}$ which are touching a Firsching cylinder and whose centers are on the sphere radius $R = \sqrt{4.836}$. Obviously, the intersection of such spheres and the concentric sphere of radius R are a spherical caps contained by the $\frac{k}{2}$-neighbourhood of Firsching patch.



Blue cylinder has radius r.
Red cylinder has radius $r + \frac{k}{4}$.
Blue sphere has radius R.
Red sphere has radius $R - \frac{k}{4}$.
Distance between red planes is $\frac{k}{2}$.

Figure 4.6: Placing spheres of radius $\frac{k}{4}$ between planes perpendicular to the cylinders.

By considering Figure 4.7, in the upper half of the neighbourhood, from the left line to the right line we can locate maximum M=48 such spheres by following calculation;

$$M = \left\lfloor \frac{2\sqrt{3.836}}{k/2} \right\rfloor = 48$$

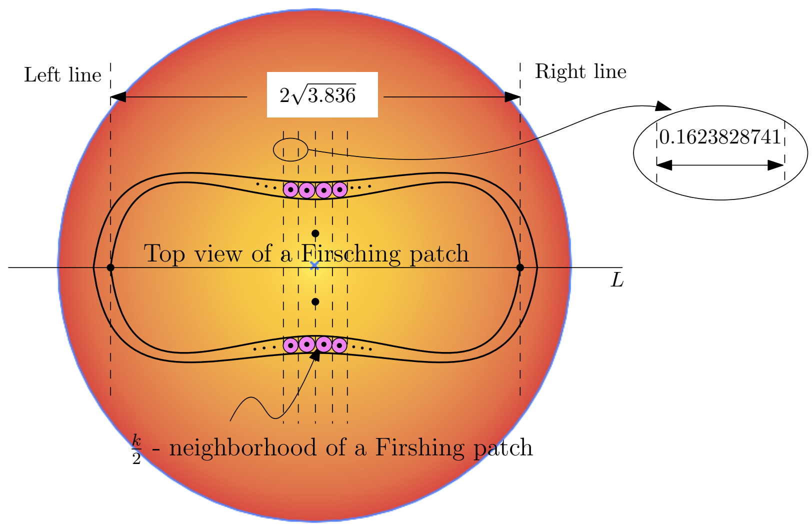


Figure 4.7: A lower bound for the area of the $\frac{k}{2}$-neighbourhood of a Firsching patch area.

Also by considering Figure 4.7, in the lower half of the neighbourhood, we have additionally 48 such touching spheres. Therefore the total surface area of the sphere of radius R inside such spheres is

$$96\frac{k^2}{16}\pi = \frac{96}{16}k^2\pi = 6P$$

(ii) A method similar to what we used in (i) gives the following result for M;

$$M = \left\lfloor \frac{2\sqrt{3.836}}{0.6k} \right\rfloor = 40$$

so total number of discs placed in the $0.6k$ -neighbourhood of a Firsching patch is 80, then

$$80\frac{(3k)^2}{100}\pi = \frac{80 \cdot 9}{100}k^2\pi = 7.2P$$

(iii) Similar method of (i) gives the following result for M;

$$M = \left\lfloor \frac{2\sqrt{3.836}}{0.724k} \right\rfloor = 33$$

so total number of discs placed in the $0.724k$ -neighbourhood of a Firsching pathcis 66, then

$$66(0.362k)^2\pi = 66 \cdot (0.362)^2 k^2 \pi \approx 8.6P$$

$\square$

4.8 Some Lemmas on Disjointness of Special Discs and Firsching Patches

We assigned two discs of radii $\frac{k}{2}$ to each pair of overlapping Firsching neighbourhoods. Let us denote them with d_{ij} (and d'_{ij}). The notation is so that the double index of d_{ij} tells us that centers are intersection points of a_i and a_j, i.e. they are assigned to Firsching cylinders of indices i and j. We make a couple of simple observations; some in the form of remarks (if they are very simple), some in the form of Lemmas (if we want to refer to them later).

Remark 8. *The sphere of radius $\frac{k}{2}$ whose center is the same as that of the disc d_{ij} is tangent to the Firsching cylinders with indices i and j. We cannot say that the spherical disc d_{ij} is touching the corresponding Firsching patch. With another words d_{ij} lies in the gaps on the concentric sphere of radius R, but is not necessarily tangent to the Firsching patches with index i and j. Similar statements hold if $\frac{k}{2}$ is replaced with other constants like $0.6k$ or $0.724k$.*

Lemma 4.6. *Assume $\frac{k}{2}$-neighbourhoods are considered and we already choose the discs d_{ij} of radii $\frac{k}{2}$. We will prove that d_{ij} and d_{im} are disjoint for distinct i, j, m. This holds also when $\frac{k}{2}$ is replaced with $0.6k$.*

Proof of Lemma 4.6: Lemma 4.6 says that if two discs have their center on a_j, but they are not assigned to the same pair of cylinders, then they must be disjoint. Indirect assume that two such discs overlap. Let O be the center of d_{ij}. Notice that the distances from center O to the c_i and to the $c_j$cylinders are equal to $\frac{k}{2}$, while to the c_m cylinder is less than $3 \times \frac{k}{2}$. Thus the sum of the distances is $< 5 \times \frac{k}{2}$, contradicting Lemma 4.1, which says the sum should be at least $3k$. In fact for any radius $\leq \frac{3}{5}k$ a similar argument holds, where under indirect assumption, the sum of the three distances turns out to be $< 3k$.

$\square$

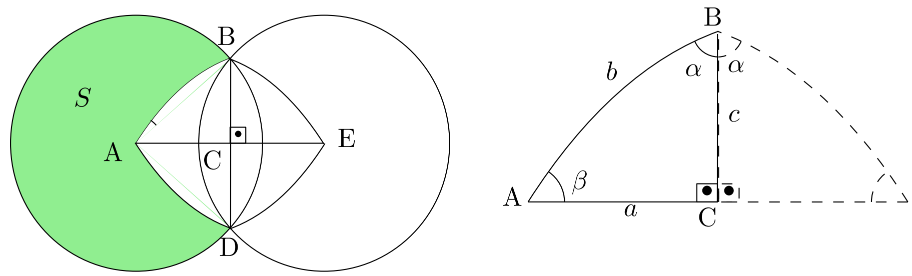
We will have a separate lemma here addressing the situation of Lemma 4.6 for radius $0.724k$.

Lemma 4.7. *Assume* $0.724k$*-neighbourhoods are considered and we already choose the discs* d_{ij} *of radii* $0.724k$ *on the surface of the concentric sphere of radius* R*. We consider the discs* d_{ij} *and* d_{im} *which were selected for a given triplet of different indices* i, j, m *and prove that the centers of* d_{ij} *and* d_{im} *are at a distance at least* $0.828k$*. A Maple code will show that the total area of the two discs is at least* $0.8838124575P$*.*

Proof of Lemma 4.7: Lemma 4.7 says that Lemma 4.6 does not hold for radius $0.724k$. If two discs have their center on a_j, but they are not assigned to the same pair of cylinders, then they do not have to be disjoint. With other words making the radius of the discs larger we loose the property of disjointness. Let x be the distance between the centers of d_{ij} and d_{im}. Just like in the proof of Lemma 4.6, we estimate the sum of the distances of the center O of d_{ij} to the three Firsching cylinders. Notice that the distances from center O to the c_i and to the c_j cylinders are equal to $0.724k$, while to the c_m cylinder is less than $0.724k + x$. Thus we have get the inequality $3 \times 0.724k + x \geq 3k$, which gives $x \geq 0.828k$.

Now let us show how to calculate the area of the union of two overlapping discs. The top view of the two overlapping discs is as in Figure 4.8. We use trigonometric formulas to find the central angles a and b.

$$a = \arcsin\left(\frac{0.414k}{\sqrt{4.836}}\right) \text{ and } b = 2\arcsin\left(\frac{0.362k}{\sqrt{4.836}}\right)$$



Top view of two overlapping discs of radius $0.724k$ on the surface of the sphere radius of $\sqrt{4.836}$

Enlarging the spherical triangle ABC.

Figure 4.8: Calculation of the area of the union of two overlapping discs of radius $0.724k$

Then by spherical trigonometric formulas we find the value of α and β as following;

$$\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\pi/2)}{\sin(b)} \Rightarrow \alpha = \arcsin\left(\frac{\sin(a)}{\sin(b)}\right)$$

and

$$\frac{\sin(2\alpha)}{\sin(2a)} = \frac{\sin\beta}{\sin b} \Rightarrow \beta = \arcsin\left(\frac{\sin(2\alpha)\sin b}{\sin(2a)}\right)$$

Then the total area of the spherical triangle ABD is $(2\alpha + 2\beta - \pi)4.836$, so the total area of spherical quadrilateral $ABED$ is $2(2\alpha + 2\beta - \pi)4.386$. The area of a spherical sector S is

$$S = \frac{2\pi - 2\beta}{2\pi}(0.724k)^2\pi$$

So the area of the union of overlapping discs is

$$A = 2(2\alpha + 2\beta - \pi)4.386 + \frac{2\pi - 2\beta}{2\pi}(0.724k)^2\pi$$

A Maple computation in Appendix C shows that $A \geq 0.8838124574P$.

$\square$

Lemma 4.8. *Assume $\frac{k}{2}$-neighbourhoods are considered and we already choose the discs d_{ij} of radii $\frac{k}{2}$. We will prove that d_{ij} and d_{mn} are disjoint for distinct i, j, m, n. This holds also when $\frac{k}{2}$ is replaced with $0.6k$ or with $0.724k$.*

Proof of Lemma 4.8: Lemma 4.8 says that if two discs are assigned to two distinct pairs of cylinders (i.e. to four different cylinders), then they still must be disjoint. Sadi Abu-Saymeh and Mowaffaq Hajja considered the family of tetrahedra with all edges greater than equal to 1 and proved that the sum of the distances from a given point to the vertices is smallest, if the tetrahedron is the regular one of edge length 1 and the point is the center of the tetrahedron [SH97]. For our Lemma we consider the family of larger tetrahedra - tetrahedra with edges longer than twice of the Firsching radius, where we place Firsching spheres centered at the vertices to get four disjoint spheres. The problem of minimizing the sum of distances from a given point to the surfaces of the spheres is equivalent to the problem of Abu-Saymeh and Hajja.

Case of $\frac{k}{2}$-neighbourhoods: Indirect assume that the discs are not disjoint. Let O be the center of d_{ij}. We estimate the sum of the distances of the center O of d_{ij} to the four Firsching cylinders. Notice that the distances from center O to the c_i and to the c_j cylinders are equal to $\frac{k}{2}$, while the distance to the c_m cylinder is less than $3\frac{k}{2}$. Thus, the sum of the distances to the surfaces of the Firsching cylinders is $\leq 8\frac{k}{2}$.

Case of $0.6k$-neighbourhoods: Similar argument gives that the sum of the distances to the surfaces of the Firsching cylinders is $\leq 8 \cdot 0.6k$.

Case of $0.724k$-neighbourhoods: Similar argument gives that the sum of the distances to the surfaces of the Firsching cylinders is $\leq 8 \cdot 0.724k$.

We will get a contradiction once we check that numerically all three of these sums are less than the sum in the alleged extremal case. We need to verify the inequality only for the largest radius. The computation is simpler, if we scale back to arrangement to tetrahedra with edge length 2. Thus, all we need to check is that

$$8 \cdot 0.724 \cdot \left(\frac{2}{\sqrt{3}} - 1 \right) < 4 \left(\frac{3}{4} \sqrt{3 - \frac{1}{3}} - 1 \right)$$

.

A simple Maple calculation shows that this inequality is

$$0.896025519 < 0.898979486$$

$\square$

Lemma 4.9. *Assume $\frac{k}{2}$-neighbourhoods are considered and we already choose the discs of radii $\frac{k}{2}$. We will prove that d_{ij} and is disjoint from any boundary curve a_k for k different from i, j. This holds also when $\frac{k}{2}$ is replaced with $0.6k$ or with $0.724k$*

Proof of Lemma 4.9: Lemma 4.9 says that if a disc d_{ij} assigned two boundary curves a_i and a_j then the disc and a third boundary curve a_k does not have common point. Indirect, assume that the disc and third boundary curve overlap. Let O be the center of d_{ij}. Notice that the distances from the center O to the curve FP_i and FP_j is $k/2$. On the other hand the distance from O to FP_k is at most k so that the sum of distances is at most $2k$ so that this contradict Theorem 4.1.

Case of $0.6k$-neighbourhoods: Similar argument shows that the distances from the center O to FP_i and FP_j are $0.6k$. And the distance from the center to FP_k is at most $1.2k$ so that the sum of the distances is at most $2.4k$ which contradicts our assumption because of Theorem 4.1.

Case of $0.724k$: Similar argument shows that the distances from the center O to FP_i and FP_j are $0.72k$. And the distance from the center to FP_k is at most $1.448k$ so that the sum of the distances is at most $2.896k$ which contradicts our assumption because of Theorem 4.1.

$\square$

The following table summarizes our knowledge on disjointness of discs, patches and lower bounds of neighbourhoods.

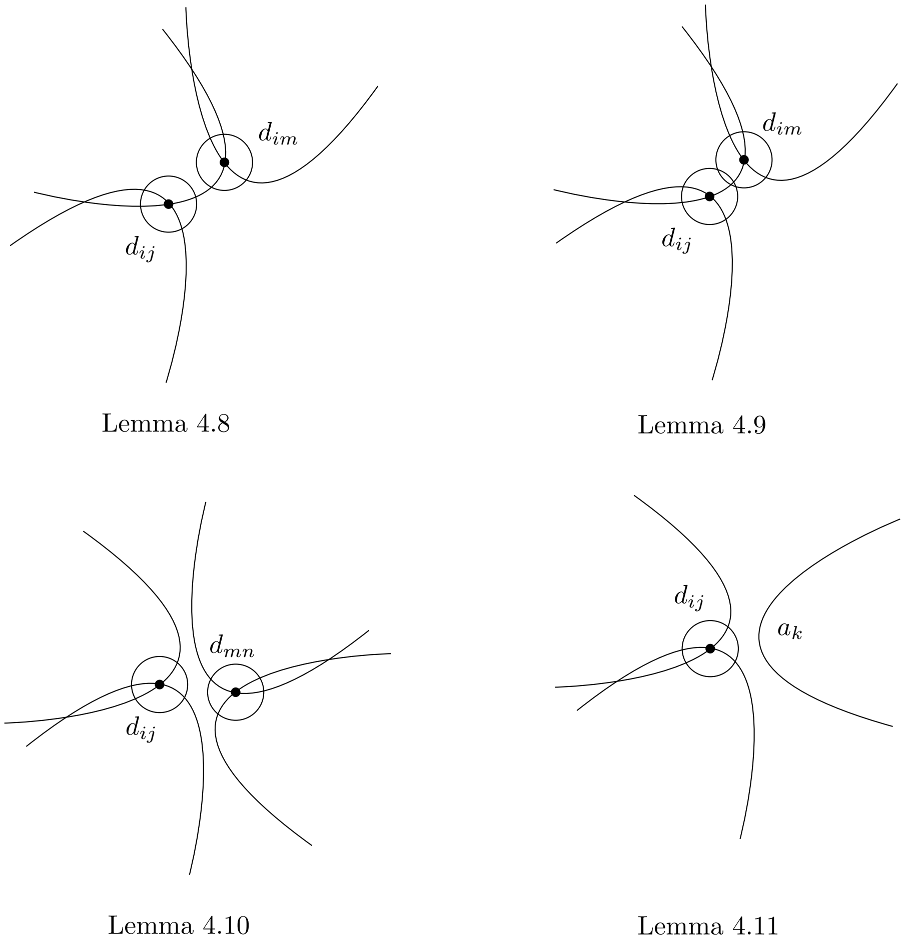


Figure 4.9: Summary of disjointness of discs placed in the gaps

4.9 Estimating uncovered area of Firsching Neighbourhood

Definition 6. *By considering Figure 4.11, we define Firsching core cap as the largest spherical cap inside the Firsching patch centered at the center of the Firsching patch.*

Lemma 4.10. *Consider three Firsching cylinders which are parallel then at least 50 of* $k/4$ *radius discs can fit in the* $k/2$ *neighbourhood of Firsching patch of each cylinder. Similarly, at least 42 of* $0.3k$ *radius discs can fit in* $0.6k$ *neighbourhood of Firsching patch, and at least 34 of* $0.362k$ *radius discs can fit in* $0.724k$ *neighbourhood of Firsching patch.*

Proof of Lemma 4.10: Consider a unit ball centered at the origin. Consider a Firsching cylinder which is tangent to the unit ball at the point $(0, 0, 1)$ and whose axis is parallel to

r	Lemmas on disjointness of discs placed in the gaps		Lemma on disjointness of disc and ring	Lower bound of the area of r-neighbourhood of Firsching patch		
Radius r of disc to be placed in gaps	Discs are disjoint from other discs in the same ring. (Lemma 4.6)	Discs are disjoint from other discs of different ring. (Lemma 4.8)	Discs are disjoint from a ring to which they are not assigned. (Lemma 4.9)	Number of discs of radii $\frac{r}{2}$ in the r-neighbourhood Lemma 4.5	Estimates by number of $\frac{r}{2}$ discs. Lemma 4.5	Calculation by maple code Lemma 4.3
$0.5k$	Yes	Yes	Yes	96	$6P$	$11.34P$
$0.6k$	Yes	Yes	Yes	80	$7.2P$	$13.65P$
$0.724k$	**	Yes	Yes	66	$8.6P$	$16.54P$

Table 4.1: A summary of disjointness of discs, patches and lower bounds of neighborhoods.

y-axis. Now consider two planes which are parallel to yz-coordinate plane and tangent to the Firsching core cap. Notice that the annulus between these planes on the surface on the sphere of radius R is not invaded by a $\frac{k}{2}$-neighbourhood of parallel Firsching cylinders.

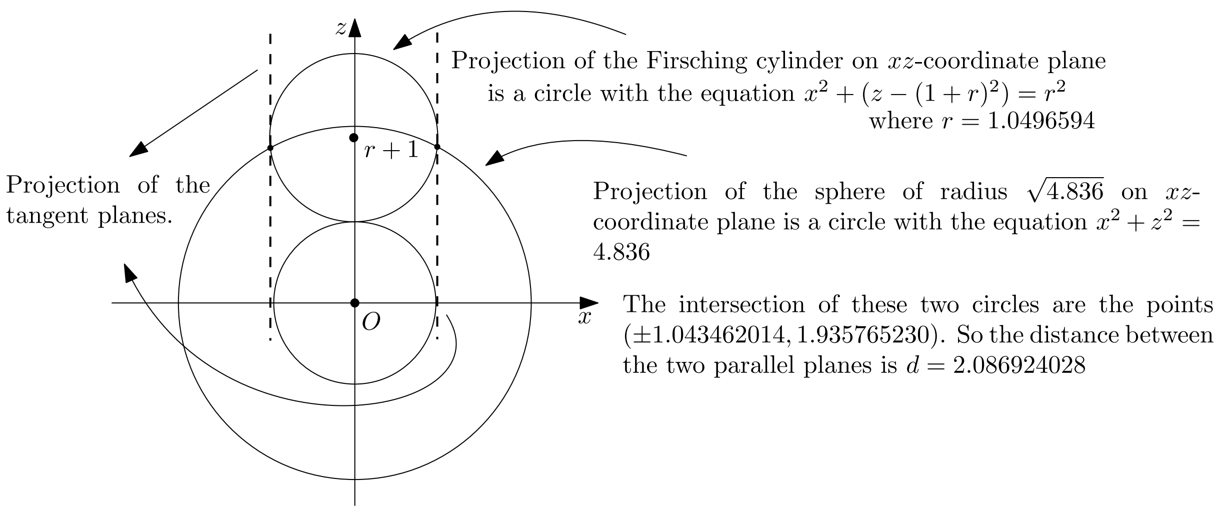


Figure 4.10: Calculation of the distance between the planes

By considering Figure 4.11, we can place at least $N = \left\lfloor \dfrac{2.086924028}{\rho} \right\rfloor$ discs of $\rho/2$ radius at the left end of the ρ-neighbourhood of a Firsching patch which is bounded by the planes.

For $\rho = k/2$, we can place $N = \left\lfloor \dfrac{2.086924028}{k/2} \right\rfloor = 25$ discs of $k/4$ radius in the described part of the $k/2$-neighbourhood of Firsching patch. Since there exists also right side of the described patch, then the total number of $k/4$ radius of discs makes 50. By similar method we can place $N = \left\lfloor \dfrac{2.086924028}{0.6k} \right\rfloor = 21$ discs of $0.3k$ radius in the left side of the described

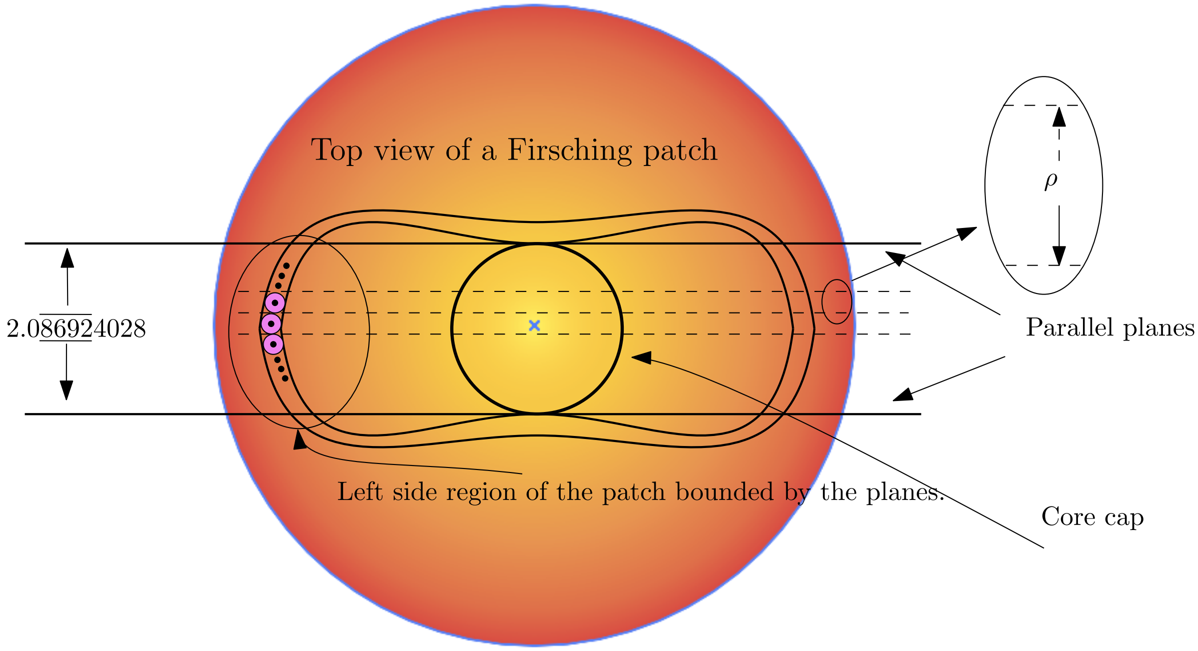


Figure 4.11: Placing discs in a Firsching neighbourhood

part of $0.6k$-neighbourhood of Firsching patch and by a similar argument the total number of $0.3k$ radius of discs turns out to equal to 42. By the same argument we can place $N = \left\lfloor \dfrac{2.086924028}{0.724k} \right\rfloor = 17$ discs of $0.362k$ radius in the left side of the described part of $0.724k$-neighbourhood of Firsching patch and by the similar argument the total number of $0.362k$ radius of discs is 34.

□

We will use the following generalization of Lemm 4.10.

Lemma 4.11. *Consider three Firsching cylinders in general position then Lemma 4.10 remains true with the constants 25, 21, and 17 instead of 50, 42 and 34.*

4.10 Rules to Assign and Distribute disc Areas to Firsching Patches

Our main goal is to give a lower estimate of the total area of gaps. The following few paragraphs, printed in the italics, will explain how this estimation is done.

We will distribute certain pieces of the gaps between the Firsching cylinders, and then find a lower bound for the area each cylinder must get at this distribution. This lower bound will be expressed in the form of constant times P. The total area of the gaps must be at least seven times this lower bound. The area assigned to a particular Firsching cylinder will come from two sources:

i) recall that we already assigned two discs of radii $\frac{k}{2}$ to each pair of overlapping Firsching neighbourhoods. We denoted them with d_{ij} and d'_{ij}. We will assign half of the area of the union of the discs d_{ij} and d'_{ij} to the i-th Firsching cylinder and the other half to j-th Firsching cylinder. Lemmas 4.6-4.9 imply that no portion of the union of the pair of discs is assigned twice.

ii) Once we guarantee that a certain portion (say half, or third) of the ρ-neighbourhood of a fixed, say i-th Firsching cylinder is disjoint from all of the assigned discs and from the ρ-neighbourhood of any other Firsching cylinders, then that area gets assigned to the ith Firsching cylinder.

In order to estimate the total area assigned to a particular Firsching cylinder we will classify the ways its ρ-neighbourhood is overlapped by others. First of all, according to Figure 4.12, we will say that two overlapping r-neighbourhoods can either long or short. An over lap is said to be long, if the two assigned discs are disjoint, and it is said to be short, if the two assigned discs overlap each other. Figures 4.14, 4.17 show the different ways how a fixed Firsching neighbourhoods can be overlapped by other neighbourhood of Firsching cylinder. The basic classification is according to the number of overlapping neighbourhoods (0, 1, 2, or at least 3). Within a single type we distinguish sub cases depending on how many of the overlapping neighbourhoods are a short overlaps.

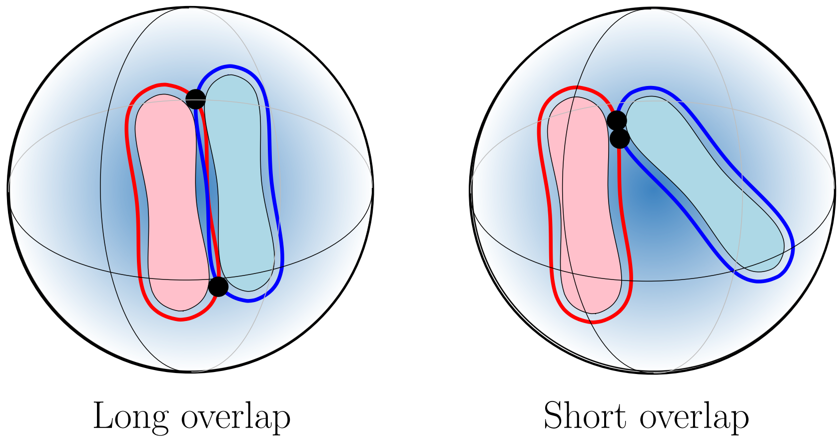


Figure 4.12: Discs assigned to a pair of patches can be disjoint or overlapping

Lemma 4.12. *Assume the $\frac{k}{2}$-neighbourhoods of the Firsching cylinders c_i and c_j overlap each other and overlap is a short overlap. Then the two discs assigned to the pair c_i, c_j overlap each other and these two discs cannot have common points with more then* 5 *of those discs of radii $\frac{k}{4}$ which we placed in the $\frac{k}{2}$-neighbourhood of cylinder c_i while proving Lemma 4.5. The same holds if $\frac{k}{2}$ is replaced with* $0.6k$ *or with* $0.724k$.

Proof of Lemma 4.12: Consider the case of $\frac{k}{2}$ first. Recall that in the proof of Lemma 4.5 we used equally placed parallel planes. These planes form slabs of with width equal $\frac{k}{2}$. By considering Figure 4.13, it becomes obvious that the two discs of radii $\frac{k}{2}$ can interfere with at most 5 slabs, which implies Lemma 4.12. The same argument works if $\frac{k}{2}$ is replaced with $0.6k$ or with $0.724k$.

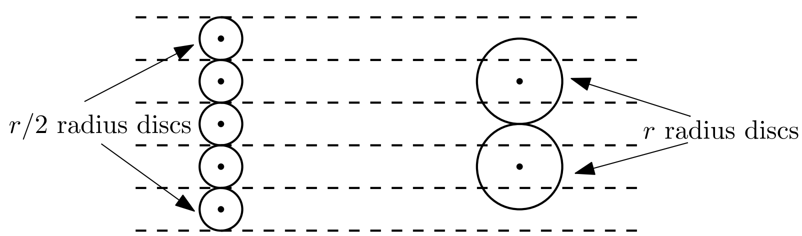


Figure 4.13: Number of interfered $r/2$ radius discs by discs of radii r

□

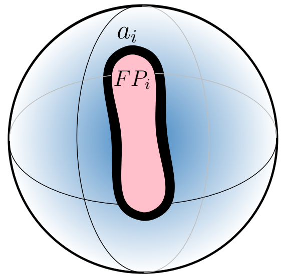


Figure 4.14: The boundary curve a_i is not overlapped by any other boundary curve

4.11 Estimating the Area Which is Assigned to a Single Firsching Patch

The following is the estimate (lower bound) for the total area assigned to a Firsching cylinder c_i of Type O (Figure 4.14):

The entire area between the Firsching patch and the curve a_i is assigned to the cylinder c_i. According to Lemma 4.5 this area is

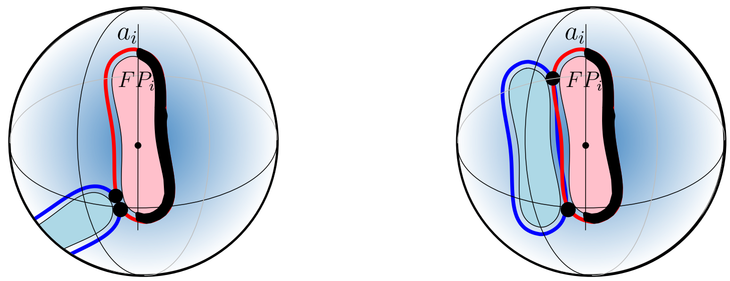
- at least $6P$ if $\frac{1}{2}$-neighbourhoods
- at least $7.2P$ if 0.6-neighbourhoods,

are considered.

We get larger lower bounds if we compute the area of this neighbourhood (also called ring) with mapple code. Namely, this area is

- at least $11.34P$ if $\frac{1}{2}$-neighbourhoods
- at least $13.65P$ if 0.6-neighbourhoods

are considered.



Type 1A

No. of short overlaps = 1
No. of long overlaps = 0

Type 1B

No. of short overlaps = 0
No. of long overlaps = 1

Figure 4.15: The boundary curve a_i is overlapped by exactly one other boundary curve

The following is the estimation (lower bound) for the total area assigned to a Firsching cylinder c_i of Types 1A and 1B (Figure 4.15):

We present our estimates for the case of $\frac{k}{2}$. Lemma 4.4 says that half of the $\frac{k}{2}$-neighbourhood of the Firsching patch can be assigned to cylinder c_i. At first it looks like that in order to improve the lower bound

- in case of short overlap (Type 1A) we could add $\frac{1}{2} \cdot 1 \cdot \frac{1}{4}P$,
- in case of long overlap (Type 1B) we could add $\frac{1}{2} \cdot 2 \cdot \frac{1}{4}P$

Notice that we cannot guarantee that the discs which are assigned to the overlapping pair of cylinders do not overlap the disjoint "half ring" (shaded region on the figure) so we do not add these terms. The same half ring estimate works if $\frac{k}{2}$ is replaced with $0.6k$.

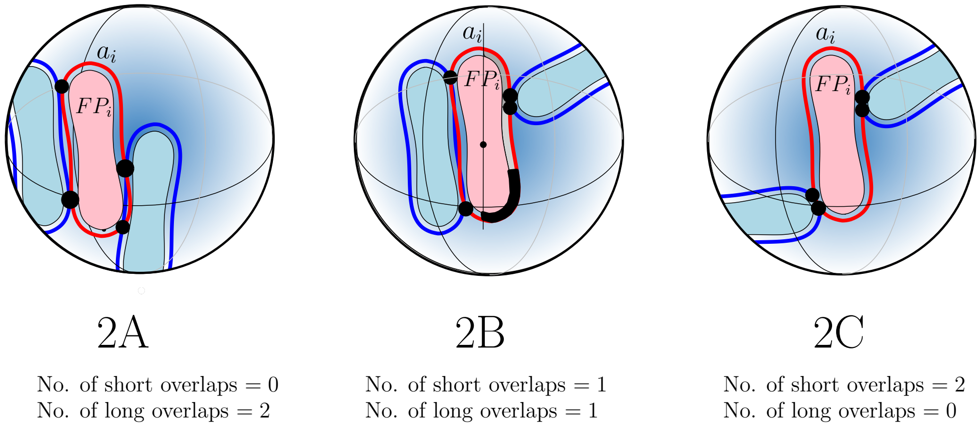


Figure 4.16: The boundary curve a_i is overlapped by exactly two other boundary curves

The following is the estimation (lower bound) for the total area assigned to a Firsching cylinder c_i of Types 2A, 2B and 2C (Figure 4.16):

We present our estimates for the case of $\frac{k}{2}$. Here we address the three subcases separately. Case of Type 2A: Let c_j be the cylinder which gives the long overlap. Recall Lemma 4.6 which says that if two discs have their center on a_j, but they are not assigned to the same pair of cylinders, then they must be disjoint. Thus all four of the assigned discs are disjoint. Moreover, according to Lemma 4.11 at least $25 - 2 = 23$ of $k/4$ radius discs can fit in the $k/2$-neighbourhood of the Firsching cylinder c_i, so that they are disjoint of the four discs assigned to the overlapping pair of cylinders. Please note that we subtracted 2 from the constant proved in Lemma 4.11, allowing the possibility that two of the assigned discs of radii $\frac{1}{2}k$ has a common point with one of the small discs. Similarly, at least 19 of $0.3k$ radius discs can fit in $0.6k$ Firsching patch neighbourhood. Also for $k/2$ and $3/5k$ cases, we have four disjoint discs with radius $k/2$, this four discs will be shared with another patch that is why we add 2 such discs. So that we have the following lower bounds for the total area of gap assigned to c_i;

- $23 \cdot \frac{1}{16}P + \frac{1}{2} \cdot 4 \cdot \frac{1}{4}P = 1.9375P$ in case of $k/2$ -neighbourhood
- $19 \cdot \frac{9}{100}P + \frac{1}{2} \cdot 4 \cdot 0.36P = 2.43P$ in case of $0.6k$ -neighbourhood

Case of Type 2B: Let c_j be the cylinder which gives the long overlap. Lemma 4.4 says that half of the $\frac{k}{2}$-neighbourhood of the Firsching patch is disjoint from overlapping neighbourhood of Firsching cylinder. Divide this half ring into three equal area parts along a_i. The two discs assigned to the pair with short overlap cannot invade all three of these parts thus $\frac{1}{6}$ of the area of the ring can be assigned to c_i. Notice that we can improve this lower bound by adding $2 \cdot \frac{1}{2} \cdot \frac{1}{4}P$, the reason is the half of it is coming form short overlapping cylinder and the other half of it coming from the one of the two assigned discs.

So that we have the following lower bounds for the total area of gap assigned to c_i

- $1/4P + 1/6\text{Full Ring} = 2.14P$ in case of $k/2$ -neighbourhood

in case of $0.6k$, it is handled the same way of $1/2k$.

- $0.36P + 1/6\text{Full Ring} = 2.635P$ in case of $0.6k$ -neighbourhood

Case of Type 2C: Lemma 4.12 says that in case of short overlap, the union of the assigned discs cannot have common points with more than 5 of those discs of radii $\frac{k}{4}$ which we placed in the $\frac{k}{2}$-neighbourhood of cylinder c_i while proving Lemma 4.5. Lemma 4.5 says that the area of the ring is at least $6P$. Thus we can guarantee that

- at least an area of $2 \cdot \frac{1}{2} \cdot \frac{1}{4}P + 6P - 10 \cdot \frac{P}{16} = 5.625P$ is assigned to the cylinder c_i.

Similarly if we change $k/2$ with $0.6k$ then we have the following lower bounds;

- at least an area of $2 \cdot \frac{1}{2} \cdot 0.36P + 7.2P - 10 \cdot 0.09P = 6.66P$ is assigned to the cylinder c_i for $0.6k$-neighborhood.

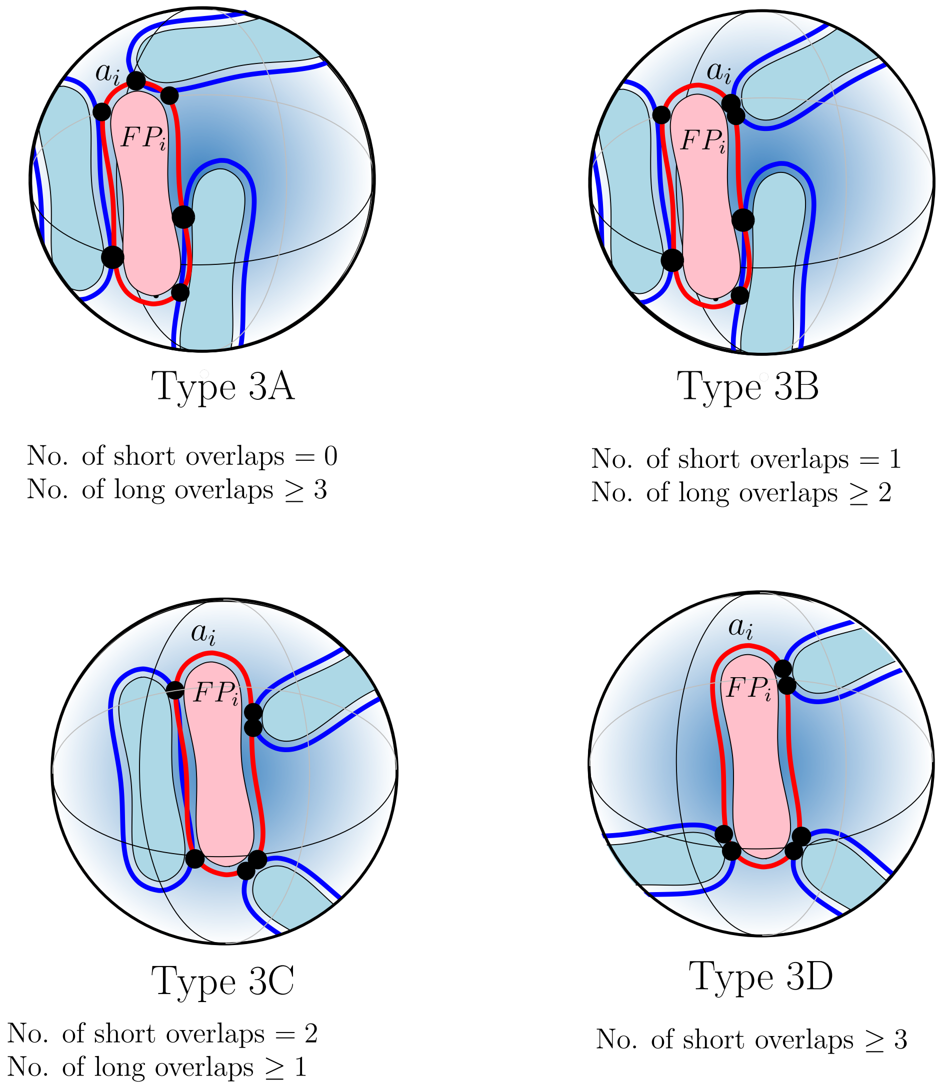


Figure 4.17: The boundary curve a_i is overlapped by at least three boundary curves

The following area is an estimation (lower bounds) for the total areas assigned to a Firsching cylinder c_i of Types 3A, 3B, 3C and 3D (Figure 4.17): Here we address the four sub cases separately.

Case of Type 3A: First we present our estimates for the case of $\frac{k}{2}$. Here we have at least three overlapping cylinders with long overlap, thus Lemmas 4.6, 4.8 and 4.9 give a lower bound $\frac{1}{2}\frac{6}{4}P = \frac{3}{4}P$ for the total area of gaps assigned to cylinder c_i.

Similarly if we change $k/2$ with $0.6k$ then we have the following lower bounds;

- at least an area of $\frac{1}{2} \cdot 6 \cdot 0.36P = 1.08P$ is assigned to the cylinder c_i for $0.6k$-neighbourhood

Case of Type 3B: First we present our estimates for the case of $\frac{k}{2}$. If there are at least three overlapping cylinders with long overlap, then Lemmas 4.6, 4.8 and 4.9 give a lower bound $\frac{1}{2} \cdot 7 \cdot \frac{1}{4}P$ for the total area of gaps assigned to cylinder c_i.

So assume that exactly two cylinders make long overlaps. This case is similar to the case of 2A. The only difference is that to the lower bound we can add $\frac{1}{2} \cdot 1 \cdot \frac{1}{4}P$, because of the contribution of the short overlapping cylinder, but subtract $5 \cdot \frac{1}{16}P$, because these two assigned discs can interfere with at most five small discs (see Lemma 4.5). So the lower bound of this sub case is $23 \cdot \frac{1}{16}P + \frac{1}{2} \cdot 4 \cdot \frac{1}{4}P + \frac{1}{2} \cdot 1 \cdot 0.25P - 5 \cdot \frac{1}{16}P = 1.75P$.

- the minimum of $0.875P$ and $1.75P$ is $0.875P$ so that this is the lower bound for case of $k/2$ -neighborhood

If we replace $1/2k$-neighbourhood with $0.6k$-neighbourhood than, it has similar argument. So that we have following lower bound $19 \cdot \frac{9}{100}P + \frac{1}{2} \cdot 4 \cdot 0.36P + \frac{1}{2} \cdot 1 \cdot 0.36P - 5 \cdot 0.09P = 2.34P$.

- the minimum of $\frac{1}{2} \cdot 7 \cdot 0.36 \cdot P = 1.26P$ and $2.34P$ is $1.26P$ so that this is the lower bound for case of $0.6k$ -neighbourhood

Case of Type 3C: First we present our estimates for the case of $\frac{k}{2}$. If there are at least two overlapping cylinders with long overlap, then Lemmas 4.6, 4.8 and 4.9 give a lower bound $\frac{1}{2}\frac{4+2}{4}P$ for the total area of gaps assigned to cylinder c_i.

So assume that exactly one cylinder which makes a long overlaps, this distributes $\frac{1}{2} \cdot 2 \cdot 0.25P$ and the two short overlaps distribute $\frac{1}{2} \cdot 2 \cdot 0.25P$. Also the amount of the small discs is $25 - 10$ so that the available area from them is $15 \cdot \frac{1}{16}P$. Then the total is $4 \cdot 0.25P + 15/16P = 1.9375P$.

- the minimum of $0.75P$ and $1.9375P$ is $0.75P$ so that this is the lower bound for case of $k/2$ -neighborhood

If we replace $1/2k$-neighbourhood with $0.6k$-neighbourhood than, it has similar argument. So that we have following lower bound $4 \cdot 0.36P + 15 \cdot 0.09P = 2.79P$.

- the minimum of $\frac{1}{2}(4+2)0.36P = 1.08P$ and $2.79P$ is $1.08P$ so that this is the lower bound for case of $0.6k$ -neighbourhood

Case of Type 3D: First we present our estimates for the case of $\frac{k}{2}$. If there is at most one overlapping cylinder with long overlap, then half of the ring is invaded by the other short overlap cylinders. So that the lower bound is $2 \cdot \frac{1}{2} \cdot \frac{1}{4}P + 3P - 5 \cdot 5 \cdot \frac{1}{16}P \approx 1.68P$ the total area of gaps assigned to cylinder c_i. So assume that there is no cylinder which makes a long overlap. This case is very similar to the Case 2C. Lemma 4.12 says that in case of short overlap, the union of the assigned discs cannot have common points with more then 5 of those discs of radii $\frac{k}{4}$ which we placed in the $\frac{k}{2}$-neighbourhood of cylinder c_i while proving Lemma 4.5. Lemma 4.5 says that the area of the ring is at least $6P$. Thus we can guarantee that at least an area of $6P - 6 \cdot \frac{1}{16}P = 4.125P$ is assigned to the cylinder c_i.

- the minimum of $1.68P$ and $4.125P$ is $1.68P$ so that this is the lower bound for case of $k/2$ -neighborhood

If we replace $1/2k$-neighbourhood with $0.6k$-neighbourhood than, it has similar argument. So that we have following lower bounds $2 \cdot \frac{1}{2} \cdot 0.36P + 3.6P - 5 \cdot 5 \cdot 0.09P = 1.71P$ and $7.2P - 6 \cdot 5 \cdot 0.09P = 4.5P$.

- the minimum of $1.71P$ and $4.5P$ is $1.71P$ so that this is the lower bound for case of $0.3k$ -neighborhood

Case	$0.5k$	$0.6k$
Type O	$11.34P$	$13.65P$
Type 1A	$5.67P$	$6.825P$
Type 1B	$5.67P$	$6.825P$
Type 2A	$1.9375P$	$2.43P$
Type 2B	$2.14P$	$2.635P$
Type 2C	$5.625P$	$6.66P$
Type 3A	$0.75P$	$1.08P$
Type 3B	$0.875P$	$1.26P$
Type 3C	$0.75P$	$1.08P$
Type 3D	$1.68P$	$1.71P$

Table 4.2: Lower bounds for $0.5k$ and $0.6k$-neighborhoods of the Firsching patch

Table 4.2 immediately implies the following theorem;

Theorem 4.13. *If seven Firsching cylinders touch a unit ball, a lower bound of the gap area on the surface of the sphere radius* $\sqrt{4.836}$ *is* $7.56P$.

Although theorem 4.13 gives fairly large lower bound but it is not enough to prove theorem 5.1. Remember we need $9.28P$ instead of $7.56P$.

Chapter 5

On Six Cylinders of Radii 1.049, Which are Tangnet to a Unit Ball

In this section we will prove a result which is weaker than Kuperberg's conjecture. He had several arrangements where 6 unit cylinders touched a unit sphere and he asked for showing that 7 unit cylinders cannot do the same. Recently, Firsching showed an arrangement of 6 larger tangent cylinders (each of radii $1.0496594\ldots$), and we set out showing that 7 such disjoint cylinders cannot touch a unit sphere. In a sense we wanted to settle Kuperberg's problem for cylinders used by Firsching.

Our main theorem will say,

Theorem 5.1. *At most 6 disjoint infinitely long cylinders of radii* 1.0496594 *can touch a unit ball .*

5.1 Step 1 of Proof of Theorem 5.1

This step is about choosing the right radius for the discs which we place in the gap around the Firsching patches.

In section 4, first we placed discs of radii $.5k$ and proved a relatively week lower bound for the total area of gaps. Then we raised the radius to $.6k$ and got a better bound. Raising the radius of the discs we put in the gaps, have a drawback. We had four lemmas addressing disjointness of Firsching discs. As the radii of the Firsching discs increase one, then two or more proofs of these lemmas will fail. The Lemmas still might be true, but their proof has to be changed. We did not see simple ways for improvement. Thus we had to compromise. We choose $0.724k$ for the radii, since that is the largest value when we can guarantee that the Firsching discs assigned to the pairs cylinders $\{c_i, c_j\}$ and $\{c_m, c_n\}$, where I, j, m, and n

are distinct integers, do not overlap each other, and use Lemma 4.7 instead of Lemma 4.6. Fortunately the other two lemmas adressing disjointness (Lemma 4.8 and 4.9 remained true.

5.2 Step 2 of Proof of Theorem 5.1

This step is about modifying the rules for how the area, of the Firsching disc will be assigned the Firsching patches.

We do not change the definition of the long and short overlaps. The new assignment will be dictated by Lemma 4.7 which says that the centers of two Firsching discs which are assigned to, say cylinder c_i but belong to two different pairs of cylinders might overlap each other. Fortunately we have a minimum for the area of their union. This allows us to say that each disc can give 1/4th of the area of the union (call it T, where $T = \frac{0.8838124575P}{4}P$) to each of the cylinders they are assigned to. After experimenting with the numerical values we saw that the number T will lead to the desired lower bound. Although we will have to deal with a couple of complications (for example there will be a need to find lower bounds for the union of three Firsching discs), it turns out that the following relatively simple rules for assigning areas of Firsching discs will do the job.

- If a Firsching disc is part of a short overlap, then it will not contribute to any of the cylinders.
- If a Firsching disc is part of a long overlap, then it will give an area equal to T to each of the cylinders.
- After careful disjointness analysis we will also assign portions of the Firsching rings to their Firsching patches.

As we progress and make the final analysis for establishing a lower bound for the total gap area, the case analysis still includes discs assigned to short overlaps. Although according to a) such discs will not contribute towards the total gap area. The knowledge of the their existence will tell us what portion of the ring area should be disregarded to avoid double assignments.

5.3 Step 3 of Proof of Theorem 5.1

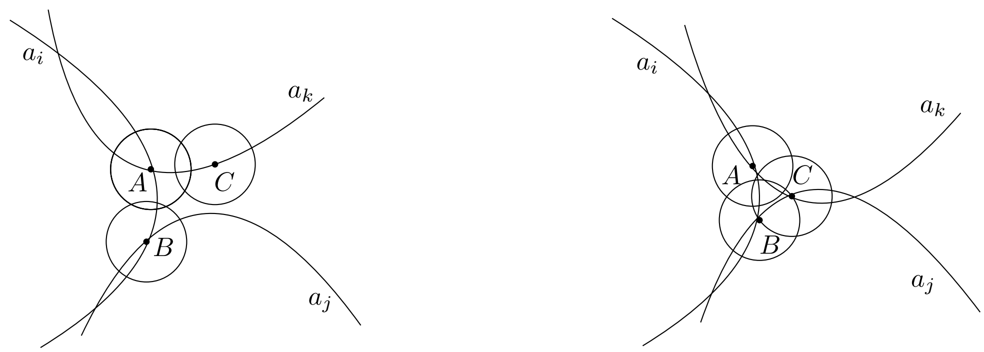
According to Step 2, some of the discs we place in the gaps might overlap each other. We say that finitely many Firsching discs form a cluster, if each of the discs was assigned to a long overlapping pair of cylinders, and their union form a maximal connected set. We need to know that the union of discs of a cluster have area large enough to allow the area distribution defined in Step 2. It turns out that the exact same proof, which was used in Lemma 4.8 shows that there is no cluster of more than 3 discs which are assigned to Firsching patches with long overlap. Again numerical values dictate a need of a lemma that the area of the union of three discs forming a cluster is at least 6 times $2T$. This way there is enough area for each participating disc to give an area T to the cylinders to which they were assigned. Step 4 and 5 contains this lemma.

5.4 Step 4 of Proof of Theorem 5.1

This Step is about the geometry of clusters of three discs. We will prove that

Lemma 5.2. *Consider three discs of radii* $0.724k$ *on the surface of the sphere of radius* $\sqrt{4.836}$ *such that the centers of these discs are at the intersections of the boundary curves of the* $0.724k$*-neighbourhoods of Firsching patches. Also assume that these discs were assigned to Firsching patches with long overlaps. We distinguish two cases according to Figure 5.1. In both cases, the area of the union of the three discs is at least* $2.5296 \cdot (0.724k)^2\pi \approx 6T$*.*

Proof Lemma 5.2 in Case 1. If $AB > 1.882k$, then by the triangular inequality we have $AC + AB > 1.882k$. When the centers of the three discs are collinear, then the minimum area of the union of discs occurs when the center of the middle disc has the same distance to the other two centers. This is true, because of the following observation: based on Figure



Case 1: $AB > 1.882k$ Case 2: $AB < 1.882k$

Figure 5.1: Overlap cases

5.2, indirect assume that the middle disc is not in symmetric position and its center is closer to the center of the disc to the right. When we move the middle disc from the right to the left, the area of the union is losses more, than what it gains.

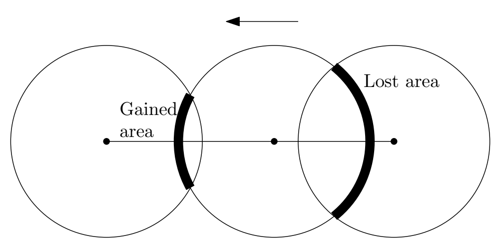


Figure 5.2: Minimum area of the union of three collinear discs

On the other hand, when distance between the left and right discs is $1.882k$ the area of the union of discs is $2.5296(0.724k)^2\pi$.

Proof Lemma 5.2 in Case 2. By minimizing the area of the union of three mutually overlapping spherical discs of radii $0.724k$ under the condition that the sides have length at least $\frac{1.882k}{2} = 0.941k$ is not going to give large enough area. We need to use more geometry of the cylinders to extract certain constraints for the cluster, and minimize the area of the union of the discs under these additional constraints. It turns out that the fact that the boundary curves of the Firsching patches belong to disjoint cylinders (each tangent to the unit ball) will imply that the all three heights of the spherical triangle formed by the centers of the discs are all greater than a certain constant. So Lemmas 5.3 and 5.4 give the proof of case 2. To be precise the following is true:

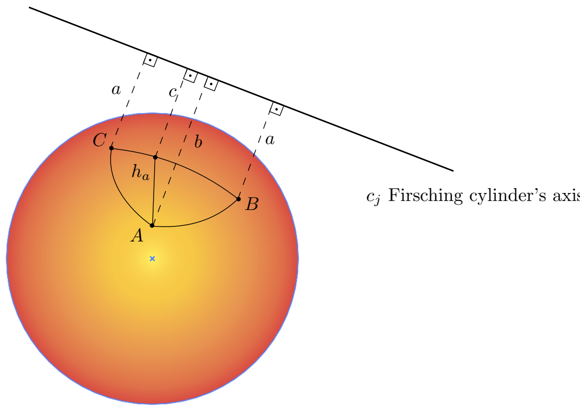


Figure 5.3: Centers of three discs of radius $0.724k$ on the sphere of radius $\sqrt{4.836}$

Lemma 5.3. *Let A, B, C be the centers of the discs d_{ij}, d_{jm} and d_{mi} on the surface of a the concentric sphere of radius R. Also assume that each two of the $0.72k$-neighbourhoods of the Firsching patches form a long overlap. Let AA', BB', CC' be the heights of the spherical triangle ABC. Let Let $a = r + 0.724k$ be the distance of the centers A, B from the axis of the Firsching cylinder c_j. Let b be the distance of the center C from the same axis. Then*

1. $b \geq (0.724 + 0.828)k + r$

2. *If every point of the side BC is at most at a distance a from the axis of the cylinder c_j, then the Euclidean distance $h_a = AA' > b - a$*

3. *No point of the side BC is further than $a + \epsilon$ from the axis of the cylinder c_j, where $\epsilon = 0.0212347208$ (Appendix F).*

4. *In general we have that the Euclidean distance $h_c = CC' > b - a - \epsilon = 0.1132182995$ (Appendix F)*

5. *If the Euclidean lengthes of the heights of the triangle ABC are listed in increasing order, i.e. $h_a < h_b < h_c$, then $h_a > 0.1132182995$, $h_b > 0.828k$ and $h_c > 0.828k$.*

Proof of Lemma 5.3:

Item 1 is best explained Figure 5.3, it was proved in Lemma 4.1 that the sum of distances from the point A to the axes of three Firsching cylinders c_i, c_j, and c_k is at least $3k+3r$;

$$(0.724k + r) + (0.724k + r) + b \geq 3k + 3r \ \Rightarrow \ b \geq 1.552k + r = (0.828k + 0.724)k + r$$

Item 2 is best explained using Figure 5.3. Figure 5.4 will help at deciding if indeed every point of the spherical segment BC has a distance to the axis of c_j at most a then $c \leq a$. On Figure 5.4, the angle of view is changed. We are looking at vertices B, C from the direction of the axis of c_j. The fact that if on this front view a point is inside of the circle (which is the cross-section of c_j) then its distance to the axis is less than a. First note that if both B, C are on the front hemisphere then the connecting great circular arc is inside of the projection of the cylinder. If B belongs to the front hemisphere and C to the back hemisphere then the connecting arc is not necessarily inside of the projection. Item 3 is about this situation. By the triangular in equality, we have

$$h_a + c \geq b \text{ so } h_a \geq b - c \geq b - a \ \Rightarrow \ h_a \geq b - a$$

.

item 3 is best explained Figure 5.7, the locations of points B and C are could be that one is on the front hemisphere and the other one is on the back hemisphere. So that the distance from the point of the side BC could be at most $a + \epsilon$. Where ϵ is calculated in Figure 5.5.

Item 4 is best explained Figure 5.6 and Figure 5.7. From item 3 we have that no point of BC is further than $a + \epsilon$ from the axis of the cylinder c_j so that the height h_c should be greater than $b - a - \epsilon = 0.828k - \epsilon$, that is 0.1132182995.

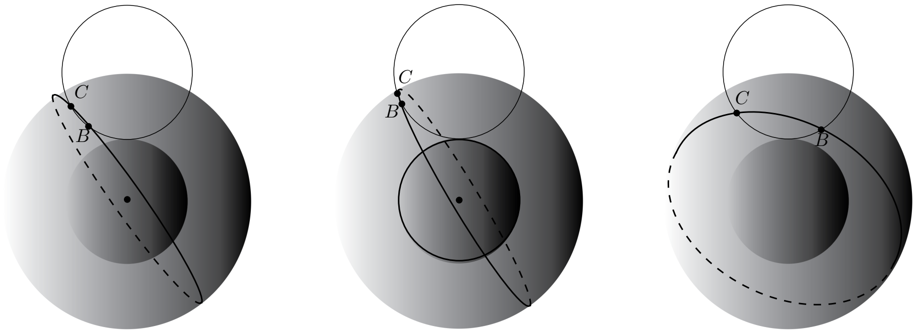
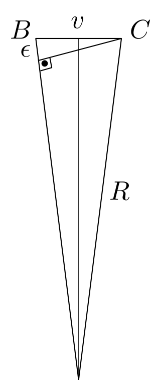


Figure 5.4: Both A and B belong to the front hemisphere on the right view



$$\frac{\epsilon}{v} = \frac{v/2}{R} \Rightarrow \epsilon = \frac{v^2}{2R}$$

Figure 5.5: Calculation of ϵ

Item 5 Since only one height could be 0.1132182995 and other two heights must be greater or equal to $0.828k$.

$\square$

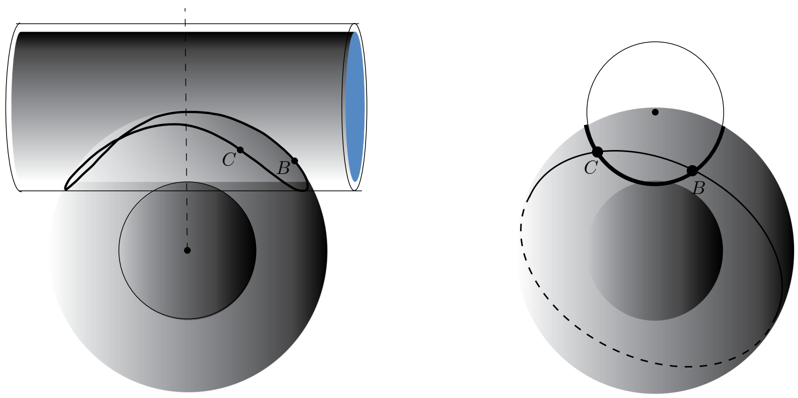


Figure 5.6: Right view shows the actual distances to the axis c_j.

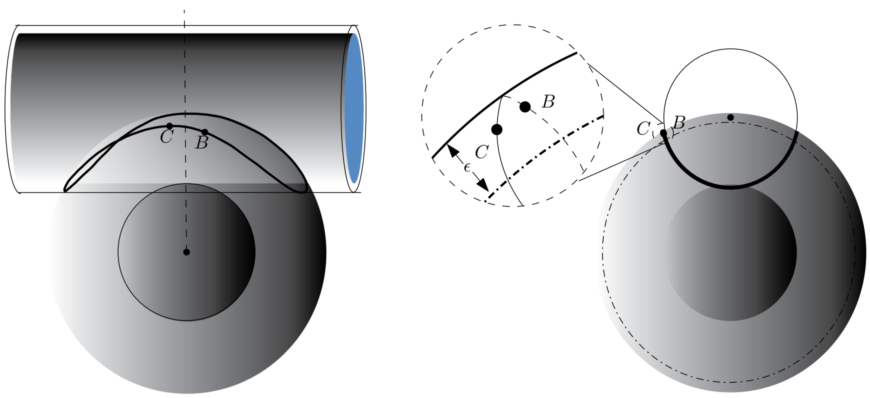


Figure 5.7: A belongs to the front, B belongs to the back hemisphere on the right view

The following is the estimate (lower bound) for the total area assigned to a Firsching cylinder c_i of Type O (Figure 5.11):

Lemma 5.4. *Consider three discs of radius* $0.724k$ *whose centers* A, B *and* C *are on the surface of the sphere of radius* $\sqrt{4.836}$. *Let* AA', BB' *and* CC' *be the heights of spherical triangle* ABC. *Assume the Euclidean distances* $h_a = AA' \geq 0.828k$, $h_b = BB' \geq 0.828k$ *and* $h_c = CC' \geq 0.1132182995$. *Then the area of union of the discs centered* A, B *and* C *is the smallest when* $h_A = 0.828k$, $h_B = 0.828k$ *and* $h_C = 0.1132182995$.

Proof Lemma 5.4 Consider three discs of radius $0.724k$ on the sphere of radius $R = \sqrt{4.836}$ in general case such that a formed Euclidean triangle ABC with the centers of the discs has all the heights more than given heights

B. Bollobas proved that if the distances of the centers of a set of discs continuously are decreased then the area of the union of discs decreases. [BB68].

Consider that the distances of the centers are continuously decreased, so some of the heights also will decrease. Consider that as the result of the continuously decreasing of the distances we have two cases as following;

Case 1: $h_A = 0.828k$, $h_B = 0.828k$ and $h_C > 0.1132182995$ or

Case 2: $h_A = 0.828k$, $h_B = 0.1132182995$ and $h_C > 0.828k$

So for both of the cases we will show that the minimum area of the union of discs occurs when the highs are $0.828k$, $0.828k$ and 0.1132182995.

For Case 1 follow the calculation: Here we use spherical trigonometry so the following values will be based on central angle and we convert the given heights in terms of central angle as follow

$$h_1 = 2\arcsin\left(\frac{0.0828k}{2R}\right) \quad \text{and} \quad r = 2\arcsin\left(\frac{0.724k}{2R}\right)$$

From the triangle ABD we have

$$\frac{\sin(c)}{\sin(\pi/2)} = \frac{\sin(h_1)}{\sin(x)} \;\Rightarrow\; \sin(c) = \frac{\sin(h_1)}{\sin(x)} \text{ so that } c = \arcsin\left(\frac{\sin(h_1)}{\sin(x)}\right) \tag{1}$$

From the triangle ABE we have

$$\frac{\sin(a/2)}{\sin(x/2)} = \frac{\sin(c)}{\sin(\pi/2)} = \sin(c) \text{ we know from equation } (\mathbf{1}) \text{ that } \sin(c) = \frac{\sin(h_1)}{\sin(x)}$$

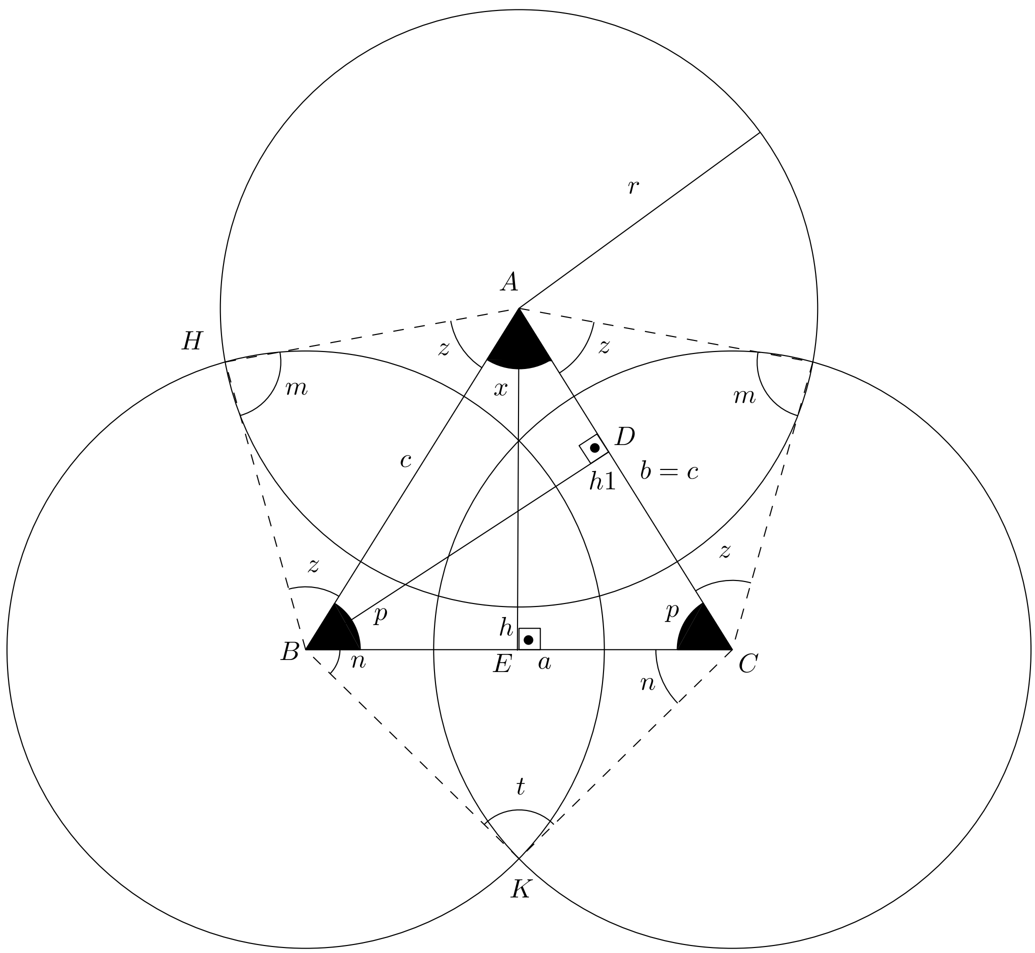


Figure 5.8: The area of the union of three discs - Case 1

so that we have

$$\frac{\sin(a/2)}{\sin(x/2)} = \frac{\sin(h_1)}{\sin(x)} = \frac{\sin(h_1)}{2\sin(x/2)\cos(x/2)} \Rightarrow a = 2\arcsin\left(\frac{\sin(h)}{2\cos(x/2)}\right) \qquad (\mathbf{2})$$

By using the spherical law of cosine to triangle ABE we have

$$\cos(c) = \cos(a/2)\cos(h) + \sin(a/2)\sin(h)\cos(\pi/2) \Rightarrow h = \arccos\left(\frac{\cos(c)}{\cos(a/2)}\right) \qquad (\mathbf{3})$$

When we substitute value of c and a from equation (**1**) and (**2**) we have

$$h = \arccos\left(\frac{\cos\left(\arcsin\left(\frac{\sin(h_1)}{\sin(x)}\right)\right)}{\cos\left(\arcsin\left(\frac{\sin(h_1)}{2\cos(x/2)}\right)\right)}\right) \qquad (\mathbf{4})$$

By using the spherical law of cosine to triangle AHB we can find angle z and m as following;

$$\cos(r) = \cos(r)\cos(c) + \sin(r)\sin(c)\cos(z) \;\Rightarrow\; z = \arccos\left(\frac{\cos(r) - \cos(r)\cos(c)}{\sin(r)\sin(c)}\right) \qquad (\mathbf{5})$$

$$\cos(c) = \cos^2(r) + \sin^2(r)\cos(m) \;\Rightarrow\; m = \arccos\left(\frac{\cos(c) - \cos^2(r)}{\sin^2(r)}\right) \qquad (\mathbf{6})$$

Similarly, by using the spherical law of cosine to triangle BKC we can find angle n and t as following;

$$\cos(r) = \cos(r)\cos(a) + \sin(r)\sin(a)\cos(n) \;\Rightarrow\; n = \arccos\left(\frac{\cos(r) - \cos(r)\cos(a)}{\sin(r)\sin(a)}\right) \qquad (\mathbf{7})$$

$$\cos(a) = \cos^2(r) + \sin^2(r)\cos(t) \;\Rightarrow\; t = \arccos\left(\frac{\cos(a) - \cos^2(r)}{\sin^2(r)}\right) \qquad (\mathbf{8})$$

By using the spherical law of cosine to triangle ABC we can find the angle p as following;

$$\cos(c) = \cos(a)\cos(c) + \sin(a)\sin(c)\cos(p) \;\Rightarrow\; p = \arccos\left(\frac{\cos(c) - \cos(a)\cos(c)}{\sin(a)\sin(c)}\right) \qquad (\mathbf{9})$$

Then the area of the union of the discs is

$$\begin{aligned} S =& ((m + 2z - \pi) + (m + 2z - \pi) + (t + 2n - \pi) + (x + 2p - \pi))R^2 \\ &+ \left(\frac{2\pi - (2z + x)}{2\pi} + \frac{2\pi - (z + p + n)}{2\pi} + \frac{2\pi - (z + p + n)}{2\pi}\right)(r^2)\pi \end{aligned} \qquad (\mathbf{10})$$

For Case 2, lets follow the calculation:

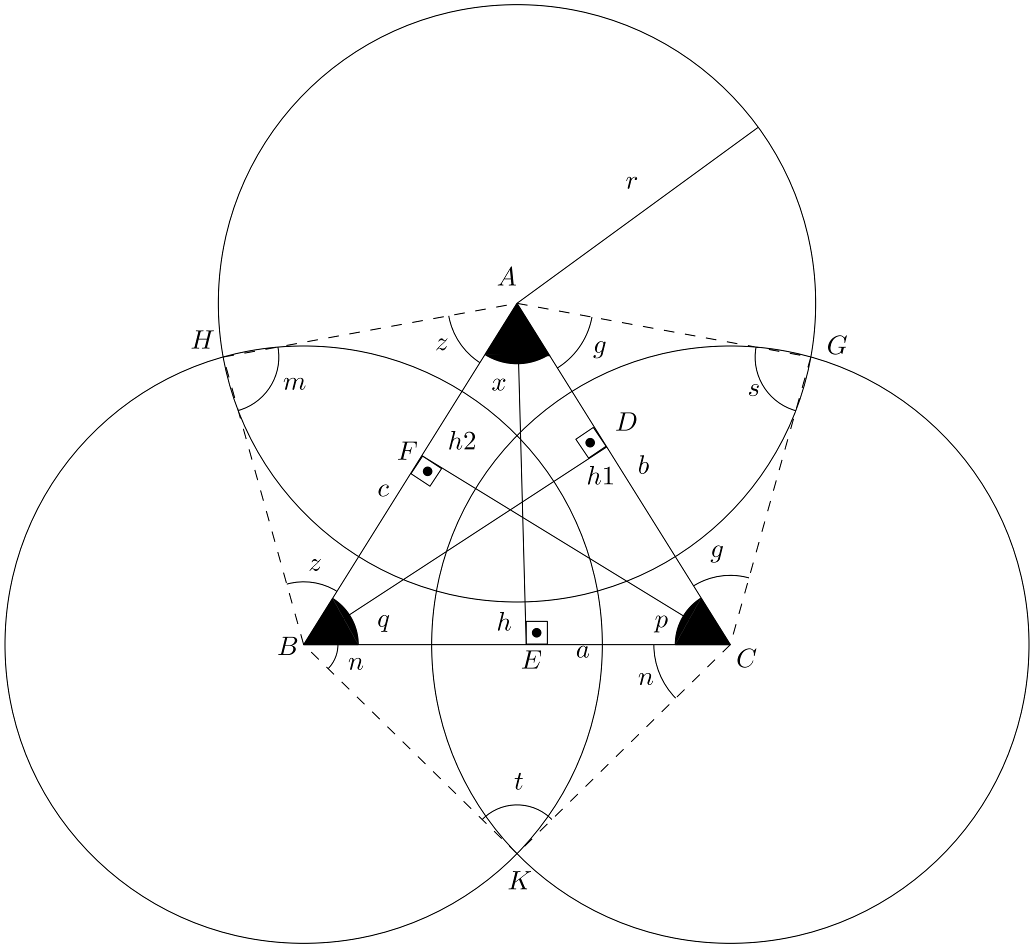


Figure 5.9: The area of the union of three discs - Case 2

$$h_1 = 2\arcsin\left(\frac{0.0828k}{2R}\right), \quad h_2 = 2\arcsin\left(\frac{0.1132182995}{2R}\right) \quad \text{and} \quad r = 2\arcsin\left(\frac{0.724k}{2R}\right)$$

From the triangle ABD we have

$$\frac{\sin(c)}{\sin(\pi/2)} = \frac{\sin(h_1)}{\sin(x)} \;\Rightarrow\; \sin(c) = \frac{\sin(h_1)}{\sin(x)} \text{ so that } c = \arcsin\left(\frac{\sin(h_1)}{\sin(x)}\right) \qquad (\mathbf{1})$$

From the triangle ACF we have

$$\frac{\sin(b)}{\sin(\pi/2)} = \frac{\sin(h_2)}{\sin(x)} \;\Rightarrow\; \sin(b) = \frac{\sin(h_2)}{\sin(x)} \text{ so that } b = \arcsin\left(\frac{\sin(h_2)}{\sin(x)}\right) \qquad (\mathbf{2})$$

From the triangle ABC we have

$$\cos(a)=\cos(b)\cos(c)+\sin(b)\sin(c)\cos(x) \text{ so that } a=\arccos(\cos(b)\cos(c)+\sin(b)\sin(c)\cos(x)) \quad (\mathbf{3})$$

By using the spherical law of cosine to triangle ABC we have

$$\cos(b)=\cos(a)\cos(c)+\sin(a)\sin(c)\cos(q) \;\Rightarrow\; q=\arccos\left(\frac{\cos(b)-\cos(a)\cos(c)}{\sin(a)\sin(c)}\right) \quad (\mathbf{4})$$

By applying the spherical law of sine to triangle ABE we have

$$\frac{\sin(h)}{\sin(q)}=\frac{\sin(c)}{\sin(\pi/2)} \quad\Rightarrow\quad h=\arcsin(\sin(q)\sin(c)) \quad (\mathbf{5})$$

By using the spherical law of cosine to triangle AHB we can find the angles z and m as following;

$$\cos(r)=\cos(r)\cos(c)+\sin(r)\sin(c)\cos(z) \;\Rightarrow\; z=\arccos\left(\frac{\cos(r)-\cos(r)\cos(c)}{\sin(r)\sin(c)}\right) \quad (\mathbf{6})$$

$$\cos(c)=\cos^2(r)+\sin^2(r)\cos(m) \;\Rightarrow\; m=\arccos\left(\frac{\cos(c)-\cos^2(r)}{\sin^2(r)}\right) \quad (\mathbf{7})$$

Similarly, by using the spherical law of cosine to triangle BKC we can find the angles n and t as following;

$$\cos(r)=\cos(r)\cos(a)+\sin(r)\sin(a)\cos(n) \;\Rightarrow\; n=\arccos\left(\frac{\cos(r)-\cos(r)\cos(a)}{\sin(r)\sin(a)}\right) \quad (\mathbf{8})$$

$$\cos(a)=\cos^2(r)+\sin^2(r)\cos(t) \;\Rightarrow\; t=\arccos\left(\frac{\cos(a)-\cos^2(r)}{\sin^2(r)}\right) \quad (\mathbf{9})$$

Similarly, by using the spherical law of cosine to triangle AGC we can find the angles g and s as following;

$$\cos(r) = \cos(r)\cos(b) + \sin(r)\sin(b)\cos(g) \;\Rightarrow\; g = \arccos\left(\frac{\cos(r) - \cos(r)\cos(b)}{\sin(r)\sin(b)}\right) \qquad \mathbf{(10)}$$

$$\cos(b) = \cos^2(r) + \sin^2(r)\cos(s) \;\Rightarrow\; t = \arccos\left(\frac{\cos(b) - \cos^2(r)}{\sin^2(r)}\right) \qquad \mathbf{(11)}$$

By using the spherical law of cosine to triangle ABC we can find the angle p as following;

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(p) \;\Rightarrow\; p = \arccos\left(\frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}\right) \qquad \mathbf{(12)}$$

Then, the area of the union of the discs is

$$\begin{aligned} S =& ((m + 2z - \pi) + (s + 2g - \pi) + (t + 2n - \pi) + (x + p + q - \pi))R^2 \\ &+ \left(\frac{2\pi - (z + x + g)}{2\pi} + \frac{2\pi - (g + p + n)}{2\pi} + \frac{2\pi - (z + q + n)}{2\pi}\right)(r^2)\pi \qquad \mathbf{(13)} \end{aligned}$$

In cases 1 and 2, we have the height h and the area of the union of discs S in terms of the angle x. We want to show that when the height decreases then the area decreases as well. We show that fact by following observation:

Let's show the height h and the area S as functions of x, so $h = f(x)$ and $S = g(x)$. Then $S(h) = g(f^{-1}(h))$ so

$$\frac{dS(h)}{dh} = \frac{dg}{dx}\Big|_{x=f^{-1}(h)} \frac{df^{-1}(h)}{dh}$$

We know that $\dfrac{df^{-1}(h)}{dh} = \dfrac{1}{f'(f^{-1}(h))}$ and since $f^{-1}(h) = x$ then

$$\frac{dS}{dh} = g^{'}(x)\frac{1}{f'(x)}$$

From appendices D and E we see that h and S decreasing with respect to x on the same domain. That means for each x dS/dh is a ratio of two negative numbers, therefore $S(h)$ is an increasing function.

$\square$

Lemma 5.5. *The area of the union of discs which is described in Lemma 5.4 is more than* $2.67(0.724k)^2\pi$.

Proof of Lemma 5.5, suppose that we have a triangle ABC whose all heights are equal to 0.1132182995. Consider Figure 5.10, and apply spherical law of cosines to triangle ABD; we

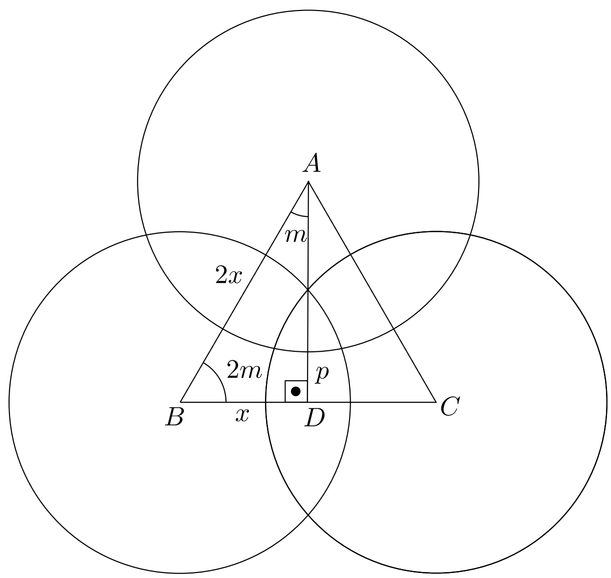


Figure 5.10: The boundary curve a_i is not overlapped by any other boundary curve

have

$$\cos 2x = \cos x \cos h \;\Rightarrow\; 2\cos^2 x - \cos x \cos h - 1 = 0$$

From here we have the values of $\cos x$ as 0.99955582956 and -0.50022044498, but the value of $\cos x$ must be positive because x is between 0 and $\pi/2$. Then by applying spherical law of sinus to triangle ABD we have

$$\frac{\sin m}{\sin x} = \frac{\sin \pi/2}{\sin 2x} \;\Rightarrow\; \sin m = \frac{1}{2\cos x}$$

So that the area of the union of discs is

$$A = R^2(6m - \pi) + 3\left(\frac{2\pi - 2m}{2\pi}\right)(0.724k)^2\pi \approx 2.67(0.724k)^2\pi$$

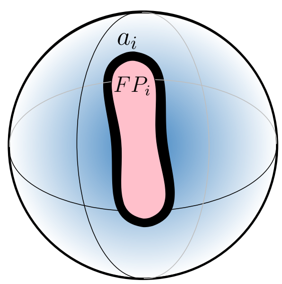
Calculation of the area is given in Appendix F.

□

So by Lemma 5.5, proof of Lemma 5.2 is completed.

5.5 A Lower bound for the area of the $0.724k$ neighbourhooods of Firsching Patches

The following estimations give us lower bounds for $0.724k$-neighbourhood of a Firsching Patch.

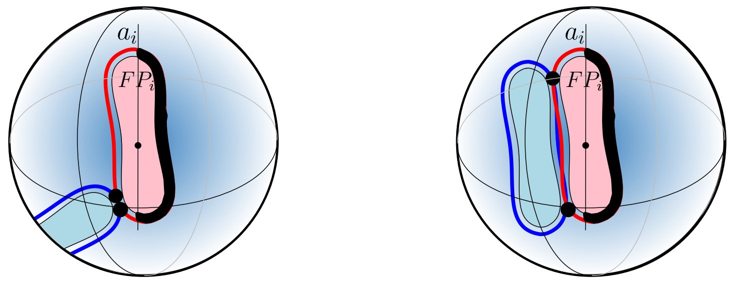


Type O

Figure 5.11: The boundary curve a_i is not overlapped by any other boundary curve

The entire area between the Firsching patch and the curve a_i is assigned to the cylinder c_i According to Lemma 4.5 this area is at least $8.6P$.

We get larger lower bounds if we compute the area of this neighbourhood (also called ring) with Maple code then this area turns out to be at least $16.54P$.



Type 1A

No. of short overlaps = 1
No. of long overlaps = 0

Type 1B

No. of short overlaps = 0
No. of long overlaps = 1

Figure 5.12: The boundary curve a_i is overlapped by exactly one other boundary curve

The lower bound for the total area assigned to a Firsching cylinder c_i of Types 1A and 1B (Figure 5.12):

Lemma 4.4 says that half of the $0.724k$-neighbourhood of the Firsching patch can be assigned to cylinder c_i. At first it looks like that in order to improve the lower bound in case of the short overlap (Type 1A) the discs do not contribute any area to the patches, in case of long overlap (Type 1B) we could add $2 \cdot \frac{0.88381245 75P}{4}$ according to our assignment rules.

However notice that we cannot guarantee that the discs which are assigned to the overlapping pair of cylinders do not overlap the disjoint "half ring" (shaded region on the figure) so we do not add this area. Thus at least an area $8.27P$ is assigned to cylinder c_i.

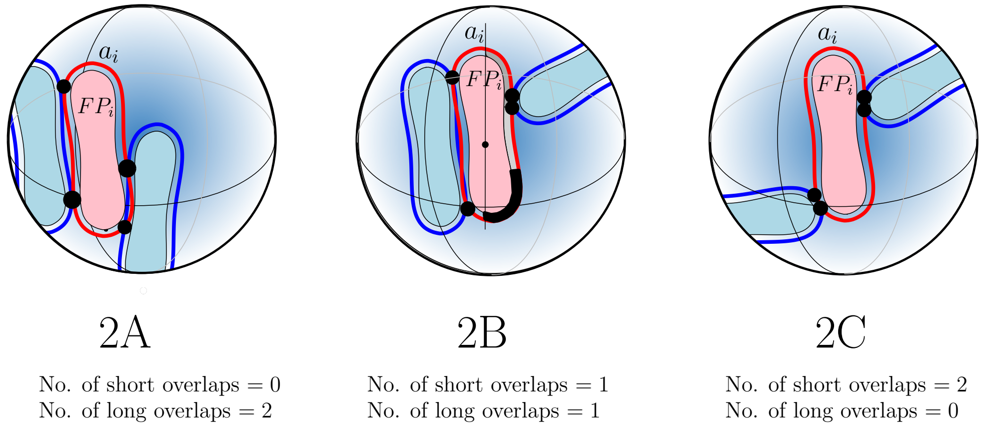


Figure 5.13: The boundary curve a_i is overlapped by exactly two other boundary curves

The following is the estimate (lower bound) for the total area assigned to a Firsching cylinder c_i of Types 2A, 2B and 2C (Figure 5.13):

Case of Type 2A: Let c_j be the cylinder which gives the long overlap. According to Lemma 4.11 at least 15 of $0.362k$ radius discs can fit in the $0.724k$ neighbourhood. Each discs contributes an area $\frac{0.88381245 75P}{4}$. So the lower bound for the total area of gap assigned to c_i

$$15 \cdot (0.362)^2 P + 4 \cdot \frac{0.88381245765P}{4} = 3.64566P.$$

Case of Type 2B: Let c_j be the cylinder which gives the long overlap. Lemma 4.4 says that half of the $0.724k$-neighbourhood of the Firsching patch is disjoint from overlapping Firsching neighbourhood. Divide this half ring into three equal area parts along a_i. The two discs assigned to the pair with short overlap cannot invade all three of these parts thus $\frac{1}{6}$ of the area of the ring can be assigned to c_i. Notice that we can improve this lower bound by adding $\frac{0.8838124575P}{4}$, this is coming from the one of the two assigned long overlap discs.

So that we have the following lower bound for the total area of gap assigned to c_i (Appendix C) so

$$\frac{0.8838124575P}{4} + 1/6\text{Full Ring} \approx 1.65P.$$

Case of Type 2C: Lemma 4.12 says that in case of short overlap, the union of the assigned discs cannot have common points with more than 5 of those discs of radii $0.362k$ which we placed in the $0.724k$-neighbourhood of cylinder c_i while proving Lemma 4.5. Lemma 4.5 says that the area of the ring is at least $8.6P$. Thus we can guarantee that

$$8.6P - 10 \cdot (0.362)^2 P \approx 7.28P$$

is assigned to the cylinder c_i.

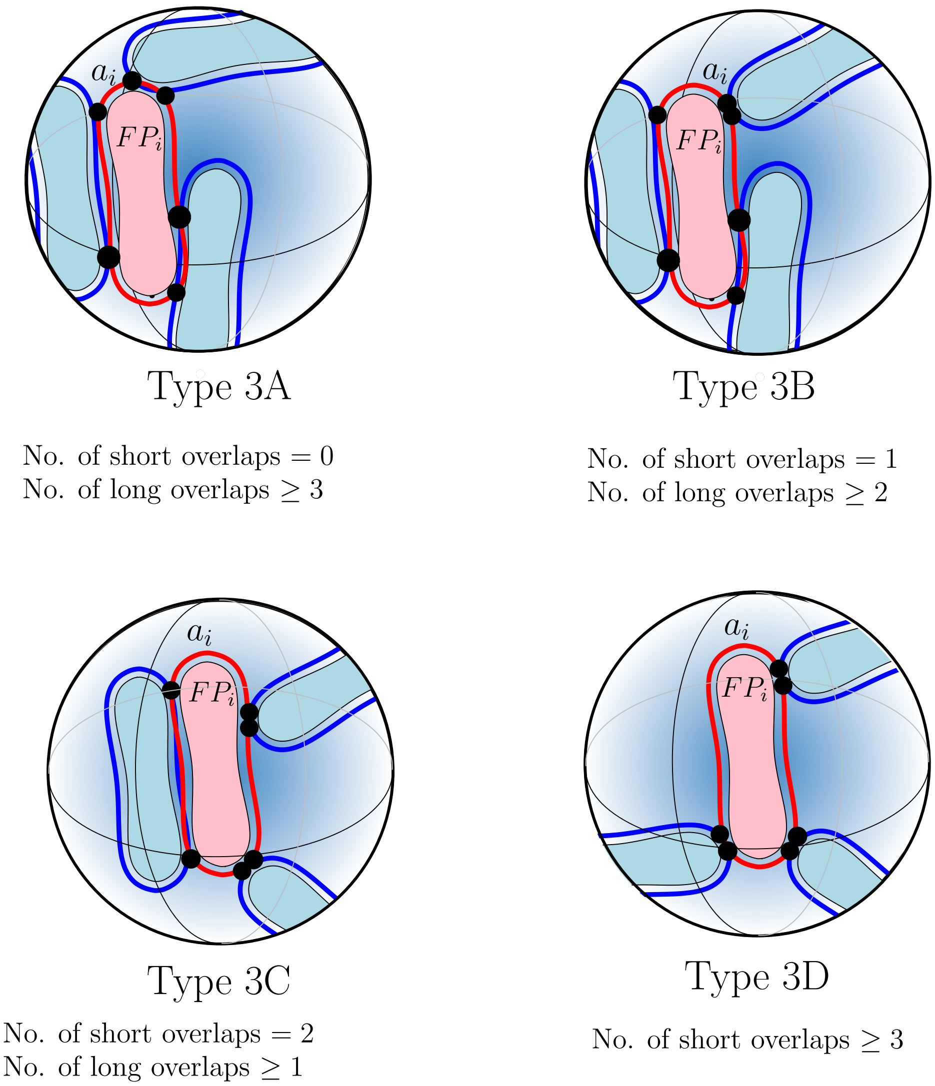


Figure 5.14: The boundary curve a_i is overlapped by at least three boundary curves

The following are the estimates (lower bounds) for the total areas assigned to a Firsching cylinder c_i of Types 3A, 3B, 3C and 3D (Figure 5.14): Here we address the four sub cases separately.

Case of Type 3A: Each of the discs contributes $\frac{0.8838124575P}{4}$ area to the patch so that at least an area of

$$6 \cdot \frac{0.8838124575P}{4} = 1.3257186863P$$

is assigned to the cylinder c_i .

Case of Type 3B: Here each long overlap discs contribute $\frac{0.8838124575P}{4}$ and the short overlap we do not add anything but subtract $5 \cdot \frac{1}{16}P$, these two assigned discs can interfere with at most five small discs (see Lemma 4.5). So that we have $17 - 5$ small discs, that makes $12 \cdot (0.362)^2 P$. So that at least an area of

$$4 \cdot \frac{0.8838124575P}{4} + 12 \cdot (0.362)^2 P = 2.4563P$$

is assigned to the cylinder c_i.

Case of Type 3C: Here we consider these case under two sub cases: Case 1: We have two long overlaps and two short overlaps. Each of two overlaps discs contributes an area $\frac{0.8838124575P}{4}$ and for each short overlaps, we substract an area $5 \cdot (0.362)^2 P$. So that the assigned area is at least

$$4 \cdot \frac{0.8838124575P}{4} + (17 - 10)(0.362)^2 P = 1.8011P$$

Case 2: We have one long overlap and two short over laps, by similar argument of Case 1, the assigned area is at least

$$2 \cdot \frac{0.8838124575P}{4} + (17 - 10)(0.362)^2 P = 1.3592P$$

Case of Type 3D: In this case we assume that there is no cylinder which makes a long overlap. Lemma 4.12 says that in case of short overlap, the union of the assigned discs cannot have common points with more then 5 of those discs of radii $0.362k$ which we placed in the $0.724k$-neighbourhood of cylinder c_i while proving Lemma 4.5. Lemma 4.5 says that the area of the ring is at least $8.6P$. If we consider the worst case that there are six short overlaps than we guarantee that at least an area of $8.6P - 30 \cdot (0.362)^2 P = 4.66868P$ is assigned to the cylinder c_i.

Case	$0.724k$
Type O	$16.54P$
Type 1A	$8.27P$
Type 1B	8.27
Type 2A	$3.64P$
Type 2B	$1.65P$
Type 2C	$7.28P$
Type 3A	$1.3257186863P$
Type 3B	$2.45P$
Type 3C	$1.80P$
Type 3D	$4.66P$

Table 5.1: A lower bound for the sum of the gaps in $0.724k$-neighbourhood of the Firsching patch

Proof of Theorem 5.1 : According to Lemma 4.2, it is enough to show that $9.28P$ is a lower bound of gap between Firsching patches on surface of the concentric sphere radius $\sqrt{4.836}$. The last column of the table constitute for a proof. Indeed seven times of the smallest value of that column is over all lower bound of that gap area.

$$7 \cdot 1.3257186863P = 9.2800308041P$$

Since this number is greater than the needed gap $9.28P$, the proof is complete. $\square$

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Appendices

Appendix A

Calculation of the Last Row of Theorem 3.3

> $R := 3.14049555$

$$R := 3.14049555 \quad \textbf{(1)}$$

> $r := 0.591478922$

$$r := 0.591478922 \quad \textbf{(2)}$$

> $y_1 := -\mathrm{sqrt}\left(R - x^2 - \left(\frac{R - x^2 + 2\,r + 1}{2 + 2\,r}\right)^2\right)$

$$y_1 := -\sqrt{3.14049555 - x^2 - (-0.3141731839\,x^2 + 1.672486302)^2} \quad \textbf{(3)}$$

> $y_2 := \mathrm{sqrt}\left(R - x^2 - \left(\frac{R - x^2 + 2\,r + 1}{2 + 2\,r}\right)^2\right)$

$$y_2 := \sqrt{3.14049555 - x^2 - (-0.3141731839\,x^2 + 1.672486302)^2} \quad \textbf{(4)}$$

> $x_1 := -\mathrm{sqrt}(R - 1)$

$$x_1 := -1.463043250 \quad \textbf{(5)}$$

> $x_2 := \mathrm{sqrt}(R - 1)$

$$x_2 := 1.463043250 \quad \textbf{(6)}$$

> $A := evalf\left(Int\left(\left(\mathrm{sqrt}(R)\cdot\arcsin\left(\frac{y_2}{\mathrm{sqrt}(R - x^2)}\right) - \mathrm{sqrt}(R)\cdot\arcsin\left(\frac{y_1}{\mathrm{sqrt}(R - x^2)}\right)\right), x = x_1 ..x_2, method = \_d01ajc\right)\right)$

$$A := 3.582911289 \quad \textbf{(7)}$$

> $k := \dfrac{r\cdot\left(1 - \cos\left(\frac{\mathrm{Pi}}{6}\right)\right)}{\cos\left(\frac{\mathrm{Pi}}{6}\right)}$

$$k := 0.3943192813\left(1 - \frac{\sqrt{3}}{2}\right)\sqrt{3} \quad \textbf{(8)}$$

> $k1 := evalf(k)$

$$k1 := 0.09150210756 \quad \textbf{(9)}$$

> $cap := \mathrm{Pi}\cdot k1^2$

$$cap := 0.02630341077 \quad \textbf{(10)}$$

> $rate1 := \dfrac{11\,A + 2\cdot cap}{4\cdot\mathrm{Pi}\cdot R}$

$$rate1 := 1.000000000 \quad \textbf{(11)}$$

>

Appendix B

Proofs of Lemma 4.2 and Lemma 4.3

> $R := 4.836$

$$R := 4.836 \tag{1}$$

> $r := 1.0496594$

$$r := 1.0496594 \tag{2}$$

> $y_1 := -\mathrm{sqrt}\left(R - x^2 - \left(\frac{R - x^2 + 2\,r + 1}{2 + 2\,r}\right)^2\right)$

$$y_1 := -\sqrt{4.836 - x^2 - (-0.2439429693\,x^2 + 1.935765230)^2} \tag{3}$$

> $y_2 := \mathrm{sqrt}\left(R - x^2 - \left(\frac{R - x^2 + 2\,r + 1}{2 + 2\,r}\right)^2\right)$

$$y_2 := \sqrt{4.836 - x^2 - (-0.2439429693\,x^2 + 1.935765230)^2} \tag{4}$$

> $x_1 := -\mathrm{sqrt}(R - 1)$

$$x_1 := -1.958570908 \tag{5}$$

> $x_2 := \mathrm{sqrt}(R - 1)$

$$x_2 := 1.958570908 \tag{6}$$

> $FP := \mathit{evalf}\left(\mathit{Int}\left(\left(\mathrm{sqrt}(R)\cdot\arcsin\left(\frac{y_2}{\mathrm{sqrt}(R - x^2)}\right) - \mathrm{sqrt}(R)\cdot\arcsin\left(\frac{y_1}{\mathrm{sqrt}(R - x^2)}\right)\right), x = x_1 \,..x_2, \mathit{method} = \_d01ajc\right)\right)$

$$FP := 8.571747969 \tag{7}$$

> $k := \dfrac{r\cdot\left(1 - \cos\left(\frac{\mathrm{Pi}}{6}\right)\right)}{\cos\left(\frac{\mathrm{Pi}}{6}\right)}$

$$k := 0.6997729333\left(1 - \frac{\sqrt{3}}{2}\right)\sqrt{3} \tag{8}$$

> $P := \mathrm{Pi}\cdot k^2$

$$P := 4.615145614\left(1 - \frac{\sqrt{3}}{2}\right)^2 \tag{9}$$

> $\mathit{evalf}(P)$

$$0.08283813644 \tag{10}$$

> $\mathit{evalf}(k)$

$$0.1623828741 \tag{11}$$

> $N := \dfrac{-7\cdot FP + 4\,\mathrm{Pi}\cdot R}{P}$

$$N := \frac{0.1665673381}{\left(1 - \frac{\sqrt{3}}{2}\right)^2} \tag{12}$$

> $evalf(N)$

$$9.279934976 \tag{13}$$

> $\text{rho} := \frac{k}{2}$

$$\rho := 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3} \tag{14}$$

> $q := r + \text{rho}$

$$q := 1.0496594 + 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3} \tag{15}$$

> $y_1 := -\text{sqrt}\left(R - x^2 - \left(\frac{R - q^2 + r^2 + 1 - x^2 + 2r}{2+2r}\right)^2\right)$

$$y_1 := -\left(4.836 - x^2 - \left(-0.2439429693\left(1.0496594 + 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3}\right)^2 + 2.204537900 - 0.2439429693\,x^2\right)^2\right)^{1/2} \tag{16}$$

> $y_2 := \text{sqrt}\left(R - x^2 - \left(\frac{R - q^2 + r^2 + 1 - x^2 + 2r}{2+2r}\right)^2\right)$

$$y_2 := \left(4.836 - x^2 - \left(-0.2439429693\left(1.0496594 + 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3}\right)^2 + 2.204537900 - 0.2439429693\,x^2\right)^2\right)^{1/2} \tag{17}$$

> $x_1 := -\text{sqrt}\left(R - (1 - (\text{rho}))^2\right)$

$$x_1 := -\sqrt{4.836 - \left(1 - 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3}\right)^2} \tag{18}$$

> $x_2 := \text{sqrt}\left(R - (1 - (\text{rho}))^2\right)$

$$x_2 := \sqrt{4.836 - \left(1 - 0.3498864666\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{3}\right)^2} \tag{19}$$

> $FA := evalf\left(Int\left(\left(\text{sqrt}(R)\cdot\arcsin\left(\frac{y_2}{\text{sqrt}(R-x^2)}\right) - \text{sqrt}(R)\cdot\arcsin\left(\frac{y_1}{\text{sqrt}(R-x^2)}\right)\right), x = x_1 .. x_2, method = \_d01ajc\right)\right)$

$$FA := 9.511813817 \tag{20}$$

> $P1 := \frac{FA - FP}{P}$

$$P1 := \frac{0.2036914816}{\left(1-\frac{\sqrt{3}}{2}\right)^2} \tag{21}$$

> $evalf(P1)$

$$11.34822544 \tag{22}$$

Appendix C

Proof of Lemma 4.7

> $R := \mathrm{sqrt}(4.836)$

$$R := 2.199090721 \tag{1}$$

> $k := 0.1623828741$

$$k := 0.1623828741 \tag{2}$$

> $s := 0.724$

$$s := 0.724 \tag{3}$$

> $x := 3(1-s)$

$$x := 0.828 \tag{4}$$

> $a := \arcsin\left(\frac{\frac{x}{2}k}{R}\right)$

$$a := 0.03057490287 \tag{5}$$

> $b := 2\arcsin\left(\frac{\frac{s}{2}k}{R}\right)$

$$b := 0.05346719194 \tag{6}$$

> $\mathrm{alpha} := \arcsin\left(\frac{\sin(a)}{\sin(b)}\right)$

$$\alpha := 0.6089757098 \tag{7}$$

> $\mathrm{beta} := \arcsin\left(\frac{\sin(b)\cdot\sin(2\,\mathrm{alpha})}{\sin(2\,a)}\right)$

$$\beta := 0.9624914271 \tag{8}$$

> $A := 2\cdot R^2\cdot(2\,\mathrm{alpha}+2\,\mathrm{beta}-\mathrm{Pi})+2\left(\frac{2\cdot\mathrm{Pi}-2\cdot\mathrm{beta}}{2\cdot\mathrm{Pi}}\right)\cdot(s\cdot k)^2\mathrm{Pi}$

$$A := 0.07321337700 \tag{9}$$

> $rate := \frac{A}{k^2\cdot\mathrm{Pi}}$

$$rate := 0.8838124575 \tag{10}$$

> $rate1 := \frac{A}{(s\,k)^2\mathrm{Pi}}$

$$rate1 := 1.686098673 \tag{11}$$

>

Appendix D

Calculation of the Area of the Union of Three Discs- Case1

> $R := \mathrm{sqrt}(4.836)$

$$R := 2.199090721 \quad \textbf{(1)}$$

> $k := 0.1623$

$$k := 0.1623 \quad \textbf{(2)}$$

> $r1 := 0.724 \cdot k$

$$r1 := 0.1175052 \quad \textbf{(3)}$$

> $r := 2 \arcsin\left(\frac{r1}{2 \cdot R}\right)$

$$r := 0.05343989780 \quad \textbf{(4)}$$

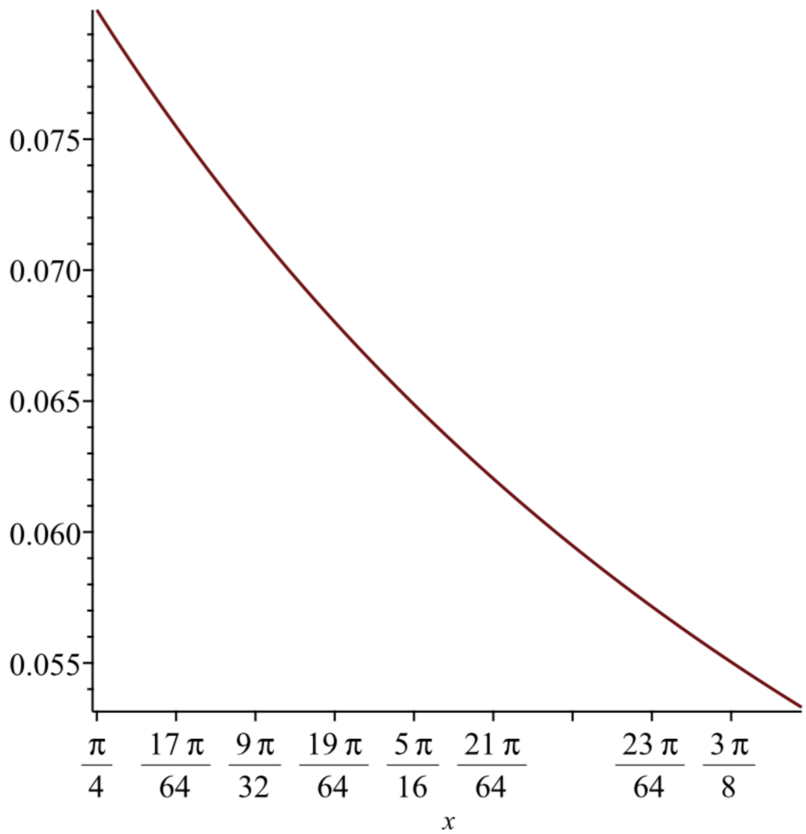
> $h1 := 2 \arcsin\left(\frac{0.828 \cdot k}{2 \cdot R}\right)$

$$h1 := 0.06111858746 \quad \textbf{(5)}$$

> $h := \arccos\left(\frac{\cos\left(\arcsin\left(\frac{\sin(h1)}{\sin(x)}\right)\right)}{\cos\left(\arcsin\left(\frac{\sin(h1)}{2\cos\left(\frac{x}{2}\right)}\right)\right)}\right)$

$$h := \arccos\left(\frac{\sqrt{1 - \frac{0.003730832775}{\sin(x)^2}}}{\sqrt{1 - \frac{0.0009327081937}{\cos\left(\frac{x}{2}\right)^2}}}\right) \quad \textbf{(6)}$$

> $plot\left(h, x = \frac{\mathrm{Pi}}{4} .. \frac{7\,\mathrm{Pi}}{18}\right)$



> $c := \arcsin\left(\frac{\sin(h1)}{\sin(x)}\right)$

$$c := \arcsin\left(\frac{0.06108054334}{\sin(x)}\right) \tag{7}$$

> $a := 2\arcsin\left(\frac{\sin(h1)}{2\cdot\cos\left(\frac{x}{2}\right)}\right)$

$$a := 2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right) \tag{8}$$

> $z := \arccos\left(\frac{\cos(r) - \cos(r)\cdot\cos(c)}{\sin(r)\cdot\sin(c)}\right)$

$$z := \arccos\left(306.5054635\left(0.9985724285 - 0.9985724285\sqrt{1 - \frac{0.003730832775}{\sin(x)^2}}\right)\sin(x)\right) \tag{9}$$

$$> \ m := \arccos\left(\frac{\cos(c) - \cos(r)\cdot\cos(r)}{\sin(r)\cdot\sin(r)}\right)$$

$$m := \arccos\left(350.4953206\sqrt{1 - \frac{0.003730832775}{\sin(x)^2}} - 349.4953206\right) \quad \textbf{(10)}$$

$$> \ n := \arccos\left(\frac{\cos(r) - \cos(r)\cdot\cos(a)}{\sin(r)\cdot\sin(a)}\right)$$

$$n := \arccos\left(\frac{1}{\sin\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)}\left(18.72152025\left(0.9985724285 - 0.9985724285\cos\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)\right)\right)\right) \quad \textbf{(11)}$$

$$> \ t := \arccos\left(\frac{\cos(a) - \cos(r)\cdot\cos(r)}{\sin(r)\cdot\sin(r)}\right)$$

$$t := \arccos\left(350.4953206\cos\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right) - 349.4953206\right) \quad \textbf{(12)}$$

$$> \ p := \arcsin\left(\frac{\sin(c)\cdot\sin(x)}{\sin(a)}\right)$$

$$p := \arcsin\left(\frac{0.06108054334}{\sin\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)}\right) \quad \textbf{(13)}$$

$$> \ S := (4\cdot z + 2\cdot m + 2\cdot n + 2\cdot p + t + x - 4\cdot \mathrm{Pi})\cdot R^2 + \frac{6\cdot\mathrm{Pi} - (4\cdot z + 2\cdot p + 2\cdot n + x)}{2}\cdot(r1)^2$$

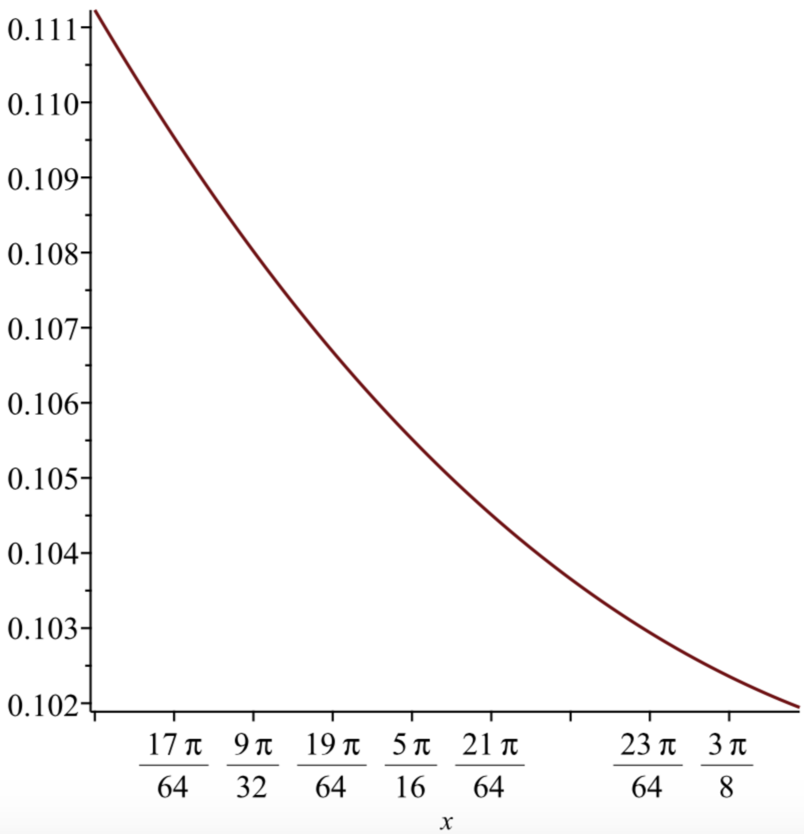
$$S := 19.31638506\arccos\left(306.5054635\left(0.9985724285 - 0.9985724285\sqrt{1 - \frac{0.003730832775}{\sin(x)^2}}\right)\sin(x)\right) + 9.671999998\arccos\left(350.4953206\sqrt{1 - \frac{0.003730832775}{\sin(x)^2}} - 349.4953206\right) + 9.658192526\arccos\left(\frac{1}{\sin\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)}\left(18.72152025\left(0.9985724285 - 0.9985724285\cos\left(2\arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)\right)\right)\right) \quad \textbf{(14)}$$

$$+ 9.658192526 \arcsin\left(\frac{0.06108054334}{\sin\left(2 \arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right)}\right)$$

$$+ 4.835999999 \arccos\left(350.4953206 \cos\left(2 \arcsin\left(\frac{0.03054027167}{\cos\left(\frac{x}{2}\right)}\right)\right) - 349.4953206\right)$$

$$+ 4.829096263\, x - 60.64083594$$

> $plot\left(S, x = \frac{\mathrm{Pi}}{4} .. \frac{7\,\mathrm{Pi}}{18}\right)$



Appendix E

Calculation of the Area of the Union of Three Discs- Case2

> $R := \text{sqrt}(4.836)$

$$R := 2.199090721 \quad \textbf{(1)}$$

> $k := 0.1623828741$

$$k := 0.1623828741 \quad \textbf{(2)}$$

> $r1 := 0.724 \cdot k$

$$r1 := 0.1175652008 \quad \textbf{(3)}$$

> $r := 2 \arcsin\left(\frac{r1}{2 \cdot R}\right)$

$$r := 0.05346719194 \quad \textbf{(4)}$$

> $h1 := 2 \arcsin\left(\frac{0.828 \cdot k}{2 \cdot R}\right)$

$$h1 := 0.06114980572 \quad \textbf{(5)}$$

> $h2 := 2 \arcsin\left(\frac{0.1132182995}{2 \cdot R}\right)$

$$h2 := 0.05148982998 \quad \textbf{(6)}$$

> $c := \arcsin\left(\frac{\sin(h1)}{\sin(x)}\right)$

$$c := \arcsin\left(\frac{0.06111170328}{\sin(x)}\right) \quad \textbf{(7)}$$

> $b := \arcsin\left(\frac{\sin(h2)}{\sin(x)}\right)$

$$b := \arcsin\left(\frac{0.05146708133}{\sin(x)}\right) \quad \textbf{(8)}$$

> $a := \arccos(\cos(c) \cdot \cos(b) + \sin(c) \cdot \sin(b) \cdot \cos(x))$

$$a := \arccos\left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}} \sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} + \frac{0.003145241003 \cos(x)}{\sin(x)^2}\right) \quad \textbf{(9)}$$

> $q := \arccos\left(\frac{\cos(b) - \cos(a) \cdot \cos(c)}{\sin(a) \cdot \sin(c)}\right)$

$$q := \arccos\left(\left(16.36347780 \left(\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} - \left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}} \sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} + \frac{0.003145241003 \cos(x)}{\sin(x)^2}\right) \sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\right) \sin(x)\right) \Big/ \left(1 - \left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}} \sqrt{1 - \frac{0.002648860461}{\sin(x)^2}}\right.\right.\right. \quad \textbf{(10)}$$

$$+\frac{0.003145241003\cos(x)}{\sin(x)^2}\Bigg)^2\Bigg)^{1/2}\Bigg)$$

> $h := \arcsin(\sin(q)\cdot\sin(c))$

$$h := \tag{11}$$

$$\arcsin\Bigg(\frac{1}{\sin(x)}\Bigg(0.06111170328\Bigg(1-\Bigg(267.7634057\Bigg(\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}$$

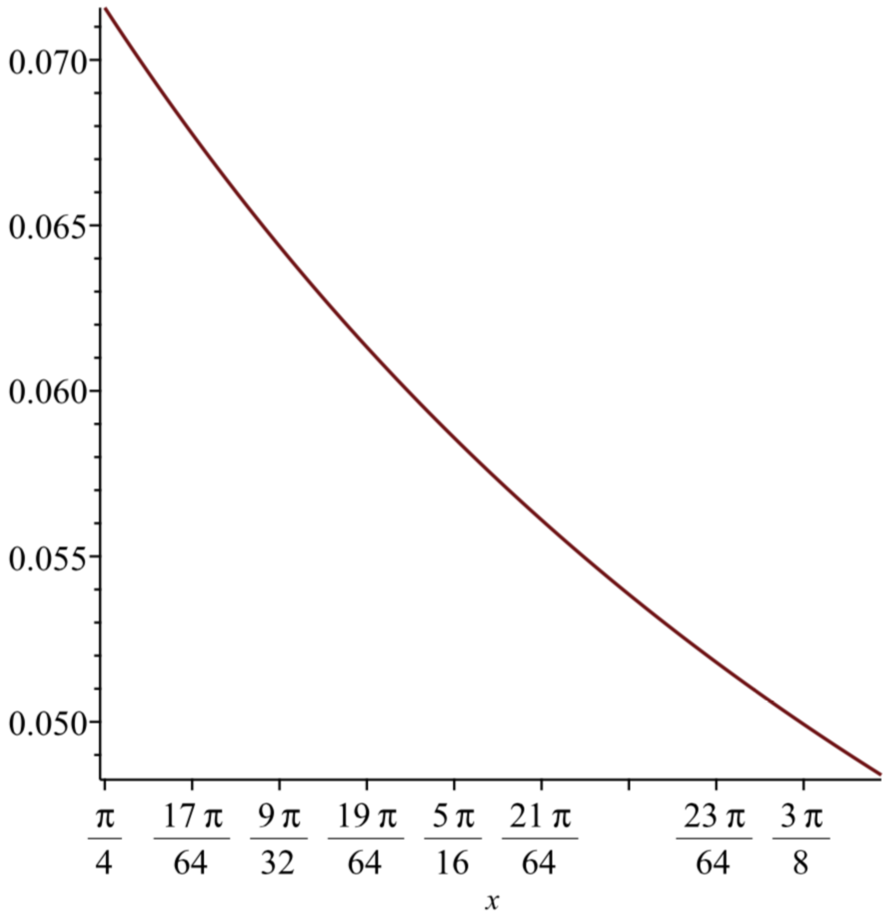
$$-\Bigg(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}$$

$$+\frac{0.003145241003\cos(x)}{\sin(x)^2}\Bigg)\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\Bigg)^2\sin(x)^2\Bigg)\Bigg/\Bigg(1$$

$$-\Bigg(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}+\frac{0.003145241003\cos(x)}{\sin(x)^2}\Bigg)^2\Bigg)$$

$$\Bigg)^{1/2}\Bigg)\Bigg)$$

> $plot\left(h, x=\frac{\text{Pi}}{4}..\frac{7\,\text{Pi}}{18}\right)$



$$> z := \arccos\left(\frac{\cos(r) - \cos(r)\cdot\cos(c)}{\sin(r)\cdot\sin(c)}\right)$$

$$z := \arccos\left(306.1929437\left(0.9985709702 - 0.9985709702\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\right)\sin(x)\right) \quad \textbf{(12)}$$

$$> m := \arccos\left(\frac{\cos(c) - \cos(r)\cdot\cos(r)}{\sin(r)\cdot\sin(r)}\right)$$

$$m := \arccos\left(350.1379081\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}} - 349.1379081\right) \quad \textbf{(13)}$$

$$> g := \arccos\left(\frac{\cos(r) - \cos(r)\cdot\cos(b)}{\sin(r)\cdot\sin(b)}\right)$$

$$g := \arccos\left(363.5716624\left(0.9985709702 - 0.9985709702\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}}\right)\sin(x)\right) \quad \textbf{(14)}$$

> $s := \arccos\left(\frac{\cos(b) - \cos(r)\cdot\cos(r)}{\sin(r)\cdot\sin(r)}\right)$

$$s := \arccos\left(350.1379081\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} - 349.1379081\right) \quad \textbf{(15)}$$

> $n := \arccos\left(\frac{\cos(r) - \cos(r)\cdot\cos(a)}{\sin(r)\cdot\sin(a)}\right)$

$$n := \arccos\left(\left(18.71197232\left(0.9985709702 - 0.9985709702\,\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} - \frac{0.003140746360\cos(x)}{\sin(x)^2}\right)\right)\Bigg/\left(1 - \left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} + \frac{0.003145241003\cos(x)}{\sin(x)^2}\right)^2\right)^{1/2}\right) \quad \textbf{(16)}$$

> $t := \arccos\left(\frac{\cos(a) - \cos(r)\cdot\cos(r)}{\sin(r)\cdot\sin(r)}\right)$

$$t := \arccos\left(350.1379081\,\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} + \frac{1.101268105\cos(x)}{\sin(x)^2} - 349.1379081\right) \quad \textbf{(17)}$$

> $p := \arccos\left(\frac{\cos(c) - \cos(a)\cdot\cos(b)}{\sin(a)\cdot\sin(b)}\right)$

$$p := \arccos\left(\left(19.42989527\left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}} - \left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}} + \frac{0.003145241003\cos(x)}{\sin(x)^2}\right)\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}}\right)\sin(x)\right)\Bigg/\left(1 - \left(\sqrt{1 - \frac{0.003734640278}{\sin(x)^2}}\,\sqrt{1 - \frac{0.002648860461}{\sin(x)^2}}\right.\right.\right. \quad \textbf{(18)}$$

$$+\frac{0.003145241003\cos(x)}{\sin(x)^2}\Bigg)^2\Bigg)^{1/2}\Bigg)$$

> $S := (2\cdot z + 2\cdot g + 2\cdot n + m + s + t + p + q + x - 4\cdot \mathrm{Pi})\cdot R^2 + \frac{6\cdot \mathrm{Pi} - (2\cdot g + 2\cdot n + 2\cdot z + p + q + x)}{2}\cdot (r1)^2$

$$S := 9.658178422\arccos\Bigg(306.1929437\Bigg(0.9985709702 - 0.9985709702\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\Bigg)\sin(x)\Bigg) \tag{19}$$

$$+9.658178422\arccos\Bigg(363.5716624\Bigg(0.9985709702 - 0.9985709702\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\Bigg)\sin(x)\Bigg)$$

$$+9.658178422\arccos\Bigg(\Bigg(18.71197232\Bigg(0.9985709702 - 0.9985709702\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\sqrt{1-\frac{0.002648860461}{\sin(x)^2}} - \frac{0.003140746360\cos(x)}{\sin(x)^2}\Bigg)\Bigg)\Bigg/$$

$$\Bigg(1-\Bigg(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\sqrt{1-\frac{0.002648860461}{\sin(x)^2}} + \frac{0.003145241003\cos(x)}{\sin(x)^2}\Bigg)^2\Bigg)^{1/2}\Bigg)$$

$$+4.835999999\arccos\Bigg(350.1379081\sqrt{1-\frac{0.003734640278}{\sin(x)^2}} - 349.1379081\Bigg)$$

$$+4.835999999\arccos\Bigg(350.1379081\sqrt{1-\frac{0.002648860461}{\sin(x)^2}} - 349.1379081\Bigg)$$

$$+4.835999999\arccos\Bigg(350.1379081\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\sqrt{1-\frac{0.002648860461}{\sin(x)^2}} + \frac{1.101268105\cos(x)}{\sin(x)^2} - 349.1379081\Bigg)$$

$$+4.829089211\arccos\Bigg(\Bigg(19.42989527\Bigg(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}$$

$$-\left(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\;\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\right.$$

$$\left.+\frac{0.003145241003\,\cos(x)}{\sin(x)^2}\right)\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\;\sin(x)\Bigg)\Bigg/$$

$$\left(1-\left(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\;\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\right.\right.$$

$$\left.\left.+\frac{0.003145241003\,\cos(x)}{\sin(x)^2}\right)^2\right)^{1/2}\Bigg)$$

$$+4.829089211\,\arccos\Bigg(\Bigg(16.36347780\left(\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\right.$$

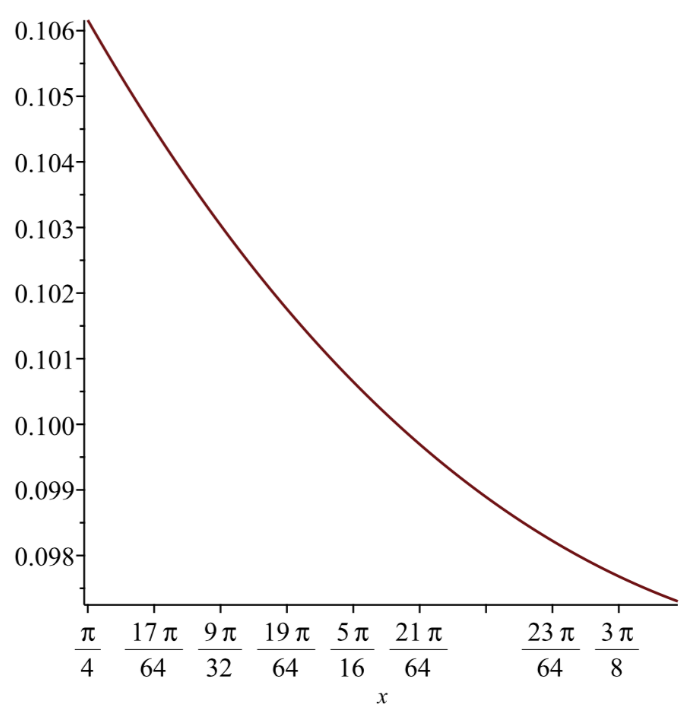
$$-\left(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\;\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\right.$$

$$\left.+\frac{0.003145241003\,\cos(x)}{\sin(x)^2}\right)\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\;\sin(x)\Bigg)\Bigg/$$

$$\left(1-\left(\sqrt{1-\frac{0.003734640278}{\sin(x)^2}}\;\sqrt{1-\frac{0.002648860461}{\sin(x)^2}}\right.\right.$$

$$\left.\left.+\frac{0.003145241003\,\cos(x)}{\sin(x)^2}\right)^2\right)^{1/2}\Bigg)+4.829089211\,x-60.64070301$$

> $plot\left(S, x=\frac{\mathrm{Pi}}{4}\,..\,\frac{7\,\mathrm{Pi}}{18}\right)$



Appendix F

Proofs of Lemmas 5.2 and 5.5

> $R := \mathrm{sqrt}(4.836)$

$$R := 2.199090721 \tag{1}$$

> $r := 1.0496594$

$$r := 1.0496594 \tag{2}$$

> $k := 0.1623828741$

$$k := 0.1623828741 \tag{3}$$

> $s := 0.724$

$$s := 0.724 \tag{4}$$

$$\beta := 0.8637942973 \tag{5}$$

> $v := 1.882\,k$

$$v := 0.3056045691 \tag{6}$$

> $t := \dfrac{v^2}{2 \cdot R}$

$$t := 0.02123472028 \tag{7}$$

> $g := 0.828\,k$

$$g := 0.1344530198 \tag{8}$$

> $h := g - t$

$$h := 0.1132182995 \tag{9}$$

> $p := 2 \cdot \arcsin\left(\dfrac{h}{2 \cdot R}\right)$

$$p := 0.05148982996 \tag{10}$$

> $solve(2 \cdot f^2 - f \cdot \cos(p) - 1 = 0, f)$

$$0.9995582956, \ -0.5002209498 \tag{11}$$

> $m := \arcsin\left(\dfrac{1}{2 \cdot 0.9995582956}\right)$

$$m := 0.5238539252 \tag{12}$$

> $A := \dfrac{R^2 \cdot (6 \cdot m - \mathrm{Pi}) + 3 \cdot \dfrac{(2 \cdot \mathrm{Pi} - 2 \cdot m)}{2 \cdot \mathrm{Pi}} \cdot (s\,k)^2 \mathrm{Pi}}{(s\,k)^2 \mathrm{Pi}}$

$$A := 2.670256518 \tag{13}$$

>