List-edge coloring planar graphs with bounded maximum degree

by

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Abstract

In this thesis we prove that triangulations of maximum degree 5 are 6-list-edge-colorable. We also find necessary conditions for maximum degree to extend a list-edge-precoloring to $E(G)$ for a planar graph G. The techniques used for these two results are the kernel method, the quantitative combinatorial nullstellensatz, and the discharging method.

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Chapter 1

Introduction

An *edge-coloring* of a graph G is an assignment of colors to the edges of G so that adjacent edges receive different colors; if at most k colors are used we say such a coloring is a *k-edge-coloring*. The main focus of our work is on a special type of edge coloring called a *list-edge coloring*, where the color assigned to an edge must come from a previously defined list of available colors. Formally, an *edge list assignment* is a function L that assigns to each edge $e \in E(G)$ a list of colors $L(e)$. An *L-edge-coloring* of G is an edge-coloring of G such that every edge e is given a color from $L(e)$. Note that a classical k-edge-coloring of G can be viewed as an L-edge-coloring for the list assignment L defined by $L(e) = \{1, \ldots, k\}$ for all $e \in E(G)$.

In general, we want to know what size the lists of L must be in order to guarantee an L-edge coloring. We say a graph G is *k-list-edge-colorable* if it is L-edge-colorable for every edge list assignment L such that $|L(e)| \geq k$ for all $e \in E(G)$. The *list-chromatic index* of G, denoted $\chi'_\ell(G)$, is the minimum k such that G has a k-list-edge coloring. The *chromatic index* of G, denoted $\chi'(G)$, is the minimum k such that G is k-edge-colorable. Given our previous comment, we know that $\chi'_\ell(G) \geq \chi'(G)$ for every graph G.

If we let $\Delta := \Delta(G)$ be the maximum degree of G, then we see a vertex of degree Δ is incident to Δ distinct edges so $\chi'(G) \geq \Delta$. In this thesis we consider every graph to be simple (without loops or parallel edges) unless stated otherwise, hence Vizing's Theorem [34] says that $\chi'(G) \leq \Delta + 1$ for all graphs G. Vizing [31] conjectured that this upper bound also holds for list-edge coloring.

Conjecture 1.1 (Vizing [31])**.** *If G is a graph, then $\chi'_\ell(G) \leq \Delta + 1$.*

The study of list-edge colorings has now become dominated by a strengthening of Vizing's conjecture known as the List-Edge Coloring Conjecture, or LECC. The LECC is attributed to many sources, some as early as 1975 (see eg. [18]).

Conjecture 1.2 (LECC)**.** *If G is a graph, then $\chi'(G) = \chi'_\ell(G)$.*

Progress on Conjectures 1.1 and 1.2 has been somewhat limited for general graphs, although Conjecture 1.1 has been verified for all graphs with $\Delta \leq 4$. The $\Delta = 3$ case was proved by Vizing [31] in 1976 and independently by Erdős, Rubin, and Taylor [12] in 1979. The $\Delta = 4$ case of Vizing's conjecture was proved in 1998 by Juvan, Moher, Škrekovski [19]. Since there are graphs with $\Delta \leq 4$ having $\chi'(G) = \Delta + 1$, these results are tight.

In 1994 Galvin [13] showed that if G is a bipartite graph, then $\chi'_\ell(G) = \Delta$. This landmark result verified the LECC for bipartite graphs and is commonly regarded as the best progress towards the conjecture.

In this thesis our focus is on *planar graphs* which are graphs that can be drawn in the plane without edge-crossings. Both edge-coloring and list-edge-coloring planar graphs are somewhat simpler. In [17], Holyer showed it is NP-complete to decide whether a graph has chromatic index Δ or $\Delta + 1$, but this does not appear to be the case for planar graphs. For $\Delta = 2, 3, 4, 5$ there are examples of planar graphs with chromatic index Δ and $\Delta + 1$. However, Vizing [33] showed that every planar G with $\Delta \geq 8$ is Δ-edge-colorable which was then strengthed to $\Delta = 7$ independently by Grünewald [15], Sanders and Zhao [26] and Zhang [38]. The collective work of the above is summarized by the following theorem.

Theorem 1.1. *If G is planar with $\Delta \geq 7$, then $\chi'(G) = \Delta$.*

Note that the case $\Delta = 6$ is still open as it is unknown whether or not there exists a planar graph with $\Delta = 6$ and chromatic index $\Delta + 1$.

Borodin, Kostochka, Woodall [6] proved the following theorem in 1997 which verified the LECC for planar graphs with $\Delta \geq 12$.

Theorem 1.2 (Borodin, Kostochka, Woodall [6])**.** *If G is planar with $\Delta \geq 12$, then $\chi'_\ell(G) = \Delta$.*

Theorem 1.1 leads us to expect all planar graphs with $\Delta \geq 7$ to have $\chi'_\ell(G) = \Delta$, but this is only known for $\Delta \geq 12$. Ellingham and Goddyn [11] were able to verify the LECC for regular planar graphs with $\chi'(G) = \Delta$ without imposing restrictions on Δ, but little other work on the LECC does so.

Borodin verified Conjecture 1.1 for planar graphs with $\Delta \geq 9$ in 1990 [5], and Cohen and Havet provided an alternate proof in 2010 [8], which we shall discuss later. The latest contribution to Conjecture 1.1 was made by Bonamy [4] in 2013 who showed that planar graphs with $\Delta \geq 8$ have $\chi'_\ell(G) \leq \Delta + 1$. This leaves the conjecture open for planar graphs with $5 \leq \Delta \leq 7$. In chapter 3 we will prove the conjecture for *triangulations* (planar graphs where every face is a triangle) with $\Delta = 5$. In this result we encounter some small graphs that must be list-edge-colored and we use two different techniques to this end; the kernel method, initially developed by Galvin [13] in his aforementioned work and the Quantitative Combinatiorial Nullstellensatz, initially developed by Alon [1] in 1993 . Both techniques are described in detail in the first two sections of chapter 2.

In chapter 4 we look to edge-color or list-edge color a graph G, but with the additional constraint that some edges have already been colored and cannot be changed. In this scenario we have no control over the edge-precoloring – if the edge-precolored subgraph is H, then it will certainly have at least $\chi'(H)$ colors, but it could have many more, perhaps even more than $\chi'(G)$ colors. If we are looking to extend the edge-precoloring to a k-edge-coloring of G, then we will certainly need that k is at least the maximum degree of G, and that the edge-coloring of H uses at most k colors (i.e. is a k-edge-coloring).

In general we consider the following question, first posed by Marcotte and Seymour [22],
Given a graph G with maximum degree Δ and a subgraph H of G that has been $(\Delta + t)$-edge-colored, can the edge-precoloring of H be extended to a $(\Delta + t)$-edge-coloring of G?

Marcotte and Seymour's main result in [22] is a necessary condition for the answer to their question to be "yes"; they prove that this condition is also sufficient when G is a multiforest (the condition is rather technical, so we do not state it here). The above question was shown to be NP-complete by Colbourn [9], and Marx [23] showed that this is true even when G is a planar 3-regular bipartite graph. Given Holyer's above-mentioned result, the special case $t = 0$ of the question is also NP-complete for general graphs.

In chapter 4 we focus on Marcotte and Seymour's question for planar graphs. Our result extends previous work on this problem by Edwards, Girão, van den Heuvel, Kang, Sereni and Puleo [10], who considered the case when $\Delta(H) = t = 1$. As Edwards et al. observe, extending an edge-precoloring to an edge-coloring is closely related to list-edge coloring. In particular, if we are trying to k-edge color a graph G which has a precolored subgraph H, then we can think of each edge in $G - H$ as having a list made by starting with $\{1, 2, ..., k\}$ and deleting the colors of any adjacent edges in H. Even with this connection, we were surprised to be able to prove our main result in chapter 4 for both edge-coloring and list-edge-coloring. We roughly prove for a given graph G and edge-list assignment L with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$, that if $H \subseteq G$ has been L-edge-colored, then the edge-precoloring can be extended to an L-edge-coloring of G, provided that $\Delta(H) \leq t$ and either $\Delta(H)$ is small enough or $\Delta(G)$ is large enough. It is worth noting that as a corollary of this result, we get Borodin's Theorem 2.8.

The content of chapter 4, which is joint work with Greg Puleo (in addition to my advisor Jessica McDonald), requires the so-called Discharging Method. We will discuss this technique in general in the third section of chapter 2. As a demonstration of the discharging method we will present Cohen and Havet's proof of Borodin's result, which played a large role in influencing this thesis as we will discuss later.

In the conclusion, chapter 5, we will discuss our two results with emphasis on their relation to Vizing's Conjecture and how the result could be extended. For graph theoretic definitions not stated here, we follow the conventions of West [37].

Chapter 2

Three Techniques for List-Edge Coloring

We will now discuss three techniques used for list-edge-coloring: the kernel method, the quantitative combinatorial nullstellensatz, and the discharging method.

2.1 The Kernel Method

Although the focus of this thesis is on simple graphs, in this section we shall discuss multigraphs, which are graphs where parallel edges are permitted. In particular, a clique is considered to be a set of vertices such that any two vertices are joined by at least one edge. We say a multigraph G contains a clique as a submultigraph if the clique is induced by a subset of $V(G)$.

We define an orientation of $G = (V, E)$ to be the digraph $D = (V, A)$, where every edge $uv \in E(G)$ is either oriented from the vertex u to the vertex v or oriented from the vertex v to the vertex u. We note that this definition of orientation is in keeping with Borodin, Kostochka, and Woodall in [7] as opposed to many other papers surveying the kernel method which allow orientations to contain bidirected edges.

A *kernel* in a digraph is an independent set of vertices K such that every vertex outside K has at least one edge into K. We say a digraph is *kernel-perfect* if every induced subdigraph has a kernel.

The study of kernels is quite rich, but we will focus on what is known for line multigraphs as this will lead us to conclusions about edge-colorings. As noted by Alon and Tarsi in Remark 2.4 in [3], Bondy, Boppana, and Siegel provided a wonderful implication for kernel-perfect orientations of line multigraphs. Their theorem is as follows.

Theorem 2.1. *(Bondy, Boppana, Siegel) Let G be a multigraph and suppose $L(G)$ has a kernel-perfect orientation where $d^+(e) \leq k-1$ for all $e \in V(L(G)) = E(G)$. Then G is k-list-edge-colorable.*

We should note this theorem was actually proved for orientations which allow bidirected edges. In his aforementioned result Galvin constructed a special orientation D of the line graph of an arbitrary bipartite graph G by edge-coloring G and then directing edges according to their colors. He then showed that an easy consequence of this orientation was $d^+(e) \leq k-1$ for all $e \in V(D)$. Lastly, Galvin showed D must contain a kernel due to a result by Maffray [21] and proceeded by induction to yield that D is kernel-perfect.

Galvin's Theorem was generalized by Borodin, Kostochka, and Woodall who characterized when all line multigraphs are kernel-perfect. Their theorem defines a *pseudochord* in a directed cycle $v_1, ..., v_t$ to be a directed edge $v_i v_{i-1}$ for some $1 \leq i \leq t$.

Theorem 2.2 (Borodin, Kostochka, Woodall [6]). *Let G be a bipartite graph and let L be an edge list assignment on G. If $|L(xy)| \geq \max\{\deg(x), \deg(y)\}$ for every edge $xy \in E(G)$, then G is L-edge-colorable.*

Theorem 2.3. *(Borodin, Kostochka, Woodall [7]) An orientation of a line multigraph is kernel-perfect iff every clique has a kernel and every directed odd cycle has a chord or pseudochord.*

Using this characterization in conjunction with Theorem 2.1 we get the following corollary:

Corollary 2.1. *The multigraph G is k-list-edge-colorable if there is an orientation of $L(G)$, call it D_L, such that the following conditions are true:*

1. *Every clique in D_L has a kernel.*

2. *Every directed odd cycle in D_L has a chord or pseudochord.*

3. $d^+(e) \leq k-1$ *for all* $e \in V(D_L)$.

Proof. If conditions 1 and 2 hold, then D_L is a kernel-perfect orientation of a line multigraph by Theorem 2.3. If condition 3 holds, then D_L is a kernel-perfect orientation of the line multigraph of G where $d^+(e) \leq k-1$ for all $e \in V(D_L)$. So by Theorem 2.1 we see G is k-list-edge-colorable. □

The above corollary gives us a list-edge-coloring of G provided $L(G)$ has a special type of orientation. In the actual application of the corollary it will be helpful for us to simply look at G rather than $L(G)$.

Let $\mathcal{L}(G)$ be a *labeling* of G which is an assignment of labels to the ends of every edge in G which orients the edges of the line multigraph $L(G)$. If two edges, e_1 and e_2, are incident to the same vertex v in G, we consider their labels at v which we call $\mathcal{L}_v(e_1)$ and $\mathcal{L}_v(e_2)$ respectively. If e_1 has a smaller label than e_2, then we say $\mathcal{L}_v(e_1) < \mathcal{L}_v(e_2)$ and e_1e_2 is the directed edge in $L(G)$. If e_2 has a smaller label than e_1, then e_2e_1 is our directed edge in $L(G)$. If an edge has ends labeled 1 and 2, then we say it is a $(1,2)$-edge.

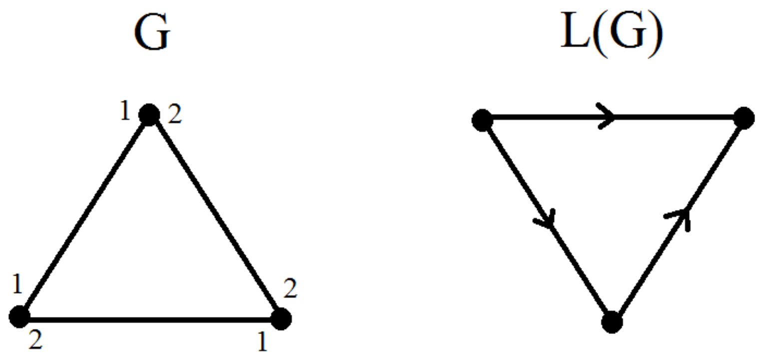


Figure 2.1: A labeling of G and it's resulting line multigraph.

By putting conditions on a labeling of G we implicitly put conditions on the orientation of $L(G)$ which gives us a way to describe list-edge-coloring in terms of G rather than $L(G)$. We present the following corollary to better understand list-edge-coloring as conditions on G rather than on $L(G)$.

Definition 2.1. *A k-kernel-perfect-labeling of G is a labeling of G which orients $L(G)$ so that the following hold:*

1. *For all $v \in G$, the set of labels $\{\mathcal{L}_v(e) : e \text{ incident to } v\}$ has a total ordering.*
2. *Every directed odd cycle in $L(G)$ that corresponds to an odd cycle in G has a pseudo-chord.*

3. $d^+(e) \le k - 1$ *for all* $e \in V(L(G))$.

Corollary 2.2. *The multigraph G is k-list-edge-colorable if there is a k-kernel-perfect-labeling of G.*

Proof. We will show the conditions of a k-kernel-perfect-labeling imply the conditions of Corollary 2.1.

The final condition of Corollary 2.1 is equivalent to the final condition of Definition 2.1. So let us show the first condition of Corollary 2.1 is satisfied by the first condition of Definition 2.1. Every clique in $L(G)$ comes from either the edges of a single vertex in G or a triangle in G. If the set of labels $\{\mathcal{L}_v(e) : e \textit{ incident to } v\}$ has a total ordering, then the edge with the highest label at v will be a sink in the resulting clique of $L(G)$ so every other edge of the resulting clique will be directed to it. This means the resulting clique has a kernel. The only other cliques in $L(G)$ must come from triangles in G. We know triangles in G correspond to triangles in $L(G)$ that are either directed or undirected. If they correspond to a directed triangle in $L(G)$, then condition (2) of Definition 2.1 tells us that directed triangle has a pseudochord. Otherwise a triangle in G corresponds to an undirected triangle in $L(G)$, either case provides a sink in the induced submultigraph.

We now show every directed odd cycle in $L(G)$ has a chord or pseudochord to complete the proof. An odd cycle in $L(G)$ comes from either an odd cycle in G or a circuit in G. If a directed odd cycle C in $L(G)$ comes from an odd cycle in G, then C must have a pseudochord by condition (2) of Definition 2.1. If a directed odd cycle C in $L(G)$ comes from a circuit H in G which is not an odd cycle, then H has a repeated vertex v which serves as a cut-vertex having two ends in each component of $H - v$. Since these ends meet at v they must form a clique in $L(G)$ meaning C has a chord. □

Although conditions (2) and (3) of Definition 2.1 are not completely in terms of G we will see they are able to be verified without having to inspect $L(G)$. Condition (2) is the most difficult to verify but verification is simplified by noting every odd cycle in G must correspond to an odd cycle in $L(G)$. If the corresponding cycle is undirected we have nothing more to show, so it suffices to show that correspoding directed cycles of $L(G)$ must contain a pseudochord.

This can be shown using two generic steps: find a large bipartite submultigraph H of G and iteratively show the edges of $G - H$ cannot be contained in directed odd cycles of $L(G)$.

To illustrate this process we now present an example of how kernel-perfect-labelings can easily show the Petersen Graph is 4-list-edge-colorable.

Theorem 2.4. *The Peterson graph is* 4*-list-edge-colorable.*

Proof. Let P be the Petersen graph and let $\mathcal{L}$ be the labeling of P given by figure 2.2. We will show $\mathcal{L}$ is a 3-kernel-perfect-labeling.

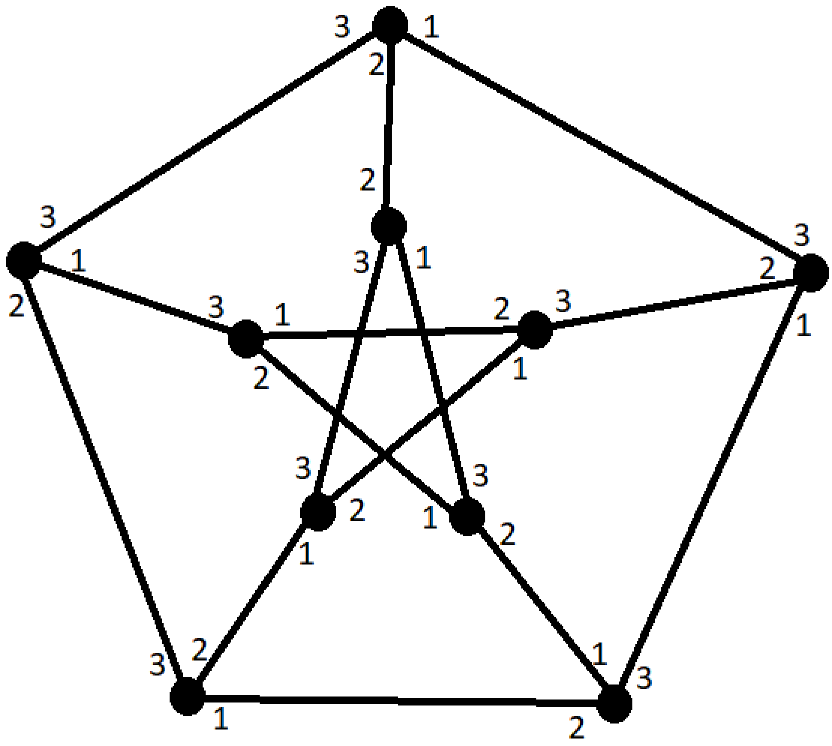


Figure 2.2: How to orient $L(P)$

We see every vertex has a label set which has a total ordering so condition 1 of Definition 2.1 is satisfied.

Let us consider a $(1,2)$-edge, $e \in P$, which has labels 1 and 2 on its ends. The 1 will direct two edges out of e in $L(P)$ and the 2 label will direct one edge out of e in $L(P)$. So we see $d^+(e) \leq 4 - 1 = 3$. We also see the only way for an edge of P to have out-degree 4 in $L(P)$ is for both of its ends to be labeled with 1. A quick check verifies no edge of P was labeled in such a way, so condition 3 of Definition 2.1 is satisfied.

To finish the proof we show that our labeling of P has no odd cycle in G corresponding to a directed odd cycle in $L(P)$. We illustrate this is so with the figure below.

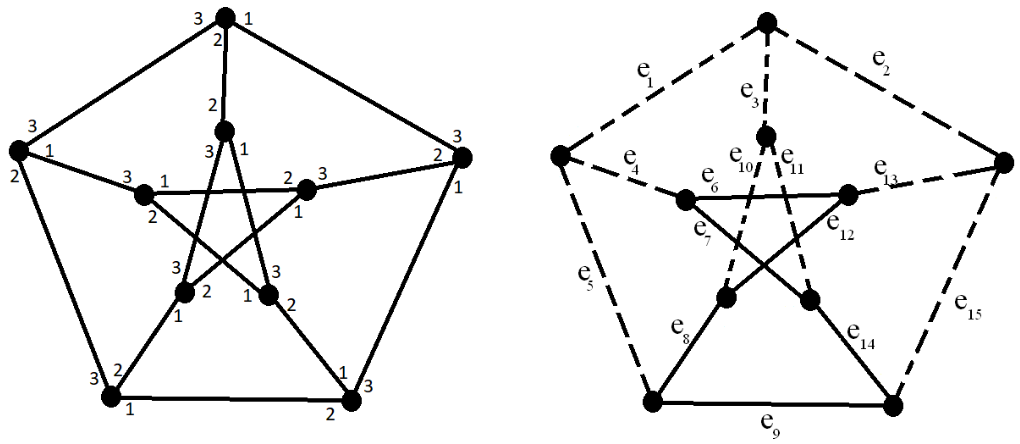


Figure 2.3: Such a labeling of P yields no directed cycles in $L(P)$

The dashed edges of P in Figure 2.3 correspond to vertices of $L(P)$ which cannot be contained in directed cycles. Any $(3, 3)$-edge of P is a sink in $L(P)$ and cannot be contained in a directed cycle, so we dash through e_1 and e_{10}. The only edge-labels e_3 is adjacent to which are not $(3, 3)$ are 1's, meaning it cannot be contained in a directed cycle of $L(G)$. Dashing through e_3 leaves e_2 and e_{11} ends which are adjacent to only dashed edges meaning they are not contained in a directed odd cycle of $L(G)$ so we may dash through e_2 and e_{11}. Both e_5 and e_{13} are $(2, 3)$-edges such that the end labeled 2 is adjacent to only undashed edges of label 1. This means a directed cycle would have to come in through the 2 label and leave through the 3 label which cannot happen. Dashing through e_5 and e_{13} leaves e_4 and e_{15} ends which are adjacent to only dashed edges meaning they are not in a directed cycle. Dashing through e_4 and e_{15} yields the picture in figure 2.3. Only a C_6 is not dashed which cannot yield a directed odd cycle in $L(P)$.

$\square$

2.2 The Quantitative Combinatorial Nullstellensatz

Hilbert's Nullstellensatz or "Root Theorem" is a well known result in algebraic geometry which concerns polynomials [35]. In 1999, Alon and Tarsi used Hilbert's result to yield a combinatorial version of the Nullstellensatz [2]. This Combinatorial Nullstellensatz has been used in many graph theoretic arguments. In particular it was used by Ellingham and Goddyn in their previously mentioned result which verified the List-Edge-Coloring Conjecture for regular planar graphs with $\chi'(G) = \Delta$.

In 2008, Schauz generalized Alon and Tarsi's work, see [29], into what was named the Quantitative Combinatorial Nullstellensatz, or QCN. Then in 2014 Schauz used the QCN to show infinitely many 1-factorable complete graphs of prime degree satisfy the LECC [28].

Let G be a k-regular graph on the vertices $v_1, \ldots, v_{2n}$ and let $F = \{e_1, e_2, \ldots, e_n\}$ be a 1-factor of G. Label the vertices so that $e_\ell = v_{i_\ell} v_{j_\ell}$ with $i_\ell < j_\ell$ for all $\ell \in \{1, 2, \ldots, n\}$. We say that an edge $e_\ell \in F$ *intersects* another edge $e_h \in F$ if $i_\ell < i_h < j_\ell < j_h$ or $i_h < i_\ell < j_h < j_\ell$. We define

$$int(e_\ell, e_h) = \begin{cases} 1 & \text{if } e_\ell \text{ instersects } e_h \\ 0 & \text{otherwise,} \end{cases}$$

and define

$$int(F) = \sum_{1 \le \ell < h \le n} int(e_\ell, e_h) \quad \text{and} \quad sign(F) = (-1)^{int(F)}.$$

Note that if the 2n vertices are positioned consecutively around a cycle and the edges are drawn as straight lines, then an intersection is an actual intersection between the lines.

Schauz introduced the above definitions in [27] and proved the following in 2018.

Theorem 2.5. (Schauz [27]) *Let G be a k-regular graph on the vertices $v_1, v_2, \ldots, v_{2n}$. Let OF(G) be the set of all 1-factorizations of G. For each $\mathcal{F} = (F_1, F_2, \ldots, F_n) \in OF(G)$, let*

$$sign(\mathcal{F}) = \prod_{1 \le i \le k} sign(F_i).$$

Then

$$\sum_{\mathcal{F} \in OF(G)} sign(\mathcal{F}) \neq 0 \quad \Rightarrow \quad G \text{ is } k\text{-list-edge-colorable.}$$

Schauz proves Theorem 2.5 using his Quantitive Combinatorial Nullstellenstaz from [29]. In [27], Schauz also provides an algorithm that computes the value of $\sum_{\mathcal{F} \in OF(G)} sign(\mathcal{F})$ when G is a small (up to about 10-vertex) regular graph on an even number of vertices. This algorithm, which was implemented in SageMath [25] using only python commands, is printed as

Algorithm 1 in the appendix of this paper. We shall apply Algorithm 1 and Theorem 2.5 together to several specific graphs in this section in order to show 6-list-edge-colorability.

To illustrate the calculation of this sum of signs we provide the following example also found in example 3 of [27].

Example 2.1. *Let* $G = K_{3,3}$ have partitions $U = \{u_1,\ u_2,\ u_3\}$ and $W = \{w_1,\ w_2,\ w_3\}$. Let the edges of G be drawn with straight lines as in figure 2.4 and let $w_1 < w_2 < w_3 < u_1 < u_2 < u_3$ be our cyclic arrangement of the vertices, then G is a 3-regular graph on an even number of vertices and the sign of an edge-coloring is $(-1)^{int(c)}$.

G has only two edge colorings which we label F' and F" and illustrate in figure 2.4. First we calculate $sgn(F')$ by counting the number of times edges of the same color class intersect. We see the blue edges have 0 intersections, the green edges have 2 intersections, and the red edges have 2 intersections. So $sgn(F') = (-1)^4 = 1$. Now we calculate $sgn(F")$ the same way. We see the blue edges have 1 intersection, the green edges have 3 intersections, and red edges have 1 intersection. So $sgn(F") = (-1)^5 = -1$.

This means $\sum_{F \epsilon OF(G)} sgn(F) = 1 - 1 = 0$ and a conclusion cannot be reached via Theorem 2.5.

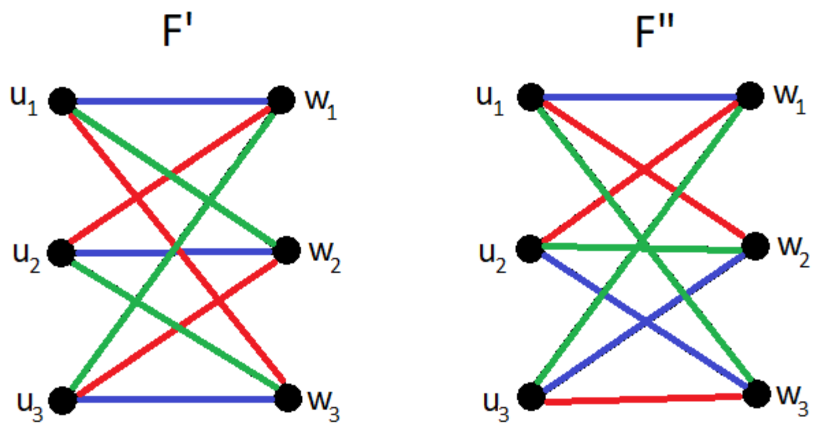


Figure 2.4: The two edge-colorings of $K_{3,3}$

Our use of the algorithm was to verify three specific triangulations of $\Delta = 5$ have $\chi'(G) \leq 6$. To this end we find 6-regular graphs on an even number of vertices which contain our triangulations as subgraphs and prove that the 6-regular graphs are 6-list-edge colorable via Algorithm 1 as mentioned in chapter 3, with computations in Appendix A.

2.3 The Discharging Method

The Discharging Method was originally developed to aid in the proof of the Four Color Theorem and has been used frequently since to prove a variety of results especially coloring results for the family of planar graphs.

We can summarize the method into three basic steps which are:

1. Choose a set X of elements of G to charge. (e.g. $X = V(G)$ or $X = V(G) \cup F(G)$)
2. Assign an *initial charge*, $\alpha(x)$ for all $x \in X$
3. Use *Discharging Rules* to distribute charge among the elements of X.
4. Find the *final charge*, $\alpha'(x)$ for all $x \in X$.
5. Remark that the *sum of charges* is preserved $\alpha(X) = \alpha'(X)$ or $\alpha(G) = \alpha'(G)$.

There are many proofs which assign an initial charge to all vertices, edges, and faces of a graph. There are also many proofs which assign charges to only one of these structures. The following proof of the degree-sum formula illustrates the method in its most basic form:

Theorem 2.6. *If G is a graph, then $\sum_{v \in V(G)} deg(v) = 2e(G)$.*

Proof. Define the initial charges: $\alpha(e) = 2 \ \forall e \in E(G)$ and $\alpha(v) = 0 \ \forall v \in V(G)$. Discharge by having every edge give a charge of 1 to each of its endpoints. After discharging we have $\alpha'(v) = deg(v) \ \forall v \in V(G)$ and $\alpha'(e) = 0 \ \forall e \in E(G)$. So $2e(G) = \alpha(G) = \alpha'(G) = \sum deg(v)$. □

At its core the discharging method is a counting argument which allows us to quantify structures within the graphs we are concerned with. The more structure a graph has the better use we can make out of discharging which is why the method has been used so frequently for planar graphs. In particular, many proofs utilize the structure of planar graphs given in Euler's Theorem.

Theorem 2.7 (Euler)**.** *Let G be a connected planar graph with v vertices, f faces, and e edges. Then $v - e + f = 2$.*

Most discharging arguments start with a minimum counterexample, show $\alpha(G) > 0$ via Euler's Formula, and then show discharging along specific structure in G forces $\alpha'(G) < 0$ to yield a contradiction.

We end this chapter with an example of the discharging method for list-edge coloring example. As mentioned in the introduction Borodin verified Vizing's conjecture for planar graphs if $\Delta \geq 9$, see [5]. Nearly ten years later Cohen and Havet provided the following alternate proof of the result in [8] which makes clever use of discharging reducing the argument to little more than a page long.

Theorem 2.8 (Borodin [5])**.** *If G is a planar graph with $\Delta(G) \geq 9$, then G is $\Delta + 1$-list-edge-colorable.*

Proof. (Cohen & Havet [8])

Let $V_i := \{v \in V(G) | deg(v) = i\}$ and let $V_{[a,b]} := \{v \in V(G) | a \leq deg(v) \leq b\}$. We consider an edge-minimal counterexample G, that is for some list assignment L which assigns lists of size $\Delta + 1$ to the edges of G we assume there is no L-edge-coloring of G but there is an L-edge-coloring for $G - e$ for all $e \in E(G)$.

Claim 2.1. *If $uv \in E(G)$, then $deg(u) + deg(v) - 2 \geq \Delta + 1$.*

Proof of Claim. Assume for contradiction there is an edge $uv \in E(G)$ such that $deg(u) + deg(v) - 2 < \Delta + 1$. We know $G - uv$ is L-edge-colorable and we see the edge uv is adjacent to $deg(u) + deg(v) - 2$ edges meaning it sees at most Δ colors which is a contradiction. □

Note this claim implies that $\delta(G) \leq 3$ and $N(V_3) \subseteq V_\Delta$.

Claim 2.2. $|V_3| < \frac{1}{2}|V_\Delta|$

Proof of Claim. Let F be the bipartite subgraph of G induced by $V_3 \cup V_\Delta$. If F is acyclic, then we have $3|V_3| = e(F) < v(F) = |V_3| + |V_\Delta|$. So to prove our claim it will be sufficient to show F is acyclic.

Assume for contradiction that F is not acyclic. Since F is bipartite this means F must contain an even cycle C. Let $uv \in C$ where $u \in V_3$ and $v \in V_\Delta$. We know $G - C$ is L-edge-colorable and we see uv is adjacent to $\Delta - 1$ edges in $G - C$ meaning every edge of C has two available colors. This is a contradiction as even cycles are 2-list-edge colorable. $\square$

We now introduce a discharging argument. For every vertex v of G let $\alpha(v) = deg(v) - 4$. For every face f of G let $\alpha(f) = \ell(f) - 4$. We also define an artificial structure P and let $\alpha(P) = 0$.

Using the Degree-Sum Formula and Euler's Formula we note that

$$\alpha(G) = \sum_{v \in V(G)} \alpha(v) + \sum_{f \in F(G)} \alpha(f) = 2e(G) - 4v(G) + 2e(G) - 4f(G) = -8$$

We discharge along the following rules

(a) If $v \in V_\Delta$, then v gives $\frac{1}{2}$ charge to P.

(b) If $v \in V_3$, then v takes 1 unit of charge from P.

(c) If $v \in V_{[8,\Delta]}$, then v gives $\frac{1}{2}$ charge to incident triangles.

(d) If $v \in V_{[5,7]}$, then v gives $\frac{deg(v)-4}{deg(v)}$ to each triangle incident to v.

We will now show that the final charge of every vertex, face, and P is nonnegative which will yield a contradiction.

We know $\alpha(P) = 0$ and rules (a) and (b) manipulate this charge, however $|V_3| < \frac{1}{2}|V_\Delta|$ by our second claim means more is given to P than is taken which implies $\alpha'(P) > 0$.

Let $v \in V_d$ for some $3 \leq d \leq \Delta$. We see $\alpha'(v) = deg(v) - 4 + p_v - t_v$ where p and t are the amounts of charge v gives or receives from P and incident triangles respectively.

If $v \in V_3$, then v receives 1 charge from P and gives no charge to incident triangles so $\alpha'(v) = 3 - 4 + 1 - 0 = 0$.

If $v \in V_{[5,7]}$, then v gives no charge to P and gives $\frac{deg(v)-4}{deg(v)}$ to incident triangles. So $\alpha'(v) \geq deg(v) - 4 + 0 - \frac{deg(v)-4}{deg(v)} = deg(v) - 5 + \frac{4}{deg(v)} > 0$ since $5 \leq deg(v) \leq 7$.

If $v \in V_{[8,\Delta-1]}$, then v gives no charge to P and gives $\frac{1}{2}$ charge to incident triangles. So if v lays in t' triangles, then $\alpha'(v) = deg(v) - 4 - \frac{1}{2}t' \geq \frac{1}{2}deg(v) - 4 \geq 0$ since $t' \leq deg(v)$.

If $v \in V_{\Delta}$, then v gives $\frac{1}{2}$ charge to P and gives $\frac{1}{2}$ charge to incident triangles. So if v lays in t' triangles, then $\alpha'(v) = deg(v) - 4 - \frac{1}{2} - \frac{1}{2}t' \geq \frac{1}{2}deg(v) - \frac{1}{2}(9) \geq 0$ since $deg(v) \geq t'$ and $deg(v) \geq 9$.

We have now shown that every vertex of G has final nonnegative charge. We will now show every face has final nonnegative charge to complete the proof.

If a face f of G has $\ell(f) = \ell$, then $\alpha'(f) = \ell - 4 + r + \frac{1}{2}d$ where r is the charge received from vertices in $V_{[5,7]}$ and d is the number of vertices incident to f in $V_{[8,\Delta]}$.

If $\ell(f) \geq 4$, then f receives no charge from any vertex and $\alpha'(f) = \ell - 4 \geq 0$.

If $\ell(f) = 3$, then f receives $\frac{deg(v)-4}{deg(v)}$ from vertices in $V_{[5,7]}$ and receives $\frac{1}{2}$ from vertices in $V_{[8,\Delta]}$.

If f contains a vertex in $V_{[3,4]}$, then the other two vertices of f must be in $V_{[\Delta-1,\Delta]}$ by our first claim. This means $\alpha'(f) = 3 - 4 + 2(\frac{1}{2}) = 0$.

If f contains a vertex in V_5, then the other two vertices of f must be in $V_{[\Delta-2,\Delta]}$ where $\Delta - 2 \geq 7$. So $\alpha'(f) \geq 3 - 4 + \frac{1}{5} + 2(\frac{3}{7}) > 0$.

If f contains vertices in $V_{[6,\Delta]}$, then $\alpha'(f) \geq 3 - 4 + 3(\frac{1}{3}) = 0$ which concludes the proof.

□

Chapter 3

List-Edge-Coloring Triangulations with $\Delta = 5$

We will primarily discuss triangulations with maximum degree 5 in this chapter. We begin exploring the properties of such graphs in the following section before list-edge-coloring them in later sections.

3.1 Triangulations with $\Delta = 5$

Lemma 3.1. *Let G be a triangulation with $\Delta(G) = 5$. If $v \in V(G)$, then $3 \leq deg(v) \leq 5$.*

Proof. There are no triangulations which contain leafs. The only triangulation with a vertex of degree 2 is the triangle. Since $\Delta(G) = 5$ we get $3 \leq deg(v) \leq \Delta(G) = 5$. □

Lemma 3.2. *Let G be a triangulation with $\Delta(G) = 5$ and let $V_x = \{v \in V(G) | deg(v) = x\}$. It follows that* $12 = 3|V_3| + 2|V_4| + |V_5|$

Proof. Let n and e be the number of vertices and edges in G respectively. Since G is a triangulation we know $e = 3n - 6$. By the Degree-Sum formula we get the following:

$$e = 3n - 6$$

$$\Leftrightarrow \frac{1}{2} \sum_{v \in V(G)} deg(v) = 3n - 6$$

$$\Leftrightarrow \sum_{v \in V(G)} deg(v) = 6n - 12$$

$$\Leftrightarrow 12 = 6n - \sum_{v \in V(G)} deg(v)$$

$$\Leftrightarrow 12 = 6(|V_3| + |V_4| + |V_5|) - (3|V_3| + 4|V_4| + 5|V_5|)$$

$$\Leftrightarrow 12 = 3|V_3| + 2|V_4| + |V_5|$$

□

From the count given by Lemma 3.2 we see there is a small set of linear combinations of $|V_3|, |V_4|, |V_5|$ which give 12. The different linear combinations are organized into cases and listed in Table 3.1.

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\lvert V_3\rvert$	4	3	3	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0
$\lvert V_4\rvert$	0	0	1	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5	6
$\lvert V_5\rvert$	0	3	1	6	4	2	0	9	7	5	3	1	12	10	8	6	4	2	0

Table 3.1: Linear combinations of $|V_3|$, $|V_4|$, $|V_5|$ *satisfying* $12 = 3|V_3| + 2|V_4| + |V_5|$.

If G is a triangulation with $\Delta(G) = 5$, then we will show G is 6-list-edge-colorable. We see Cases 1, 7, and 19 are not possible for G. In the next section we will show that a number of cases have a unique embedding in the plane which is 6-list-edge colorable.

3.2 6-list-edge-coloring case 4 and cases 13 through 18

Lemma 3.3 (Case 4)**.** *There is a unique 8-vertex triangulation with* $|V_3| = 2$ *and* $|V_5| = 6$. *Moreover, this graph is* G_4 *(pictured on the left-hand side of Figure 3.1), and* G_4 *is 6-list-edge-colorable.*

Proof. Let G be an 8-vertex triangulation with $|V_3| = 2$ and $|V_5| = 6$. Consider a 5-vertex v in G and observe that since G is a triangulation, the neighbourhood of v contain a 5-cycle C. Let U consist of the two vertices of G that are not in $C \cup \{v\}$.

If C contains both 3-vertices, then there are at most 6 edges between C and U (at most two from each of the three 5-vertices on C). On the other hand, U consists of two 5-vertices in this case, meaning that are at least 8 edges between U and C. Hence C contains at most one 3-vertex (and in fact any 5-vertex in G is adjacent to at most one 3-vertex).

Suppose now that C contains only 5-vertices, i.e, U consists of two 3-vertices. Since 5-vertices are adjacent to at most one 3-vertex, each 5-vertex must be adjacent to at least one

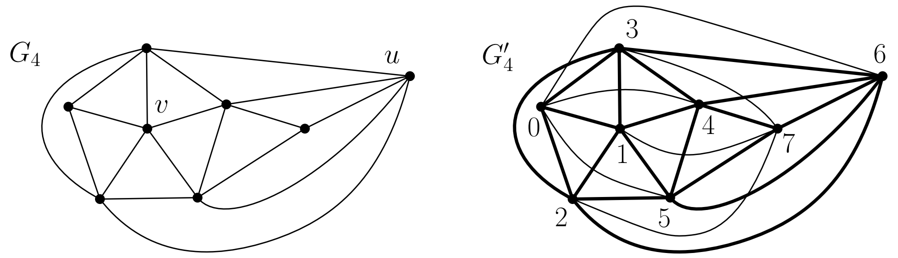


Figure 3.1: The graphs G_4 and G'_4 from the proof of Lemma 3.3. The bolded edges in G'_4 are a copy of G_4, and the numerical labelling corresponds to the input for Computation A.1.

non-consecutive vertex on C. However, planarity makes this impossible. Hence C contains exactly one 3-vertex.

Let u be the 5-vertex in U. In order to have enough degree, it must be adjacent to four of the vertices on C (i.e. all the 5-vertices on C), and to the other vertex in U as well. The fact that G is a triangulation forces the two neighbours of the 3-vertex on C to also be adjacent. The final edges of G (between the 3-vertex in U and the two 5-vertices on C still in need of degree) are thus forced, and we get that G is the graph G_4 depicted on the left-hand side of Figure 3.1.

The graph G'_4 on the right-hand side of Figure 3.1 is a 6-regular graph on an even number of vertices which contains G_4 as a subgraph. Labelling the vertices of G'_4 as $0, 1, \ldots, 7$ (as indicated in the figure), we can input G'_4 into Algorithm 1 and get that $\sum_{F \in OF(G')} sgn(F) \neq 0$ (see Computation A.1 in the Appendix). Hence, by Theorem 2.5, G'_4 (and hence G_4) is 6-list-edge-colorable. $\square$

Lemma 3.4 (Case 18)**.** *There is a unique 7-vertex triangulation with* $|V_4| = 5$ *and* $|V_5| = 2$*. Moreover, this graph is* G_{18} *(pictured in the center of Figure 3.2), and* G_{18} *is 6-list-edge-colorable.*

Proof. Let G be a 7-vertex triangulation with $|V_4| = 5$ and $|V_5| = 2$. Let $v \in V(G)$ have degree 5, and consider a 5-cycle C contained in $N(v)$. There is only one vertex outside the $C \cup \{v\}$, call it u.

Suppose first that there are consecutive vertices x, y, z on C such that x and z are adjacent. Consider the separating cycle xvz in the plane (see the left-most picture in Figure 3.2, where this cycle is bolded). In order for u to have degree at least 4, it must be on the opposite side of this cycle as compared to y. However that means that y has degree only 3, contradiction.

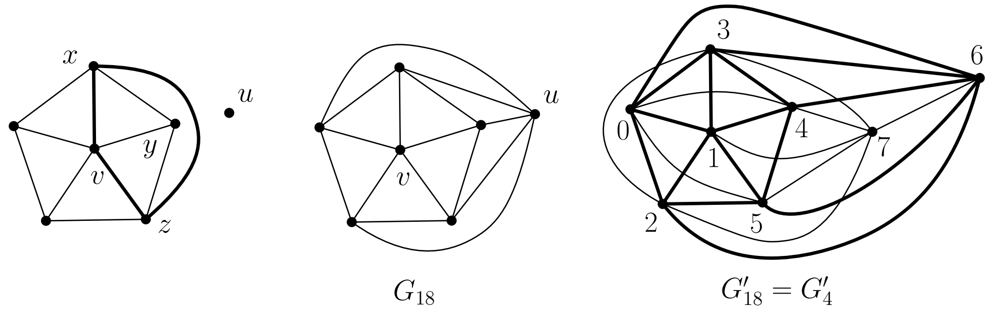


Figure 3.2: Three graphs from the proof of Lemma 3.4. The bolded edges in G'_{18} are a copy of $G_1 8$, and the numerical labelling shows that G'_{18} = G'_4.

We now know that no non-consecutive vertices on C are adjacent. Hence they must all be adjacent to u, forcing the graph G_{18} in Figure 3.2). The graph G'_{18} in Figure 3.2 is a 6-regular graph on an even number of vertices which contains G_{18} as a subgraph. In fact, labelling the vertices of G'_{18} as $0, 1, \ldots, 7$ (as indicated in the figure), we can compare it to Figure 3.1 and observe that G'_{18} = G'_4 (although $G_4 \nsubseteq G_{18}$, $G_{18} \nsubseteq G_4$). Since we showed that G'_4 is 6-list-edge-colorable in the proof of Lemma 3.3, we now also know that G_{18} is 6-list-edge-colorable.

Alternatively we can show that G_{18} is 6-list-edge-colorable by providing the 6-kernel-perfect-labeling illustrated in figure 3.3. We see the labels at each vertex have a total ordering. Note edges with ends labeled with one of the following pairs has outdegree at least 6; $\{(1,3),(1,2),(1,1),(1,b),(1,a),(2,2),(2,a),(a,a)\}$. A quick check will verify every edge in figure 3.3 has outdegree at most 5. We will now show our labeling of G_{18} yields no directed odd cycle in $L(G_{18})$. We will do so by iteratively deleting edges which cannot be in an odd cycle until we are left with the bipartite graph H, illustrated by the blue edges of figure 3.3. Note that the edge with ends labeled $(5, d)$, call it e_1 will have all adjacent edges directed towards it in the line graph, so it cannot be in a directed odd cycle. By deleting e_1 we may also delete the edge with ends labeled $(5, c)$ since the edge it is directed towards is e_1. This means the edges with ends labeled $(4, d)$ can now be deleted. Doing so leaves the edge with ends labeled $(3, c)$ and the edge with ends labeled (c, d) free to be deleted. So we are left with the bipartite subgraph H which cannot yield a directed odd cycle in the line graph. By definition 2.1 we see our labeling in figure 3.3 is a 6-kernel-perfect-labeling and by corollary 2.2, G_{18} is 6-list-edge-colorable. $\square$

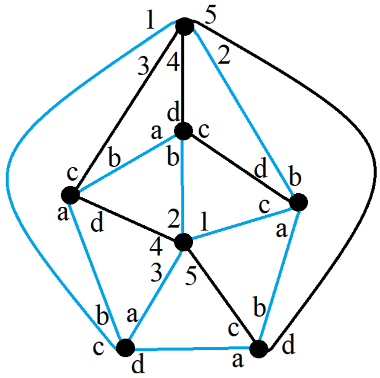


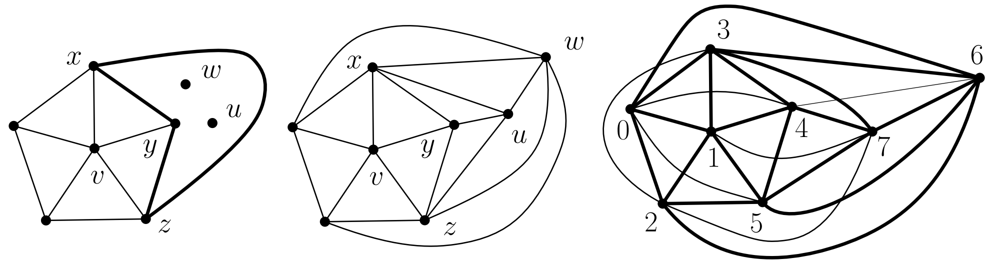
Figure 3.3: A 6-kernel-perfect-labeling of G_{18} where the blue edges form the bipartite subgraph H.

Lemma 3.5. (Case 17) *There is a unique 8-vertex triangulation with $|V_4| = 4$ and $|V_5| = 4$. Moreover, this graph is G_{17} (pictured in the center of Figure 3.4), and G_{17} is 6-list-edge-colorable.*

Proof. Let G be an 8-vertex triangulation with $|V_4| = 4$ and $|V_5| = 4$. Let v be a 5-vertex in G. Since G is a triangulation, $N(v)$ contains a 5-cycle, C. Let u, w be the two vertices of G not in $C \cup \{v\}$.

Suppose first that there are consecutive vertices x, y, z on C such that x and z are adjacent. Consider the separating cycle xyz in the plane (see the left-most picture in Figure 3.4, where this cycle is bolded). In order for y to have degree at least 4, at least one of u, w (without loss, say u) must be on the opposite side of this cycle as compared to v. However, since u must also have degree at least 4, in fact w must also be on the opposite side of xyz as compared to v. Moreover, in order to have degree at least four, both u, w are adjacent to all of x, y, z. However this implies that x is adjacent to v, the two vertices before and after it on C, z, as well as w, u. That is, x has degree at least 6, contradiction

Now suppose that there are two consecutive 5-vertices on C, say x, y. Since neither can be adjacent to any non-consecutive vertex on C, both x and y must be adjacent to both u and w. It must either be the case that the cycle xyw separates v and u in the plane, or that the cycle xyu separates v and w in the plane; suppose, without loss, that it is the former. However, then the only possible neighbors of u are x, y, w, contradicting the fact that u must have degree at least 4.



G_{17} $\quad$ $G'_{17} = G'_{18} = G'_4$

Figure 3.4: Three graphs from the proof of Lemma 3.5. The bolded edges in G'_{17} are a copy of G_{17}, and the numerical labelling shows that G'_{17} = G'_{18} = G'_4.

We now know that there are at most two 5-vertices on C. In fact, we claim that there must be exactly two 5-vertices on C. If there is only one 5-vertex on C, then both u, w have degree 5, and so there must be at least 8 edges between C and $\{u, w\}$. On the other hand, C contains only one 5-vertex along with four 4-vertices, so there are at most (in fact, exactly) 6 edges from C to $\{u, w\}$, which is a contradiction.

We now know that there are exactly two 5-vertices on C, and they are non-consecutive. Say x, y, z are consecutive vertices on C, x, z are 5-vertices. Since x, z cannot be adjacent to any no-consectuive vertices on C, they must both be adjacent to both u and w. It must either be the case that the cycle $xyzw$ separates v and u in the plane, or that the cycle $xyzu$ separates v and w in the plane; suppose, without loss, that it is the former. Since u has degree at least 4 it must be adjacent to all of x, y, z, w, and must be a 4-vertex. This forces w to be a 5-vertex that is adjacent to all vertices on C except for y, giving the graph G_{17} in Figure 3.4.

The graph G'_{17} in Figure 3.4 is a 6-regular graph on an even number of vertices which contains G_{17} as a subgraph. In fact, labelling the vertices of G'_{17} as $0, 1, \ldots, 7$ (as indicated in the figure), we can compare it to Figure 3.1 and observe that G'_{17} = G'_4 = G'_{18} (although none of G_{17}, G_{18}, G_4 are subgraphs of one another). Since we showed that G'_4 is 6-list-edge-colorable in the proof of Lemma 3.3, we now also know that G_{17} is 6-list-edge-colorable.

Alternatively we can show that G_{17} is 6-list-edge-colorable by providing the 6-kernel-perfect-labeling illustrated in figure 3.5. We see the labels at each vertex have a total ordering. Note edges with ends labeled with one of the following pairs has outdegree at least 6; $\{(1, 3), (1, 2), (1, 1), (1, b), (1, a), (2, 2), (2, a), (a, a)\}$. A quick check will verify every edge in figure 3.3 has outdegree at most 5. We will now show our labeling of G_{17} yields no directed

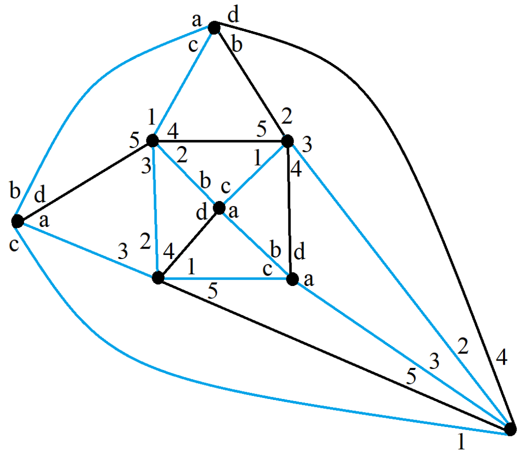


Figure 3.5: A 6-kernel-perfect-labeling of G_{17} where the blue edges form the bipartite subgraph H.

odd cycle in $L(G_{17})$. We will do so by iteratively deleting edges which cannot be in an odd cycle until we are left with the bipartite graph H, illustrated by the blue edges of figure 3.5. We first delete the edge with ends labeled $(5, d)$ and the edge with ends labeled $(5, 5)$. This will free the edge labeled $(4, 5)$ to be deleted. Last, we can delete all edges with ends labeled $(4, d)$. So we are left with the bipartite subgraph H which cannot yield a directed odd cycle in the line graph. By definition 2.1 we see our labeling in figure 3.5 is a 6-kernel-perfect-labeling and by corollary 2.2, G_{17} is 6-list-edge-colorable.

□

Lemma 3.6. *If G is a triangulation with $\Delta(G) = 5$ which is in case 16, then G is 6-list-edge colorable.*

Proof. Let G be a 9-vertex triangulation with $|V_4| = 3, |V_5| = 6$. Let v be a 5-vertex in G. Since G is a triangulation, $N(v)$ contains a 5-cycle, C. Let U denote the set of 3 vertices not in $C \cup \{v\}$.

Suppose first that there are consecutive vertices x, y, z on C such that x and z are adjacent. Consider the separating cycle xyz in the plane (as in the left-most picture in Figure 3.4, where this cycle is bolded). In order for y to have degree at least 4, at least one vertex from U must be on the opposite side of this cycle as compared to v. However, since all vertices in G have degree at least 4, in fact at least two vertices from U must also be on the opposite side of xyz as compared to v. If just two of the U-vertices are there, then as argued above in the proof of

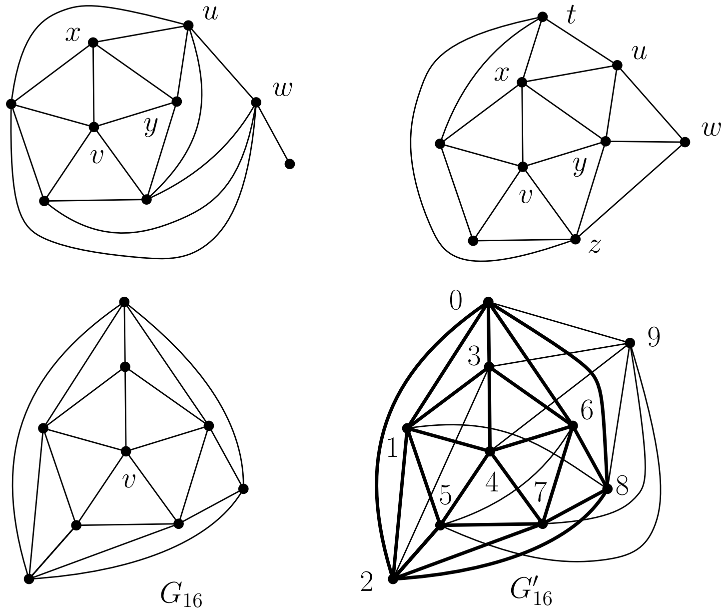


Figure 3.6: Four graphs from the proof of Lemma 3.6. The bolded edges in G'_{16} are a copy of G_{16}, and the numerical labelling corresponds to the input for Computation A.3.

Lemma 3.5, this means that x has degree at least 6, contradiction. So, in fact, all three vertices of U must be on the opposite side of xyz as compared to v. However this means that it is not possible for both of the two other vertices on C (besides x, y, z) to have degree at least four, due to planarity. Hence, no non-consecutive vertices on C are adjacent. In fact, since we choose v arbitrarily, this means that the neighbourhood of any 5-vertex in G induces a 5-cycle.

Suppose now that there are two consecutive 4-vertices on C, x, y. Since G is a triangulation, x, y must be adjacent to a common $u \in U$, and moreover, u must be adjacent to the two other neighbours of x and y on C (see the top-left picture in Figure 3.6). Since there is at most one 4-vertex in U, we can choose $w \in U$, $w \neq u$ such that $deg(w) = 5$. The vertex w, since it is not adjacent to x and y, must be adjacent to all of the other three vertices on C, as well as u, and as well as the third vertex in U. However, this means that the third vertex in U cannot be adjacent to v, x, y, u (since u already has degree 5 now). So, in order for this vertex to have degree at least four, it must be adjacent to all three vertices on C besides x, y, which is impossible by planarity. Hence, C has no consecutive 4-vertices.

Suppose now that C has three consecutive 5-vertices, x, y, z. Since C is an induced cycle in G, x must have two neighbours in U, say t, u. However we also know that $N(x)$ induces a 5-cycle, forcing $t \sim u$, and two edges from t, u to C, including say uy (see the top-right picture in Figure 3.6). The neighborhood of y also induces a 5-cycle, and since $y \nsim t$ (otherwise $N(x)$ would not be an induced cycle), we get that $y \sim w$, where $U = \{t, u, w\}$. The vertex z cannot be adjacent to u (otherwise $N(y)$ would not be an induced cycle), so since z is a 5-vertex and C is an induced cycle, z must be adjacent to t. Since the two vertices on C besides x, y, z must both have degree at least 4, in fact they both have degree exactly four. However, this contradicts the fact that C cannot have two consecutive 4-vertices.

We now know that C cannot have three consecutive 5-vertices, so in particular it has at most three 5-vertices. Since no 4-vertices on C can be adjacent, C must in fact have exactly three 5-vertices, with its two 4-vertices being non-consecutive. Since G is a triangulation, this forces G to be the graph G_{16} pictured on the bottom-left in Figure 3.6. The graph G'_{16}, pictured on the bottom-right of Figure 3.6, is a 6-regular graph on an even number of vertices which contains G_{16} as a subgraph. Labelling the vertices of G'_{16} as $0, 1, \ldots, 9$ (as indicated in the figure), we can input G'_{16} into Algorithm 1 and get that $\sum_{F \in OF(G')} sgn(F) \neq 0$ (see Computation A.3 in the Appendix). Hence, by Theorem 2.5, G'_{16} (and hence G_{16}) is 6-list-edge-colorable.

$\square$

Lemma 3.7. (Case 15) *There is a unique 10-vertex triangulation with $|V_4| = 2$ and $|V_5| = 8$. Moreover, this graph is G_{15} (pictured in the bottom-left of Figure 3.8), and G_{15} is 6-list-edge-colorable.*

Proof. Let G be a 10-vertex triangulation with $|V_4| = 2, |V_5| = 8$. Let v be a 4-vertex in G, and note that its neighbours contain a 4-cycle, C. By planarity, C must contain a pair of non-adjacent vertices, say x, y. We claim that we can choose x, y so that the other pair of vertices on C, say u, w, are both 5-vertices. If not, then u, w are adjacent and $deg(u) = 4$, without loss (see the left-most picture in Figure 3.7). Note that this means that x has degree 5, since G has only two 4-vertices. Hence, x must have two neighbours that are separated from v by the triangle uxw (bolded in the picture). Since u has no more neighbours however, it is not possible to do this, since G is a triangulation. Hence we may indeed assume that u, w are both 5-vertices.

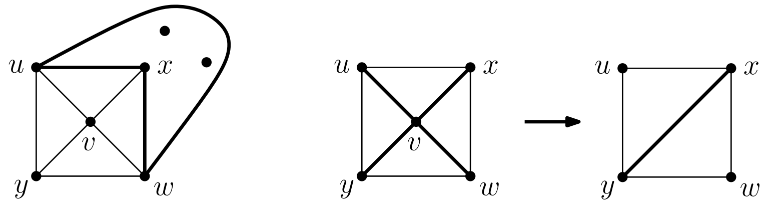


Figure 3.7: Three graphs from the proof of Lemma 3.7. The two right-most images show the transition from G to G'.

Define G' to be the triangulation obtained from G by deleting v and joining x and y inside the 4-face created by the deletion of v (see the right two pictures in Figure 3.7 for this transition). Note that in G', all vertices have the same degree as in G, except for u, w, which both went from degree 5 to degree 4. Hence G' is a 9-vertex triangulation with $|V_4| = 3$ (lost v, gained u, w) and $|V_5| = 6$. Hence, by Lemma 3.6, $G' = G_{16}$. In the top-left of Figure 3.8, see a copy of G_{16} with three edges labelled e_1, e_2, e_3. These are the only 3 edges in G_{16} that could be the edge xy in G', given that after deletion of the edge xy, all four vertices on the 4-face created would have degree at most 4. The version of G that would result from each of e_1, e_2, e_3 being xy, respectively, are also pictured in Figure 3.8. It is not hard to see that these three graphs are isomorphic, so we indeed get that G is unique (call it G_{15}). The graph G'_{15} pictured in Figure 3.8 is a 6-regular graph on an even number of vertices which contains G_{15} as a subgraph. Labelling the vertices of G'_{15} as $0, 1, \ldots, 11$ (as indicated in the figure), we can input G'_{15} into Algorithm 1 and get that $\sum_{F \in OF(G')} sgn(F) \neq 0$ (see Computation A.2 in the Appendix). Hence, by Theorem 2.5, G'_{15} (and hence G_{15}) is 6-list-edge-colorable. $\square$

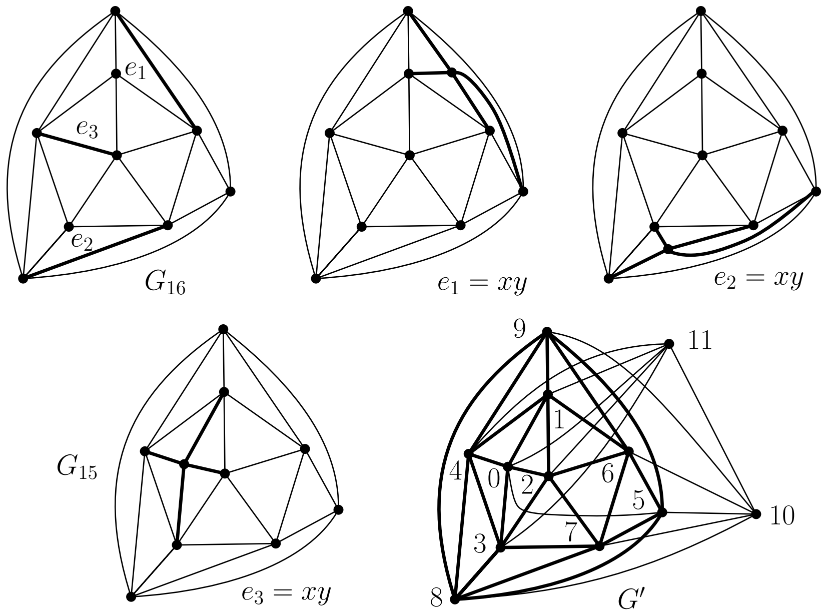


Figure 3.8: Five graphs from the proof of Lemma 3.7. The bolded edges in G' are a copy of G (in particular, the e_3 = xy image). The numerical labelling of G' corresponds to the input for Computation A.2.

Lemma 3.8. (Case 14) *There is no 11-vertex triangulation with* $|V_4|$ = 1 *and* $|V_5|$ = 10.

Proof. Let G be an 11-vertex triangulation with $|V_4|$ = $1, |V_5|$ = 10. Let v be the 4-vertex in G, and note that its neighbours induce a 4-cycle, C. By planarity, C must contain a pair of non-adjacent vertices, say x, y. Define G' to be the triangulation obtained from G by deleting v and joining x and y inside the 4-face created by the deletion of x (see the two right-most images in Figure 3.7). Note that in G', all vertices have the same degree as in G, except for two which went from degree 5 to degree 4. Hence G' is a 10-vertex triangulation with $|V_4|$ = 2 and $|V_5|$ = 8. Hence, by Lemma 3.7, G' = G_{15}. However, looking at the image of G_{15} in Figure 3.8 we see that it does not have two 4-vertices on a 4-cycle, contradiction. □

Lemma 3.9. *If G is a triangulation with* $\Delta(G)$ = 5 *which is in case 13, then G is* 6*-list-edge-colorable.*

Proof. Observe that if G is a triangulation in case 13, then it is a regular planar graph of maximum degree 5 which is known to have $\chi'(G)$ = 5 and therefore is 5-list-edge-colorable by the previsouly mentioned result of Ellingham and Goddyn [11]. □

3.3 Proof of our result via minimality

We will now show that the remaining cases cannot be an edge minimal triangulation with $\Delta = 5$. It is worth noting that, as part of this argument, we will need to appeal to the fact that planar graphs cannot contain $K_{3,3}$, nor can they contain a subdivision of $K_{3,3}$.

Theorem 3.3. *If G is a triangulation with $\Delta(G) = 5$, then $\chi'_\ell(G) \leq \Delta(G) + 1$.*

Proof. Let G be an edge-minimal counterexample. So, in particular, there is an edge list assignment L of G with $|L(e)| \leq 6$ for all $e \in E(G)$, such that G is not L-edge-colorable. By Lemmas 3.3, 3.4, 3.5, 3.6, 3.7, 3.8 in the previous section and our comments at the end of the introduction, it is sufficient to show that G cannot have the degree sequence prescribed in any of the cases 2, 3, 5, 6, or 8–12 (as listed in Table 3.1).

Claim 3.1. *If $v \in V_3$, then $N(v) \subseteq V_5$*

Proof of Claim. Let $v \in V_3$, and let $x \in N(v)$. Since G is a triangulation $N(v)$ induces a triangle, hence $G' = G - v$ is also a triangulation. We know that $deg_G(x) = \{3, 4, 5\}$ by Lemma 3.2.

If $deg_G(x) = 3$, then $deg_{G'}(x) = 2$. However G' is a triangulation, which implies that $G' = K_3$, and hence that G has no vertex of degree 5. So, we may assume that $deg_G(x) = 4$.

By the minimality of G, we know that $\chi'_\ell(G') \leq 6$. Let ϕ be an L-edge-coloring of G' (where L is restricted to G'). For the three edges e incident to v in G, let $L^-(e)$ be obtained from $L(e)$ by removing all colors used by ϕ on the edges of G' adjacent to e. Since $deg_G(x) = 4$, we get that $deg_{G'}(x) = 3$. This means that vx sees at most 3 colors in ϕ, leaving 3 available colors for $L^-(vx)$. Let y, z be the other two vertices in $N_G(v)$ (aside from x). Since $\Delta(G) = 5$ we know $deg_G(y),\ deg_G(z) \leq 5$. Hence vy and vz see at most 4 colors in ϕ, leaving to 2 available colors for each of $L^-(vy)$ and $L^-(vz)$. In order to extend ϕ to an L-edge-coloring of G, we can first choose distinct colors from $L^-(vy)$ and $L^-(vz)$, and then, since $|L^-(vx)| \geq 3$, there will be at least one color left that we can use on vx. Hence, G is not a counterexample. □

Claim 3.1 automatically precludes G having the degree sequence prescribed by any of the cases 3, 6, or 12, since each has a 3-vertex, but less than three 5-vertices. If G has the degree

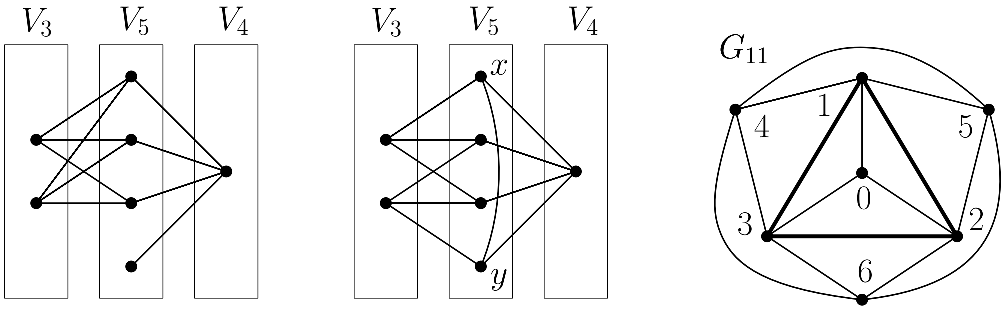


Figure 3.9: Three graphs from the proof of Theorem 4.2. The triangle T is bolded in G_{11}, and the numerical labelling corresponds to the labels of G'_4 in Figure 3.1.

sequence of case 2 then $|V_3| = |V_5| = 3$, but then Lemma 3.1 implies that G contains a copy of $K_{3,3}$, which contradicts planarity. We can make a similar argument for case 5, as follows.

Claim 3.2. *G cannot have the degree sequence prescribed by case 5.*

Proof of Claim. Suppose that G is a 7-vertex triangulation with $|V_3| = 2$, $|V_4| = 1$, $|V_5| = 4$. By Claim 3.1 the two 3-vertices are only adjacent to 5-vertices. This also means that the neighbourhood of the single 4-vertex must consist of all the 5-vertices. If the two 3-vertices share the same three 5-vertices as neighbors, then G has a copy of $K_{3,3}$ (see the left-most image in Figure 3.9). So, there are two 5-vertices (say, x, y) that are each adjacent to only one 3-vertex each (with these 3-vertices being distinct); see the center image in Figure 3.9. Then x, y must therefore be adjacent, in order to have enough degree. By deleting the edge between x and V_4, and then suppressing x, we again get a $K_{3,3}$. $\square$

We will now deal with each of the remaining possible degree sequences for G: those prescribed by cases 8–11.

Claim 3.3. *G cannot have the degree sequence prescribed by any of the cases 8–11.*

Proof of Claim. Suppose, on the contrary, that G is a triangulation which has a degree sequence prescribed by one of the cases 8–11. In each case, this means G has a single 3-vertex, say v. Since G is a triangulation, the neighbourhood of the 3-vertex induces a triangle T, and by Claim 3.1, the three vertices in T are all 5-vertices in G. Consider the triangulation $G' = G - v$.

Suppose first that G falls into one of cases 8, 9, or 10. In moving from G' to G, we lost our one 3-vertex, we lost three 5-vertices, and we gained three 4-vertices (which induce T). So the degree sequence of G' is now, respectively (for cases 8, 9, 10): $|V_4| = 3, |V_5| = 6$; $|V_4| = 4$, $|V_5| = 4$, or; $|V_4| = 5$, $|V_5| = 2$. By Lemmas 3.6, 3.5, and 3.4, respectively, this means that G' must be either G_{16}, G_{17}, or G_{18}, as pictured in Figures 3.6, 3.4, 3.2. However, none of these three graphs contain a triangle induced by 4-vertices. Since T is a part of G, this is a contradiction.

We may now assume that G has the degree sequence prescribed by case 11, meaning that G is a 7-vertex triangulation with $|V_3| = 1$, $|V_4| = 3$, and $|V_5| = 3$. In order to have enough degree, each of the three 5-vertices must be adjacent to precisely two of the 4-vertices. Since this means exactly 6 edges between T and the 4-vertices, the three 4-vertices must themselves induce a graph with $\frac{3(4)-6}{2} = 3$ edges. Hence, the 4-vertices induce a triangle, and each must be adjacent to exactly two vertices on T. Hence G must be the graph G_{11} pictured on the right of Figure 3.9 (the edges of T are in bold). However, by labelling the vertices of G_{11} to correspond to the labels of G'_4 in Figure 3.1, we see that G_{11} is actually a subgraph of G_4. By Lemma 3.3, G_{11} is therefore L-edge-colorable. □

We have now eliminated all possible cases and shown there is no edge-minimal counterexample to our hypothesis.

□

As mentioned in the introduction Vizing conjectured that for any graph G it should be true that $\chi'_\ell(G) \leq \Delta + 1$. This conjecture is currently open for planar graphs with $5 \leq \Delta \leq 7$. Our result makes progress on the $\Delta = 5$ case but is limited to triangulations. In the conclusion we will discuss ideas for extending our result for all planar graphs with $\Delta = 5$.

Chapter 4

List-edge-coloring graphs with precolored subgraphs

In this chapter we present our contribution to Marcotte and Seymour's precoloring question and discuss its connection to list-edge-coloring.

4.1 Marcotte & Seymour's Question and our results

Recall the question of Marcotte and Seymour [22] discussed in the introduction:

Question 4.1. *"Given a graph G with maximum degree Δ and a subgraph H of G that has been $(\Delta + t)$-edge-colored, can the edge-precoloring of H be extended to a $(\Delta + t)$-edge-coloring of G?"*

If t is huge – say at least $\Delta - 1$ – then the answer is *yes*, and moreover, the extension can be done greedily. This is because an edge in G sees at most $2(\Delta - 1)$ other edges, and when $t \geq \Delta - 1$, this value is at most $\Delta + t - 1$. If the maximum degree of H is Δ then this threshold for t is actually sharp. To see this, consider the graph G shown in Figure 4.1, formed by taking a copy of $K_{1,\Delta}$ with one edge colored Δ and the rest uncolored, and joining each leaf to $\Delta - 1$ distinct new vertices via edges colored $1, 2, \ldots, \Delta - 1$. Then G has maximum degree Δ, as does its edge-precolored subgraph. However, in order to extend the edge-precoloring to a $(\Delta + t)$-edge-coloring of G, we need $\Delta - 1$ new colors, which forces $t \geq \Delta - 1$.

Given the above paragraph, Question 4.1 is only interesting when $d := \Delta(H)$ is strictly less than Δ. Here, we get a natural barrier to extension when $d > t$, via nearly the same example as above. Let G be the graph shown in Figure 4.2, formed by taking an (uncolored) copy of $K_{1,\Delta}$ and joining each leaf to $d < \Delta$ distinct new vertices, via edges colored $1, 2, \ldots, d$. The resulting

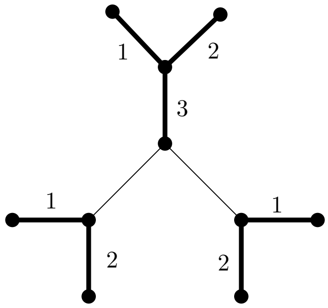


Figure 4.1: A graph G with maximum degree $\Delta = 3$ with a precolored subgraph of maximum degree Δ. In order to extend the edge-precoloring to a $(\Delta + t)$-edge-coloring of G we need $t \geq \Delta - 1$.

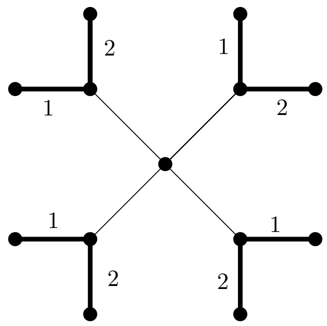


Figure 4.2: A graph G with maximum degree $\Delta = 4$ and a precolored subgraph of maximum degree $d = 2$. In order to extend the edge-precoloring to a $(\Delta + t)$-edge-coloring of G we need $t \geq d$.

graph G has maximum degree Δ, and contains a precolored subgraph H with maximum degree d. However, in order to extend the edge-precoloring to G, we need Δ new colors, meaning that for a $(\Delta + t)$-edge-coloring of G, we need $d \leq t$.

If it happened that H was edge-colored efficiently (i.e. using at most $\chi'(H)$ colors), then our problem would be significantly reduced. In this special situation, one could use a completely new set of $\chi'(G - E(H))$ colors to extend to an edge-coloring of G with at most the following number of colors (according to Vizing's Theorem):

$$\chi'(G - E(H)) + \chi'(H) \leq \chi'(G) + \chi'(H) \leq \Delta + d + 2. \tag{4.1}$$

That is, when H has been edge-colored efficiently, the answer to Question 4.1 is *yes* whenever $d \leq t - 2$. Since extension can be impossible when $d > t$ (according to the above paragraph), this makes $d \in \{t - 1, t\}$ the only interesting values in this case, with further restrictions if any of the inequalities in (4.1) are strict. For example, if both G and H have chromatic index equal to their maximum degrees, then the coloring described above works whenever $d \le t$, and hence we get a sharp threshold. Of course, this only works when H has been edge-precolored efficiently, and in general we have no control over the edge-precoloring on H.

We make progress on Question 4.1 in this chapter by focusing on planar graphs. In particular, we prove that the answer to Question 4.1 is *yes* whenever $d \le t$, provided d is small enough or Δ is large enough. As discussed above, the $d \le t$ assumption is sharp.

Theorem 4.1. *Let G be a planar graph of maximum degree at most Δ, let t be a positive integer, and let H be a subgraph of G that has been $(\Delta + t)$-edge-colored. If H has maximum degree at most d, then the edge-precoloring can be extended to a $(\Delta + t)$-edge-coloring of G provided that either:*

1. $d \le t - 4$, *or*
2. $t - 3 \le d \le t$ *and*

$$\Delta \ge \begin{cases} 16 + d, & \textit{if } d = t, \\ 9 + d, & \textit{if } d = t - 1, \\ 8 + d, & \textit{if } d = t - 2, \\ 7 + d, & \textit{if } d = t - 3. \end{cases}$$

Theorem 4.1 does not include the case $t = 0$, however the requirement of $d \le t$ means that would correspond to H being edgeless. Then the problem is not about precoloring at all, but simply about edge-coloring planar graphs as discussed above.

The case $d = t = 1$ of Theorem 4.2 was previously established by Edwards, Girão, van den Heuvel, Kang, Sereni and the third author [10], with the slightly stronger assumption of $\Delta \ge 19$. (Note that the restriction of our proof for Theorem 4.1 to this case provides a somewhat new proof; both arguments use global discharging, but we discharge in a different way). After the seminal work of Marcotte and Seymour [22], the vertex-version of the precoloring extension problem received much more attention than Question 4.1. Edwards et al. [10] re-initiated this study in their paper, with planar graphs being only one of the many families they considered. The main concern in [10] however is when H is a matching, and in order to guarantee extensions they often impose distance conditions on the edges in the precolored matching. In particular, this means avoiding the issues with t being too small as exhibited in Figures 4.1 and 4.2. Specifically, in addition to the aforementioned result for $d = t = 1$, they showed that if H is an edge-precolored matching in a planar graph G where edges are at distance at least 3 from one another, then any Δ-edge-coloring on H can be extended to G provided $\Delta \geq 20$. More recently, Girão and Kang [14] studied extension from precolored matchings in general graphs, proving that if H is a matching in a (not necessarily planar) graph G where edges are distance at least 9 from each other, then any $(\Delta + 1)$-edge-coloring on H can be extended to a $(\Delta + 1)$-edge-coloring of G.

As state in the introduction we have in fact proved the list-edge-coloring analog of Theorem 4.1. This stronger result is as follows.

Theorem 4.2. *Let G be a planar graph of maximum degree at most Δ, let L be an edge list assignment on G with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$, where t is a positive integer, and let H be a subgraph of G that has been L-edge-colored. If H has maximum degree at most d, then the edge-precoloring can be extended to an L-edge-coloring of G provided that either:*

1. *$d \leq t - 4$, or*

2. *$t - 3 \leq d \leq t$ and*

$$\Delta \geq \begin{cases} 16 + d, & \text{if } d = t, \\ 9 + d, & \text{if } d = t - 1, \\ 8 + d, & \text{if } d = t - 2, \\ 7 + d, & \text{if } d = t - 3. \end{cases}$$

We again omit the case $t = 0$, however the required $d \leq t$ condition means that H is edgeless and hence the best result is that of Theorem 1.2 above. Theorem 4.2 does have something meaningful to say when H is edgeless however: the case $t = 1$ and $d = 0$ gives Theorem 2.8 precisely.

The following section contains some technical results needed for our proof of Theorem 4.2, which comprises Section 5.3. The final section of this chapter, Section 5.4, extends Theorem 4.2 beyond planar graphs. We show that requiring $G - E(H)$ to be planar is sufficient, and in fact "planar" can be replaced by "non-negative Euler characteristic".

4.2 Technical Lemmas

In this section, we gather some technical lemmas that will be needed for the proof of Theorem 4.2.

Edwards et al. [10] applied Theorem 2.2 to obtain a precoloring extension result for bipartite graphs (Theorem 15 of [10]), which we will use as part of our proof. While the result as stated in [10] only applies to classical edge-precoloring, a list-edge-coloring version can be obtained using essentially the same proof:

Theorem 4.3. *Let G be a bipartite multigraph, and let L be an edge list assignment on G with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$. Let H be a subgraph of G that has been L-edge-colored. If H has maximum degree at most d, then the edge-precoloring can be extended to an L-edge-coloring of G provided that $t \geq d$.*

Proof. Let $G' = G - E(H)$. For each edge $e \in E(G')$, let $L'(e)$ be obtained from $L(e)$ by removing all colors used on the edges of H incident to e. Let xy be an arbitrary edge of G'. Now

$$|L'(xy)| \geq |L(xy)| - \deg_H(x) - \deg_H(y) \geq \Delta + t - \deg_H(x) - \deg_H(y).$$

Since $t \geq d \geq \Delta(H)$, this implies that

$$|L'(xy)| \geq \Delta - \deg_H(x), \quad \text{and}$$
$$|L'(xy)| \geq \Delta - \deg_H(y).$$

On the other hand,

$$\deg_{G'}(x) = \deg_G(x) - \deg_H(x) \le \Delta - \deg_H(x), \quad \text{and}$$
$$\deg_{G'}(y) = \deg_G(y) - \deg_H(y) \le \Delta - \deg_H(y).$$

Thus, $|L'(xy)| \ge \max\{\deg_{G'}(x), \deg_{G'}(y)\}$, this inequality holds for all $xy \in E(G')$. By Theorem 2.2, it follows that G' is L'-edge-colorable, and any L'-edge-coloring of G' gives the desired L-edge-coloring of G. $\square$

In what follows and in the main argument, given a graph G, we define $V_i(G) = V_i$ as the set of all vertices $v \in V(G)$ with $\deg(v) = i$, and we define $V_{[a,b]}(G) = V_{[a,b]}$ as $\cup_{i \in [a,b]} V_i$.

Lemma 4.1. *Let G be a graph of maximum degree at most Δ, and let L be an edge list assignment on G with $|L(e)| \ge \Delta + t$ for all $e \in E(G)$. Let H be a subgraph of G with maximum degree at most d. Suppose that H has been L-edge-colored, and that this extends to an L-edge-coloring of $G - e$ for all $e \in E(G) \setminus E(H)$, but not to G.*

Let $A = V_{[a_0,a]}$ and $B = V_{[b_0,\Delta]}$, where a_0, a, b_0 are positive integers with $a_0 \ge t + 1$, $b_0 > a$, and $a + b_0 \ge \Delta + t + 1$. Let X be the bipartite subgraph of $G - E(H)$ induced by the bipartition (A, B). If every vertex $u \in A$ has the property that

$$\deg_X(u) \ge \deg_G(u) - d,$$

then

$$(t + 1 - d)|A| \le \sum_{i=b_0}^{\Delta} (a + i - 1 - (\Delta + t))|V_i|.$$

Moreover, if $a_0 > t + 1$ and $a + b_0 > \Delta + t + 1$ then the above inequality is strict.

Proof. Say that an induced subgraph $J \subseteq X$ is *bad* if

- $\deg_J(u) \ge \deg_G(u) - t$ for all $u \in A \cap V(J)$, and
- $\deg_J(v) \ge a + \deg_G(v) - (\Delta + t)$ for all $v \in B \cap V(J)$.

Notice that for all $u \in A$, $v \in B$,

$$\deg_G(u) - t \geq a_0 - t \geq 1 \tag{4.2}$$

and

$$a + \deg_G(v) - (\Delta + t) \geq a + b_0 - (\Delta + t) \geq 1, \tag{4.3}$$

so that if a bad induced subgraph exists, it has no isolated vertices, and in particular has at least one edge. We will first show that X has no bad induced subgraph, and then show that this implies the desired claim.

Suppose that X has a bad induced subgraph J. Let $G' = G - E(J)$. Since $E(J)$ is nonempty, G' is a proper subgraph of G, so by assumption, the edge-precoloring on H extends to an L-edge-coloring φ of G'. We derive a contradiction by showing we can further extend to an L-edge-coloring of G. To this end, let L^J be the edge list assignment on J defined as follows: for each edge $uv \in E(J)$, $L^J(uv)$ is the set of colors from $L(uv)$ that do not appear on any G'-edge adjacent to uv. Observe that for each $uv \in E(J)$, we have

$$|L^J(uv)| \geq \Delta + t - \deg_G(u) - \deg_G(v) + \deg_J(u) + \deg_J(v).$$

Since J is bad, we have $\deg_J(u) \geq \deg_G(u) - t$, so that

$$|L^J(uv)| \geq \Delta - \deg_G(v) + \deg_J(v) \geq \deg_J(v),$$

and likewise $\deg_J(v) \geq \deg_G(v) + a - (\Delta + t)$ so that

$$|L^J(uv)| \geq a - \deg_G(u) + \deg_J(u) \geq \deg_J(u).$$

Hence, for every $uv \in E(J)$, we have $|L^J(uv)| \geq \max\{d(u), d(v)\}$. By Theorem 2.2, J is L^J-edge-colorable. Now any proper L^J-edge-coloring of J, combined with the L-edge-coloring φ of G', yields a proper L-edge-coloring of G that extends the edge-precoloring of H as desired; contradiction.

Hence, X contains no bad induced subgraph, and so every induced subgraph J of X contains a vertex violating the definition of a "bad" subgraph. By iteratively removing these vertices and counting the edges removed when each vertex is deleted, we see that

$$\begin{aligned} |E(X)| &\le \sum_{u\in A}[\deg_G(u) - t - 1] + \sum_{v\in B}[a + \deg_G(v) - (\Delta + t) - 1] \qquad (4.4)\\ &\le \sum_{u\in A}[(\deg_X(u) + d) - t - 1] + \sum_{v\in B}[a + \deg_G(v) - (\Delta + t) - 1] \\ &= |E(X)| + \sum_{u\in A}[d - t - 1] + \sum_{i=b_0}^{\Delta}(a + i - (\Delta + t) - 1)|V_i|. \end{aligned}$$

Rearranging the last inequality yields

$$(t + 1 - d)|A| \le \sum_{i=b_0}^{\Delta}(a + i - 1 - (\Delta + t))|V_i|,$$

which is the desired conclusion. If we additionally know that $a_0 > t + 1$ and $a + b_0 > \Delta + t + 1$, then inequalities (4.2) and (4.3) become strict. Hence each $u \in A$ and $v \in B$ is contributing a positive amount to the right-hand-side of (4.4). Since the last vertex removed is isolated, this is an overcount, and hence we get a strict inequality. □

4.3 Proof of Theorem 4.2

For fixed values of Δ, t, d, we choose a counterexample (G, H) where the quantity $3|E(G)| + |V_{[2,t+1]}(G)|$ is as small as possible.

Claim 4.1. *The edge-precoloring on H can be extended to an L-edge-coloring of $G - e$ for any $e \in E(G) \smallsetminus E(H)$.*

Proof of Claim. Let any $e \in E(G) \smallsetminus E(H)$ be given, and let $G' = G - e$. Note that (G', H) satisfies the hypotheses of the theorem with Δ, t, d. Exactly two vertices in G' have lower degrees than in G, so $|V_{[2,t+1]}(G')|$ may be as large as $|V_{[2,t+1]}(G)| + 2$. However, since G' has one edge less than G, we still get that

$$3|E(G')| + |V_{[2,t+1]}(G')| < 3|E(G)| + |V_{[2,t+1]}(G)|.$$

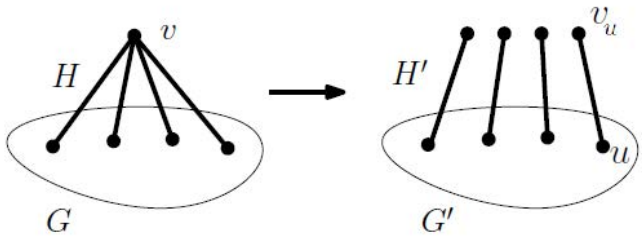


Figure 4.3: Moving from (G,H) to (G',H') in the proof of Claim 4.4.

Hence, by our choice of counterexample, the edge-precoloring of H extends to an L-edge-coloring of G'. □

Claim 4.2. *If $uv \in E(G) \setminus E(H)$, then $\deg_G(u) + \deg_G(v) \geq \Delta + t + 2$.*

Proof of Claim. By Claim 4.1, the edge-precoloring of H can be extended to an L-edge-coloring φ of $G - uv$. The edge uv sees at most $\deg_G(u) + \deg_G(v) - 2$ different colors in φ, so since (G, H, t) is a counterexample, it must be that $\deg_G(u) + \deg_G(v) - 2 \geq \Delta + t$. □

Claim 4.3. *If $v \in V_{[1,t+1]}$, then every edge incident to v in G is also in H.*

Proof of Claim. Assume for contradiction that $v \in V_{[1,t+1]}$ and v is incident to an edge not in H, say uv. By Claim 4.2, we know that $\deg_G(u) + \deg_G(v) \geq \Delta + t + 2$. However, since $\deg_G(v) \leq t + 1$, this implies that $\deg_G(u) \geq \Delta + 1$, a contradiction. □

Claim 4.4. $V_{[2,t+1]} = \varnothing$.

Proof of Claim. Suppose not, and take $v \in V_{[2,t+1]}$. By Claim 4.3, every edge uv incident to v must lie in H.

Let G' and H' be the graphs obtained from G and H, respectively, by deleting v and, for each $u \in N_G(v)$, adding a new vertex v_u adjacent only to u. We precolor each edge uv_u with the same color received by the edge uv in the precoloring of H. See Figure 4.3. Observe that the edge-precoloring of H' extends to G' if and only if the edge-precoloring of H extends to G.

Now G' has the same number of edges as G, and has one fewer vertex in $V_{[2,t+1]}$. As $\Delta(G') \leq \Delta$ and $\Delta(H') \leq d$, our choice of counterexample implies that the edge-precoloring of H' extends to G', but this means that the edge-precoloring of H extends to G as well. □

Claim 4.5. *Every vertex of G is either a leaf incident to an edge in H, or of degree at least $t+2$.*

Proof. This follows by combining Claim 4.3 and Claim 4.4. □

Let F_m be the set of faces in G with exactly m vertices on its boundary having degree 3 or higher in G.

Claim 4.6. $F_0 = F_1 = F_2 = \varnothing$.

Proof of Claim. Suppose that $f \in F_0 \cup F_1 \cup F_2$; we will show a contradiction. We know that $V_2 = \varnothing$ by Claim 4.5, since $t \geq 1$. So, if the boundary of f contains a cycle, then it contains at least three vertices of degree at least three, yielding a contradiction. Thus, the boundary of f contains no cycle. This means that G is a forest, and f is its one face. In particular, G is bipartite. By Theorem 4.3, this implies that the precoloring of H extends to all of G, contradicting our choice of G as a counterexample. □

We now introduce a discharging argument. To each vertex in G assign an initial charge of $\alpha(v) = 3\deg_G(v) - 6$. To each face in G assign an initial charge of $\alpha(f) = -6$. We also define an additional structure P (a "global pot") and assign to it an initial charge of $\alpha(P) = 0$. We discharge along the following rules:

(a) For each m, every face $f \in F_m$ takes $\frac{6}{m}$ from each vertex of degree 3 or higher on its boundary.

(b) Every vertex $v \in V_1$ takes 3 from its neighbor.

In the special case where $t = d + \ell$ for $\ell \in \{0, 1, 2, 3\}$, we also add the following rules:

(c) For every vertex $v \in V_i$, where $i \in \{t+2, \ldots, t+5-\ell\}$:

$$v \text{ takes } t + 6 - \ell - i \text{ from } P.$$

(d) For every vertex $v \in V_j$, where $j \in \{\Delta - 3 + \ell, \ldots, \Delta\}$:

$$v \text{ gives } \tfrac{q(j)(q(j)+1)}{2(\ell+1)} \text{ to } P, \text{ where } q(j) = j - \Delta + 4 - \ell.$$

While it is not immediately obvious, discharging rules (c) and (d) never apply to the same vertex, due to the following claim.

Claim 4.7. *If* $t = d + \ell$ *for some* $\ell \in \{0, 1, 2, 3\}$, *then* $\Delta - 3 + \ell > t + 5 - \ell$.

Proof of Claim. We get the desired inequality if and only if $\Delta + 2\ell > 8 + t$. If $\ell = 0$, then we have $d = t$, so the hypothesis of Theorem 4.2 yields

$$\Delta + 2\ell = \Delta \geq 16 + d = 16 + t > 8 + t.$$

If $\ell \in \{1, 2, 3\}$ we may rewrite hypothesis of Theorem 4.2 as

$$\Delta \geq 10 + d - \ell = 10 + (t - \ell) - \ell = 10 + t - 2\ell, so$$

$$\Delta + 2\ell \geq 10 + t > 8 + t.$$

$\square$

Using Euler's formula for planar graphs, the sum of initial charges is at most -12:

$$\begin{aligned} \alpha(P) + \sum_{v \in V(G)} \alpha(v) + \sum_{f \in F(G)} \alpha(f) &= 0 + \sum_{v \in V(G)} (3\deg_G(v) - 6) + \sum_{f \in F(G)} (-6) \\ &= 6|E(G)| - 6|V(G)| - 6|F(G)| \leq 6(-2) = -12. \end{aligned} \tag{4.5}$$

For each graph element x (either a vertex, a face, or the global pot), let $\alpha'(x)$ denote the final charge of x. Since each discharging rule conserves the total charge, we see that $\sum_x \alpha'(x) = \sum_x \alpha(x) = -12$. We will achieve our desired contradiction by showing that the final charge of each element is nonnegative.

First consider a face f. By Claim 4.6, $f \in F_m$ for $m \geq 3$. So according to discharging rule (a) (the only rule affecting f),

$$\alpha'(f) = (-6) + m(\tfrac{6}{m}) = 0.$$

Now consider the global pot P. We know $\alpha(P) = 0$ and that the charge of P is unaffected when $d \le t - 4$, so the following claim precisely amounts to showing showing that $\alpha'(P) > 0$ when $t - 3 \le d \le t$.

Claim 4.8. *If $t = d + \ell$ for some $\ell \in \{0, 1, 2, 3\}$, then*

$$\sum_{i=t+2}^{t+5-\ell} (t + 6 - \ell - i)|V_i| < \sum_{j=\Delta-3+\ell}^{\Delta} \frac{q(j)(q(j) + 1)}{2(\ell + 1)} |V_j|. \tag{4.6}$$

Proof of Claim. For each $k \in \{0, \ldots, 3-\ell\}$, define $A_k = V_{[t+2,t+5-\ell-k]}$ and $B_k = V_{[\Delta-3+\ell+k,\Delta]}$ and let X_k be the bipartite subgraph of $G - E(H)$ induced by the partition (A_k, B_k). We will show we can apply Lemma 4.1 for each value of k, and then we will sum the resulting inequalities to get our desired result. For fixed k, this means we want to apply Lemma 4.1 with parameter choices

$$a_0 = t + 2, \qquad a = t + 5 - \ell - k,$$
$$b_0 = \Delta - 3 + \ell + k,$$

and hence to do so we must verify that $a_0 \ge t + 1$ (true) and that $a + b_0 \ge \Delta + t + 1$, which is true since

$$(t + 5 - \ell - k) + (\Delta - 3 + \ell + k) = t + 2 + \Delta.$$

In fact, since both these inequalities hold strictly, we will apply the strict version of Lemma 4.1. Of course, there are several other hypotheses we must check. In particular, we must verify that $b_0 > a$, which is equivalent to showing that $\Delta > t + 8 - 2\ell$. Since $t = d + \ell$, we get this inequality by Claim 4.7. By Claim 4.1, we can therefore apply Lemma 4.1 for k provided that every vertex $u \in A_k$ has the property that

$$\deg_{X_k}(u) \ge \deg_G(u) - d.$$

Consider such a vertex u with incident edge uv in $E(G) \setminus E(H)$. Since $u \in A_k$, and by Claim 4.2, we know that

$$\deg_G(v) \geq \Delta + t + 2 - \deg_G(u) \geq \Delta + t + 2 - (t + 5 - \ell - k) = \Delta - 3 + \ell + k.$$

This means, by definition of X_k, that the edge uv is in X_k. So $deg_{X_k} \geq deg_G(u) - deg_H(u) \geq deg(u) - d$, as desired.

For any fixed k, we can now apply Lemma 4.1 to get

$$(\ell + 1)|A_k| < \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j|, \tag{4.7}$$

since $t + 1 - d = \ell + 1$ by the hypothesis of Claim 8, and since, for our choices of parameters,

$$\begin{aligned} a + j - 1 - (\Delta + t) &= (t + 5 - \ell - k) + j - 1 - (\Delta + t) \\ &= j - \Delta + 4 - \ell - k \\ &= q(j) - k. \end{aligned}$$

Dividing (4.7) by $(\ell + 1)$ and summing over all k yields

$$\sum_{k=0}^{3-\ell} |A_k| < \left(\tfrac{1}{\ell+1}\right) \sum_{k=0}^{3-\ell} \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j|. \tag{4.8}$$

The left-hand-side of (4.8) is

$$\begin{aligned} \sum_{k=0}^{3-\ell} |V_{[t+2,t+5-\ell-k]}| &= |V_{[t+2,t+5-\ell]}| + |V_{[t+2,t+4-\ell]}| + \cdots + |V_{[t+2,t+2]}| \\ &= (4 - \ell)|V_{t+2}| + \cdots + 2|V_{t+4-\ell}| + |V_{t+5-\ell}| \\ &= \sum_{i=t+2}^{t+5-\ell} (t + 6 - \ell - i)|V_i|, \end{aligned}$$

matching the left-hand side of (4.6). It remains only to show that the right-hand-side of (4.8) equals the right-hand side of (4.6). To this end, note that

$$j \geq \Delta - 3 + \ell + k \iff k \leq j - \Delta + 3 - \ell = q(j) - 1, \text{and so}$$

$$\sum_{k=0}^{3-\ell} \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j| = \sum_{j=\Delta-3+\ell}^{\Delta} \left(\sum_{k=0}^{q(j)-1} (q(j) - k) \right) |V_j|.$$

Now the bracketed sum can be rewritten as

$$\sum_{k=0}^{q(j)-1} (q(j) - k) = q(j) + (q(j) - 1) + (q(j) - 2) + \cdots + 1 = \tfrac{q(j)(q(j)+1)}{2},$$

which is precisely what we needed to prove. □

We have now shown $\alpha'(P) > 0$, so it remains only to consider the final charge of an arbitrary vertex v. If $v \in V_1$, then only discharging rule (b) affects v, and we get

$$\alpha'(v) = (-3) + 3 = 0.$$

By Claim 4.5, we may now assume that $\deg_G(v) \geq t + 2$.

Suppose v lies on the boundary of x distinct faces and is incident to y leaves. We know that x is no more than $\deg_G(v) - y$, so $x + y \leq \deg_G(v)$. We also know that $y \leq d$, by Claim 4.5 and by definition of d. By doubling the first inequality and adding the result to the second inequality we get

$$2x + 3y \leq 2\deg_G(v) + d. \tag{4.9}$$

Since $F_0, F_1, F_2 = \varnothing$ by Claim 4.6, each of the x distinct faces incident to v has at least 3 vertices of degree at least 3 on their boundary. This means that each of these x faces takes charge at most 2 from v, according to discharging rule (a). Each of the y leaves incident to v takes exactly 3 from v, according to discharging rule (b). Hence by inequality (4.9), after applying discharging rules (a) and (b) (but before considering discharging rules (c) or (d)), the charge of v is at least

$$3\deg_G(v) - 6 - (2x + 3y) \geq \deg_G(v) - 6 - d. \tag{4.10}$$

Note that since $d \leq t$, the additional discharging rules (c) and (d) are applied precisely when $d \geq t - 3$. If $d \leq t - 4$, then we do not apply them, and by inequality (4.10),

$$\alpha'(v) \geq \deg_G(v) - 6 - d \geq \deg_G(v) - 6 - (t - 4) = \deg_G(v) - (t + 2) \geq 0.$$

We may now assume that $t = d + \ell$ for $\ell \in \{0, 1, 2, 3\}$. Let p denote the total charge transferred from P to v according to discharging rules (c) and (d); note that p may be positive, negative, or zero. In all cases, by inequality (4.10), we have that

$$\alpha'(v) \geq \deg_G(v) - 6 - d + p. \tag{4.11}$$

If neither discharging rule (c) nor (d) applies to v, then we know that $t + 5 - \ell < \deg_G(v)$ and therefore (4.11) says that

$$\alpha'(v) \geq (t + 5 - \ell + 1) - 6 - d + (0) = (t - d) - \ell = 0,$$

as desired.

Now suppose that discharging rule (c) applies to v (and hence (d) does not, according to Claim 4.7). In this situation, (4.11) implies that

$$\alpha'(v) \geq \deg_G(v) - 6 - d + (t + 6 - \ell - \deg_G(v)) = 0.$$

Finally, we may assume that discharging rule (d) applies to v (and hence (c) does not, according to Claim 4.7). In this case, we have $t = d + \ell$, where $\ell \in \{0, 1, 2, 3\}$, and $\deg_G(v) \in$

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$\deg_G(v) = \Delta - 3 + \ell$	$\Delta - d - 10$	$\Delta - d - 17/2$	$\Delta - d - 22/3$	$\Delta - d - 25/4$
$\deg_G(v) = \Delta - 2 + \ell$	$\Delta - d - 11$	$\Delta - d - 17/2$	$\Delta - d - 7$	$\ast$
$\deg_G(v) = \Delta - 1 + \ell$	$\Delta - d - 13$	$\Delta - d - 9$	$\ast$	$\ast$
$\deg_G(v) = \Delta - 0 + \ell$	$\Delta - d - 16$	$\ast$	$\ast$	$\ast$

Table 4.1: Lower bounds on $\alpha'(v)$ when discharging rule (d) applies. Starred entries are impossible due to $\deg_G(v) \le \Delta$.

$\{\Delta - 3 + \ell, \ldots, \Delta\}$. By (4.11),

$$\alpha'(v) \ge \deg_G(v) - 6 - d - \left(\frac{(\deg_G(v) - (\Delta - 4 + \ell))(\deg_G(v) - (\Delta - 5 + \ell))}{2(\ell + 1)}\right).$$

Writing $\deg_G(v)$ as $\Delta - h + \ell$, where $h \in \{\ell, \ldots, 3\}$, we can rewrite this lower bound as

$$\begin{aligned}
\alpha'(v) &\ge \Delta - h + \ell - 6 - d - \left(\frac{(\Delta - h + \ell - (\Delta - 4 + \ell))(\Delta - h + \ell) - (\Delta - 5 + \ell)}{2(\ell + 1)}\right) \\
&= \Delta - h + \ell - 6 - d - \left(\frac{(4 - h)(5 - h)}{2(\ell + 1)}\right) \\
&= \Delta - d - \left(6 + h - \ell + \frac{(4 - h)(5 - h)}{2(\ell + 1)}\right).
\end{aligned}$$

Table 4.1 computes the bracketed quantity for each permissible combination of $\deg_G(v)$ and ℓ. For each possible value of ℓ, the hypothesis of Theorem 4.2 ensures that this lower bound is always nonnegative.

We have proved that $\alpha'(x) \ge 0$ for every graph element x, and this completes the proof of Theorem 4.2.

4.4 Extensions

In the proof of Theorem 4.2 our initial charges sum to at most -12, and after discharging the vertices and faces all have nonnegative charge and the global pot has a strictly positive charge. In fact, when we examine inequality (4.5), we see that the sum of initial charges is at most -6ε, where ε is the Euler characteristic of the plane. Hence our argument works identically well for any surface of non-negative Euler characteristic; namely G may be embedded on the plane, torus, Klein bottle, or projective plane. Moreover, this embedding requirement need not concern the edges of the precolored H: imagine applying Theorem 4.2 to the graph obtained by replacing every edge $e = uv$ in H with a pair of edges $e_u = uu'$ and $e_v = vv'$ where u', v' are new leaves, and e_u and e_v retain the precoloring (and lists) of e. Given these observations, we can strengthen Theorem 4.2 by removing the assumption that "G is planar" and replacing it by the somewhat milder "$G - E(H)$ can be embedded in a surface of nonnegative Euler characteristic".

Chapter 5

Conclusion

In chapter 3 we prove (in Theorem 3.3) that every triangulation with $\Delta = 5$ is 6-list-edge-colorable. This is a step towards answering Conjecture 1.1 which is open for planar graphs with $5 \leq \Delta \leq 7$. One natural extension to our result would be to prove that every triangulation with $\Delta = 6$ is 7-list-edge-colorable, however such an attempt would require techniques beyond what we employed in chapter 3. This is primarily due to our use of Lemma 3.2. Recall the last three lines of that proof read as follows:

$$\Leftrightarrow 12 = 6n - \sum_{v \in V(G)} deg(v)$$

$$\Leftrightarrow 12 = 6(|V_3| + |V_4| + |V_5|) - (3|V_3| + 4|V_4| + 5|V_5|)$$

$$\Leftrightarrow 12 = 3|V_3| + 2|V_4| + |V_5|$$

By allowing a triangulation to have $\Delta = 6$ we require the set V_6 to be nonempty; however it will still vanish in the equations above. This means we cannot develop a short list of cases for triangulations with $\Delta = 6$, though we could still use Lemma 3.2 to limit vertices with $3 \leq deg(v) \leq 5$ and use an argument similar to claim 3.1 to say there are more vertices in V_6 than in V_3.

The other natural extension to Theorem 3.3 would be to show that all planar graphs with $\Delta = 5$ are 6-list-edge-colorable. Cohen and Havet's proof of Theorem 2.8 redistributes the charge of high degree vertices to low-degree vertices and to triangular faces, proving that every element which was assigned charge ends with nonnegative charge. This means high degree vertices must have sufficient charge for the other elements of the graph, which is why Cohen and Havet's proof requires $\Delta \geq 9$. If one could show that high-degree vertices lie in few triangles or have few low-degree neighbors, then one could possibly lower their maximum degree condition. This is to some extent what Bonamy [4] accomplishes by finding special configurations which cannot occur in a minimal counterexample.

There is some benefit to working with $\Delta = 5$ in the discharging argument of Cohen and Havet, we can think of our Theorem 3.3 as the base case of an induction on the number of nontriangular faces in planar graphs with $\Delta = 5$. If we assume G is a nontriangular-face minimal graph, then we can show that every face has at most 2 vertices which are not maximum degree. We then proceed with the typical discharging argument assigning charges $\alpha(v) = deg(v) - 4$ and $\alpha(f) = \ell(f) - 4$ to all vertices and faces respectively. Since $\Delta = 5$, we need only worry about some very specific configurations. Unfortunately, we need to reduce such configurations in order to make progress. Even if we assume G is 5-regular each vertex must give more charge than it has to since we have not limited the number of triangles each vertex lies in.

Although Algorithm 1 is able to deal with the cases of Theorem 3.3 we are still interested in the kernel method. If one could extend Theorem 2.3 to allow for bidirected edges in the orientation, then we believe most if not all of the cases of Theorem 3.3 could be addressed. This brings us to question for what other families of graphs could we employ the kernel method to yield list-edge coloring results.

We do not expect any improvement to Theorem 4.2 without the use of techniques beyond those discussed in chapter 4. However, in the proofs of both our Theorem 4.2 and Borodin's Theorem 2.8 a list-edge coloring result for bipartite graphs is exploited in order to show there are more high-degree vertices than low-degree vertices in a graph. In particular Theorem 2.8 uses the result that even cycles are 2-list-edge-colorable and Theorem 4.2 uses Lemma 4.1. This relation is then used in a discharging argument to yield a list-edge coloring result for all planar graphs of a bounded maximum degree. This means that list-edge coloring results for well-known families of graphs, or even other list-edge coloring results on bipartite graphs, could be used to further build connections between list-edge-coloring and list-edge-precoloring or used to extend Theorem 3.3.

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Appendix A

Algorithm 1

The following code is an algorithm due to Schauz [27]. It takes as an input a k-regular graph on an even number of vertices and outputs $\sum_{F \epsilon OF(G)} sgn(F)$ as mentioned in chapter 2. The code is designed for SageMath and uses only python commands. Further details can be found in the appendix of [27].

Algorithm 1

```
def weighted_sum(Graph, previous_Unmatched = [-1..9], \
           next_Unmatched = [1..11]): # 2 optional param.
  # by default, start = next_Unmatched[-1] = 11 > len(Graph)
  # next_Unmatched[j] is the unmatched vertex after j
  # previous_Unmatched[j] is the unmatched vertex before j
  to_match = next_Unmatched[-1] # next_Unmatched[-1] is start
  if to_match < len(Graph): # 1-factor under construction
      neighbors = Graph[to_match]
  elif len(Graph[0]) <> 0: # start next 1-factor
      to_match = 0 # 0 shall be matched first
      neighbors = [Graph[0][0]] # to avoid color permutations
      previous_Unmatched = [-1..9] # fresh bootstrapping
      next_Unmatched = [1..11]
  else: return 1 # 1-factorization complete, edgeless graph
  um = next_Unmatched[to_match]
  previous_Unmatched[um] = -1 # bypass to_match
  next_Unmatched[-1] = um # bypass to_match
  w_sum = 0 # subtotal of weighted_sum()
  sgn = 1 # initial sign of edge {to_match,nbr}
  for i in range(len(neighbors)):
      nbr = neighbors[i] # i^th neighbor of to_match
```

```
    while um < nbr: # um is bridged by {to_match,nbr}
        sgn = -sgn # bridged unmatched vertices flip sgn
        um = next_Unmatched[um]
    if um == nbr: # match to_match with nbr
        gr = [[n for n in lst] for lst in Graph] # deepcopy
        del gr[to_match][i] # remove edge {to_match,nbr}
        p_um = [n for n in previous_Unmatched] # deepcopy
        n_um = [n for n in next_Unmatched] # deepcopy
        p_um[n_um[nbr]] = p_um[nbr] # bypass nbr
        n_um[p_um[nbr]] = n_um[nbr] # bypass nbr
        w_sum = w_sum + sgn * weighted_sum(gr,p_um,n_um)
return w_sum # output w_sum

graph = [[1,2,3,4,5],[2,3,4,5],[3,4,5],[4,5],[5],[]] # K6
# vertex 0 is adjacent to vertices 1,2,3,4,5; 1 adjacent to 2,3,4,5
  (and 0); etc.
weighted_sum(graph) # the initial call of weighted_sum()
# returns the sum of all signs of all 1-factorizations of graph
```

We use Algorithm 1 to compute $weighted\_sum(graph)$ for the following three graphs as mentioned in chapter 3.

Computation A.1 (Case 4)
Input: graph = [[1,2,3,4,5,6],[2,3,4,5,7],[3,5,6,7],[4,6,7],[5,6,7],[6,7],[7],[]]
Output: $weighted\_sum(graph)$ = −288

Computation A.2 (Case 15)
Input: graph = [1,2,3,4,5,11],[2,4,6,9,11],[3,6,7,11],[4,7,8,11],[8,9,11],[6,7,8,9,10],[7,9,10],[8,10],[9,10],[10],[11],[]]
Output: $weighted\_sum(graph)$ = −384

Computation A.3 (Case 16)
Input: graph = [[1,2,3,6,8,9],[2,3,4,5,8],[3,5,7,8],[4,6,9],[5,6,7,9],[6,7,9],[7,8],[8,9],[9],[]]
Output: $weighted\_sum(graph)$ = 256