Study of Stochastic Differential Equation Driven by Time-Changed Lévy Noise

by

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Abstract

This dissertation is composed of two parts. The first part studies stabilities of the solution of stochastic differential equation (SDE) driven by time-changed Lévy noise in probability, moment, and path sense. This provides more flexibility in modeling schemes in application areas including physics, biology, engineering, finance and hydrology. Necessary conditions for solution of time-changed SDE to be stable in different senses will be established. Connection between stability of solution to time-changed SDE and that to corresponding original SDE will be disclosed.

The second part studies a time-changed stochastic control problem, where the underlying stochastic process is a Lévy noise time-changed by an inverse subordinator. We establish a maximum principle theory for the time-changed stochastic control problem. We also prove the existence and uniqueness of the corresponding time-changed backward stochastic differential equation involved in the stochastic control problem. Some examples are provided for illustration.

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Chapter 1

Introduction

1.1 Time-Changed Stochastic Differential Equations

Study of stochastic differential equations (SDE) is a mature field of research. Numerous types of SDEs have been used to model different phenomena in various areas, such as unstable stock prices in finance [24], dynamics of biological systems [12], and Kalman filter in navigation control. Lyapunov [19] introduced the concept of stability of a dynamical system. Since then, the concept of stability have been studied widely in different senses, including stochastical stability, almost sure stability, exponential stability, etc. In [21], Mao investigated various types of stabilities for the following SDE

$$dX(t) = f(X(t))dt + g(X(t))dB(t),\ t \geq 0, \tag{1.1}$$

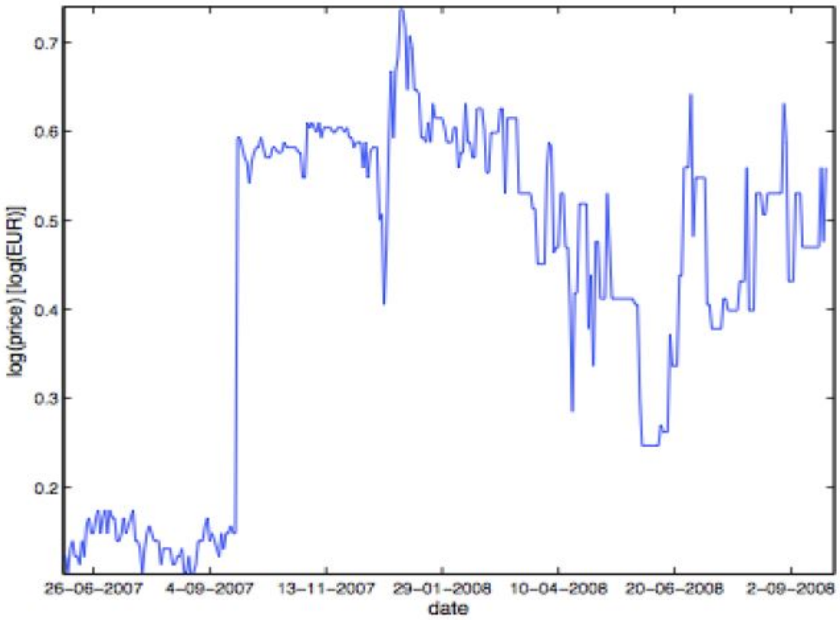
with $X(0) = x_0$, where B is the standard Brownian motion.

Siakalli [30] extended Mao's results to SDEs driven by Lévy noise

$$dX(t) = f(X(t-))dt + g(X(t-)dB(t) + \int_{|y|<c} h(X(t-), y)\tilde{N}(dt, dy),\ t \geq 0, \tag{1.2}$$

with $X(0) = x_0$, where $\tilde{N}$ is the compensated Poisson measure. This type of SDEs provide as a tool of modeling the price of financial assets with continuous change. However, we also observe such special behavior in financial market that prices are on the same level during a period of time, see Figure 1.1. But this phenomena can be modeled by the time-changed SDEs, tantgraph

Figure 1.1: Log price of the Kalev stock [11]



which allow more flexibility in modelling and thus become popular among researchers, see [29] and [32].

Kobayashi [15] introduced the duality theorem between time-changed SDEs

$$
\begin{aligned}
&dX(t) = f(E_t, X(t-))dE_t + g(E_t, X(t-))dB_{E_t},\\
&X(0) = x_0,
\end{aligned}
\tag{1.3}
$$

equk

and the corresponding non-time-changed SDEs

$$
\begin{aligned}
&dY(t) = f(t, Y(t-))dt + g(t, Y(t-))dB_t,\\
&Y(0) = x_0,
\end{aligned}
\tag{1.4}
$$

equk2

where E_t is the inverse of a strictly increasing subordinator $D(t)$: if a process $Y(t)$ satisfies SDE (1.4), then $X(t) := Y(E_t)$ satisfies the time-changed SDE (1.3); if a process $X(t)$ satisfies the time-changed SDE (1.3), then $Y(t) := X(D(t))$ satisfies SDE (1.4).

In light of time-changed Itô formula, recent paper [32] analyzes the SDE driven by time-changed Brownian motion

$$\begin{aligned} &dX(t) = f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t}, \\ &X(0) = x_0, \end{aligned} \tag{1.5}$$

equw

where E_t is specified as an inverse of a stable subordinator of index β in $(0, 1)$, and discusses the stability of solution to above SDE in probability sense, including stochastically stable, stochastically asymptotically stable and globally stochastically asymptotically stable.

In this paper, we focus on the following time-changed SDE

$$\begin{aligned} dX(t) = f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t} \\ + \int_{|y|<c} h(t, E_t, X(t-), y)\tilde{N}(dE_t, dy), \end{aligned} \tag{1.6}$$

SDE

with $X(t_0) = x_0$, where E_t is the inverse of a strictly increasing subordinator, and discuss stability of its solution in probability, moment and path senses, including stochastically stability, stochastically asymptotic stability, global stochastic asymptotic stability, pth moment exponential stability, pth moment asymptotic stability, almost surely exponentially path stable, and almost surely path stable. We also extend our analysis regarding path stabilities of (1.6) to linear large jumps

$$\begin{aligned} dX(t) = f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t} \\ + \int_{|y|<c} h(y)X(t-)\tilde{N}(dE_t, dy) + \int_{|y|\geq c} H(y)X(t-)N(dE_t, dy). \end{aligned} \tag{1.7}$$

aimfinal

with $X(t_0) = x_0$.

1.2 Time-Changed Stochastic Control Problem

Uncertainty is inherent in the real world and changes over time, putting people's decisions at risk. A decision maker wants to select the best choice among all possible ones. The stochastic control theory serves as a tool to such dynamic optimization problem. The world has witnessed many applications of stochastic control theory in various fields such as biology [31], economics [8], and finance [28].

A well known approach to stochastic control problem is based on the maximum principle method. Such method for Itô diffusion case is first studied by Kushner [17], Bismut [6] and further developed by Bensoussan [4], Peng [27], and others. The jump diffusion case is formulated by Framstad, Øksendal and Sulem [9]. The idea of the maximum principle approach is to formulate a Hamiltonian function and derive the adjoint equations, which involve the backward stochastic differential equation. Under sufficient conditions, the optimal control is the solution of a coupled system of forward and backward stochastic differential equations.

As time-changed stochastic processes have been adopted in more and more areas, the traditional stochastic control problem framework needs updates to fit the time-changed cases. For example, a mutual fund manager, whose investment portfolios consist of stocks whose prices follow time-changed Brownian motions as shown in Figure 1.1, will find the time-changed stochastic control a better tool to manager the portfolio than the traditional stochastic control. A biologist, who investigates how outside interferences affect the movements of insects, may find the time-changed stochastic control problem better describe the experiment since some insects sometimes move and sometimes stay still. Because the time-changed stochastic process better describe many phenomena and people seek the optimal choice based on them, we believe it is necessary to study the stochastic control problem based on the time-changed stochastic process, which will build up a framework to solve potential optimization problems.

We investigate the time-changed stochastic control problem using the maximum principle method. Specifically, we consider the time-changed stochastic process (1.6) and the corresponding performance function

$$J(u) = \mathbb{E}\Big[\int_0^T U_1(t, E_t, X(t), u(t))dE_t + U_2(X(T))\Big],\ u \in \mathcal{A}, \tag{1.8}$$ 14

where $u(t) = u(t, w)$ is the control and $\mathcal{A}$ denotes the set of *admissible* controls. A maximum principle theory for the stochastic control problem is established to find $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{1.9}$$ 13

In (1.8), the performance function can be utility function, energy consumption function that we care about. For example, the performance function in Example 4.2.3 is the utility function $\exp(-\delta t)u(t)^2$, where $u(t)$ is the consumption rate. Given the wealth level described by the time-changed process $X(t)$, we seek the optimal consumption rate $u^*(t)$, as indicated in (4.3) , that maximize the overall utility performance $J(u) = \mathbb{E}\Big[\int_0^\tau \exp(-\delta t)u(t)^2 dt\Big]$.

In the remaining parts of this paper, further needed concepts and related background will be given in Chapter 2. In Chapter 3, the conditions for the solution to our target time-changed SDEs to be stable in various senses will be given. In Chapter 4, we reveal the maximum principle method for time-changed stochastic control problems. Examples are provided.

Chapter 2

Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space satisfying usual hypotheses of completeness and right continuity. Assume that $\mathcal{F}_t$-adapted Poisson random measure N on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ is independent of the drift and the standard Brownian motion, define its compensator $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$, where ν is a Lévy measure satisfying $\int_{\mathbb{R}-\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$.

Let $\{D(t), t \geq 0\}$ be a right continuous with left limits (RCLL) increasing Lévy process that is called subordinator starting from 0 with Laplace transform

$$\mathbb{E}e^{-\lambda D(t)} = e^{-t\psi(\lambda)}, \tag{2.1}$$

where Laplace exponent $\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(dx)$. Define its inverse

$$E_t := \inf\{\tau > 0 : D(\tau) > t\}. \tag{2.2}$$

inverse-E

The concept of regular variation is needed to introduce the mixed stable subordinator. A measurable function R is regularly varying at infinity with exponent $\gamma \in \mathbb{R}$, denoted by $R \in RV_\infty(\gamma)$, if R is eventually positive and $R(ct)/R(t) \to c^\gamma$ as $t \to \infty$, for any $c > 0$. Similarly, a measurable function R is regularly varying at zero with exponent $\gamma \in \mathbb{R}$, denoted by $R \in RV_0(\gamma)$, if R is positive in some neighborhood of zero and $R(ct)/R(t) \to c\gamma$ as $t \to 0$, for any $c > 0$.

Given a measurable function $p : (0, 1) \to \mathbb{R}_+$ such that $p \in RV_0(\gamma - 1)$ for some $\gamma > 0$, let $L(u) = C \int_0^1 u^{-\alpha} p(\alpha) d\alpha$ and $C^{-1} = \int_0^1 p(\alpha) d\alpha$. Without loss of generality, let $C = 1$, then

p is a probability density of Lévy measure of the α-stable subordinators. Let $\{D(t)\}_{t\geq 0}$ be a subordinator such that $D(1)$ has Lévy-Khinchin representation $[0, 0, \phi]$ and the Lévy measure ϕ is defined as $\phi(u, \infty) = L(u)$, then $\{D(t)\}_{t\geq 0}$ is the so called "mixed" stable subordinator. In this case the Laplace exponent is given by

$$\psi(\lambda) = \int_0^1 \Gamma(1-\beta)\lambda^\beta p(\beta)d\beta \tag{2.3}$$

laplace-e

By Theorem 3.9 in [22], there exists a function $L \in RV_\infty(0)$ such that

$$\mathbb{E}[E(t)] \sim (\log t)^\gamma L(\log t)^{-1} \ \ as\ t \to \infty. \tag{2.4}$$

expofet

We require f, k, g, h, H in (1.6) and (1.7) to be real-valued functions and satisfy the following Lipschitz condition in Assumption 2.0.1, growth condition in Assumption 2.0.2 and Assumption 2.0.3. Under these assumptions, by Lemma 4.1 in [15], both of the equations (1.6) and (1.7) have unique $\mathcal{G}_t = \mathcal{F}_{E_t}$-adapted solution processes $X(t)$.

lip **Assumption 2.0.1** *(Lipschitz condition) There exists a positive constant K_1 such that*

$$\begin{aligned} &\Big|f(t_1,t_2,x) - f(t_1,t_2,y)\Big|^2 + \Big|k(t_1,t_2,x) - k(t_1,t_2,y)\Big|^2 + \Big|g(t_1,t_2,x) - g(t_1,t_2,y)\Big|^2 \\ &+ \int_{|z|<c} \Big|h(t_1,t_2,x,z) - h(t_1,t_2,y,z)\Big|^2 \nu(dz) \leq K_1|x-y|^2, \end{aligned} \tag{2.5}$$

for all $t_1, t_2 \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

linear **Assumption 2.0.2** *(Growth condition) There exists a positive constant K_2 such that, for all $t_1, t_2 \in \mathbb{R}_+$ and $x \in \mathbb{R}$,*

$$|f(t_1,t_2,x)|^2 + |k(t_1,t_2,x)|^2 + |g(t_1,t_2,x)|^2 + \int_{|y|<c} |h(t_1,t_2,x,y)|^2\nu(dy) \leq K_2(1+|x|^2). \tag{2.6}$$

tec **Assumption 2.0.3** *If $X(t)$ is right continuous with left limits (rcll) and a $\mathcal{G}_t$-adapted process, then*

$$f(t, E_t, X(t)), k(t, E_t, X(t)), g(t, E_t, X(t)), h(t, E_t, X(t), y) \in \mathcal{L}(\mathcal{G}_t), \tag{2.7}$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of left continuous with right limits and $\mathcal{G}_t$-adapted processes.

Next, we define different types of stability.

Definition 2.0.4 *(1) The trivial solution of the time-changed SDE* (1.6) *is said to be stochastically stable or stable in probability if for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r) > 0$ such that*

$$P\{|X(t, x_0)| < r \; for \; all \; t \geq 0\} \geq 1 - \epsilon \tag{2.8}$$

whenever $|x_0| < \delta$.

(2) The trivial solution of the time-changed SDE (1.6) *is said to be stochastically asymptotically stable if for every $\epsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that*

$$P\{\lim_{t\to\infty} X(t, x_0) = 0\} \geq 1 - \epsilon \tag{2.9}$$

whenever $|x_0| < \delta_0$.

(3) The trivial solution of the time-changed SDE (1.6) *is said to be globally stochastically asymptotically stable or stochastically asymptotically stable in the large if it is stochastically stable and for all $x_0 \in \mathbb{R}$*

$$P\{\lim_{t\to\infty} X(t, x_0) = 0\} = 1. \tag{2.10}$$

Definition 2.0.5 *(1) The trivial solution of the time-changed SDE* (1.6) *is said to be pth moment exponentially stable if there are positive constants λ and C such that*

$$E[|X(t)|^p] \leq C|x_0|^p \exp(-\lambda t), \; \forall t \geq 0, \; \forall x_0 \in \mathbb{R}, \; p > 0. \tag{2.11}$$

(2) The trivial solution of the time-changed SDE (1.6) *is said to be pth moment asymptotically stable if there is a function $v(t) : [0, +\infty) \to [0, \infty)$ decaying to 0 as $t \to \infty$ and a*

positive constant C such that

$$E[|X(t)|^p] \leq C|x_0|^p v(t),\ \forall t \geq 0,\ \forall x_0 \in \mathbb{R},\ p > 0. \tag{2.12}$$

Definition 2.0.6 *(Definition 3.1 in [21]) (1) The trivial solution of the time-changed SDE* (1.6) *is said to be almost surely exponentially path stable if*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t; t_0, x_0)| < 0 \ \ a.s. \tag{2.13}$$

for all $x_0 \in R$.

(2)The trivial solution of the time-changed SDE (1.6) *is said to be almost surely path stable if there exists a function* $\nu(t) : [0, \infty) \to [0, \infty)$ *such that*

$$\lim_{t\to\infty} \nu(t) = \infty, \tag{2.14}$$

and

$$\limsup_{t\to\infty} \frac{1}{\nu(t)} \log |X(t; t_0, x_0)| < 0 \ \ a.s. \tag{2.15}$$

for all $x_0 \in R$.

Chapter 3

Stability of Time-Changed Stochastic Differential Equations

In this chapter, we discuss stability of the solution to SDE (1.6) in probability, moment and path senses, including stochastically stability, stochastically asymptotic stability, global stochastic asymptotic stability, pth moment exponential stability, pth moment asymptotic stability, almost surely exponentially path stable, and almost surely path stable. In particular, we discover the conditions under which the solutions of time-changed SDEs are stable in various senses. We also provide examples to illustrate our theories.

The Itô formula is heavily used in our proofs. We derive the following Itô formula for time-changed Lévy noise and will utilize it frequently in the remaining sections.

itofor **Lemma 3.0.1** *(Itô formula for time-changed Lévy noise) Let $D(t)$ be a RCLL subordinator and E_t its inverse process as* (2.2). *Define a filtration $\{\mathcal{G}_t\}_{t\geq 0}$ by $\mathcal{G}_t = \mathcal{F}_{E_t}$. Let X be a process defined as following:*

$$X(t) = x_0 + \int_0^t f(t, E_t, X(t-))dt + \int_0^t k(t, E_t, X(t-))dE_t + \int_0^t g(t, E_t, X(t-))dB_{E_t} + \int_0^t \int_{|y|<c} h(t, E_t, X(t-), y)\tilde{N}(dE_t, dy), \tag{3.1}$$

sdelevy

where f, k, g, h are measurable functions such that all integrals are defined. Here c is the maximum allowable jump size.

Then, for all $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ in $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, with probability one,

$$
\begin{aligned}
F(t, E_t, X(t)) - F(0,0,x_0) &= \int_0^t L_1 F(s, E_s, X(s-))ds + \int_0^t L_2 F(s, E_s, X(s-))dE_s \\
&+ \int_0^t \int_{|y|<c} \Big[F(s, E_s, X(s-) + h(s, E_s, X(s-), y)) - F(s, E_s, X(s-))\Big] \tilde{N}(dE_s, dy) \\
&+ \int_0^t F_x(s, E_s, X(s-)) g(s, E_s, X(s-)) dB_{E_s},
\end{aligned}
\tag{3.2}
$$

itolevy

where

$$
\begin{aligned}
L_1 F(t_1, t_2, x) &= F_{t_1}(t_1, t_2, x) + F_x(t_1, t_2, x) f(t_1, t_2, x), \\
L_2 F(t_1, t_2, x) &= F_{t_2}(t_1, t_2, x) + F_x(t_1, t_2, x) k(t_1, t_2, x) + \frac{1}{2} g^2(t_1, t_2, x) F_{xx}(t_1, t_2, x) \\
&+ \int_{|y|<c} \Big[F(t_1, t_2, x + h(t_1, t_2, x, y)) - F(t_1, t_2, x) - F_x(t_1, t_2, x) h(t_1, t_2, x, y)\Big] \nu(dy).
\end{aligned}
\tag{3.3}
$$

linearop

Proof: This proof is a direct application of multidimensional Itô formula, which is established in Corollary 3.4 in [15], to $F(t, E_t, X(t))$ in $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$.

$$
\begin{aligned}
F(t, E_t, X(t)) - F(0,0,x_0) &= \int_0^t F_{t_1}(s, E_s, X(s-))ds + \int_0^t F_{t_2}(s, E_s, X(s-))dE_s \\
&+ \int_0^t F_x(s, E_s, X(s-)) \Big[f(s, E_s, X(s-))ds + k(s, E_s, X(s-))dE_s \\
&+ g(s, E_s, X(s-))dB_{E_s}\Big] + \frac{1}{2}\int_0^t F_{xx}(s, E_s, X(s-)) g(s, E_s, X(s-)) dE_s \\
&+ \int_0^t \int_{|y|<c} \Big[F(s, E_s, X(s-) + h(s, E_s, X(s-), y)) - F(s, E_s, X(s-))\Big] \tilde{N}(dE_s, dy) \qquad (3.4) \\
&+ \int_0^t \int_{|y|<c} \Big[F(s, E_s, X(s-) + h(s, E_s, X(s-), y)) - F(s, E_s, X(s-)) \\
&\qquad - F_x(s, E_s, X(s-)) h(s, E_s, X(s-), y)\Big] \nu(dy) dE_s \\
&= \int_0^t L_1 F(s, E_s, X(s-))ds + \int_0^t L_2 F(s, E_s, X(s-))dE_s \\
&+ \int_0^t \int_{|y|<c} \Big[F(s, E_s, X(s-) + h(s, E_s, X(s-), y)) - F(s, E_s, X(s-))\Big] \tilde{N}(dE_s, dy) \\
&+ \int_0^t F_x(s, E_s, X(s-)) g(s, E_s, X(s-)) dB_{E_s}.
\end{aligned}
$$

□

Lemma 3.0.2 *Let $D(t)$ be a RCLL subordinator and E_t be its inverse process as in* (2.2)*. Define a filtration $\{\mathcal{G}_t\}_{t\geq 0}$ by $\mathcal{G}_t = \mathcal{F}_{E_t}$. Let $\tilde{N}$ be a compensated Poisson measure defined on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ with intensity measure ν, where ν is a Lévy measure such that $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ and $\int_{\mathbb{R}-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$. Then, for any $A \in \mathcal{B}(\mathbb{R} - \{0\})$ bounded below, the time-changed process $\tilde{N}(E_t, A)$ is a martingale.*

Proof: Let $\tau_n = \inf\{t \geq 0; |\tilde{N}(t, A)| \geq n\}$, it is obvious that $\tau_n \to \infty$ as $n \to \infty$. Then $|\tilde{N}(\tau_n \wedge t, A)| \leq n + 1$, for all $t \in \mathbb{R}_+$, thus $\tilde{N}(\tau_n \wedge t, A)$ is a bounded martingale.

By optional stopping theorem, for any $0 \leq s < t$,

$$\mathbb{E}\Big[\tilde{N}(\tau_n \wedge E_t, A)|\mathcal{G}_s\Big] = \tilde{N}(\tau_n \wedge E_s, A). \tag{3.5}$$

The right hand side $\tilde{N}(\tau_n \wedge E_s, A)$ converges to $\tilde{N}(E_s, A)$, as $n \to \infty$. For the left hand side, we have

$$|\tilde{N}(\tau_n \wedge E_t, A)| \leq \sup_{0\leq u\leq t} |\tilde{N}(E_u, A)|, \tag{3.6}$$

thus, by Hölder's inequality, Doob's martingale inequality,

$$\begin{aligned}
\mathbb{E}\Big[\Big|\tilde{N}(\tau_n \wedge E_t, A)\Big|\Big] &\leq \mathbb{E}\Big[\Big|\sup_{0\leq u\leq t} |\tilde{N}(E_u, A)\Big|\Big] = \mathbb{E}\Big[\Big|\sup_{0\leq u\leq E_t} |\tilde{N}(u, A)\Big|\Big] \\
&= \int_0^\infty \mathbb{E}\Big[\Big|\sup_{0\leq u\leq \tau} |\tilde{N}(u, A)\Big|\Big|\tau = E_t\Big] f_{E_t}(\tau)d\tau \\
&\leq \int_0^\infty \mathbb{E}\Big[\Big|\sup_{0\leq u\leq \tau} |\tilde{N}(u, A)\Big|^2\Big|\tau = E_t\Big]^{\frac{1}{2}} f_{E_t}(\tau)d\tau \\
&\leq \int_0^\infty 2\mathbb{E}\Big[\Big||\tilde{N}(\tau, A)\Big|^2\Big|\tau = E_t\Big]^{\frac{1}{2}} f_{E_t}(\tau)d\tau \\
&= 2\int_0^\infty [\nu(A)\tau]^{\frac{1}{2}} f_{E_t}(\tau)d\tau \\
&= 2\nu(A)^{\frac{1}{2}}\mathbb{E}[E_t^{\frac{1}{2}}]. \\
&\leq 2\nu(A)^{\frac{1}{2}}\mathbb{E}[E_t]^{\frac{1}{2}},
\end{aligned} \tag{3.7}$$

where the last inequality follows from Jensen's inequality.

For any $t \geq 0$ and $x > 0$, by Markov's inequality, we have

$$P(E_t > s) \leq P(D(s) < t) = P(e^{-xD(s)} \geq e^{-xt}) \leq e^{xt}\mathbb{E}[e^{-xD(s)}] = e^{xt}e^{-s\phi(x)}, \tag{3.8}$$

it follows that

$$\mathbb{E}[E_t] = \int_0^\infty P(E_t > s)ds = e^{xt}\frac{1}{\phi(x)} < \infty. \tag{3.9}$$

Then, by dominated convergence theorem, we have

$$\mathbb{E}\Big[\tilde{N}(\tau_n \wedge E_t, A)|\mathcal{G}_s\Big] \to \mathbb{E}\Big[\tilde{N}(E_t, A)|\mathcal{G}_s\Big], \tag{3.10}$$

as $n \to \infty$. So

$$\mathbb{E}\Big[\tilde{N}(E_t, A)|\mathcal{G}_s\Big] = \tilde{N}(E_s, A). \tag{3.11}$$

Also,

$$\mathbb{E}\Big[|\tilde{N}(E_t, A)|\Big] \leq \mathbb{E}\Big[\sup_{0\leq u\leq t}|\tilde{N}(E_u, A)|\Big] < \infty, \tag{3.12}$$

thus $\tilde{N}(E_t, A)$ is a martingale.

□

3.1 Stability in Probability

Let $\mathcal{K}$ denote the family of all nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(x) > 0$ for all $x > 0$.

tm1 **Theorem 3.1.1** *Assume that there exists a function* $V(t_1, t_2, x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times S_h, \mathbb{R})$ *with* $h \geq 2c$ *and* $\mu \in \mathcal{K}$ *such that for all* $(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S_h$

$$
\begin{aligned}
&1.\ V(t_1,t_2,0)=0,\\
&2.\ \mu(|x|)\le V(t_1,t_2,x),\\
&3.\ L_1V(t_1,t_2,x)\le 0,\\
&4.\ L_2V(t_1,t_2,x)\le 0,
\end{aligned}
\tag{3.13}
$$

then the trivial solution of the time-changed SDE (1.6) *is stochastically stable or stable in probability.*

roofoftm1

Proof: Let $\epsilon \in (0,1)$ and $r \in (0,h)$ be arbitrary. By continuity of $V(t_1,t_2,x)$ and the fact $V(t_1,t_2,0)=0$, we can find a $\delta=\delta(\epsilon,r,0)>0$ such that

$$
\frac{1}{\epsilon}\sup_{x\in S_\delta} V(0,0,x_0)\le \mu(r). \tag{3.14}
$$

equini

By (3.14) and condition (2), $\delta < r$. Fix initial value $x_0 \in S_\delta$ arbitrarily and define the stopping time

$$
\tau_r=\inf\{t\ge 0: |X(t,x_0)|\ge r\}, \tag{3.15}
$$

stoptime

where $r\le \frac{h}{2}$, and

$$
\begin{aligned}
U_k =&k\wedge \inf\{t\ge 0; \left|\int_0^{\tau_r\wedge t} V_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s}\right|\ge k\},\\
W_k =&k\wedge \inf\{t\ge 0; \left|\int_0^{\tau_r\wedge t}\int_{|y|<c}\Big[V(s,E_s,X(s-)+H(s,E_s,X(s-),y))\right.\\
&\left.\qquad -V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)\right|\ge k\},
\end{aligned}
\tag{3.16}
$$

stoptime2

for k=1,2,.... It is easy to see that $U_k \to \infty$ and $W_k \to \infty$ as $k \to \infty$. Apply Itô formula (3.2) to $V(t_1, t_2, x)$ associated with SDE (1.6), then for any $t \geq 0$,

$$
\begin{aligned}
&V(t \wedge \tau_r \wedge U_k \wedge W_k, E_{t\wedge\tau_r\wedge U_k\wedge W_k}, X(t \wedge \tau_r \wedge U_k \wedge W_k)) - V(0, 0, x_0) \\
&= \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} L_1 V(s, E_s, X(s-))ds + \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} L_2 V(s, E_s, X(s-))dE_s \\
&+ \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \\
&+ \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} \int_{|y|<c} \Big[V(s, E_s, X(s-) + H(s, E_s, X(s-), y)) \\
&\qquad - V(s, E_s, X(s-)) \Big] \tilde{N}(dE_s, dy).
\end{aligned} \tag{3.17}
$$

By [20] and [16], both

$$
\int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \tag{3.18}
$$

and

$$
\int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} \int_{|y|<c} \Big[V(s, E_s, X(s-) + H(s, E_s, X(s-), y)) - V(s, E_s, X(s-)) \Big] \tilde{N}(dE_s, dy) \tag{3.19}
$$

are mean zero martingales.

Taking expectations on both sides, we have

$$
\mathbb{E}[V(t \wedge \tau_r \wedge U_k \wedge W_k, E_{t\wedge\tau_r\wedge U_k\wedge W_k}, X(t \wedge \tau_r \wedge U_k \wedge W_k))] \leq V(0, 0, x_0).
$$

Letting $k \to \infty$,

$$
\mathbb{E}[V(t \wedge \tau_r, E_{t\wedge\tau_r}, X(t \wedge \tau_r))] \leq V(0, 0, x_0).
$$

Now, $|X(t \wedge \tau_r)| < r$ for $t < \tau_r$. For all $w \in \{\tau_r < \infty\}$, $|X(\tau_r)(w)| \leq r + c \leq h$. Since $V(t_1, t_2, x) \geq \mu(|x|)$ for all $x \in S_h$, we have for all $w \in \{\tau_r < \infty\}$

$$V(\tau_r, E_{\tau_r}, X(\tau_r)(w)) \geq \mu(|X(\tau_r)(w)|) \geq \mu(r). \tag{3.20}$$

Also,

$$V(0,0,x_0) \geq E[V(t\wedge\tau_r, E_{t\wedge\tau_r}, X(t\wedge\tau_r))1_{\{\tau_r<t\}}] \geq E[\mu(r)1_{\{\tau_r<t\}}] = \mu(r)P(\tau_r < t), \tag{3.21}$$

thus, combined with (3.14),

$$P(\tau_r < t) \leq \frac{V(0,0,x_0)}{\mu(r)} \leq \frac{\epsilon\mu(r)}{\mu(r)} = \epsilon. \tag{3.22}$$

Then, letting $t \to \infty$, we have

$$P(\tau_r < \infty) \leq \epsilon, \tag{3.23}$$

equivalently,

$$P(|X(t, x_0)| < r \; for \; all \; t \geq 0) \geq 1 - \epsilon, \tag{3.24}$$

so $X(t, x_0)$ is stochastically stable. □

tm2 **Theorem 3.1.2** *Assume that there exists a function* $V(t_1, t_2, x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times S_h, \mathbb{R})$ *with* $h \geq 2c$ *and* $\mu \in \mathcal{K}$ *such that for all* $(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S_h$

1. $V(t_1, t_2, 0) = 0,$
2. $\mu(|x|) \leq V(t_1, t_2, x),$
3. $L_1V(t_1, t_2, x) \leq -\gamma_1(\alpha) \; a.s. \; and \; L_2V(t_1, t_2, x) \leq -\gamma_2(\alpha) \; a.s., \; for \; any \; \alpha \in (0, h),$

$$where \; \gamma_1(\alpha) \geq 0 \; and \; \gamma_2(\alpha) \geq 0 \; but \; not \; equal \; to \; zero \; at \; the \; same \; time, \; x \in S_h - \bar{S}_\alpha, \tag{3.25}$$

then the trivial solution of the time-changed SDE (1.6) *is stochastically asymptotically stable.*

Proof: By Theorem 3.1.1, trivial solution of (1.6) is stochastically stable. For any fixed $\epsilon \in (0,1)$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$P(|X(t,x_0)| < h) \geq 1 - \frac{\epsilon}{5} \tag{3.26}$$

when $x_0 \in S_\delta$. Fix $x_0 \in S_\delta$ and let $0 < \alpha < \beta < |x_0|$ arbitrarily. Define the following stopping times

$$\begin{aligned}
&\tau_h = \inf\{t \geq 0; |X(t,x_0)| > h\} \\
&\tau_\alpha = \inf\{t \geq 0; |X(t,x_0)| < \alpha\} \\
&U_k = k \wedge \inf\{t \geq 0; \left| \int_0^{t\wedge\tau_h\wedge\tau_\alpha} V_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s} \right| \geq k\}, \\
&W_k = k \wedge \inf\{t \geq 0; \left| \int_0^{t\wedge\tau_h\wedge\tau_\alpha} \int_{|y|<c} \big[V_x(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \right. \\
&\qquad\qquad\qquad \left. - V_x(s,E_s,X(s-))\big]\tilde{N}(dE_s,dy) \right| \geq k\}.
\end{aligned} \tag{3.27}$$

By Itô's formula (3.2), we have

$$\begin{aligned}
0 &\leq \mathbb{E}\big[V(t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k, E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k}, X(t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k))\big] \\
&= V(0,0,x_0) + \mathbb{E}\int_0^{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k} L_1V(s,E_s,X(s-))ds \\
&\qquad\qquad + \mathbb{E}\int_0^{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k} L_2V(s,E_s,X(s-))dE_s \\
&\leq V(0,0,x_0) - \gamma_1(\alpha)\mathbb{E}[t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k] - \gamma_2(\alpha)\mathbb{E}[E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k}].
\end{aligned} \tag{3.28}$$

Letting $k \to \infty$ and $t \to \infty$, we have

$$\gamma_1(\alpha)\mathbb{E}[\tau_h\wedge\tau_\alpha] + \gamma_2(\alpha)\mathbb{E}[E_{\tau_h\wedge\tau_\alpha}] \leq V(0,0,x_0), \tag{3.29}$$

By condition (3) and $E_t \to \infty$ a.s. as $t \to \infty$, see proof of Theorem 3.2.1, we have

$$P(\tau_h \wedge \tau_\alpha < \infty) = 1. \tag{3.30}$$

Since $P(\tau_h = \infty) > 1 - \frac{\epsilon}{5}$, it follows that $P(\tau_h < \infty) \leq \frac{\epsilon}{5}$, thus

$$1 = P(\tau_h \wedge \tau_\alpha < \infty) \leq P(\tau_h < \infty) + P(\tau_\alpha < \infty) \leq P(\tau_\alpha < \infty) + \frac{\epsilon}{5}, \tag{3.31}$$

that's,

$$P(\tau_\alpha < \infty) \geq 1 - \frac{\epsilon}{5}. \tag{3.32}$$

Choose θ sufficiently large for

$$P(\tau_\alpha < \theta) \geq 1 - \frac{2\epsilon}{5}. \tag{3.33}$$

Then

$$\begin{aligned} P(\tau_\alpha < \tau_h \wedge \theta) &\geq P(\{\tau_\alpha < \theta\} \cap \{\tau_h = \infty\}) = P(\tau_\alpha < \theta) - P(\{\tau_\alpha < \theta\} \cap \{\tau_h < \infty\}) \\ &\geq P(\tau_\alpha < \theta) - P(\tau_h < \infty) \geq 1 - \frac{2\epsilon}{5} - \frac{\epsilon}{5} = 1 - \frac{3\epsilon}{5} \end{aligned} \tag{3.34}$$

Now define some stopping times

$$\sigma = \begin{cases} \tau_\alpha, & if\ \tau_\alpha < \tau_h \wedge \theta \\ \infty, & otherwise \end{cases} \tag{3.35}$$

$$\begin{aligned} &\tau_\beta = \inf\{t \geq \sigma; |X(t, x_0)| \geq \beta\}, \\ &S_i = \inf\{t \geq \sigma; \big| \int_\sigma^{\tau_\beta \wedge t} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \big| \geq i\}, \\ &T_i = \inf\{t \geq \sigma; \big| \int_\sigma^{\tau_\beta \wedge t} \int_{|y|<c} \big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) \\ &\qquad\qquad - V(s, E_s, X(s-))\big] \tilde{N}(dE_s, dy) \big| \geq i\}. \end{aligned} \tag{3.36}$$

Again, by Itô's formula,

$$
\begin{aligned}
&\mathbb{E}\Bigg[V(t\wedge\tau_\beta\wedge S_i\wedge T_i, E_{t\wedge\tau_\beta\wedge S_i\wedge T_i}, X(t\wedge\tau_\beta\wedge S_i\wedge T_i))\Bigg]\\
\leq&\mathbb{E}\Bigg[V(t\wedge\sigma, E_{t\wedge\sigma}, X(t\wedge\sigma\wedge))\Bigg]+\mathbb{E}\Bigg[\int_{t\wedge\sigma\wedge}^{t\wedge\tau_\beta\wedge S_i\wedge T_i} L_1V(s,E_s,X(s-))ds\Bigg]\\
&+\mathbb{E}\Bigg[\int_{t\wedge\sigma\wedge}^{t\wedge\tau_\beta\wedge S_i\wedge T_i} L_2V(s,E_s,X(s-))dE_s\Bigg]\\
\leq&\mathbb{E}\Bigg[V(t\wedge\sigma, E_{t\wedge\sigma}, X(t\wedge\sigma))\Bigg].
\end{aligned}
\tag{3.37}
$$

Letting $i\to\infty$,

$$
\mathbb{E}\Bigg[V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))\Bigg]\geq\mathbb{E}\Bigg[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\Bigg],
\tag{3.38}
$$

that is,

$$
\begin{aligned}
\mathbb{E}\Bigg[&V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))[\mathbb{1}_{\{\sigma<\infty\}}+\mathbb{1}_{\{\sigma=\infty\}}]\Bigg]\\
&\geq\mathbb{E}\Bigg[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))[\mathbb{1}_{\{\sigma<\infty\}}+\mathbb{1}_{\{\sigma=\infty\}}]\Bigg].
\end{aligned}
\tag{3.39}
$$

For $w\in\{\tau_\alpha\geq\tau_h\wedge\theta\}$, we have $\sigma=\infty$, then $\tau_\beta=\infty$, thus

$$
V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))=V(t,E_t,X(t))
\tag{3.40}
$$

and

$$
V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))=V(t,E_t,X(t))
\tag{3.41}
$$

Thus,

$$
\mathbb{E}\Bigg[V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Bigg]\geq\mathbb{E}\Bigg[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Bigg].
\tag{3.42}
$$

a

Now, focus on the right hand side of (3.42), by definition of τ_β, $\tau_\beta \geq \sigma$, thus $\mathbb{1}_{\{\sigma<\infty\}} \geq \mathbb{1}_{\{\tau_\beta<\infty\}}$, then

$$\mathbb{E}\Big[V(\tau_\beta \wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta \wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big] \geq \mathbb{E}\Big[V(\tau_\beta \wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta \wedge t))\mathbb{1}_{\{\tau_\beta<\infty\}}\Big]. \quad (3.43)$$ b

Combining (3.42) and (3.43), we have

$$\mathbb{E}\Big[V(\sigma \wedge t, E_{\sigma\wedge t}, X(\sigma \wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big] \geq \mathbb{E}\Big[V(\tau_\beta \wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta \wedge t))\mathbb{1}_{\{\tau_\beta<\infty\}}\Big]. \quad (3.44)$$

Since $P(\sigma < \infty) = P(\tau_\alpha < \tau_h \wedge \theta)$ and $P(\tau_\beta < \infty) \geq P(\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\})$, it follows that

$$\mathbb{E}\big[V(\tau_\beta, E_{\tau_\beta}, X(\tau_\beta))\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] \leq \mathbb{E}\big[V(\tau_\alpha, E_{\tau_\alpha}, X(\tau_\alpha))\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big]. \quad (3.45)$$ c

By condition (2)

$$0 \leq \mu(|x|) \leq V(t_1, t_2, x), \quad (3.46)$$

for all $(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, and $|X(\tau_\beta)| \geq \beta > 0$.

Then, for the left hand side of (3.45), we have

$$\begin{aligned}\mathbb{E}\big[V(\tau_\beta, E_{\tau_\beta}, X(\tau_\beta))\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] &\geq \mathbb{E}\big[\mu(|X(\tau_\beta)|)\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] \\ &\geq \mathbb{E}\big[\mu(\beta)\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] \\ &= \mu(\beta)\mathbb{E}\big[\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] \\ &= \mu(\beta)P(\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\}).\end{aligned} \quad (3.47)$$ d

Let

$$B_\alpha = \sup_{t_1\times t_2\times x\in\mathbb{R}_+\times\mathbb{R}_+\times\bar{S_\alpha}} V(t_1, t_2, x), \quad (3.48)$$

then $B_\alpha \to 0$ as $\alpha \to 0$, that's, $\frac{B_\alpha}{\mu(\beta)} < \frac{\epsilon}{5}$ for some α.

For the right hand side of (3.45),

$$\begin{aligned}\mathbb{E}\big[V(\tau_\alpha, E_{\tau_\alpha}, X(\tau_\alpha))\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big] &\le \mathbb{E}\big[B_\alpha \mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big] \\ &= B_\alpha \mathbb{E}\big[\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big] \\ &= B_\alpha P(\tau_\alpha < \tau_h \wedge \theta).\end{aligned} \tag{3.49}$$

Combining (3.47) and (3.49), we have

$$P(\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\})\mu(\beta) \le B_\alpha P(\tau_\alpha < \tau_h \wedge \theta), \tag{3.50}$$

thus

$$P(\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\}) \le \frac{B_\alpha}{\mu(\beta)} P(\tau_\alpha < \tau_h \wedge \theta) < \frac{\epsilon}{5}. \tag{3.51}$$

Also,

$$P(\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\}) \ge P(\tau_\beta < \infty) - P(\tau_h < \infty) > P(\tau_\beta < \infty) - \frac{\epsilon}{5}, \tag{3.52}$$

so,

$$P(\tau_\beta < \infty) < \frac{2\epsilon}{5}. \tag{3.53}$$

Next

$$\begin{aligned}P(\{\sigma < \infty\} \cap \{\tau_\beta = \infty\}) &\ge P(\sigma < \infty) - P(\tau_\beta < \infty) \\ &> P(\tau_\alpha < \tau_h \wedge \theta) - \frac{2\epsilon}{5} \\ &\ge 1 - \frac{3\epsilon}{5} - \frac{2\epsilon}{5} \\ &= 1 - \epsilon.\end{aligned} \tag{3.54}$$

Hence,

$$P\{\omega; \limsup_{t\to\infty} \big|X(t, x_0)\big| \le \beta\} > 1 - \epsilon. \tag{3.55}$$

Since β is arbitrary, we have

$$P\{\omega; \limsup_{t\to\infty} \big|X(t,x_0)\big| = 0\} > 1-\epsilon, \tag{3.56}$$

as desired.

□

tm3 **Theorem 3.1.3** *Assume that there exists a function* $V(t_1,t_2,x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ *and* $u \in \mathcal{K}$ *such that for all* $(t_1,t_2,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$

1. $V(t_1,t_2,0) = 0,$
2. $\mu(|x|) \le V(t_1,t_2,x),$
3. $L_1V(t_1,t_2,x) \le -\gamma_1(x)$ *a.s. and* $L_2V(t_1,t_2,x) \le -\gamma_2(x)$ *a.s.,* (3.57)

   where $\gamma_1(x) \ge 0$ *and* $\gamma_2(x) \ge 0$ *but not equal to zero at the same time,*
4. $\lim_{|x|\to\infty} \inf_{t_1,t_2\ge 0} V(t_1,t_2,x) = \infty,$

then the trivial solution of the time-changed SDE (1.6) *is globally stochastically asymptotically stable.*

Proof: This proof has similar idea as Theorem 4.2.4 in [21], so we omit the details here. □

Example 3.1.4 *Consider the following SDE driven by time-changed Lévy noise*

$$\begin{aligned} dX(t) =& f(t,E_t)X(t)dt + k(t,E_t)X(t)dE_t \\ &+ g(t,E_t)X(t)dB_{E_t} + \int_{|y|<c} h(t,E_t,y)X(t)d\tilde{N}(dE_s,dy) \end{aligned} \tag{3.58}$$

example1

with $X(0) = x_0$*, where* k, f, g, h *are* $\mathcal{G}_t$*-measurable real-valued functions satisfying Lipschitz condition 2.0.1, growth condition 2.0.2 and assumption 2.0.3. Define Lyapunov function*

$$V(t_1,t_2,x) = |x|^\alpha \tag{3.59}$$

on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ *for some* $\alpha \in (0,1)$. *Then*

$$L_1V(t_1,t_2,x) = \alpha f(t_1,t_2)|x|^\alpha \tag{3.60}$$

and

$$L_2V(t_1,t_2,x) = \Big[\alpha k(t_1,t_2) + \frac{\alpha(\alpha-1)}{2}g^2(t_1,t_2) + \int_{|y|<c}\Big[|1+h(t_1,t_2,y)|^\alpha - 1 - \alpha h(t_1,t_2,y)\Big]\nu(dy)\Big]|x|^\alpha. \tag{3.61}$$

Thus, if

$$\alpha f(t,E_t) \leq 0 \;\; a.s. \tag{3.62}$$

and

$$\alpha k(t,E_t) + \frac{\alpha(\alpha-1)}{2}g^2(t,E_t) + \int_{|y|<c}\Big[|1+h(t,E_t,y)|^\alpha - 1 - \alpha h(t,E_t,y)\Big]\nu(dy) \leq 0 \;\; a.s. \tag{3.63}$$

for all $t, E_t \in \mathbb{R}_+$, *the trivial solution of SDE* (3.188) *is stochastically stable, by Theorem 3.1.1.*

Let $\alpha = 0.5,\; c = 1$ *and* $f(t_1,t_2) = -1,\; k(t_1,t_2) = 0.25,\; g(t_1,t_2) = 1,\; h(t_1,t_2,y) = y$ *for all* $t_1, t_2 \in \mathbb{R}_+$, *then*

$$L_1V(t_1,t_2,x) = -\frac{|x|^\alpha}{2} \leq 0 \tag{3.64}$$

and

$$L_2V(t_1,t_2,x) = \int_{|y|<1}\Big[|1+y|^{\frac{1}{2}} - 1 - \frac{1}{2}y\Big]\nu(dy) < 0. \tag{3.65}$$

Therefore, by Theorem 3.1.3, trivial solution of SDE

$$dX(t) = -X(t)dt + 0.25X(t)dE_t + X(t)dB_{E_t} + \int_{|y|<1} yX(t)d\tilde{N}(ds,dy) \tag{3.66}$$

with $X(0) = x_0$ *is globally stochastically asymptotically stable.*

Remark 3.1.5 *Note that* $V(t_1,t_2,x) = |x|^\alpha$ *with* $\alpha \in (0,1)$ *is not a* C^2 *function with respect to* x *in* $\mathcal{R}$, *but it is sufficient in this case since* $X(t) \neq 0$ *if* $X(0) \neq 0$ *for* $t \geq 0$, *see the following* *for details. That is, $V(t_1,t_2,x)=|x|^\alpha$ is a C^2 function with respect to x in the domain of $X(t)$ for $t\geq 0$.*

By Itô formula for time-changed Lévy noise, we have, for $x_0\neq 0$,

$$\begin{aligned}
\ln(|X(t)|) &= \ln(|x_0|)+\int_0^t \frac{1}{X(s-)}f(s,E_S)X(s-)ds+\int_0^t \frac{1}{X(s-)}g(s,E_S)X(s-)dB_{E_s}\\
&+\int_0^t \Big[\frac{1}{X(s-)}k(s,E_S)X(s-)+\frac{1}{2}g(s,E_s)^2X(s-)^2\frac{-1}{X(s-)^2}\\
&+\int_{|y|<c}\big[\ln(|X(s-)+h(s,E_s,y)X(s-)|)-\ln(|X(s-)|)\\
&\qquad\qquad -\frac{1}{X(s-)}h(s,E_s,y)X(s-)\big]\nu(dy)\Big]dE_s\\
&+\int_0^t\int_{|y|<c}\big[\ln(|X(s-)+h(s,E_s,y)X(s-)|)-\ln(|X(s-)|)\big]\tilde{N}(dE_s,dy)\\
&=\ln(|x_0|)+\int_0^t f(s,E_S)ds+\int_0^t g(s,E_S)dB_{E_s}\\
&+\int_0^t\int_{|y|<c}\big[\ln(|1+h(s,E_s,y)|)\big]\tilde{N}(dE_s,dy)\\
&+\int_0^t\Big[k(s,E_s)-\frac{g(s,E_s)^2}{2}+\int_{|y|<c}\big[\ln(|1+h(s,E_s,y)|)-h(s,E_s,y)\big]\nu(dy)\Big]dE_s.
\end{aligned}\tag{3.67}$$

Let

$$\begin{aligned}
M(t)&=\int_0^t f(s,E_S)ds+\int_0^t g(s,E_S)dB_{E_s}+\int_0^t\int_{|y|<c}\big[\ln(|1+h(s,E_s,y)|\big]\tilde{N}(dE_s,dy)\\
&+\int_0^t\Big[k(s,E_S)-\frac{1}{2}g(s,E_s)^2+\int_{|y|<c}\big[\ln(|1+h(s,E_s,y)|-h(s,E_s,y)\big]\nu(dy)\Big]dE_s,
\end{aligned}\tag{3.68}$$

then $|X(t)|=|x_0|\exp(M(t))>0$ for all $t\geq 0$.

Similar argument applies to Example 3.2.2.

3.2 Stability in Moment

tm4 **Theorem 3.2.1** *Let $p,\alpha_1,\alpha_2,\alpha_3$ be positive constants. If $V\in C^2(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R};\mathbb{R}_+)$ satisfies*

$$\begin{aligned}
&1.\ V(t_1,t_2,0)=0, \quad 2.\ \alpha_1|x|^p\leq V(t_1,t_2,x)\leq\alpha_2|x|^p,\\
&3.\ L_2V(t_1,t_2,x)\leq 0,\ \ 4.\ L_1V(t_1,t_2,x)\leq -\alpha_3V(t_1,t_2,x),
\end{aligned}\tag{3.69}$$

$\forall(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, *then the trivial solution of the time-changed SDE* (1.6) *is pth moment exponentially stable with*

$$\mathbb{E}|X(t, x_0)|^p \leq \frac{\alpha_2}{\alpha_1}|x_0|^p \exp(-\alpha_3 t). \tag{3.70}$$

roofoftm4

Proof: Define a function $Z : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$Z(t_1, t_2, x) = exp(\alpha_3 t_1)V(t_1, t_2, x). \tag{3.71}$$

Fix any $x_0 \neq 0$ in $\mathbb{R}$. For each $n \geq |x_0|$, define

$$\tau_n = \inf\{t \geq 0 : |X(t)| \geq n\},$$

and

$$\begin{aligned}
U_k =& k \wedge \inf\{t \geq 0; \left| \int_0^{\tau_n \wedge t} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \right| \geq k\}, \\
W_k =& k \wedge \inf\{t \geq 0; \left| \int_0^{\tau_n \wedge t} \int_{|y|<c} \Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) \right. \\
& \left. - V(s, E_s, X(s-)) \Big] \tilde{N}(ds, dy) \right| \geq k\},
\end{aligned} \tag{3.72}$$

stoptime2

for k=1,2,.... It is easy to see that $U_k \to \infty$ and $W_k \to \infty$ as $k \to \infty$.

Apply Itô formula (3.2) to $Z(\tau_n \wedge U_k \wedge W_k, E_{\tau_n \wedge U_k \wedge W_k}, X(\tau_n \wedge U_k \wedge W_k))$, then we have

$$
\begin{aligned}
& Z(t \wedge \tau_n \wedge U_k \wedge W_k, E_{t \wedge \tau_n \wedge U_k \wedge W_k}, X(t \wedge \tau_n \wedge U_k \wedge W_k)) - Z(0, 0, x_0) \\
= & \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \exp(\alpha_3 s) \Big[\alpha_3 V(s, E_s, X(s-)) + V_s(s, E_s, X(s-) \Big] ds \\
& + \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \exp(\alpha_3 s) V_{E_s}(s, E_s, X(s-)) dE_s \\
& + \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \exp(\alpha_3 s) V_x(s, E_s, X(s-)) \Big[f(s, E_s, X(s-)) dt \\
& \qquad + k(s, E_s, X(s-)) dE_t + g(s, E_s, X(s-)) dB_{E_t} \Big] \\
& + \frac{1}{2} \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \exp(\alpha_3 s) V_{xx}(s, E_s, X(s-)) g^2(s, E_s, X(s-)) dE_s \\
& + \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \int_{|y|<c} \exp(\alpha_3 s) \Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) \\
& \qquad - V(s, E_s, X(s-)) \Big] \tilde{N}(dE_s, dy) \\
& + \int_0^{t \wedge \tau_n \wedge U_k \wedge W_k} \int_{|y|<c} \exp(\alpha_3 s) \Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) - V(s, E_s, X(s-)) \\
& \qquad - V_x(s, E_s, X(s-)) h(s, E_s, X(s-), y) \Big] \nu(dy) dE_s
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
&= \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s,E_s,X(s-)) + V_s(s,E_s,X(s-)) \\
&\qquad\qquad\qquad\qquad\qquad + V_x(s,E_s,X(s-))f(s,E_s,X(s-))\Big]ds \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[V_{E_s}(s,E_s,X(s-)) + V_x(s,E_s,X(s-))k(s,E_s,X(s-)) \\
&\quad + \frac{1}{2}V_{xx}(s,E_s,X(s-))g^2(s,E_s,X(s-)) + \int_{|y|<c}\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad - V(s,E_s,X(s-)) - V_x(s,E_s,X(s-))h(s,E_s,X(s-),y)\Big]\nu(dy)\Big]dE_s \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)g(s,E_s,X(s-))V_x(s,E_s,X(s-))dB_{E_s} \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \int_{|y|<c} \exp(\alpha_3 s)\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad\qquad\qquad\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s,E_s,X(s-)) + L_1 V(s,E_s,X(s-)\Big]ds \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)L_2 V(s,E_s,X(s-))dE_s \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)g(s,E_s,X(s-))V_x(s,E_s,X(s-))dB_{E_s} \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \int_{|y|<c} \exp(\alpha_3 s)\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad\qquad\qquad\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)
\end{aligned}
$$

By similar ideas as in the proof of (3.1), we have that

$$
\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)g(s,E_s,X(s-))V_x(s,E_s,X(s-))dB_{E_s}
$$

and

$$
\begin{aligned}
&\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \int_{|y|<c} \exp(\alpha_3 s)\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad\qquad\qquad\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)
\end{aligned} \tag{3.74}
$$

are mean zero martingales. Taking expectations on both sides, we have

$$
\begin{aligned}
&\mathbb{E}[\exp(\alpha_3(t\wedge\tau_n\wedge U_k\wedge W_k))V(t\wedge\tau_n\wedge U_k\wedge W_k, E_{t\wedge\tau_n\wedge U_k\wedge W_k}, X(t\wedge\tau_n\wedge U_k\wedge W_k))]\\
\leq&\mathbb{E}\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k}\exp(\alpha_3 s)\Big[\alpha_3 V(s,E_s,X(s-))+L_1V(s,E_s,X(s-)\Big]ds+V(0,0,x_0)\\
\leq&V(0,0,x_0).
\end{aligned}
\tag{3.75}
$$

Letting $k\to\infty$ and $n\to\infty$, $\mathbb{E}[\exp(\alpha_3 t)V(t,E_t,X(t))]\leq V(0,0,x_0)$. By condition (2),

$$\alpha_1|X(t)|^p\leq V(t,E_t,X(t)), \tag{3.76}$$

then

$$\alpha_1\mathbb{E}(\exp(\alpha_3 t)|X(t)|^p)\leq\mathbb{E}(\exp(\alpha_3 s)V(t,E_t,X(t)))\leq V(0,0,x_0)\leq\alpha_2|x_0|^p, \tag{3.77}$$

that's

$$\mathbb{E}(|X(t)|^p)\leq\frac{\alpha_2}{\alpha_1}\exp(-\alpha_3 t)|x_0|^p, \tag{3.78}$$

as desired. □

3.10 **Example 3.2.2** *Consider the following SDE driven by time-changed Lévy noise*

$$dX(t)=-X(t)dt+X(t)E_t^2dB_{E_t}+\int_{|y|<1}\Big[X(t)y^2-X(t)\Big]\tilde{N}(dE_t,dy) \tag{3.79}$$

with $X(0)=x_0$ and ν is a Lévy measure. Let $V(t_1,t_2,x)=|x|$, then

$$L_1V(t_1,t_2,x)=-|x| \tag{3.80}$$

and

$$L_2V(t_1,t_2,x) = \frac{1}{2}x^2t_2^4\big(-\frac{1}{x^2}\big) + \int_{|y|<1}\Big[|x+xy^2-x| - |x| - sgn(x)(xy^2-x)\Big]\nu(dy)$$
$$= -\frac{t_2^4}{2} + \int_{|y|<1}\Big[(|y^2|-y^2)|x|\Big]\nu(dy) \le 0. \tag{3.81}$$

By Theorem 3.2.1, X(t) is first moment exponentially stable, that is,

$$\mathbb{E}|X(t,x_0)| \le |x_0|\exp(-t), \forall t \ge 0. \tag{3.82}$$

3.3 Duality Property

Next, we reduce SDE (1.6) by setting $f(t,E_t,X(t-)) = 0$,

$$dX(t) = k(E_t,X(t-))dE_t + g(E_t,X(t-))dB_{E_t} + \int_{|y|<c} h(E_t,X(t-),y)\tilde{N}(dE_t,dy), \tag{3.83}$$

redSDE

with $X(0) = x_0$.

Kobayashi [15] mentioned duality related to (3.83) and the following SDE

$$dY(t) = k(t,Y(t-))dt + g(t,Y(t-))dB_t + \int_{|y|<c} h(t,Y(t-),y)\tilde{N}(dt,dy), Y(0) = x_0, \tag{3.84}$$

redSDEsim

with $Y(0) = x_0$, stating that

1. If a process $Y(t)$ satisfies SDE (3.84), then $X(t) := Y(E_t)$ satisfies the time-changed SDE (3.83);

2. If a process $X(t)$ satisfies the time-changed SDE (3.83), then $Y(t) := X(D(t))$ satisfies SDE (3.84).

Corollary 3.3.1 *Let $Y(t)$ be a stochastically stable (stochastically asymptotically stable, globally stochastically asymptotically stable) process satisfying SDE* (3.84)*, then the trivial solution $X(t)$ of SDE* (3.83) *is a stochastically stable (stochastically asymptotically stable, globally stochastically asymptotically stable) process, respectively.*

Proof: This proof has similar idea as Corollary 3.1 in[32], thus we omit details.

Though the conclusion of Corollary 3.1 in[32] is correct, there is a minor problem in the proof. We correct it as following

$$\begin{aligned}
P\Big\{|X(t,x_0)| < h, \forall t \geq 0\Big\} &= P\Big\{|Y(E_t,x_0)| < h, \forall t \geq 0\Big\} \\
&= P\Big\{\sup_{0\leq t<\infty} |Y(E_t,x_0)| < h\Big\} \\
&= P\Big\{\sup_{\{E_t:\, 0\leq t<\infty\}} |Y(E_t,x_0)| < h\Big\} \\
&= P\Big\{\sup_{0\leq \tau<\infty} |Y(\tau,x_0)| < h\Big\} \\
&= P\Big\{|Y(t,x_0)| < h, \forall t \geq 0\Big\} \\
&= 1-\epsilon.
\end{aligned} \tag{3.85}$$

Here, we use the fact that the image of $[0,\infty)$ under E_t process is almost surely equal to $[0,\infty)$.

□

tm5 **Corollary 3.3.2** *Let $Y(t)$ be a pth moment exponentially stable process satisfying SDE* (3.84)*, the $X(t)$ is a pth moment asymptotically stable satisfying SDE* (3.83)*.*

Proof: If $Y(t)$ satisfies SDE (3.84), by Theorem 4.2 in [15], $X(t) = Y(E_t)$ satisfies (3.83). Since $Y(t)$ is pth moment exponentially stable, there exist two positive constants λ and C such that

$$\mathbb{E}[|X(t)|^p] \leq C|x_0|^p \exp(-\lambda t),\ \forall t \geq 0,\ \forall x_0 \in \mathbb{R},\ p > 0, \tag{3.86}$$

then

$$\begin{aligned}
\mathbb{E}[|Y(t)|^p] &= \mathbb{E}[|X(E_t)|^p] \\
&= \int_0^\infty \mathbb{E}[|X(s)|^p \exp(\lambda s)\exp(-\lambda s)|E_t = s] f_{E_t}(s)ds \\
&= \int_0^\infty \mathbb{E}[|X(s)|^p \exp(\lambda s)|E_t = s] \exp(-\lambda s) f_{E_t}(s)ds \\
&\leq \int_0^\infty C|x_0|^p \exp(-\lambda s) f_{E_t}(s)ds \\
&= C|x_0|^p \mathbb{E}[\exp(-\lambda E_t)].
\end{aligned} \tag{3.87}$$

Since E_t is nondecreasing and $E_0 = 0$, by definition of E_t, we claim that $\lim_{t\to\infty} E_t = \infty$ a.s.. Assume to the contrary that there exists $B > 0$ such that $E_t < B$ for all $t > 0$ with positive probability, then $D(B) > t$ for all $t > 0$ with positive probability. However, by Lemma 12.1 of [5], $D(B)$ is bounded, which results in a contradiction. Consequently, $\mathbb{E}[\exp(-\lambda E_t)] \to 0$ as $t \to \infty$, as desired. □

Remark 3.3.3 *Existence of pth moment stability of the solution of SDE* (3.84) *has been proved by Theorem 4.1 in [2].*

Remark 3.3.4 *Our results can not be easily extended to time-changed stochastic differential equation with large jumps, this is because that stochastic integral against Poisson process is not automatically to be local martingale. Thus, the normal method to prove stability of solutions of time-changed stochastic differential equation as used in this paper does not work. It is possible to apply stricter conditions to derive similar results for time-changed stochastic differential equation with large jumps, but the strength of the results has to be compromised.*

Remark 3.3.5 *The Lyapunov functions V in our main results above vary from case to case, but under certain conditions it is possible to construct Lyapunov function by a general formula, see [3] as an example.*

3.4 Stability in Path

To perform future analysis regarding path stability, we need some conditions under which the solutions of (1.6) can not reach the origin after certain time t_0 given that $X(t_0) \neq 0$.

preass1 **Assumption 3.4.1** *For any $\theta > 0$ there exists $K_\theta > 0$, such that*

$$|k(t_1,t_2,x)| + |g(t_1,t_2,x)| + 2\int_{|y|<c} \frac{|h(t_1,t_2,x,y)|(|x| + |h(t_1,t_2,x,y)|)}{|x + h(t_1,t_2,x,y)|}\nu(dy) \leq K_\theta |x| \tag{3.88}$$

preassfor

and

$$|f(t_1,t_2,x)| \leq K_\theta |x|^2 \text{ , for } 0 < |x| \leq \theta \text{ and } t_1, t_2 \in \mathbb{R}_+. \tag{3.89}$$

preassfor

Lemma 3.4.2 *Given that the assumption* (3.4.1) *holds, the solution of* (1.6) *satisfies*

$$P(X(t) \neq 0 \; for \; all \; t \geq t_0) = 1, \qquad (3.90)$$

preassfor

if $x_0 \neq 0$.

Proof: We follow the idea in the proof of Lemma 3.4.4 in [30] and prove this result by contradiction. Suppose that (3.90) is not true, that is, there exists initial condition $x_0 \neq 0$ and stopping time τ with $P(\tau < \infty) > 0$ where

$$\tau = \inf\{t \geq t_0 : |X(t)| = 0\}. \qquad (3.91)$$

Since the paths of $X(t)$ are right continuous with left limit (rcll), there exist $T > 0$ and $\theta > 1$ sufficiently large such that $P(B) > 0$, where

$$B = \{w \in \Omega : \tau(w) \leq T \; and \; |X(t)(w)| \leq \theta - 1 \; for \; all \; t_0 < t < \tau(w)\}. \qquad (3.92)$$

Next, define another stopping time

$$\tau_\epsilon = \inf\{t \geq t_0 : |X(t)| \leq \epsilon \; or \; |X(t)| \geq \theta\} \qquad (3.93)$$

for each $0 < \epsilon < |X(t_0)|$.

Let $\lambda = 2K_\theta + K_\theta^2$ be a constant and define $Z(t) = e^{-\lambda E_t}|X(t)|^{-1}$. Since $F(t_1, t_2, x) = e^{-\lambda t_2}|x|^{-1}$ is in $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times (\mathbb{R}\backslash 0))$, and by definition of τ_ϵ, $X(t)$ will not reach 0 for $t_0 \leq t \leq \tau_\epsilon \wedge T$, so Itô formula can be applied to $e^{-\lambda(E_{\tau_\epsilon \wedge T})}|X(\tau_\epsilon \wedge T)|^{-1}$.

By (3.88) and (3.89),

$$
\begin{aligned}
& e^{-\lambda(E_{\tau_\epsilon\wedge T})}|X(\tau_\epsilon\wedge T)|^{-1}-|x_0|^{-1} \\
=&\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}[-\frac{X(s-)f(s,E_s,X(s-))}{|X(s-)|^3}]ds+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\frac{g(s,E_s,X(s-))^2}{|X(s-)|^3}dE_s \\
&+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\frac{-1}{|X(s-)|^3}\Big[\lambda|X(s-)|^2dE_s+k(s,E_s,X(s-))X(s-)dE_s \\
&\qquad\qquad +g(s,E_s,X(s-))X(s-)dB_{E_s}\Big] \\
&+\int_{t_0}^{\tau_\epsilon\wedge T}\int_{|y|<c} e^{-\lambda E_s}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|}\Big]\tilde{N}(dE_s,dy) \\
&+\int_{t_0}^{\tau_\epsilon\wedge T}\int_{|y|<c} e^{-\lambda E_s}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|} \\
&\qquad\qquad +\frac{X(s-)h(s,E_s,X(s-),y)}{|X(s-)|^3}\Big]\nu(dy)dE_s \\
\le&\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}K_\theta ds+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\frac{-g(s,E_s,X(s-))X(s-)}{|X(s-)|^3}dB_{E_s} \\
&+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\Bigg[\frac{-\lambda}{|X(s-)|}+\frac{-k(s,E_s,X(s-))X(s-)}{|X(s-)|^3}+\frac{g(s,E_s,X(s-))^2}{|X(s-)|^3} \\
&+\int_{|y|<c}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|} \\
&\qquad\qquad +\frac{X(s-)h(s,E_s,X(s-),y)}{|X(s-)|^3}\Big]\nu(dy)\Bigg]dE_s \\
&+\int_{t_0}^{\tau_\epsilon\wedge T}\int_{|y|<c} e^{-\lambda E_s}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|}\Big]\tilde{N}(dE_s,dy) \\
\le& K_\theta\tau_\epsilon\wedge T+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\Big[\frac{-\lambda}{|X(s-)|}+\frac{2K_\theta+K_\theta^2}{|X(s-)|}\Big]dE_s \\
&+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\frac{-g(s,E_s,X(s-))X(s-)}{|X(s-)|^3}dB_{E_s} \\
&+\int_{t_0}^{\tau_\epsilon\wedge T}\int_{|y|<c} e^{-\lambda E_s}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|}\Big]\tilde{N}(dE_s,dy) \\
\le& K_\theta T+\int_{t_0}^{\tau_\epsilon\wedge T} e^{-\lambda E_s}\frac{-g(s,E_s,X(s-))X(s-)}{|X(s-)|^3}dB_{E_s} \\
&+\int_{t_0}^{\tau_\epsilon\wedge T}\int_{|y|<c} e^{-\lambda E_s}\Big[\frac{1}{|X(s-)+h(s,E_s,X(s-),y)|}-\frac{1}{|X(s-)|}\Big]\tilde{N}(dE_s,dy)
\end{aligned}
\tag{3.94}
$$

ques

The penultimate inequality is derived from lemma 3.4.2 on page 54 of [30], which states that $\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{xy}{|x|^3} \leq \frac{2|y|}{|x|^2}\frac{(|y|+|x|)}{|x+y|}$ for $x, y, x+y \neq 0$, thus

$$\begin{aligned}
&\int_{|y|<c} \Big[\frac{1}{|X(s-) + h(s, E_s, X(s-), y)|} - \frac{1}{|X(s-)|} + \frac{X(s-)h(s, E_s, X(s-), y)}{|X(s-)|^3}\Big]\nu(dy) \\
\leq &\int_{|y|<c} \frac{2|h(X(s, E_s, s-), y)|}{|X(s-)|^2}\Big[\frac{|h(s, E_s, X(s-), y)| + |X(s-)|}{|h(s, E_s, X(s-), y) + X(s-)|}\Big]\nu(dy) \\
=&\frac{1}{|X(s-)|^2}\int_{|y|<c} \frac{2|h(s, E_s, X(s-), y)|(|h(s, E_s, X(s-), y)| + |X(s-)|)}{|h(s, E_s, X(s-), y) + X(s-)|}\nu(dy) \\
\leq&\frac{K_\theta|X(s-)|}{|X(s-)|^2} = \frac{K_\theta}{|X(s-)|}.
\end{aligned} \tag{3.95}$$

Observe that the last two terms in the last line of the inequality (3.94) are martingales. Then by taking expectations of both sides, we derive that

$$\mathbb{E}\Big[e^{-\lambda(E_{\tau_\epsilon \wedge T})}|X(\tau_\epsilon \wedge T)|^{-1}\Big] \leq |x_0|^{-1} + K_\theta T. \tag{3.96}$$

If $w \in B$, then $\tau_\epsilon(w) \leq T$ and $|X(\tau_\epsilon(w))| \leq \epsilon$, then

$$\mathbb{E}\Big[e^{-\lambda E_{\tau_\epsilon \wedge T}}\epsilon^{-1}\mathbb{1}_B\Big] \leq \mathbb{E}\Big[\frac{e^{-\lambda E_{\tau_\epsilon \wedge T}}}{|X(\tau_\epsilon(w))|}\mathbb{1}_B\Big] \leq \mathbb{E}\Big[\frac{e^{-\lambda E_{\tau_\epsilon \wedge T}}}{|X(\tau_\epsilon(w))|}\Big] \leq |x_0|^{-1} + K_\theta T. \tag{3.97}$$

Recall the reverse Hölder's inequality: for all $p > 1$

$$\mathbb{E}(|XY|) \geq (\mathbb{E}|X|^{1/p})^p(\mathbb{E}(|Y|^{-1/(p-1)}))^{-(p-1)}.$$

We use the reverse Hölder's inequality with $p = 2$, $X = \mathbb{1}_B$ and $Y = e^{-\lambda E_{\tau_\epsilon \wedge T}}$. Since $X^{1/2} = X$, this gives

$$[\mathbb{P}(B)]^2\Bigg[E(e^{\lambda E_{\tau_\epsilon \wedge T}})\Bigg]^{-1} \leq \mathbb{E}\Big[e^{-\lambda E_{\tau_\epsilon \wedge T}}\mathbb{1}_B\Big] \leq \epsilon(|x_0|^{-1} + K_\theta T),\ \textit{for all } \epsilon \geq 0$$

Since the inverse subordinator has finite exponential moment, $E(e^{\lambda E_{\tau_\epsilon \wedge T}})$ is finite for any fixed time T, see Lemma 8 in [14]. Then, letting $\epsilon \to 0$, we obtain $P(B) = 0$, which contradicts the assumption, thus the desired result is correct. $\square$

Remark 3.4.3 *When the Laplace exponent of the subordinator is given by* (2.3)*, an alternative method to show that the expectation $E(e^{\lambda E_{\tau_\epsilon \wedge T}})$ is finite is to use the moments of E_t. Since $\{E_t, t \geq 0\}$ is nonnegative and nondecreasing, we have $\tau_\epsilon \wedge T \leq T$. Because $\lambda > 0$, e^x is a strictly positive and increasing function, $E(e^{\lambda E_{\tau_\epsilon \wedge T}}) \leq E(e^{\lambda E_T})$. Thus, it is sufficient to show that $E(e^{\lambda E_T})$ is finite. By Theorem 3.9 in [22], there exists a function $L \in RV_\infty(0)$ such that for any $n > 0$,$\gamma > 0$ and sufficiently large t,*

$$\mathbb{E}[E_t^n] \sim (\log t)^{\gamma n} L(\log t)^{-n}. \tag{3.98}$$

By Taylor expansion and Fubini theorem,

$$\begin{aligned} \mathbb{E}[\exp(\lambda E_t)] = \mathbb{E}[\sum_{n=0}^{\infty} \frac{\lambda^n {E_t}^n}{n!}] = \sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}[{E_t}^n]}{n!} \sim \sum_{n=0}^{\infty} \frac{\lambda^n (\log t)^{\gamma n} L(\log t)^{-n}}{n!} \\ = \sum_{n=0}^{\infty} \frac{(\lambda (\log t)^{\gamma} L(\log t)^{-1})^n}{n!} = \exp(\lambda (\log t)^{\gamma} L(\log t)^{-1}). \end{aligned} \tag{3.99}$$

Hence, for fixed large t, $\mathbb{E}[\exp(\lambda E_t)] \sim \exp(\lambda (\log t)^{\gamma} L(\log t)^{-1})$ is finite.

A similar method applies when the Laplace exponent of the subordinator $D(t)$ is given by

$$\psi(s) = \sum_{i=1}^{k} c_i s^{\beta_i}, \tag{3.100}$$

laplace-e

where $\sum_{i=1}^k c_i = 1$ and $0 < \beta_1 < \beta_2 < ... < \beta_k < 1$. Then the Laplace transform of the $n-th$ moment of E_t is $\mathcal{L}(\mathbb{E}(E_t^n))(s) = \frac{n!}{s(\sum_{i=1}^k c_i s^{\beta_i})^n}$; see Lemma 8 in [14]. Using the Karamata Tauberian Theorem (see [7], Theorem 1 and Lemma on pp. 443-446) we can deduce that for large t, $\mathbb{E}(E_t^n) \approx C_n t^{n\beta_1}$

Lemma 3.4.4 *(Time-Changed Exponential Martingale Inequality) Let $D(t)$ be a rcll subordinator and its inverse process $E_t := \inf\{\tau > 0 : D(\tau) > t\}$. Let T, λ, κ be any positive* *numbers,* $B_c = \{y \in \mathbb{R} : |y| < c\}$*. Assume* $g : \mathbb{R}^+ \to \mathbb{R}$ *and* $h : \mathbb{R}^+ \times B_c \to \mathbb{R}$ *satisfy* $\mathbb{E}[\int_0^T |g(t)|^2 dE_t] < \infty$ *and* $\mathbb{E}[\int_0^T \int_{|y|<c} |h(t,y)|^2 \nu(dy) dE_t] < \infty$*, then*

$$\begin{aligned} P\Big[\sup_{0\le t\le T} \Big\{ \int_0^t g(s)dB_{E_s} - \frac{\lambda}{2}\int_0^t |g(s)|^2 dE_s + \int_0^t \int_{|y|<c} h(s,y)\tilde{N}(dE_s,dy) \\ - \frac{1}{\lambda}\int_0^t \int_{|y|<c} \Big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\Big]\nu(dy)dE_s \Big\} > \kappa\Big] \le \exp(-\lambda\kappa) \end{aligned} \tag{3.101}$$

expmarine

Proof: Define a sequence of stopping times $(\tau_n, n \ge 1)$ as below

$$\begin{aligned} \tau_n =& \inf\Big\{t \ge 0 : \Big|\int_0^t g(s)dB_{E_s}\Big| + \frac{\lambda}{2}\int_0^t |g(s)|^2 dE_s + \Big|\int_0^t \int_{|y|<c} h(s,y)\tilde{N}(dE_s,dy)\Big| \\ &+ \frac{1}{\lambda}\Big|\int_0^t \int_{|y|<c} \Big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\Big]\nu(dy)dE_s\Big| \ge n\Big\},\ for\ n \ge 1. \end{aligned} \tag{3.102}$$

Note that $\tau_n \to \infty$ as $n \to \infty$ a.s.

Define the following Itô process

$$\begin{aligned} X_n(t) =& \lambda\int_0^t g(s)\mathbb{1}_{[0,\tau_n]}(s)dB_{E_s} - \frac{\lambda^2}{2}\int_0^t |g(s)|^2\mathbb{1}_{[0,\tau_n]}(s)dE_s \\ &+ \lambda\int_0^t \int_{|y|<c} h(s,y)\mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s,dy) \\ &- \int_0^t \int_{|y|<c} \Big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\Big]\mathbb{1}_{[0,\tau_n]}(s)\nu(dy)dE_s, \end{aligned} \tag{3.103}$$

with $X_n(0) = 0$ for all $n \ge 0$. Then for all $0 \le t \le T$

$$\begin{aligned} |X_n(t)| \le& \lambda\Big|\int_0^t g(s)\mathbb{1}_{[0,\tau_n]}(s)dB_{E_s}\Big| + \Big|\lambda\int_0^t \int_{|y|<c} h(s,y)\mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s,dy)\Big| \\ +& \frac{\lambda^2}{2}\int_0^t |g(s)|^2\mathbb{1}_{[0,\tau_n]}(s)dE_s + \Big|\int_0^t \int_{|y|<c} \Big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\Big]\mathbb{1}_{[0,\tau_n]}(s)\nu(dy)dE_s\Big| \\ \le& \lambda n. \end{aligned} \tag{3.104}$$

Let $Z(t) = \exp(X_n(t))$, by the time-changed Itô's formula (3.2),

$$
\begin{aligned}
&\exp(X_n(t)) - \exp(x_0) \\
&= \int_0^t \exp(X_n(s))\Big[-\frac{\lambda^2}{2}|g(s)|^2 \mathbb{1}_{[0,\tau_n]}(s) \\
&\quad - \int_{|y|<c} \big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\big] \mathbb{1}_{[0,\tau_n]}(s)\nu(dy) \\
&\quad + \int_{|y|<c} \big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)\big] \mathbb{1}_{[0,\tau_n]}(s)\nu(dy) + \frac{\lambda^2}{2}|g(s)|^2 \mathbb{1}_{[0,\tau_n]}(s)\Big] dE_s \\
&\quad + \int_0^t \int_{|y|<c} \big[\exp(X_n(s) + \lambda h(s,y)) - \exp(X_n(s))\big] \mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s, dy) \\
&\quad + \lambda \int_0^t \exp(X_n(s)) g(s) \mathbb{1}_{[0,\tau_n]}(s) dB_{E_s} \\
&= \int_0^t \int_{|y|<c} \big[\exp(X_n(s) + \lambda h(s,y)) - \exp(X_n(s))\big] \mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s, dy) \\
&\quad + \lambda \int_0^t \exp(X_n(s)) g(s) \mathbb{1}_{[0,\tau_n]}(s) dB_{E_s},
\end{aligned}
\tag{3.105}
$$

thus $\{\exp(X_n(t)), 0 \le t \le T\}$ is a local martingale. Since we have

$$
\sup_{t\in[0,T]} \exp(X_n(t)) \le \exp(\lambda n) \;\; a.s. \tag{3.106}
$$

there exists a sequence of stopping times $(T_m, m \in \mathbb{N})$ with $(T_m \to \infty)(a.s.)$ as $n \to \infty$ such that for all $0 \le s \le t \le T$

$$
\mathbb{E}[\exp(X_n(t \wedge T_m))|\mathcal{F}_s] = \exp(X_n(s \wedge T_m)) \le \exp(\lambda n) \;\; a.s. \tag{3.107}
$$

By Dominated Convergence Theorem, we have

$$
\mathbb{E}[\exp(X_n(t))|\mathcal{F}_s] = \lim_{m\to\infty} \mathbb{E}[\exp(X_n(t \wedge T_m))|\mathcal{F}_s] = \lim_{m\to\infty} \exp(X_n(s \wedge T_m)) = \exp(X_n(s)), \tag{3.108}
$$

that is, $Z(t) = \exp(X_n(t))$ is a martingale for all $0 \le t \le T$ with $\mathbb{E}[\exp(X_n(t))] = 1$.

Apply Doob's martingale inequality

$$\mathbb{P}\Big[\sup_{0\le t\le T}\exp(X_n(t))\ge\exp(\lambda\kappa)\Big]\le\exp(-\lambda\kappa)\mathbb{E}[\exp(X_n(T))]=\exp(-\lambda\kappa),\tag{3.109}$$

equivalently,

$$\mathbb{P}\Big[\sup_{0\le t\le T}\frac{X_n(t)}{\lambda}\ge\kappa\Big]\le\exp(-\lambda\kappa),\tag{3.110}$$

writing $\exp(X_n(t))$ explicitly, we have

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le t\le T}\Big\{&\int_0^t g(s)\mathbb{1}_{[0,\tau_n]}(s)dB_{E_s}-\frac{\lambda}{2}\int_0^t|g(s)|^2\mathbb{1}_{[0,\tau_n]}(s)dE_s\\&+\int_0^t\int_{|y|<c}h(s,y)\mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s,dy)\\&-\frac{1}{\lambda}\int_0^t\int_{|y|<c}\Big[\exp(\lambda h(s,y))-1-\lambda h(s,y)\Big]\mathbb{1}_{[0,\tau_n]}(s)\nu(dy)dE_s\Big\}\ge\kappa\Big]\le\exp(-\lambda\kappa)\end{aligned}\tag{3.111}$$

Define

$$\begin{aligned}A_n=\Big\{w\in\Omega:\sup_{0\le t\le T}\Big\{&\int_0^t g(s)\mathbb{1}_{[0,\tau_n]}(s)dB_{E_s}-\frac{\lambda}{2}\int_0^t|g(s)|^2\mathbb{1}_{[0,\tau_n]}(s)dE_s\\&+\int_0^t\int_{|y|<c}h(s,y)\mathbb{1}_{[0,\tau_n]}(s)\tilde{N}(dE_s,dy)\\&-\frac{1}{\lambda}\int_0^t\int_{|y|<c}\Big[\exp(\lambda h(s,y))-1-\lambda h(s,y)\Big]\mathbb{1}_{[0,\tau_n]}(s)\nu(dy)dE_s\Big\}\ge\kappa\Big\},\end{aligned}\tag{3.112}$$

then $\mathbb{P}(A_n)\le\exp(-\lambda\kappa)$.

Since

$$\mathbb{P}[\liminf_{n\to\infty}A_n]\le\liminf_{n\to\infty}\mathbb{P}(A_n)\le\limsup_{n\to\infty}\mathbb{P}(A_n)\le\mathbb{P}[\limsup_{n\to\infty}A_n]\tag{3.113}$$

and

$$\limsup_{n\to\infty}\mathbb{P}(A_n)\le\exp(-\lambda\kappa),\tag{3.114}$$

also

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A, \tag{3.115}$$

where

$$\begin{aligned} A = \Big\{ w \in \Omega : \sup_{0\le t\le T} \Big\{ \int_0^t g(s)dB_{E_s} - \frac{\lambda}{2}\int_0^t |g(s)|^2 dE_s + \int_0^t \int_{|y|<c} h(s,y)\tilde{N}(dE_s, dy) \\ - \frac{1}{\lambda}\int_0^t \int_{|y|<c} \Big[\exp(\lambda h(s,y)) - 1 - \lambda h(s,y) \Big] \nu(dy) dE_s \Big\} \ge \kappa \Big\}, \end{aligned} \tag{3.116}$$

thus

$$\mathbb{P}(A) = \mathbb{P}[\liminf_{n\to\infty} A_n] \le \limsup_{n\to\infty} P(A_n) \le \limsup_{n\to\infty} \exp(-\lambda\kappa) = \exp(-\lambda\kappa). \tag{3.117}$$

□ The next result can be considered as a strong law of large numbers for the inverse subordinator.

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Lemma 3.4.5 *Let* $\{E_t\}_{t\ge 0}$ *be the inverse of the mixed stable subordinator* $D(t)$ *with laplace exponent given in* (2.3) *as defined in* (2.2), *then*

$$\lim_{t\to\infty} \frac{E_t}{t} = 0, \ a.s. \tag{3.118}$$

Proof: Fix $\epsilon > 0$ and define

$$A_n = \Big\{ \sup_{2^n < t < 2^{n+1}} \Big| \frac{E_t}{t} \Big| > \epsilon \Big\}, \tag{3.119}$$

then, by Markov's inequality and equation (2.4), as $n \to \infty$, for some $\gamma > 0$,

$$\begin{aligned} \epsilon \mathbb{P}(A_n) \le \mathbb{E}\Big[\sup_{2^n < t < 2^{n+1}} \Big| \frac{E_t}{t} \Big| \Big] \le \mathbb{E}\Big[\Big| \frac{E_{2^{n+1}}}{2^n} \Big| \Big] \sim \frac{[\log(2^{n+1})]^\gamma L(\log(2^{n+1}))^{-1}}{2^n} \\ = \frac{(n+1)^\gamma (\log 2)^\gamma L(\log(2^{n+1}))^{-1}}{2^n} \sim \frac{C(n+1)^\gamma}{2^n}. \end{aligned} \tag{3.120}$$

By the ratio test, $\sum_{n=1}^\infty \mathbb{P}(A_n) < \infty$. Applying Borel-Cantelli lemma, we have

$$\lim_{t\to\infty} \frac{E_t}{t} = 0, \ a.s. \tag{3.121}$$

□

Remark 3.4.6 *Lemma 3.4.5 can also be proved for discrete case with the help of Laplace transform. Let* E_t *be an inverse of the subordinator with Laplace exponent* $\psi(s) = \sum_{i=1}^{k} c_i s^{\beta_i}$, *where* $\sum_{i=1}^{k} c_i = 1$ *and* $0 < \beta_1 < \beta_2 < ... < \beta_k < 1$. *Then the Laplace transform of the* nth *moment of* E_t *is* $\mathcal{L}(\mathbb{E}(E_t^n))(s) = \frac{n!}{s(\sum_{i=1}^{k} c_i s^{\beta_i})^n}$.

By a Karamata Tauberian theorem (see [7], Theorem 1 and Lemma on pp. 443-446), since $\mathcal{L}(\mathbb{E}(E_t))(s) \sim cs^{-(1+\beta_1)}$ *as* $s \to 0$ *then* $\mathbb{E}(E_t^n) \sim Ct^{\beta_1}$. *Utilizing this result,* $\epsilon\mathbb{P}(A_n) \leq \mathbb{E}\Big[\Big|\frac{E_{2^n+1}}{2^n}\Big|\Big] \sim \frac{(2^n+1)^{\beta_1}}{2^n} = 2^{\beta_1} 2^{-(1-\beta_1)n}$, *thus* $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. *Applying Borel-Cantelli lemma, we have* $\lim_{t\to\infty} \frac{E_t}{t} = 0,\ a.s.$

3.4.1 Path Stability with Small Jumps

In this subsection, we will analyze conditions for almost sure exponential path stability and almost sure path stability for the SDEs in equations (1.6) , followed by some examples.

1stthm **Theorem 3.4.7** *Suppose that Assumption 3.4.1 holds. Let* $V \in C^2(\mathbb{R}; \mathbb{R}^+)$ *and let* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \in \mathbb{R}, c_4 \geq 0, c_5 > 0$ *such that for all* $x_0 \neq 0$ *and* $t_1, t_2 \in \mathbb{R}^+$,

$$
\begin{aligned}
&(i)c_1|x|^p \leq V(x),\ \ (ii)L_1V(x) \leq c_2V(x),\ \ (iii)L_2V(x) \leq c_3V(x),\\
&(iv)|(\partial_x V(x))g(t_1,t_2,x)|^2 \geq c_4(V(x))^2,\\
&(v)\int_{|y|<c}\Big[\log\Big(\frac{V(x+h(t_1,t_2,x,y))}{V(x)}\Big) - \frac{V(x+h(t_1,t_2,x,y))-V(x)}{V(x)}\Big]\nu(dy) \leq -c_5.
\end{aligned}
\tag{3.122}
$$

Then when $f \neq 0$ *and* $\lim_{t\to\infty} \frac{E_t}{t} = 0$ *a.s.,*

$$
\limsup_{t\to\infty} \frac{1}{t}\log|X(t)| \leq \frac{c_2}{p} \quad a.s. \tag{3.123}
$$

and if $c_2 < 0$, *the trivial solution of* (1.6) *is almost surely exponentially path stable; when* $f = 0$ *(i.e. no time drift in the SDE),*

$$
\limsup_{t\to\infty} \frac{1}{E_t}\log|X(t))| \leq \frac{1}{2p}\Big(c_3 - \frac{1}{2}c_4 - c_5\Big) \quad a.s., \tag{3.124}
$$

and if $c_3 < \frac{1}{2}c_4 + c_5$*, the trivial solution of* (1.6) *is almost surely path stable.*

Proof: Define $Z(t) = \log|V(X(t))|$ and apply time-changed Itô formula (3.2) to it, then for all $t \geq t_0$,

$$
\begin{aligned}
&\log|V(X(t))| \\
=&\log|V(x_0)| + \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} f(s, E_s, X(s-))ds + \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} k(s, E_s, X(s-)) \\
&+ \frac{1}{2}\frac{\partial_x^2 V(X(s-))g^2(s, E_s, X(s-))}{V(X(s-))} - \frac{1}{2}\frac{(\partial_x V(X(s-))g(s, E_s, X(s-)))^2}{V(X(s-))^2} \\
&+ \int_{|y|<c} \Big[\log(V(X(s-) + h(s, E_s, X(s-), y))) - \log(V(X(s-)) \\
&\qquad\qquad - \frac{\partial_x V(X(s-))}{V(X(s-))} h(s, E_s, X(s-), y)\Big]\nu(dy)dE_s \\
&+ \int_{t_0}^{t}\int_{|y|<c} \Big[\log(V(X(s-) + h(s, E_s, X(s-), y))) - \log(V(X(s-))\Big]\tilde{N}(dE_s, dy) \\
&+ \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} g(s, E_s, X(s-))dB_{E_s}
\end{aligned}
\tag{3.125}
$$

$$
\begin{aligned}
=& \log|V(x_0)| + \int_{t_0}^{t} \frac{\partial_x V(X(s-))f(s,E_s,X(s-))}{V(X(s-))} ds \\
&+ \int_{t_0}^{t} \frac{\partial_x V(X(s-))k(s,E_s,X(s-))}{V(X(s-))} + \frac{\partial_x^2 V(X(s-)g^2(s,E_s,X(s-)))}{2V(X(s-))} \\
&+ \int_{|y|<c} \Big[\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} - 1 \\
&\qquad\qquad - \frac{\partial_x V(X(s-))}{V(X(s-))} h(s,E_s,X(s-),y)\Big]\nu(dy)dE_s \\
&+ \int_{t_0}^{t}\int_{|y|<c} \Big[\log(V(X(s-)+h(s,E_s,X(s-),y))) - \log(V(X(s-)) \\
&\qquad\qquad - \frac{\partial_x V(X(s-))}{V(X(s-))} h(s,E_s,X(s-),y)\Big]\nu(dy)dE_s \\
&- \int_{t_0}^{t}\int_{|y|<c} \Big[\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} - 1 \\
&\qquad\qquad - \frac{\partial_x V(X(s-))}{V(X(s-))} h(s,E_s,X(s-),y)\Big]\nu(dy)dE_s \\
&- \int_{t_0}^{t} \frac{1}{2}\frac{(\partial_x V(X(s-))g(s,E_s,X(s-)))^2}{V(X(s-))^2} dE_s \\
&+ \int_{t_0}^{t}\int_{|y|<c} \Big[\log(V(X(s-)+h(s,E_s,X(s-),y))) - \log(V(X(s-))\Big]\tilde{N}(dE_s,dy) \\
&+ \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} g(s,E_s,X(s-))dB_{E_s} \\
=& \log|V(x_0)| + \int_{t_0}^{t} \frac{L_1 V(X(s-))}{V(X(s-))} ds + \int_{t_0}^{t} \frac{L_2 V(X(s-))}{V(X(s-))} dE_s \\
&+ \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} g(s,E_s,X(s-))dB_{E_s} - \frac{1}{2}\int_{t_0}^{t} \frac{(\partial_x V(X(s-))g(s,E_s,X(s-)))^2}{V(X(s-))^2} dE_s \\
&+ \int_{t_0}^{t}\int_{|y|<c} \Big[\log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))}\Big)\Big]\tilde{N}(dE_s,dy) + I_2(t),
\end{aligned}
\tag{3.126}
$$

where

$$
\begin{aligned}
I_2(t) = \int_{t_0}^{t}\int_{|y|<c} \Big[&\log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))}\Big) \\
&- \frac{V(X(s-)+h(s,E_s,X(s-),y)) - V(X(s-))}{V(X(s-))}\Big]\nu(dy)dE_s.
\end{aligned}
\tag{3.127}
$$

Define

$$M(t) = \int_{t_0}^{t} \frac{\partial_x V(X(s-))}{V(X(s-))} g(s, E_s, X(s-)) dB_{E_s} + \int_{t_0}^{t} \int_{|y|<c} \Big[\log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big) \Big] \tilde{N}(dE_s, dy), \tag{3.128}$$

then, applying conditions (ii) and (iii),

$$\log |V(X(t))| \leq \log |V(x_0)| + c_2(t - t_0) + c_3(E_t - E_{t_0}) + M(t) + I_2(t) - \frac{1}{2} \int_{t_0}^{t} \frac{(\partial V(X(s-))g(s, E_s, X(s-)))^2}{V(X(s-))^2} dE_s. \tag{3.129}$$

By exponential martingale inequality (3.101), for $T = n, \lambda = \epsilon, \kappa = \epsilon n$ where $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$. Then for every integer $n \geq t_0$, we find that

$$\begin{aligned} P\Big[\sup_{t_0 \leq t \leq n} \Big\{ M(t) - \frac{\epsilon}{2} \int_{t_0}^{t} \frac{(\partial V(X(s-))g(s, E_s, X(s-)))^2}{V(X(s-))^2} dE_s \\ - \frac{1}{\epsilon} \int_{t_0}^{t} \int_{|y|<c} \Big[\exp \Big(\log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big)^{\epsilon} \Big) - 1 \\ - \epsilon \log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big) \Big] \nu(dy) dE_s \Big\} > \epsilon n \Big] \leq \exp(-\epsilon^2 n) \end{aligned} \tag{3.130}$$

Since $\sum_{n=1}^{\infty} \exp(-\epsilon^2 n) < \infty$, by Borel-Cantelli lemma , we have

$$\begin{aligned} P\Big[\limsup_{n \to \infty} \frac{1}{n} \Big[\sup_{t_0 \leq t \leq n} \Big\{ M(t) - \frac{\epsilon}{2} \int_{t_0}^{t} \frac{(\partial V(X(s-))g(s, E_s, X(s-)))^2}{V(X(s-))^2} dE_s \\ - \frac{1}{\epsilon} \int_{t_0}^{t} \int_{|y|<c} \Big[\exp \Big(\log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big)^{\epsilon} \Big) - 1 \\ - \epsilon \log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big) \Big] \nu(dy) dE_s \Big\} \Big] \leq \epsilon \Big] = 1 \end{aligned} \tag{3.131}$$

Hence for almost all $w \in \Omega$ there exists an integer N such that for all $n \geq N$, $t_0 \leq t \leq n$,

$$\begin{aligned} M(t) \leq & \frac{\epsilon}{2} \int_{t_0}^{t} \frac{(\partial V(X(s-))g(s, E_s, X(s-)))^2}{V(X(s-))^2} dE_s + \epsilon n \\ & + \frac{1}{\epsilon} \int_{t_0}^{t} \int_{|y|<c} \Big[\exp \Big(\log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big)^{\epsilon} \Big) - 1 \\ & + \epsilon \log \Big(\frac{V(X(s-) + h(s, E_s, X(s-), y))}{V(X(s-))} \Big) \Big] \nu(dy) dE_s \end{aligned} \tag{3.132}$$

Thus,

$$
\begin{aligned}
\log|V(X(t))| \leq& \log|V(x_0)| + c_2(t-t_0) + c_3(E_t - E_{t_0}) + I_2(t) \\
&- \frac{1}{2}\int_{t_0}^{t} \frac{(\partial V(X(s-))g(s,E_s,X(s-)))^2}{V(X(s-))^2} dE_s \\
&+ \frac{\epsilon}{2}\int_{t_0}^{t} \frac{(\partial V(X(s-))g(s,E_s,X(s-)))^2}{V(X(s-))^2} dE_s + \epsilon n \\
&+ \frac{1}{\epsilon}\int_{t_0}^{t}\int_{|y|<c} \Big[\exp\Big(\log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} \Big)^{\epsilon} \Big) - 1 \\
&+ \epsilon \log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} \Big) \Big] \nu(dy) dE_s \\
\leq& \log|V(x_0)| + c_2(t-t_0) + c_3(E_t - E_{t_0}) + I_2(t) - \frac{1-\epsilon}{2} c_4(E_t - E_{t_0}) + \epsilon n \\
&+ \frac{1}{\epsilon}\int_{t_0}^{t}\int_{|y|<c} \Big[\exp\Big(\log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} \Big)^{\epsilon} \Big) - 1 \\
&+ \epsilon \log\Big(\frac{V(X(s-)+h(s,E_s,X(s-),y))}{V(X(s-))} \Big) \Big] \nu(dy) dE_s
\end{aligned}
\tag{3.133}
$$

for $n \geq N,\ t_0 \leq t \leq n$.

Letting $\epsilon \to 0$, we have

$$
\log|V(X(t))| \leq \log|V(x_0)| + c_2(t-t_0) + c_3(E_t - E_{t_0}) - \frac{1}{2}c_4(E_t - E_{t_0}) + I_2(t) \tag{3.134}
$$

The details can be found in Theorem 3.4.8 in Siakalli's [30] with certain simple modifications. By condition (v), $I_2(t) \leq -c_5(E_t - E_{t_0})$, thus applying condition (i)

$$
\log|X(t)| \leq \frac{1}{p}\log|\frac{V(X(t))}{c_1}| \leq \frac{1}{p}\Big[\log|V(x_0)| - \log(c_1) + c_2(t-t_0) + (c_3 - \frac{1}{2}c_4 - c_5)(E_t - E_{t_0}) \Big]. \tag{3.135}
$$

When $f \neq 0$, then $c_2 \neq 0$, thus, for almost all $w \in \Omega$, $n-1 \leq t \leq n$, $n \geq N$,

$$
\frac{1}{t}\log|V(X(t))| \leq \frac{1}{p}\Big[\frac{\log|V(x_0)| - \log(c_1)}{t} + \frac{c_2(t-t_0)}{t} + \frac{(c_3 - \frac{1}{2}c_4 - c_5)(E_t - E_{t_0})}{t} \Big], \tag{3.136}
$$

then by Lemma 3.4.5

$$\limsup_{t\to\infty} \frac{1}{t} \log |V(X(t))| \leq \frac{c_2}{p} \quad a.s. \tag{3.137}$$

When $f = 0$, then $c_2 = 0$, thus

$$\log|X(t)| \leq \frac{1}{p}\log|\frac{V(X(t))}{c_1}| \leq \frac{1}{p}\Big[\log|V(x_0)| - \log(c_1) + (c_3 - \frac{1}{2}c_4 - c_5)(E_t - E_{t_0})\Big], \tag{3.138}$$

consequently,

$$\limsup_{t\to\infty} \frac{1}{E_t} \log|X(t)| \leq \frac{1}{2p}\Big(c_3 - \frac{1}{2}c_4 - c_5\Big) \quad a.s.. \tag{3.139}$$

□

Remark 3.4.8 *From the proof of the previous theorem, when $f = 0$, we can deduce the following. When $\lim_{t\to\infty} \frac{E_t}{t} = 0$ a.s., the following estimation is also true.*

$$\limsup_{t\to\infty} \frac{1}{t} \log|X(t)| \leq 0 \quad a.s.. \tag{3.140}$$

exmp0 **Example 3.4.9** *Consider the following stochastic differential equation*

$$dX(t) = -X(t-)^{\frac{3}{2}} dE_t + X(t-) dB_{E_t} + \int_{|y|\leq 1} X(t-) y^2 \tilde{N}(dE_t, dy), \tag{3.141}$$ example0

with $X(0) = 1$, ν is uniform distribution $[0, 1]$.

Choose the Lyapunov function as $V(x) = x^{\frac{3}{2}}$ which satisfies the conditions (i) and (ii) in Theorem 3.4.7. Furthermore,

$$\begin{aligned} L_2V(x) &= -\frac{3}{2}x^2 + \frac{3}{8}x^{\frac{3}{2}} + \Big[\int_{|y|\leq 1} [(1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2]\nu(dy)\Big]x^{\frac{3}{2}} \\ &= x^{\frac{3}{2}}\Big[-\frac{3}{2}x^{\frac{1}{2}} + \frac{3}{8} + \int_{|y|\leq 1} [(1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2]\nu(dy)\Big] \\ &\leq x^{\frac{3}{2}}\Big[\frac{3}{8} + \int_{|y|\leq 1} [(1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2]\nu(dy)\Big] \\ &\leq V(x). \end{aligned} \tag{3.142}$$

The last inequality is derived by the following argument, Let $f(y) = (1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2$, *then* $f'(y) > 0$ *for* $0 \le y \le 1$ *and* $f'(y) < 0$ *for* $-1 \le y \le 0$. *Thus* $f(y) \le f(1) = f(-1) = .33$, *for* $-1 \le y \le 1$. *Since* ν *is assumed to be the standard normal distribution,* $\int_{|y|\le 1}[(1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2]\nu(dy) = \int_{|y|\le 1} f(y)\nu(dy) \le .33\int_{|y|\le 1}\nu(dy) < .33$. *Thus,* $x^{\frac{3}{2}}\Big[\frac{3}{8} + \int_{|y|\le 1}[(1+y^2)^{\frac{3}{2}} - 1 - \frac{3}{2}y^2]\nu(dy)\Big] \le x^{\frac{3}{2}}[\frac{3}{8} + .33] \le x^{\frac{3}{2}} = V(x)$.

In addition, $|V_x(x)g(x)^2| = |\frac{3}{2}x^{\frac{1}{2}}x|^2 = \frac{9}{4}V(x)^2$ *and*

$$\begin{aligned}&\int_{|y|\le 1}\Big[\log\Big(\frac{(x+xy^2)}{x}\Big)^{\frac{3}{2}} - \frac{(x+xy^2)^{\frac{3}{2}} - x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\Big]\nu(dy)\\ =&\int_{|y|\le 1}\Big[\frac{3}{2}\log(1+y^2) - (1+y^2)^{\frac{3}{2}} + 1\Big]\nu(dy) < -.018.\end{aligned} \tag{3.143}$$

Similar as above, the last inequality can be proved as following. Let $f(y) = \frac{3}{2}\log(1+y^2) - (1+y^2)^{\frac{3}{2}} + 1$, *then* $f'(y) < 0$ *for* $0 \le y \le 1$ *and* $f'(y) > 0$ *for* $-1 \le y \le 0$. *Thus*

$$\begin{aligned}&\int_{|y|\le 1}\Big[\frac{3}{2}\log(1+y^2) - (1+y^2)^{\frac{3}{2}} + 1\Big]\nu(dy) = \int_{|y|\le 1} f(y)\nu(dy)\\ \le&\int_{.5\le|y|\le 1} f(y)\nu(dy) = 2\int_{.5\le y\le 1} f(y)\nu(dy) \le 2\int_{.5\le y\le 1} f(.5)\nu(dy)\\ <&2\int_{.5\le y\le 1} -.062\nu(dy) = -.124\int_{.5\le y\le 1}\nu(dy) = -.124[\Phi(1) - \Phi(.5)]\\ =&-.124(.8413 - .6915) < -.018\end{aligned} \tag{3.144}$$

The constants of Theorem 3.4.7 are $c_3 = 1,\ c_4 = 2.25,\ c_5 = .018$, *then* $\frac{1}{2\times\frac{3}{2}}\Big(c_3 - \frac{1}{2}c_4 - c_5\Big) = -.0477 < 0$, *thus the trivial solution of stochastic differential equation* (3.141) *is almost surely path stable. A simulation of a path of SDE in equation* (3.141) *is given in* **Figure 3.1**, *it can be observed that* $\frac{\log(X(t))}{E_t}$ *is strictly below* 0 *when* t *is large, which illustrates our analysis above.*

Remark 3.4.10 *Note that* $f(x) = x^{\frac{3}{2}}$ *fails to be a Lipschitz function and does not have linear growth condition. However, existence of unique solution to* (3.141) *is guaranteed by Theorem 3.5 on page 58 of Mao [21].*

Figure 3.1: $\log(X(t))/E_t$ of SDE (3.141)

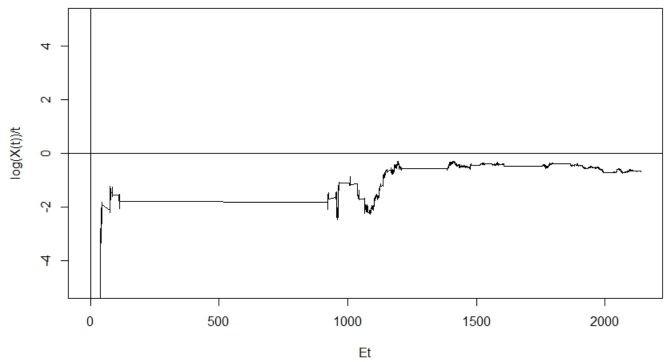


fig:figur

Remark 3.4.11 *In the figures of all examples, we assume that $E(t)$ is the inverse of stable subordinator with parameter $\alpha = .8$.*

3.4.2 Path Stability with Large Jumps

In this section, we will analyze conditions for almost sure exponential path stability and almost sure path stability for the SDEs in equations (1.7), followed by some examples. First, let us discuss exponential stability of the following time-changed SDE with noise that has only small linear jump

$$\begin{aligned} dX(t) &= f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t} \\ &\quad + \int_{|y|<c} h(y)X(t-)\tilde{N}(dE_t, dy), \end{aligned} \tag{3.145}$$

aim11

with $X(t_0) = x_0$, which is a special case of (1.6) when $h(t_1, t_2, x, y) = h(x)y$. Then we extend (3.145) to (1.7) by adding large jumps $\int_{|y|\geq c} H(y)X(t-)N(dE_t, dy)$.

sumption3 **Assumption 3.4.12**

$$Z_c = \int_{|y|<c} (|h(y)| \bigvee |h(y)|^2)\nu(dy) < \infty, \tag{3.146}$$

for all $t_1, t_2 \in \mathbb{R}^+$.

2ndthm **Theorem 3.4.13** *Given Assumptions 3.4.1 and 3.4.12, suppose that there exist* $\xi > 0, \gamma \geq 0, \delta \geq 0, K_1, K_2 \in \mathbb{R}$ *such that the following conditions*

$$\begin{aligned} &(1)\gamma|x|^2 \leq |g(t_1,t_2,x)|^2 \leq \xi|x|^2,\ (2)\int_{|y|<c} h(y)\nu(dy) \geq \delta \\ &(3)f(t_1,t_2,x)x \leq K_1|x|^2,\ (4)k(t_1,t_2,x)x \leq K_2|x|^2 \end{aligned} \tag{3.147}$$

are satisfied for all $x \in \mathbb{R}$ *and* $t_1, t_2 \in \mathbb{R}^+$. *Then when* $f \neq 0$ *and* $\lim_{t\to\infty} \frac{E_t}{t} = 0$ *a.s., we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log|X(t)| \leq K_1 \ a.s. \tag{3.148}$$

for any $x_0 \neq 0$*, the trivial solution of* (3.145) *is almost surely exponential path stable if* $K_1 < 0$*; when* $f = 0$*, we have*

$$\limsup_{t\to\infty} \frac{1}{E_t} \log|X(t)| \leq -\Big(\gamma - K_2 - \frac{\xi}{2} - \int_{|y|<c} \log(1+|h(y)|)\nu(dy) + \delta\Big) \ a.s. \tag{3.149}$$

for any $x_0 \neq 0$*, the trivial solution of* (3.145) *is almost surely path stable if* $\gamma > K_2 + \frac{\xi}{2} + \int_{|y|<c} \log(1+|h(y)|)\nu(dy) - \delta$.

Proof: [Proof of Theorem 3.4.13] Fix $x_0 \neq 0$, then by Itô formula for time-changed SDE, see Lemma 3.1 in [25], we have

$$\begin{aligned} \log(|X(t)|^2) =& \log(|x_0|^2) + \int_{t_0}^t L_1 \log(|X(s-)|^2)ds + \int_{t_0}^t L_2 \log(|X(s-)|^2)dE_s \\ &+ \int_{t_0}^t \int_{|y|<c} \Big[\log(|X(s-) + X(s-)h(s,E_s,y)|^2) - \log(|X(s-)|^2)\Big]\tilde{N}(dE_s,dy) \\ &+ \int_{t_0}^t \int_{|y|<c} \frac{d}{dx}\log(|X(s-)|^2)g(s,E_s,X(s-))dB_{E_s}, \end{aligned} \tag{3.150}$$

where

$$L_1 \log(|X(s-)|^2) = \frac{2X(s-)}{|X(s-)|^2} f(s,E_s,X(s-)) \leq 2K_1, \tag{3.151}$$

$$L_2\log(|X(s-)|^2)dE_s = \frac{2X(s-)}{|X(s-)|^2}k(s,E_s,X(s-)) - \frac{|g(s,E_s,X(s-))|^2}{|X(s-)|^2} + \int_{|y|<c}\Big[\log(|X(s-)+h(y)X(s-)|^2) - \log(|X(s-)|^2) - 2h(y)\Big]\nu(dy). \tag{3.152}$$ L2

Applying condition (2) and Assumption 3.4.12 to (3.152),

$$\begin{aligned}\int_{t_0}^t L_2\log(|X(s-)|^2)dE_s = &\int_{t_0}^t\Big[\frac{2X(s-)}{|X(s-)|^2}k(s,E_s,X(s-)) - \frac{|g(s,E_s,X(s-))|^2}{|X(s-)|^2}\Big]dE_s\\ &+\int_{t_0}^t\Big[\int_{|y|<c}\big[\log(|X(s-)+h(y)X(s-)|^2)-\log(|X(s-)|^2)-2h(y)\big]\nu(dy)\Big]dE_s\\ \le&\int_{t_0}^t\Big[\frac{2K_2|X(s-)|^2}{|X(s-)|^2}+(\xi-2\gamma)\Big]dE_s + \int_{t_0}^t\Big[\int_{|y|<c}\big[\log((1+|h(y)|)^2)\big]\nu(dy)-2\delta\Big]dE_s\\ \le&\int_{t_0}^t 2K_2 dE_s + 2(E_t-E_{t_0})\int_{|y|<c}\Big[\log((1+|h(y)|))\Big]\nu(dy)\\ &-(2\gamma+2\delta-\xi)(E_t-E_{t_0})\\ \le&(E_t-E_{t_0})\Big[2\int_{|y|<c}\log(1+|h(y)|)\nu(dy)+2K_2+\xi-2\gamma-2\delta\Big]\end{aligned} \tag{3.153}$$

Note that both

$$M_1(t) = \int_{t_0}^t\int_{|y|<c}\frac{d}{dx}\log(|X(s-)|^2)g(s,E_s,X(s-))dB_{E_s} \tag{3.154}$$

and

$$M_2(t) = \int_{t_0}^t\int_{|y|<c}\Big[\log(|X(s-)+X(s-)h(y)|^2)-\log(|X(s-)|^2)\Big]\tilde{N}(dE_s,dy) \tag{3.155}$$

are martingales.

Now,

$$\begin{aligned}\log(|X(t)|^2) \le&\log(|x_0|^2)+2K_1(t-t_0)+M_1(t)+M_2(t)\\ &+(E_t-E_{t_0})\Big(2\int_{|y|<c}\log(1+|h(y)|)\nu(dy)+2K_2+\xi-2\gamma-2\delta\Big).\end{aligned} \tag{3.156}$$

Define corresponding non-time-changed stochastic process $\{z_t\}_{t\geq 0}$ by

$$z(t) = z(t_0) + \int_{t_0}^{t} f(s, z(s-))dt + \int_{t_0}^{t} g(s, z(s-))dB(t) + \int_{t_0}^{t}\int_{|y|<c} h(y)z(s-)\tilde{N}(ds, dy), \tag{3.157}$$

with $z(t_0) = x_0$. By the duality theorem 4.2 in [15], $X(t) = z(E_t)$ for $t \geq t_0$.

By the result on page 282 in Mao [21],

$$\begin{aligned}\langle M_1\rangle(t) =& \langle 2\int_{E_{t_0}}^{E_t} \frac{z(s-)g(s, z(s-))}{|z(s-)|^2} dB_k(s)\rangle \\ =& 4\int_{E_{t_0}}^{E_t} \frac{|z(s-)g(s, z(s-))|^2}{|z(s-)|^4} ds \\ \leq& 4\xi(E_t - E_{t_0}).\end{aligned} \tag{3.158}$$

Define $\rho_{M_1}(t) = \int_{t_0}^{t} \frac{d\langle M_1\rangle(s)}{(1+E_s)^2}$, then

$$\rho_{M_1}(t) \leq 4\xi\int_{t_0}^{t} \frac{dE_s}{(1+E_s)^2} = 4\xi\int_{E_{t_0}}^{E_t} \frac{ds}{(1+s)^2} = \frac{-4\xi}{1+s}\Big|_{E_{t_0}}^{E_t} = 4\xi\big[\frac{1}{1+E_{t_0}} - \frac{1}{1+E_t}\big], \tag{3.159}$$

then

$$\lim_{t\to\infty} \rho_{M_1}(t) \leq \lim_{t\to\infty} 4\xi\big[\frac{1}{1+E_{t_0}} - \frac{1}{1+E_t}\big]) \leq 4\xi < \infty. \tag{3.160}$$

By Theorem 10 of Chapter 2 in [**?**],

$$\lim_{t\to\infty} \frac{M_1(t)}{E_t} = 0,\ a.s.. \tag{3.161}$$

Similarly,

$$\begin{aligned}\langle M_2\rangle(t) =& \int_{E_{t_0}}^{E_t}\int_{|y|<c} [\log(\frac{z(s-)+z(s-)h(y)}{|z(s-)|^2})]^2\nu(dy)ds \\ \leq& \int_{E_{t_0}}^{E_t}\int_{|y|<c} [\log((1+|h(y)|)^2)]^2\nu(dy)ds \\ \leq& 4\int_{E_{t_0}}^{E_t}\int_{|y|<c} |h(y)|^2\nu(dy)ds \\ \leq& 4Z_c(E_t - E_{t_0}),\end{aligned} \tag{3.162}$$

so

$$\lim_{t\to\infty}\rho_{M_2}(t) \le \lim_{t\to\infty} 4Z_c \int_{t_0}^{t} \frac{dE_s}{(1+E_s)^2} < \infty \; a.s.. \tag{3.163}$$

As a result,

$$\lim_{t\to\infty}\frac{M_2(t)}{E_t} = 0, \; a.s.. \tag{3.164}$$

In the end, since

$$\lim_{t\to\infty}\frac{E_t}{t} = 0, \; a.s., \tag{3.165}$$

and

$$\begin{aligned}\frac{\log|X(t)|}{t} \le & \frac{(E_t - E_{t_0})(\int_{|y|<c}\log(1+|h(y)|)\nu(dy) + K_2 + \frac{\xi}{2} - \gamma - \delta)}{t} \\ & + \frac{\log|x_0|}{t} + \frac{2K_1(t-t_0)}{t} + \frac{M_1(t)}{2E_t}\frac{E_t}{t} + \frac{M_2(t)}{2E_t}\frac{E_t}{t}\end{aligned} \tag{3.166}$$

thus,

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{t} \le K_1 \; a.s.. \tag{3.167}$$

When $f = 0$,

$$\begin{aligned}\frac{\log|X(t)|}{E_t} \le & \frac{(E_t - E_{t_0})(\int_{|y|<c}\log(1+|h(y)|)\nu(dy) + K_2 + \frac{\xi}{2} - \gamma - \delta)}{E_t} \\ & + \frac{\log|x_0|}{E_t} + \frac{M_1(t)}{2E_t} + \frac{M_2(t)}{2E_t}\end{aligned} \tag{3.168}$$

thus,

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{E_t} \le \int_{|y|<c}\log(1+|h(y)|)\nu(dy) + K_2 + \frac{\xi}{2} - \gamma - \delta \; a.s.. \tag{3.169}$$

□

Other than the direct proof above, the following is an alternative proof utilizing Theorem 3.4.7.

Proof: [Alternate Proof of Theorem 3.4.13]

Let $V(x) = |x|^2$, then $V \in C^2(\mathbb{R}, \mathbb{R}^+)$ and condition (i) in Theorem 3.4.7 is satisfied.

Next, by applying the time-changed Itô formula to $V(X(t))$, $L_1V(x) = f(t_1,t_2,x)2x \leq 2K_1V(x)$, thus condition (ii) in Theorem 3.4.7 is satisfied;

$$
\begin{aligned}
L_2V(x) &= k(t_1,t_2,x)2x + |g(t_1,t_2,x)|^2 + \int_{|y|<c}\Big[|x+h(y)x|^2 - |x|^2 - h(y)x2x\Big]\nu(dy)\\
&\leq 2K_2|x|^2 + |g(t_1,t_2,x)|^2 + \int_{|y|<c}|x|^2\Big[(1+h(y))^2 - 1 - 2h(y)\Big]\nu(dy)\\
&\leq \Big[2K_2 + \xi + \int_{|y|<c}|h(y)|^2\nu(dy)\Big]|x|^2 < \infty,
\end{aligned} \tag{3.170}
$$

thus, condition (iii) in Theorem 3.4.7 is satisfied by Assumption 3.4.12 and setting $c_3 = 2K_2 + \xi + \int_{|y|<c}|h(y)|^2\nu(dy)$.

Condition (iv) is satisfied since

$$
|(\partial_x V(x))g(t_1,t_2,x)|^2 = |2xg(t_1,t_2,x)|^2 \geq 4\gamma|x|^4. \tag{3.171}
$$

For the last condition (v), by denoting $c_5 = -\int_{|y|<c}\Big[\log(1+|h(y)|) - |h(y)|^2\Big]\nu(dy) - 2\delta$ we have

$$
\begin{aligned}
&\int_{|y|<c}\Big[\log\Big(\frac{V(x+h(y)x)}{V(x)}\Big) - \frac{V(x+h(y)x)-V(x)}{V(x)}\Big]\nu(dy)\\
&= \int_{|y|<c}\Big[\log\Big(\frac{|x+h(y)x|^2}{|x|^2}\Big) - \frac{|x+h(y)x|^2 - |x|^2}{|x|^2}\Big]\nu(dy)\\
&\leq \int_{|y|<c}\Big[\log(1+|h(y)|) - \frac{2xh(y)x + |h(y)x|^2}{|x|^2}\Big]\nu(dy)\\
&\leq \int_{|y|<c}\Big[\log(1+|h(y)|) - |h(y)|^2\Big]\nu(dy) - 2\delta < 0.
\end{aligned} \tag{3.172}
$$

Since all five conditions in Theorem 3.4.7 are satisfied, we have that when $f \neq 0$,

$$
\limsup_{t\to\infty}\frac{1}{t}\log|X(t)| \leq K_1 \;\; a.s.; \tag{3.173}
$$

and that when $f = 0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{E_t} \log |X(t)| \\
&\leq \frac{1}{2}\Big(2K_2 + \xi - \int_{|y|<c} |h(y)|^2 \nu(dy) - \frac{4\gamma}{2} - \int_{|y|<c} \Big[\log(1+|h(y)|) - |h(y)|^2\Big]\nu(dy) - 2\delta\Big) \\
&= -\Big(- K_2 - \frac{\xi}{2} + \gamma - \int_{|y|<c} \Big[\log(1+|h(y)|_1)\Big]\nu(dy)) + \delta\Big) \;\; a.s.
\end{aligned} \tag{3.174}$$

as desired.

□

exmp00 **Example 3.4.14** *Consider the following stochastic differential equation*

$$dX(t) = -sin(X(t-))X(t-)dE_t + \frac{X(t-)}{E_t+1} dB_{E_t} + \int_{|y|\leq 1} 16X(t-)y^2 \tilde{N}(dE_t, dy), \tag{3.175}$$ example00

with $X(0) = 1$, ν *is uniform distribution* $[0, 1]$.

Applying Theorem 3.4.13, $0 \leq |g(x, t_1, t_2)^2| \leq |x|^2$, $\int_{|y|\leq 1} h(y)\nu(dy) \geq \frac{16}{3}$ *and* $k(t_1, t_2, x)x \leq |x|^2$, *thus* $\gamma = 0,\ \xi = 1,\ \delta = \frac{16}{3},\ K_2 = 1$.

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{E_t} \log |X(t)| &\leq -\Big(\gamma - K_2 - \frac{\xi}{2} - \int_{|y|<c} \log(1+|h(y)|)\nu(dy) + \delta\Big) \\
&= -(0 - 1 - \frac{1}{2} - \log(17) + \frac{16}{3}) < 0 \; a.s..
\end{aligned} \tag{3.176}$$

Hence, stochastic differential equation (3.175) *is almost surely path stable. The simulated path of SDE* (3.175) *is given in* **Figure 3.2.** *The ratio of* $\frac{\log|X(t)|}{E_t}$ *is strictly below* 0 *for large time t, this is consistent with above analysis.*

Next, we analyze the following time-changed stochastic differential equation involving large jumps,

$$dX(t) = \int_{|y|\geq c} H(y)X(t-)N(dE_t, dy), \tag{3.177}$$ sdelarge

with $X(t_0) = x_0 \in \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ is a measurable function.

Before stating the next theorem, we need another assumption, see [30].

Figure 3.2: $\log(X(t))/E_t$ of SDE (3.175)

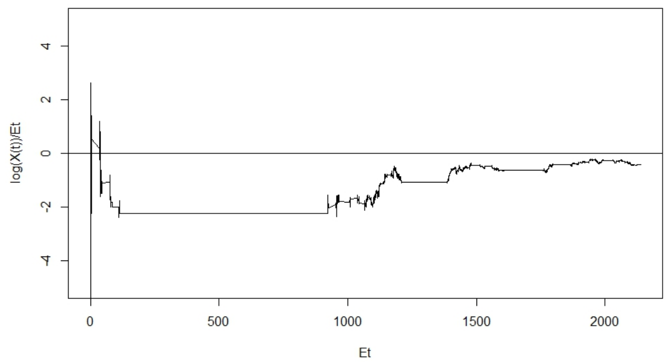


figure00

sumptionE **Assumption 3.4.15** *Assume that*

$$\int_{|y|\geq c} ||H(y)||_1^2 \nu(dy) < \infty \tag{3.178}$$

and that $H(y) \neq -1$ *for* $|y| \geq c$.

By above assumption, the function $H(y)x$ satisfies Lipschitz and growth conditions, assuring the existence and uniqueness of solution to equation (3.177). In addition, $H(y) \neq -1$ implies that $P(X(t) \neq 0 \; for \; all \; t \geq t_0) = 1$, this is an application of interlacing technique in [1], details can be found in Lemma 4.3.2 in [30] with simple modification.

thm2 **Theorem 3.4.16** *If*

$$\sup_{x \in \mathbb{R}-0} \int_{|y|\geq c} \Big[\log(|x + H(y)x|) - \log(|x|) \Big] \nu(dy) < -K, \tag{3.179}$$

for some $K > 0$*, then the sample Lyapunov exponent of solution of* (3.177) *exists and satisfies*

$$\limsup_{t\to\infty} \frac{1}{E_t} \log |X(t)| \leq -2K \;\; a.s., \tag{3.180}$$

for any $x_0 \neq 0$*, that is, the trivial solution of* (3.177) *is almost surely path stable.*

Proof: Fix $x_0 \neq 0$, apply Itô formula (3.2) to $\log(|X(t)|^2)$, then for any $t \geq 0$,

$$
\begin{aligned}
\log(|X(t)|^2) =& \log(x_0^2) + \int_{t_0}^t \int_{|y|\geq c} \Big[\log(|X(s)+H(y)X(s)|^2) - \log(|X(s)|^2)\Big] N(dE_s, dy) \\
=& \log(x_0^2) + \int_{t_0}^t \int_{|y|\geq c} \Big[\log(|X(s)+H(y)X(s)|^2) - \log(|X(s)|^2)\Big] \tilde{N}(dE_s, dy) \\
&+ \int_{t_0}^t \int_{|y|\geq c} \Big[\log(|X(s)+H(y)X(s)|^2) - \log(|X(s)|^2)\Big] \nu(dy) dE_s.
\end{aligned}
\tag{3.181}
$$

Let $M_3(t) = \int_{t_0}^t \int_{|y|\geq c} \Big[\log(|X(s)+H(y)X(s)|^2) - \log(|X(s)|^2)\Big] \tilde{N}(dE_s, dy)$, similar ideas as in the proof of the corresponding inequality for $M_2(t)$ in the proof of Theorem (3.4.13), we have

$$
\lim_{t\to\infty} \frac{M_3(t)}{E_t} = 0, \ \ a.s., \tag{3.182}
$$

thus

$$
\begin{aligned}
\frac{\log(|X(t)|^2)}{E_t} \leq& \frac{\log(x_0^2)}{E_t} + \frac{(E_t - E_{t_0}) \sup_{0\leq s\leq t} \int_{|y|\geq c} \Big[\log(\frac{|X(s)+H(y)X(s)|^2)}{\log(|X(s)|^2}\Big] \nu(dy)}{E_t} \\
&\to \sup_{0\leq s\leq t} \int_{|y|\geq c} \Big[\log(|X(s)+H(y)X(s)|^2) - \log(|X(s)|^2)\Big] \nu(dy) \leq -2K, \\
&as\ t \to \infty.
\end{aligned}
\tag{3.183}
$$

□

Next, by similar ideas as the proof of Theorem 4.6.1 in [30], it is not difficult to derive the following theorem for the following time-changed SDE

$$
\begin{aligned}
dX(t) =& f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t} \\
&+ \int_{|y|<c} h(y)X(t-)\tilde{N}(dE_t, dy) + \int_{|y|\geq c} H(y)X(t-)N(dE_t, dy).
\end{aligned}
\tag{3.184}
$$

with $X(t_0) = x_0$.

finalthm

Theorem 3.4.17 *Given assumptions 3.4.1, 3.4.12 and 3.4.15, suppose that there exist* $\xi > 0, \gamma \geq 0, \delta \geq 0, K_1, K_2 \in \mathbb{R}$ *such that the following conditions*

$$
\begin{aligned}
&(1)\gamma|x|^2 \le |g(t_1,t_2,x)|^2 \le \xi|x|^2,\ (2)\int_{|y|<c} h(y)\nu(dy) \ge \delta \\
&(3)f(t_1,t_2,x)x \le K_1|x|^2,\ (4)k(t_1,t_2,x)x \le K_2|x|^2
\end{aligned}
\tag{3.185}
$$

are satisfied for all $x \in \mathbb{R}$ and $t_1, t_2 \in \mathbb{R}^+$. Then when $f \ne 0$ and $\lim_{t\to\infty} \frac{E_t}{t} = 0$ a.s., we have

$$
\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le K_1\ a.s., \tag{3.186}
$$

for any $x_0 \ne 0$, the trivial solution of (1.7) *is almost surely exponentially path stable if $K_1 < 0$; when $f = 0$, we have*

$$
\limsup_{t\to\infty} \frac{1}{E_t} \log |x(t)| \le -\Big(\gamma - K_2 - \frac{\xi}{2} - \int_{|y|<c} \log(1+|h(y)|)\nu(dy) + \delta - M(c)\Big)\ a.s., \tag{3.187}
$$

where $M(c) = \sup_{x\in\mathbb{R}-\{0\}} \int_{|y|\ge c} \Big[\log(|x + H(y)x|) - \log(|x|)\Big]\nu(dy) < \infty$, for any $x_0 \ne 0$, and the trivial solution of (1.7) *is almost surely path stable if $\gamma > K_2 + \frac{\xi}{2} + \int_{|y|<c} \log(1 + |h(y)|)\nu(dy) - \delta + M(c)$.*

Proof: Application of Theorem 3.4.7 and Theorem 3.4.16. □

Remark 3.4.18 *The Theorems 3.4.7 and 3.4.17 show that the coefficient of "dt" (i.e. the drift term) plays the dominating role in determining the almost sure exponential path stabilities. In absence the of "dt" part, almost sure path stability is the result of the coefficients of the other components.*

Next, we list some examples to illustrate the results of above theorems.

Example 3.4.19 *Consider the following two stochastic differential equations*

$$
\begin{aligned}
dX(t) = X(t-)dt + X(t-)dB_{E_t} + \int_0^t \int_{|y|\le 1} X(t-)y^2 \tilde{N}(dE_t, dy) \\
+ \int_0^t \int_{|y|>1} X(t-)y^2 N(dE_t, dy)
\end{aligned}
\tag{3.188}
$$

example1

with $X(0) = .1$ and ν is standard normal distribution,

Figure 3.3: $\log(X(t))/t$ of SDE (3.188)

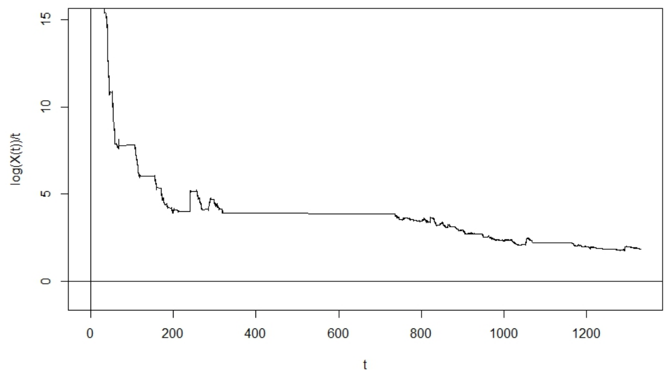


figure1

Figure 3.4: $\log(X(t))/t$ of SDE (4.2.3)

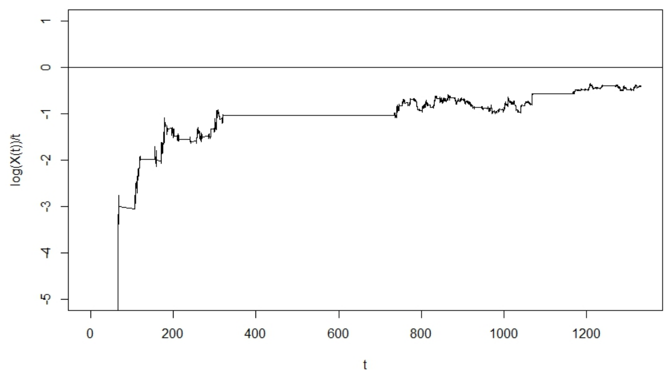


figure2

and

$$dX(t) = -X(t-)dt + X(t-)dB_{E_t} + 2\int_0^t \int_{|y|\leq 1} X(t-)y^2 \tilde{N}(dE_t, dy) + 2\int_0^t \int_{|y|>1} X(t-)y^2 N(dE_t, dy) \tag{3.189}$$

example2

with $X(0) = .1$ *and* ν *is standard normal distribution.*

Figure 3.3 *illustrates that stochastic differential equation* (3.188) *is not almost surely exponentially path stable, this is because "dt" component exists in the linear stochastic system,* *such component plays dominant role in determining almost sure exponential path stability and has positive scalar 1, thus* $\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq 1$, *this is not enough for almost sure exponential path stability.*

In contrast, as illustrated in the **Figure 3.4**, *(also verified by Theorem 3.4.17) stochastic differential equation* (4.2.3) *is almost surely exponentially stable. This is because that coefficient for* dt *in* (4.2.3) *is -1, thus* $\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq -1$, *this implies almost sure exponential path stability.*

Example 3.4.20 *Consider following two stochastic differential equations*

$$dX(t) = -X(t-)dE_t + X(t-)dB_{E_t} + \int_0^t\int_{|y|\leq 1} X(t-)y^2\tilde{N}(dE_t, dy) + \int_0^t\int_{|y|>1} X(t-)y^2 N(dE_t, dy) \tag{3.190}$$

example3

with $X(0) = -3$, *and*

$$dX(t) = -X(t-)dE_t + 2X(t-)dB_{E_t} + \int_0^t\int_{|y|\leq 1} X(t-)y^2\tilde{N}(dE_t, dy) + \int_0^t\int_{|y|>1} X(t-)y^2 N(dE_t, dy) \tag{3.191}$$

example4

with $X(0) = -3$.

figure3

Figure 3.5: $\log(X(t))/E_t$ of SDE (3.190)

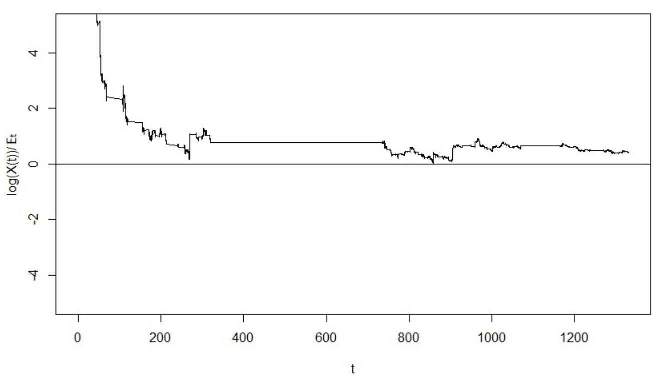
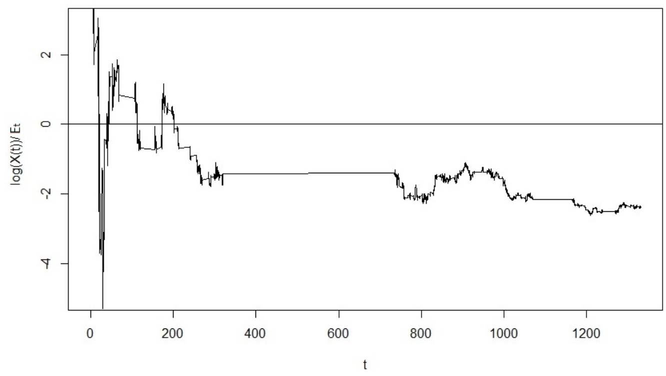


figure4

Figure 3.6: $\log(X(t))/E_t$ of SDE (3.191)



In both of the equations (3.190) *and* (3.191)*, "dt" component is missing, thus almost sure exponential path stability is no longer possible. However, almost sure path stability is possible, depending on the scalars of time-changed drift, Brownian motion, and Possion jump.*

In stochastic differential equations (3.190)*, the corresponding parameters are* $K_2 = \xi = \gamma = 1$, $\delta = .2$, $h(y) = H(y) = y^2$ *and* $0 \le \delta \le \int_{|y|<1} y^2 \nu(dy) < 1$, *by Theorem 3.4.17*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{E_t} \log |X(t)| \\
&\quad < -\Big(1 - 1 - \frac{1}{2} - \int_{|y|<1} \log(1+y^2)\nu(dy) + .2 - \sup_{x\in\mathbb{R}^d - 0} \int_{|y|<1} \log(1+y^2)\nu(dy)\Big) \\
&\quad \le \int_{|y|<1} \log(1+y^2)\nu(dy) + .3 \quad a.s.,
\end{aligned} \tag{3.192}$$

which is not enough to conclude the almost sure path stability of stochastic differential equations (3.190).

However, in stochastic differential equations (3.191) *corresponding parameters are* $K_2 = 1,\ \delta = .2,\ \gamma = \xi = 4$, $h(y) = H(y) = y^2$ *and* $0 \leq \delta \leq \int_{|y|<1} y^2 \nu(dy) < 1$, *by Theorem 3.4.17*

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{E_t} \log |X(t)| \\ &< -\Big(4 - 1 - 2 - \int_{|y|<1} \log(1+y^2)\nu(dy) + .2 - \sup_{x\in\mathbb{R}^d - 0} \int_{|y|<1} \log(1+y^2)\nu(dy)\Big) \\ &\leq -.8 + 2\int_{|y|<1} \log(1+y^2)\nu(dy) \leq -.8 + 2\int_{|y|<1} y^2\nu(dy) \leq 0 \quad a.s.,\end{aligned} \tag{3.193}$$

thus the solution of stochastic differential equation (3.191) *is almost surely path stable.*

Chapter 4

Time-Changed Stochastic Control Problem

In this paper, we investigate the time-changed stochastic control problem using the maximum principle method. Specifically, we consider the following time-changed stochastic process, see [14, 25]:

$$\begin{aligned} dX(t) &= b(t, E_t, X(t-), u(t))dE_t + \sigma(t, E_t, X(t-), u(t))dB_{E_t} \\ &\quad + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\tilde{N}(dE_t, dy), \end{aligned} \tag{4.1}$$

simSDE

with $X(0) = x_0 \neq 0$ and the corresponding performance function

$$J(u) = \mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big],\ u \in \mathcal{A}, \tag{4.2}$$

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where $u(t) = u(t, w) \in U \subset \mathbb{R}$ is the control and $\mathcal{A}$ denotes the set of $admissible$ controls. We establish a maximum principle theory for the stochastic control problem to find $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{4.3}$$

13

The process $u(t) = u(t, w) \in U \subset \mathbb{R}$ is the control. Assume that u is adapted and RCLL, and that the corresponding equation (4.1) has a unique strong solution $X^{(u)}(t), t \in [0, T]$. Such controls are called $admissible$. The set of admissible controls is denoted by $\mathcal{A}$.

Then we extend such result to a more general time-changed stochastic process involving time drift term dt:

$$dX(t) = \mu(t, E_t, X(t-), u(t))dt + b(t, E_t, X(t-), u(t))dE_t + \sigma(t, E_t, X(t-), u(t))dB_{E_t} + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\tilde{N}(dE_t, dy), \tag{4.4}$$

with $X(0) = x_0 \neq 0$, and the corresponding performance function

$$J(u) = \mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))dt + \int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big], \; u \in \mathcal{A}. \tag{4.5}$$

exunbsde **Lemma 4.0.1** *(Existence and Uniqueness of BSDE)*

Consider the following time-changed Backward stochastic differential equation

$$dX(t) = -\mu(t, E_t, X(t-), u(t))dE_t + u(t)dB_{E_t} + \int_{\mathbb{R}\backslash\{0\}} h(t, z)\tilde{N}(dE_t, dz), \tag{4.6}$$ BSDE

with $X(T) = X$, where $\mu \in L^2(\mathbb{R}_+, \mathbb{R}_+, \mathbb{R}, \mathbb{R}), h \in L^2(\mathbb{R}_+, \mathbb{R})$. If there exists a positive constant $L_\mu > 0$ such that $|\mu(t_1, t_2, x_1, u_1) - \mu(t_1, t_2, x_2, u_2)| \leq L_\mu\Big(|x_1 - x_2| + |u_1 - u_2|\Big)$, then there exists a unique solution $(X(t), u(t))$ of (4.6).

Proof: To prove the uniqueness, suppose $(X_1(t), u_1(t))$ and $(X_2(t), u_2(t))$ are two solutions to (4.6) in $L^2(\Omega \times \mathbb{R}_+) \times L^2(\Omega \times \mathbb{R}_+)$. By Itô formula,

$$\Big|X_1(T) - X_2(T)\Big|^2 - \Big|X_1(t) - X_2(t)\Big|^2 = \int_t^T |u_1(s) - u_2(s)|^2 dE_s + \int_t^T 2(X_1(s) - X_2(s))\Big[- \Big(\mu(s, E_s, X_1(s), u_1(s)) - \mu(s, E_s, X_2(s), u_2(s))\Big)dE_s + \big(u_1(s) - u_2(s)\big)dB_{E_s}\Big] \tag{4.7}$$ uniu

Thus,

$$
\begin{aligned}
&\Big|X_1(t)-X_2(t)\Big|^2+\int_t^T|u_1(s)-u_2(s)|^2dE_s+\int_t^T 2(X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)dB_{E_s}\\
=&\int_t^T 2(X_1(s)-X_2(s))\Big(\mu(s,E_s,X_1(s),u_1(s))-\mu(s,E_s,X_2(s),u_2(s))\Big)dE_s\\
\le&\int_t^T 2L_\mu|X_1(s)-X_2(s)|\Big(|X_1(s)-X_2(s)|+|u_1-u_2|\Big)dE_s\\
\le&\int_t^T 2L_\mu\Big[|X_1(s)-X_2(s)|^2+\frac{L_\mu}{2}|X_1(s)-X_2(s)|^2+\frac{1}{2L_\mu}|u_1(s)-u_2(s)|^2\Big]dE_s\\
=&(2L_\mu+L_\mu^2)\int_t^T|X_1(s)-X_2(s)|^2dE_s+\int_t^T|u_1(s)-u_2(s)|^2dE_s.
\end{aligned}
\tag{4.8}
$$

Take expectations on both sides,

$$
\mathbb{E}\Big[\Big|X_1(t)-X_2(t)\Big|^2\Big]\le(2L_\mu+L_\mu^2)\mathbb{E}\Big[\int_t^T|X_1(s)-X_2(s)|^2dE_s\Big]. \tag{4.9}
$$

inequ

Note that we apply Martingale property to derive inequality (4.9) and lay some details below.

$$
\begin{aligned}
&\int_t^T(X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)dB_{E_s}\\
&=\int_0^\infty 1\{t\le s\le T\}(X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)dB_{E_s} \qquad (4.10)\\
&=\int_0^\infty 1_{\{t\le D(s-)\le T\}}(X_1(D(s-))-X_2(D(s-)))\big(u_1(D(s-))-u_2(D(s-))\big)dB_s,
\end{aligned}
$$

since $(X_1(t),u_1(t))$ and $(X_2(t),u_2(t))$ are in $L^2(\Omega\times\mathbb{R}_+)$,

$$
\begin{aligned}
&\mathbb{E}\int_0^\infty\Big|1_{\{t\le D(s-)\le T\}}(X_1(D(s-))-X_2(D(s-)))\big(u_1(D(s-))-u_2(D(s-))\big)\Big|^2ds\\
&\le\mathbb{E}\int_0^\infty\Big|(X_1(D(s-))-X_2(D(s-)))\big(u_1(D(s-))-u_2(D(s-))\big)\Big|^2ds<\infty,
\end{aligned}
\tag{4.11}
$$

we have

$$\begin{aligned}
&\mathbb{E}\int_t^T (X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)dB_{E_s} \\
&=\mathbb{E}\int_0^\infty 1_{\{t\le D(s-)\le T\}}(X_1(D(s-))-X_2(D(s-)))\big(u_1(D(s-))-u_2(D(s-))\big)dB_s \\
&=0.
\end{aligned} \tag{4.12}$$

Next we apply time-changed Gronwall's method by Lemma 3.1 in [33]. Define $F(t)=\int_t^T|X_1(s)-X_2(s)|^2dE_s$, then $F(T)=0$ and

$$\begin{aligned}
-d\Big(F(t)\exp(kE_t)\Big) &= -\exp(kE_t)dF(t)-k\exp(kE_t)F(t)dE_t \\
&=\exp(kE_t)\Big(\Big|X_1(t)-X_2(t)\Big|^2-k\int_t^T\Big|X_1(s)-X_2(s)\Big|^2dE_s\Big)dE_t,
\end{aligned} \tag{4.13}$$

thus

$$\begin{aligned}
&-F(T)\exp(kE_T)+F(t)\exp(kE_t) \\
&\qquad=\int_t^T\Big[\exp(kE_s)\Big(\Big|X_1(s)-X_2(s)\Big|^2-k\int_s^T\Big|X_1(u)-X_2(u)\Big|^2dE_u\Big)\Big]dE_s.
\end{aligned} \tag{4.14}$$

Taking expectations and letting $k=2L_\mu+L_\mu^2$ imply that

$$\begin{aligned}
&\mathbb{E}\Big[F(t)\exp(kE_t)\Big] \\
=&\mathbb{E}\Big[\int_t^T\exp(kE_s)\Big(\Big|X_1(s)-X_2(s)\Big|^2-k\int_s^T\Big|X_1(u)-X_2(u)\Big|^2dE_u\Big)dE_s\Big] \\
=&\mathbb{E}\Big[\mathbb{E}\Big[\int_t^T\exp(kE_s)\Big(\Big|X_1(s)-X_2(s)\Big|^2-k\int_s^T\Big|X_1(u)-X_2(u)\Big|^2dE_u\Big)dE_s\Big]\Big|\sigma\{E_s,s\in(t,T)\}\Big] \\
=&\mathbb{E}\Big[\int_t^T\exp(kE_s)\mathbb{E}\Big(\Big|X_1(s)-X_2(s)\Big|^2-k\int_s^T\Big|X_1(u)-X_2(u)\Big|^2dE_u\Big)dE_s\Big|\sigma\{E_s,s\in(t,T)\}\Big] \\
\le&0
\end{aligned} \tag{4.15}$$

It follows that

$$\mathbb{E}\Big[F(t)\Big]\le\mathbb{E}\Big[F(t)\exp(kE_t)\Big]\le 0, \tag{4.16}$$

so $X_1(s) = X_2(s)$ a.s. for $\forall s \in (t,T)$. By (4.7), since $X_1(s) = X_2(s)$ a.s. for $\forall s \in (t,T)$, we have $\int_t^T |u_1(s) - u_2(s)|^2 dE_s = 0$, thus $u_1(s) = u_2(s)$ a.s. for $\forall s \in (t,T)$. The uniqueness is proved.

To prove the existence, let $u_0(t) = 0$, $\{(X_n(t), u_n(t)); 0 \le t \le T\}_{n\ge 1}$ be a sequence defined recursively by

$$\begin{aligned} X_n &= X_{n-1}(t) \\ &+ \int_t^T \mu(s, E_s, X_{n-1}(s), u_{n-1}(s)) dE_s - \int_t^T u_{n-1}(s) dB_{E_s} - \int_t^T \int_{\mathbb{R}\setminus\{0\}} h(s,z)\tilde{N}(dE_s, dz). \end{aligned} \tag{4.17}$$

Then

$$\begin{cases} dX_n(t) = -\mu(t, E_t, X_{n-1}(t), u_{n-1}(t)) dE_t + u_{n-1}(t) dB_{E_t} + \int_{\mathbb{R}\setminus\{0\}} h(t,z)\tilde{N}(dE_t, dz), \\ dX_{n+1}(t) = -\mu(t, E_t, X_n(t), u_n(t)) dE_t + u_n(t) dB_{E_t} + \int_{\mathbb{R}\setminus\{0\}} h(t,z)\tilde{N}(dE_t, dz), \\ X_n(T) = X_{n+1}(T) = X. \end{cases} \tag{4.18}$$

By Itô formula in Lemma 3.0.1, there exists $k > 0$ such that

$$\begin{aligned} &\Big|X_{n+1}(t) - X_n(t)\Big|^2 + \int_t^T (u_n(s) - u_{n-1}(s))^2 dE_s + 2\int_t^T (X_{n+1}(s) - X_n(s))(u_n(s) - u_{n-1}(s)) dB_{E_s} \\ &= 2\int_t^T (X_{n+1}(s) - X_n(s))\Big(\mu(s, E_s, X_n(s), u_n(s)) - \mu(s, E_s, X_{n-1}(s), u_{n-1}(s))\Big) dE_s \\ &\le 2L_\mu \int_t^T |X_{n+1}(s) - X_n(s)|\Big(|X_n(s) - X_{n-1}(s)| + |u_n(s) - u_{n-1}(s)|\Big) dE_s \\ &\le k\Big[\int_t^T |X_{n+1}(s) - X_n(s)|^2 dE_s + \int_t^T |X_n(s) - X_{n-1}(s)|^2 dE_s\Big] + \frac{1}{2}\int_t^T |u_n(s) - u_{n-1}(s)|^2 dE_s. \end{aligned} \tag{4.19}$$

Taking expectation on both sides implies

$$\begin{aligned} &\mathbb{E}\Big|X_{n+1}(t) - X_n(t)\Big|^2 + \frac{1}{2}\mathbb{E}\int_t^T |u_n(s) - u_{n-1}(s)|^2 dE_s \\ &\qquad \le k\mathbb{E}\Big[\int_t^T |X_{n+1}(s) - X_n(s)|^2 dE_s + \int_t^T |X_n(s) - X_{n-1}(s)|^2 dE_s\Big]. \end{aligned} \tag{4.20}$$

exist1

Define $F_n(t) = \int_t^T \Big|X_n(s) - X_{n-1}(s)\Big|^2 dE_s$ for all $n \geq 1$, then $F_n(T) = 0$ and

$$
\begin{aligned}
-d\Big(F_{n+1}(t)\exp(kE_t)\Big) &= -\exp(kE_t)dF_{n+1}(t) - k\exp(kE_t)F_{n+1}(t)dE_t \\
&= \exp(kE_t)\Big[\Big|X_{n+1}(t) - X_n(t)\Big|^2 - k\int_t^T \Big|X_{n+1}(s) - X_n(s)\Big|^2 dE_s\Big]dE_t,
\end{aligned}
\tag{4.21}
$$

By a similar argument for uniqueness and using (4.20),

$$
\begin{aligned}
&\mathbb{E}\Big[F_{n+1}(t)\exp(kE_t)\Big] \\
=&\mathbb{E}\Big[\int_t^T \exp(kE_s)\Big[\Big|X_{n+1}(s) - X_n(s)\Big|^2 - k\int_s^T \Big|X_{n+1}(l) - X_n(l)\Big|^2 dE_l\Big]dE_s\Big] \\
=&\mathbb{E}\Big[\mathbb{E}\Big[\int_t^T \exp(kE_s)\Big[\Big|X_{n+1}(s) - X_n(s)\Big|^2 - k\int_s^T \Big|X_{n+1}(l) - X_n(l)\Big|^2 dE_l\Big]dE_s\Big]\Big|\{\sigma(E_s, s \in (t,T))\}\Big] \\
=&\mathbb{E}\Big[\int_t^T \exp(kE_s)\mathbb{E}\Big[\Big|X_{n+1}(s) - X_n(s)\Big|^2 - k\int_s^T \Big|X_{n+1}(l) - X_n(l)\Big|^2 dE_l\Big]dE_s\Big|\{\sigma(E_s, s \in (t,T))\}\Big] \\
\leq&\mathbb{E}\Big[\int_t^T \exp(kE_s)k\mathbb{E}\Big[\int_s^T |X_n(l) - X_{n-1}(l)|^2 dE_l\Big]dE_s\Big|\{\sigma(E_s, s \in (t,T))\}\Big] \\
=&\mathbb{E}\Big[\int_t^T k\exp(kE_s)\mathbb{E}\Big[F_n(s)\Big]dE_s\Big|\{\sigma(E_s, s \in (t,T))\}\Big] \\
=&\mathbb{E}\Big[\int_t^T k\exp(kE_s)F_n(s)dE_s\Big],
\end{aligned}
\tag{4.22}
$$

letting $t = 0$,

$$
\mathbb{E}F_{n+1}(0) \leq \mathbb{E}\int_0^T ke^{kE_s}F_n(s)dE_s \leq \mathbb{E}\Big[\Big(e^{kE_T}\Big)^n \frac{F_1(0)}{n!}\Big] \to 0,\ as\ n \to \infty. \tag{4.23}
$$

Thus, $\{X_n\}$ is a Cauchy sequence in $L^2(\Omega \times \mathbb{R}_+)$. Taking (4.20) into consideration, $\{u_n\}$ is also a Cauchy sequence in $L^2(\Omega \times \mathbb{R}_+)$. Thus, the existence of solution to (4.6) is proved. □

4.1 Maximum Principle Method

In this section, we solve the time-changed stochastic control problem through the maximum principle method. An example is provided to illustrate how our method works for a particular time-changed stochastic problem.

We consider a performance criterion $J = J(u)$ of the form

$$J(u) = \mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big],\ u \in \mathcal{A}, \tag{4.24}$$

performan

where $g : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times U \to \mathbb{R}$ is continuous, $h : \mathbb{R} \to \mathbb{R}$ is $C^1, T < \infty$ is a fixed deterministic time and

$$\mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big] < \infty, \forall u \in \mathcal{A}. \tag{4.25}$$

The stochastic control problem is to find the optimal control $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{4.26}$$

supj

Since E_t is right continuous and nondecreasing, $\frac{dE_t}{dt}$ exists for $t \geq 0$ a.e.

Define the $Hamiltonian$ $H : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R}$ by

$$\begin{aligned} H(t_1, t_2, x, u, p, q, r) =& g(t_1, t_2, x, u) + pb(t_1, t_2, x, u) + q\sigma(t_1, t_2, x, u) \\ &+ \int_{\mathbb{R}} \gamma(t_1, t_2, x, u, z)r(t_2, z)\nu(dz), \end{aligned} \tag{4.27}$$

hamil

or

$$\begin{aligned} H(t, E_t, X(t), u(t), p(t), q(t), r(t, z)) &= g(t, E_t, X(t), u(t)) + p(t)b(t, E_t, X(t), u(t)) \\ &+ q(t)\sigma(t, E_t, X(t), u(t)) + \int_{\mathbb{R}} \gamma(t, E_t, X(t), u(t), z)r(E_t, z)\nu(dz), \end{aligned} \tag{4.28}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that the integrals in (4.27) exists.

Define the adjoint equation in the unknown processes $p(t) \in \mathbb{R},\ q(t) \in \mathbb{R}$, and $r(t, z) \in \mathbb{R}$ in the backward stochastic differential equations

$$\begin{aligned} dp(t) =& - H_x(t, E_t, X(t), u(t), p(t), q(t), r(t, \cdot))dE_t \\ & + q(t)dB_{E_t} + \int_{\mathbb{R}} r(E_t, z)\tilde{N}(dE_t, dz), t < T \\ p(T) =& h_x(X(T)). \end{aligned} \tag{4.29}$$

adj1

tcmpt **Theorem 4.1.1** *(Time-Changed Maximum Principle Theorem) Let* $\hat{u} \in \mathcal{A}$ *with corresponding solution* $\hat{X} = X^{(\hat{u})}$ *of* (4.1) *and suppose there exists a solution* $(\hat{p}(t), \hat{q}(t), \hat{r}(t, z))$ *of the corresponding adjoint equation* (4.29) *satisfying*

$$\mathbb{E}\Big[\int_0^T (\hat{X}(t) - X^{(u)}(t))^2 \Big(\hat{q}^2(t) + \int_{\mathbb{R}} \hat{r}^2(E_t, z)\nu(dz)\Big)dE_t\Big] < \infty \tag{4.30}$$

and

$$\mathbb{E}\Big[\int_0^T \hat{p}^2(t)\Big(\sigma^2(t, E_t, X^{(u)}(t), u(t)) + \int_{\mathbb{R}} \gamma^2(t, E_t, X^{(u)}(t), u(t), z)\nu(dz)\Big)dE_t\Big] < \infty, \forall u \in \mathcal{A}. \tag{4.31}$$

Moreover, suppose that

$$H(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = \sup_{v \in U} H(t, E_t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \tag{4.32}$$

for all t*, that* $h(x)$ *in* (4.24) *is a concave function of* x *and that*

$$\hat{H}(x) := \max_{v \in U} H(t_1, t_2, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \tag{4.33}$$

exists and is a concave function of x *for all* $t \in [0, T]$*. Then* $\hat{u}$ *is an optimal control of stochastic control problem* (4.26).

Proof: Let $u \in \mathcal{A}$ be an admissible control with corresponding state process $X(t) = X^{(u)}(t)$. We would like to show that

$$J(\hat{u})-J(u) = \mathbb{E}\Big[\int_0^T g(t,E_t,\hat{X}(t),\hat{u}(t)) - g(t,E_t,X(t),u(t))dt + h(\hat{X}(T)) - h(X(T))\Big] \geq 0. \tag{4.34}$$

Since g is concave, using Itô formula (3.2),

$$\begin{aligned}
&\mathbb{E}[h(\hat{X}(T)) - h(X(T))] \geq \mathbb{E}[h_x(\hat{X}(T))(\hat{X}(T) - X(T))] = \mathbb{E}[(\hat{X}(T) - X(T))\hat{P}(T)] \\
&= \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{p}(t) + \int_0^T \hat{p}(t)d(\hat{X}(t) - X(t)) + \int_0^T d\hat{p}(t)d(\hat{X}(t) - X(t))\Big] \\
&= \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{p}(t) + \int_0^T \hat{p}(t)d(\hat{X}(t) - X(t)) \\
&\quad + \int_0^T \hat{q}(t)\Big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t)) - \sigma(t,E_t,X(t),u(t))\Big)dE_t \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t)) - \gamma(t,E_t,X(t),u(t))\Big)\nu(dz)dE_t\Big].
\end{aligned} \tag{4.35}$$

Among above terms,

$$\mathbb{E}\Big[\int_0^T \hat{p}(t)d(\hat{X}(t) - X(t))\Big] = \mathbb{E}\Big[\int_0^T \hat{p}(t)\Big(b(t,E_t,\hat{X}(t),u(t)) - b(t,E_t,X(t),u(t))\Big)dE_t\Big] \tag{4.36}$$

Thus,

$$\begin{aligned}
J(\hat{u}) - J(u) =& \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{p}(t) + \int_0^T g(t,E_t,\hat{X}(t),\hat{u}(t)) - g(t,E_t,X(t),u(t))dE_t \\
&+ \int_0^T \hat{p}(t)\Big(b(t,E_t,\hat{X}(t),\hat{u}(t)) - b(t,E_t,X(t),u(t))\Big)dE_t \\
&+ \int_0^T \hat{q}(t)\Big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t)) - \sigma(t,E_t,X(t),u(t))\Big)dE_t \\
&+ \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t)) - \gamma(t,E_t,X(t),u(t))\Big)\nu(dz)dE_t\Big].
\end{aligned} \tag{4.37}$$

equation1

In addition,

$$
\begin{aligned}
&H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t)) - H(t,E_t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t))\\
=&\Big(g(t,E_t,\hat{X}(t),\hat{u}(t)) - g(t,E_t,X(t),u(t))\Big) + \hat{p}(t)\Big(b(t,E_t,\hat{X}(t),\hat{u}(t)) - b(t,E_t,X(t),u(t))\Big)\\
&+\hat{q}(t)\Big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t)) - \sigma(t,E_t,X(t),u(t))\Big)\\
&+\int_{\mathbb{R}}\hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t)) - \gamma(t,E_t,X(t),u(t))\Big)\nu(dz),
\end{aligned}
\tag{4.38}
$$

equation2

and by (4.29) we have

$$
\begin{aligned}
&(\hat{X}(t)-X(t))d\hat{p}(t) = \hat{X}(t)d\hat{p}(t) - X(t)d\hat{p}(t)\\
=&\hat{X}(t)\Big[-H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dE_t + \hat{q}(t)dB_{E_t} + \int_{\mathbb{R}}\hat{r}(t,z)\tilde{N}(dE_t,dz)\Big]\\
&-X(t)\Big[-H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dE_t + \hat{q}(t)dB_{E_t} + \int_{\mathbb{R}}\hat{r}(t,z)\tilde{N}(dE_t,dz)\Big]\\
=&-(\hat{X}(t)-X(t))H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dE_t\\
&+(\hat{X}(t)-X(t))(\hat{q}(t)dB_{E_t} + \int_{\mathbb{R}}\hat{r}(t,z)\tilde{N}(dE_t,dz)).
\end{aligned}
\tag{4.39}
$$

equation3

Then, since H is concave in x, putting equations (4.38) and (4.39) into (4.37) and following the proof in [9], we get

$$
\begin{aligned}
&J(\hat{u}) - J(u) = \mathbb{E}\Big[\int_0^T -(\hat{X}(t)-X(t))H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dE_t\\
&+\int_0^T H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H(t,E_t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dE_t\\
&\geq 0.
\end{aligned}
\tag{4.40}
$$

□

Remark 4.1.2 *The maximum principle suggests that the optimal control can be solved using Hamiltonian framework, which is a boundary value problem and a maximum condition of a function called the Hamiltonian. The application of the maximum principle lies in that maximizing the Hamiltonian is easier and more feasible than directly solving the original stochastic* *control problem. This leads to the closed form solutions for certain classes of optimal control problems.*

Example 4.1.3 *(The Time-Changed Stochastic Linear Regulator Problem)*
The Linear Regulator Problem aims to reduce the amount of work or energy consumed by the control system to optimize the controller. In this example, we consider the following time-changed stochastic linear regulator problem:

$$\Phi(x_0) = \inf_{u\in\mathcal{A}} \mathbb{E}\Big[\int_0^T \frac{X^2(t)+u^2(t)}{2} dE_t + \lambda X^2(T)\Big], \tag{4.41}$$

where

$$dX(t) = u(t)dE_t + \sigma dB_{E_t} + \int_{\mathbb{R}} z\tilde{N}(dE_t, dz),\ X(0) = x_0. \tag{4.42}$$

Construct the $Hamiltonian$*:*

$$H(t_1, t_2, x, u, p, q, r) = \frac{x^2+u^2}{2} + pu + \sigma q + \int_{\mathbb{R}} \gamma z \nu(dz). \tag{4.43}$$

The adjoint equations are

$$\begin{cases} dp(t) = -X(t)dE_t + q(t)dB_{E_t} + \int_{\mathbb{R}} r(E_t, z)\tilde{N}(dE_t, dz), \\ p(T) = 2\lambda X(T). \end{cases} \tag{4.44}$$ ex1p1

The first and second order condition implies that $Hamiltonian:\ H(t_1, t_2, x, u, p, q, r)$ *achieves the minimum at* $u^*(t) = -p(t)$*.*

To find an explicit solution of $u^*(t)$*, suppose* $p(t) = h(E_t)X(t)$*, where* $h : \mathbb{R}_+ \to \mathbb{R}_+$*. Then* $u^*(t) = -h(E_t)X(t)$ *and*

$$\begin{aligned} dp(t) &= h(E_t)dX(t) + h'(E_t)X(t)dE_t \\ &= h(E_t)\Big(u(t)dE_t + \sigma dB_{E_t} + \int_{\mathbb{R}} z\tilde{N}(dE_t, dz)\Big) + h'(E_t)X(t)dE_t \\ &= X(t)(-h^2(E_t) + h'(E_t))dE_t + h(E_t)\sigma dB_{E_t} + h(E_t)\int_{\mathbb{R}} z\tilde{N}(dE_t, dz). \end{aligned} \tag{4.45}$$ ex1p2

Compare (4.44) *and* (4.45), $-h^2(E_t) + h'(E_t) = -1$ *and* $h(E_T) = 2\lambda$. *The general solution to this ordinary differential equation gives*

$$h(E_t) = -\frac{2\lambda - 1 + (2\lambda + 1)e^{2(E_t - E_T)}}{2\lambda - 1 - (2\lambda + 1)e^{2(E_t - E_T)}}. \tag{4.46}$$

Thus, we have the explicit formula for the optimal control $u^*(t) = -h(E_t)X(t)$. *Similarly,* $q(t) = h(E_t)\sigma$ *and* $r(E_t, z) = h(E_t)z$. *A simulation of the optimal control* $u^*(t)$ *with* $\lambda = -\frac{1}{2}, \sigma = 1, x_0 = -.01$, *standard normal distribution* ν, *and inverse stable subordinator* $E(t)$ *having* $\alpha = .9$ *is displayed in Figure 4.1.*

Keeping all others parts the same as in the figure 4.1, we also simulate the optimal control $u^*(t)$ *for* $\alpha = .7$ *and* $\alpha = .5$ *in Figure 4.2 and 4.3, respectively. Overall, replacing* t *by* E_t *would only insert some constant periods into the original process. As* α *gets closer to* 1, *the constant periods vanish gradually.*

Figure 4.1: Simulation of $u^*(t)$ for Example 1, $\alpha = .9$

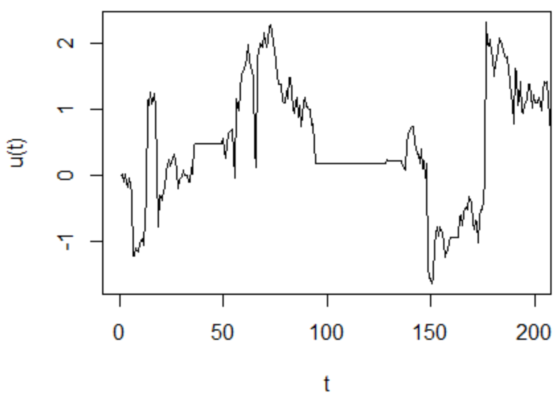


fig:EXAMPLE1_9

Remark 4.1.4 *To demonstrate above example in an intuitive way, we simplify the specification by letting* $\lambda = \frac{1}{2}$, $\sigma = 1$, *and* $z = 0$. *The example problem becomes seeking the optimal control*

Figure 4.2: Simulation of $u^*(t)$ for Example 1, $\alpha = .7$

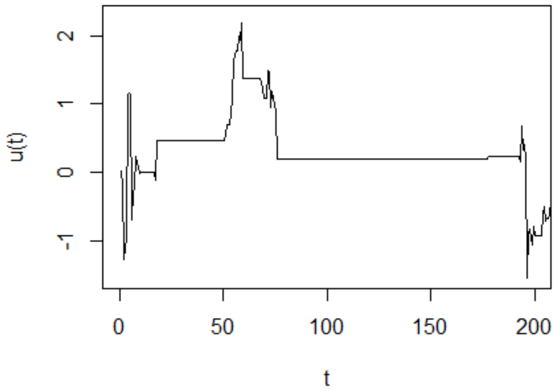


fig:EXAMPLE1_7

Figure 4.3: Simulation of $u^*(t)$ for Example 1, $\alpha = .5$

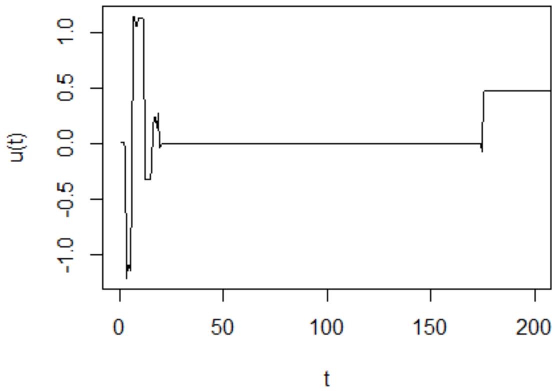


fig:EXAMPLE1_5

of the energy consumption system:

$$\Phi(x_0) = \inf_{u \in \mathcal{A}} \mathbb{E}\Big[\int_0^T \frac{X^2(t) + u^2(t)}{2} dE_t + \frac{X^2(T)}{2}\Big], \tag{4.47}$$

where

$$dX(t) = u(t)dE_t + dB_{E_t},\ X(0) = x_0. \tag{4.48}$$

In this case, $h(E_t) = 1$ and $u^*(t) = -X(t)$. Thus, the optimal control is $du^*(t) = -u^*(t)dE_t - dB_{E_t}$, which means that the optimal control keeps the energy consumption constant over time.

4.2 A More General Time-changed Stochastic Control Problem

Now we extend the time-changed SDE (4.1) to a more general case by adding a time drift term as below,

$$
\begin{aligned}
dX(t) =& \mu(t, E_t, X(t-), u(t))dt + b(t, E_t, X(t-), u(t))dE_t + \sigma(t, E_t, X(t-), u(t))dB_{E_t} \\
&+ \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\tilde{N}(dE_t, dy),
\end{aligned}
\tag{4.49}
$$

with $X(0) = x_0 \neq 0$, where μ, b, σ, γ are real-valued functions satisfying the Lipschitz condition 2.0.1 and assumption 2.0.3.

Suppose the performance function is given by

$$
J(u) = \mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))dt + \int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big],\ u \in \mathcal{A},
\tag{4.50}
$$

performan

where the function $f, g : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times U \to \mathbb{R}$ are continuous, $h : \mathbb{R} \to \mathbb{R}$ is $C^1, T < \infty$ is a fixed deterministic time and

$$
\mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))dt + \int_0^T g(t, E_t, X(t), u(t))dE_t + h(X(T))\Big] < \infty, \forall u \in \mathcal{A}.
\tag{4.51}
$$

The stochastic control problem is to find the optimal control $u^* \in \mathcal{A}$ such that

$$
J(u^*) = \sup_{u \in \mathcal{A}} J(u).
\tag{4.52}
$$

supjex

Remark 4.2.1 *Performance functions* (4.24) *and* (4.50) *are slightly different in terms of their integral kernels. This difference results in different $Hamiltonians$ and adjoint equations.*

Define the $Hamiltonian\ H$ $: [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$ by

$$H(t_1,t_2,x,u,p,q,r) =\Big(p\mu(t_1,t_2,x,u)+f(t_1,t_2,x,u)\Big) + \Big(pb(t_1,t_2,x,u)+q\sigma(t_1,t_2,x,u)+g(t_1,t_2,x,u)\Big)\frac{dt_2}{dt_1} + \int_{\mathbb{R}}\gamma(t_1,t_2,x,u,z)r(t,z)\nu(dz)\frac{dt_2}{dt}, \tag{4.53}$$

or

$$H(t,E_t,X(t),u(t),p(t),q(t),r(t,z)) = \Big(p(t)\mu(t,E_t,X(t),u(t))+f(t,E_t,X(t),u(t))\Big) + \Big(p(t)b(t,X(t),u(t))+q(t)\sigma(t,E_t,X(t),u(t))+g(t,E_t,X(t),u(t))\Big)\frac{dE_t}{dt} + \int_{\mathbb{R}}\gamma(t,E_t,X(t),u(t),z)r(t,z)\nu(dz)\frac{dE_t}{dt}. \tag{4.54}$$

Define the adjoint equation

$$dp(t) =-H_x(t,E_t,X(t),u(t),p(t),q(t),r(t,\cdot))dt + q(t)dB_{E_t}+\int_{\mathbb{R}}r(t,z)\tilde{N}(dE_t,dz), t<T \tag{4.55}$$ adj2

$$p(T) =h_x(X(T))$$

tcmptex **Theorem 4.2.2** *(Time-Changed Maximum Principle Theorem) Let $\hat{u}\in\mathcal{A}$ with corresponding solution $\hat{X}=X^{(\hat{u})}$ and suppose there exists a solution $(\hat{p}(t),\hat{q}(t),\hat{r}(t,z))$ of the corresponding adjoint equation* (4.29) *satisfying*

$$\mathbb{E}\Big[\int_0^T(\hat{X}(t)-X^{(u)}(t))^2\Big(\hat{q}^2(t)+\int_{\mathbb{R}}\hat{r}^2(t,z)\nu(dz)\Big)dE_t\Big]<\infty \tag{4.56}$$

and

$$\mathbb{E}\Big[\int_0^T\hat{p}^2(t)\Big(\sigma^2(t,E_t,X^{(u)}(t),u(t))+\int_{\mathbb{R}}\gamma^2(t,E_t,X^{(u)}(t),u(t),z)\nu(dz)\Big)dE_t\Big]<\infty, \forall u\in\mathcal{A}. \tag{4.57}$$

Moreover, suppose that

$$H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) = \sup_{v\in U} H(t,E_t,\hat{X}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \tag{4.58}$$

for all $t > 0$*, that* $h(x)$ *in* (4.50) *is a concave function of* x *and that*

$$\hat{H}(x) := \max_{v\in U} H(t_1,t_2,x,v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \tag{4.59}$$

exists and is a concave function of x *for all* $t \in [0,T]$*. Then* $\hat{u}$ *is an optimal control of stochastic control problem* (4.52).

Proof: Let $u \in \mathcal{A}$ be an admissible control with the corresponding state process $X(t) = X^{(u)}(t)$. We would like to show that

$$\begin{aligned} J(\hat{u}) - J(u) = \mathbb{E}\Big[\int_0^T f(t,E_t,\hat{X}(t),\hat{u}(t)) - f(t,E_t,X(t),u(t))dt \\ + \int_0^T g(t,E_t,\hat{X}(t),\hat{u}(t)) - g(t,E_t,X(t),u(t))dE_t + h(\hat{X}(T)) - h(X(T))\Big] \geq 0. \end{aligned} \tag{4.60}$$

Since h is concave, using Itô formula (3.2),

$$\begin{aligned} &\mathbb{E}[h(\hat{X}(T)) - g(X(T))] \geq \mathbb{E}[h_x(\hat{X}(T))(\hat{X}(T) - X(T))] = \mathbb{E}[(\hat{X}(T) - X(T))\hat{p}(T)] \\ =&\mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{p}(t) + \int_0^T \hat{p}(t)d(\hat{X}(t) - X(t)) + \int_0^T d\hat{p}(t)d(\hat{X}(t) - X(t))\Big] \\ =&\mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{p}(t) + \int_0^T \hat{p}(t)d(\hat{X}(t) - X(t)) \\ &+ \int_0^T \hat{q}(t)\Big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t)) - \sigma(t,E_t,X(t),u(t))\Big)\hat{q}(t)dE_t \\ &+ \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t)) - \gamma(t,E_t,X(t),u(t))\Big)\nu(dz)dE_t\Big]. \end{aligned} \tag{4.61}$$

Among above terms,

$$\mathbb{E}\Big[\int_0^T \hat{p}(t)d(\hat{X}(t)-X(t))\Big] = \mathbb{E}\Bigg[\int_0^T \hat{p}(t)\bigg(\Big(\mu(t,E_t,\hat{X}(t),\hat{u}(t))-\mu(t,E_t,X(t),u(t))\Big)dt + \Big(b(t,E_t,\hat{X}(t),\hat{u}(t))-b(t,E_t,X(t),u(t))\Big)dE_t\bigg)\Bigg] \tag{4.62}$$

Thus,

$$\begin{aligned} J(\hat{u})-J(u) =& \mathbb{E}\Bigg[\int_0^T (\hat{X}(t)-X(t))d\hat{p}(t) + \int_0^T f(t,E_t,\hat{X}(t),\hat{u}(t))-f(t,E_t,X(t),u(t))dt \\ &+ \int_0^T g(t,E_t,\hat{X}(t),\hat{u}(t))-g(t,E_t,X(t),u(t))dE_t \\ &+ \int_0^T \hat{p}(t)\Big[\Big(\mu(t,E_t,\hat{X}(t),\hat{u}(t))-\mu(t,E_t,X(t),u(t))\Big)dt \\ &+ \Big(b(t,E_t,\hat{X}(t),\hat{u}(t))-b(t,E_t,X(t),u(t))\Big)dE_t\Big] \\ &+ \int_0^T \hat{q}(t)\Big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t))-\sigma(t,E_t,X(t),u(t))\Big)dE_t \\ &+ \int_0^T\int_{\mathbb{R}} \hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t))-\gamma(t,E_t,X(t),u(t))\Big)\nu(dz)dE_t\Bigg]. \end{aligned} \tag{4.63}$$

In addition,

$$\begin{aligned} &(H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t))-H(t,E_t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t)))dt \\ =&\Big[\hat{p}(t)\mu(t,E_t,\hat{X}(t),\hat{u}(t))-\hat{p}(t)\mu(t,E_t,X(t),u(t)) \\ &\qquad + f(t,E_t,\hat{X}(t),\hat{u}(t))-f(t,E_t,X(t),u(t))\Big]dt \\ &+ \Big(g(t,E_t,\hat{X}(t),\hat{u}(t))-g(t,E_t,X(t),u(t))\Big)dE_t \\ &+ \Big(\hat{p}(t)b(t,E_t,\hat{X}(t),\hat{u}(t))+\hat{q}(t)\sigma(t,E_t,\hat{X}(t),\hat{u}(t))\Big)dE_t \\ &- \Big(\hat{p}(t)b(t,E_t,X(t),u(t))+\hat{q}(t)\sigma(t,E_t,X(t),u(t))\Big)dE_t \\ &+ \int_{\mathbb{R}} \hat{r}(t,z)\Big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t))-\gamma(t,E_t,X(t),u(t))\Big)\nu(dz)dE_t, \end{aligned} \tag{4.64}$$

and

$$
\begin{aligned}
&(\hat{X}(t) - X(t))d\hat{p}(t) = \hat{X}(t)d\hat{p}(t) - X(t)d\hat{p}(t) \\
=&\hat{X}(t)\Big[- H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,))dt + \hat{q}(t)dB_{E_t} + \int_{\mathbb{R}} r(t,z)\tilde{N}(dE_t, dz)\Big] \\
&- X(t)\Big[- H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,))dt + \hat{q}(t)dB_{E_t} + \int_{\mathbb{R}} r(t,z)\tilde{N}(dE_t, dz)\Big] \\
=& - (\hat{X}(t) - X(t))H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt \\
&+ (\hat{X}(t) - X(t))\Big(\hat{q}(t)dB_{E_t} + \int_{\mathbb{R}} \hat{r}(t,z)\tilde{N}(dE_t, dz)\Big).
\end{aligned} \tag{4.65}
$$

Then, by concavity of H and following the proof in [9],

$$
\begin{aligned}
&J(\hat{u}) - J(u) = \mathbb{E}\Big[\int_0^T -(\hat{X}(t) - X(t))H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt \\
&+ \int_0^T H(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) - H(t, E_t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt\Big] \\
&\geq 0.
\end{aligned} \tag{4.66}
$$

□

example2 **Example 4.2.3** *(Income and Consumption Optimization) Consider the stochastic control problem*

$$\Phi(x_0) = \sup_{u\in\mathcal{A}} \mathbb{E}\Big[\int_0^\tau \exp(-\delta t)u(t)^2 dt\Big], \tag{4.67}$$

where

$$\tau = \inf\{t > 0; X(t) \leq 0\} \tag{4.68}$$

and

$$dX(t) = -u(t)dt + X(t)\Big(bdE_t + \sigma dB_{E_t} + \theta \int_{\mathbb{R}} z\tilde{N}(dz, dE_t)\Big),\ X(0) = x_0 > 0, \tag{4.69}$$

where $\delta > 0, \sigma$, and θ are constants and $b = -\frac{\sigma^2+\theta^2\int_{\mathbb{R}} z^2\nu(dz)}{2}$.

We can interpret $u(t)$ as the consumption rate, $X(t)$ as the corresponding wealth, and τ as the bankruptcy time. Then Φ represents the maximal expected total quadratic utility of the consumption up to bankruptcy time.

Define the $Hamiltonian\ H$:

$$H(t) = -p(t)u(t) + \exp(-\delta t)u(t)^2 + X(t)\Big(p(t)b + q(t)\sigma + \int_{\mathbb{R}} \theta z r(t,z)\nu(dz)\Big)\frac{dE_t}{dt}, \quad (4.70)$$

and the adjoint equation

$$\begin{aligned} dp(t) =& -\Big(p(t)b + q(t)\sigma + \int_{\mathbb{R}} \theta z r(t,z)\nu(dz)\Big)dE_t \\ &+ q(t)dB_{E_t} + \int_{\mathbb{R}} r(t,z)\tilde{N}(dE_t, dz), t < \tau, \qquad (4.71) \\ p(T) =& 0. \end{aligned}$$

adje1

Let $\frac{\partial H}{\partial u} = (-p(t) + 2u(t)\exp(-\delta t)) = 0$, we have $u^*(t) = \frac{p(t)}{2}\exp(\delta t)$. Suppose that $p(t) = h(t)X(t)$, then $u^*(t) = \frac{h(t)X(t)}{2}\exp(\delta t)$, thus

$$\begin{aligned} dp(t) &= X(t)h(t)'dt + h(t)dX(t) \\ &= X(t)h(t)'dt + (-u(t)h(t))dt + h(t)X(t)\Big(bdE_t + \sigma dB_{E_t} + \theta\int_{\mathbb{R}} z\tilde{N}(dz, dE_t)\Big) \\ &= X(t)\Big(h(t)' - \frac{h(t)}{2}\exp(\delta t)\Big)dt + h(t)X(t)\Big(bdE_t + \sigma dB_{E_t} + \theta\int_{\mathbb{R}} z\tilde{N}(dz, dE_t)\Big) \end{aligned} \quad (4.72)$$

adje2

Comparing (4.71) and (4.72), we derive that $h'(t) = \frac{h(t)}{2}e^{\delta t}$, equivalently, $h(t) = \exp(\frac{1}{2\delta}e^{\delta t})$, thus

$$u(t)^* = \exp(\frac{1}{2\delta}e^{\delta t} + \delta t)\frac{X(t)}{2}. \quad (4.73)$$

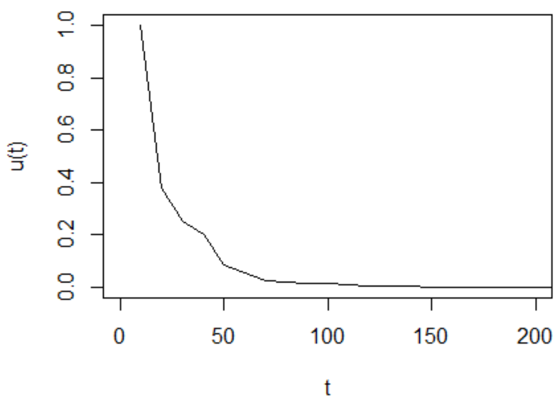
Moreover,

$$\begin{aligned} h(t)X(t)\sigma &= q(t), \\ h(t)X(t)\theta z &= r(t,z). \end{aligned} \quad (4.74)$$

Some algebra implies that

$$\begin{aligned} q(t) &= 2\exp(-\delta t)u(t)\sigma, \\ r(t,z) &= 2\exp(-\delta t)u(t)\theta z. \end{aligned} \quad (4.75)$$

Figure 4.4: Simulation of $u^*(t)$ for Example 2



gure2home

A simulation of the optimal control $u^*(t)$ with $\delta = -.001, \sigma = 1, \theta = 1, x_0 = 1$, standard normal distribution ν, and inverse stable subordinator $E(t)$ having $\alpha = .9$ is displayed in Figure 4.4.

Because of the existence of dt term in the underlying process $X(t)$, the simulated process $u^*(t)$ has no periods of constant value. Compared with dE_t terms, dt term plays the dominating role in the evolution of corresponding wealth $X(t)$, see [26] for a detailed discussion. More specifically, the increasing trend $bX(t)dE_t$ is dominated by the consumption rate $-u(t)dt$. Consequently, the optional consumption rate declines as the wealth shrinks in the long term.

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