A Study of Pólya's Enumeration Theorem

Except where reference is made to the work of others, the work described in this thesis is my own or was done in collaboration with my advisory committee. This thesis does not include proprietary or classified information.

Elizabeth C. Williams

Certificate of Approval:

Greg A. Harris
Associate Professor
Department of Mathematics and Statistics

Edward E. Slaminka, Chair
Associate Professor
Department of Mathematics and Statistics

Stewart L. Baldwin
Professor
Department of Mathematics and Statistics

Stephen L. McFarland
Dean
Graduate School

A Study of Pólya's Enumeration Theorem

Elizabeth C. Williams

A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
August 8, 2005

A Study of Pólya's Enumeration Theorem

Elizabeth C. Williams

Signature of Author

Date

Copy sent to:

Name	Date

Vita

Elizabeth Craig Williams, daughter of Kenneth Neal and Jane (Thompson) Williams was born on April 20, 1976 in Montgomery, Alabama. She is a 1994 graduate of the Lanier High School Academic Motivational Program. She entered LaGrange College, LaGrange, Georgia in the fall of that year and graduated with a Bachelor of Science degree in Mathematics on 6 June, 1998. In September, 1998 she entered the Graduate School of Auburn University as a Graduate Assistant in Mathematics.

Thesis Abstract

A Study of Pólya's Enumeration Theorem

Elizabeth C. Williams

Master of Science, August 8, 2005
(B.S., LaGrange College, 1998)

22 Typed Pages

Directed by Edward E. Slaminka

The controversial lemma, known most commonly as Burnside's Lemma, is stated and proven. The influential Pólya's Enumeration Theorem is stated and proven. Computations using Pólya's Enumeration Theorem are discussed and examples of these computations are given.

Acknowledgments

The author would like to thank her hard-working, über-understanding advisor Dr. Edward Slaminka for helping her finally find a topic to keep her interest – math and jewelry. For his patience, knowledge and advice she will forever be indebted.

For their unconditional love and unscripted entertainment, the author acknowledges her dogs. Who knew tossing an empty water bottle could provide mathematical inspiration? Thank you for a life full of love and laughter.

The author thanks her parents for raising her in an academic environment and teaching her to never shy from learning. Their encouragement, understanding and support (in all forms) has made her life better than most, and for that she will be infinitely thankful. You are my role models as a couple, as parents, as educators and as humans.

Finally, the author wishes to acknowledge her late grandfather Dr. Ernest Williams for nurturing her imagination and for leaving a legacy of which she is extremely proud.

Style manual or journal used Transactions of the American Mathematical Society (together with the style known as “auphd”). Bibliograpy follows van Leunen’s *A Handbook for Scholars.*

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `auphd.sty`.

Table of Contents

Chapter 1

Introduction

Consider a necklace that consists of n colored beads. On paper this can be represented as a word of length n over an alphabet of k colors. For example, one such necklace could be represented as the word $bbgbrr$ (b for blue, g for green and r for red). Now, two words that differ purely by a cyclic rotation must represent the same necklace and thus are equivalent. For our above example, $bbgbrr$, $rbbgbr$, $rrbbgb$, $brrbbg$, $gbrrbb$ and $bgbrrb$ are all equivalent. So, an (n, k)-necklace is an equivalence class of words of length n over an alphabet of size k under rotation and reflection.

This raises a commonly-known enumeration question: For a given n and k, how many unique (n, k)-necklaces can be made [1]? On a broader scale the question becomes, "How many orbits does the dihedral group D_n have on the set of all necklaces with n-beads over k-colors?"

Published in 1934, Georg Pólya's Enumeration Theorem answers the complete question of how many necklaces can be formed using n beads of k colors, accounting for symmetry of rotation and of reflection. Having applications in Combinatorics and Chemistry, Pólya's Enumeration Theorem has led to a new branch of Graph Theory, Enumerative Graph Theory.

This theorem is stated as: Let A and B be finite sets and let the finite group G act on A. Let $c_k(G)$ denote the number of permutations in G that have exactly k cycles in their cycle decomposition on A. Then the number of orbits of G on the set B^A of all mappings $f : A \to B$ is $\dfrac{1}{|G|}\sum_{k=1}^{\infty} c_k(G) \cdot |B|^k$ [3].

Chapter two defines the terms and notations that will be used in the paper. Chapter three presents and proves LaGrange's Theorem, Burnside's Lemma and other theorems used in the process of proving the main theorem, which is found at the end of the chapter. Chapter four provides several examples of applications of Pólya's Enumeration Theorem.

Chapter 2

Definitions and Notation

Definition: Given any set X, a bijection from X to itself is a *permutation.*

Definition: For a finite set X, $|X|$ is the number of elements of X.

Definition: S_n is the set whose elements are permutations of the set of positive integers $\{1, 2, ..., n\}$

Definition: *Symmetry* refers to a rigid motion of a geometric figure. In this paper, we will specifically look at how rotations and reflections affect vertices of geometric figures. Any symmetry determines a permutation in S_n by specifying where each vertex goes under the symmetry.

Definition: A *dihedral group*, D_n, is the group of symmetries of the regular n-gon.

Definition: For a subgroup, H, of a group G, the number of distinct left cosets of H in G is the *index* of H in G. It is written as $|G : H|$.

Definition: A *partition* of a set S is a decomposition of S into nonempty disjoint subsets such that every element of S is in exactly one of the subsets. The subsets of a partition are called *cells* of the partition.

Definition: Each cell in the natural partition arising from an equivalence relation is an *equivalence class.*

Definition: Let X be a set and G a group. An *action of G on X* is a map $\phi : G \times X \to X$ (where $\phi(g, x)$ is denoted by gx) such that:

1. $ex = x$ for all $x \in X$

2.$(g_1 g_2)(x) = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

Definition: Let X be a set, let x be an element of X and let G be any subgroup of D_n. Then the set of all elements of G which fix x is a subgroup of G and is called the *stabilizer* of x in G, denoted by $G_x = \{g \in G | gx = x\}$.

Definition: Let X be a set, let x be an element of X and let G be any subgroup of D_n. Then the set of all elements x which remain fixed by an element of G is the set of *fixed points* of g in X and is denoted by $X_g = \{x \in X | gx = x\}$.

Definition: Let σ be a permutation of a set A. The *orbits of* σ are the equivalence classes in A determined by the following equivalence relation: For $a, b \in A$, let $a \sim b$ if and only if $b = \sigma^n(a)$ for some $n \in \{1, 2, 3, ...\}$. Another way of stating this definition is: Given $x \in X$, the *orbit of x in G*, $\text{Orb}(x)$, is the set of all images y mapped to by some ϕ in G. $\text{Orb}(x) = \{y \in X | \phi(x) = y \text{ for } \phi \in G\}$.

Chapter 3

Pólya's Enumeration Theorem

Theorem 3.1 *Let H be a subgroup of a group G. Left cosets of H form a partition of G.*

Proof: Let aH and bH be two left cosets in H. Suppose that aH and bH have at least one element in common, say $c \in aH \cap bH$. Then for some $h_1, h_2 \in H$, $c = ah_1$ and $c = bh_2$. Thus, $ah_1 = c = bh_2 \Rightarrow ah_1 = bh_2 \Rightarrow a = bh_2h_1^{-1}$. Since H is a subgroup, $h_2h_1^{-1}$ must be in H. Let $h_3 = h_2h_1^{-1} \in H$. Thus, $a = bh_3$ and for every $h \in H$, $ah = bh_3h = bh_4$ for $h_3 \cdot h = h_4 \in H$. Therefore $ah \in bH$ for all $h \in H$ and $aH \subseteq bH$. The opposite containment, $bH \subseteq aH$, can be shown using a similar argument. If $ah \subseteq bH$ and $bH \subseteq aH$ then $aH = bH$, proving that two left cosets that are not disjoint must be equal . Thus, distinct left cosets of H separate elements of G into disjoint subsets.

Theorem 3.2 *(LaGrange's Theorem) If G is a finite group and H is a subgroup of G, then $|G| = |H| \cdot |G : H|$*

Proof: Let G be a finite group of order n. Let H be a subgroup of G with order k. By Theorem 3.1, the left cosets of H separate the elements of G into mutually disjoint subsets. Let $m = |G : H| =$ the number of distinct left cosets of H in G. Let aH be any left coset of H. A mapping $\phi : H \to aH$ defined by $\phi(h) = ah$, for $h \in H$, is injective because the cancellation law holds for G. It is also surjective since any $x \in aH$ can be written as $x = ah$, $h \in H$. Thus ϕ is bijective and $|aH| = |H| = k$. So, we have the n elements of G divided into m disjoint subsets, each with k elements. Therefore $n = km \Rightarrow |G| = |H| \cdot |G : H|$.

Theorem 3.3 *Let G be a subgroup of D_n which acts on a finite set X and let $x \in X$. Then $|Orb(x)|$=$|G : G_x|$=$\frac{|G|}{|G_x|}$.*

Proof: By LaGrange's Theorem, $|G : G_x| = \frac{|G|}{|G_x|}$. To show that this is the same as $|\text{Orb}(x)|$, we construct a bijection $\phi : \text{Orb}(x) \rightarrow \frac{G}{G_x}$ defined by $\phi(y)$=$\{g \in G | y = gx\}$. Now, $\phi(y)$ is not empty because if $y \in \text{Orb}(x)$ then there must be at least one $g \in G$ such that $y = gx$. Choose an element, $g_0 \in G$, such that $y = g_0 x$. Let $g_0 h \in g_0 G_x$ (recall that $h \in G_x$ means $hx = x$). So $g_0 hx = g_0 x = y \Rightarrow g_0 h \in \phi(y)$, by the definition of ϕ. Now, for the opposite inclusion, let $g \in \phi(y)$. Then $y = gx$; however, since $y = g_0 x$, we have that $gx = g_0 x \Rightarrow g_0^{-1} gx = x$. Thus, $g_0^{-1} g \in G_x \Rightarrow g \in g_0 G_x$ and ϕ(y) is well-defined in $\frac{G}{G_x}$. Suppose that $y_1, y_2 \in \text{Orb}(x)$ such that $y_1 \neq y_2$. Then $\phi(y_1) = \{g \in G | gx = y_1\}$ and $\phi(y_2) = \{g \in G | gx = y_2\}$. Clearly these two sets are disjoint. Thus ϕ is one-to-one. Take any coset $gG_x \in \frac{G}{G_x}$. Then $gG_x = \phi(gx)$. Thus ϕ is onto. Therefore the mapping $\phi : \text{Orb}(x) \rightarrow \frac{G}{G_x}$ is bijective and $|\text{Orb}(x)| = |G : G_x| = \frac{|G|}{|G_x|}$.

Theorem 3.4 *The number of orbits of an action of the group G on an element $x \in X$ is equal to $\sum_{x \in X} \frac{1}{|Orb(x)|}$*

Proof: Since X is a disjoint union of orbits, terms in the sum can be collected as $\sum_{x \in X} \frac{1}{|Orb(x)|} = \sum_{i=1}^{N} \left(\sum_{x \in Orb(x_i)} \frac{1}{|Orb(x_i)|} \right) = \sum_{i=1}^{N} (1) = N.$

Theorem 3.5 *(Burnside's Lemma) Let the finite group G act on a finite set X. Then the number of orbits is $\frac{1}{|G|} \sum_{g \in G} |X_g|$.*

Proof: Let $S = \{(g,x) \in G \times X | gx = x\}$. So, $|S| = \sum_{g \in G} |\{x \in X | gx = x\}| = \sum_{g \in G} |X_g|$ and $|S| = \sum_{x \in X} |\{g \in G | gx = x\}| = \sum_{x \in X} |G_x|$. By Theorem 3.3, $|\text{Orb}(x)| = |G : G_x| = \frac{|G|}{|G_x|}$, so $|G_x| = \frac{|G|}{|Orb(x)|}$. Thus, $\sum_{g \in G} |X_g| = |S| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|Orb(x)|}$. Divide by $|G|$ to get $\frac{1}{|G|} \sum_{g \in G} |X_g| = \sum_{x \in X} \frac{1}{|Orb(x)|}$. By Theorem 3.4, $\sum_{x \in X} \frac{1}{|Orb(x)|} = N$. Therefore, $\frac{1}{|G|} \sum_{g \in G} |X_g| = N$.

Theorem 3.6 *(Pólya's Enumeration Theorem) Let A and B be finite sets and let the finite group G act on A. Let $c_k(G)$ denote the number of permutations in G that have exactly k cycles in their cycle decomposition on A. Then the number of orbits of G on the set B^A of all mappings $f : A \to B$ is* $\frac{1}{|G|} \sum_{k=1}^{\infty} c_k(G) \cdot |B|^k$.

Proof: From Theorem 3.5, the number of orbits is given by $N = \frac{1}{|G|} \sum_{g \in G} |X_g|$. Let $|X_g|$ denote the number of mappings, f, left fixed by a permutation $g \in G$. This holds true if and only if f is constant on each of the cycles of g. These mappings are obtained by assigning an element of B to each cycle of g. If g has k cycles, then $|B|^k$ is the number of mappings fixed by g.

Chapter 4

Applications of Pólya's Enumeration Theorem

Pólya's Enumeration Theorem (restated) The number of orbits of a finite set G on the set B^A of all mappings $f : A \to B$ is $\dfrac{1}{|G|}\sum_{k=1}^{\infty} c_k(G)\cdot |B|^k$.

Let A be a finite set denoting the vertices of an n-gon, or the possible positions of beads on a necklace.

Let B be the finite set of colors available.

Let G be the appropriate dihedral group: D_3, D_4, D_5 or D_6.

$D_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ where ρ_i corresponds to rotating the triangle clockwise $\dfrac{2\pi i}{3}$ radians and μ_i corresponds to reflecting the triangle about the three angle bisectors.

$D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ where ρ_i corresponds to rotating the square clockwise $\dfrac{\pi i}{2}$ radians; μ_i corresponds to reflecting the square about the m_i axes; δ_i corresponds to reflecting the square about the diagonals, d_i.

$D_5 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ where ρ_i corresponds to rotating the pentagon clockwise $\dfrac{2\pi i}{5}$ radians and μ_i corresponds to reflecting the pentagon about the lines joining angles and the midpoints of their opposite sides.

$D_6 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where ρ_i corresponds to rotating the hexagon clockwise $\dfrac{2\pi i}{6}$ radians and μ_i corresponds to reflecting the hexagon about the lines connecting opposite vertices and the lines joining midpoints of opposite sides.

The preferred method for representing permutations in S_n is disjoint cycle notation. With this notation, start with a left parentheses, '(', followed by some number in the domain $\{1, 2, ..., n\}$. The next number to the right is the image of the first under the mapping. This process continues until the first number is reached. As soon as one gets back to the starting number, this string of numbers is closed off with a right parentheses, ')'. Repeat this entire process, beginning with '(' and a number not yet listed. If a number is fixed by the permutation (if it is equal to its image) then it is denoted as a single number in parentheses.

We begin with an example of a plain necklace made of three beads with two color options. As we are dealing with three beads, the set A, positions of the beads, is the vertex set of a triangle; $A = \{1, 2, 3\}$. Let G be the dihedral group D_3; therefore, $|G|$=6. Let B be the set of possible colors; in this example $|B|$=2. Label the vertices clockwise, beginning at the top of the triangle. We first write the cycles formed by permuting the necklace into itself. In other words, we apply the elements of D_3 to the set A and write down, using the disjoint cycle notation detailed at the beginning of this chapter, the results of each permutation. Doing so for this example yields the following:

ρ_0	(1)(2)(3)
ρ_1	(123)
ρ_2	(132)
μ_1	(1)(23)
μ_2	(2)(13)
μ_3	(3)(12)

Next, we use these cycle decompositions to find the number of permutations in D_3 that have exactly k cycles; simply, we count the number of permutations above that have one set of parentheses, two sets of parentheses, three sets, etc. We denote each number as $c_k(G)$. Thus, $c_1(G) = 2, c_2(G) = 3$ and $c_3(G) = 1$. Now, apply Pólya's Enumeration Theorem, with A, G and B as defined earlier and N, the number of orbits, being the number of possible unique necklaces.

$$N = \frac{1}{|G|}\sum_{k=1}^{3} c_k(G)\cdot 2^k = \frac{1}{6}[c_1(G)\cdot 2^1 + c_2(G)\cdot 2^2 + c_3(G)\cdot 2^3] = \frac{1}{6}[2\cdot 2^1 + 3\cdot 2^2 + 1\cdot 2^3] = 4.$$

Thus there are four possible plain, unique necklaces to be made with three beads and two color options.

Now, if we add one more color option, obviously the number of possible unique necklaces increases. However, since we are still using three beads, the only value from the preceding example that changes is $|B|$=3. Thus, using Póyla's Enumeration Theorem, the formula becomes:

$$N = \frac{1}{6}\sum_{k=1}^{3} c_k(G)\cdot 3^k = \frac{1}{6}[2\cdot 3^1 + 3\cdot 3^2 + 1\cdot 3^3] = 10.$$

Thus there are ten possible unique necklaces to made using three beads of three colors. Continuing in this fashion, we see that using three beads and having 1,2,3,4,5,6,... color options there are 1,4,10,20,35,56,... unique necklaces possible.

Using four beads increases not only the size and aesthetics of the necklace, but also the calculation of the number of possibilities. Instead of working with the vertices and symmetries of a triangle as before, we now work with the vertices and symmetries of a square. Let A be the set of vertices of a square; $A = \{1, 2, 3, 4\}$. Let $G = D_4$ with $|G|$=8. For illustration purposes, let $|B|$=4, or let there be four color options for this necklace.

As in the last example, use the disjoint cycle notation to list the affect of each permutation of D_4:

ρ_0	(1)(2)(3)(4)
ρ_1	(1234)
ρ_2	(13)(24)
ρ_3	(1432)
μ_1	(12)(34)
μ_2	(14)(23)
δ_1	(13)(2)(4)
δ_2	(24)(1)(3)

Using these decompositions, we get that $c_1(G) = 2, c_2(G) = 3, c_3(G) = 2$ and $c_4(G) = 1$. Applying all of this information to Pólya's Enumeration Theorem, we see: N= $\frac{1}{8}\sum_{k=1}^{4} c_k(G) \cdot 4^k$=$\frac{1}{8}[c_1(G) \cdot 4^1 + c_2(G) \cdot 4^2 + c_3(G) \cdot 4^3 + c_4(G) \cdot 4^4]$=$\frac{1}{8}[2 \cdot 4^1 + 3 \cdot 4^2 + 2 \cdot 4^3 + 1 \cdot 4^4]$=55.

Thus, there are 55 possible unique necklaces to be made using four beads and choosing from four colors. In fact, using four beads and 1,2,3,4,5,6,7,... colors of beads, there are 1,6,21,55,120,231,406,... possible unique necklaces to design.

Now, some of us have expensive tastes, so let us calculate how many unique necklaces using five beads there are from which to choose. As in the previous two examples, we let A be the set of vertices of a pentagon, making $A = \{1, 2, 3, 4, 5\}$. Label the vertices clockwise from one to five. This means $G = D_5$ and thus $|G|$=10. For calculation purposes, let $|B|$=3. The cycle decomposition is as follows:

ρ_0	(1)(2)(3)(4)(5)
ρ_1	(12345)
ρ_2	(13524)
ρ_3	(14253)
ρ_4	(15432)
μ_1	(1)(25)(34)
μ_2	(2)(13)(45)
μ_3	(3)(24)(15)
μ_4	(4)(12)(35)
μ_5	(5)(14)(23)

Thus $c_1(G) = 4, c_2(G) = 0, c_3(G) = 5, c_4(G) = 0$ and $c_5(G) = 1$. Applying Pólya's Enumeration Theorem for three colors, we get:

N=$\frac{1}{10}\sum_{k=1}^{5} c_k(G)\cdot 3^k$=$\frac{1}{10}[c_1(G)\cdot 3^1 + c_2(G)\cdot 3^2 + c_3(G)\cdot 3^3 + c_4(G)\cdot 3^4 + c_5(G)\cdot 3^5]$=$\frac{1}{10}[4\cdot 3^1 + 0 + 5\cdot 3^3 + 0 + 1\cdot 3^5]$=39.

Thus there are 39 gorgeous, unique necklaces to be designed using five beads and three possible colors. In general, for five beads and 1,2,3,4,5,6,7,8,... colors, there are 1,8,39,136,377,888,1855,3536,... unique necklaces to be made.

Finally, we will determine how many really glamorous, unique necklaces can be made using six beads. Let $A = \{1, 2, 3, 4, 5, 6\}$, the set of vertices of a hexagon. Label the vertices clockwise one to six. Let $G = D_6$ making $|G| = 12$. Let us make this necklace worthy of giving Momma by using nine colors in the design, thus making $|B|$=9. The cycle decompositions are as follows:

ρ_0	(1)(2)(3)(4)(5)(6)
ρ_1	(123456)
ρ_2	(135)(246)
ρ_3	(14)(25)(36)
ρ_4	(153)(264)
ρ_5	(165432)
μ_1	(16)(25)(34)
μ_2	(1)(26)(35)(4)
μ_3	(12)(36)(45)
μ_4	(13)(2)(5)(46)
μ_5	(14)(23)(56)
μ_6	(6)(15)(24)(3)

Thus $c_1(G) = 2$, $c_2(G) = 2$, $c_3(G) = 4$, $c_4(G) = 3$, $c_5(G) = 0$ and $c_6(G) = 1$. Applying Pólya's Enumeration Theorem, the formula becomes:

$$N=\frac{1}{12}\sum_{k=1}^{6} c_k(G)\cdot 5^k=\frac{1}{12}[c_1(G)\cdot 5^1 + c_2(G)\cdot 5^2 + c_3(G)\cdot 5^3 + c_4(G)\cdot 5^4 + c_5(G)\cdot 5^5 + c_6(G)\cdot 5^6] = \frac{1}{12}[2\cdot 5^1 + 2\cdot 5^2 + 4\cdot 5^3 + 3\cdot 5^4 + 0 + 1\cdot 5^6] = 1505.$$

Therefore, we have 1;13;92;430;1,505;4,291;10,528;23,052;46,185;... options for designing an elegant and expensive necklace using six beads (or gems) of 1,2,3,4,5,6,7,8,9,... colors.

Bibliography

[1] Davis, T., "Polya's Counting Theory",http://www.geometer.org/mathcircles (September 12, 2001).

[2] Fraleigh, J. B., *A First Course in Abstract Algebra* ,6th ed., Reading, MA: Addison-Wesley,(1999).

[3] Van Lint, J. H. and Wilson, R. M., *A Course in Combinatorics* , New York, NY: Cambridge University Press, (1992).