Hopfield Neural Lattice Models

by

Basiru Usman

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 8, 2020

Keywords: Hopfield neural network, neural lattice model, random ordinary differential equation, random dynamical system, random attractor, lattice dynamical system, global attractor, upper semi-continuous convergence.

Approved by

Xiaoying Han, Chair, Marguerite Scharnagle Endowed Professor of Mathematics and Statistics
Georg Hetzer, Professor of Mathematics and Statistics
Wenxian Shen, Professor of Mathematics and Statistics
Peter E. Kloeden, Affiliate Professor of Mathematics and Statistics

Abstract

Hopfield neural network model is a continuous deterministic model proposed by **John J. Hopfield** in the early **1980's**. The model was proposed in an attempt to produced an artificial neural network architecture that mimic the activity of the brain neurons. The continuous deterministic model was developed so that it incorporate one of the basic properties of biological neurons which is a **continuous input-output relations**. The model captured the attention of many researchers and since then, the model has been extensively studied, modified and extended. As of the time of writing this dissertation, the paper where the model was first introduced has received more than eight thousand citations.

In this dissertation, we start by looking at the very first motivational part of the model, which is **the brain**. An average human brain has about 86 billion neurons, with this fact in mind, we have created a Hopfield neural network model where the number of neurons is increasingly large, that is a Hopfield neural lattice model is developed as the infinite dimensional extension of the classical finite dimensional Hopfield model. In addition, random external inputs are considered to incorporate environmental noise. The resulting random lattice dynamical system is first formulated as a random ordinary differential equation on the space of square summable bi-infinite sequences. Then the existence and uniqueness of solutions, as well as long term dynamics of solutions are investigated.

In the second part of the dissertation, we study infinite dimensional extension of the classical Hopfield model and its corresponding finite dimensional approximations. The existence of global attractors is established for both the lattice system and its finite dimensional approximations. Moreover, the global attractors for the finite dimensional approximations are shown to converge to the attractor for the infinite dimensional lattice system upper semi-continuously.

In the final part of the dissertation we present the ongoing works, which are the stochastic Hopfiled neural lattice model and the non autonomous Hopfield lattice model.

Acknowledgments

First and foremost I will like to thank the almighty God for giving me strength, courage and patience to complete this dissertation. My deepest appreciation goes to my advisor Professor Xiaoying Han for her untiring support, advice and guidance with which without her effort this will not have been possible. I will also like to thank my co-advisor Dr. Peter E. Kloeden for his insightful comments and suggestions. My appreciation goes Professor Georg Hetzer, Professor Wenxian Shen for agreeing to be on my Ph.D. committee, and Dr. Yang Zhou of computer science and software engineering department for reading the draft and making valuable comments. I will like to thank Professor Charles E. Chidume, Professor Geraldo De-souza, Professor Lawal Sulaiman Bilbis, Dr. Usman Bello AbdulMalik and Dr. Ma'aruf Shehu Minjibir for their motivation and encouragement to further my studies.

Finally, I will like to thank my parents late Alhj. Usman Sama'ila and Haj. Amina Usman for their love, prayers and unending support all through my life. I want to thank my wife Zainab Aliyu Abubakar, for her patience and for most of this time, taking care of our son Usman Adil Basiru(2). I want thank all my friends and family for their support and prayers.

Table of Contents

Abstract ii

Acknowledgments iii

1 Introduction 1

2 Long term behavior of a random Hopfield neural lattice model 3

2.1 Introduction 3

2.2 Preliminaries 4

2.3 Basic properties of solutions 7

2.4 Existence of random attractors 14

2.5 Closing remarks 25

3 Upper semi-continuous convergence of attractors for a Hopfield-type lattice model . . 26

3.1 Introduction 26

3.2 Properties of solutions 28

3.2.1 Existence and uniqueness of solutions 28

3.2.2 Existence of attractors 31

3.2.3 Special structure of the global attractor 39

3.3 Upper-semi continuity of global attractors 41

3.4 Closing remarks 56

4 On going work 58

4.1 Attractors for Hopfield lattice model in weighted spaces 58

4.1.1 Random dynamical systems generated by stochastic lattice 58

4.1.2 Existence and uniqueness of solutions of random lattice systems 64

4.2 Non autonomous Hopfield-type lattice model 77

References 78

List of Figures

3.1 Replacements of neurons $-N-\mathfrak{n}$ through $-N-1$ under assumption (**A7**). . . 43

3.2 Replacements of neurons $N+1$ through $N+\mathfrak{n}$ under assumption (**A7**). 43

3.3 Replacements of neurons $-N-\mathfrak{n}$ through $-N-1$ under assumption (**A8**). . . 44

3.4 Replacements of neurons $N+1$ through $N+\mathfrak{n}$ under assumption (**A8**). 44

List of Tables

Chapter 1

Introduction

Neural networks have been gaining increasing attention of researchers, due to their wide range of applications such as character recognition (see, e.g., [56]), image compression (see, e.g., [29]), stock market prediction (see, e.g., [35, 48]), traveling saleman's problem (see, e.g., [32]), etc. In particular, one of the most popular mathematical models for (artificial) neural network is the Hopfield neural model proposed by John Hopfield in 1984 [46], described by the following system of n ordinary differential equations (ODEs):

$$\mu_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=1}^{n} \lambda_{i,j} g_j(u_j(t)) + I_i, \qquad i = 1, ..., n, \tag{1.1}$$

where u_i represents the voltage on the input of the ith neuron at time t; $\mu_i > 0$ and $\gamma_i > 0$ represents the neuron amplifier input capacitance and resistance of the ith neuron, respectively; and I_i is the constant external forcing on the ith neuron.

Here n is the total number of neurons coupled by an $n \times n$ matrix $(\lambda_{i,j})_{1 \le i,j \le n}$, where the entity $\lambda_{i,j}$ represents the connection strength between the ith and the jth the neuron. More precisely, for each pair of $i, j = 1, \ldots, n$, $\lambda_{i,j}$ is the synapse efficacy between neurons i and j, and thus $\lambda_{i,j} > 0$ ($\lambda_{i,j} < 0$, resp.) means the output of neuron j excites (inhibits, resp.) neuron i. The term $\lambda_{i,j} g_j(u_j(t))$ represents the electric current input to neuron i due to the present potential of neuron j, in which the function g_j is neuron activation functions and assumed to be a sigmoid type function.

In this dissertation we focus on

$$\mu_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j(t)) + I_i, \quad i \in \mathbb{Z}. \tag{1.2}$$

which is the infinite extension of the classical Hopfield neural network model in (1.1).

The remaining part of the dissertation is organized as follow; In chapter 2 we will study a long term behavior of (2.1) where the input function I_i is replaced by $I_i(\theta_t\omega)$ to incorporate the environmental noise, in chapter 3 we will study (2.1) together with some of its finite dimensional approximations and their relationship.

Chapter 2

Long term behavior of a random Hopfield neural lattice model

2.1 Introduction

In this chapter we are interested in studying dynamics of the above Hopfield neural network model (1.1) when its size becomes increasingly large, i.e., $n \to \infty$. To this end, we extend the n dimensional ODE system (1.1) to an infinite dimensional lattice system, that models the dynamics of an infinite number of neurons indexed by $i \in \mathbb{Z}$, in which each neuron is still connected with other neurons within its finite N neighborhood. More precisely, the ith neuron is connected to the $(i-N)$th, $\cdots$, $(i+N)$th neurons through the strength matrix $(\lambda_{i,j})_{i-N\le j\le i+N}$ and the activation functions g_j for $j = i-N, \cdots, i+N$. In addition, to take into account random perturbations of the environment, we introduce a noise in the equations in (1.1) by replacing each constant input I_i by a random forcing $I_i(\theta_t\omega)$ represented by a measure-preserving dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ acting on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. System (1.1) then becomes the following random lattice dynamical system, namely the random Hopfield neural lattice model:

$$\mu_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j(t)) + I_i(\theta_t\omega), \quad i \in \mathbb{Z}. \tag{2.1}$$

Over the past two decades, extensive studies have been done on dynamics of lattice dynamical systems (see, e.g., [9, 20, 21, 62, 64, 66, 67, 75] and references therein). However, most of the existing works consider a simple linear diffusion described by a linear operator such as $u_{i-1} - 2u_i + u_{i+1}$. Very often the lattice dynamical system under consideration arises from discretization of a partial differential equation. Notice that the terms $\sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j(t))$ in

system (2.1) model a nonlinear finite neighborhood connection structure, which is intrinsically discrete and does not arise from a continuous operator. Such models have not been studied in the literature of lattice dynamical systems.

The goal of this work is to investigate the long term dynamics of the random lattice dynamical system (2.1), in particular, the existence of random attractors. The main tool is the theory of random dynamical systems and random attractors (see, e.g., [6, 24, 25, 41, 34]). The paper is organized as follows. In Section 2.2 we provide necessary preliminaries on random dynamical systems and random attractors. In Section 3.2 we first reformulate system (2.1) as a random ordinary differential equation (RODE) on the space of bi-infinite sequences and then show that the resulting RODE has a unique solution that generates a random dynamical system (RDS). In Section 3.2.2 we investigate the existence of a random attractor for the RDS obtained in Section 3.2. Some closing remarks will be given in Section 3.4.

2.2 Preliminaries

In this section, we present some basic concepts and theory of random dynamical system (RDS) required in the sequel (for more references on RDS see, e.g., [6, 11, 15, 18, 31]).

Let $(X, \|\cdot\|_X)$ be a separable Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω (called "events") and $\mathbb{P}$ is the probability measure. To connect the state $\omega \in \Omega$ at time 0 with its state after a time of t elapses, define a flow $\Theta = \{\theta_t\}_{t\in\in\mathbb{R}}$ on Ω with each θ_t being a mapping $\theta_t : \Omega \longrightarrow \Omega$ that satisfies

(1) $\theta_0 = Id_\Omega$;

(2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$;

(3) the mapping $(t, \omega) \mapsto \theta_t\omega$ is measurable;

(4) the probability measure $\mathbb{P}$ is preserved by θ_t, i.e, $\theta_t\mathbb{P} = \mathbb{P}$.

Then $(\Omega, \mathcal{F}, \mathbb{P}, \Theta)$ is called driving dynamical system [6].

Definition 1. *A stochastic process $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ is said to be a continuous random dynamical system (RDS) over $(\Omega,\mathcal{F},\mathbb{P},\Theta)$ with state space X if $\varphi:[0,\infty)\times\Omega\times X\to X$ is $(\mathcal{B}[0,+\infty)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable, and for each $\omega\in\Omega$,*

(i) The mapping $\varphi(t,\omega,\cdot):X\to X$, $x\mapsto\varphi(t,\omega,x)$ is continuous for every $t\geq 0$;

(ii) $\varphi(0,\omega,\cdot)$ is the identity operator on X;

(iii) (cocycle property) $\varphi(t+s,\omega,\cdot)=\varphi(t,\theta_s\omega,\varphi(s,\omega,\cdot))$ for all $s,t\geq 0$.

Definition 2. *A set-valued mapping $K:\Omega\to 2^X\backslash\emptyset$ is said to be a random set if the mapping $\omega\mapsto dist_X(x,K(\omega))$ is measurable for any $x\in X$. A random set $K(\omega)\subset X$ is said to be tempered with respect to Θ if for a.e. $\omega\in\Omega$*

$$\lim_{t\to\infty}e^{-\beta t}\sup_{x\in K(\theta_{-t}\omega)}\|x\|_X=0,\quad\forall\ \beta>0.$$

A random variable $\omega\mapsto r(\omega)\in\mathbb{R}$ is said to be tempered if for a.e. $\omega\in\Omega$,

$$\lim_{t\to\infty}e^{-\beta t}\sup_{t\in\mathbb{R}}|r(\theta_{-t}\omega)|=0,\quad\forall\ \beta>0.$$

Throughout this paper, denote by $\mathscr{D}(X)$ the set of all tempered random sets of X.

Definition 3. *Let $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be an RDS over $(\Omega,\mathcal{F},\mathbb{P},\Theta)$ with state space X and let $\mathcal{A}(\omega)(\subset X)$ be a random set. Then $\mathcal{A}(\omega)$ is called a global random $\mathscr{D}$ attractor (or pullback $\mathscr{D}$ attractor) for $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ if $\omega\mapsto\mathcal{A}(\omega)$ satisfies*

(i) (random compactness) $\mathcal{A}(\omega)$ is a compact set of X for a.e. $\omega\in\Omega$;

(ii) (invariance) for a.e. $\omega\in\Omega$ and all $t\geq 0$, it holds

$$\varphi(t,\omega,\mathcal{A}(\omega))=\mathcal{A}(\theta_t\omega);$$

(iii) (attracting property) for any $K \in \mathscr{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \mathrm{dist}_X(\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where

$$\mathrm{dist}_X(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_X$$

is the Hausdorff semi-metric for $A, B \subseteq X$.

The existence of random attractors usually requires the existence of random absorbing set defined below.

Definition 4. *A random set $\Gamma(\omega) \subset X$ is called random absorbing set in $\mathscr{D}(X)$ if for any $K \in \mathscr{D}(X)$ and a.e $\omega \in \Omega$, there is $T_K(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)) \subset \Gamma(\omega) \quad \forall\ t \geq T_K(\omega).$$

The following theorem is one of the most widely used results for the existence of random attractors.

Proposition 1. *[11, 18, 31] Let $\Gamma \in \mathscr{D}(X)$ be a closed absorbing set for the continuous RDS $\{\varphi(t, \omega)\}_{t\geq 0,\omega\in\Omega}$ and satisfies the asymptotic compactness condition for $a.e.\ \omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n}, \Gamma(\theta_{-t_n}\omega))$ has a convergent subsequence in X when $t_n \to \infty$. Then the RDS φ has a unique global random attractor with component subsets*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq t_\Gamma(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega))}.$$

If the pullback absorbing set is positively invariant, i.e., $\varphi(t, \omega, \Gamma(\omega)) \subset \Gamma(\theta_t\omega))$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} \varphi(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega)).$$

2.3 Basic properties of solutions

In this section we study the existence and uniqueness of solutions to the lattice system (2.1), and show that the solutions generate a random dynamical system. To this end, we consider the separable Hilbert space of square summable bi-infinite sequences:

$$\ell^2 = \left\{ \boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} u_i^2 < \infty \right\}$$

equipped with inner product and norm

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \|\boldsymbol{u}\| = \left(\sum_{i\in\mathbb{Z}} u_i^2 \right)^{\frac{1}{2}}, \quad \forall \boldsymbol{u} = (u_i)_{i\in\mathbb{Z}}, \boldsymbol{v} = (v_i)_{i\in\mathbb{Z}} \in \ell^2.$$

For any $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, define the operators $A\boldsymbol{u} = ((A\boldsymbol{u})_i)_{i\in\mathbb{Z}}$ and $\Lambda\boldsymbol{u} = ((\Lambda\boldsymbol{u})_i)_{i\in\mathbb{Z}}$ by

$$(A\boldsymbol{u})_i = \frac{1}{\mu_i\gamma_i} u_i, \qquad (\Lambda\boldsymbol{u})_i = \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j).$$

Throughout this paper it is assumed that

(**A1**) the efficacy among neurons is finite, i.e., there exists $B_\lambda > 0$ such that

$$\sup_{i,j\in\mathbb{Z}} |\lambda_{i,j}| \le B_\lambda.$$

In addition, it is assumed that

(**A2**) for each $i \in \mathbb{Z}$, $\mu_i \in [b_\mu, B_\mu]$, $\gamma_i \in [b_\gamma, B_\gamma]$ with $b_\gamma, b_\mu > 0$;

(**A3**) for each $i \in \mathbb{Z}$, $g_i \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $g_i(0) = 0$, and there exists a continuous function $L(r) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\sup_{i\in\mathbb{Z}} \max_{s\in[-r,r]} |g_i'(s)| \le L(r) \qquad \forall\, r \in \mathbb{R}_+.$$

It follows immediately from the assumption (**A2**) that $A\boldsymbol{u} \in \ell^2$ for every $\boldsymbol{u} \in \ell^2$. In addition, notice that the assumption (**A3**) implies that given any $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, for each

$i \in \mathbb{Z}$ there exists $\xi_i \in \mathbb{R}$ with $|\xi_i| \leq |u_i|$ such that

$$|g_i(u_i)| = |g_j'(\xi_i)u_i| \leq L(|u_i|)|u_i| \leq L(\|\boldsymbol{u}\|)|u_i|. \tag{2.2}$$

Then by the assumptions (**A1**) and (**A3**) we have

$$\begin{aligned}
\sum_{i\in\mathbb{Z}}(\Lambda\boldsymbol{u})_i^2 &\leq \frac{(2N+1)}{b_\mu^2}\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N}\lambda_{i,j}^2 g_j^2(u_j) \\
&\leq \frac{(2N+1)}{b_\mu^2}B_\lambda^2 L^2(\|\boldsymbol{u}\|)\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N}u_j^2 \\
&\leq \frac{(2N+1)^2}{b_\mu^2}B_\lambda^2\cdot L^2(\|\boldsymbol{u}\|)\cdot\|\boldsymbol{u}\|^2,
\end{aligned}$$

and thus $\Lambda\boldsymbol{u} \in \ell^2$ for every $\boldsymbol{u} \in \ell^2$.

At last, define $\mathcal{I}(\theta_t\omega) = \left(\frac{I_i(\theta_t\omega)}{\mu_i}\right)_{i\in\mathbb{Z}}$ and assume that

(**A4**) $I_i(\theta_t\omega)$ is continuously differentiable for each $i \in \mathbb{Z}$ and satisfies

$$\sum_{i\in\mathbb{Z}} I_i^2(\theta_t\omega) < \infty \quad \forall\, t \in \mathbb{R}.$$

Then $\mathcal{I}(\theta_t\omega) \in \mathcal{C}^1(\mathbb{R}, \ell^2)$ and the lattice system (2.1) can be rewritten as the random ordinary differential equation (RODE):

$$\frac{\mathrm{d}\boldsymbol{u}(t,\omega)}{\mathrm{d}t} = -A\boldsymbol{u} + \Lambda\boldsymbol{u} + \mathcal{I}(\theta_t\omega). \tag{2.3}$$

The theorem (3.1) states the existence and uniqueness of a global solution to the RODE (2.3). The proof needs the following general Gronwall type Lemma.

Lemma 1. *Let $a(t)$ be non-negative and non-decreasing, $b(t)$ be non-negative and continuous, and $c(t)$ be non-negative and integrable on $[t_0, T]$, and assume that the function $w : [t_0, T] \to [0, \infty)$ satisfies*

$$w(t) \leq a(t) + \int_{t_0}^t b(s)w(s)\mathrm{d}s + \int_{t_0}^t c(s)\ln[w(s)+1]w(s)\mathrm{d}s, \quad \forall\, t \in [t_0, T]$$

Then

$$w(t) \leq \max\left\{2a(t)e^{2\int_{t_0}^{t} b(s)\mathrm{d}s}, 2^{e^{2\int_{t_0}^{t} c(s)\mathrm{d}s}} - 1\right\} \ \forall\ t \in [t_0, T].$$

Proof. Since all functions and integrals are nonnegative, we have

$$w(t) \leq 2\max\left\{a(t) + \int_{t_0}^{t} b(s)w(s)\mathrm{d}s, \int_{t_0}^{t} c(s)\ln(w(s)+1)w(s)\mathrm{d}s\right\} \tag{2.4}$$

for all $t \in [t_0, T]$.

With $w(t) \leq 2a(t) + 2\int_{t_0}^{t} b(s)w(s)\mathrm{d}s$ and the assumption that $a(t)$ is non-decreasing it follows directly from Gronwall's inequality that

$$w(t) \leq 2a(t)e^{2\int_{t_0}^{t} b(s)\mathrm{d}s}. \tag{2.5}$$

With $w(t) \leq 2\int_{t_0}^{t} c(s)\ln(w(s)+1)w(s)\mathrm{d}s$ and the assumption that $c(t)$ is nonnegative we have

$$\begin{aligned} w(t)+1 &\leq 2\int_{t_0}^{t} c(s)\ln[w(s)+1](w(s)+1)\mathrm{d}s + 1 \\ &\leq 2\left(\int_{t_0}^{t} c(s)\ln[w(s)+1](w(s)+1)\mathrm{d}s + 1\right). \end{aligned}$$

Then by using a generalized Gronwall-like integral inequality [30], we obtain

$$w(t)+1 \leq 2^{e^{2\int_{t_0}^{t} c(s)\mathrm{d}s}}.$$

which means that

$$w(t) \leq 2^{e^{2\int_{t_0}^{t} c(s)\mathrm{d}s}} - 1. \tag{2.6}$$

Inserting the inequalities (2.5) and (2.6) into (2.4) gives immediately the desired assertion. The proof is complete. □

Theorem 2.1. *Let assumptions (**A1**) – (**A4**) hold. In addition, assume that there exist $\kappa_1, \kappa_2 > 0$ such that $L(r) \leq \kappa_1 \ln(r^2+1) + \kappa_2$ for all $r \geq 0$. Then for any $\omega \in \Omega$, $t_0 \in \mathbb{R}$ and any initial*

data $\boldsymbol{u}_o = (u_{o,i})_{i\in\mathbb{Z}} \in \ell^2$*, the RODE* (2.3) *has a unique solution* $\boldsymbol{u}(\cdot; t_0, \omega, \boldsymbol{u}_o) \in \mathcal{C}([t_0, \infty), \ell^2)$ *with* $\boldsymbol{u}(t_0; t_0, \omega, \boldsymbol{u}_o) = \boldsymbol{u}_o$*. In addition, the solution* $\boldsymbol{u}(\cdot; t_0, \omega, \boldsymbol{u}_o)$ *is continuous in* $\boldsymbol{u}_o \in \ell^2$*.*

Proof. Given $\omega \in \Omega$ and $\boldsymbol{u} \in \ell^2$, let

$$F(\boldsymbol{u}, \theta_t\omega) = -A\boldsymbol{u} + \Lambda\boldsymbol{u} + \mathcal{I}(\theta_t\omega).$$

Then F is continuous in $\boldsymbol{u}$ and measurable in ω from $\ell^2 \times \Omega$ into ℓ^2. First for any $\boldsymbol{u}, \boldsymbol{v} \in \ell^2$, we have

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|^2 = \sum_{i\in\mathbb{Z}} \frac{1}{\mu_i^2\gamma_i^2}(u_i - v_i)^2 \le \frac{1}{b_\mu^2 b_\gamma^2}\|\boldsymbol{u} - \boldsymbol{v}\|^2. \tag{2.7}$$

Second, by using assumptions (**A2**) and (**A3**) we have

$$\begin{aligned}
\|\Lambda\boldsymbol{u} - \Lambda\boldsymbol{v}\|^2 &= \sum_{i\in\mathbb{Z}} \frac{1}{\mu_i^2}\left(\sum_{j=i-N}^{i+N} \lambda_{i,j}\left(g_j(u_j) - g_j(v_j)\right)\right)^2 \\
&\le \frac{1}{b_\mu^2}(2N+1)\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N} \lambda_{i,j}^2\left(g_j(u_j) - g_j(v_j)\right)^2 \\
&\le \frac{B_\lambda^2}{b_\mu^2}(2N+1)\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N} L^2(2\max\{|u_j|, |v_j|\})|u_j - v_j|^2 \\
&\le \frac{B_\lambda^2}{b_\mu^2}(2N+1)^2 L^2(2\max\{\|\boldsymbol{u}\|, \|\boldsymbol{v}\|\})\|\boldsymbol{u} - \boldsymbol{v}\|^2.
\end{aligned} \tag{2.8}$$

Hence by summing (3.3) and (3.4) we obtain

$$\begin{aligned}
\|F(\boldsymbol{u}, \theta_t\omega) - F(\boldsymbol{v}, \theta_t\omega)\|^2 &\le 2\|A\boldsymbol{u} - A\boldsymbol{v}\|^2 + 2\|\Lambda\boldsymbol{u} - \Lambda\boldsymbol{v}\|^2 \\
&\le \frac{2}{b_\mu^2}\left(\frac{1}{b_\gamma^2} + B_\lambda^2(2N+1)^2 L^2(2\max\{\|\boldsymbol{u}\|, \|\boldsymbol{v}\|\})\right)\|\boldsymbol{u} - \boldsymbol{v}\|^2,
\end{aligned}$$

which implies that $F(\boldsymbol{u}, \theta_t\omega)$ is locally Lipschitz in $\boldsymbol{u}$. Therefore the RODE (2.3) has a unique local solution.

We next show that given any $T > t_0$ fixed the solution $\boldsymbol{u}(t)$ is bounded for all $t \in [t_0, T]$. To this end, multiplying both sides of the equation (2.1) by $u_i(t)$ and sum over all $i \in \mathbb{Z}$ to give

$$\frac{1}{2}\frac{d\left\|\boldsymbol{u}(t,\omega)\right\|^2}{dt} = -\sum_i \frac{u_i^2}{\mu_i\gamma_i} + \sum_i \frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j) + \sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}I_i(\theta_t\omega). \tag{2.9}$$

First by Assumption (**A2**), we have

$$-\sum_i \frac{u_i^2}{\mu_i\gamma_i} \le -\frac{\|\boldsymbol{u}\|^2}{B_\mu B_\gamma}. \tag{2.10}$$

Then by using the fact that $xy \le \frac{1}{2}(ax^2 + \frac{1}{a}y^2)$, for some $a > 0$ (to be determined later), it holds

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}I_i(\theta_t\omega) &\le \sum_{i\in\mathbb{Z}}\frac{1}{2}\left(au_i^2 + \frac{I_i^2(\theta_t\omega)}{a\mu_i^2}\right) \le \frac{a}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2a}\sum_{i\in\mathbb{Z}}\frac{I_i^2(\theta_t\omega)}{\mu_i^2}\\ &\le \frac{a}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2ab_\mu^2}\sum_{i\in\mathbb{Z}}I_i^2(\theta_t\omega).\end{aligned} \tag{2.11}$$

Then by assumptions (**A1**) and (**A2**),

$$\begin{aligned}\sum_i\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j) &\le \frac{B_\lambda}{b_\mu}\sum_i\sum_{j=i-N}^{i+N}|u_i g(u_j)|\\ &= \frac{B_\lambda}{b_\mu}\sum_i\sum_{j=i-N}^{i+N}|u_i u_j|\cdot|g'(\xi_j)|\end{aligned}$$

for some ξ_j with $|\xi_j| \le |u_j|$. Thus by Assumption (**A2**), (**A3**) and the fact that $xy \le \frac{1}{2}(x^2+y^2)$ we have

$$\begin{aligned}\sum_i\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j) &\le \frac{B_\lambda}{2b_\mu}L(\|\boldsymbol{u}\|)\sum_i\sum_{j=i-N}^{i+N}\left(u_j^2+u_i^2\right)\\ &\le (2N+1)\frac{B_\lambda}{b_\mu}L(\|\boldsymbol{u}\|)\|\boldsymbol{u}\|^2.\end{aligned} \tag{2.12}$$

Inserting estimations (3.7) – (3.10) into (3.6) we obtain

$$\frac{\mathrm{d}\left\|\boldsymbol{u}(t)\right\|^2}{\mathrm{d}t} \leq 2\left(-\frac{1}{B_\mu B_\gamma}+\frac{(2N+1)B_\lambda L(\|\boldsymbol{u}\|)}{b_\mu}+\frac{a}{2}\right)\cdot\|\boldsymbol{u}\|^2+\frac{1}{ab_\mu^2}\sum_{i\in\mathbb{Z}}I_i^2(\theta_t\omega).$$

Pick $a>0$ such that $-\frac{1}{B_\mu B_\gamma}+\frac{a}{2}<0$. In particular, let $a=\frac{1}{B_\mu B_\gamma}$, and for simplicity denote

$$\chi_1 := \frac{1}{B_\mu B_\gamma},\quad \chi_2 := \frac{2(2N+1)B_\lambda}{b_\mu},\quad M(\theta_t\omega) := \frac{B_\mu B_\gamma}{b_\mu^2}\sum_{i\in\mathbb{Z}}I_i^2(\theta_t\omega).$$

Then the above inequality becomes

$$\frac{\mathrm{d}\left\|\boldsymbol{u}(t)\right\|^2}{\mathrm{d}t} \leq -\chi_1\|\boldsymbol{u}\|^2+\chi_2 L(\|\boldsymbol{u}\|)\|\boldsymbol{u}\|^2+M(\theta_t\omega). \tag{2.13}$$

Multiplying both sides of (2.13) by $e^{\chi_1 t}$ gives

$$\frac{\mathrm{d}\left\|e^{\chi_1 t}\boldsymbol{u}(t)\right\|^2}{\mathrm{d}t} \leq \chi_2 e^{\chi_1 t}L(\|\boldsymbol{u}(t)\|)\|\boldsymbol{u}(t)\|^2+M(\theta_t\omega)e^{\chi_1 t}.$$

Integrating the above inequality from t_0 to t results in

$$e^{\chi_1 t}\left\|\boldsymbol{u}(t)\right\|^2 \leq \|\boldsymbol{u}_o\|^2 e^{\chi_1 t_0}+\chi_2\int_{t_0}^t e^{\chi_1 s}L(\|\boldsymbol{u}(s)\|)\|\boldsymbol{u}(s)\|^2\mathrm{d}s+\int_{t_0}^t M(\theta_s\omega)e^{\chi_1 s}\mathrm{d}s,$$

and hence

$$\begin{aligned}\|\boldsymbol{u}(t)\|^2 \;&\leq\; \|\boldsymbol{u}_o\|^2 e^{-\chi_1(t-t_0)}+\int_{t_0}^t M(\theta_s\omega)e^{-\chi_1(t-s)}\mathrm{d}s\\ &\quad+\chi_2\int_{t_0}^t e^{-\chi_1(t-s)}L(\|\boldsymbol{u}(s)\|)\|\boldsymbol{u}(s)\|^2\mathrm{d}s.\end{aligned} \tag{2.14}$$

First notice that the assumption (**A4**) implies that $\int_{t_0}^t M(\theta_s\omega)e^{-\chi_1(t-s)}\mathrm{d}s<\infty$ for all $t\geq t_0$. Now using the assumption that $L(r)\leq\kappa_1\ln(r^2+1)+\kappa_2$ in (2.14) we obtain

$$\begin{aligned}\|\boldsymbol{u}(t)\|^2 \;&\leq\; \|\boldsymbol{u}_o\|^2+\int_{t_0}^t M(\theta_s\omega)e^{-\chi_1(t-s)}\mathrm{d}s+\chi_2\kappa_2\int_{t_0}^t\|\boldsymbol{u}(s)\|^2\mathrm{d}s\\ &\quad+\chi_2\kappa_1\int_{t_0}^t\ln(\|\boldsymbol{u}(s)\|^2+1)\|\boldsymbol{u}(s)\|^2\mathrm{d}s.\end{aligned}$$

Since $M(\theta_s\omega) \geq 0$ for all $s \in \mathbb{R}$, $\int_{t_0}^{t} M(\theta_s\omega)e^{-\chi_1(t-s)}\mathrm{d}s$ is non-negative and non-decreasing. It then follows directly from Lemma 1 that

$$\|\boldsymbol{u}(t)\|^2 \leq \max\left\{2(\|\boldsymbol{u}_o\|^2 + \hat{M}(\theta_t\omega))e^{2\kappa_2\chi_2(t-t_0)}, 2^{e^{2\kappa_1\chi_2(t-t_0)}} - 1\right\}. \tag{2.15}$$

for all $t \in [t_0, T]$, where

$$\hat{M}(t,\omega) := \int_{t_0}^{t} M(\theta_s\omega)e^{-\chi_1(t-s)}\mathrm{d}s < \infty \quad \forall\ t \geq t_0. \tag{2.16}$$

We have just shown that the solution exists for all $t \geq t_0$. It remains to show the continuous dependence of solutions on initial data. To this end, let $\boldsymbol{u}_o, \boldsymbol{v}_o \in \ell^2$ and consider two solutions of system (2.3) with initial value $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$ and $\boldsymbol{u}(t_0) = \boldsymbol{v}_o$, respectively, and write

$$X(t) = (X_i(t))_{i\in\mathbb{Z}} := \boldsymbol{u}(t; t_0, \omega, \boldsymbol{u}_0), \qquad Y(t) = (Y_i(t))_{i\in\mathbb{Z}} = \boldsymbol{u}(t; t_0, \omega, \boldsymbol{v}_0).$$

Let $h(t) = (h_i(t))_{i\in\mathbb{Z}} = X(t) - Y(t)$, then $h(t)$ satisfies the random lattice system

$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t} = -\frac{1}{\mu_i\gamma_i}h_i + \frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}\left(g_j(X_j) - g_j(Y_j)\right), \quad i \in \mathbb{Z}. \tag{2.17}$$

Multiplying the equation (2.17) by $h_i(t)$ and summing over $i \in \mathbb{Z}$ gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h(t)\|^2 &\leq -\sum_{i\in\mathbb{Z}}\frac{1}{\mu_i\gamma_i}h_i^2 + \sum_{i\in\mathbb{Z}}\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}\left(g_j(X_j) - g_j(Y_j)\right)h_i \\ &\leq -\frac{1}{B_\mu B_\gamma}\|h(t)\|^2 + \frac{B_\lambda}{b_\mu}\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N}|g_j(X_j) - g_j(Y_j)|\cdot|h_i|.\end{aligned}$$

By Assumption (**A3**) again, we have

$$|g_j(X_j) - g_j(Y_j)| \leq L(\max\{|X_j|, |Y_j|\})|h_j| \leq L_T|h_j|,$$

where L_T is a constant depends on T according to (2.15). Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h(t)\|^2 &\leq -\frac{2}{B_\mu B_\gamma}\|h(t)\|^2 + \frac{4B_\lambda L_T}{b_\mu}\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N} h_i^2 \\ &\leq \left(-\frac{2}{B_\mu B_\gamma} + \frac{4(2N+1)B_\lambda L_T}{b_\mu}\right)\|h(t)\|^2. \end{aligned} \tag{2.18}$$

Integrating the above inequality gives

$$\|h(t)\|^2 \leq e^{\left(-\frac{2}{B_\mu B_\gamma}+\frac{4(2N+1)B_\lambda L_T}{b_\mu}\right)(t-t_0)}\cdot\|h(0)\|^2,$$

which implies that

$$\sup_{t\in[t_0,T]}\|X(t)-Y(t)\|^2 \leq \max\left\{e^{2\left(-\frac{2}{B_\mu B_\gamma}+\frac{4(2N+1)B_\lambda L_T}{b_\mu}\right)(T-t_0)},1\right\}\cdot\|\boldsymbol{u}_0-\boldsymbol{v}_0\|^2.$$

The solution depends continuously on the initial data. The proof is now complete. □

It is straightforward to check that

$$\boldsymbol{u}(t+t_0;t_0,\omega,\boldsymbol{u}_o)=\boldsymbol{u}(t;0,\theta_{t_0}\omega,\boldsymbol{u}_o),\quad \forall\, t\geq 0,\ \boldsymbol{u}_0\in\ell^2,\ \omega\in\Omega,$$

and this allows us to define a continuous random dynamical system $\varphi(t,\omega,\cdot)$ by

$$\varphi(t,\omega,\boldsymbol{u}_0)=\boldsymbol{u}(t;0,\omega,\boldsymbol{u}_o),\quad \forall\, t\geq 0,\ \boldsymbol{u}_0\in\ell^2,\ \omega\in\Omega.$$

From now on, we will write $\boldsymbol{u}(t;\omega,\boldsymbol{u}_o)$ instead of $\boldsymbol{u}(t;0,\omega,\boldsymbol{u}_o)$.

2.4 Existence of random attractors

In this section we investigate the existence of attractors for the random dynamical system $\varphi(t,\omega)$ defined by the solutions to the RODE (2.3). To this end, we first construct a closed and bounded absorbing set for $\varphi(t,\omega)$ and then prove the asymptotic compactness of the absorbing

set. For simplification of exposition, throughout this section we assume that $L(r) \equiv L > 0$ in Assumption (**A3**). In addition, assume that the functions g_i satisfy the dissipative condition

(**A5**) there exist $\alpha \geq 0$ and $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}} \in \ell^2$ such that

$$sg_j(s) \leq -\alpha s^2 + \beta_j^2, \quad \forall\ s \in \mathbb{R}.$$

Lemma 2. *Assume that assumptions (***A1***) - (***A5***) hold. Then the continuous random dynamical system $\varphi(t,\omega)$ generated by the RODE* (2.3) *has a random absorbing set $\Gamma(\omega)$ provided*

$$b_\lambda := \inf_{i\in\mathbb{Z}} \min_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}\neq 0}} |\lambda_{i,j}| > 0; \tag{2.19}$$

$$\sigma := \frac{1}{B_\mu B_\gamma} + \frac{\alpha(2N+1)b_\lambda}{B_\mu} - (2N+1)\frac{LB_\lambda}{b_\mu}\left(\frac{9}{2} + \frac{L}{\alpha}\right) > 0. \tag{2.20}$$

Proof. We will start from the equation (3.6) and still use the inequalities (3.7) and (3.8). It then remains to estimate $\sum_{i\in\mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j)$. To this end first note that

$$\sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) = \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}>0}} \lambda_{i,j} u_i g_j(u_j) + \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}<0}} \lambda_{i,j} u_i g_j(u_j). \tag{2.21}$$

Also, by Assumption (**A5**),

$$u_j g_j(u_j) \leq -\alpha u_j^2 + \beta_j^2, \quad \forall\ j \in \mathbb{Z}.$$

Then by Assumption (**A3**) with $L(r) \equiv L$ and the fact that $xy \leq \frac{1}{2}\left(\frac{x^2}{a} + ay^2\right)$ we get

$$\begin{aligned} g_j(u_j)\cdot(u_i - u_j) &= g_j'(\xi_j)u_j(u_i - u_j) \quad \text{for some } \xi_j \text{ with } |\xi_j| \leq |u_j| \\ &\leq L \cdot \left(\frac{u_i^2}{2a} + \frac{au_j^2}{2} + u_j^2\right) \end{aligned}$$

for some $a > 0$ (to be determined later). Therefore

$$\begin{aligned} u_i g_j(u_j) &= (u_i - u_j) g_j(u_j) + u_j g_j(u_j) \\ &\leq \left(\frac{a}{2}L - \alpha\right) u_j^2 + L u_j^2 + \frac{L}{2a} u_i^2 + \beta_j^2, \quad \forall\ i, j \in \mathbb{Z}. \end{aligned} \tag{2.22}$$

Pick a such that $\frac{a}{2}L - \alpha < 0$ in (2.22). In particular, let $a = \frac{\alpha}{L}$. Then multiplying (2.22) by $\lambda_{i,j} > 0$ and summing over all j with $\lambda_{i,j} > 0$ we have

$$\begin{aligned} \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}>0}} \lambda_{i,j} u_i g_j(u_j) &\leq -\frac{\alpha}{2} \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}>0}} \lambda_{i,j} u_j^2 \\ &\quad + B_\lambda \left(L \sum_{j=i-N}^{i+N} u_j^2 + \frac{(2N+1)L^2}{2\alpha} u_i^2 + \sum_{j=i-N}^{i+N} \beta_j^2 \right) \\ &\leq -\frac{\alpha b_\lambda}{2} \sum_{j=i-N}^{i+N} u_j^2 + B_\lambda \left(L \sum_{j=i-N}^{i+N} u_j^2 + \frac{(2N+1)L^2}{2\alpha} u_i^2 + \sum_{j=i-N}^{i+N} \beta_j^2 \right). \end{aligned} \tag{2.23}$$

Next with $\lambda_{i,j} < 0$ we have

$$\begin{aligned} \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}<0}} \lambda_{i,j} u_i g_j(u_j) &\leq B_\lambda \sum_{\substack{j=i-N,\cdots,i+N \\ \lambda_{i,j}<0}} |u_i||g_j(u_j)| \\ &\leq B_\lambda \sum_{j=i-N}^{i+N} L|u_i||u_j| \\ &\leq B_\lambda L \sum_{j=i-N}^{i+N} u_j^2 + \frac{B_\lambda L}{4}(2N+1) u_i^2. \end{aligned} \tag{2.24}$$

Using (2.21), (2.23) and (2.24) we obtain

$$\begin{aligned} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) &\leq -\frac{\alpha b_\lambda}{2} \sum_{j=i-N}^{i+N} u_j^2 + 2B_\lambda L \sum_{j=i-N}^{i+N} u_j^2 \\ &\quad + B_\lambda \frac{(2N+1)L}{4\alpha}(2L+\alpha) u_i^2 + B_\lambda \sum_{j=i-N}^{i+N} \beta_j^2. \end{aligned} \tag{2.25}$$

Multiplying both sides of (3.9) by $\frac{1}{\mu_i}$ and summing over $i \in \mathbb{Z}$ we obtain

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \;\leq\; & -\frac{\alpha b_\lambda}{2B_\mu} \sum_{i\in\mathbb{Z}} \sum_{j=i-N}^{i+N} u_j^2 + \frac{2B_\lambda}{b_\mu} L \sum_{i\in\mathbb{Z}} \sum_{j=i-N}^{i+N} u_j^2 \\ & + \frac{B_\lambda}{b_\mu} \frac{(2N+1)L}{4\alpha} (2L+\alpha) \sum_{i\in\mathbb{Z}} u_i^2 + \frac{B_\lambda}{b_\mu} \sum_{i\in\mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_j^2.\end{aligned}$$

It then follows from the assumption (2.19) and a shift of index that

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \;\leq\; & (2N+1) \cdot \bigg(-\frac{\alpha b_\lambda}{2B_\mu} + \frac{B_\lambda L}{4\alpha b_\mu} (2L+\alpha) \\ & + \frac{2B_\lambda L}{b_\mu} \bigg) \|\boldsymbol{u}(t)\|^2 + (2N+1) \frac{B_\lambda}{b_\mu} \|\boldsymbol{\beta}\|^2. \end{aligned} \tag{2.26}$$

Now inserting the inequality (3.7), the inequality (3.8) with $a = \frac{1}{B_\mu B_\gamma}$, and the inequality (2.26) into (3.6) results in

$$\frac{\mathrm{d}\|\boldsymbol{u}(t,\omega)\|^2}{\mathrm{d}t} \leq -\sigma \|\boldsymbol{u}(t,\omega)\|^2 + 2(2N+1) \frac{B_\lambda}{b_\mu} \|\boldsymbol{\beta}\|^2 + M(\theta_t \omega), \tag{2.27}$$

where σ is defined as in (2.20) and $M(\theta_t\omega) = \frac{B_\mu B_\gamma}{b_\mu^2} \sum_{i\in\mathbb{Z}} I_i^2(\theta_t\omega)$.

Integrating (2.27) from 0 to t gives

$$\begin{aligned}\|\varphi(t,\omega,\boldsymbol{u}_o)\|^2 \;\leq\; & e^{-\sigma t} \|\boldsymbol{u}_o\|^2 + e^{-\sigma t} 2(2N+1) \frac{B_\lambda}{b_\mu} \|\boldsymbol{\beta}\|^2 \int_0^t e^{\sigma s} \mathrm{d}s \\ & + \int_0^t M(\theta_s\omega) e^{-\sigma(t-s)} \mathrm{d}s \\ \leq\; & e^{-\sigma t} \|\boldsymbol{u}_o\|^2 + 2(2N+1) \frac{B_\lambda}{b_\mu \sigma} \|\boldsymbol{\beta}\|^2 + \int_0^t M(\theta_s\omega) e^{-\sigma(t-s)} \mathrm{d}s.\end{aligned}$$

Then replacing ω by $\theta_{-t}\omega$ in the above inequality we finally obtain

$$\|\varphi(t,\theta_{-t}\omega,\boldsymbol{u}_o)\|^2 \leq e^{-\sigma t} \|\boldsymbol{u}_o\|^2 + 2(2N+1) \frac{B_\lambda}{b_\mu \sigma} \|\boldsymbol{\beta}\|^2 + \int_{-t}^0 M(\theta_\tau\omega) e^{\sigma\tau} \mathrm{d}\tau. \tag{2.28}$$

Note that $\int_{-t}^{0} M(\theta_\tau\omega)e^{\sigma\tau}\mathrm{d}\tau$ is a tempered random variable due to Assumption (**A4**) and the integrability of $e^{\sigma\tau}$ for $\tau < 0$. Define

$$\Gamma(\omega) := \left\{ \boldsymbol{u} \in \ell^2 : \|\boldsymbol{u}\| \leq \left(2(2N+1)\frac{B_\lambda}{b_\mu \sigma}\|\boldsymbol{\beta}\|^2 + \int_{-\infty}^{0} M(\theta_\tau\omega)e^{\sigma\tau}\mathrm{d}\tau + 1 \right)^{1/2} \right\}. \quad (2.29)$$

Then for any $K(\omega) \in \mathscr{D}(\ell^2)$ and given $\boldsymbol{u}_o(\omega) \in K(\omega)$, there exists $T_K(\omega) > 0$ such that

$$\|\varphi(t, \theta_{-t}\omega, \boldsymbol{u}_o(\theta_{-t}\omega))\|^2 \leq 2(2N+1)\frac{B_\lambda}{b_\mu \sigma}\|\boldsymbol{\beta}\|^2 + \int_{-\infty}^{0} M(\theta_\tau\omega)e^{\sigma\tau}\mathrm{d}\tau + 1$$

for all $t \geq T_K(\omega)$, i.e., $\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)) \in \Gamma(\omega)$. The proof is complete. □

Remark 1. *The dissipativity condition (***A5***) on g_i's is sufficient but not necessary. In fact, even if $\alpha < 0$, an absorbing set can still exist provided $\frac{1}{B_\mu B_\gamma}$ is large enough compared to L. But with g_i's being dissipative, the convergence rate of solutions toward the absorbing set is faster.*

We next show that the absorbing set defined in (2.29) is asymptotically compact under the RDS $\varphi(t, \omega)$ by following the techniques introduced in [9]. This will be done by a tail estimate of solutions, presented in the lemma below. Note that as a particular case of Lemma 6, there exists $T_\Gamma(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega)) \in \Gamma(\omega), \quad \forall\ t \geq T_\Gamma(\omega).$$

Lemma 3. *Assume that assumptions (***A1***) - (***A5***) and* (2.19) – (2.20) *hold. Then for any $\epsilon > 0$ there exist $\hat{T}(\epsilon, \omega, \Gamma) \geq T_\Gamma(\omega)$ and $\iota(\epsilon, \omega) \in \mathbb{N}$ such that the solution $\varphi(t, \omega, \boldsymbol{u}_o) = \boldsymbol{u}(t; \omega, \boldsymbol{u}_o) = (u_i(t; \omega, \boldsymbol{u}_o))_{i\in\mathbb{Z}}$ of the RODE* (2.3) *with $\boldsymbol{u}_o = (u_{o,i})_{i\in\mathbb{Z}} \in \Gamma(\theta_{-t}\omega)$ satisfies*

$$\sum_{|i|\geq\iota(\epsilon,\omega)} |(\varphi(t, \theta_{-t}\omega, \boldsymbol{u}_o))_i|^2 = \sum_{|i|\geq\iota(\epsilon,\omega)} |u_i(t, \theta_{-t}\omega, \boldsymbol{u}_o)|^2 \leq \epsilon \quad \forall\, t \geq \hat{T}(\epsilon, \omega, \Gamma).$$

Proof. Given $k \in \mathbb{N}$ fixed (to be determine later), let $\rho_k : \mathbb{R}_+ \to [0,1]$ be a continuous, increasing and sub-additive function such that

$$\rho_k(|i|) = \begin{cases} 0, & |i| < k, \\ \in [0,1], & |i| \in [k, 2k] \\ 1, & |i| \geq 2k, \end{cases} \qquad i \in \mathbb{Z}.$$

For any solution $\boldsymbol{u}(t;\omega,\boldsymbol{u}_o) = (u_i(t;\omega,\boldsymbol{u}_o))_{i\in\mathbb{Z}}$ of (2.3), set $\boldsymbol{v} = (v_i)_{i\in\mathbb{Z}}$ where $v_i = \rho_k(|i|)u_i$, then $\boldsymbol{v} \in \ell^2$. Taking the inner product of $\boldsymbol{v}$ with (2.3) we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2(t;\omega,\boldsymbol{u}_o) &= -\sum_{i\in\mathbb{Z}}\rho_k(|i|)\frac{u_i^2}{\mu_i\gamma_i} + \sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i\frac{1}{\mu_i}I_i(\theta_t\omega) \\ &\quad + \sum_{i\in\mathbb{Z}}\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}\rho_k(|i|)u_i g_j(u_j). \end{aligned} \tag{2.30}$$

First by Assumption (**A2**),

$$-\sum_{i\in\mathbb{Z}}\rho_k(|i|)\frac{u_i^2}{\mu_i\gamma_i} \leq -\frac{1}{B_\mu B_\gamma}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2. \tag{2.31}$$

Second, similar to (3.8) there exists some $a > 0$ such that

$$\sum_{i\in\mathbb{Z}}\rho_k(|i|)\frac{u_i}{\mu_i}I_i(\theta_t\omega) \leq \frac{a}{2}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2 + \frac{1}{2ab_\mu^2}\sum_{i\in\mathbb{Z}}\rho_k(|i|)I_i^2(\theta_t\omega). \tag{2.32}$$

It remains to estimate the last term of (3.14). To this end, multiplying (3.9) by $\frac{1}{\mu_i}\rho_k(|i|)$ and summing over $i \in \mathbb{Z}$ to get

$$\sum_{i\in\mathbb{Z}}\rho_k(|i|)\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j) \tag{2.33}$$

$$\begin{aligned} \leq\ & -\frac{\alpha b_\lambda}{2B_\mu}\sum_{i\in\mathbb{Z}}\rho_k(|i|)\sum_{j=i-N}^{i+N}u_j^2 + \frac{2B_\lambda}{b_\mu}L\sum_{i\in\mathbb{Z}}\rho_k(|i|)\sum_{j=i-N}^{i+N}u_j^2 \\ & + \frac{B_\lambda L(2N+1)(2L+\alpha)}{4b_\mu\alpha}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2 + \frac{B_\lambda}{b_\mu}\sum_{i\in\mathbb{Z}}\rho_k(|i|)\sum_{j=i-N}^{i+N}\beta_j^2. \end{aligned} \tag{2.34}$$

Choose $k > N$. Then

$$\rho_k(i) = \rho_k(j), \quad \forall\ |i-j| \le N. \tag{2.35}$$

In fact, for any $j = i-N, \cdots, i+N$, we have $\rho_k(|i-j|) = 0$ and due to the increasingness and sub-additivity of ρ_k we obtain

$$\begin{aligned}
\rho_k(|i|) &\le \rho_k(|i-j|+|j|) \le \rho_k(|i-j|) + \rho_k(|j|) = \rho_k(|j|),\\
\rho_k(|j|) &\le \rho_k(|i|+|j-i|) \le \rho_k(|i|) + \rho_k(|j-i|) = \rho_k(|i|).
\end{aligned}$$

Using (2.35) in the inequality (2.34) results in

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \rho_k(|i|)\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j)\\
\le\ & -\frac{\alpha b_\lambda}{2B_\mu}\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N}\rho_k(|j|)u_j^2 + \frac{2B_\lambda}{b_\mu}L\sum_{i\in\mathbb{Z}}\sum_{j=i-N}^{i+N}\rho_k(|j|)u_j^2\\
&\frac{B_\lambda L(2N+1)(2L+\alpha)}{4b_\mu\alpha}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2 + \frac{B_\lambda}{b_\mu}\sum_{i\in\mathbb{Z}}\rho_k(|i|)\sum_{j=i-N}^{i+N}\beta_j^2.
\end{aligned}$$

Then it follows from a shift of index that

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} \rho_k(|i|)\frac{1}{\mu_i}\sum_{j=i-N}^{i+N}\lambda_{i,j}u_i g_j(u_j) \le\ & \frac{B_\lambda}{b_\mu}\sum_{i\in\mathbb{Z}}\left(\sum_{j=i-N}^{i+N}\rho_k(|j|)\right)\beta_i^2\\
&+ (2N+1)\left(-\frac{\alpha b_\lambda}{2B_\mu} + \frac{2B_\lambda L}{b_\mu} + \frac{B_\lambda(2L^2+\alpha L)}{4b_\mu\alpha}\right)\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2.
\end{aligned} \tag{2.36}$$

Inserting estimations (3.15), (3.16) with $a = \frac{1}{B_\mu B_\gamma}$ and (3.17) in (3.14) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2(t) \le\ & -\sigma\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2 + \frac{2B_\lambda}{b_\mu}\sum_{i\in\mathbb{Z}}\left(\sum_{j=i-N}^{i+N}\rho_k(|j|)\right)\beta_i^2\\
&+\frac{B_\mu B_\gamma}{b_\mu^2}\sum_{i\in\mathbb{Z}}\rho_k(|i|)I_i^2(\theta_t\omega),
\end{aligned}$$

where σ is as defined in (2.20). Integrating the above inequality from 0 to t then replacing ω by $\theta_{-t}\omega$ gives

$$\sum_{i\in\mathbb{Z}} \rho_k(|i|)u_i^2(t;\theta_{-t}\omega,\boldsymbol{u}_o) \le e^{-\sigma t}\sum_{i\in\mathbb{Z}} \rho_k(|i|)u_{o,i}^2(\theta_{-t}\omega) + \frac{2B_\lambda}{\sigma b_\mu}\sum_{i\in\mathbb{Z}}\left(\sum_{j=i-N}^{i+N}\rho_k(|j|)\right)\beta_i^2 + \frac{B_\mu B_\gamma}{b_\mu^2}\int_{-t}^{0}\sum_{i\in\mathbb{Z}}\rho_k(|i|)I_i^2(\theta_\tau\omega)e^{\sigma\tau}d\tau. \quad (2.37)$$

First notice that $\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_{o,i}^2(\theta_{-t}\omega) \le \|\boldsymbol{u}_o(\theta_{-t}\omega)\|^2$. Then for any $\epsilon>0$, there exists $\hat{T}(\epsilon,\omega)>0$ such that

$$e^{-\sigma t}\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_{o,i}^2(\theta_{-t}\omega) < \frac{\epsilon}{3}, \quad \forall\ t\ge\hat{T}(\epsilon,\omega). \quad (2.38)$$

Second, since $\boldsymbol{\beta}\in\ell^2$, given any $\epsilon>0$ there exists $N_1(\epsilon)>0$ such that

$$\sum_{|i|\ge N_1(\epsilon)}\beta_i^2 < \frac{\sigma b_\mu}{6B_\lambda(2N+1)}\epsilon.$$

Pick k such that $k>N_1(\epsilon)+N$. Then

$$\sum_{|j|\ge k}\beta_i^2 < \frac{\sigma b_\mu}{6B_\lambda(2N+1)}\epsilon, \quad \text{for each } j=i-N,\cdots,i,\cdots,i+N,$$

and consequently

$$\begin{aligned}\frac{2B_\lambda}{\sigma b_\mu}\sum_{i\in\mathbb{Z}}\left(\sum_{j=i-N}^{i+N}\rho_k(|j|)\right)\beta_i^2 &\le \frac{2B_\lambda}{\sigma b_\mu}\left(\sum_{|i-N|\ge k}\beta_i^2+\cdots+\sum_{|i+N|\ge k}\beta_i^2\right)\\ &\le \frac{2B_\lambda}{\sigma b_\mu}(2N+1)\cdot\frac{\sigma b_\mu}{6B_\lambda(2N+1)}\epsilon=\frac{\epsilon}{3}. \quad (2.39)\end{aligned}$$

Next by using Assumption (**A4**), for any $\epsilon>0$ there exists $N_2(\epsilon,\omega)>0$ such that

$$\sum_{i\in\mathbb{Z}}\rho_k(|i|)I_i^2(\theta_t\omega) \le \sum_{|i|\ge N_2} I_i^2(\theta_t\omega) < \frac{\sigma b_\mu^2}{3B_\mu B_\gamma}\epsilon, \quad \forall\, t\in\mathbb{R}.$$

Therefore

$$\frac{B_\mu B_\gamma}{b_\mu^2}\int_{-t}^{0}\sum_{i\in\mathbb{Z}}\rho_k(|i|)I_i^2(\theta_\tau\omega)e^{\sigma\tau} \le \frac{B_\mu B_\gamma}{b_\mu^2}\cdot\frac{\sigma b_\mu^2}{3B_\mu B_\gamma}\epsilon\cdot\int_{-t}^{0}e^{\sigma\tau}\mathrm{d}\tau < \frac{\epsilon}{3}. \tag{2.40}$$

In summary, letting $k := \max\{N_1(\epsilon)+N, N_2(\epsilon,\omega)\}$, and applying (3.19) – (2.40) to (3.18) we obtain

$$\sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2(t;\theta_{-t}\omega,\boldsymbol{u}_o) < \epsilon, \qquad \forall\ t\ge\hat{T}(\epsilon,\omega),$$

which implies that

$$\sum_{|i|\ge 2k}u_i^2(t,\theta_{-t}\omega,\boldsymbol{u}_o) \le \sum_{i\in\mathbb{Z}}\rho_k(|i|)u_i^2(t,\theta_{-t}\omega,\boldsymbol{u}_o) < \epsilon \quad \forall\, t\ge\hat{T}(\epsilon,\omega).$$

The proof is complete. □

Remark 2. *The function $\rho_k(|i|)$ above serves the same purpose as e.g., the function $\theta(|i|/k)$ in [9], the function $\rho(|i|/k)$ in [17]. The difference is that we do not need the smoothness of ρ_k in this work. Instead we need a sub-additivity property to handle the nonlinear interaction term. One example for ρ_k could be*

$$\rho_k(x) = \begin{cases} 0, & x<k, \\ \frac{x}{k}-1, & x\in[k,2k] \\ 1, & x\ge 2k, \end{cases} \qquad x\in\mathbb{R}.$$

Lemma 4. *Assume that assumptions (***A1***) - (***A5***) and* (2.19) – (2.20) *hold. Then the absorbing set $\Gamma(\omega)$ defined in* (2.29) *is asymptotically compact under the RDS $\varphi(t,\omega)$ defined by solutions of the RODE* (2.3)*.*

Proof. For a sequence $\{t_n\}$ with $\lim_{n\to\infty}t_n=\infty$, let $x_n(\omega)\in\Gamma(\theta_{-t_n}\omega)\in\mathscr{D}(\ell^2)$ and

$$p_n(\omega)=\varphi(t_n,\theta_{-t_n}\omega,x_n), \quad n=1,2,\cdots,$$

where $p_{n,i}=(\varphi(t_n,\theta_{-t_n}\omega,x_n))_i$ for $i\in\mathbb{Z}$.

First since $\lim_{n\to\infty} t_n = \infty$ there exists $N_1(\omega,\Gamma) \in \mathbb{N}$ such that $t_n \geq T_\Gamma(\omega)$ if $n \geq N_1(\omega,\Gamma)$ and hence

$$p_n(\omega) = \varphi(t_n, \theta_{-t_n}\omega, x_n) \in \Gamma(\omega), \quad \forall\ n \geq N_1(\omega,\Gamma).$$

Then there is a subsequence of $\{p_n\}$ (still denoted by $\{p_n\}$), and $\boldsymbol{u}^* \in \ell^2$ such that

$$p_n = \varphi(t_n, \theta_{-t_n}\omega, x_n) \rightharpoonup \boldsymbol{u}^* \text{ weakly in } \ \ell^2.$$

We now show that this convergence is actually strong. Given any $\epsilon > 0$, by Lemma 8 there exists $\iota_1(\epsilon,\omega) > 0$ and $N_2(\epsilon,\omega) > 0$ such that

$$\sum_{|i|\geq \iota(\epsilon,\omega)} \left|\left(\varphi(t_n, \theta_{-t_n}\omega, \boldsymbol{u}_0)\right)_i\right|^2 \leq \frac{\epsilon^2}{8}, \quad \forall\ n \geq N_2(\epsilon,\omega). \tag{2.41}$$

Also, since $\boldsymbol{u}^* = (u_i^*)_{i\in\mathbb{Z}} \in \ell^2$, there exists $\iota_2(\epsilon) > 0$ such that

$$\sum_{|i|\geq \iota_2(\epsilon)} |u_i^*|^2 \leq \frac{\epsilon^2}{8}. \tag{2.42}$$

Set $\iota(\epsilon,\omega) := \max\{\iota_1(\epsilon,\omega), \iota_2(\epsilon,\omega)\}$ and since $\varphi(t_n, \theta_{-t_n}\omega, x_n) \rightharpoonup \boldsymbol{u}^*$ in ℓ^2 then

$$\left(\varphi(t_n, \theta_{-t_n}\omega, x_n)\right)_i \longrightarrow u_i^* \ \text{ for } \ |i| \leq \iota(\epsilon,\omega), \quad \text{as } n \to \infty.$$

Therefore there exists $N_3(\epsilon,\omega) > 0$ such that

$$\sum_{|i|\leq \iota(\epsilon,\omega)} \left|\left(\varphi(t_n, \theta_{-t_n}\omega, x_n)\right)_i - u_i^*\right|^2 \leq \frac{\epsilon^2}{2}, \quad \forall\ n \geq N_3(\epsilon,\omega). \tag{2.43}$$

Set $\hat{N}(\epsilon,\omega) := \max\{N_1(\epsilon,\omega), N_2(\epsilon,\omega), N_3(\epsilon,\omega)\}$, then using $(3.21) - (3.23)$ we have for $n \geq \hat{N}(\epsilon,\omega)$

$$\begin{aligned}\|\varphi(t_n,\theta_{-t_n}\omega,x_n) - \boldsymbol{u}^*\|^2 &= \sum_{|i|\leq\iota(\epsilon,\omega)} |(\varphi(t_n,\theta_{-t_n}\omega,x_n))_i - u_i^*|^2 \\ &\quad + \sum_{|i|>\iota(\epsilon,\omega)} |(\varphi(t_n,\theta_{-t_n}\omega,x_n))_i - u_i^*|^2 \\ &\leq \frac{\epsilon^2}{2} + 2\sum_{|i|>\iota(\epsilon,\omega)} |(\varphi(t_n,\theta_{-t_n}\omega,x_n))_i|^2 + |u_i^*|^2 \leq \epsilon^2.\end{aligned}$$

Hence p_n (the subsequence) is strongly convergent in ℓ^2, and therefore $\Gamma(\omega)$ is asymptotically compact. The proof is complete. □

The following theorem follows directly from Lemma 6, Lemma 4 and Proposition 1.

Theorem 2.2. *Assume that assumptions (***A1***) - (***A5***) and* (2.19) – (2.20) *hold. The random dynamical system* $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ *generated by the RODE* (2.3) *possesses a unique global random attractor with component subsets*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq\hat{T}_\Gamma(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega,\Gamma(\theta_{-t}\omega))},$$

where Γ *is defined as in* (2.29)*.*

Corollary 1. *Assume that assumptions (***A1***) - (***A4***) hold. In addition, assume that all* g_j*'s are uniformly globally Lipschitz with Lipschitz constant* L_g*. Then the attractor* $\mathcal{A}(\omega)$ *for the RDS* $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ *is a singleton provided*

$$(2N+1)B_\lambda L_g < \frac{b_\mu}{2B_\mu B_\gamma}.$$

Proof. Let $\boldsymbol{u}(t;\omega,\boldsymbol{u}_o) = (u_i(t;\omega,\boldsymbol{u}_o))_{i\in\mathbb{Z}}$ and $\boldsymbol{v}(t;\omega,\boldsymbol{v}_o) = (v_i(t;\omega,\boldsymbol{u}_o))_{i\in\mathbb{Z}}$ be two solutions of the RODE (2.3). Since g_j's are uniformly globally Lipschitz,

$$|g_j(u_j) - g_j(v_j)| \leq L_g|u_j - v_j|.$$

Thanks to (2.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|^2 \leq \left(-\frac{2}{B_\mu B_\gamma}+\frac{4(2N+1)B_\lambda L_g}{b_\mu}\right)\cdot\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|^2, \quad \forall\, t\in\mathbb{R},$$

which implies that

$$\begin{aligned}\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|^2 &\leq e^{-(\frac{2}{B_\mu B_\gamma}-\frac{4(2N+1)B_\lambda L_g}{b_\mu})t}\|\boldsymbol{u}_o-\boldsymbol{v}_o\|^2\\ &\longrightarrow 0 \qquad \text{as } t\to\infty.\end{aligned}$$

The proof is complete. □

2.5 Closing remarks

This work was motivated by the increasing size of a Hopfield neural network under the influence of random forcing. In the context of Hopfield models, there are two novelties in this work: (1) the underlying system is an infinite dimensional system of ODEs, i.e., lattice dynamical system; and (2) the forcing on each neuron is described by a random process. In the context of lattice dynamical systems, the major novelty as well as difficulty is the consideration of a nonlinear interconnection structure described by the nonlinear operator $\sum_{j=i-N}^{i+N}\lambda_{i,j}g_j(u_j(t))$. Such systems have never been studied in the past. The main contribution of this work is showing the existence of a global random attractor for the random Hopfield lattice system. Along with the construction of the random absorbing set, another interesting discovery is that inhibit relation among neurons contribute more to the convergence of solutions towards the attractor.

We also discussed special scenarios under which the attractor consists of a singleton solution. Further details of geometric structure of the attractor provide more insight of the long term dynamics of the underlying model, but are not included in this work. Nevertheless the results obtained in this work serve as an important initial step towards the study on more interesting dynamical behavior of neural networks, such as upper semi continuity of the attractors and synchronization behavior of discrete neural networks under random perturbation.

Chapter 3

Upper semi-continuous convergence of attractors for a Hopfield-type lattice model

3.1 Introduction

Extensive studies have been done on Hopfield-type neural networks during the past decades. For example, Hopfield neural networks with delays were studied in [4], [5], [14], [54], [63], [60], [65], [68], [72], etc. Complex-valued Hopfield neural networks have been used to study a wide range of applications such as image storage (see e.g., [51], [59]), associative memory (see, e.g., [27], [50], [57], [70]), land cover detection (see, e.g., [69]), agriculture (see, e.g., [61]), neurohydrodynamics and Hydraulic (see, e.g., [45], [71]). Memoristic Hopfield neural network was considered in [58], and also used in harmonic current estimation [19] and self-modeling [73]. Other modifications of Hopfield neural network include Octonionic Hopfield neural network [26], Hyperbolic Hopfield neural network [52], random reaction-diffusion Hopfield neural network [53], etc. When the size of an underlying neural network becomes increasingly large, and approaching infinity, the finite dimensional system (1.1) can be extended to the following lattice dynamical system (see [44]):

$$\mu_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j(t)) + g_i, \qquad i \in \mathbb{Z}. \tag{3.1}$$

In this lattice model, the total number of neurons are assumed to be infinitely large, but each neuron is only interacting with the neurons within its $\mathfrak{n}$-neighborhood. More precisely, the ith neuron interacts with the $(i-\mathfrak{n})$th through the $(i+\mathfrak{n})$th neurons. Notice that the operator describing interactions among neurons, $\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j)$, is nonlinear when f_j is nonlinear.

This makes the lattice system (4.4) different from most of the lattice systems considered in the literature (see, e.g., [9], [20], [21], [62], [64], [66], [67], [75] and references therein). Very recently, Han & Kloeden studied an Amari type neural field lattice model with nonlocal interactions and a Heaviside operator in [42], and Faye investigated the traveling fronts for a simplified version of the model considered in [42] with linear reaction in the absence of external input [33].

The goal of this work is to study the long term dynamics of the lattice system (4.4) and compare to the dynamics of its finite dimensional approximations. In particular, we want to investigate the relation between the attractor for the lattice system (4.4) and the attractors for its finite dimensional approximations. This will be done by proving the upper semi-continuous convergence of the attractor for the finite dimensional approximations to the attractor for the infinite dimensional lattice system. Upper semi-continuous convergence of attractors have been investigated for different types of lattice dynamical systems (see, e.g., [1], [2], [9], [36], [74]), infinite dimensional dynamical systems generated by partial differential equations (see, e.g., [7], [8], [16], [38] and reference therein), and in numerical dynamics (see, e.g., [40]). However, all existing studies on upper semi-continuity of attractors considered lattice systems with linear interactions within the nearest neighborhood such as $u_{i-1} - 2u_i + u_{i+1}$. This work is the first attempt on upper semi-continuous convergence of attractors for lattice dynamical systems with nonlinear finite neighborhood operators, and in particular the first result on upper semi-continuous convergence of attractors for lattice dynamical systems arising from neural network models.

The rest of the chapter is organize as follows. In Section 3.2 we first reformulate the lattice system (4.4) as an ordinary differential equation (ODE) on a sequence space, and then study the existence and uniqueness of global solutions, as well as the existence of a global attractor for the resulting ODE. In Section 3.3 we propose three different finite dimensional approximations of the lattice system (4.4), and show that each of them possesses a global attractor that converges upper semi-continuously to the global attractor for the lattice system (4.4) as the dimension approaches infinity. Some closing remarks will be provided in Section 3.4.

3.2 Properties of solutions

In this section we first establish the existence and uniqueness of solutions to the lattice system (4.4), and show that the solutions generate a dynamical system. We then prove that the resulting dynamical system possesses a global attractor. To this end, we first rewrite the lattice system (4.4) as an ordinary differential equation in an appropriate state space. In particular, consider the separable Hilbert space of square summable bi-infinite real-valued sequences:

$$\ell^2 = \left\{ \boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} u_i^2 < \infty \right\}$$

equipped with the inner product and the norm

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \|\boldsymbol{u}\| = \left(\sum_{i\in\mathbb{Z}} u_i^2 \right)^{\frac{1}{2}}, \qquad \forall\ \boldsymbol{u} = (u_i)_{i\in\mathbb{Z}},\ \boldsymbol{v} = (v_i)_{i\in\mathbb{Z}}.$$

3.2.1 Existence and uniqueness of solutions

Throughout this paper it is assumed that

(**A1**) the connection strength between each pair of neurons is finite, i.e., there exists $M_\lambda > 0$ such that

$$\sup_{i,j\in\mathbb{Z}} |\lambda_{i,j}| \le M_\lambda;$$

(**A2**) the neuron amplifier input capacitance and resistance of neurons are uniformly bounded, i.e., there exist positive constants m_μ, M_μ, m_γ, and M_γ, such that

$$m_\mu \le \mu_i \le M_\mu, \quad m_\gamma \le \gamma_i \le M_\gamma, \qquad \forall\ i \in \mathbb{Z};$$

(**A3**) the forcing on the whole network is square summable, i.e., letting $\boldsymbol{g} := (g_i)_{i\in\mathbb{Z}}$ then

$$\|\boldsymbol{g}\| = \left(\sum_{i\in\mathbb{Z}} g_i^2 \right)^{1/2} < \infty;$$

(**A4**) for all $i \in \mathbb{Z}$, the neuron activation function f_i is globally Lipschitz with Lipschitz constant L and satisfies $f_i(0) = 0$.

Given any $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}}$, define the operators $A\boldsymbol{u} = ((A\boldsymbol{u})_i)_{i\in\mathbb{Z}}$, $\Lambda\boldsymbol{u} = ((\Lambda\boldsymbol{u})_i)_{i\in\mathbb{Z}}$ by

$$(A\boldsymbol{u})_i = \frac{u_i}{\gamma_i\mu_i}; \qquad (\Lambda\boldsymbol{u})_i = \sum_{j=i+\mathfrak{n}}^{i+\mathfrak{n}} \frac{\lambda_{i,j}}{\mu_i} f_j(u_j),$$

and define the operator F to be

$$F(\boldsymbol{u}) = -A\boldsymbol{u} + \Lambda\boldsymbol{u} + \left(\frac{g_i}{\mu_i}\right)_{i\in\mathbb{Z}}. \tag{3.2}$$

Lemma 5. *Let the assumptions (**A1**) – (**A4**) hold. Then the operator $F : \ell^2 \to \ell^2$ is globally Lipschitz with*

$$\|F(\boldsymbol{u}) - F(\boldsymbol{v})\| \le \frac{\sqrt{2}}{m_\mu}\left(\frac{1}{m_\gamma^2} + M_\lambda^2(2\mathfrak{n}+1)^2L^2\right)^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|, \quad \forall\ \boldsymbol{u},\boldsymbol{v}\in\ell^2.$$

Proof. First given any $\boldsymbol{u}, \boldsymbol{v} \in \ell^2$, we have

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|^2 = \sum_{i\in\mathbb{Z}} \frac{1}{\mu_i^2\gamma_i^2}(u_i - v_i)^2 \le \frac{1}{m_\mu^2 m_\gamma^2}\|\boldsymbol{u}-\boldsymbol{v}\|^2. \tag{3.3}$$

Second, by assumptions (**A2**) and (**A4**) we have

$$\begin{aligned}
\|\Lambda\boldsymbol{u} - \Lambda\boldsymbol{v}\|^2 &= \sum_{i\in\mathbb{Z}} \frac{1}{\mu_i^2}\left(\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j}\left[f_j(u_j) - f_j(v_j)\right]\right)^2 \\
&\le \frac{M_\lambda^2}{m_\mu^2}\sum_{i\in\mathbb{Z}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\left[f_j(u_j) - f_j(v_j)\right]^2 \\
&\le \frac{M_\lambda^2}{m_\mu^2}\sum_{i\in\mathbb{Z}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} L^2|u_j - v_j|^2 = \frac{M_\lambda^2}{m_\mu^2}(2\mathfrak{n}+1)^2L^2\|\boldsymbol{u}-\boldsymbol{v}\|^2.
\end{aligned} \tag{3.4}$$

Hence by summing (3.3) and (3.4) we obtain

$$\begin{aligned}\|F(\boldsymbol{u})-F(\boldsymbol{v})\|^2 &\le 2\|A\boldsymbol{u}-A\boldsymbol{v}\|^2+2\|\Lambda\boldsymbol{u}-\Lambda\boldsymbol{v}\|^2\\ &\le \frac{2}{m_\mu^2}\left(\frac{1}{m_\gamma^2}+M_\lambda^2(2\mathfrak{n}+1)^2L^2\right)\|\boldsymbol{u}-\boldsymbol{v}\|^2,\end{aligned}$$

which implies that $F(\boldsymbol{u})$ is glocally Lipschitz in $\boldsymbol{u}$.

Setting $\boldsymbol{v}=\mathbf{0}$ in (3.3) and (3.4) and using $f_i(0)=0$ gives immediately

$$\begin{aligned}\|F(\boldsymbol{u})\| &\le \|A\boldsymbol{u}\|+\|\Lambda\boldsymbol{u}\|+\frac{1}{m_\mu}\left(\sum_{i\in\mathbb{Z}}g_i^2\right)^{1/2}\\ &\le \frac{1}{m_\mu}\left(\frac{1}{m_\gamma}+(2\mathfrak{n}+1)M_\lambda L\right)\|\boldsymbol{u}\|+\frac{1}{m_\mu}\|\boldsymbol{g}\|,\end{aligned}$$

which implies that F maps ℓ^2 to ℓ^2. □

The above Lemma ensures that the lattice system (4.4) can be rewritten as the following ODE on ℓ^2:

$$\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t}=F(\boldsymbol{u}),\quad \boldsymbol{u}\in\ell^2. \tag{3.5}$$

The following theorem follows directly from Lemma 5 and standard theory of ODEs on Banach spaces (see, e.g., [28]).

Theorem 3.1. *Let assumptions (**A1**) – (**A4**) hold. Then for any initial data $\boldsymbol{u}_o=(u_{o,i})_{i\in\mathbb{Z}}\in\ell^2$, Equation* (3.5) *has a unique solution $\boldsymbol{u}(\cdot;\boldsymbol{u}_o)\in\mathcal{C}([0,\infty),\ell^2)$ with $\boldsymbol{u}(0;\boldsymbol{u}_o)=\boldsymbol{u}_o$. In addition, the solution $\boldsymbol{u}(\cdot;\boldsymbol{u}_o)$ is continuous in $\boldsymbol{u}_o\in\ell^2$.*

Remark 3. *The global Lipschitz condition in Assumption (**A4**) is stronger than needed. It can be replaced by a local Lipschitz condition such as $\sum_{i\in\mathbb{Z}}\max_{s\in[-r,r]}|f_i'(s)|\le L(r)$ for some positive continuous increasing function $L(r)\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$ satisfying some growth bound. Since the main aim of this work is to study attractors, assuming $L(r)\equiv L$ is not a big restriction, as eventually we could restrict all calculations to a closed and bounded absorbing set.*

3.2.2 Existence of attractors

Due to Theorem 3.1, the solution of Equation (3.5) defines a dynamical system $\{\Phi(t)\}_{t\geq 0}$ that maps ℓ^2 to ℓ^2 by

$$\Phi(t)\boldsymbol{u}_o = \boldsymbol{u}(t;\boldsymbol{u}_o), \qquad \forall\ t \geq 0,\ \ \boldsymbol{u}_o \in \ell^2.$$

In this section we investigate the existence of attractors for the dynamical system $\{\Phi(t)\}_{t\geq 0}$ generated by the solution of (3.5). To this end, we first recall the definition of global attractors as well as a theorem on the existence of attractors below (see, e.g., [37]).

Definition 5. *Assume that $\mathcal{A}$ is a subset of metric space $\mathfrak{X}$ and $\{\Phi(t)\}_{t\geq 0}$ is a semigroup of a continuous operators in $\mathfrak{X}$. Then $\mathcal{A}$ is called a global attractor for $\{\Phi(t)\}_{t\geq 0}$ in $\mathfrak{X}$ if $\mathcal{A}$ is a compact invariant set which attracts every bounded set in $\mathfrak{X}$, that is*

(i) $\mathcal{A}$ is compact;

(ii) $\mathcal{A}$ is invariant: $\Phi(t)\mathcal{A} = \mathcal{A}$;

(iii) $\mathcal{A}$ attracts every bounded set, i.e., for any bounded set $B \subset \mathfrak{X}$,

$$dist_{\mathfrak{X}}\left(\Phi(t)B, \mathcal{A}\right)) = \sup_{b\in B}\inf_{a\in\mathcal{A}} d_{\mathfrak{X}}(\Phi(t)b, a) \to 0 \ \ as \ \ \ t \to \infty$$

where $d_{\mathfrak{X}}$ denotes the metric of $\mathfrak{X}$.

The existence of attractors for a dynamical system $\{\Phi(t)\}_{t\geq 0}$ relies on the existence of a closed absorbing set and the asymptotic compactness of the dynamical system, defined below.

Definition 6. *A dynamical system $\{\Phi(t)\}_{t\geq 0}$ is asymptotically compact if $\{\Phi(t_m)\boldsymbol{u}^m\}$ is pre-compact in $\mathfrak{X}$ for every bounded sequence $\{\boldsymbol{u}^m\}_{m\in\mathbb{N}} \subset \mathfrak{X}$ and $t_m \to \infty$.*

Remark that if the global attractor exists, then it must be unique and minimal. The following proposition (see e.g., [8]) will be used in the sequel.

Proposition 2. *Let ($\mathfrak{X}$, $d_{\mathfrak{X}}$) be a metric space and $\{\Phi(t)\}_{t\geq 0}$ be a continuous dynamical system in $\mathfrak{X}$. If $\{\Phi(t)\}_{t\geq 0}$ has a bounded absorbing set and is asymptotically compact, then $\{\Phi(t)\}_{t\geq 0}$*

has a global attractor in $\mathfrak{X}$. Furthermore, if for each $u \in \mathfrak{X}$, $\Phi(t)u$ is continuous from $(0, +\infty)$ to $\mathfrak{X}$, then the global attractor is connected.

We next prove the existence of an absorbing set for the continuous dynamical system $\{\Phi(t)\}_{t\geq 0}$ defined by the solution of (3.5).

Lemma 6. *Assume that assumptions (**A1**) - (**A4**) hold. Then the continuous dynamical system $\{\Phi(t)\}_{t\geq 0}$ generated by solutions to the system (3.5) has a positive invariant bounded absorbing set provided*

(**A5**) $\sigma := \frac{1}{M_\mu M_\gamma} - 2(2\mathfrak{n}+1)\frac{LM_\lambda}{m_\mu} > 0.$

Proof. First multiplying both sides of the equation (4.4) by $u_i(t)$ and summing over all $i \in \mathbb{Z}$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}\,\|\boldsymbol{u}(t)\|^2}{\mathrm{d}t} = -\sum_{i\in\mathbb{Z}}\frac{u_i^2}{\mu_i\gamma_i} + \sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\lambda_{i,j}f_j(u_j) + \sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}g_i. \tag{3.6}$$

By Assumption (**A2**), we have

$$-\sum_{i\in\mathbb{Z}}\frac{u_i^2}{\mu_i\gamma_i} \leq -\frac{\|\boldsymbol{u}\|^2}{M_\mu M_\gamma}. \tag{3.7}$$

Using the fact that $xy \leq \frac{1}{2}(ax^2 + \frac{1}{a}y^2)$, for some $a > 0$ (to be determined later),

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}g_i &\leq \sum_{i\in\mathbb{Z}}\frac{1}{2}\left(au_i^2 + \frac{g_i^2}{a\mu_i^2}\right) \leq \frac{a}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2a}\sum_{i\in\mathbb{Z}}\frac{g_i^2}{\mu_i^2}\\ &\leq \frac{a}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2am_\mu^2}\|\boldsymbol{g}\|^2.\end{aligned} \tag{3.8}$$

Then by Assumption (**A4**) we have

$$\lambda_{i,j}u_if_j(u_j) \leq M_\lambda L|u_i||u_j| \leq \frac{LM_\lambda}{2}(u_i^2 + u_j^2), \tag{3.9}$$

which after a shift of index implies that

$$\sum_{i\in\mathbb{Z}}\frac{u_i}{\mu_i}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\lambda_{i,j}f_j(u_j) \leq \frac{LM_\lambda}{m_\mu}(2\mathfrak{n}+1)\|\boldsymbol{u}\|^2. \tag{3.10}$$

Now inserting the inequality (3.7), the inequality (3.8) with $a = \frac{1}{M_\mu M_\gamma}$, and the inequality (3.10) into (3.6) results in

$$\frac{\mathrm{d}\|\boldsymbol{u}(t)\|^2}{\mathrm{d}t} \leq -\sigma\|\boldsymbol{u}(t)\|^2 + \frac{M_\mu M_\gamma}{m_\mu^2}\|\boldsymbol{g}\|^2, \tag{3.11}$$

where σ is defined as in (**A5**). Integrating the above differential inequality from 0 to t gives

$$\begin{aligned}\|\Phi(t)\boldsymbol{u}_o\|^2 &\leq e^{-\sigma t}\|\boldsymbol{u}_o\|^2 + \frac{M_\mu M_\gamma}{m_\mu^2}\|\boldsymbol{g}\|^2 \int_0^t e^{-\sigma(t-s)}\mathrm{d}s \\ &\leq e^{-\sigma t}\|\boldsymbol{u}_o\|^2 + \frac{M_\mu M_\gamma}{\sigma m_\mu^2}\|\boldsymbol{g}\|^2(1-e^{-\sigma t}).\end{aligned}$$

For any $\epsilon > 0$, define $\mathcal{K}_\epsilon$ to be

$$\mathcal{K}_\epsilon := \left\{\boldsymbol{u} \in \ell^2 : \|\boldsymbol{u}\| \leq \frac{\sqrt{M_\mu M_\gamma}}{\sqrt{\sigma}}\frac{\|\boldsymbol{g}\|^2}{m_\mu} + \epsilon\right\}. \tag{3.12}$$

Then given any bounded set $B \in \ell^2$, there exists $T_B = T_B(\epsilon)$ such that

$$\Phi(t)B \subset \mathcal{K}_\epsilon, \quad \forall\ t \geq T_B,$$

i.e., $\mathcal{K}_\epsilon$ is a bounded absorbing set for the dynamical system $\{\Phi(t)\}_{t\geq 0}$. As a particular case, there exists $T_{\mathcal{K}} = T_{\mathcal{K}}(\epsilon)$ such that

$$\Phi(t)\mathcal{K}_\epsilon \subset \mathcal{K}_\epsilon, \qquad \forall\ t > T_{\mathcal{K}}(\epsilon),$$

i.e., $\mathcal{K}_\epsilon$ is positive invariant.

□

The asymptotic compactness of $\{\Phi(t)\}_{t\geq 0}$ will be proved by using a tail estimate of solutions (see, e.g., [9]) to be presented in Lemma 8 below. To that end, for any given positive

integer k let $\theta_k : \mathbb{Z}_+ \to [0,1]$ be a continuous, increasing and sub-additive function such that

$$\theta_k(|i|) \in \begin{cases} \{0\}, & |i| < k, \\ [0,1], & |i| \in [k, 2k] \\ \{1\}, & |i| \geq 2k, \end{cases} \qquad i \in \mathbb{Z}. \tag{3.13}$$

Lemma 7. *Let θ_k be a function defined as in* (3.13)*. Then for any $k \geq \mathfrak{n}$ it holds that*

$$\theta_k(|j|) = \theta_k(|i|), \qquad \forall\ j = i - \mathfrak{n}, \cdots, i, \cdots, \mathfrak{n}.$$

Proof. First notice that since for any $j = i - \mathfrak{n}, \cdots, i + \mathfrak{n}$, $|i - j| \leq \mathfrak{n}$, thus $\theta_k(|i - j|) = 0$. Then using the increasingness and sub-additivity of θ_k we obtain

$$\begin{aligned} \theta_k(|i|) &\leq \theta_k(|i - j| + |j|) \leq \theta_k(|i - j|) + \theta_k(|j|) = \theta_k(|j|), \\ \theta_k(|j|) &\leq \theta_k(|i| + |j - i|) \leq \theta_k(|i|) + \theta_k(|j - i|) = \theta_k(|i|), \end{aligned}$$

i.e., $\theta_k(|j|) = \theta_k(|i|)$. □

Remark 4. *The function $\theta_k(|i|)$ defined above serves the same purpose as e.g., the function $\theta(|i|/k)$ in [9] and the function $\rho(|i|/k)$ in [17]. The difference is that we do not need the smoothness of θ_k here and instead we impose a sub-additivity property of θ_k to handle the nonlinear interaction term. One example for θ_k could be*

$$\theta_k(x) = \begin{cases} 0, & x < k, \\ \frac{x}{k} - 1, & x \in [k, 2k] \\ 1, & x \geq 2k, \end{cases} \qquad x \in \mathbb{R}_+.$$

Lemma 8. *Let the assumptions (**A1**) - (**A5**) hold. Then for any $\epsilon > 0$ there exist $T_\epsilon = T(\epsilon, \mathcal{K}_\epsilon) \geq T_{\mathcal{K}}(\epsilon)$ and $I_\epsilon \in \mathbb{N}$ such that the solution $\Phi(t)\boldsymbol{u}_o = \boldsymbol{u}(t; \boldsymbol{u}_o) = (u_i(t; \boldsymbol{u}_o))_{i \in \mathbb{Z}}$ of*

the equation (4.4) *with* $\boldsymbol{u}_o = (u_{o,i})_{i\in\mathbb{Z}} \in \mathcal{K}_\epsilon$ *satisfies*

$$\sum_{|i|\geq I_\epsilon} |(\Phi(t)\boldsymbol{u}_o)_i|^2 = \sum_{|i|\geq I_\epsilon} |u_i(t;\boldsymbol{u}_o)|^2 \leq \epsilon \qquad \forall\, t \geq T_\epsilon.$$

Proof. The proof uses the same idea as in [9], with more complex calculations due to the nonlinear structure of the operator Λ.

For any solution $\boldsymbol{u}(t;\boldsymbol{u}_o) = (u_i(t;\boldsymbol{u}_o))_{i\in\mathbb{Z}}$ of (4.4), let $v_i := \theta_k(|i|)u_i$ where θ_k is the continuous, increasing and sub-additive function as defined in (3.13) with $k \geq \mathfrak{n}$. Then $\boldsymbol{v} := (v_i)_{i\in\mathbb{Z}} \in \ell^2$. Taking the inner product of $\boldsymbol{v}$ with (4.4) we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2(t;\boldsymbol{u}_o) &= -\sum_{i\in\mathbb{Z}}\theta_k(|i|)\frac{u_i^2}{\mu_i\gamma_i} + \sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i\frac{1}{\mu_i}g_i \\ &\quad + \sum_{i\in\mathbb{Z}}\frac{1}{\mu_i}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\lambda_{i,j}\theta_k(|i|)u_i f_j(u_j). \end{aligned} \tag{3.14}$$

First by Assumption (**A2**),

$$-\sum_{i\in\mathbb{Z}}\theta_k(|i|)\frac{u_i^2}{\mu_i\gamma_i} \leq -\frac{1}{M_\mu M_\gamma}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2. \tag{3.15}$$

Second, similar to (3.8) there exists some $a > 0$ such that

$$\sum_{i\in\mathbb{Z}}\theta_k(|i|)\frac{u_i}{\mu_i}g_i \leq \frac{a}{2}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2 + \frac{1}{2am_\mu^2}\sum_{i\in\mathbb{Z}}\theta_k(|i|)g_i^2. \tag{3.16}$$

It remains to estimate the last term of (3.14). To this end, multiplying (3.9) by $\frac{1}{\mu_i}\theta_k(|i|)$ and summing over $i \in \mathbb{Z}$ to get

$$\sum_{i\in\mathbb{Z}}\theta_k(|i|)\frac{1}{\mu_i}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\lambda_{i,j}u_i f_j(u_j) \leq \frac{LM_\lambda}{2m_\mu}\sum_{i\in\mathbb{Z}}\theta_k(|i|)\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}(u_i^2+u_j^2).$$

Notice that by a shift of index and Lemma 7,

$$\sum_{i\in\mathbb{Z}}\theta_k(|i|)\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}u_j^2 = \sum_{i\in\mathbb{Z}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}}\theta_k(|j|)u_i^2 = (2\mathfrak{n}+1)\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2. \tag{3.17}$$

Inserting estimates (3.15), (3.16) with $a = \frac{1}{M_\mu M_\gamma}$, and (3.17) in (3.14) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2(t) \le -\sigma\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2 + \frac{M_\mu M_\gamma}{m_\mu^2}\sum_{i\in\mathbb{Z}}\theta_k(|i|)g_i^2,$$

where σ is as defined in (**A5**). Integrating the above inequality from 0 to t and using $\int_0^t e^{-\sigma(t-s)}\mathrm{d}s \le \frac{1}{\sigma}$ gives

$$\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2(t) \le e^{-\sigma t}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_{o,i}^2 + \frac{M_\mu M_\gamma}{\sigma m_\mu^2}\sum_{i\in\mathbb{Z}}\theta_k(|i|)g_i^2. \tag{3.18}$$

First notice that $\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_{o,i}^2 \le \|\boldsymbol{u}_o\|^2$. Then for any $\epsilon > 0$ and $\boldsymbol{u}_o \in \mathcal{K}_\epsilon$, there exists $T_\epsilon > 0$ such that

$$e^{-\sigma t}\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_{o,i}^2 < \frac{\epsilon}{2}, \qquad \forall\ t \ge T_\epsilon. \tag{3.19}$$

Second, since $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}} \in \ell^2$, for any $\epsilon > 0$ there exists $I(\epsilon) > 0$ such that

$$\sum_{|i|\ge I(\epsilon)} g_i^2 < \frac{\sigma m_\mu^2}{2M_\mu M_\gamma}\epsilon. \tag{3.20}$$

Therefore letting $k = k(\epsilon) = \max\{I(\epsilon), \mathfrak{n}\}$ and applying (3.19) and (3.20) to (3.18) we obtain

$$\sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2(t;\boldsymbol{u}_o) < \epsilon, \qquad \forall\ t \ge T_\epsilon.$$

This implies that with $I_\epsilon = 2k$,

$$\sum_{|i|\ge I_\epsilon} u_i^2(t,\boldsymbol{u}_o) \le \sum_{i\in\mathbb{Z}}\theta_k(|i|)u_i^2(t,\boldsymbol{u}_o) < \epsilon, \qquad \forall\, t \ge T_\epsilon.$$

The proof is complete. □

Theorem 3.2. *Let the assumptions (**A1**) - (**A5**) hold. Then the dynamical system $\{\Phi(t)\}_{t\ge0}$ generated by solutions to the system* (3.5) *possesses a global attractor $\mathcal{A}$ which is minimal and connected.*

Proof. Note that we have already established the existence of a bounded absorbing set for $\{\Phi(t)\}_{t\ge0}$ in Lemma 6, it remains to prove that the absorbing set $\mathcal{K}_\epsilon$ defined in (3.12) is

asymptotically compact under $\{\Phi(t)\}_{t\geq 0}$. To this end consider a sequence of time $\{t_m\}$ with $\lim_{m\to\infty} t_m = \infty$ and let $\{\boldsymbol{x}^m\}$ be a sequence in $\mathcal{K}_\epsilon$. For $m = 1, 2, \cdots$, let

$$\boldsymbol{p}^m = (p_i^m)_{i\in\mathbb{Z}} \quad \text{where} \quad p_i^m = (\Phi(t_m)\boldsymbol{x}^m)_i \quad \text{for } i \in \mathbb{Z}.$$

First since $\lim_{m\to\infty} t_m = \infty$ there exists $M_1(\mathcal{K}_\epsilon) \in \mathbb{N}$ such that $t_m \geq T_{\mathcal{K}}(\epsilon)$ when $m \geq M_1(\mathcal{K}_\epsilon)$ and hence

$$\boldsymbol{p}^m = \Phi(t_m)x_m \in \mathcal{K}_\epsilon, \quad \forall\ m \geq M_1(\mathcal{K}_\epsilon).$$

Then there is a subsequence of $\{\boldsymbol{p}^m\}$ (still denoted by $\{\boldsymbol{p}_m\}$), and $\boldsymbol{u}^* \in \ell^2$ such that

$$\boldsymbol{p}^m = \Phi(t_m)\boldsymbol{x}^m \rightharpoonup \boldsymbol{p}^* \text{ weakly in } \ \ell^2.$$

We now show that this convergence is actually strong. Given any $\epsilon > 0$, by Lemma 8 there exists $I_1(\epsilon) > 0$ and $M_2(\epsilon) > 0$ such that

$$\sum_{|i|\geq I_1(\epsilon)} |(\Phi(t_m)\boldsymbol{x}^m)_i|^2 \leq \frac{\epsilon^2}{8}, \quad \forall\ m \geq M_2(\epsilon). \tag{3.21}$$

Also, since $\boldsymbol{p}^* = (p_i^*)_{i\in\mathbb{Z}} \in \ell^2$, there exists $I_2(\epsilon) > 0$ such that

$$\sum_{|i|\geq I_2(\epsilon)} |p_i^*|^2 \leq \frac{\epsilon^2}{8}. \tag{3.22}$$

Set $I(\epsilon) := \max\{I_1(\epsilon), I_2(\epsilon)\}$. Since $\Phi(t_m)\boldsymbol{x}^m \rightharpoonup \boldsymbol{p}^*$ in ℓ^2 then

$$(\Phi(t_m)\boldsymbol{x}^m)_i \longrightarrow p_i^* \ \text{ for } \ |i| \leq I(\epsilon), \quad \text{as } m \to \infty.$$

Therefore there exists $M_3(\epsilon) > 0$ such that

$$\sum_{|i|\leq I(\epsilon)} |\,(\Phi(t_m)\boldsymbol{x}^m)_i - p_i^*|^2 \leq \frac{\epsilon^2}{2}, \quad \forall\ m \geq M_3(\epsilon). \tag{3.23}$$

Set $M(\epsilon) := \max\{M_1(\epsilon), M_2(\epsilon), M_3(\epsilon)\}$, then by using $(3.21) - (3.23)$ we have

$$\begin{aligned}\|\Phi(t_m)\mathbf{x}^m - \boldsymbol{p}^*\|^2 &= \sum_{|i|\le I(\epsilon)} |(\Phi(t_m)\boldsymbol{x}^m)_i - p_i^*|^2 + \sum_{|i|>I(\epsilon)} |(\Phi(t_m)\boldsymbol{x}^m)_i - p_i^*|^2 \\ &\le \frac{\epsilon^2}{2} + 2\sum_{|i|>I(\epsilon)} |(\Phi(t_m)\boldsymbol{x}^m)_i|^2 + |p_i^*|^2 \le \epsilon^2 \quad \forall\ m \ge M(\epsilon).\end{aligned}$$

Hence $\boldsymbol{p}^m$ (the subsequence) converges strongly in ℓ^2, and therefore $\mathcal{K}_\epsilon$ is asymptotically compact. The desired assertion follows immediately from Proposition 2. □

The invariance $\Phi(t)\mathcal{A} = \mathcal{A}$ of the global attractor $\mathcal{A}$ holds for all $t \ge 0$. In fact $\mathcal{A}$ consists of all bounded solutions defined on the whole $\mathbb{R}$, i.e., the bounded entire solutions. The next theorem shows the backward uniqueness of solutions to (3.5) on $\mathcal{A}$, and hence $\Phi(t)$ is indeed a group on $\mathcal{A}$.

Theorem 3.3. *Let the assumptions (**A1**) - (**A4**) hold. For any two solutions $\boldsymbol{u}(t) = \Phi(t)\boldsymbol{u}_o$ and $\boldsymbol{v}(t) = \Phi(t)\boldsymbol{v}_o$ of* (3.5)*, if $\boldsymbol{u}(T) = \boldsymbol{v}(T)$ for some $T > 0$, then $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for all $t \in [0, T]$.*

Proof. Let $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$ and (for the sake of contradiction) suppose there exists $t_0 \in [0, T)$ such that $\boldsymbol{w}(t_0) \ne 0$. Then by the continuity of $\boldsymbol{w}$, there exists $t_1 \in [t_0, T)$ such that $\|\boldsymbol{w}(t)\| \ne 0$ for all $t \in [t_0, t_1)$ and

$$0 < \|\boldsymbol{w}(t_1)\| < \epsilon \quad \text{for any } \epsilon > 0. \tag{3.24}$$

Write $\boldsymbol{w} = (w_i)_{i\in\mathbb{Z}}$ then $\boldsymbol{w}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} w_i(t) = -\frac{w_i}{\gamma_i\mu_i} + \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \frac{\lambda_{i,j}}{\mu_i}\left(f_j(u_j) - f_j(v_j)\right).$$

Multiplying the above equation by $-w_i$ and summing over $i \in \mathbb{Z}$ we get

$$\begin{aligned}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}(t)\|^2 &= \sum_{i\in\mathbb{Z}} \frac{w_i^2}{\gamma_i\mu_i} - \sum_{i\in\mathbb{Z}} \frac{w_i}{\mu_i} \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j}\left(f_j(v_j) - f_j(u_j)\right) \\ &\le \left(\frac{1}{M_\gamma M_\mu} + \frac{M_\lambda L}{m_\mu}(2\mathfrak{n}+1)\right)\|\boldsymbol{w}\|^2 := \hat{\sigma}\|\boldsymbol{w}\|^2, \quad \forall\ t \in [0, T)\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{1}{\|\boldsymbol{w}(t)\|}\right) \leq \hat{\sigma}, \quad \forall\ t \in [0,T) \text{ and } \boldsymbol{w}(t) \neq 0.$$

Integrating the above differential inequality from t_0 to t_1 results in

$$\ln\left(\frac{1}{\|\boldsymbol{w}(t_1)\|}\right) \leq \ln\left(\frac{1}{\|\boldsymbol{w}(t_0)\|}\right) + \hat{\sigma}(t_1 - t_0) \leq \ln\left(\frac{1}{\|w(t_0)\|}\right) + \hat{\sigma}T. \tag{3.25}$$

On the other side, due to (3.24), $\ln\left(\frac{1}{\|\boldsymbol{w}(t_1)\|}\right) > \ln\frac{1}{\epsilon}$. Now picking $\epsilon = \|\boldsymbol{w}(t_0)\|/e^{2\hat{\sigma}T}$, then

$$\ln\left(\frac{1}{\|\boldsymbol{w}(t_1)\|}\right) > \ln\frac{1}{\epsilon} = \ln\left(\frac{1}{\|w(t_0)\|}\right) + 2\hat{\sigma}T,$$

which contradicts with (3.25). Thus $\boldsymbol{w}(t) = 0$ for all $t \in [t_0, T)$.

□

3.2.3 Special structure of the global attractor

In addition to the existence of global attractors, the structure of global attractors is always of interest. For example, traveling wave solutions for lattice dynamical systems have been studied in [3, 10, 22, 23, 55, 76], amongst others. Very recently, Faye investigated the traveling fronts for a neural lattice system with linear reaction in the absence of external input [33]. While traveling wave solutions are of particular importance for neural networks, they are generally challenging to study, especially for a complex lattice system such as (4.4). In what follows we present a special case where the global attractor of the lattice system (4.4) consists of a unique steady state.

Lemma 9. *Let Assumptions (***A1***) – (***A4***) hold. Then the ODE* (3.5) *on ℓ^2 has a unique steady state $\boldsymbol{u}^* \in \ell^2$ provided*

$$(2\mathfrak{n}+1)M_\gamma^2 M_\lambda^2 L^2 < 1. \tag{3.26}$$

Proof. Define a mapping $Z : \ell^2 \to \ell^2$ such that for every $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $Z\boldsymbol{u} = ((Z\boldsymbol{u})_i)_{i\in\mathbb{Z}}$ where

$$(Z\boldsymbol{u})_i = \gamma_i \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j(t)) + \gamma_i g_i, \quad i \in \mathbb{Z}, \quad .$$

Then Z is continuous, and $F(\boldsymbol{u}) = 0$ is equivalent to $Z(\boldsymbol{u}) = \boldsymbol{u}$. It suffices to show that Z is a contraction mapping. In fact, for any $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^2$ and $\boldsymbol{v} = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, by Assumptions (**A1**), (**A2**) and (**A4**) we have

$$\begin{aligned} \|Z\boldsymbol{u} - Z\boldsymbol{v}\|^2 &= \sum_{i\in\mathbb{Z}} \gamma_i^2 \left(\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} \left(f_j(u_j) - f_j(v_j) \right) \right)^2 \\ &\leq M_\gamma^2 M_\lambda^2 \sum_{i\in\mathbb{Z}} (2\mathfrak{n}+1) \left(f_j(u_j) - f_j(v_j) \right)^2 \\ &\leq (2\mathfrak{n}+1) M_\gamma^2 M_\lambda^2 L^2 \|\boldsymbol{u} - \boldsymbol{v}\|^2. \end{aligned}$$

Condition (3.26) ensures that Z is a contraction mapping and thus has a unique fixed point $\boldsymbol{u}^*$. Then $F(\boldsymbol{u}^*) = 0$, i.e., $\boldsymbol{u}^*$ is a steady state of (3.5). □

We next show that the steady state $\boldsymbol{u}^*$ ensured by the Lemma above is asymptotic stable. To that end, set $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*$. Then $\tilde{\boldsymbol{u}} = (\tilde{u}_i)_{i\in\mathbb{Z}}$ satisfies the following system of ODEs:

$$\frac{\mathrm{d}\tilde{u}_i}{\mathrm{d}t} = -\frac{\tilde{u}_i}{\gamma_i \mu_i} + \frac{1}{\mu_i} \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} \left(f_j(u_j) - f_j(u_j^*) \right). \tag{3.27}$$

Theorem 3.4. *Let Assumptions (***A1***) - (***A5***) and condition* (3.26) *hold. Then the global attractor $\mathcal{A}$ for* (3.5) *consists of a singleton point $\boldsymbol{u}^*$.*

Proof. Multiplying both sides of the equation (3.27) by $\tilde{u}_i$ then summing over $i \in \mathbb{Z}$ gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\|\tilde{\boldsymbol{u}}\|^2}{\mathrm{d}t} &= -\sum_{i\in\mathbb{Z}} \frac{\tilde{u}_i^2}{\gamma_i \mu_i} + \sum_{i\in\mathbb{Z}} \frac{\tilde{u}_i}{\mu_i} \left(\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} (f_j(u_j) - f_j(u_j^*)) \right) \\ &\leq -\left(\frac{1}{M_\gamma M_\mu} - \frac{M_\lambda (2\mathfrak{n}+1) L}{m_\mu} \right) \|\tilde{\boldsymbol{u}}\|^2. \end{aligned}$$

Under the assumption (**A5**), $\frac{\mathrm{d}\|\tilde{\boldsymbol{u}}\|^2}{\mathrm{d}t} < 0$ for all $\tilde{\boldsymbol{u}} \neq 0$. Thus $V(\tilde{\boldsymbol{u}}) = \|\tilde{\boldsymbol{u}}\|^2$ is an appropriate Lyaponuv functional to ensure that $\tilde{\boldsymbol{u}} = 0$ is asymptotic stable for the system (3.27). Consequently, $\boldsymbol{u}^*$ is a unique asymptotic stable steady state for the original lattice system (4.4). This completes the proof. □

3.3 Upper-semi continuity of global attractors

In this section we consider finite dimensional approximations of the infinite dimensional lattice system (4.4), and investigate the upper-semi continuous convergence of the global attractor for the finite dimensional approximation to the global attractor for the system (4.4). This issue is of crucial importance when numerical simulations of system (4.4) are sought. More precisely, a numerical simulation of system (4.4) requires a simplified finite dimensional approximation of the original infinite system, and the convergence of the global attractor ensures that numerical solutions based on the simplified finite dimensional system mimic the solutions for the infinite lattice system in the limit. To this end, we first consider the following $(2N+1)$-dimensional system of ODEs obtained from directly truncating the lattice system (4.4):

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}t} u_i(t) = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j(t)) + g_i, \quad i = -N, \cdots, 0, \cdots, N. \tag{3.28}$$

Since every neuron is interacting with $2\mathfrak{n}$ neurons in its neighborhood ordered by their indices, throughout this section we assume that $N \geq \mathfrak{n}$ to capture enough dynamics of the network.

Notice that the sum term in the above system involves $u_{-N-\mathfrak{n}}$, …, u_{-N-1} and u_{N+1}, …, $u_{N+\mathfrak{n}}$, which are not governed by any equation within the system. Thus in order for the system (3.28) to be well-defined, some boundary conditions need to be imposed. There are multiple ways to construct such boundary conditions. Here we consider three most intuitive constructions: a Dirichlet type boundary condition with vanishing tails (**A6**) below; a periodic type boundary condition with period $2\mathfrak{n}+1$ (**A7**) below; and a periodic type boundary condition with period $2N+1$ (**A8**) below.

(**A6**) the neuron activation functions satisfy $f_i(u_i) = 0$ for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N+1 \cdots N+\mathfrak{n}$.

(**A7**) the neuron activation functions and connection strength among neurons satisfy

$$f_i(u_i) = f_{i+2\mathfrak{n}+1}(u_{i+2\mathfrak{n}+1}), \quad \lambda_{i,j} = \lambda_{i,j+2\mathfrak{n}+1}$$

for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N - 2\mathfrak{n} \cdots N - \mathfrak{n} - 1$.

(**A8**) the neuron activation functions and connection strength among neurons satisfy

$$f_i(u_i) = f_{i+2N+1}(u_{i+2N+1}), \quad \lambda_{i,j} = \lambda_{i,j+2N+1}$$

for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N - 2\mathfrak{n} \cdots N - \mathfrak{n} - 1$.

(**A6**) the neuron activation functions satisfy $f_i(u_i) = 0$ for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N+1 \cdots N+\mathfrak{n}$.

(**A7**) the neuron activation functions and connection strength among neurons satisfy

$$f_i(u_i) = f_{i+2\mathfrak{n}+1}(u_{i+2\mathfrak{n}+1}), \quad \lambda_{i,j} = \lambda_{i,j+2\mathfrak{n}+1}$$

for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N - 2\mathfrak{n} \cdots N - \mathfrak{n} - 1$.

(**A8**) the neuron activation functions and connection strength among neurons satisfy

$$f_i(u_i) = f_{i+2N+1}(u_{i+2N+1}), \quad \lambda_{i,j} = \lambda_{i,j+2N+1}$$

for $i = -\mathfrak{n} - N, \cdots, -N-1$ and $N - 2\mathfrak{n} \cdots N - \mathfrak{n} - 1$.

Remark 5. *Assumption (***A6***) assumes the neurons out of the index domain $i = -N, \cdots, N$ do not excite or inhibit those within the index domain. Assumptions (***A7***) and (***A8***) ensure that the activation functions for those neurons out of the index domain $i = -N, \cdots, N$ can be replicated by neurons inside the index domain. In particular, in Assumption (***A7***) each neuron outside the index domain is substituted by the neuron $2\mathfrak{n}+1$ index away within the domain (see Figures 1 and 2 below); and in Assumption (***A8***) each neuron outside the index domain is substituted by the neuron $2N+1$ indices away within the domain (see Figures 3 and 4 below).*

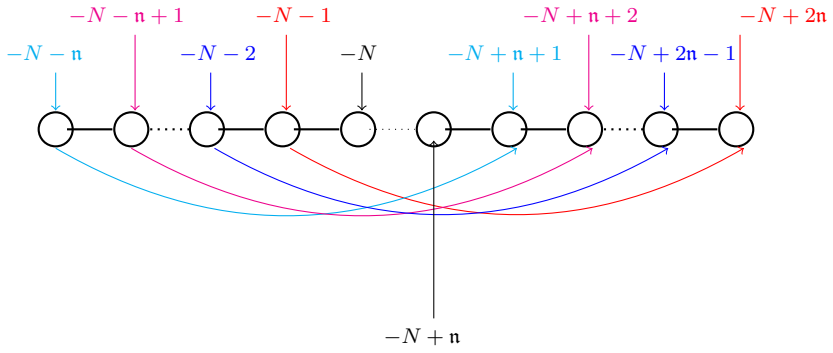

Figure 3.1: Replacements of neurons $-N-\mathfrak{n}$ through $-N-1$ under assumption (**A7**).

For any $N \in \mathbb{N}$, denote by $\mathfrak{u}^N(t) = (u_{-N}, \dots, u_0, \dots, u_N) \in \mathbb{R}^{2N+1}$, and

$$B^N := \begin{pmatrix} -\frac{1}{\mu_{-N}\gamma_{-N}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{\mu_{-N+1}\gamma_{-N+1}} & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -\frac{1}{\mu_N\gamma_N} \end{pmatrix} \in \mathbb{R}^{(2N+1)\times(2N+1)}.$$

In addition, let

$$\mathfrak{g}^N := \left(\frac{g_{-N}}{\mu_{-N}}, \dots, \frac{g_N}{\mu_N}\right) \in \mathbb{R}^{2N+1}.$$

Under Assumption (**A6**), the $(2N+1)$-dimensional ODE system (3.28) becomes

$$\frac{\mathrm{d}\mathfrak{u}^N(t)}{\mathrm{d}t} = B^N\mathfrak{u}^N + \mathfrak{f}^N(\mathfrak{u}^N) + \mathfrak{g}^N, \tag{3.29}$$

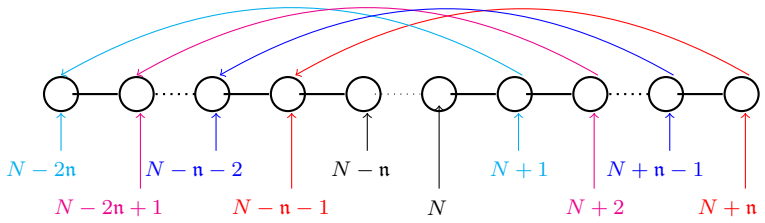

Figure 3.2: Replacements of neurons $N+1$ through $N+\mathfrak{n}$ under assumption (**A7**).

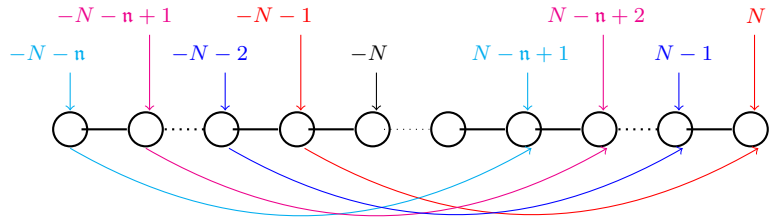

Figure 3.3: Replacements of neurons $-N-\mathfrak{n}$ through $-N-1$ under assumption (**A8**).

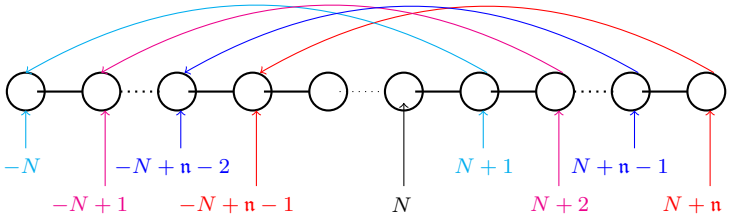

Figure 3.4: Replacements of neurons $N+1$ through $N+\mathfrak{n}$ under assumption (**A8**).

where $\mathfrak{f}^N(\cdot) = (\mathfrak{f}_i^N(\cdot))_{i=-N,\cdots N}$ is defined by

$$\mathfrak{f}_i^N(\mathfrak{u}^N) = \begin{cases} \dfrac{1}{\mu_i}\displaystyle\sum_{j=-N}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j), & i = -N, \cdots, -N+\mathfrak{n}-1 \\ \dfrac{1}{\mu_i}\displaystyle\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j), & i = -N+\mathfrak{n}, \cdots, N-\mathfrak{n} \\ \dfrac{1}{\mu_i}\displaystyle\sum_{j=i-\mathfrak{n}}^{N} \lambda_{i,j} f_j(u_j), & i = N-\mathfrak{n}+1, \cdots, N \end{cases}.$$

Under Assumption (**A7**), the $(2N+1)$-dimensional ODE system (3.28) becomes

$$\frac{\mathrm{d}\mathfrak{u}^N(t)}{\mathrm{d}t} = B^N \mathfrak{u}^N + \mathfrak{z}^N(\mathfrak{u}^N) + \mathfrak{g}^N, \tag{3.30}$$

where $\mathfrak{z}^N(\cdot) = (\mathfrak{z}_i^N(\cdot))_{i=-N,\cdots N}$ is defined by

$$\mathfrak{z}_i^N(\mathfrak{u}^N) = \begin{cases} \dfrac{1}{\mu_i} \displaystyle\sum_{j=-N}^{-N+2\mathfrak{n}} \lambda_{i,j} f_j(u_j), & i = -N, \cdots, -N+\mathfrak{n}-1 \\ \dfrac{1}{\mu_i} \displaystyle\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j), & i = -N+\mathfrak{n}, \cdots, N-\mathfrak{n} \\ \dfrac{1}{\mu_i} \displaystyle\sum_{j=N-2\mathfrak{n}}^{N} \lambda_{i,j} f_j(u_j), & i = N-\mathfrak{n}+1, \cdots, N \end{cases}.$$

Under Assumption (**A8**), the $(2N+1)$-dimensional ODE system (3.28) becomes

$$\frac{\mathrm{d}\mathfrak{u}^N(t)}{\mathrm{d}t} = B^N \mathfrak{u} + \mathfrak{y}^N(\mathfrak{u}^N) + \mathfrak{g}^N, \tag{3.31}$$

where $\mathfrak{y}^N(\cdot) = (\mathfrak{y}_i^N(\cdot))_{i=-N,\cdots N}$ is defined by

$$\mathfrak{y}_i^N(\mathfrak{u}^N) = \begin{cases} \dfrac{1}{\mu_i} \left(\displaystyle\sum_{j=-N}^{i+\mathfrak{n}} + \sum_{j=i-\mathfrak{n}+2N+1}^{N} \right) \lambda_{i,j} f_j(u_j), & i = -N, \cdots, -N+\mathfrak{n}-1 \\ \dfrac{1}{\mu_i} \displaystyle\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(u_j), & i = -N+\mathfrak{n}, \cdots, N-\mathfrak{n} \\ \dfrac{1}{\mu_i} \left(\displaystyle\sum_{j=i-\mathfrak{n}}^{N} + \sum_{-N}^{i+\mathfrak{n}-2N-1} \right) \lambda_{i,j} f_j(u_j) & i = N-\mathfrak{n}+1, \cdots, N \end{cases}.$$

Remark 6. *Notice that when $N = \mathfrak{n}$, both systems* (3.30) *and* (3.31) *are reduced to*

$$\mu_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{u_i}{\gamma_i} + \sum_{i=-\mathfrak{n}}^{\mathfrak{n}} \lambda_{i,j} f_j(u_j) + g_i, \quad i = -\mathfrak{n}, \cdots, \mathfrak{n}, \tag{3.32}$$

which is the classical finite dimensional Hopfield model (1.1)*. When $N = \mathfrak{n}$, system* (3.29) *also resembles* (3.32) *with $\lambda_{i,j} = 0$ for some i, j.*

In what follows we will investigate the existence of global attractors for each of the systems (3.29), (3.30) and (3.31), and then show that they converge to the global attractor for the lattice system (4.4) when $N \to \infty$. The following Lemma provides critical estimations to be used later.

Lemma 10. *Given any* $\mathfrak{u}^N = (u_{-N}, \dots, u_0, \dots, u_N) \in \mathbb{R}^{2N+1}$,

$$\begin{aligned}
\sum_{i=-N}^{N} u_i \mathfrak{f}_i^N(\mathfrak{u}^N) &\leq \frac{LM_\lambda}{m_\mu}(5\mathfrak{n}+1)\sum_{i=-N}^{N} u_i^2;\\
\sum_{i=-N}^{N} u_i \mathfrak{z}_i^N(\mathfrak{u}^N) &\leq \frac{LM_\lambda}{m_\mu}(5\mathfrak{n}+2)\sum_{i=-N}^{N} u_i^2;\\
\sum_{i=-N}^{N} u_i \mathfrak{y}_i^N(\mathfrak{u}^N) &\leq \frac{LM_\lambda}{m_\mu}(7\mathfrak{n}+2)\sum_{i=-N}^{N} u_i^2.
\end{aligned}$$

Proof. First by assumptions (**A1**), (**A2**) and (**A4**) and the definitions of $\mathfrak{f}_i^N$, $\mathfrak{z}_i^N$ and $\mathfrak{y}_i^N$ we have

$$\sum_{i=-N}^{N} u_i \mathfrak{f}_i^N \leq \frac{LM_\lambda}{m_\mu}\Big(\underbrace{\sum_{i=-N}^{-N+\mathfrak{n}-1}\sum_{j=-N}^{i+\mathfrak{n}} |u_iu_j|}_{(\mathbf{i})} + \underbrace{\sum_{-N+\mathfrak{n}}^{N-\mathfrak{n}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} |u_iu_j|}_{(\mathbf{ii})} + \underbrace{\sum_{i=N-\mathfrak{n}+1}^{N}\sum_{j=i-\mathfrak{n}}^{N} |u_iu_j|}_{(\mathbf{iii})} \Big), \tag{3.33}$$

$$\sum_{i=-N}^{N} u_i \mathfrak{z}_i^N \leq \frac{LM_\lambda}{m_\mu}\Big(\underbrace{\sum_{i=-N}^{-N+\mathfrak{n}-1}\sum_{j=-N}^{-N+2\mathfrak{n}} |u_iu_j|}_{(\mathbf{iv})} + \underbrace{\sum_{-N+\mathfrak{n}}^{N-\mathfrak{n}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} |u_iu_j|}_{(\mathbf{ii})} + \underbrace{\sum_{i=N-\mathfrak{n}+1}^{N}\sum_{j=N-2\mathfrak{n}}^{N} |u_iu_j|}_{(\mathbf{v})} \Big), \tag{3.34}$$

$$\begin{aligned}
\sum_{i=-N}^{N} u_i \mathfrak{y}_i^N \leq \frac{LM_\lambda}{m_\mu}\Big(&\underbrace{\sum_{i=-N}^{-N+\mathfrak{n}-1}\Big(\sum_{j=-N}^{i+\mathfrak{n}} + \sum_{j=i-\mathfrak{n}+2N+1}^{N}\Big)|u_iu_j|}_{(\mathbf{vi})} + \underbrace{\sum_{-N+\mathfrak{n}}^{N-\mathfrak{n}}\sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} |u_iu_j|}_{(\mathbf{ii})} \Big)\\
&+\frac{LM_\lambda}{m_\mu}\underbrace{\sum_{i=N-\mathfrak{n}+1}^{N}\Big(\sum_{j=i-\mathfrak{n}}^{N} + \sum_{-N}^{i+\mathfrak{n}-2N-1}\Big)|u_iu_j|}_{(\mathbf{vii})}.
\end{aligned} \tag{3.35}$$

Then using $|u_iu_j| \leq \frac{1}{2}(u_i^2+u_j^2)$ and shifting of indices repeatedly to obtain

$$\begin{aligned}
(\mathbf{i}) &\leq \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1} u_i^2 \sum_{j=-N}^{i+\mathfrak{n}} 1 + \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1}\sum_{j=-N}^{i+\mathfrak{n}} u_j^2\\
&\leq \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1}(i+N+1+\mathfrak{n})u_i^2 + \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1}\big(\mathfrak{n}u_i^2 + (-N+\mathfrak{n}-i)u_{i+\mathfrak{n}}^2\big)\\
&\leq \frac{3\mathfrak{n}}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1} u_i^2 + \frac{\mathfrak{n}}{2}\sum_{i=-N+\mathfrak{n}}^{-N+2\mathfrak{n}-1} u_i^2 \leq \frac{3\mathfrak{n}}{2}\sum_{i=-N}^{-N+2\mathfrak{n}-1} u_i^2;
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{ii}) \;&\le\; \frac{1}{2}\sum_{i=-N+\mathfrak{n}}^{N-\mathfrak{n}} u_i^2 \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} 1 + \frac{1}{2}\sum_{k=0}^{\mathfrak{n}} \sum_{i=-N+k}^{N-2\mathfrak{n}+k} u_i^2 \\
&\le\; \frac{2\mathfrak{n}+1}{2}\sum_{i=-N+\mathfrak{n}}^{N-\mathfrak{n}} u_i^2 + \frac{\mathfrak{n}+1}{2}\sum_{i=-N}^{N} u_i^2 \le \frac{3\mathfrak{n}+2}{2}\sum_{i=-N}^{N} u_i^2;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{iii}) \;&\le\; \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2 \sum_{j=i-\mathfrak{n}}^{N} 1 + \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} \sum_{j=i-\mathfrak{n}}^{N} u_j^2 \\
&\le\; \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} (N-i+\mathfrak{n}+1)u_i^2 + \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} \big(\mathfrak{n}u_i^2 + (i-N+\mathfrak{n})u_{i-\mathfrak{n}}^2 \\
&\le\; \frac{3\mathfrak{n}}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2 + \frac{\mathfrak{n}}{2}\sum_{i=N-2\mathfrak{n}+1}^{N-\mathfrak{n}} u_i^2 \le \frac{3\mathfrak{n}}{2}\sum_{i=N-2\mathfrak{n}+1}^{N} u_i^2;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{iv}) \;&\le\; \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1} u_i^2 \sum_{j=-N}^{-N+2\mathfrak{n}} 1 + \frac{1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1} \sum_{j=-N}^{-N+2\mathfrak{n}} u_j^2 \\
&=\; \frac{2\mathfrak{n}+1}{2}\sum_{i=-N}^{-N+\mathfrak{n}-1} u_i^2 + \frac{\mathfrak{n}}{2}\sum_{i=-N}^{-N+2\mathfrak{n}} u_i^2 \le \frac{3\mathfrak{n}+1}{2}\sum_{i=-N}^{-N+2\mathfrak{n}} u_i^2;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{v}) \;&\le\; \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2 \sum_{j=N-2\mathfrak{n}}^{N} 1 + \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} \sum_{j=N-2\mathfrak{n}}^{N} u_j^2 \\
&=\; \frac{2\mathfrak{n}+1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2 + \frac{\mathfrak{n}}{2}\sum_{i=N-2\mathfrak{n}}^{N} u_i^2 \le \frac{3\mathfrak{n}+1}{2}\sum_{i=N-2\mathfrak{n}}^{N} u_i^2;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{vi}) \;&\le\; \frac{1}{2}\sum_{i=-N}^{-N+2\mathfrak{n}-1} u_i^2\Big(\sum_{j=-N}^{i+\mathfrak{n}} 1 + \sum_{j=i-\mathfrak{n}+2N+1}^{N} 1\Big) + \frac{1}{2}\sum_{i=-N}^{-N+2\mathfrak{n}-1} \Big(\sum_{j=-N}^{i+\mathfrak{n}} u_j^2 + \sum_{j=i-\mathfrak{n}+2N+1}^{N} u_j^2\Big) \\
&\le\; \frac{2\mathfrak{n}+1}{2}\sum_{i=-N}^{-N+2\mathfrak{n}-1} u_i^2 + \mathfrak{n}\sum_{i=-N}^{N} u_i^2 \le \frac{5\mathfrak{n}+1}{2}\sum_{i=-N}^{N} u_i^2;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{vii}) \;&\leq\; \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2\Big(\sum_{j=i-\mathfrak{n}}^{N} 1 + \sum_{-N}^{i+\mathfrak{n}-2N-1} 1\Big) + \frac{1}{2}\sum_{i=N-\mathfrak{n}+1}^{N}\Big(\sum_{j=i-\mathfrak{n}}^{N} u_j^2 + \sum_{-N}^{i+\mathfrak{n}-2N-1} u_j^2\Big)\\
&\leq\; \frac{2\mathfrak{n}+1}{2}\sum_{i=N-\mathfrak{n}+1}^{N} u_i^2 + \mathfrak{n}\sum_{i=-N}^{N} u_i^2 \leq \frac{5\mathfrak{n}+1}{2}\sum_{i=-N}^{N} u_i^2.
\end{aligned}
$$

Inserting estimations (**i**) - (**iii**) in (3.33) implies the first assertion of the Lemma, estimations (**ii**), (**iv**) and (**v**) in (3.34) implies the second assertion of the Lemma, and estimations (**ii**), (**vi**) and (**vii**) in (3.35) implies the last assertion of the Lemma. □

Let the initial condition for systems (3.29), (3.30) and (3.31) be $\mathfrak{u}^N(0) = \mathfrak{u}_o^N$ where $\mathfrak{u}_o^N$ is the $(2N+1)$-dimensional truncation of the initial value $\boldsymbol{u}_o$ of the infinite dimensional system (3.5), i.e., $\mathfrak{u}_o^N = (u_{o,i})_{i=-N,\cdots,N}$. Then for every $\mathfrak{u}_o^N \in \mathbb{R}^{2N+1}$, each of the systems (3.29), (3.30) and (3.31) has a unique solution that generates a dynamical system from $\mathbb{R}^{2N+1}$ to $\mathbb{R}^{2N+1}$. More precisely, denote by $\{\phi^N(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{S}^N(t)\}_{t\in\mathbb{R}}$ and $\{\psi^N(t)\}_{t\in\mathbb{R}}$ the dynamical systems generated by solutions of the system (3.29), (3.30) and (3.31), respectively, i.e.,

$$
\begin{aligned}
\phi^N(t)\mathfrak{u}_o^N &= \mathfrak{u}^N(t;\mathfrak{u}_o^N) \text{ where } \mathfrak{u}^N(t,\mathfrak{u}_o^N) \text{ is the solution to (3.29) with } \mathfrak{u}^N(0) = \mathfrak{u}_o^N,\\
\mathcal{S}^N(t)\mathfrak{u}_o^N &= \mathfrak{u}^N(t;\mathfrak{u}_o^N) \text{ where } \mathfrak{u}^N(t,\mathfrak{u}_o^N) \text{ is the solution to (3.30) with } \mathfrak{u}^N(0) = \mathfrak{u}_o^N,\\
\psi^N(t)\mathfrak{u}_o^N &= \mathfrak{u}^N(t;\mathfrak{u}_o^N) \text{ where } \mathfrak{u}^N(t,\mathfrak{u}_o^N) \text{ is the solution to (3.31) with } \mathfrak{u}^N(0) = \mathfrak{u}_o^N.
\end{aligned}
$$

In the calculations below and throughout the rest of this section, $\|\cdot\|$ still denotes the norm of ℓ^2, and denote by $|\cdot|_N$ the Euclidean norm of $\mathbb{R}^{2N+1}$. Moreover, for $i = -N,\cdots,N$ denote by $u_i^{N,\phi}(t)$, $u_i^{N,\mathcal{S}}(t)$ and $u_i^{N,\psi}(t)$ the ith component of $\phi^N(t)\mathfrak{u}_o^N$, $\mathcal{S}^N(t)\mathfrak{u}_o^N$, and $\psi^N(t)\mathfrak{u}_o^N$, respectively. In addition, to avoid redundant computations, the notation $\natural$ is used to denote dynamical system ϕ, $\mathcal{S}$, or ψ when they share some estimations.

Lemma 11. *Let (**A1**) - (**A5**) hold. In addition, assume that*

$$
\tilde{\sigma} := \frac{1}{M_\mu M_\gamma} - 2\frac{LM_\lambda}{m_\mu}(7\mathfrak{n}+2) < 0. \tag{3.36}
$$

*Then the dynamical system $\{\phi^N(t)\}_{t\in\mathbb{R}}$ has an absorbing set in $\mathbb{R}^{2N+1}$ provided (**A6**) holds; the dynamical system $\{\mathcal{S}^N(t)\}_{t\in\mathbb{R}}$ has an absorbing set in $\mathbb{R}^{2N+1}$ provided (**A7**) holds; and the dynamical system $\{\psi^N(t)\}_{t\in\mathbb{R}}$ has an absorbing set in $\mathbb{R}^{2N+1}$ provided (**A8**) holds.*

Proof. First taking the inner product on $\mathbb{R}^{2N+1}$ of equations (3.29), (3.30), (3.31) with $\phi^N(t)\mathfrak{u}_o^N$, $\mathcal{S}^N(t)\mathfrak{u}_o^N$ and $\psi^N(t)\mathfrak{u}_o^N$, respectively, gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\phi^N(t)\mathfrak{u}_o^N|_N^2 = -\sum_{i=-N}^{N}\frac{(u_i^{N,\phi})^2}{\mu_i\gamma_i}+\sum_{i=-N}^{N}u_i^{N,\phi}\mathfrak{f}_i^N(\phi^N(t)\mathfrak{u}_o^N)+\sum_{i=-N}^{N}\frac{g_iu_i^{N,\phi}}{\mu_i}, \tag{3.37}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathcal{S}^N(t)\mathfrak{u}_o^N|_N^2 = -\sum_{i=-N}^{N}\frac{(u_i^{N,\mathcal{S}})^2}{\mu_i\gamma_i}+\sum_{i=-N}^{N}u_i^{N,\mathcal{S}}\mathfrak{z}_i^N(\mathcal{S}^N(t)\mathfrak{u}_o^N)+\sum_{i=-N}^{N}\frac{g_iu_i^{N,\mathcal{S}}}{\mu_i}, \tag{3.38}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\psi^N(t)\mathfrak{u}_o^N|_N^2 = -\sum_{i=-N}^{N}\frac{(u_i^{N,\psi})^2}{\mu_i\gamma_i}+\sum_{i=-N}^{N}u_i^{N,\psi}\mathfrak{y}_i^N(\psi^N(t)\mathfrak{u}_o^N)+\sum_{i=-N}^{N}\frac{g_iu_i^{N,\psi}}{\mu_i}. \tag{3.39}$$

Then by assumption (**A2**),

$$-\sum_{i=-N}^{N}\frac{(u_i^{N,\natural})^2}{\mu_i\gamma_i} \le -\frac{1}{M_\mu M_\gamma}|\natural^N(t)\mathfrak{u}_o^N|^2, \qquad \natural=\phi,\mathcal{S},\psi, \tag{3.40}$$

$$\sum_{i=-N}^{N}\frac{g_iu_i^{N,\natural}}{\mu_i} \le \frac{1}{2M_\mu M_\gamma}|\natural^N(t)\mathfrak{u}_o^N|^2+\frac{M_\mu M_\gamma}{2m_\mu^2}\|\boldsymbol{g}\|^2, \qquad \natural=\phi,\mathcal{S},\psi. \tag{3.41}$$

In addition, it follows immediately from Lemma 10 that

$$\sum_{i=-N}^{N}\phi_i^N\mathfrak{f}_i^N(\phi^N(t)\mathfrak{u}_o^N) \le \frac{LM_\lambda}{m_\mu}(5\mathfrak{n}+1)|\phi^N(t)\mathfrak{u}_o^N|_N^2; \tag{3.42}$$

$$\sum_{i=-N}^{N}\mathcal{S}_i^N\mathfrak{z}_i^N(\mathcal{S}^N(t)\mathfrak{u}_o^N) \le \frac{LM_\lambda}{m_\mu}(5\mathfrak{n}+2)|\mathcal{S}^N(t)\mathfrak{u}_o^N|_N^2; \tag{3.43}$$

$$\sum_{i=-N}^{N}\psi_i^N\mathfrak{y}_i^N(\psi^N(t)\mathfrak{u}_o^N) \le \frac{LM_\lambda}{m_\mu}(7\mathfrak{n}+2)|\psi^N(t)\mathfrak{u}_o^N|_N^2. \tag{3.44}$$

Collecting (3.37) – (3.44) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}|\natural^N(t)\mathfrak{u}_o^N|_N^2 \le -\frac{1}{M_\mu M_\gamma}+\frac{LM_\lambda}{m_\mu}(7\mathfrak{n}+2)|\natural^N(t)\mathfrak{u}_o^N|_N^2+\frac{M_\mu M_\gamma}{m_\mu^2}\|\boldsymbol{g}\|^2, \qquad \natural=\phi,\mathcal{S},\psi.$$

Integrating the above inequality from 0 to t gives immediately

$$|\natural^N(t)\mathfrak{u}_o^N|_N^2 \leq e^{-\tilde{\sigma}t}|\mathfrak{u}_o^N|_N^2 + \frac{M_\mu M_\gamma}{\tilde{\sigma}m_\mu^2}\|\boldsymbol{g}\|^2(1-e^{-\tilde{\sigma}t}), \quad \natural = \phi, \mathcal{S}, \psi,$$

where $\tilde{\sigma}$ is defined as in (3.36). Therefore, given any $\epsilon > 0$,

$$\mathcal{K}_\epsilon^N = \left\{\mathfrak{u}^N \in \mathbb{R}^{2N+1} : |\mathfrak{u}^N|_N \leq \frac{M_\mu M_\gamma}{\sqrt{\tilde{\sigma}}}\frac{\|\boldsymbol{g}\|^2}{m_\mu} + \epsilon\right\} \tag{3.45}$$

is an absorbing set for the dynamical systems $\{\phi^N(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{S}^N(t)\}_{t\in\mathbb{R}}$ and $\{\psi^N(t)\}_{t\in\mathbb{R}}$ respectively. The proof is complete. □

Remark 7. *It is worth noting that the Assumption (***A5***) along with the condition* (3.36) *are equivalent to*

$$2\mathfrak{n} + 1 < \frac{m_\mu}{M_\mu}\frac{1}{2LM_\lambda M_\gamma} < 7\mathfrak{n} + 2. \tag{3.46}$$

This implies that the existence of absorbing sets for each of the finite dimensional approximating dynamical systems depends on a direct comparison of the Lipschitz constant for the activation functions, the maximum connection strength among neurons, and the maximum neuron amplifier input resistance of neurons with the size of interacting neighborhoods, i.e., the number of neurons in the neighborhood that each neuron interacts with. The magnitudes of neuron amplifier input capacitance, on the other hand, plays a different role. In fact, the assumption (3.46) *essentially puts a restriction on the range of the input capacitance, i.e., the existence of absorbing sets depends on the ratio between the minimum and maximum input capacitance, instead of depending on each of them separately.*

It follows immediately from the lemma above that the dynamical systems $\{\phi^N(t)\}_{t\in\mathbb{R}}$, $\{\varphi^N(t)\}_{t\in\mathbb{R}}$ and $\{\psi^N(t)\}_{t\in\mathbb{R}}$ each possesses a global attractor, denoted by $\mathcal{A}_\phi^N$, $\mathcal{A}_\varphi^N$ and $\mathcal{A}_\psi^N$, respectively. We next show that natural embeddings of the global attractors $\mathcal{A}_\phi^N$, $\mathcal{A}_\varphi^N$ and $\mathcal{A}_\psi^N$ into ℓ^2 all approach the global attractor $\mathcal{A}$ for the lattice system (4.4) upper-semi continuously as $N \to \infty$. Recall that for every $\mathfrak{a} = (a_i)_{i=-N,\cdots,N} \in \mathbb{R}^{2N+1}$, its natural extension $\boldsymbol{a} = (\tilde{a}_i)_{i\in\mathbb{Z}}$

in ℓ^2 is defined by

$$\tilde{a}_i = \begin{cases} a_i, & i = -N, \cdots, N, \\ \\ 0, & \text{otherwise} \end{cases}.$$

Lemma 12. *Let assumptions (**A1**) - (**A5**) hold. Then given any $\epsilon > 0$ there exists $I(\epsilon) > 0$ depending on $\mathbf{g}$, m_μ, M_λ, M_μ, M_γ, but independent of N, such that every $\mathfrak{a} = (a_i)_{i=-N,\cdots,N} \in \mathcal{A}_\phi^N, \mathcal{A}_\mathcal{S}^N, \mathcal{A}_\psi^N$ under (**A6**), (**A7**), (**A8**) (resp.) satisfies*

$$\sum_{I(\epsilon) \le |i| \le N} |a_i|^2 < \epsilon.$$

Proof. The proof is similar to that of Lemma 8. Similar to the fact that $\Phi(t)$ is a group on $\mathcal{A}$, we have

$$\phi^N(t)\mathcal{A}_\phi^N = \mathcal{A}_\phi^N, \quad \mathcal{S}^N(t)\mathcal{A}_\mathcal{S}^N = \mathcal{A}_\mathcal{S}^N, \quad \psi^N(t)\mathcal{A}_\phi^N = \mathcal{A}_\psi^N \qquad \forall\ t \in \mathbb{R}.$$

Thus given any $\mathfrak{u}_o^{N,\phi} \in \mathcal{A}_\phi^N$, $\mathfrak{u}_o^{N,\mathcal{S}} \in \mathcal{A}_\mathcal{S}^N$ and $\mathfrak{u}_o^{N,\psi} \in \mathcal{A}_\psi^N$, the solution $\phi^N(t)\mathfrak{u}_o^{N,\phi}$ of (3.29), the solution $\mathcal{S}^N(t)\mathfrak{u}_o^{N,\mathcal{S}}$ of (3.30) and the solution $\psi^N(t)\mathfrak{u}_o^{N,\psi}$ of (3.31) are defined for all $t \in \mathbb{R}$ and

$$\phi^N(t)\mathfrak{u}_o^{N,\phi} \subset \mathcal{K}_\epsilon^N, \quad \mathcal{S}^N(t)\mathfrak{u}_o^{N,\mathcal{S}} \subset \mathcal{K}_\epsilon^N, \quad \psi^N(t)\mathfrak{u}_o^{N,\psi} \subset \mathcal{K}_\epsilon^N \qquad \forall\ t \in \mathbb{R},$$

where $\mathcal{K}_\epsilon^N$ is defined as in (3.45).

For simplicity denote by $\left(u_i^{N,\phi}(t)\right)_{i=-N,\cdots,N}$, $\left(u_i^{N,\varphi}(t)\right)_{i=-N,\cdots,N}$, $\left(u_i^{N,\psi}(t)\right)_{i=-N,\cdots,N}$ the component wise solutions of (3.29), (3.30), and (3.31), respectively. Given a fixed $k(\le N)$ which will be determine later, define

$$\mathfrak{x}^\phi = (x_i^\phi)_{|i|\le N}, \quad \mathfrak{x}^\varphi = (x_i^\varphi)_{|i|\le N}, \quad \mathfrak{x}^\psi = (x_i^\psi)_{|i|\le N},$$

where

$$x_i^\phi = \theta_k(|i|)u_i^{N,\phi}, \quad x_i^\mathcal{S} = \theta_k(|i|)u_i^{N,\mathcal{S}}, \quad x_i^\psi = \theta_k(|i|)u_i^{N,\psi}, \quad i = -N, \cdots, N,$$

and θ_k is defined as in (3.13).

Taking the inner product of (3.29) with $\mathfrak{x}^{\phi}$, (3.30) with $\mathfrak{x}^{\varphi}$, and (3.31) with $\mathfrak{x}^{\psi}$, respectively, in $\mathbb{R}^{2N+1}$, results in

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|i|\le N}\theta_k(|i|)(u_i^{N,\phi})^2 &\le -\sum_{|i|\le N}\theta_k(|i|)\frac{(u_i^{N,\phi})^2}{\gamma_i\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{\mathfrak{f}_i^N(\mathfrak{u}^{N,\phi})}{\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{u_i^{N,\phi}g_i}{\mu_i},\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|i|\le N}\theta_k(|i|)(u_i^{N,\mathcal{S}})^2 &\le -\sum_{|i|\le N}\theta_k(|i|)\frac{(u_i^{N,\varphi})^2}{\gamma_i\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{\mathfrak{z}_i^N(\mathfrak{u}^{N,\mathcal{S}})}{\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{u_i^{N,\varphi}g_i}{\mu_i},\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|i|\le N}\theta_k(|i|)(u_i^{N,\psi})^2 &\le -\sum_{|i|\le N}\theta_k(|i|)\frac{(u_i^{N,\psi})^2}{\gamma_i\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{\mathfrak{y}_i^N(\mathfrak{u}^{N,\psi})}{\mu_i}+\sum_{|i|\le N}\theta_k(|i|)\frac{u_i^{N,\psi}g_i}{\mu_i}.
\end{aligned}$$

Then by using Lemma 10 and proceeding as in the proof of Lemma 8 we obtain

$$\sum_{|i|\le N}\theta_k(|i|)(u_i^{N,\natural})^2\le e^{-\tilde{\sigma}t}\sum_{|i|\le N}\theta_k(|i|)(u_{o,i}^{N,\natural})^2+\frac{M_\mu m_\gamma}{\sigma m_\mu^2}\sum_{|i|\le N}\theta_k(|i|)g_i^2,\quad \natural=\phi,\varphi,\psi.$$

Therefore as in the proof of Lemma 8, given any $\epsilon>0$ there exists $T_\epsilon>0$ and $I(\epsilon)>0$ such that with $\frac{k}{2}\ge I(\epsilon)$, we have

$$\sum_{I(\epsilon)\le|i|\le N}(u_i^{\natural})^2(t)\le\sum_{|i|\le N}\theta_k(|i|)(u_i^{\natural})^2(t)<\epsilon\quad\text{for all } t\ge T_\epsilon,\quad \natural=\phi,\varphi,\psi. \tag{3.47}$$

For $\natural=\phi,\varphi,\psi$ consider $\mathfrak{a}=(a_i)_{i=-N,\cdots,N}\in\mathcal{A}_\natural^N$. Then by the fact that $\natural^N(T_\epsilon)\mathcal{A}_\natural^N=\mathcal{A}_\natural^N$ there exists $\mathfrak{u}_o^\natural\in\mathcal{A}_\natural^N$, such that $\mathfrak{a}=\natural^N(T_\epsilon)\mathfrak{u}_o^\natural=(u_i^\natural(T_\epsilon))_{|i|\le N}$. It then follows from (3.47) that

$$\sum_{I(\epsilon)\le|i|\le N}|a_i|^2=\sum_{I(\epsilon)\le|i|\le N}(u_i^\natural)^2(t)\le\epsilon\qquad\text{for all }\mathfrak{a}\in\mathcal{A}_\natural^N,\quad\natural=\phi,\varphi,\psi.$$

□

Lemma 13. *Let (**A1**) - (**A5**) hold. Let $\mathfrak{a}^{N,\phi}\in\mathcal{A}_\phi^N$ (under (**A6**)), $\mathfrak{a}^{N,\varphi}\in\mathcal{A}_\varphi^N$ (under (**A7**)) and $\mathcal{A}_\psi^N$ (under (**A8**)). Then for $\natural=\phi,\varphi,\phi$ there exists a subsequence $\{\mathfrak{a}^{N_k,\natural}\}_{k\in\mathbb{N}}$ of $\{\mathfrak{a}^{N,\natural}\}_{N\in\mathbb{N}}$ and $\boldsymbol{a}^{*,\natural}\in\mathcal{A}$ such that $\{\mathfrak{a}^{N_k,\natural}\}$ converges to $\boldsymbol{a}^{*,\natural}$ in ℓ^2 as $k\to\infty$.*

Proof. For $\natural=\phi,\mathcal{S},\psi$ let $\mathfrak{a}_o^{N,\natural}\in\mathcal{A}_\natural^N$ and $\mathfrak{a}^{N,\natural}(t)=\natural^N(t)\mathfrak{a}_o^{N,\natural}$ be the solution of (3.29), (3.30) and (3.31), respectively. Since $\mathfrak{a}_o^{N,\natural}\in\mathcal{A}_\natural^N$, then $\mathfrak{a}^{N,\natural}(t)\in\mathcal{A}_\natural^N$ and $\mathfrak{a}^{N,\natural}(t)$ is define for all

$t \in \mathbb{R}$. For $N \in \mathbb{N}$ and $\natural = \phi, \mathcal{S}, \psi$, let $\boldsymbol{a}^{N,\natural}(t)$ be the natural extension of $\mathfrak{a}^{N,\natural}(t)$ in ℓ^2. Then due to Lemma 11

$$\left\|\boldsymbol{a}^{N,\natural}(t)\right\| \leq \frac{M_\mu M_\gamma}{\sqrt{\tilde{\sigma}}} \frac{\|\boldsymbol{g}\|^2}{m_\mu} + \epsilon := R_\epsilon \quad \text{for } \natural = \phi, \mathcal{S}, \psi, \quad t \in \mathbb{R}, \quad N \in \mathbb{N}. \tag{3.48}$$

In addition, by (3.29), (3.30) and (3.31) we have

$$\begin{aligned}
\left\|\frac{\mathrm{d}\boldsymbol{a}^{N,\phi}(t)}{\mathrm{d}t}\right\| &\leq \left|B^N \mathfrak{a}^{N,\phi}(t)\right|_N + \left|\mathfrak{f}^N(\mathfrak{a}^{N,\phi}(t))\right|_N + \left|\mathfrak{g}^N\right|_N, \\
\left\|\frac{\mathrm{d}\boldsymbol{a}^{N,\mathcal{S}}(t)}{\mathrm{d}t}\right\| &\leq \left|B^N \mathfrak{a}^{N,\mathcal{S}}(t)\right|_N + \left|\mathfrak{z}^N(\mathfrak{a}^{N,\mathcal{S}}(t))\right|_N + \left|\mathfrak{g}^N\right|_N, \\
\left\|\frac{\mathrm{d}\boldsymbol{a}^{N,\psi}(t)}{\mathrm{d}t}\right\| &\leq \left|B^N \mathfrak{a}^{N,\psi}(t)\right|_N + \left|\mathfrak{y}^N(\mathfrak{a}^{N,\psi}(t))\right|_N + \left|\mathfrak{g}^N\right|_N.
\end{aligned}$$

First notice that

$$\left|\mathfrak{g}^N\right|_N \leq \|\boldsymbol{g}\|, \qquad \left|B^N \mathfrak{a}^{N,\natural}(t)\right|_N \leq \frac{1}{m_\mu m_\gamma} \left\|\boldsymbol{a}^{N,\natural}(t)\right\| \text{ for } \natural = \phi, \mathcal{S}, \psi.$$

Then by definitions of $\mathfrak{f}^N$, $\mathfrak{z}^N$ and $\mathfrak{y}^N$, and following computations similar to those in the proof of Lemma 10 to obtain

$$\begin{aligned}
\left|\mathfrak{f}^N(\mathfrak{a}^{N,\phi}(t))\right|_N^2 &\leq \frac{M_\lambda^2 L^2}{m_\mu^2} \left(\sum_{i=-N}^{-N+\mathfrak{n}-1} \sum_{j=-N}^{i+\mathfrak{n}} \mathfrak{a}_j^{N,\phi}(t)^2 + \sum_{i=-N+\mathfrak{n}}^{N-\mathfrak{n}} \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \mathfrak{a}_j^{N,\phi}(t)^2 + \sum_{i=N-\mathfrak{n}+1}^{N} \sum_{j=i-\mathfrak{n}}^{N} \mathfrak{a}_j^{N,\phi}(t)^2 \right) \\
&\leq (3\mathfrak{n}+1) \left\|\boldsymbol{a}^{N,\phi}(t)\right\|^2, \\
\left|\mathfrak{z}^N(\mathfrak{a}^{N,\mathcal{S}}(t))\right|_N^2 &\leq \frac{M_\lambda^2 L^2}{m_\mu^2} \left(\sum_{i=-N}^{-N+\mathfrak{n}-1} \sum_{j=-N}^{-N+2\mathfrak{n}} \mathfrak{a}_j^{N,\mathcal{S}}(t)^2 + \sum_{i=-N+\mathfrak{n}}^{N-\mathfrak{n}} \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \mathfrak{a}_j^{N,\mathcal{S}}(t)^2 + \sum_{i=N-\mathfrak{n}+1}^{N} \sum_{j=N-2\mathfrak{n}}^{N} \mathfrak{a}_j^{N,\mathcal{S}}(t)^2 \right) \\
&\leq (3\mathfrak{n}+1) \left\|\boldsymbol{a}^{N,\mathcal{S}}(t)\right\|^2, \\
\left|\mathfrak{y}^N(\mathfrak{a}^{N,\psi}(t))\right|_N^2 &\leq \frac{M_\lambda^2 L^2}{m_\mu^2} \left(\sum_{i=-N}^{-N+\mathfrak{n}-1} \Big(\sum_{j=-N}^{i+\mathfrak{n}} + \sum_{j=i-\mathfrak{n}+2N+1}^{N} \Big) \mathfrak{a}_j^{N,\psi}(t)^2 + \sum_{i=-N+\mathfrak{n}}^{N-\mathfrak{n}} \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \mathfrak{a}_j^{N,\psi}(t)^2 \right) \\
&\quad + \frac{M_\lambda^2 L^2}{m_\mu^2} \sum_{i=N-\mathfrak{n}+1}^{N} \Big(\sum_{j=i-\mathfrak{n}}^{N} + \sum_{-N}^{i+\mathfrak{n}-2N-1} \Big) \mathfrak{a}_j^{N,\psi}(t)^2 \leq (5\mathfrak{n}+1) \left\|\boldsymbol{a}^{N,\psi}(t)\right\|^2.
\end{aligned}$$

In summary, there exists $\tilde{R}_\epsilon$ dependent on L, M_λ, m_μ, m_γ, $\mathfrak{n}$, $\|\boldsymbol{g}\|$ but independent of N such that

$$\left\|\frac{\mathrm{d}\boldsymbol{a}^{N,\natural}(t)}{\mathrm{d}t}\right\| \le \tilde{R}_\epsilon \quad \text{for } \natural = \phi, \mathcal{S}, \psi, \quad t \in \mathbb{R}, \quad N \in \mathbb{N}. \tag{3.49}$$

Let J_k be a sequence of compact intervals in $\mathbb{R}$ such that $J_k \subset J_{k+1}$ and $\bigcup_k J_k = \mathbb{R}$. Then it follows from (3.49) that for $\natural = \phi, \mathcal{S}, \psi$, $\{\boldsymbol{a}^{N,\natural}\}_{N\in\mathbb{N}}$ is equicontinuous in $\mathcal{C}(J_k, \ell^2)$, that is, for each $t \in J_k$ and $\epsilon > 0$, there is a neighborhood $\mathcal{N}(t)$ of t such that

$$\|\boldsymbol{a}^{N,\natural}(t) - \boldsymbol{a}^{N,\natural}(s)\| \le \epsilon, \qquad \text{for all } s \in \mathcal{N}(t), \quad \text{and } N \in \mathbb{N}.$$

We next show that for each fixed $t \in J_k$, the set $\{\boldsymbol{a}^{N,\natural}\}_{N\in\mathbb{N}}$ is precompact in ℓ^2. First of all by (3.48) the sequence $\{\boldsymbol{a}^{N,\natural}\}_{N\in\mathbb{N}}$ is bounded in ℓ^2, and hence there is a subsequence of $\{\boldsymbol{a}^{N,\natural}\}_{N\in\mathbb{N}}$, still denoted by $\{\boldsymbol{a}^{N,\natural}\}_{N\in\mathbb{N}}$, and $\alpha_t^\natural \in \ell^2$ such that

$$\boldsymbol{a}^{N,\natural}(t) \rightharpoonup \alpha_t^\natural \qquad \text{weakly in } \ell^2, \quad \natural = \phi, \mathcal{S}, \psi. \tag{3.50}$$

Following similar arguments as in Theorem 3.2 we can show that the above convergence is in fact strong and hence by the Arzelà-Ascoli theorem $\{\boldsymbol{a}^{N,\natural}(t)\}_{N\in\mathbb{N}}$ is precompact in $\mathcal{C}(J_k, \ell^2)$ for each $k = 1, 2, \ldots$. Therefore there exists a subsequence $\{\boldsymbol{a}^{N_1,\natural}\}$ of $\{\boldsymbol{a}^{N,\natural}\}$ and $\boldsymbol{a}^{*,\natural} \in \mathcal{C}(J_1, \ell^2)$ such that $\{\boldsymbol{a}^{N_1,\natural}\}$ converges to $\boldsymbol{a}^{*,\natural}$ in $\mathcal{C}(J_1, \ell^2)$.

Using the Arzelà-Ascoli theorem inductively it can be shown that there is a subsequence $\{\boldsymbol{a}^{N_{k+1},\natural}\}$ of $\{\boldsymbol{a}^{N_k,\natural}\}$ such that $\{\boldsymbol{a}^{N_{k+1},\natural}\}$ converges to $\boldsymbol{a}^{*,\natural}$ in $\mathcal{C}(J_{k+1}, \ell^2)$. Taking a diagonal sequence in the usual way we obtain a subsequence $\{\boldsymbol{a}^{N_N,\natural}\}$ of $\{\boldsymbol{a}^{N,\natural}\}$ and $\boldsymbol{a}^{*,\natural} \in \mathcal{C}(\mathbb{R}, \ell^2)$ such that

$$\boldsymbol{a}^{N_N,\natural} \overset{N\to\infty}{\longrightarrow} \boldsymbol{a}^{*,\natural} \quad \text{in } \mathcal{C}(J, \ell^2) \quad \text{for any compact interval } J \subset \mathbb{R}, \quad \natural = \phi, \mathcal{S}, \psi. \tag{3.51}$$

It follows immediately from (3.48) that

$$\|\boldsymbol{a}^{*,\natural}(t)\| \le R_\epsilon \quad \forall\, t \in \mathbb{R}, \quad \natural = \phi, \mathcal{S}, \psi. \tag{3.52}$$

We next show that that $\boldsymbol{a}^{*,\natural}(t)$ is a solution of the original lattice system (4.4) and that $\boldsymbol{a}^{*,\natural}(0) \in \mathcal{A}$, the global attractor for (4.4) for $\natural = \phi, \mathcal{S}, \psi$. To simplify notation denote the diaganol subsequence in (3.51) by $\{\boldsymbol{a}^{N,\natural}\}$. Then by (3.49) and the Banach-Alaoglu theorem we have that

$$\dot{\boldsymbol{a}}^{N,\natural} \overset{N\to\infty}{\longrightarrow} \dot{\boldsymbol{a}}^{*,\natural} \qquad \text{weak star in } \mathcal{L}^\infty(\mathbb{R}, \ell^2). \tag{3.53}$$

For $i \in \mathbb{Z}$ fixed, let $N > |i| + \mathfrak{n}$. Since $\mathfrak{a}^{N,\phi}(t)$, $\mathfrak{a}^{N,\mathcal{S}}(t)$, and $\mathfrak{a}^{N,\psi}(t)$ are solutions to (3.29), (3.30) and (3.31), respectively, then

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}t} a_i^{N,\natural}(t) = -\frac{1}{\gamma_i} a_i^{N,\phi}(t) + \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(a_j^{N,\natural}) + g_i, \quad i = -N+\mathfrak{n}, \cdots, N-\mathfrak{n}, \quad \natural = \phi, \mathcal{S}, \psi.$$

Consider $\xi \in \mathcal{C}_0^\infty(J)$, then for each $i = -N+\mathfrak{n}, \cdots, N-\mathfrak{n}$

$$\int_J \frac{\mathrm{d}}{\mathrm{d}t} a_i^{N,\natural}(t)\xi(t)\mathrm{d}t = -\frac{1}{\mu_i\gamma_i}\int_J a_i^{N,\natural}(t)\xi(t)\mathrm{d}t + \frac{1}{\mu_i}\int_J \sum_{i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(a_j^{N,\natural})\xi(t)\mathrm{d}t + \frac{1}{\mu_i}\int_J g_i\xi(t)\mathrm{d}t, \quad \natural = \phi, \mathcal{S}, \psi,$$

and using Assumption (**A4**) and (3.51) to obtain

$$\begin{aligned}
&\left| \int_J \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(a_j^{N,\natural})\xi(t)\mathrm{d}t - \int_J \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \lambda_{i,j} f_j(a_j^{*,\natural})\xi(t)\mathrm{d}t \right| \\
\leq\ & M_\lambda \int_J \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \left| f_j(a_j^{N,\natural}) - f_j(a_j^{*,\natural}) \right| |\xi(t)|\mathrm{d}t \\
\leq\ & M_\lambda L \sup_{t\in J} \max_{j=i-\mathfrak{n},\cdots,i+\mathfrak{n}} |\boldsymbol{a}_j^{N,\natural} - a_j^{*,\natural}| \int_J |\xi(t)|\mathrm{d}t \overset{N\to\infty}{\longrightarrow} 0, \quad \natural = \phi, \mathcal{S}, \psi.
\end{aligned}$$

As a result

$$\frac{\mathrm{d}}{\mathrm{d}t} a_i^{*,\natural}(t) = -\frac{a_i^{*,\natural}(t)}{\gamma_i\mu_i} + \sum_{j=i-\mathfrak{n}}^{i+\mathfrak{n}} \frac{\lambda_{i,j}}{\mu_i} f_j(a_j^{*,\natural}(t)) + \frac{g_i}{\mu_i}, \quad \forall\ i \in \mathbb{Z}, \quad \natural = \phi, \mathcal{S}, \psi.$$

Since J was arbitrarily chosen the above equation hold for all $t \in \mathbb{R}$, and hence $\boldsymbol{a}^{*,\natural}$ is a bounded solution of (4.4) for all $t \in \mathbb{R}$. Therefore $\boldsymbol{a}^{*,\natural}(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$ and finally by (3.51) we obtain

$$\boldsymbol{a}^{N_N,\natural}(0) \overset{N\to\infty}{\longrightarrow} \boldsymbol{a}^{*,\natural}(0), \quad \natural = \phi, \mathcal{S}, \psi,$$

which completes the proof. □

Theorem 3.5. *Let (**A1**) - (**A5**) hold. Then the attractors $\mathcal{A}_\phi^N$, $\mathcal{A}_\mathcal{S}^N$, $\mathcal{A}_\psi^N$ all converge to $\mathcal{A}$ upper semi-continuously as $N \to \infty$, i.e.,*

$$\begin{aligned}
\lim_{N\to\infty} \mathit{dist}_{\ell^2}(\mathcal{A}_\phi^N, \mathcal{A}) &= 0 \quad \textit{(under (}\mathbf{A6}\textit{))},\\
\lim_{N\to\infty} \mathit{dist}_{\ell^2}(\mathcal{A}_\mathcal{S}^N, \mathcal{A}) &= 0 \quad \textit{(under (}\mathbf{A7}\textit{))},\\
\lim_{N\to\infty} \mathit{dist}_{\ell^2}(\mathcal{A}_\psi^N, \mathcal{A}) &= 0 \quad \textit{(under (}\mathbf{A8}\textit{))}.
\end{aligned}$$

Proof. Suppose (for contradiction) that the assertions are not true. Then there exists subsequences $\boldsymbol{a}^{N_k,\natural} \in \mathcal{A}_\natural^{N_k}$ and $\delta > 0$ such that

$$\mathrm{dis}_{\ell^2}(\boldsymbol{a}^{N_k,\natural}, \mathcal{A}) \geq \delta > 0, \quad \natural = \phi, \mathcal{S}, \psi. \tag{3.54}$$

On the other hand due to lemma13 for $\natural = \phi, \mathcal{S}, \psi$ there exists subsequences $\{\mathfrak{a}^{N_{k_l},\natural}\}$ of $\{\boldsymbol{a}^{N_k,\natural}\}$ such that

$$\mathrm{dist}_{\ell^2}(\mathfrak{a}^{N_{k_l},\natural}, \mathcal{A}) \longrightarrow 0 \quad \natural = \phi, \mathcal{S}, \psi,$$

which contradict (3.54) and complete the proof. □

3.4 Closing remarks

The main aim of this work is to study the upper semi-continuity of global attractors for a Hopfield type lattice system. To this end, global attractors for three different finite dimensional approximations of the Hopfield lattice system were studied, and shown to converge to the global attractor for the infinite lattice system upper semi-continuously as their dimension approaches ∞.

One difference between the models considered here and most of those in the literature is the nonlinear structure of the interaction term. Moreover, the interaction between a pair of neurons could be either excitation or inhibition, and thus a dissipativity condition in the

activation function may not result in an absorbing set. Instead, an assumption on the relations among the neuron amplifier input capacitance and resistance is needed.

The construction of the absorbing set and the asymptotic compactness is done by using the classical approach initiated by Bates, Lu, & Wang [9], but with significantly different computations. More specifically, for a Laplacian leading operator with nearest neighborhood interaction, the periodic boundary conditions were imposed to replace a single element out of the index domain by one within the domain to obtain a finite dimensional truncation. Whereas in our work with finite $\mathfrak{n}$-neighborhood interactions the finite dimensional truncations require a replacement of $\mathfrak{n}$ elements instead of just one.

Numerical simulations provide deeper insight into the behaviour of the lattice systems shown in the above theorems, especially for differing values of the parameters within the stated ranges. Since computations can only be undertaken for finite dimensional systems they must be applied to one of the various finite dimensional approximating systems introduced above (see, e.g., [43]). The proven theorems provide justification in a qualitative sense of the validity of such an approximation, but give little information in a quantitative or comparative sense which numerical simulations can provide. Of course, a numerical scheme is itself an approximation and care needs to be taken in interpreting the results. This issue has been studied under the name numerical dynamics, see Han & Kloeden [41].

Chapter 4

On going work

4.1 Attractors for Hopfield lattice model in weighted spaces

In this section we introduce the one of the ongoing work. We are interested in investigating the long term behavior of the following stochastic differential equations.

$$du_i = \left(-\alpha_i u_i + \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t \omega) f_j(u_j) + g_i \right) dt + u_i \circ dw(t) \tag{4.1}$$

$$du_i = \left(-\alpha_i u_i + \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t \omega) f_j(u_j) + g_i \right) dt + a_i dw_i(t) \tag{4.2}$$

where $i \in \mathbb{Z},\ \alpha_i > 0\ \ \forall i,\ u_i, g_i \in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$ and satisfy some boundedness condition (to be specify later).

4.1.1 Random dynamical systems generated by stochastic lattice

In this section we show that (4.1) and (4.2) generated a random dynamical system in weighted space ℓ^2_ρ. First the stochastic lattice systems (4.1) and (4.2) are transform into random lattice systems in ℓ^2_ρ, then we show that (4.1) and (4.2) generate a random dynamical system.

Mathematical setting

Here we transform the systems (4.1) and (4.2) to random lattice systems. First of all, we mention here that throughout the rest of this paper we consider a positive weighted function

$\rho : \mathbb{Z} \longrightarrow \mathbb{R}^+$ is chosen such that

(A0) $0 < \rho(i) \leq M_0$, $\rho(i) \leq c_0\rho(i \pm l)$ for some positive constants M_0, c_0, and$l = 1, \cdots, n$ for $n \in \mathbb{N}$.

Observe that $\ell^2 \subset \ell^2_\rho$ and ℓ^2 is dense in ℓ^2_ρ. If $\sum \rho(i) < \infty$, then $\ell^2 \subset \ell^\infty \subset \ell^2_\rho$ ([]). Now, let

$$\Omega_0 = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} = C_0(\mathbb{R}, \mathbb{R}) \tag{4.3}$$

$\mathcal{F}_0$ be the Borel σ-algebra on Ω_0 generated by the compact open topology (see [], []), and $\mathbb{P}_0$ be the corresponding Wiener measure on $\mathcal{F}_0$. consider the stochastic lattice system (4.1)

$$du_i = \left(-\alpha_i u_i + \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega) f_j(u_j) + g_i \right) dt + u_i \circ dw(t)$$

where $i \in \mathbb{Z}$, $\alpha_i > 0 \ \ \forall i$, $u_i, g_i \in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$ and satisfy $|f_i'(s)| \leq L(s)$ for some positive increasing function L. $\lambda_{i,j}(\omega)$ $i, j \in \mathbb{Z}$ are random variable on the probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$,

$$\theta_t\omega(.) = \omega(. + t) - \omega(t), t \in \mathbb{R}$$

and $\circ$ denote the Stratonovich sense of stochastic term, $w(t)$ is the Brownian motion on $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Then (4.1) can wrutten as

$$d\boldsymbol{u} = (-\alpha\boldsymbol{u} + A(\theta_t\omega)\boldsymbol{u} + g)\, dt + \boldsymbol{u} \circ dw(t) \tag{4.4}$$

where $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}}$, $g = (g_i)_{i\in\mathbb{Z}}$, $\alpha\boldsymbol{u} = (\alpha_i u_i)_{i\in\mathbb{Z}}$, $(A(\omega)\boldsymbol{u})_i = \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega) f_j(u_j)$

Note! $A(\omega)$ is not linear, but if

(A1) $\sup_i \max_{i-n\leq j\leq i+n} |\lambda_{i,j}(\omega)| = \lambda(\omega) < \infty$ then $A(\omega) : \ell^2_\rho \longrightarrow \ell^2_\rho$

To transform (4.1) into random differential equation, let

$$\delta(\theta_t\omega) = -\int_{-\infty}^{0} e^s \theta_t\omega(s) ds, \ \ t \in \mathbb{R}, \omega \in \Omega_0.$$

then, $\delta(\theta_t\omega)$ is an Ornstein-Uhlenbeck process on $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0, (\theta_t)_{t\in\mathbb{R}})$, and solves the following Ornstein-Uhlenbeck equation

$$d\delta + \delta dt = dw(t), \;\; t \geq 0 \tag{4.5}$$

where $w(t)(\omega) = w(t,\omega) = \omega(t)$ see [.,.] for details.

Lemma 14. *(see [], []) There exists a θ_t-invariant set $\tilde{\Omega}_0 \in \mathcal{F}_0$ of Ω_0 of full $\mathbb{P}_0$ measure such that for $\omega \in \tilde{\Omega}_0$, we have*

(i) the random variable $|\delta(\omega)|$ is tempered: for $\omega \in \tilde{\Omega}_0$, $\lim\limits_{t\to\infty} e^{-\beta t} \sup\limits_{t\in\mathbb{R}} |\delta(\theta_{-t}\omega)| = 0$ for all $\beta > 0$;

(ii) the mapping

$$(t,\omega) \mapsto \delta(\theta_t\omega) = -\int_{-\infty}^{0} e^s \omega(t+s)ds + w(t),$$

is a stationary solution of Ornstein-Uhlenbeck equation (4.5) *with continuous trajectories ;*

(iii)

$$\lim_{t\to\pm\infty} \frac{|\delta(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \delta(\theta_s\omega)ds = 0 \tag{4.6}$$

Set $\Pi(\theta_t\omega) := e^{\delta(\theta_t\omega)}\mathrm{Id}_{\ell^2_\rho}$, which is a homeomorphism in ℓ^2_ρ and the inverse of Π is defined as $\Pi^{-1}(\theta_t\omega) := e^{-\delta(\theta_t\omega)}\mathrm{Id}_{\ell^2_\rho}$. By (4.6) we have that $||\Pi(\theta_t\omega||$ and $||\Pi^{-1}(\theta_t\omega)||$ have sub-exponential growth as $t \to \pm\infty$ for $\omega \in \tilde{\Omega}_0$, thus $||\Pi||$ and $||\Pi^{-1}||$ are tempered. The mapping θ posses the same properties as the original one if we choose trace σ- algebra with respect to $\tilde{\Omega}_0$, denote that σ- algebra also by $\mathcal{F}_0$. Hence we change the metric dynamical system with respect to $\tilde{\Omega}_0$ and still denote it by $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0, (\theta_t)_{t\in\mathbb{R}})$.

Let

$$v(t,\omega) = \Pi^{-1}(\theta_t\omega) := e^{-\delta(\theta_t\omega)}u(t,\omega), \qquad \omega \in \Omega_0 \tag{4.7}$$

where $u(t,\omega)$ is a solution of (4.4), therefore system (4.4) can be written as

$$\frac{dv}{dt} = -\alpha v + e^{-\delta(\theta_t\omega)}\left(A(\theta_t\omega)e^{\delta(\theta_t\omega)}v\right) + e^{-\delta(\theta_t\omega)}g + \delta(\theta_t\omega)v \tag{4.8}$$

Let

$$\Omega_1 = \left\{\omega \in C(\mathbb{R}, \ell^2) : \omega(0) = 0\right\}$$

$\mathcal{F}_1$ be the Borel σ-algebra on Ω_1 generated by the compact open topology (see [39]), and $\mathbb{P}_1$ be the corresponding Wiener measure on $\mathcal{F}_1$.

$$\theta_t\omega(.) = \omega(. + t) - \omega(t). \qquad t \in \mathbb{R}$$

For each $i \in \mathbb{Z}$ we consider

$$z_i(\theta_t\omega) = -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\theta_t\omega_i(s)ds, \qquad t \in \mathbb{R}, \omega \in \Omega_1$$

Note that each $z_i(\theta_t\omega)$ is an Ornstein-Uhlenbeck process on $(\Omega_1, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and solve the following Ornstein-Uhlenbeck equation

$$dz_i = -\alpha_i z_i dt + \sigma_i dw_i(t) \tag{4.9}$$

where $w_i(t)(\omega) = w_i(t, \omega) = \omega_i(t)$ for any $\omega \in \Omega_1$ and $t \in \mathbb{R}$. Moreover using similar argument as in ([]) we get the following lemma.

Lemma 15. *There exists a $(\theta_t)_{t\in\mathbb{R}}$ invariant subset $\tilde{\Omega}_1 \in \mathcal{F}$ of Ω_1 of full measure such that for $\omega \in \tilde{\Omega}_1$.*

(i) For each $i \in \mathbb{Z}$,

$$\lim_{t\to\pm\infty} \frac{|\omega_i(t)|}{t} = 0.$$

(ii) For each $i \in \mathbb{Z}$ the random variable given by

$$z_i(\omega) = -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\theta_t\omega_i(s)ds, \qquad t \in \mathbb{R}, \omega \in \Omega$$

is well defined, and the mapping

$$(t,\omega) \mapsto z_i(\theta_t\omega) = -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\theta_t\omega_i(s)ds,$$
$$= -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\omega_i(t+s)ds + \alpha_i\sigma_i\omega_i(t)$$

is a stationary solution of (4.9) *with continuous trajectories.*

(iii) For each $i \in \mathbb{Z}$

$$\lim_{t\to\pm\infty} \frac{|z_i(\theta_t\omega)|}{|t|} = 0,$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z_i(\theta_s\omega)ds = 0,$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z_i(\theta_s\omega)|ds = \mathbb{E}|z_i| < \infty.$$

Lemma 16. *There exists a* $(\theta_t)_{t\in\mathbb{R}}$ *invariant subset* $\tilde{\Omega}_1 \in \mathcal{F}$ *of* Ω_1 *of full measure such that for* $\omega \in \tilde{\Omega}_1$.

(i) For each $i \in \mathbb{Z}$*,*

$$\lim_{t\to\pm\infty} \frac{|\omega_i(t)|}{t} = 0.$$

(ii) For each $i \in \mathbb{Z}$ *the random variable given by*

$$z_i(\omega) = -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\theta_t\omega_i(s)ds, \qquad t \in \mathbb{R}, \omega \in \Omega$$

is well defined, and the mapping

$$(t,\omega) \mapsto z_i(\theta_t\omega) = -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\theta_t\omega_i(s)ds,$$
$$= -\sigma_i\alpha_i \int_{-\infty}^{0} e^{\alpha_i s}\omega_i(t+s)ds + \alpha_i\sigma_i\omega_i(t)$$

is a stationary solution of (4.9) *with continuous trajectories.*

(iii) For each $i \in \mathbb{Z}$

$$\lim_{t\to\pm\infty} \frac{|z_i(\theta_t\omega)|}{|t|} = 0,$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z_i(\theta_s\omega)ds = 0,$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t |z_i(\theta_s\omega)|ds = \mathbb{E}|z_i| < \infty.$$

Proof. the proof of (i) and (iii) is similar as in ([]), to show (ii) we proceed as follows ;

$$(t,\omega)\mapsto z_i(0,\omega)e^{-\alpha_i t} + \sigma_i\int_0^\infty e^{-\alpha_i(t-s)}dw_i(s,\omega)$$

is a solution of (4.9), thus replacing ω by $\theta_{-t}\omega$ we have

$$\begin{aligned}
(t,\theta_{-t}\omega) &\mapsto z_i(0,\theta_{-t}\omega)e^{-\alpha_i t} + \sigma_i\int_0^t e^{-\alpha_i(t-s)}dw_i(s,\theta_{-t}\omega)\\
&= z_i(0,\theta_{-t}\omega)e^{-\alpha_i t} + \sigma_i\left[e^{-\alpha_i(0)}w_i(t,\theta_{-t}\omega) - e^{-\alpha_i t}w_i(0,\theta_{-t}\omega) - \alpha_i\int_0^t w_i(s,\theta_{-t}\omega)e^{-\alpha_i(t-s)}ds\right]\\
&= z_i(0,\theta_{-t}\omega)e^{-\alpha_i t} - \sigma_i w_i(-t,\omega) - \sigma_i\alpha_i\int_0^t e^{-\alpha_i(t-s)}\left(w_i(s-t,\omega) - w_i(-t,\omega)\right)ds\\
&= z_i(0,\theta_{-t}\omega)e^{-\alpha_i t} - \sigma_i w_i(-t,\omega) - \sigma_i\alpha_i\int_0^t e^{-\alpha_i(t-s)}w_i(s-t,\omega)ds + \sigma_i w_i(-t,\omega)e^{-\alpha_i t}\left[e^{\alpha_i t}-1\right]
\end{aligned}$$

using integral transformation $\tau = s - t$ and let $t\longrightarrow\infty$ the right hand side tend to $z_i(\omega)$ for all ω satisfying the above growth condition. The stationarity of this solution follows by the invariance of the Wiener measure $\mathbb{P}$ with respect to the flow θ. The rest of (ii) follows as in []. □

We remark here that in what follows we consuder θ define on $\tilde{\Omega}_1$ instead of Ω_1. This mapping posses the same properties as the original one if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\tilde{\Omega}_1$ still denoted by $\mathcal{F}$.

Let

$$\tilde{v}(t,\omega) = u(t,\omega) - z(\theta_t\omega), \omega\in\Omega_1 t\in\mathbb{R}$$

Then, (4.2) can be written as ;

$$\frac{d\tilde{v}}{dt} = -\alpha\tilde{v} + A(\theta_t\omega)(\tilde{v}+z) + g \tag{4.10}$$

where $A, \alpha(\tilde{v})$ are define as before.

Now we make the following assumptions;

(A1) $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2_\rho$

(A2) $\sup\limits_i \max\limits_{i-n\le j\le i+n} |\lambda_{i,j}(\omega)| = \lambda(\omega) < \infty$, for $n \in \mathbb{N}, \lambda(\theta_t\omega) \in L^1_{loc}(\mathbb{R})$

(A3) there exists a positive, increasing function L such that $|f'(s)| \le L(s), s \in \mathbb{R}, j \in \mathbb{Z}$ and $L(r) \le \kappa_1 \ln(r) + \kappa_2,\ r \ge 0$.

(A4) $\alpha_- = \inf\limits_{i\in\mathbb{Z},\lambda_{i,j}\neq 0} \min\limits_{i-n\le j\le i+n} |\alpha_{i,j}| > 0, \sup\limits_i |\alpha_i| < \infty$

4.1.2 Existence and uniqueness of solutions of random lattice systems

For multiplicative noise.

$$F(v,\omega) = -\alpha v + e^{\omega}(A(\omega)e^{\delta(\omega)}v) + e^{-\delta(\omega)}g + \delta v$$

Suppose $v^1, v^2 \in \ell^2, g \in \ell^2, \omega \in \Omega_0$.

$$\begin{aligned}\big\|F(v^1,\omega) - F(v^2,\omega)\big\| &= \big\|-\alpha(v^1-v^2) + e^{-\delta(\omega)}\left((A(\omega)e^{\omega})v^1) - (A(\omega)e^{\delta(\omega)}v^2)\right) + \delta(v^1-v^2)\big\| \\ &\le \big\|\alpha(v^1-v^2)\big\| + \big\|e^{-\delta(\omega)}\left((A(\omega)e^{\delta(\omega)}v^1) - (A(\omega)e^{\delta(\omega)}v^2)\right)\big\| + \big\|\delta(v^1-v^2)\big\|\end{aligned}$$

Now, $\left\|\alpha(v^1-v^2)\right\| \leq \sup_{i\in\mathbb{Z}} |\alpha_i| \left\|v^1-v^2\right\|$, $\|\delta(v^1-v^2)\| \leq |\delta(\omega)| \|v^1-v^2\|$ and

$$\begin{aligned}
\left\|e^{-\delta(\omega)}\left(A(\omega)e^{\delta(\omega)}v^1(A(\omega)e^{\delta(\omega)}v^2)\right)\right\| &= e^{-\delta(\omega)}\left\|\left((A(\omega)e^{\delta(\omega)}v^1(A(\omega)e^{\delta(\omega)}v^2)\right)\right\| \\
&= e^{-\delta(\theta_t\omega)}\left[\sum_{i\in\mathbb{Z}}\left[\sum_{j=i-n}^{i+n}\lambda_{i,j}(\omega)[f_j(u_j^1)-f_j(u_j^2)]\right]^2\right]^{\frac{1}{2}} \\
&\leq e^{-\delta(\omega)}\sup_{i\in\mathbb{Z}}\max_{i-n\leq j\leq i+n}|\lambda_{i,j}(\omega)|\left[\sum_{i\in\mathbb{Z}}\left[\sum_{j=i-n}^{i+n}[f_j(u_j^1)-f_j(u_j^2)]\right]^2\right]^{\frac{1}{2}} \\
&\leq e^{-\delta(\omega)}\lambda(\omega)\left[\sum_{i\in\mathbb{Z}}\left[\sum_{j=i-n}^{i+n}f_j'(\xi_j)(u_j^1-u_j^2)\right]^2\right]^{\frac{1}{2}} \qquad \text{where } \xi_j \\
&\text{is between } v_j^1 \text{ and } v_j^2.
\end{aligned}$$

Thus,

$$\left\|e^{-\delta(\omega)}\left(A(\omega)e^{\delta(\omega)}v^1(A(\omega)e^{\delta(\omega)}v^2)\right)\right\| \leq (2n+1)^{\frac{1}{2}}e^{-\delta(\omega)}\lambda(\omega)L(||v^1||+||v^2||)\|v^1-v^2\|$$

Therefore,

$$\left\|F(v^1,\omega)-F(v^2,\omega)\right\| = \left(\alpha_\infty+(2n+1)^{\frac{1}{2}}\lambda(\omega)e^{-\delta(\omega)}L(||v^1||+||v^2||)+|\delta(\omega)|\right)\left\|v^1-v^2\right\|.$$

For any set $B\subset\ell^2$ with

$$\sup_{u\in B}||u|| \leq r$$

Defined

$$\xi_B(\omega) := \left(\alpha_\infty+(2n+1)\lambda(\omega)e^{-\delta(\omega)}L(2r)+|\delta(\omega)|2r+e^{-\delta(\omega)}||g||\right)$$

Then, $\xi_B(\omega)\geq 0$ and

$$\int_\tau^{\tau+1}\xi_B(\theta\omega)dt = \int_\tau^{\tau+1}\left(\alpha_\infty+(2n+1)\lambda(\omega)e^{-\delta(\omega)}L(2r)+|\delta(\omega)|2r+e^{-\delta(\omega)}||g||\right)dt<\infty \qquad \forall\tau\in\mathbb{R}$$

For any $v, v^1, v^2 \in B$.

$$\|F(v,\omega)\| \leq \xi_B(\omega).$$

and

$$\left\|F(v^1,\omega) - F(v^2,\omega)\right\| \leq \xi_B(\omega)\left\|v^1 - v^2\right\|.$$

Therefore, by [13], (4.8) has a unique mild solution $v\left(.;\omega,v_0,g\right) \in C\left([0,T_{max}),\ell^2\right)$ with $0 < T_{max} \leq T$, satisfying

$$v(t) = v_0 + \int_0^t \left(-\alpha v + e^{-\delta(\theta_s\omega)}\left(A(\theta_s\omega)e^{\delta(\theta_s\omega)}v\right) + e^{-\delta(\theta_s\omega)}g + \delta(\theta_s\omega)v\right)ds \tag{4.11}$$

To show that $T_{max} = T$, we proceed as follows:

Let $\left(\lambda_{i,j}^m(t,\omega)\right)_{m\in\mathbb{N}}$ be a of function in t such that for each $\omega \in \Omega_0$;

$$\lim_{m\to\infty}\int_0^\tau \left|\lambda_{i,j}^m(t,\omega) - \lambda_{i,j}(\theta_s\omega)\right|ds = 0. \qquad \forall\tau > 0 \qquad \text{(P1)}$$

and

$$\left|\lambda_{i,j}^m(t,\omega)\right| \leq \left|\lambda_{i,j}(\theta_t\omega)\right| \leq \left|\lambda(\theta_t\omega)\right|; \forall t \in \mathbb{R} \qquad \text{(P2)}$$

Consider the following RODE with $v_0 \in \ell^2$.

$$\frac{dv^m}{dt} = -\alpha v^m + e^{-\delta(\theta_t\omega)}\left(A_m(t,\omega)e^{\delta(\theta_t\omega)}v^m\right) + e^{\delta(\theta_t\omega)}g + \delta(\theta_t\omega)v^m \tag{4.12}$$

with $v^m(0) = v_0$ where

$$\left(A_m(t,\omega)v^m\right)_i = \sum_{j=i-n}^{i+n}\lambda_{i,j}^m(t,\omega)f_j(v_j^m)$$

Using the same argument as in (4.8), (4.12) has a uniques mild solution $v^m\left(.;\omega,v_0,g\right) \in C\left([0,T_{max}^m),\ell^2\right)$ with $0 < T_{max}^m \leq T$, satisfying

$$v^m(t) = v_0 + \int_0^t \left(-\alpha v^m + e^{-\delta(\theta_s\omega)}\left(A_m(s,\omega)e^{\delta(\theta_s\omega)}v^m\right) + e^{-\delta(\theta_s\omega)}g + \delta(\theta_s\omega)v^m\right)ds \tag{4.13}$$

by continuity of $A_m(s,\omega)$ in s we have that

$$\frac{dv_i^m}{dt} = -\alpha_i v_i^m + e^{-\delta(\theta_t\omega)}\left(A_m(s,\omega)(e^{\delta(\theta_t\omega)}v^m\right)_i + e^{-\delta(\theta_t\omega)})g_i + \delta(\theta_t\omega)v_i^m, \qquad t \in [0, T_{max}^m)$$

Multiplying by v_i^m and summing over $i \in \mathbb{Z}$ we get

$$\begin{aligned}
\frac{d}{dt}||v^m(t)||^2 &\leq -2\sum_{i\in\mathbb{Z}} \alpha_i (v_i^m)^2 + 2\sum_{i\in\mathbb{Z}} e^{-\delta(\theta_t\omega)} v_i^m \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(t,\omega) f_j(e^{\delta(\theta_t\omega)}v_j^m) \\
&\quad + 2e^{-\delta(\theta_t\omega)}\sum_{i\in\mathbb{Z}} g_i v_i^m + 2\delta(\theta_t\omega)\sum_{i\in\mathbb{Z}}(v_i^m)^2 \\
&\leq (2\alpha_\infty + a + 2\delta(\theta_t\omega))\,\|v^m\|^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g\|^2 \\
&\quad + 2\sum_{i\in\mathbb{Z}} e^{-\delta(\theta_t\omega)} v_i^m \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(t,\omega) f_j'(e^{\delta(\theta_t\omega)}\xi_j^m) e^{\delta(\theta_t\omega)} v_j^m, \\
&\quad \text{for some } a > 0 \text{ and } |\xi_j^m| \leq |v_j^m| \\
&\leq (2\alpha_\infty + a + 2\delta(\theta_t\omega))\,\|v^m\|^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g\|^2 \\
&\quad + 2(2n+1)\lambda(\theta_t\omega) L(e^{\delta(\theta_t\omega)}||v^m||)\,\|v^m\|^2
\end{aligned}$$

Using **(A3)** we have

$$\begin{aligned}
\frac{d}{dt}||v^m(t)||^2 &\leq (2\alpha_\infty + a + 2\delta(\theta_t\omega) + 2k_2(2n+1)\lambda(\theta_t\omega))\,\|v^m\|^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g\|^2 \\
&\quad + 2k_1(2n+1)\lambda(\theta_t\omega)\ln\left(e^{2\delta(\theta_t\omega)}||v^m||^2 + 1\right)\|v^m\|^2 \\
&\leq (2\alpha_\infty + a + 2\delta(\theta_t\omega) + 2k_2(2n+1)\lambda(\theta_t\omega))\,\|v^m\|^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g\|^2 \\
&\quad + 2k_1(2n+1)\lambda(\theta_t\omega)\ln\left(e^{2|\delta(\theta_t\omega)|}(||v^m||^2 + 1)\right)\|v^m\|^2 \\
&= (2\alpha_\infty + a + 2\delta(\theta_t\omega) + 2k_2(2n+1)\lambda(\theta_t\omega))\,\|v^m\|^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g\|^2 \\
&\quad + 2k_1(2n+1)\lambda(\theta_t\omega)\left(2|\delta(\theta_t\omega)| + \ln\left(||v^m||^2 + 1\right)\right)\|v^m\|^2
\end{aligned}$$

Therefore,

$$
\begin{aligned}
||v^m(t)||^2 \leq ||v^m(0)||^2 &+ \frac{1}{a}\int_0^t ||g||^2 e^{-2\delta(\theta_s\omega)}ds \\
&+ \int_0^t \left(2\alpha_\infty + a + 2\delta(\theta_s\omega) + 2(2n+1)\lambda(\theta_s\omega)(k_2 + 2|\delta(\theta_s\omega)|)\right) \|v^m\|^2 \, ds \\
&+ 2k_1(2n+1)\int_0^t \lambda(\theta_s\omega)\ln\left(||v^m||^2 + 1\right)\|v^m\|^2 \, ds
\end{aligned}
$$

Using lemma 3.1 in [46], we have

$$
\begin{aligned}
||v^m(t)||^2 \leq \max\Bigg\{ & 2\left(||v^m(0)||^2 + \frac{1}{a}\int_0^t ||g||^2 e^{-2\delta(\theta_s\omega)}ds\right) \\
& e^{2\int_0^t (2\alpha_\infty + a + 2\delta(\theta_s\omega) + 2(2n+1)\lambda(\theta_s\omega)(k_2+2|\delta(\theta_s\omega)|))ds}, \\
& 2^{e^{2\int_0^t 2k_1(2n+1)\lambda(\theta_s\omega)ds}} - 1 \Bigg\}
\end{aligned}
$$

Define

$$
\begin{aligned}
\eta(t)^2 := \max\Bigg\{ & 2\left(||v_0||^2 + \frac{1}{a}\int_0^t ||g||^2 e^{-2\delta(\theta_s\omega)}ds\right) \\
& e^{2\int_0^t (2\alpha_\infty + a + 2\delta(\theta_s\omega) + 2(2n+1)\lambda(\theta_s\omega)(k_2+2|\delta(\theta_s\omega)|))ds}, \\
& 2^{e^{2\int_0^t 2k_1(2n+1)\lambda(\theta_s\omega)ds}} - 1 \Bigg\}
\end{aligned}
$$

Then,

$$
||v^m(t)||^2 \leq \eta(t)^2 \text{ for } t \in [0, T^m_{\max}) \tag{4.14}
$$

where $\eta(t) \in C([0, T0, \mathbb{R})$ independent of m, which implies the interval of existence of $v^m(t)$ is $[0, T)$.

So,

$$
|v_i^m| \leq \eta(t) \text{ for all } m \in \mathbb{N}, t \in [0, T) \qquad \textbf{(P3)}
$$

and

$$|F_i^m(v^m,\theta_t\omega)| = |-\alpha_i v_i^m + e^{-\delta(\theta_t\omega)}(A_m(s,\omega)(e^{\delta(\theta_t\omega)}v^m))_i + e^{-\delta(\theta_t\omega)}g_i + \delta(\theta_t\omega)v_i^m|$$
$$\leq K(T,\omega), t \in [0,T) \text{ for some } K(T\omega) \geq 0$$

For any $t, \tau \in [0,T)$ we have

$$|v_i^m(t) - v_i^m(\tau)| \leq \int_\tau^t |F_i^m(v^m(s),\theta_t\omega)|ds$$
$$\leq K(T,\omega)|t-\tau|$$

by Azéla Ascoli theorem there is $\{v_i^{m_k}(t)\} \subset \{v_i^m(t)\}$ such that

$$v_i^{m_k} \longrightarrow \tilde{v}(t) \text{ as } k \to \infty \text{ for } t \in [0,T)$$

and $\tilde{v}_i(t)$ is continuous in $t \in [0,T)$, with

$$|\tilde{v}_i(t)| \leq \eta(t) \text{ for } t \in [0,T)$$

Therefore, by **(P1),(P2), (P3),(A2)** and the lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty}\int_0^t (\lambda_{i,j}^{m_k}(s,\omega) - \lambda_{i,j}(\theta_t\omega))ds = \int_0^t \lim_{k\to\infty}(\lambda_{i,j}^{m_k}(s,\omega) - \lambda_{i,j}(\theta_t\omega))ds$$
$$= 0,$$

$$\lim_{k\to\infty}(\lambda_{i,j}^{m_k}(s,\omega) - \lambda_{i,j}(\theta_s\omega)) = 0 \text{ a.e } s \in [0,T),$$

and

$$\lim_{k\to\infty}\int_0^t (\lambda_{i,j}^{m_k}(s,\omega)v_i^{m_k}(s) - \lambda_{i,j}(\theta_s\omega)\tilde{v}_i(s))ds = \int_0^t \lim_{k\to\infty}(\lambda_{i,j}^{m_k}(s,\omega)v_i^{m_k}(s) - \lambda_{i,j}(\theta_s\omega)\tilde{v}_i(s))ds$$
$$= 0$$

replacing m by m_k in (4.13) and letting $k \to \infty$ we get

$$\bar{v}_i(t) = v_i(0) + \int_0^t \left(-\alpha_i \bar{v}_i + (e^{-\delta(\theta_s\omega)} A(\theta_s\omega)(e^{\delta(\theta_s\omega)}\bar{v})_i + e^{\delta(\theta_s\omega)} g_i + \delta(\theta_s\omega)\bar{v}_i\right) ds$$
$$\text{for } t \in [0, T)$$

by uniqueness of mild solution of (4.8) we have

$$\bar{v}_i(t) = v_i(t) \qquad \text{for } t \in [0, T)$$

and by

$$\|v^m(t)\|^2 \leq \eta^2(t) \qquad \text{we get } \|v_i(t)\| \leq \eta(t)$$

which implies $T_{\max} = T$.

Now we show that there is a continuous map

$$\Phi : (\ell^2, ||.||_\rho) \times (\ell^2, ||.||_\rho) \to C([0, T), \ell^2_\rho)$$

Let $v_0^1, v_0^2, g^1, g^2 \in \ell^2$ and let $v^{1,m}, v^{2,m}$ be solutions of (4.12) with initials v_0^1, v_0^2 respectively. define

$$h^m(t) := v^{1,m}(t) - v^{2,m}(t) \in \ell^2 \subset \ell^2_\rho$$

Thus, we have

$$\frac{\mathrm{d}}{dt} h_i^m(t) = -\alpha_i h_i^m(t) + e^{-\delta(\theta_t\omega)} \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(\theta_t\omega) \left(f_j(e^{\delta(\theta_t\omega)} v_j^{1,m}) - f_j(e^{\delta(\theta_t\omega)} v_j^{1,m})\right)$$
$$+ e^{-\delta(\theta_t\omega)}(g_i^1 - g_i^2) + \delta(\theta_t\omega) h_i^m$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} &\leq -2\sum_i \alpha_i\rho_i(h_i^m)^2 + 2e^{-\delta(\theta_t\omega)}\sum_i \rho_i \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(\theta_t\omega)L(e^{\delta(\theta_t\omega)}(||v1,m||_\rho + ||v^{2,m}||_\rho))|h_j^m|h_i^m \\ &+ 2\sum_i e^{-\delta(\theta_t\omega)}\rho_i(g_i^1 - g_i^2)h_i^m + 2\delta(\theta_t\omega)\|h^m\|_\rho^2 \\ &\leq (2\alpha_\infty + 2\delta(\theta_t\omega) + a)\,\|h^m\|_\rho^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g^1 - g^2\|_\rho^2 \\ &+ 2(2n+1)\lambda(\theta_t\omega)L(e^{\delta(\theta_t\omega)}(||v1,m||_\rho + ||v^{2,m}||)_\rho)\|h^m\|_\rho^2\end{aligned}$$

using (4.14) we have

$$L(e^{\delta(\theta_t\omega)}(||v1,m||_\rho + ||v^{2,m}||)_\rho) \leq L_T$$

where L_T depends on $T, g^1, g^2, k_1, k_2, \delta$ and λ. Thus we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^m(t)\|_\rho^2 &\leq (2\alpha_\infty + 2\delta(\theta_t\omega) + a + 2(2n+1)L_T\lambda(\theta_t\omega))\,\|h^m\|_\rho^2 \\ &+ \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g^1 - g^2\|_\rho^2 \end{aligned} \tag{4.15}$$

from (4.15) and Gronwall's inequality it follows that

$$\|v^{1,m}(t) - v^{2,m}(t)\|^2 \leq C_T(\|v_0^1 - v_0^2||_\rho + \|g^1 - g^2\|_\rho) \qquad \forall t \in [0,T) \tag{4.16}$$

where C_T depends on T. Using the same argument as above we have that, there is $m_k \to \infty$ such that

$$\begin{aligned}\lim_{k\to\infty} v^{1,m_k}(t) &= v(t;\omega,v_0^1,g^1) \\ \lim_{k\to\infty} v^{2,m_k}(t) &= v(t;\omega,v_0^2,g^2) \qquad \forall t \in [0,T)\end{aligned} \tag{4.17}$$

by (4.16) and(4.17) we get

$$\|v(t;\omega,v_0^1,g^1) - v(t;\omega,v_0^2,g^2)\|_\rho \leq C_T(\|v_0^1 - v_0^1\|_\rho + \|g^1 - g^2\|_r ho) \qquad \forall t \in [0,T) \tag{4.18}$$

Thus, from (4.18) we see that for any given $T > 0$, there is a continuous map

$$\Phi : (\ell^2, ||.||_\rho) \times (\ell^2, ||.||_\rho) \to C([0,T), \ell^2_\rho)$$
$$(v_0, g) \mapsto v(\cdot; \omega, v_0, g)$$

Since $(\ell^2, ||.||_\rho) \times (\ell^2, ||.||_\rho)$ is dense in $\ell^2_\rho \times \ell^2_\rho$, Φ can be extended uniquely to a continuous function

$$\tilde{\Phi} : \ell^2_\rho \times \ell^2_\rho \to C([0,T), \ell^2_\rho) \qquad \text{for } T > 0$$

Next we need to show that for $v_0, g \in \ell^2_\rho$, $v(\cdot; \omega, v_0, g)$ satisfies the integral equation (4.11). Now, since $(\ell^2, ||.||_\rho) \times (\ell^2, ||.||_\rho)$ is dense in $\ell^2_\rho \times \ell^2_\rho$ we have $\exists\ \{v_0^l\}, \{g^l\} \subset (\ell^2, ||\cdot||_\rho)$ such that

$$\|v_0^l - v_0\|_\rho \to 0, \|g^l - g\|_\rho \to 0 \qquad \text{as } l \to \infty$$

Let $v^{l,m}(t) = v(t; \omega, v_0^l, g^l) \in (\ell^2, ||\cdot||_\rho)$ be the solution of (4.12) then we have

$$v_i^{l,m}(t) = v_{0,i}^l + \int_0^t -\alpha_i v_i^{l,m} + e^{-\delta(\theta_t\omega)} \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(\theta_t\omega) f_j(e^{\delta(\theta_t\omega)} v_j^{l,m})$$
$$+ \delta(\theta_t\omega) v_i^{i,m} + e^{\delta(\theta_t\omega)} g_i^l \tag{4.19}$$

multiplying (4.19) by $\rho_i v_i^{l,m}$ and summing over all $i \in \mathbb{Z}$ we get for some $a > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v^{l,m}(t)\|_\rho^2 \le (2\alpha_\infty + 2\delta(\theta_t\omega) + a)\|v^{l,m}\|_\rho^2 + \frac{1}{a}e^{-2\delta(\theta_t\omega)}\|g^l\|_\rho^2$$
$$+ 2\sum_i \rho_i v_i^{l,m} \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega) f_j(e^{\delta(\theta_t\omega)} v_j^{l,m})$$

using similar argument as in proof of (4.14) we have

$$\|v^{l,m}(t)\| \le \eta^2(t)$$

and $\eta(t)$ is independent of l, m. Using similar previous arguments we have that $v(t;\omega,v_0,g)$ is the limiting function of the subsequence of the $\{v^{l,m}\}$ in ℓ^2_ρ and $v(t;\omega,v_0,g)$ satisfies (4.11) for $t \in [0,T)$.

For additive noise (4.10) we have

$$F(v,\omega) = -\alpha v + A(\theta_t\omega)(v+z) + g$$

using **(A2)** through **(A4)** we have that F is continuous in v, measurable in ω, locally bounded and Lipschitz continuous in v.

Thus, for $v_0, g \in \ell^2$ (4.10) possess a unique mild solution $v(\cdot;\omega,v_0,g) \in C([0,T_{\max}),\ell^2)$ which satisfies

$$v(t) = v_0 + \int_0^t (-\alpha v + A(\theta_s\omega)(v+z) + g)ds \qquad t \in [0,T_{\max})$$

similar to what we did in (4.8) we consider the following

$$\frac{\mathrm{d}}{\mathrm{d}t}v^m(t) = -\alpha v^m + A_m(t,\omega)(v^m+z) + g \qquad v^m(0) = v_0 \tag{4.20}$$

where z is as in (4.9), and $A_m(t,\omega)(v^m+z) = \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega) f_j(v_j^m + z_j)$.

Then, (4.20) has a unique mild solution $v^m(\cdot;\omega,v_0,g) \in C([0,T^m_{\max}),\ell^2) \cap C^1([0,T^m_{\max}),\ell^2)$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|v^m\|^2 &\le 2\Big(\alpha_\infty\|v^m\|^2 + \sum_i \sum_{j=i-n}^{i+n} \lambda_{i,j}^m(t,\omega) f f_j(v_j^m + z_j) + (v^m, g)\Big) \\
&\le (2\alpha_\infty + a)\|v^m\|^2 + \frac{1}{a}\|g\|^2 + 2\lambda(\theta_t\omega)\sum_i \sum_{j=i-n}^{i+n} f'(\xi_j^m)|v_i^m||v_j^m + z_j| \\
&\text{for } a > 0, |\xi_j^m| \le |v_j^m + z_j| \\
&\le (2\alpha_\infty + a)\|v^m\|^2 + \frac{1}{a}\|g\|^2 + 2\lambda(\theta_t\omega)\sum_i \sum_{j=i-n}^{i+n} L(||v^m|| + ||z||)|v_i^m|(|v_j^m| + |z_j|) \\
&\le (2\alpha_\infty + a)\|v^m\|^2 + \frac{1}{a}\|g\|^2 + 2\lambda(\theta_t\omega)\sum_i \sum_{j=i-n}^{i+n} L(||v^m|| + ||z||)|v_i^m||v_j^m| \\
&+ 2\lambda(\theta_t\omega)\sum_i \sum_{j=i-n}^{i+n} L(||v^m|| + ||z||)|v_i^m||z_j| \\
&\le (2\alpha_\infty + a)\|v^m\|^2 + \frac{1}{a}\|g\|^2 + 3(2n+1)\lambda(\theta_t\omega)L(||v^m|| + ||z||)\|v^m\|^2 \\
&+ \lambda(\theta_t\omega)L(||v^m|| + ||z||)\|z\|^2 \\
&\le (2\alpha_\infty + a)\|v^m\|^2 + \frac{1}{a}\|g\|^2 + 3(2n+1)\lambda(\theta_t\omega)k_2\|v^m\|^2 \\
&+ 3(2n+1)k_1\lambda(\theta_t\omega)\ln((||v^m|| + ||z||)^2 + 1)\|v^m\|^2 \\
&+ k_2\lambda(\theta_t\omega)\|z\|^2 + k_1\lambda(\theta_t\omega)\ln((||v^m|| + ||z||)^2 + 1)\|z\|^2 \\
&= (2\alpha_\infty + a + 3(2n+1)\lambda(\theta_t\omega)k_2)\|v^m\|^2 + \frac{1}{a}\|g\|^2 \\
&+ k_2\lambda(\theta_t\omega)\|z\|^2 \\
&+ (k_1\lambda(\theta_t\omega)\|z\|^2 + 3(2n+1)k_1\lambda(\theta_t\omega)\|v^m\|^2)\ln((\|v^m\| + \|z\|)^2 + 1)
\end{aligned}$$

Let

$$w(t) = \frac{1}{a}\|g\|^2 + k_2\lambda(\theta_t\omega)\|z\|^2,\ \gamma(t) = 2\alpha_\infty + a + 3(2n+1)\lambda(\theta_t\omega)k_2$$

and

$$\nu(t) = 3(2n+1)k_1\lambda(\theta_t\omega)$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v^m\|^2 \leq \nu(t)\ln((\|v^m\|+\|z\|)^2+1)(\|v^m\|^2+\|z\|^2)+\gamma(t)\|v^m\|^2+w(t)$$

therefore,

$$\begin{aligned}
\|v^m\|^2 &\leq \|v_0\|^2+\int_0^t\Big(\nu(s)\ln((\|v^m\|+\|z\|)^2+1)(\|v^m\|^2+\|z\|^2)\Big)ds+\int_0^t\gamma(s)\|v^m\|^2ds\\
&+\int_0^t w(s)ds\\
&\leq \|v_0\|^2+\int_0^t\Big(\nu(s)\ln(2(\|v^m\|^2+\|z\|^2+1))(\|v^m\|^2+\|z\|^2)\Big)ds\\
&+\int_0^t\gamma(s)(\|v^m\|^2+\|z\|^2)ds+\int_0^t w(s)ds\\
&= \|v_0\|^2+\int_0^t\Big(\nu(s)\ln(\|v^m\|^2+\|z\|^2+1)(\|v^m\|^2+\|z\|^2)\Big)ds\\
&+\int_0^t(\gamma(s)+\ln(2)\nu(s))(\|v^m\|^2+\|z\|^2)ds+\int_0^t w(s)ds
\end{aligned}$$

which implies

$$\begin{aligned}
\|v^m\|^2+\|z\|^2 \leq \|v_0\|^2+\|z\|^2+\int_0^t&\Big(\nu(s)\ln(\|v^m\|^2+\|z\|^2+1)(\|v^m\|^2+\|z\|^2)\Big)ds\\
&+\int_0^t(\gamma(s)+\ln(2)\nu(s))(\|v^m\|^2+\|z\|^2)ds+\int_0^t w(s)ds \qquad (4.21)
\end{aligned}$$

Let $h(t)=\|v^m\|^2+\|z\|^2$ then (4.21) becomes

$$\begin{aligned}
h(t) &\leq \|v_0\|^2+\|z\|^2+\int_0^t\nu(s)\ln(h(s)+1)h(s)ds\\
&+\int_0^t(\gamma(s)+\ln(2)\nu(s))h(s)ds+\int_0^t w(s)ds \qquad (4.22)
\end{aligned}$$

applying lemma 3.1 in [46], we have

$$h(t) \leq 2\max\Big\{(\|v_0\|^2 + \|z\|^2 + \int_0^t w(s)ds)e^{2\int_0^t(\gamma(s)+\ln(2)\nu(s))ds}, \\ 2^{e^2\int_0^t \nu(s)} - 1\Big\}$$

hence

$$\|v^m(t)\|^2 \leq \eta_1^2(t) \qquad \forall t \in [0, T^m_{\max})$$

for some $\eta_1^2(t)$ independent of m, which implies (using the same argument as in the proof of the solution of (4.8)) the interval of existence of $v^m(t)$ is $[0, T)$.

To complete the proof we need to show that for $\tilde{v}_0, g \in \ell^2_\rho$, (4.10) has a mild solution $v(t; \omega, \tilde{v}_0, g)$ on $[0, T)$ that satisfies $v(0; \omega, \tilde{v}_0, g) = \tilde{v}_0$. Like in the previous case (4.8) it suffices to proof Lipschitz continuity of the solution on bounded subset of $(\ell^2, ||\cdot||_\rho) \times (\ell^2, ||\cdot||_\rho)$.
Let $g^1, g^2, v_0^1, v_0^2 \in \ell^2$ and $v^{1,m}(t; \omega, v_0^1, g^1) \& v^{2,m}(t; \omega, v_0^2, g^2)$ be the solution of (4.20). Let $h^m(t) = v^{1,m} - v^{2,m} \in \ell^2 \subset \ell^2_\rho$ then,

$$\frac{\mathrm{d}}{\mathrm{d}t}h_i^m(t) = -\alpha h_i^m \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega)(f_j(v_j^{1,m} + z_j) - f_j(v^{2,m} + z_j)) + g_i^1 - g_i^2$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h^m\|_\rho^2 \leq 2(\alpha_\infty + 1)\|h^m\|_\rho^2 + \|g^1 - g^2\|_\rho^2 + 2\sum_i \rho_i \sum_{j=i-n}^{i+n} \lambda_{i,j}(\theta_t\omega) f_j'(\xi_j^m) h_j^m h_i^m$$
$$\leq 2(\alpha_\infty + 1 + 2(2n+1)\lambda(\theta_t\omega)L_T)\|h^m\|_\rho^2 + \|g^1 - g^2\|_\rho^2 \tag{4.23}$$

where L_T is similar as before. From (4.23) and Gronwall's inequality we get

$$\|v^{1,m}(t) - v^{2,m}(t)\|_\rho^2 \leq C_1(\|v_0^1 - v_0^2\|_\rho^2 + \|g^1 - g^2\|_\rho^2) \tag{4.24}$$

where C_1 depends on $T, \lambda(\omega), v_0^1, v_0^2$.

4.2 Non autonomous Hopfield-type lattice model

Recently, the dynamical behavior of (1.1) has been studied when the number of neurons is increasingly large (that is $n \to \infty$) see [46]. Our aim in this section is to study the dynamical behavior of (1.1) when, in addition to $n \to \infty$, the forcing is dependent on time (that is $I = I(t)$).

To this end our interest is in the dynamics of the following nonautonomous lattice model

$$\mu_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{\gamma_i} + \sum_{j\in\mathbb{Z}} \lambda_{i,j} g_j(u_j(t)) + I_i(t), \quad i \in \mathbb{Z}. \tag{4.25}$$

Nonautonomuous differential equation has shown to be very useful in application for example see ([21],[47],[12], [49], [37], [15]). The rest of the work is to investigate the existence of pullback, forward and pullback exponential attractors.

References

[1] A. Y. Abdallah. Upper semicontinuity of the attractor for a second order lattice dynamical system. *Disc. Cont. Dyn. Syst. Ser. B*, 5(4):899–916, 2005.

[2] A. Y. Abdallah. Upper semicontinuity of the attractor for lattice dynamical systems of partly dissipative reaction-diffusion systems. *J. Appl. Math.*, 3:273–288, 2005.

[3] V. S. Afraimovich and V. I. Nekorkin. Chaos of traveling waves in a discrete chain of diffusively coupled maps. *Int. J. Bifur. Chaos*, 4:631–637, 1994.

[4] C. Aouiti. Oscillation of impulsive neutral delay generalized high-order Hopfield neural networks. *Neural Comput. and Appl.*, 29:477–495, 2018.

[5] C. Aouiti, M. S. Mahamdi, J. Cao, and A. Alsaedi. Piecewise pseudo almost periodic solution for impulsive generalised high-order Hopfiled neural networks with leakage delays. *Neural Process Lett.*, 45:615–648, 2017.

[6] L. Arnold. *Random Dynamical system*. Springer-Verlag, Berlin, 1998.

[7] J. M. Arrieta. Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions. *J. Diff. Eqs.*, 168:33–59, 2000.

[8] J. M. Ball. Continuity properties and global attractors of generalized semiflows and the Navier- Stokes equations. *J. Nonl. Sci.*, 7:475–502, 1997.

[9] P. W. Bate, K. Lu, and B. Wang. Attractors for lattice systems. *Int. J. Bifur. Chaos*, 11:143–153, 2001.

[10] P. W. Bates and A. Chmaj. On a discrete convolution model for phase transitions. *Archive for Rational Mechanics and Analysis*, 150:281–368, 1999.

[11] P. W. Bates, H. Lisei, and K. Lu. Attractors for stochastic lattice dynamical systems. *Stochastic and Dynamics*, 6:1–21, 2006.

[12] W. J. Beyn and S. Yu Piyugin. Attractors of reaction difussion system of infinite lattice. *J. Dyn. Diff. Eqs.*, 15:485–515, 2003.

[13] F. E. Browder. Nonlinear mappings of nonexpansive and accretive type in banach spaces. *Bull. Amer. Math. Soc.*, 72:875–882, 1967.

[14] J. Cao and Q. Song. Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays. *London Mathematical Society: Nonlinearity*, 19:1601–1617, 2006.

[15] T. Caraballo and X. Han. *Applied Nonautonomous and Random Dynamical Systems*. BCAM SpringerBrief, Springer-Verlag, Cham, 2016.

[16] T. Caraballo and J. A. Langa. On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems. *Dynamics of continuous, Discrete and Impulsive systems, Series A: Mathematical Analysis*, 10:491–513, 2003.

[17] T. Caraballo and K. Lu. Attractors for stochastic lattice dynamical systems with a multiplicative noise. *Front. Math. China*, 3:317–335, 2008.

[18] T. Caraballo, G. Lukaszewicz, and J. Real. Pullback attractor for asymptotically compact nonautonomous dynamical systems. *Nonlinear Analysis TMA*, 6:484–498, 2006.

[19] P. Chittora, A. Singh, and M. Singh. Application of Hopfield neural network for harmonic current estimation and shunt compensation. *Electric Power Components and Systems*, 46:290–301, 2018.

[20] S.-N. Chow. *Lattice Dynamical Systems, in: Lecture notes in Math. Vol. 1822*. Springer-Verlag, Berlin, 2003.

[21] S.-N. Chow and J. Mallet-Paret. Pattern formation and spatial chaos in lattice dynamical systems. *IEEE Trans. Circuits Syst.*, 42:746–751, 1995.

[22] S.-N. Chow and J. Mallet-Paret. Pattern formation and spatial chaos in lattice dynamical systems, i, ii. *IEEE Trans. Circuits Systems*, 42:746–751, 1995.

[23] S.-N. Chow, J. Mallet-Paret, and W. Shen. Traveling waves in lattice dynamical systems. *J. Diff. Eqs.*, 149:248–291, 1998.

[24] I. Chueshov. *Monotone Random Systems Theory and Applications*. Springer-Verlag, Berlin, Heidelberg, 2002.

[25] H. Crauel and F. Flandoli. Attractors for random dynamical systems. *Probab. Related Fields*, 100:365–393, 1994.

[26] F. Z. de Castro and M. E. Valle. Continuous-valued octonionic Hopfield neural network. *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics*, 6, 2018.

[27] T. Deb, A. K. Ghosh, and A. Mukherjee. Singular value decomposition applied to associative memory of Hopfield neural network. *Matrialstoday*, 5:2222–2228, 2018.

[28] K. Demling. *Differential Equations on Banach Spaces*. Springer-Verlag, Hielberg, 1977.

[29] R. D. Dony and S. Haykin. Neural networks approaches to image compression. *IEEE*, 83 No. 2:288–303, 1995.

[30] H. Engler. Global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity. *Mathematische Zeitschrift*, 202:251–259, 1989.

[31] F. Fandoli and B. Schmalfuss. Random attractor for the 3d stochastic navier-stokes equation with multiplicative noise. *Stochastics and Stochastics Reports*, 59:21–45, 1996.

[32] F. Favata and R. Walker. A study of the application of Kohonen-type neural networks to the traveling salesman problem. *Biological Cybernetics*, 64:463–468, 1991.

[33] G. Faye. Traveling fronts for lattice neural field equations. *Physica D*, 378–379:20–32, 2018.

[34] J. Zabczyk G. Da Prato. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, Cambridge, 1992.

[35] E. Grusen, G. Kayakutlu, and T. U. Daim. Using artificial neural network models in stock market index prediction. *Expert System with Application*, 38:10389–10397, 2011.

[36] A. Gu and Y. Li. A combined criterion for existence and continuity of random attractors for stochastic lattice dynamical systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 27(2):1750019, 14, 2017.

[37] J. K. Hale. *Asymptotic Behavior of Dissipative Systems*. AMS, Province, RI, 1988.

[38] J. K. Hale, X. Lin, and G. Raugel. Upper semicontinuity of attractors for approximations of semigroups and partial differential equations. *Mathematics of Computations*, 50:89–123, 1988.

[39] X. Han. Asymptotic behavior of stochastic partly dissipative lattice systems in weighted space. *international journal of differential equation*, 2011:23, 2011.

[40] X. Han and P. E. Kloeden. *Discretisation of Attractors*. SpringerBrief, Springer, Cham, 2017.

[41] X. Han and P. E. Kloeden. *Random Ordinary Differential Equations and Their Numerical Solution*. Springer, Singapore, 2017.

[42] X. Han and P. E. Kloeden. Asymptotic behaviour of a neural field lattice model with a Heaviside operator. *Phys. D*, page doi.org/10.1016/j.physd.2018.09.004, 2018.

[43] X. Han, P. E. Kloeden, and S. Sonner. Discretisation of global attractors for lattice dynamical systems. *J. Dyn. Diff. Eqs.*, pages 1–18, 2019.

[44] X. Han, P. E. Kloeden, and B. Usman. Long term behavior of a random Hopfield neural lattice model. *Comm. Pure Appl. Anal.*, 18:809–824, 2019.

[45] L. Hardy. An application of neurohydrodynamics to a Hopfield neural network. *International Joint Conference on Neural Networks*, 2015.

[46] J. J. Hopfield. Neurons with graded response have collective computational properties like those of two-stage neurons. *Proc. Nat. Acad. Sci. U.S.A*, 81:3088–3092, 1984.

[47] Q. Jia, S. Zhou, and F. Yin. Kolmogorov entropy of global attractor for dissipative lattice dynamical systems. *J. Math. Phys.*, 44:5804–5810, 2003.

[48] T. Kimoto, K. Asakawa, M. Yoda, and M. Takeoka. Stock market prediction system with modular neural networks. *IEE Xplore*, pages 1–6, 1990.

[49] Peter E. Kloeden and Thomas Lorenz. construction of nonautonomous forward attractors. *Proceedings of the American Mathematical Society*, 144:459–468, 2016.

[50] M. Kobayashi. Hybrid quaternionic Hopfield neural network. *IEICE Trans. Fundam. Electron., Commun. Comput. Sci.*, E98-A, no. 7:1512–1518, 2015.

[51] M. Kobayashi. Decomposition of rotor Hopfield neural networks using complex numbers. *IEEE Transactions on Neural Networks and Learning Systems*, 29:1366–1370, 2018.

[52] M. Kobayashi. Hyperbolic Hopfield neural networks with directional multistate activation function. *Elsevier: Neurocomputing*, 275:2217–2226, 2018.

[53] X. Liang, L. Wang, and R. Wang. Random attractor of reaction-diffusion Hopfield neural networks driven by wiener processes. *Mathematical Problems in Engineering*, 2018:1–11, 2018.

[54] X. Liao and J. Yu. Robust stability for interval Hopfield neural networks with time delay. *IEEE Transactions on Neural Network*, 9:1042–1045, 1998.

[55] J. Mallet-Paret. The global structure of traveling waves in spatially discrete dynamical systems. *J. Dyn. Diff. Eqs.*, 11:49–127, 1999.

[56] N. Mani and B. Srinivasan. Application of artificial neural network model for optical character recognition. *IEEE Xplore*, 100:2517–2520, 1997.

[57] T. Minemoto, T. Isokawa, H. Nishimura, and N. Matsui. Quaternionic multistate Hopfield neural network with extended projection rule. *Artif Life Robotics*, 21:106–111, 2016.

[58] Z. T. Njitacke, J. Kengne, and H. B. Fotsin. A plethora of behaviors in a memristor based Hopfield neural networks (hnns). *International Journal of Dynamics and Control*, pages 1–17, 2018.

[59] N. Ozdemir, B. B. Iskender, and N. Y. Ozgur. Complex valued neural network with Mabius activation function. *Commun Nonlinear Sci Numer Simulat*, 16:4698–4703, 2011.

[60] R. Rakkiyappan, G. Velmurugan, and J. Cao. Stability analysis of fractional-order complex-valued neural networks with time delays. *Chaos, Soliton and Fractals*, 78:297–316, 2015.

[61] R. Sammouda, N. Adgaba, A. Touir, and A. Al-Ghamdi. Agriculture satellite image segmentation using a modified artificial Hopfield neural network. *Computers in Human Behavior*, 30:436–441, 2014.

[62] W. Shen. Lifted lattice, hyperbolic structure, and topological disorder in coupled map lattices. *SIAM J. Appl. Math.*, 56:1379–1399, 1996.

[63] P. van den Driessche and X. Zou. Global attractivity in delayed Hopfield neural network model. *SIAM J. Appl. Math*, 58:1878–1890, 1998.

[64] E. Van Vleck and B. Wang. Attractors for lattice fitzHugh-nagumo systems. *Phys. D*, 221:317–336, 2005.

[65] L. Wan, Q. Zhou, and J. Liu. Delay-dependent attractor analysis of Hopfield neural networks with time-varying delays. *Chaos, Soliton and Fractals*, 101:68–72, 2017.

[66] B. Wang. Dynamics of system of infinite lattices. *J. Diff. Eqs.*, 221:224–245, 2006.

[67] B. Wang. Asymptotic behavior of non-autonomous lattice systems. *J. Math. Anal. Appl.*, 331:121–136, 2007.

[68] H. Wang, Y. Yu, and G. Wen. Stability analysis of fractional-order Hopfield neural networks with time delays. *Neural Networks*, 55:98–109, 2014.

[69] Q. Wang, W. Shi, P. M. Atkinson, and Z. Li. Land cover change detection at subpixel resolution with a Hopfield neural network. *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing,*, 8:1339–1352, 2015.

[70] J. Yang, L. Wang, Y. Wang, and T. Guo. A novel memristive Hopfield neural network with application in associative memory. *Neurocomputing*, 227:142–148, 2017.

[71] J. Yao, T. Wang, Z. Wan, S. Chen, Q. Niu, and L. Zhang. Identification of acceleration harmonics for a hydraulic shaking table by using Hopfield neural network. *Scientia Iranica*, 25:299–310, 2018.

[72] Q. Yao, L. Wang, and Y. Wang. Existence-uniqueness and stability of reaction-diffusion stochastic Hopfield neural networks with s-type distributed time delays. *Elsevier: Neurocomputing*, 275:470–477, 2018.

[73] M. Zarco and T. Froese. Self-modeling in Hopfield neural networks with continuous activation function. *Procedia Computer Science*, 123:573–578, 2018.

[74] S. Zhou. Attractors and approximations for lattice dynamical systems. *J. Diff. Eqs.*, 200(2):342–368, 2004.

[75] S. Zhou. Attractors for second order lattice dynamical systems. *J. Diff. Eqs.*, 179:605–624, 2007.

[76] B. Zinner. Existence of traveling wavefront solutions for the discrete Nagumo equation. *J. Diff. Eqs.*, 96:1–27, 1992.