Nonlocal Dispersal Equations with Almost Periodic Dependence

by

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A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 6, 2022

Keywords: Nonlocal dispersal, almost periodic, top Lyapunov exponents, generalized principal eigenvalues, spreading speeds.

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Abstract

Nonlocal dispersal equations are used to model the population dynamics of species that exhibit long-range dispersal mechanisms. This model of spatial spread is obtained by replacing the Laplacian in the usual reaction-diffusion equation with an integral operator. Most studies on this model were done for temporally homogeneous or periodic environments. However, nature is typically heterogeneous, and even when seasonal, the variations could exhibit disproportionate periods. Therefore, it seems more appropriate to incorporate the time-dependent variability of these factors using almost periodicity. The asymptotic behavior of solutions with strictly positive initials is among the fundamental issues for such population models and the stability of the zero solution is crucial in investigating these asymptotic dynamics. Thus the principal spectral theory of the linearization of the model at the zero solution is important in its own right but vital for investigating the asymptotic dynamics. This dissertation is devoted to the study of the spectral theory and asymptotic dynamics of nonlocal dispersal equations with almost periodic dependence. First, the principal spectral theory of linear nonlocal dispersal equations is investigated from three aspects: top Lyapunov exponents, principal dynamical spectrum point, and generalized principal eigenvalues. Among others, we established the equality of the top Lyapunov exponents and the principal dynamical spectrum point, provided various characterizations of the top Lyapunov exponents and generalized principal eigenvalues, established the relations between them, and studied the effect of time and space variations on them. Secondly, employing the principal spectral theory developed in the first part, we studied the asymptotic dynamics of nonlinear nonlocal dispersal equations with almost periodic dependence. In particular, we established the existence, uniqueness, and stability of a strictly positive, bounded, entire, almost periodic solution of the Fisher-KPP equation with nonlocal dispersal and almost periodic reaction term. Finally, when the domain is the whole space $\mathbb{R}^N$, we investigated the spatial spreading speeds of positive solutions with front-like initials.

Acknowledgments

Words are insufficient to articulate my sentiments of gratitude; first, to the Almighty God who has shown me so much mercy and to my blessed mother Mary who has been my closest companion in this PhD journey. Having Dr. Wenxian Shen as my PhD advisor has been one of the big blessings of my academic life. It is just like having an academic mother. I will be eternally grateful for her compassionate mentoring. Her enthusiasm for learning is quite contagious and her generosity with ideas is both inspiring and edifying. I believe the best way to thank her is to work hard to be like her ”when I grow up.” To my PhD committee members, Dr. Dmitry Glotov, Dr. Hans Werner Van Wyk, and Dr. Selim Sukhtaiev and my University reader Dr. Ahyi, Ayayi Claude, I say thank you so much for your generous availability, kind suggestions, and support. To my academic ”big brothers” in the Dr. Shen academic family: Dr. Rachidi Salako and Dr. Tahir Issa, I cannot thank you enough for the great privilege of having your continual guidance and help during the ”PhD grind.”

I present a basket of thanks to my beloved department of Mathematics and Statistics, Auburn University for providing such a conducive environment and great opportunity for comprehensive growth both in teaching and research. To the faculty, staff and colleagues who made Parker Hall such a welcoming environment I say thank you so much. I cannot forget my special classmates and friends: Dr. Mariam Khachatryan, Dr. Fauziya Yakassai, Dr. Yuzhen Liu and Dr. Mark Uzochukwu who made this journey quite memorable. I feel blessed to have started with you and formed beautiful friendships that will last a lifetime. Thank you for your invaluable moral support.

To my families both natural and religious, I say a very big thank you for your encouragement. Special thanks to my mother Mrs. Julia Onyido, my prime mentor and heroine, whose foresight and sacrifices instilled in me the love for learning and my uncle and auntie late Mr and Mrs. Linus and Augustina Aladi whose endorsement imbued me with the desire to work hard academically. To my siblings especially my sister Mrs. Chidimma Egbule (Nee Onyido), thank you for always being there for me, especially for being my spiritual backup and uncompromising cheerleader. I feel so blessed to have you. May God bless and keep you all for me. To my religious sisters of the Immaculate Heart of Mary mother of Christ, especially the North-American region, I say a very big thank you for your love and support. It meant so much to me. I cannot fail to mention in a special way Sr. Maureen Jovita Nwajiobi who has been such a pillar of support, you're such a treasure to have. May God bless and reward you. To Chinedu Eleh, my brother from a different mother, I say thank you so much for making Auburn feel like home.

I have been exceptionally blessed with amazing friends whose push have provided me with the courage and stability to prevail. To all my ”accountability partners” I say thank you for providing the drive to make this a reality. I will remain eternally grateful to the St. Michaels' Catholic community in Auburn; may God bless you for affording me the opportunity to enjoy such a holy Catholic environment. It has been a great privilege to enjoy this loveliest village in the plains. To the entire Auburn University - the unique Auburn family, I say a very big thank you for that priceless Auburn spirit - a spirit that is not afraid. ...I believe in Auburn and love it. ...War Eagle!!!

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Chapter 1

Introduction

Understanding the changes in a species' population over time is an essential biological issue. The two major factors that influence these changes are the species' dispersal mechanisms (through which a species expands the distribution of its population) and environmental conditions (like resource availability, growth or proliferation rate, and other limiting factors). Environmental factors could be homogeneous or exhibit seasonal variations. However, nature is typically heterogeneous, and the factors that influence the evolution of populations are roughly but not exactly periodic. For instance, some of these factors may depend on weather cycles which may exhibit different disproportionate periods. Therefore, it seems more appropriate to incorporate the time-dependent variability of these factors using almost periodicity. This dissertation is devoted to the study of the spectral theory of the linear nonlocal dispersal equation

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D} \tag{1.1}$$

and the asymptotic dynamics of the nonlinear nonlocal dispersal equation

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + uf(t,x,u), \quad x \in \bar{D}, \tag{1.2}$$

where $D \subset \mathbb{R}^N$ is a bounded domain or $D = \mathbb{R}^N$, and $\kappa(\cdot)$, $a(\cdot,\cdot)$ and $f(\cdot,\cdot,\cdot)$ satisfy

(H1) $\kappa(\cdot) \in C^1(\mathbb{R}^N,[0,\infty))$, $\kappa(0) > 0$, $\int_{\mathbb{R}^N} \kappa(x)dx = 1$, *and there are* $\mu, M > 0$ *such that* $\kappa(x) \le e^{-\mu|x|}$ *and* $|\nabla\kappa| \le e^{-\mu|x|}$ *for* $|x| \ge M$.

(H2) $a(t,x)$ *is uniformly continuous in* $(t,x) \in \mathbb{R} \times \bar{D}$*, and is almost periodic in* t *uniformly with respect to* $x \in \bar{D}$ *(see Definition 2.1 for the definition of almost periodic functions).*

(H3) *$f(t,x,u)$ is C^1 in u; $f(t,x,u)$ and $f_u(t,x,u)$ are uniformly continuous and bounded on $(\mathbb{R}\times\bar{D}\times E)$ for any bounded set $E\subset\mathbb{R}$; $f(t,x,u)$ is almost periodic in t uniformly with respect to $x\in\bar{D}$ and u in bounded sets of $\mathbb{R}$; $f(t,x,u)$ is also almost periodic in x uniformly with respect to $t\in\mathbb{R}$ and u in bounded sets when $D=\mathbb{R}^N$; $f(t,x,u)+1<0$ for all $(t,x)\in\mathbb{R}\times\bar{D}$ and $u\gg 1$; and $\sup_{t\in\mathbb{R},x\in\bar{D}} f_u(t,x,u)<0$ for each $u\geq 0$.*

Typical examples of the kernel function $\kappa(\cdot)$ satisfying **(H1)** include the probability density function of the normal distribution $\kappa(x)=\frac{1}{\sqrt{(2\pi)^N}}e^{-\frac{|x|^2}{2}}$ and any C^1 convolution kernel function supported on a bounded ball $B(0,r)=\{x\in\mathbb{R}^N\,|\,|x|<r\}$. A prototype of the function f is given by $f(t,x,u)=a(t,x)-b(t,x)u(t,x)$

Equation (1.2) is called the nonlocal Fisher-KPP equation due to the pioneering works by Fisher [19] (1937) , and Kolmogorov, Petrowsky, Piscunov [32] (1937) on the following equation

$$\partial_t u=\Delta u+u(a-bu).$$

Most continuous models that incorporate dispersal are based upon reaction-diffusion equations such as

$$\begin{cases} u_t=\Delta u+ug(t,x,u), & x\in\Omega\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.3}$$

$$\begin{cases} u_t=\Delta u+ug(t,x,u), & x\in\Omega\\ \frac{\partial u}{\partial n}=0, & x\in\partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded smooth domain, or

$$u_t=\Delta u+ug(t,x,u),\quad x\in\mathbb{R}^N. \tag{1.5}$$

In such equations, the dispersal is represented by the Laplacian and is governed by random walk. It is referred to as random dispersal and is essentially a local behavior describing the movement of cells or organisms between adjacent spatial locations.

In reality, the movements of some organisms can occur between non-adjacent spatial locations. For such a model species, one can think of trees whose seeds and pollens are disseminated on a wide range. Reaction-diffusion equations are inadequate to model such dispersal.

Recently there has been extensive investigation on the dynamics of such populations having a long range dispersal strategy (see [1, 2, 3, 6, 7, 8, 11, 14, 21, 31, 35, 36, 37, 49, 57, 56, 58, 63, 64], etc.). The following nonlocal reaction diffusion equations are commonly used models to integrate the long range dispersal for these populations (see [17, 22, 29, 38, 61], etc):

$$\partial_t u = \int_\Omega \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u), \quad x \in \bar{\Omega}, \tag{1.6}$$

$$\partial_t u = \int_\Omega \kappa(y-x)(u(t,y) - u(t,x))dy + ug(t,x,u), \quad x \in \bar{\Omega}, \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and

$$\partial_t u = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u), \quad x \in \mathbb{R}^N. \tag{1.8}$$

In equations (1.6), (1.7), and (1.8), the function $u(t,x)$ represents the population density of the species at time t and location x. The dispersal kernel $\kappa(y-x)$ describes the probability of jumping from location x to location y and the support of $\kappa(\cdot)$ can be considered the range of dispersion. Thus $\int_D \kappa(y-x)u(t,y)dy$ gives the rate at which individuals are arriving at position y from all other places x, and $-u(t,x) = -\int_\Omega \kappa(y-x)u(x,t)dy$ is the rate at which they are leaving location x. The function $g(t,x,u)$ accounts for growth/decay or proliferation rates, self limitations and other environmental factors. Note that (1.6) is the nonlocal dispersal counterpart of the reaction-diffusion equation with Dirichlet boundary condition given by (1.3) since (1.6) can be written as

$$\partial_t u = \int_{\mathbb{R}^N} \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u), \quad x \in \bar{\Omega} \tag{1.9}$$

complemented with the following Dirichlet-type boundary condition

$$u(t,x) = 0, \quad x \in \mathbb{R}^N \setminus \bar{\Omega}. \tag{1.10}$$

Similarly, (1.7) is the nonlocal dispersal counterpart of the reaction-diffusion equation with Neumann boundary condition given by (1.4). See [9, 10, 54] for the relation between (1.6) and (1.3), and the relation between (1.7) and (1.4). Equations (1.6) and (1.7) can be viewed as nonlocal dispersal models for populations with growth function $ug(t,x,u)$ and with Dirichlet- and Neumann-type boundary conditions, respectively. Observe that (1.6) (respectively (1.7), (1.8)) can be written as (1.2) with $D = \Omega$ and $f(t,x,u) = -1 + g(t,x,u)$ (respectively $D = \Omega$ and $f(t,x,u) = -\int_D \kappa(y-x)dy + g(t,x,u)$, $D = \mathbb{R}^N$ and $f(t,x,u) = -1 + g(t,x,u)$). Thus, for conciseness and simplicity, we shall study (1.2).

The fundamental dynamical issues for (1.2) include the asymptotic behavior of solutions with strictly positive initials, propagation phenomena of solutions with compactly supported or front-like initials when the underlying environment is unbounded, and the effects of dispersal strategy and spatial-temporal variations on the population dynamics. These dynamical issues have been extensively studied for the population models described by reaction diffusion equations and are quite well understood in many cases. Recently there has also been extensive investigation on these dynamical issues for the nonlocal dispersal population models (see [1, 2, 3, 6, 7, 8, 11, 12, 14, 21, 31, 35, 36, 37, 48, 49, 57, 56, 58, 59, 63, 64], etc.). However, the understanding of these issues for nonlocal dispersal equations is much less, and, to our knowledge they have been essentially investigated in specific situations such as time independent and space heterogeneous media or time and space periodic media.

Observe that $u(t,x) \equiv 0$ is a solution of (1.2), referred to as the *trivial solution* of (1.2). With $a(t,x) = f(t,x,0)$, then (1.1) is the linearization of (1.2) at the trivial solution. The stability of the zero solution of (1.2) is crucial in investigating its asymptotic dynamics. This naturally leads to the study of the spectral theory of the linearization of the model at the zero solution, which is of independent research interest. Therefore, the first part of this dissertation (Chapter 3) is devoted to the study of the principal spectral theory of (1.1).

The principal spectrum for linear random dispersal or reaction diffusion equations has been extensively studied and is quite well understood in many cases. For example, consider the following random dispersal counterpart of (1.1) on a bounded smooth domain D with Dirichlet boundary condition,

$$\begin{cases} u_t = \Delta u + a(t,x)u, \quad x \in D \\ u = 0 \quad x \in \partial D. \end{cases} \tag{1.11}$$

For the periodic case ($a(t+T,x) = a(t,x)$ for all $x \in D$ and $t \in \mathbb{R}$), there is well-known theory (see [23]) yielding the existence of a principal eigenvalue $\lambda(a)$ and eigenfunction $\phi(t,x)$, that is,

$$\begin{cases} -\phi_t(t,x) + \Delta\phi(t,x) + a(t,x)\phi(t,x) = \lambda(a)\phi(t,x), \quad x \in D \\ \phi(t,x) = 0 \quad x \in \partial D \\ \phi(t+T,x) = \phi(t,x) > 0 \quad \forall\, t \in \mathbb{R},\ x \in D. \end{cases}$$

Note that the principal eigenvalue of (1.11) in the time periodic case is a notion related to the existence of an eigenpair: an eigenvalue associated with a positive eigenfunction. The principal eigenvalue theory for (1.11) in the time periodic case has been well extended to general time dependent case with the principal eigenvalue and eigenfunction in the time periodic case replaced by the principal Lyapunov exponent and principal Floquet bundle, respectively (see [27, 28, 40, 41, 55], etc.)

It is pertinent at this point to mention that due to the lack of regularity and compactness of the solution operator of nonlocal dispersal equations, some difficulties which are not encountered in the study of the spectral theory of random dispersal equations, show up in the study of the spectral theory of nonlocal dispersal equations. Several authors have established the fact that nonlocal dispersal operators may not have principal eigenvalues unlike their random dispersal counterparts (for instance, see [7], [57] for some examples). In the case where the function a depends only on the space variable, the authors [2], [7], [31], [52], and [57] established some sufficient conditions for the existence of the principal eigenvalue and its dependence on the underlying parameters. Subsequently, the authors in [48] investigated the following nonlocal eignevalue problem:

$$\begin{cases} u_t = \nu[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] + a(t,x)u, \quad x \in D \\ u(t+T,x) = u(t,x), \end{cases} \tag{1.12}$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain and $a(t,x)$ is a continuous function with $a(t+T;x) = a(t;x)$. They established some criteria for the existence of the principal eigenvalue of (1.12). In [30], the authors investigated the influence of time periodicity/almost periodicity on the principal eigenvalue and also considered the relationships between the principal eigenvalue and some other equivalent concepts like the principal Lyapunov exponent and principal dynamical spectrum point. They established the equality of these concepts in the time periodic case, and for the time almost periodic case, they obtained that the top Lyapunov exponent is always larger than or equal to the principal eigenvalue of the corresponding time-averaged equation.

The concept of generalized principal eigenvalues of (1.1), a natural extension of principal eigenvalues, was introduced in [7] for the case when the function $a(t,x) \equiv a(x)$ and studied in [7, 5, 11]. They established some of their properties and criteria for their equality. The authors in [59] studied these concepts in the time periodic case. However, there is not much study on the aspects of the spectral theory for (1.1) when $a(t,x)$ is not periodic in t.

In this dissertation, we investigate the spectral theory of (1.1) in the time almost periodic case via the following quantities:

- top Lyapunov exponents (see Definition 3.1 for detail);
- principal dynamical spectrum point (see Definition 3.3 for detail);
- generalized principal eigenvalues (see Definition 3.4 for detail).

In particular, we study the following aspects related to the above quantities.

- relations between the top Lyapunov exponents, principal dynamical spectrum point, and generalized principal eigenvalues of (1.1) (see Theorems 3.1 and 3.2);
- effects of time and space variations of $a(t,x)$ on the top Lyapunov exponents and generalized principal eigenvalues of (1.1) (see Theorems 3.3 and 3.4);
- characterizations of the generalized principal eigenvalues of (1.1) (see Theorem 3.5).

We refer to the theory of the top Lyapunov exponents, principal dynamical spectrum point, and generalized principal eigenvalues of (1.1) as *the principal spectral theory* for the linear nonlocal dispersal equation (1.1). The results on the spectral theory of (1.1) established in this dissertation recovered and extended some of the previous results obtained in the time independent and time periodic cases to the time almost periodic case and established new results on these concepts in the time almost periodic case.

Exploiting the spectral theory thus developed, we study the existence, uniqueness and stability of a strictly positive bounded entire solution of (1.2). An *entire solution* $u(t,x)$ of (1.2) is a solution defined for all $t \in \mathbb{R}$. Such a solution is said to be *strictly positive* if $\inf_{t\in\mathbb{R}, x\in\bar{D}} u(t,x) > 0$. A strictly positive entire solution $u(t,x)$ of (1.2) is called an *almost periodic solution* if it is almost periodic in t uniformly with respect to $x \in \bar{D}$ in the case that D is bounded and is almost periodic in both t and x when $D = \mathbb{R}^N$ (See Definition 2.1 for the definition of almost periodicity).

We established in [46] that for the time almost periodic Fisher-KPP equation with nonlocal dispersal (1.2)

- Equation (1.2) has at most one strictly positive, bounded entire solution and any such solution is almost periodic (see Theorem 4.1 for details).

- Equation (1.2) has a strictly positive bounded almost periodic entire solution if and only if the generalized principal eigenvalue λ_{PE} given in Definition 3.4 is positive (see Theorem 4.2 for details).

- The strictly positive bounded entire solution of (1.2) attracts every other solution whose initial has strictly positive infimum (see Theorem 4.1(c) for details).

- The frequency module of the strictly positive almost periodic solution is contained in the frequncy module of the function f in (1.2)(see Theorem 4.1(d) for details).

- The zero solution of (1.2) is globally asymptotically stable if the top Lyapunov exponent is negative (see Theorem 4.2(b) for details).

The above results extends the persistence and extinction results in Theorems E and F of [48] from the time periodic case to the almost periodic case. We note that the existence, uniqueness and stability results obtained has been established based solely on the signs of the generalized eigenvalue and top Lyapunov exponent (which always exist). Hence even when the existence of the principal eigenvalue cannot be determined, we still have information on the survival and extinction of the species.

Another essential issue on the asymptotic dynamics of (1.2) is the spatial spreading speeds of solutions with compactly supported or front-like initials when the underlying environment is unbounded. This is concerned with the following:

- If $D = \mathbb{R}^N$, how fast does the population invade into a region with no initial population?

In Chapter 5, we shall present the definition of the spreading speed interval introduced in [26], and establish the following:

- If the zero solution of (1.2) is unstable, then the spreading speed interval is finite, with a precise upper bound (See Theorem 5.1(i) for details)
- If the almost periodic function is bounded below by the sum of a time and space periodic function $a_T(t, x)$ and a time almost periodic function $a_0(t)$ then we obtain both upper and lower bounds for the spreading speed interval (See Theorem 5.1(ii) for details).

The first result above extends the results in [26, Theorems 2.1(i) and 2.3(1)] from the random dispersal case to the non-local dispersal case and the second result recovers the existing result on the spreading speed in the time periodic case.

The rest of the dissertation is organised as follows: In Chapter 2, we present some preliminary materials needed in the entire subsequent discussion. These are the important theorems and properties of almost periodic functions and the comparison principles. We study the spectral theory of (1.1) in Chapter 3, establishing the relationships between the top Lyapunov exponents and the generalized principal eigenvalues, their monotonicity, characterizations and dependence on time and space variations. Chapters 4 and 5 are devoted to the asymptotic dynamics of the nonlocal Fisher-KPP equations with almost periodic dependence. We shall present the existence, uniqueness and stability of positive almost periodic solutions in chapter 4 and discuss the spatial spreading properties of solutions with front-like initials in chapter 5. Chapter 6 presents some future projects and concludes the dissertation

Chapter 2

Almost periodic functions and comparison principles

Most of our results are established based on the technique of sub- and super-solutions using comparison principles. In this chapter, we present the comparison principles used (which is more general than what is in the literature) and collect some important facts about almost periodic functions which will be employed in establishing the main results.

2.1 Definition and basic properties of almost periodic functions

First, we present the definitions of almost periodic and limiting almost periodic functions, and some basic properties of almost periodic functions.

Definition 2.1. *(1) Let $E \subset \mathbb{R}^N$ and $f \in C(\mathbb{R} \times E, \mathbb{R})$. $f(t,x)$ is said to be* almost periodic in t uniformly with respect to $x \in E$ *if it is uniformly continuous in $(t,x) \in \mathbb{R} \times E$ and for any $\epsilon > 0$, $T(\epsilon)$ is relatively dense in $\mathbb{R}$, where*

$$T(\epsilon) = \{\tau \in \mathbb{R} \,|\, |f(t+\tau,x) - f(t,x)| \le \epsilon \ \forall\, t \in \mathbb{R},\, x \in E\}.$$

(2) Let $E \subset \mathbb{R}^N$ and $f \in C(\mathbb{R} \times E, \mathbb{R})$. f is said to be limiting almost periodic in t uniformly with respect to $x \in E$ *if there is a sequence $f_n(t,x)$ of uniformly continuous functions which are periodic in t such that*

$$\lim_{n\to\infty} f_n(t,x) = f(t,x)$$

uniformly in $(t,x) \in \mathbb{R} \times E$.

(3) *Let* $f \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. $f(t,x)$ *is said to be* almost periodic in x uniformly with respect to $t \in \mathbb{R}$ *if* f *is uniformly continuous in* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ *and for each* $1 \le i \le N$, $f(t, x_1, x_2, \cdots, x_N)$ *is almost periodic in* x_i *uniformly with respect to* $t \in \mathbb{R}$ *and* $x_j \in \mathbb{R}$ *for* $1 \le j \le N, j \ne i$.

(4) *Let* $f(t,x) \in C(\mathbb{R} \times E, \mathbb{R})$ *be an almost periodic function in* t *uniformly with respect to* $x \in E \subset \mathbb{R}^N$. *Let* Λ *be the set of real numbers* λ *such that*

$$a(x, \lambda, f) := \lim_{T \to \infty} \int_0^T f(t,x) e^{-i\lambda t}\, dt$$

is not identically zero for $x \in E$. *The set consisting of all real numbers which are linear combinations of elements of the set* Λ *with integer coefficients is called the* frequency module of $f(t,x)$, *which we denote by* $\mathcal{M}(f)$.

Proposition 2.1. *(1) If* $f(t,x)$ *is almost periodic in* t *uniformly with respect to* $x \in E$, *then for any sequence* $\{t_n\} \subset \mathbb{R}$, *there is a subsequence* $\{t_{n_k}\}$ *such that limit* $\lim_{k \to \infty} f(t + t_{n_k}, x)$ *exists uniformly in* $(t,x) \in \mathbb{R} \times E$.

(2) *If* $f(t,x)$ *is almost periodic in* t *uniformly with respect to* $x \in E$, *then the limit*

$$\hat{f}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t,x) dt$$

exists uniformly with respect to $x \in E$. *If* $E = \mathbb{R}^N$ *and for each* $1 \le i \le N$, $f(t, x_1, x_2, \cdots, x_N)$ *is also almost periodic in* x_i *uniformly with respect to* $t \in \mathbb{R}$ *and* $x_j \in \mathbb{R}$ *for* $1 \le j \le N$, $j \ne i$, *then the limit*

$$\bar{f} := \lim_{q_1, q_2, \cdots, q_N \to \infty} \frac{1}{q_1 q_2 \cdots q_N} \int_0^{q_N} \cdots \int_0^{q_2} \int_0^{p_N} \hat{f}(x_1, x_2, \cdots, x_N) dx_1 dx_2 \cdots dx_N$$

exists.

(3) Given an almost periodic function $f(t)$, for any $\epsilon > 0$, there exists a trigonometric polynomial $P_\epsilon(t) = \sum_{k=1}^{N_\epsilon} b_{k,\epsilon} e^{i\lambda_{k,\epsilon} t}$ such that

$$\sup_{t\in\mathbb{R}} \|f(t) - P_\epsilon(t)\| < \epsilon.$$

Proof. (1) It follows from [18, Theorem 2.7]

(2) It follows from [18, Theorem 3.1]

(3) It follows from [18, Theorem 3.17]. □

Proposition 2.2. *A function $f(t, x)$ is almost periodic in t uniformly with respect to $x \in E \subset \mathbb{R}^K$ if and only if it is uniformly continuous on $\mathbb{R} \times E$ and for every pair of sequences $\{s_n\}_{n=1}^\infty$, $\{r_m\}_{m=1}^\infty$, there are subsequences $\{s_n'\}_{n=1}^\infty \subset \{s_n\}_{n=1}^\infty$, $\{r_m'\}_{m=1}^\infty \subset \{r_m\}_{m=1}^\infty$ such that for each $(t, x) \in \mathbb{R} \times \mathbb{R}^K$,*

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_n' + r_m', x) = \lim_{n\to\infty} f(t + s_n' + r_n', x).$$

Proof. See [18, Theorems 1.17 and 2.10]. □

Proposition 2.3. *Let $f,\ g \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be two almost periodic functions in t uniformly with respect to x in bounded sets. $\mathcal{M}(g) \subset \mathcal{M}(f)$ if and only if for any sequence $\{t_n\} \subset \mathbb{R}$, if $\lim_{n\to\infty} f(t + t_n, x) = f(t, x)$ uniformly for $t \in \mathbb{R}$ and x in bounded sets, then there is $\{t_{n_k}\}$ a subsequence of $\{t_n\}$ such that $\lim_{k\to\infty} g(t+t_{n_k}, x) = g(t, x)$ uniformly for $t \in \mathbb{R}$ and x in bounded sets.*

Proof. See [18, Theorem 4.5] □

2.2 Comparison principle

In this section, we introduce super- and sub-solutions of (1.2) and (1.1) in some general sense and present a comparison principle.

Let

$$X = X(D) := C^b_{\rm unif}(\bar{D}) = \{u \in C(\bar{D}) \,|\, u \ \text{ is uniformly continuous and bounded}\} \tag{2.1}$$

equipped with the norm $\|u\|_X = \sup_{x\in D} |u(x)|$ for $u(\cdot) \in X$, and

$$X^+ = \{u \in X, | \, u(x) \geq 0 \quad \forall \, x \in \bar{D}\}. \tag{2.2}$$

and

$$X^{++} = \{u \in X^+ \, | \, \inf_{x\in \bar{D}} u(x) > 0\}.$$

For any given $s \in \mathbb{R}$ and $u_0 \in X$, $u(t, x; s, u_0)$ denotes the unique solution of (1.2) with $u(s, x; s, u_0) = u_0(x)$. Let $T_{\max}(s, u_0) \in (0, \infty]$ be the largest number such that $u(t, x; s, u_0)$ exists on $[s, s + T_{\max}(s, u_0))$. To indicate the dependence of $u(t, x; s, u_0)$ on D, we may write it as $u(t, x; s, u_0, D)$. For given $u^1, \ u^2 \in X$, we define

$$u^1 \leq u^2, \text{ if } u^1(x) \leq u^2(x) \quad \forall \, x \in \bar{D}.$$

Definition 2.2. *A continuous function* $u(t, x)$ *on* $[t_0, t_0 + \tau) \times \bar{D}$ *is called a* super-solution (or sub-solution) *of* (1.2) *on* $[t_0, t_0 + \tau)$ *if for any* $x \in \bar{D}$, $u(\cdot, x) \in W^{1,1}(t_0, t_0 + \tau)$*, and satisfies,*

$$\frac{\partial u(t,x)}{\partial t} \geq (or \leq) \int_D \kappa(y - x)u(t, y)dy + u(t, x)f(t, x, u) \quad \text{for a.e. } t \in (t_0, t_0 + \tau). \tag{2.3}$$

Super- and sub-solutions of (1.1) are defined similarly. Note that, in the literature, super-solutions (or sub-solutions) of (1.2) on $[t_0, t_0+\tau)$ are defined to be functions $u(\cdot, \cdot) \in C^{1,0}([t_0, t_0+\tau) \times \bar{D})$ satisfying (2.3) for all $t \in (t_0, t_0 + \tau)$ and $x \in \bar{D}$. Super-solutions (sub-solutions) of (1.2) defined in the above are more general. Nevertheless, we still have the following comparison principle.

Proposition 2.4. *(Comparison Principle)*

(1) If $u^1(t, x)$ *and* $u^2(t, x)$ *are bounded sub- and super-solutions of* (1.1) *on* $[0, \tau)$ *and* $u^1(0, \cdot) \leq u^2(0, \cdot)$*, then* $u^1(t, \cdot) \leq u^2(t, \cdot)$ *for* $t \in [0, \tau)$.

(2) For given $u_0 \in X^+$ *and* $a^1, \ a^2 \in \mathcal{X}$*, if* $a^1 \leq a^2$ *then* $u(t, \cdot; s, u_0, a^1) \leq u(t, \cdot; s, u_0, a^2)$*, where* $u(t, \cdot; s, u_0, a_i)$ *is the solution of* (1.1) *with* a *being replaced by* a_i *and* $u(s, \cdot; s, u_0, a_i) = u_0(\cdot)$ *for* $i = 1, 2$.

(3) If $u^1(t,x)$ and $u^2(t,x)$ are bounded sub and super-solutions of (1.2) on $[0,\tau)$ and $u^1(0,\cdot)\le u^2(0,\cdot)$, then $u^1(t,\cdot)\le u^2(t,\cdot)$ for $t\in[0,\tau)$.

(4) For every $u_0\in X^+$, $u(t,x;s,u_0)$ exists for all $t\ge s$.

Proof. (1) Set $v(t,x)=e^{ct}(u^2(t,x)-u^1(t,x))$. Then for each $x\in\bar{D}$, $v(t,x)$ satisfies

$$\frac{\partial v}{\partial t}\ge\int_D\kappa(y-x)v(t,y)dy+p(t,x)v(t,x)\ \text{ for } a.e.\ t\in[0,\tau), \tag{2.4}$$

where

$$p(t,x)=a(t,x)+c, \tag{2.5}$$

and $c>0$ is such that $p(t,x)>0$ for all $t\in\mathbb{R}$ and $x\in D$. Since $u^i(\cdot,x)\in W^{1,1}(0,\tau)$ for each $x\in\bar{D}$, by [13, Theorem 2, Section 5.9], we have that

$$\begin{aligned}v(t,x)-v(0,x)&=\int_0^t v_t(s,x)ds\\&\ge\int_0^t\Big(\int_D\kappa(y-x)v(s,y)dy+p(s,x)v(s,x)\Big)ds\quad\forall\,t\in(0,\tau),\ x\in\bar{D}.\end{aligned}$$

Let $p_0=\sup\limits_{t\in\mathbb{R},x\in D}p(t,x)$ and $T_0=\min\{\tau,\frac{1}{p_0+1}\}$. Assume that there exist $\bar{t}\in(0,T_0)$ and $\bar{x}\in D$ such that $v(\bar{t},\bar{x})<0$. Then there exists $t_0\in(0,T_0)$ such that $v_{inf}:=\inf\limits_{(t,x)\in[0,t_0)\times D}v(t,x)<0$. We can then find $t_n\in[0,t_0),\ x_n\in D$ such that $v(t_n,x_n)\to v_{inf}$ as $n\to\infty$. By (2.4), we have

$$v(t_n,x_n)-v(0,x_n)\ge\int_0^{t_n}[\int_D\kappa(y-x_n)v(t,y)dy+p(t,x_n)v(t,x_n)]dt.$$

By $v(0,x_n)\ge0$, we have

$$v(t_n,x_n)\ge\int_0^{t_n}[\int_D\kappa(y-x)v_{inf}dy+p_0v_{inf}]dt+v(0,x_n)\ge t_n(1+p_0)v_{inf}.$$

This implies that

$$v_{inf}\ge t_0(1+p_0)v_{inf}>v_{inf},$$

which is a contradiction. Hence $v(t,x)\ge0$ for all $t\in[0,T_0)$ and for all $x\in D$.

Let $k \geq 1$ be such that $kT_0 \leq \tau$ and $(k+1)T_0 > \tau$. Repeating the above arguments yield,

$$v(t,x) \geq 0 \quad \forall\, t \in [(i-1)T_0, iT_0),\ x \in \bar{D},\ i = 1, 2, \cdots k,$$

and $v(t,x) \geq 0 \quad \forall\, t \in [kT_0, \tau),\ x \in \bar{D}$. It then follows that

$$v(t,x) \geq 0 \quad \forall\, t \in [0, \tau),\ x \in \bar{D}.$$

This implies that $u^1(t,x) \leq u^2(t,x)$ for all $t \in [0,\tau),\ x \in D$.

(2) By (1),

$$u(t,x;s,u_0,a^i) \geq 0 \quad \forall\, t \geq 0,\ x \in \bar{D},\ i = 1, 2.$$

This together with $a^1 \leq a^2$ implies that

$$u_t(t,x;s,u_0,a^1) \leq \int_D \kappa(y-x)u(t,y;s,u_0,a^1)dy + a^2(t,x)u(t,x;s,u_0,a^1) \quad \forall\, t \geq 0,\ x \in \bar{D}.$$

Then by (2) again,

$$u(t,x;s,u_0,a^1) \leq u(t,x;s,u_0,a^2) \quad \forall\, t \geq 0,\ x \in \bar{D}.$$

(3) Follows similarly as in (1) where the function $a(t,x)$ in (2.5) is given by

$$a(t,x) = \int_0^1 \frac{\partial}{\partial s}\Big((su^2(t,x) + (1-s)u^1(t,x))f(t,x,su^2(t,x) + (1-s)u^1(t,x))\Big)ds.$$

(4) Note that $u \equiv 0$ is an entire solution of (1.2) and $u \equiv M$ is a super-solution of (1.2) when $M \gg 1$. By (1),

$$0 \leq u(t,x;s,u_0) \leq M \quad \forall\ t \in [s, s + T_{\max}(s,u_0)),\ x \in \bar{D},\ M \gg 1.$$

This implies that $T_{\max}(s,u_0) = \infty$ and (4) follows. The proposition is thus proved. $\square$

Proposition 2.5. *Let $D_0 \subset D$. Then*

$$u(t,x;s,u_0|_{D_0},D_0) \le u(t,x;s,u_0,D) \quad \forall t \ge s,\ x \in \bar{D}_0,$$

where $u_0 \in C^b_{\rm unif}(\bar{D})$, $u_0 \ge 0$.

Proof. Observe that $u(t,x;s,u_0,D)$ solves

$$\begin{aligned} u_t &= \int_D \kappa(y-x)u(t,y)dy + u(t,x)f(t,x,u), \quad x \in \bar{D}. \\ &\ge \int_{D_0} \kappa(y-x)u(t,y)dy + u(t,x)f(t,x,u), \quad x \in \bar{D}_1. \end{aligned}$$

Since $u_0|_{D_0} \le u_0$ the inequality follows from Proposition 2.4. □

Chapter 3

Principal Spectral Theory of Nonlocal Dispersal Equations

This chapter is devoted to the study of the principal spectral theory of the linear nonlocal dispersal equation (1.1).

The principal spectrum for various special cases of (1.1) has been studied by many authors. For example, when D is bounded and $a(t,x)$ is independent of t or periodic in t, the principal spectrum of (1.1) has been studied in [7, 20, 25, 30, 34, 48, 51, 52, 53, 57, 58, 59]. Subsequently, [5, 12, 48, 57] studied it when $D = \mathbb{R}^N$ and $a(t,x)$ is periodic in both t and x, or $a(t,x) \equiv a(x)$. Unlike the random dispersal operators, even when $a(t,x) \equiv a(x)$ is independent of t, the operator $L : C(\bar{D}) \to C(\bar{D})$, $(Lu)(x) = \int_D \kappa(y-x)u(y)dy + a(x)u(x)$, may not have an eigenvalue associated with a positive eigenfunction when $a(x)$ is not a constant function (see [7, 57] for examples). Because of this, to study the aspects of the spectral theory for nonlocal dispersal operators, the concept of principal spectrum point for nonlocal dispersal operators was introduced in [30] (see also [48, 53]), and the concept of generalized principal eigenvalues for nonlocal dispersal operators was introduced in [5] (see also [7]). Some criteria have been established in [48, 57] for the principal spectrum point of a time periodic dispersal operator to be an eigenvalue with a positive eigenfunction. In [7], some criteria were established for the generalized principal eigenvalue of a time independent dispersal operator to be an eigenvalue with a positive eigenfunction.

However, there is not much study on the aspects of spectral theory for (1.1) when $a(t,x)$ is not periodic in t. Here, we investigate the spectral theory of (1.1) in the time almost periodic case from three aspects: top Lyapunov exponents, principal dynamical spectrum point, and generalized principal eigenvalues. In particular, we provide various characterizations of the top Lyapunov exponents and generalized principal eigenvalues of (1.1), discuss the relations between them, and study the effects of time and space variations of $a(t,x)$ on them. The theory of the top Lyapunov exponents and generalized principal eigenvalues is referred to as *the principal spectral theory* for the nonlocal dispersal operators.

3.1 Notations, definitions, and main results

3.1.1 Notations and definitions

Let $X = X(D)$ and X^+ be as in (2.1) and (2.2), respectively. For any $s \in \mathbb{R}$ and $u_0 \in X$, let $u(t,x;s,u_0)$ be the unique solution of (1.1) with $u(s,x;s,u_0) = u_0(x)$ (the existence and uniqueness of solutions of (1.1) with given initial function $u_0 \in X$ follow from the general semigroup theory, see [47]). Denote

$$\Phi(t,s;a)u_0 = u(t,\cdot;s,u_0). \tag{3.1}$$

Definition 3.1. *Let*

$$\lambda_{PL}(a) = \limsup_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)\|}{t-s}, \quad \lambda'_{PL}(a) = \liminf_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)\|}{t-s}. \tag{3.2}$$

$\lambda_{PL}(a)$ *and* $\lambda'_{PL}(a)$ *are called the* top Lyapunov exponents *of* (1.1).

For given $\lambda \in \mathbb{R}$, define

$$\Phi_\lambda(t,s;a) = e^{-\lambda(t-s)}\Phi(t,s;a),$$

where $\Phi(t,s;a)$ is as in (3.1).

Definition 3.2. *Given* $\lambda \in \mathbb{R}$, $\{\Phi_\lambda(t,s;a)\}_{s,t\in\mathbb{R},s\le t}$ *is said to admit an* exponential dichotomy (ED) for short *on* X *if there exist* $\beta > 0$, $C > 0$, *and continuous projections* $P(s): X \to X$ $(s \in \mathbb{R})$ *such that for any* $s,t \in \mathbb{R}$ *with* $s \le t$ *the following holds:*

(1) $\Phi_\lambda(t,s;a)P(s) = P(t)\Phi_\lambda(t,s;a)$*;*

(2) $\Phi_\lambda(t,s;a)|_{R(P(s))} : R(P(s)) \to R(P(t))$ *is an isomorphism for* $t \ge s$ *(hence* $\Phi_\lambda(s,t;a) := \Phi_\lambda(t,s;a)^{-1} : R(P(t)) \to R(P(s))$ *is well defined);*

(3)

$$\|\Phi_\lambda(t,s;a)(I-P(s))\| \le Ce^{-\beta(t-s)}, \quad t \ge s$$

$$\|\Phi_\lambda(t,s;a)P(s)\| \le Ce^{\beta(t-s)}, \quad t \le s.$$

Definition 3.3. *(1)* $\lambda \in \mathbb{R}$ *is said to be in the* dynamical spectrum*, denoted by* $\Sigma(a)$*, of* (1.1) *or* $\{\Phi(t,s;a)\}_{s\le t}$ *if* $\Phi_\lambda(t,s;a)$ *does not admit an ED.*

(2) $\lambda_{PD}(a) = \sup\{\lambda \in \Sigma(a)\}$ *is called the* principal dynamical spectrum point *of* $\{\Phi(t,s;a)\}_{s\le t}$.

Let

$$\mathcal{X}(D) = C^b_{\rm unif}(\mathbb{R}\times\bar{D}) := \{u \in C(\mathbb{R}\times\bar{D}) \,|\, u \text{ is uniformly continuous and bounded}\} \quad (3.3)$$

with the norm $\|u\| = \sup_{(t,x)\in\mathbb{R}\times\bar{D}} |u(t,x)|$. In the absence of possible confusion, we may write

$$\mathcal{X} = \mathcal{X}(D). \quad \mathcal{X}^+ = \{u \in \mathcal{X} \,|\, u(t,x) \ge 0, \quad t \in \mathbb{R},\ x \in \bar{D}\}, \quad \text{and}$$

$$\mathcal{X}^{++} = \{u \in \mathcal{X}^+ \,|\, \inf_{t\in\mathbb{R},x\in\bar{D}} u(t,x) > 0\}.$$

Let $L(a) : \mathcal{D}(L(a)) \subset \mathcal{X} \to \mathcal{X}$ be defined as follows,

$$(L(a)u)(t,x) = -\partial_t u(t,x) + \int_D \kappa(y-x)u(t,y)dy + a(t,x)u(t,x).$$

Let

$$\Lambda_{PE}(a,D) = \big\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X},\ \inf_{t\in\mathbb{R}} \phi(t,x) \ge\not\equiv 0,\ \text{for each } x \in \bar{D},\ \phi(\cdot,x) \in W^{1,1}_{\rm loc}(\mathbb{R}) \text{ and}$$
$$(L(a)\phi)(t,x) \ge \lambda\phi(t,x) \text{ for } a.e.\, t \in \mathbb{R}\big\} \quad (3.4)$$

and

$$\Lambda'_{PE}(a,D) = \big\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X},\ \inf_{t\in\mathbb{R},x\in\bar{D}} \phi(t,x) > 0,\ \text{for each } x \in \bar{D},\ \phi(\cdot,x) \in W^{1,1}_{\rm loc}(\mathbb{R}) \text{ and}$$
$$(L(a)\phi)(t,x) \le \lambda\phi(t,x) \text{ for } a.e.\, t \in \mathbb{R}\big\}. \quad (3.5)$$

Definition 3.4. *Define*

$$\lambda_{PE}(a) = \sup\{\lambda \mid \lambda \in \Lambda_{PE}(a)\} \tag{3.6}$$

and

$$\lambda'_{PE}(a) = \inf\{\lambda \mid \lambda \in \Lambda'_{PE}(a)\}. \tag{3.7}$$

Both $\lambda_{PE}(a)$ *and* $\lambda'_{PE}(a)$ *are called the* generalized principal eigenvalues *of* (1.1).

Let

$$\hat{a}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T a(t,x)dt \tag{3.8}$$

((see Proposition 2.1 for the existence of $\hat{a}(\cdot)$). Let

$$\bar{a} = \frac{1}{|D|}\int_D \hat{a}(x)dx \tag{3.9}$$

when D is bounded, and

$$\bar{a} = \lim_{q_1,q_2,\cdots,q_N\to\infty} \frac{1}{q_1q_2\cdots q_N}\int_0^{q_N}\cdots\int_0^{q_2}\int_0^{q_1} \hat{a}(x_1,x_2,\cdots,x_N)dx_1dx_2\cdots dx_N \tag{3.10}$$

when $D = \mathbb{R}^N$ and $a(t,x)$ is almost periodic in x uniformly with respect to $t \in \mathbb{R}$ (see Proposition 2.1 for the existence of $\bar{a}$). Note that $\hat{a}(x)$ is the time average of $a(t,x)$, and $\bar{a}$ is the space average of $\hat{a}(x)$. To discuss the monotonicity of $\lambda_{PL}(a)$, $\lambda_{PE}(a)$, and $\lambda'_{PE}(a)$ with respect to the domain D, we may put

$$\Phi(t,s;a,D) = \Phi(t,s;a), \quad \Lambda_{PE}(a,D) = \Lambda_{PE}(a), \quad \Lambda'_{PE}(a,D) = \Lambda'_{PE}(a).$$

and

$$\lambda_{PL}(a,D) = \lambda_{PL}(a),\ \lambda_{PE}(a,D) = \lambda_{PE}(a),\ \lambda'_{PE}(a,D) = \lambda'_{PE}(a).$$

3.1.2 Main results

In this subsection, we state the main theorems of this chapter. Throughout this subsection, we assume that $a(t,x)$ satisfies **(H2)**. Sometimes, we may also assume the following:

(H2)$'$ $a(t,x)$ is limiting almost periodic in t with respect to x and is also limiting almost periodic in x when $D=\mathbb{R}^n$ (see Definition 2.1).

The first theorem is on the relation between $\lambda'_{PL}(a)$, $\lambda_{PL}(a)$, and $\lambda_{PD}(a)$.

Theorem 3.1 (Relations between $\lambda'_{PL}(a)$, $\lambda_{PL}(a)$ and $\lambda_{PD}(a)$)**.**

(1) For any $u_0\in X$ *with* $\inf_{x\in D}u_0(x)>0$,

$$\lambda'_{PL}(a)=\lambda_{PL}(a)=\lim_{t-s\to\infty}\frac{\ln\|\Phi(t,s;a)\|}{t-s}=\lim_{t-s\to\infty}\frac{\ln\|\Phi(t,s;a)u_0\|}{t-s}.$$

(2) $\lambda_{PL}(a)=\lambda_{PD}(a)$.

The second theorem is on the relations between $\lambda_{PE}(a)$, $\lambda'_{PE}(a)$, and $\lambda_{PL}(a)$.

Theorem 3.2 (Relations between $\lambda_{PE}(a)$, $\lambda'_{PE}(a)$, and $\lambda_{PL}(a)$)**.**

(1) $\lambda'_{PE}(a)=\lambda_{PL}(a)$.

(2) $\lambda_{PE}(a)\le\lambda_{PL}(a)$. *If* $a(t,x)$ *satisfies (H2)*$'$*, then* $\lambda_{PE}(a)=\lambda_{PL}(a)$.

(3) If $a(t,x)\equiv a(t)$*, then* $\lambda_{PE}(a)=\lambda'_{PE}(a)=\lambda_{PL}(a)=\hat{a}+\lambda_{PL}(0)$.

The third theorem is on the effects of time and space variations on $\lambda_{PE}(a)$.

Theorem 3.3 (Effects of time and space variations on $\lambda_{PE}(a)$)**.**

(1) $\lambda_{PE}(a)\ge\sup_{x\in D}\hat{a}(x)$. *If* $a(t,x)$ *satisfies (H2)*$'$*, then* $\lambda_{PE}(a)\ge\lambda_{PE}(\hat{a})\ge\sup_{x\in D}\hat{a}(x)$.

(2) If D *is bounded,* $a(t,x)\equiv a(x)$*, and* $\kappa(\cdot)$ *is symmetric, then*

$$\lambda_{PE}(a)\ge\bar{a}+\frac{1}{|D|}\int_D\int_D\kappa(y-x)dydx,$$

where $|D|$ *is the Lebesgue measure of* D.

(3) If $D=\mathbb{R}^N$, $a(t,x)\equiv a(x)$ *is almost periodic in* x*, and* $\kappa(\cdot)$ *is symmetric, then*

$$\lambda_{PE}(a)\ge\bar{a}+1.$$

The fourth theorem is on the effects of time and space variations on $\lambda_{PL}(a)$.

Theorem 3.4 (Effects of time and space variations on $\lambda_{PL}(a)$)**.**

(1) If D is bounded or $D = \mathbb{R}^N$ and a satisfies (H2)$'$, then $\lambda_{PL}(a) \geq \lambda_{PL}(\hat{a}) \geq \sup_{x\in D} \hat{a}(x)$.

(2) If D is bounded and $\kappa(\cdot)$ is symmetric, then

$$\lambda_{PL}(a) \geq \bar{a} + \frac{1}{|D|}\int_D \int_D \kappa(y-x)dydx.$$

(3) If $D = \mathbb{R}^N$, $a(t,x)$ is almost periodic in x uniformly with respect to $t \in \mathbb{R}$, and $\kappa(\cdot)$ is symmetric, then

$$\lambda_{PL}(a) \geq \bar{a} + 1.$$

The last theorem is on the characterization of $\lambda_{PE}(a)$ and $\lambda'_{PE}(a)$ when $a(t,x)$ is independent of t or periodic in t.

Theorem 3.5 (Characterization of $\lambda_{PE}(a)$ and $\lambda'_{PE}(a)$)**.** *Assume that a satisfies (H2)$'$.*

(1) If $a(t,x) \equiv a(x)$, then

$$\lambda_{PE}(a) = \sup\{\lambda \,|\, \lambda \in \tilde{\Lambda}_{PE}(a)\} = \inf\{\lambda \,|\, \lambda \in \tilde{\Lambda}'_{PE}(a)\} = \lambda'_{PE}(a),$$

where

$$\tilde{\Lambda}_{PE}(a) = \{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in X,\ \phi(x) \geq\not\equiv 0, \\ \int_D \kappa(y-x)\phi(y)dy + a(x)\phi(x) \geq \lambda\phi(x)\ \forall\, x \in \bar{D}\}$$

and

$$\tilde{\Lambda}'_{PE}(a) = \{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in X,\ \inf_{x\in D}\phi(x) > 0, \\ \int_D \kappa(y-x)\phi(y)dy + a(x)\phi(x) \leq \lambda\phi(x)\ \forall\, x \in \bar{D}\}.$$

(2) If $a(t+T,x) \equiv a(t,x)$*, then*

$$\lambda_{PE}(a) = \sup\{\lambda \,|\, \lambda \in \hat{\Lambda}_{PE}(a)\} = \inf\{\lambda \,|\, \lambda \in \hat{\Lambda}'_{PE}(a)\} = \lambda'_{PE}(a),$$

where

$$\hat{\Lambda}_{PE}(a) = \{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X}_T,\ \inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0,\ \text{for each } x \in \bar{D},\ \phi(\cdot,x) \in W^{1,1}(\mathbb{R}) \text{ and}$$
$$(L(a)\phi)(t,x) \geq \lambda\phi(t,x) \text{ for } a.e.\, t \in \mathbb{R}\},$$

$$\hat{\Lambda}'_{PE}(a) = \{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X}_T,\ \inf_{t\in\mathbb{R}, x\in D} \phi(t,x) > 0,\ \text{for each } x \in \bar{D},\ \phi(\cdot,x) \in W^{1,1}(\mathbb{R}) \text{ and}$$
$$(L(a)\phi)(t,x) \leq \lambda\phi(t,x) \text{ for } a.e.\, t \in \mathbb{R}\}.$$

and

$$\mathcal{X}_T = \{\phi \in \mathcal{X} \,|\, \phi(t+T,x) = \phi(t,x)\}.$$

3.1.3 Remarks on the main results

In this subsection, we provide the following remarks on the main results established in this chapter.

Remark 3.1. *Spectral theory for a linear evolution equation is strongly related to the growth/decay rates of its solutions. From the point of view of dynamical systems, one usually employs the top Lyapunov exponents and principal dynamical spectrum point to characterize the largest growth rate of the solutions of a linear evolution equation. Theorem 3.1 shows that the top Lyapunov exponents and principal dynamical spectrum point of* (1.1) *are the same, which is then exactly the largest growth rate of the solutions of* (1.1).

Remark 3.2. *The notion of generalized principal eigenvalues for time independent nonlocal dispersal equations was introduced in [5, 7, 12] (see Remark 3.3 in the following for some detail). It is a natural extension of principal eigenvalues, which is related to the existence of* *eigenvalues associated with positive eigenfunctions. Theorem 3.2 shows that*

$$\lambda_{PE}(a) = \lambda'_{PE}(a) = \lambda_{PL}(a)$$

when $a(t,x)$ is limiting almost periodic in t, and in general,

$$\lambda_{PE}(a) \le \lambda'_{PE}(a) = \lambda_{PL}(a).$$

Therefore, in any case, $\lambda'_{PE}(a)$ is exactly the largest growth rate of the solutions of (1.1)*. It is definitely of great importance that the largest growth rate of the solutions of* (1.1) *can be characterized by two different approaches, one by the top Lyapunov exponent $\lambda_{PL}(a)$ and the other by the generalized principal eigenvalue $\lambda'_{PE}(a)$.*

Remark 3.3. *When $a(t,x) \equiv a(x)$, the following generalized principal eigenvalues were introduced in [5] for* (1.1)*:*

$$\begin{aligned}\lambda_p(a) = \sup\{\lambda \in \mathbb{R} \mid \exists \phi \in C(\bar{D}),\ \phi > 0 \\ \int_D \kappa(y-x)\phi(y)dy + a(x)\phi + \lambda\phi \le 0 \text{ in } D\},\end{aligned}$$

and

$$\begin{aligned}\lambda'_p(a) = \inf\{\lambda \in \mathbb{R} \mid \exists\, \phi \in C(D) \cap L^\infty(D),\ \phi \gneqq 0 \\ \int_D \kappa(y-x)\phi(y)dy + a(x)\phi + \lambda\phi \ge 0 \text{ in } D\}.\end{aligned}$$

Note that, in our definitions of $\lambda_{PL}(a)$ and $\lambda'_{PE}(a)$, we require the function ϕ in the sets $\Lambda_{PL}(a)$ and $\Lambda'_{PE}(a)$ to be uniformly continuous and bounded. By Theorem 3.5, we have the following relation between $\lambda_p(a), \lambda'_p(a)$, and $\lambda_{PE}(a), \lambda'_{PE}(a)$:

$$-\lambda_p(a) \le \lambda'_{PE}(a) = \lambda_{PE}(a) \le -\lambda'_p(a), \tag{3.11}$$

which implies that

$$\lambda_p^{'}(a) \le \lambda_p(a). \tag{3.12}$$

It should be noted that, among others, it was proved in [5] that, if $\kappa(\cdot)$ has compact support, then

$$\lambda_p(a) = \lambda_p^{'}(a) \quad \text{when } D \text{ is bounded} \tag{3.13}$$

and

$$\lambda_p^{'}(a) \le \lambda_p(a) \quad \text{when } D \text{ is unbounded} \tag{3.14}$$

(see [5, Theorem 1.1] and [5, Theorem 1.2]). It should also be pointed out that the paper [5] dealt with more general kernel functions $\kappa(x, y)$. Note that in Theorem 3.2(2), it was proved that (3.12) *holds without the assumption that $\kappa(\cdot)$ has compact support. Hence* (3.12) *is an improvement of* (3.14) *when the kernel function in [5] $\kappa(x, y) = \kappa(y - x)$.*

Remark 3.4. *Theorems 3.3 and 3.4 are on the influence of time and space variation of $a(t, x)$ on the top Lyapunov exponent $\lambda_{PL}(a)$ and the generalized principal eigenvalue $\lambda_{PE}(a)$. Theorem 3.4(1) shows that time variation does not reduce the top Lyapunov exponent $\lambda_{PL}(a)$. Since $\lambda_{PE}^{'}(a) = \lambda_{PL}(a)$, this also holds for $\lambda_{PE}^{'}(a)$. Theorem 3.3(2) indicates that space variation of $a(t, x) \equiv a(x)$ does not reduce the generalized principal eigenvalue $\lambda_{PE}(a)$ when* (1.1) *is viewed as a nonlocal dispersal equation with Neumann type boundary condition on the bounded domain D. To be more precise, write* (1.1) *with $a(t, x) \equiv a(x)$ as*

$$u_t = \int_D \kappa(y - x)[u(t, y) - u(t, x)]dy + \tilde{a}(x)u(t, x), \quad x \in \bar{D}, \tag{3.15}$$

where $\tilde{a}(x) = \int_D \kappa(y-x)dy + a(x)$. (3.15) *can then be viewed as a nonlocal dispersal equation with reaction term $\tilde{a}(x)u$ and Neumann-type boundary condition. Theorem 3.3(2) then follows from the arguments of [53, Theorem 2.1]. Theorem 3.3(3) indicates that the space variation of $a(t, x) \equiv a(x)$ does not reduce the generalized principal eigenvalue $\lambda_{PE}(a)$ when* (1.1) *with $a(t, x) \equiv a(x)$ is viewed as the following nonlocal dispersal equation on $\mathbb{R}^N$ with reaction term $\tilde{a}(x)u$,*

$$u_t = \int_{\mathbb{R}^N} \kappa(y - x)(u(t, y) - u(t, x))dy + \tilde{a}(x)u, \quad x \in \mathbb{R}^N, \tag{3.16}$$

where $\tilde{a}(x) = 1 + a(x)$. When $a(x)$ is periodic in x, Theorem 3.3(3) follows from the arguments of [25, Theorem 2.1]. When $a(x)$ is almost periodic in x, Theorem 3.3(3) is new. Note that Theorem 3.4(2), (3) follow from Theorem 3.3(2), (3) and the fact that $\lambda_{PL}(a) \geq \lambda_{PE}(a)$.

Remark 3.5. *There are several interesting open problems. For example, it remains open whether $\lambda_{PE}(a) = \lambda'_{PE}(a)$ for any a satisfying* **(H2)***. If $\lambda_{PE}(a) = \lambda'_{PE}(a)$, under what condition there is a positive function $\phi(t,x)$, such that*

$$-\phi_t + \int_D \kappa(y-x)\phi(t,y)dy + a(t,x)\phi(t,x) = \lambda_{PE}(a)\phi(t,x) \quad \forall\ t \in \mathbb{R},\ x \in \bar{D}.$$

If there is such $\phi(t,x)$, we may call $\lambda_{PE}(a)$ the principal eigenvalue of (1.1)*. It remains open whether $\lambda_{PE}(a) \geq \lambda_{PE}(\hat{a})$ for any a satisfying* **(H2)***.*

Remark 3.6. *It should be pointed out that the definitions of top Lyapunov exponents, principal dynamical spectrum point, and generalized principal eigenvalues can be applied to* (1.1) *when $a(t,x)$ is a general time dependent function. But some results in the above theorems may not hold when $a(t,x)$ is not almost periodic in t, for example, $\lambda'_{PL}(a) = \lambda_{PL}(a)$ may not be true when $a(t,x)$ is not almost periodic in t. For such general $a(t,x)$, by the arguments of Theorem 3.2 we have the following relations between $\lambda_{PL}(a)$, $\lambda'_{PL}(a)$, $\lambda_{PE}(a)$, and $\lambda'_{PE}(a)$,*

$$\lambda_{PE}(a) \leq \lambda'_{PL}(a) \leq \lambda_{PL}(a) \leq \lambda'_{PE}(a).$$

We will not discuss the aspects of spectral theory of (1.1) *with general time dependent $a(t,x)$.*

3.2 Relations between the top Lyapunov exponents and principal dynamical spectrum point

In this section, we prove Theorem 3.1. We first prove a lemma on the continuity of $\lambda_{PL}(a)$, $\lambda'_{PL}(a)$, $\lambda_{PE}(a)$, and $\lambda'_{PE}(a)$ in a.

Lemma 3.1. *$\lambda_{PL}(a)$, $\lambda'_{PL}(a)$, $\lambda_{PE}(a)$, and $\lambda'_{PE}(a)$ are continuous in $a \in \mathcal{X}$ satisfying (H2).*

Proof. First, we prove the continuity of $\lambda_{PL}(a)$ and $\lambda'_{PL}(a)$ in a. For any $a_1, a_2 \in \mathcal{X}$ satisfying (H2),

$$a_2(t,x) - \|a_2 - a_1\| \leq a_1(t,x) \leq a_2(t,x) + \|a_2 - a_1\| \quad \forall\, t \in \mathbb{R},\, x \in \bar{D}.$$

This implies that for any $u_0 \in X$ with $u_0 \geq 0$, using Proposition 2.4 we have,

$$e^{-\|a_2 - a_1\|(t-s)}\Phi(t,s;a_2)u_0 \leq \Phi(t,s;a_1)u_0 \leq e^{\|a_2-a_1\|(t-s)}\Phi(t,s;a_2)u_0.$$

It then follows that

$$-\|a_2 - a_1\| + \lambda_{PL}(a_2) \leq \lambda_{PL}(a_1) \leq \|a_2 - a_1\| + \lambda_{PL}(a_2),$$

and

$$-\|a_2 - a_1\| + \lambda'_{PL}(a_2) \leq \lambda'_{PL}(a_1) \leq \|a_2 - a_1\| + \lambda'_{PL}(a_2).$$

Hence $\lambda_{PL}(a)$ and $\lambda'_{PL}(a)$ are continuous in a.

Next, we prove that $\lambda_{PE}(a)$ is continuous in a. For any $a_1, a_2 \in \mathcal{X}$ and any $\lambda \in \Lambda_{PE}(a_1)$, it is clear that $\lambda - \|a_2 - a_1\| \in \Lambda_{PE}(a_2)$. Hence

$$\lambda_{PE}(a_2) \geq \lambda_{PE}(a_1) - \|a_2 - a_1\|.$$

Conversely, for any $\lambda \in \Lambda_{PE}(a_2)$, $\lambda - \|a_2 - a_1\| \in \Lambda_{PE}(a_1)$. Hence

$$\lambda_{PE}(a_1) \geq \lambda_{PE}(a_2) - \|a_2 - a_1\|.$$

Therefore,

$$-\|a_2 - a_1\| + \lambda_{PE}(a_2) \leq \lambda_{PE}(a_1) \leq \|a_2 - a_1\| + \lambda_{PE}(a_2)$$

and $\lambda_{PE}(a)$ is continuous in a.

Similarly, it can be proved that $\lambda'_{PE}(a)$ is continuous in a. □

3.3 Equality of the top Lyapunov exponents and the principal dynamical spectrum point

In this section, we examine the equality of the two top Lyapunov exponents and the principal dynamical spectrum point and prove Theorem 3.1.

Proof of Theorem 3.1. (1) First, we introduce the hull $H(a)$ of a,

$$H(a) = \mathrm{cl}\{\sigma_t a(\cdot,\cdot) := a(t+\cdot,\cdot) \,|\, t \in \mathbb{R}\}$$

with the open compact topology, where the closure is taken under the open compact topology. Note that, by the almost periodicity of $a(t,x)$ in t uniformly with respect to $x \in \bar{D}$ (see **(H2)**) and Proposition 2.1(1), for any sequence $\{t_n\} \subset \mathbb{R}$, there is a subsequence $\{t_{n_k}\}$ such that the limit $\lim_{n_k\to\infty} a(t_{n_k}+t,x)$ exists uniformly in $(t,x) \in \mathbb{R}\times\bar{D}$. Hence the open compact topology of $H(a)$ is equivalent to the topology of uniform convergence. Let

$$\Phi(t,b)u_0 = u(t,\cdot;b,u_0), \tag{3.17}$$

where $u(t,\cdot;b,u_0)$ is the solution of (1.1) with a being replaced by $b \in H(a)$ and $u(0,\cdot;b,u_0) = u_0(\cdot) \in X$.

Note that $(H(a),\sigma_t)$ is a compact minimal flow and ν is the unique invariant ergodic measure of $(H(a),\sigma_\tau)$, where ν is the Haar measure of $H(a)$. It is clear that the map $[0,\infty) \ni t \mapsto \ln\|\Phi(t,b)\|$ is subadditive. By the subadditive ergodic theorem, there are $\lambda_0(a) \in \mathbb{R}$ and $H_0(a) \subset H(a)$ with $\nu(H_0(a)) = 1$ such that $\sigma_t(H_0(a)) = H_0(a)$ for any $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} \frac{1}{t}\ln\|\Phi(t,b)\| = \lambda_0(a) \tag{3.18}$$

for any $b \in H_0(a)$.

Next, we prove that (3.18) holds for any $b \in H(a)$ and the limit is uniform in $b \in H(a)$. Assume that this does not hold. Then there are $\epsilon_0 > 0$, $t_n \to \infty$, and $b_n \in H(a)$ such that

$$\left|\frac{1}{t_n}\ln\|\Phi(t_n,b_n)\| - \lambda_0(a)\right| \geq \epsilon_0. \tag{3.19}$$

By the compactness of $H(a)$, there is $b^* \in H(a)$ and a subsequence of b_n, which, without loss of generality, we still denote as b_n, such that

$$b_n(t,x) \to b^*(t,x) \quad \text{as} \quad n \to \infty$$

uniformly in $t \in \mathbb{R}$ and $x \in \bar{D}$. Then

$$|b^*(t,x) - b_n(t,x)| \le \frac{\epsilon_0}{4} \quad \forall\, t \in \mathbb{R},\ x \in D,\ n \gg 1.$$

Note that $H_0(a)$ is dense in $H(a)$. Therefore there is $b^{**} \in H_0(a_1)$ such that

$$|b^{**}(t,x) - b^*(t,x)| \le \frac{\epsilon_0}{4} \quad \forall t \in \mathbb{R},\ x \in D,\ n \gg 1.$$

This implies that

$$|b^{**}(t,x) - b_n(t,x)| \le \frac{\epsilon_0}{2} \quad \forall\, t \in \mathbb{R},\ x \in D,\ n \gg 1.$$

Then by the comparison principle (see Proposition 2.4), we have

$$\begin{aligned} e^{-\frac{\epsilon_0}{2}t}\Phi(t,b_n)u_0 = \Phi(t,b_n - \frac{\epsilon_0}{2})u_0 \le \Phi(t,b^{**})u_0 \\ \le \Phi(t,b_n + \frac{\epsilon_0}{2})u_0 = e^{\frac{\epsilon_0}{2}t}\Phi(t,b_n)u_0 \end{aligned}$$

for any $u_0 \in X$ with $u_0(x) \ge 0$. This implies that

$$-\frac{\epsilon_0}{2}t + \ln\|\Phi(t,b_n)\| \le \ln\|\Phi(t,b^{**})\| \le \frac{\epsilon_0}{2}t + \ln\|\Phi(t,b_n)\| \quad \forall t \ge 0,\ n \gg 1. \tag{3.20}$$

By (3.19) and (3.20), we have

$$|\lim_{t\to\infty}\frac{1}{t}\ln\|\Phi(t,b^{**})\| - \lambda_0(a)| \ge \frac{\epsilon_0}{2}.$$

This is a contradiction. Hence (3.18) holds for any $b \in H(a)$ and the limit is achieved uniformly in $b \in H(a)$.

Now we prove that $\lambda_{PL}(a) = \lambda'_{PL}(a) = \lambda_0(a)$. By the definition of $\Phi(t, s; a)$ (see (3.1)) and $\Phi(t; b)$ (see (3.17)), we have

$$\Phi(t, s; a) = \Phi(t - s; \sigma_s a).$$

Then, by the above arguments, we have

$$\lim_{t-s\to\infty} \frac{\ln \|\Phi(t, s; a)\|}{t - s} = \lim_{t-s\to\infty} \frac{\ln \|\Phi(t - s; \sigma_s a)\|}{t - s}.$$

Hence $\lambda_{PL}(a) = \lambda'_{PL}(a) = \lambda_0(a)$. Moreover, we have

$$\lambda_{PL}(a) = \lambda'_{PL}(a) = \lim_{t-s\to\infty} \frac{\ln \|\Phi(t, s; a)\|}{t - s} = \lim_{t-s\to\infty} \frac{\ln \|\Phi(t, s; a)u_0\|}{t - s}$$

for any $u_0 \in X$ with $\inf_{x\in D} u_0(x) > 0$. This proves (1).

(2) First, observe that $\Phi(t, s; a)$ is exponentially bounded from above as well as from below. That is, there exist $M, m > 0$ and $\omega_\pm \in \mathbb{R}$ such that

$$me^{\omega_-(t-s)} \le \|\Phi(t, s; a)\| \le Me^{\omega_+(t-s)}.$$

In fact, let

$$\mathcal{K} : X \to X, \quad (\mathcal{K}u)(x) = \int_D \kappa(y - x)u(y)dy \quad \forall\, x \in \bar{D}$$

and

$$a_{\min} = \inf_{t\in\mathbb{R}, x\in\bar{D}} a(t, x),\ \ a_{\max} = \sup_{t\in\mathbb{R}, x\in\bar{D}} a(t, x).$$

Then we have

$$e^{a_{\min}(t-s)}e^{\mathcal{K}(t-s)}u_0 \le \Phi(t, s)u_0 \le e^{a_{\max}(t-s)}e^{\mathcal{K}(t-s)}u_0$$

for all $t \ge s$ and $u_0 \in X$ with $u_0 \ge 0$. Note that

$$u_0 \le e^{\mathcal{K}(t-s)}u_0 \le e^{\|\mathcal{K}\|(t-s)}\|u_0\|$$

for any $t \geq s$ and $u_0 \in X$ with $u_0 \geq 0$. It then follows that

$$e^{a_{\min}(t-s)} \leq \|\Phi(t,s;a)\| \leq e^{(a_{\max}+\|\mathcal{K}\|)(t-s)} \quad \forall\, t \geq s.$$

Therefore $\Phi(t,s;a)$ is exponentially bounded from above and below.

Next, we prove that $\lambda_{PL}(a) \leq \lambda_{PD}(a)$. To this end, for any given $\epsilon > 0$, let $\lambda_* = \lambda_{PD}(a)+\epsilon$. Then we can find $M > 0$ such that;

$$\|\Phi_{\lambda_*}(t,s;a)\| = \|e^{-\lambda_*(t-s)}\Phi(t,s;a)\| \leq M \quad \forall\, t \geq s.$$

That is

$$\|\Phi(t,s;a)\| \leq Me^{\lambda_*(t-s)} \quad \forall\, t \geq s.$$

It then follows that,

$$\limsup_{t-s\to\infty} \frac{\ln\|\Phi(t,s;a)\|}{t-s} \leq \lambda_*,$$

which implies $\lambda_{PL}(a) \leq \lambda_{PD}(a) + \epsilon$. Letting $\epsilon \to 0$. we conclude that $\lambda_{PL}(a) \leq \lambda_{PD}(a)$.

Now, we prove that $\lambda_{PD}(a) \leq \lambda_{PL}(a)$. To this end, for any $\epsilon > 0$, let $\bar{\lambda} = \lambda_{PL}(a) + \epsilon$. We have

$$\|\Phi_{\bar{\lambda}}(t,s;a)\| = e^{-(\lambda_{PL}(a)+\epsilon)(t-s)}\|\Phi(t,s;a)\| \to 0$$

as $t-s \to \infty$. This implies that $\Phi_{\lambda_{PL}(a)+\epsilon}(t,s;a)$ admits an exponential dichotomy with $P = 0$. So $\lambda_{PL}(a)+\epsilon \in \mathbb{R}\setminus\Sigma(a)$, and then $\lambda_{PD}(a) \leq \lambda_{PL}(a)+\epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\lambda_{PD}(a) \leq \lambda_{PL}(a)$. Hence $\lambda_{PL}(a) = \lambda_{PD}(a)$. □

3.4 Relations between the generalized principal eigenvalues and the top Lyapunov exponent

Having established the equality of the two top Lyapunov exponents, we can now simply refer to $\lambda_{PL}(a)$ as *the top Lyapunov exponent*. In this section, we discuss the relations between $\lambda_{PE}(a)$, $\lambda'_{PE}(a)$ and $\lambda_{PL}(a)$ and prove Theorem 3.2.

Before presenting the results of this section, we outline some preliminary propositions and lemmas to be used in obtaining the results. First, let us recall some existing results on the principal eigenvalue theory for (1.1) when D is bounded and $a(t,x)$ is T-periodic in t (i.e. $a(t+T,x) = a(t,x)$) or $D = \mathbb{R}^N$ and $a(t,x)$ is T-periodic in t and P-periodic in x, where $P = (p_1, p_2, \cdots, p_N)$ and $p_i \geq 0$ for $i = 1, 2, \cdots, N$) (i.e. $a(t+T,x) = a(t, x+p_i\mathbf{e_i}) = a(t,x)$ for $i = 1, 2, \cdots, N$).

Let

$$X_P = \begin{cases} X \quad \text{if} \quad D \quad \text{is bounded} \\ \{u \in X \,|\, u \text{ is } P\text{−periodic in } x\} \quad \text{if} \quad D = \mathbb{R}^N \end{cases}$$

and

$$\mathcal{X}_P = \begin{cases} \{u \in \mathcal{X} \,|\, u \text{ is } T\text{−periodic in } t\} \quad \text{if} \quad D \quad \text{is bounded} \\ \{u \in \mathcal{X} \,|\, u \text{ is is } T\text{−periodic in } t \text{ and } P\text{−periodic in } x\} \quad \text{if} \quad D = \mathbb{R}^N. \end{cases}$$

For given $a \in \mathcal{X}_p$, define $L_p(a) : \mathcal{D}(L_p(a)) \subset \mathcal{X}_P \to \mathcal{X}_P$ by

$$L_p(a)u = -u_t + \int_D \kappa(y-x)u(t,y)dy + a(t,x)u.$$

Definition 3.5. *For given $a \in \mathcal{X}_p$, let*

$$\lambda_s(a) = \sup\{\text{Re}\lambda \,|\, \lambda \in \sigma(L_p(a))\},$$

where $\sigma(L_p(a))$ is the spectrum of $L_p(a)$. $\lambda_s(a)$ is called the principal spectrum point *of $L_p(a)$. If $\lambda_s(a)$ is an isolated eigenvalue of $L_p(a)$ with a positive eigenfunction ϕ (i.e. $\phi \in \mathcal{X}_p$ with $\phi(t,x) > 0$), then $\lambda_s(a)$ is called the* principal eigenvalue *of $L_p(a)$ or it is said that $L_p(a)$* has a principal eigenvalue.

Proposition 3.1. *For given $a \in \mathcal{X}_p$, the following hold.*

(1) $\lambda_s(a) = \lambda_{PL}(a)$.

(2) The principal eigenvalue of $L_p(a)$ exists if $\hat{a}(\cdot)$ is C^N, there is some $x_0 \in \text{Int}(D)$ satisfying $\hat{a}(x_0) = \max_{x\in\bar{D}} \hat{a}(x)$, and the partial derivatives of $\hat{a}(x)$ up to order $N-1$ at x_0 are zero.

(3) For any $\epsilon > 0$, there is $a_\epsilon \in \mathcal{X}_P$ satisfying that

$$\|a - a_\epsilon\|_{\mathcal{X}} < \epsilon;$$

$\hat{a}_\epsilon$ is C^N; $\hat{a}_\epsilon$ attains its maximum at some point $x_0 \in \text{Int}(D)$; and the partial derivatives of $\hat{a}_\epsilon$ up to order $N-1$ at x_0 are zero, where $\hat{a}_\epsilon(x) = \frac{1}{T}\int_0^T a_\epsilon(t,x)dt$.

(4) $\lambda_s(a) \geq \lambda_s(\hat{a}) \geq \sup_{x\in D} \hat{a}(x)$.

Proof. (1) It follows from [30, Theorem 3.2].

(2) It follows from [48, Theorem B(1)].

(3) It follows from [48, Lemma 4.1].

(4) It follows from [48, Theorem C]. □

Next, we present four important lemmas.

Lemma 3.2. *For any $x \in D$ and $\epsilon > 0$, there is $A_{x,\epsilon} \in W^{1,\infty}(\mathbb{R})$ such that*

$$a(t,x) + A^{'}_{x,\epsilon}(t) \geq \hat{a}(x) - \epsilon \quad \textit{for} \;\; a.e. \; t \in \mathbb{R}.$$

Proof. It follows from [42, Lemma 3.2]. □

Lemma 3.3. *If $D_1 \subset D_2$, then $\lambda_{PL}(D_1) \leq \lambda_{PL}(D_2)$.*

Proof. For $u_0(x) \equiv 1$ on D_2, we have

$$\Phi(t,s;a,D_1)u_0|_{D_1} \leq \Phi(t,s;a,D_2)u_0 \quad \text{on} \quad D_1, \;\; \forall\, t \geq s.$$

This implies that

$$\lambda_{PL}(a,D_1) = \lim_{t-s\to\infty} \frac{\ln|\Phi(t,s;a,D_1)u_0|_{D_1}\|}{t-s} \leq \lim_{t-s\to\infty} \frac{\ln\|\Phi(t,s;a,D_2)u_0\|}{t-s} = \lambda_{PL}(a,D_2).$$

□

Lemma 3.4. $\lambda_{PL}(a) \geq \sup_{x\in D} \hat{a}(x)$.

Proof. Note that this lemma follows from $\lambda_{PL}(a) \geq \lambda_{PE}(a)$ (see Theorem 3.2(2)) and $\lambda_{PE}(a) \geq \sup_{x\in D}\hat{a}(x)$ (see Theorem 3.3(1)), whose proofs are independent of each other and do not require the conclusion in this lemma. In the following, we give a direct proof of this lemma.

For any $\epsilon > 0$, let $x_0 \in D$ be such that

$$\hat{a}(x_0) \geq \sup_{x\in D} \hat{a}(x) - \epsilon.$$

By Lemma 3.2, there are $\delta > 0$ and $A_0 \in W^{1,\infty}(\mathbb{R})$ such that

$$a(t,x_0) + A_0^{'}(t) \geq \hat{a}(x_0) - \epsilon \quad \text{for } \ a.e. \ t \in \mathbb{R} \tag{3.21}$$

and

$$a(t,x) \geq a(t,x_0) - \epsilon \quad \forall\, t \in \mathbb{R}, \ x \in D_1, \tag{3.22}$$

where

$$D_1 = D_1(x_0,\delta) = \{x \in D \,|\, |x - x_0| \leq \delta\}.$$

Let $u(t,x;D_1)$ be the solution of

$$u_t = \int_{D_1} \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}_1$$

with $u(0,x;D_1) = 1$. Let $v(t,x;D_1) = e^{A_0(t)}u(t,x;D_1)$. Then

$$v_t = \int_{D_1} \kappa(y-x)v(t,y;D_1)dy + (a(t,x) + A_0^{'}(t))v(t,x;D_1) \quad \text{for } \ a.e. \ t \geq 0, \ \forall\, x \in \bar{D}_1.$$

This together with Proposition 2.4, (3.21), and (3.22) implies that

$$v(t,x;D_1) \geq e^{A_0(0)}e^{(\hat{a}(x_0)-2\epsilon)t} \quad for \ a.e. \ t \geq 0, \ \forall\, x \in D_1.$$

Hence

$$\lambda_{PL}(D_1) \geq \hat{a}(x_0) - 2\epsilon \geq \sup_{x\in D} \hat{a}(x) - 3\epsilon.$$

By Lemma 3.3, we have

$$\lambda_{PL}(D) \geq \sup_{x\in D} \hat{a}(x) - 3\epsilon$$

for any $\epsilon > 0$. Letting $\epsilon \to 0$, the lemma follows. $\square$

Let $a(t,x), g(\cdot,\cdot) \in \mathcal{X}$ and $a(t,x)$ be almost periodic in t uniformly with respect to $x \in \bar{D}$. Consider

$$\frac{d\phi}{dt} = a(t,x)\phi(t) - \lambda\phi(t) + g(t,x), \tag{3.23}$$

where $\lambda \in \mathbb{R}$ is a constant and $x \in \bar{D}$. (3.23) can be viewed as a family of ODEs with parameter $x \in \bar{D}$.

Lemma 3.5. *If $\lambda > \sup_{x\in D} \hat{a}(x)$, then for any $x \in \bar{D}$,*

$$\phi^*(t; x, g) = \int_{-\infty}^{t} e^{\int_s^t a(\tau,x)d\tau - \lambda(t-s)} g(s,x)ds$$

is a unique bounded solution of (3.23) *on* $\mathbb{R}$. *Moreover, $\phi^*(t;x,g)$ is uniformly continuous in $(t,x) \in \mathbb{R}\times\bar{D}$. If $\inf_{t\in\mathbb{R},x\in\bar{D}} g(t,x) > 0$, then $\inf_{t\in\mathbb{R},x\in\bar{D}} \phi^*(t;x,g) > 0$.*

Proof. First, since $\lambda > \sup_{x\in D} \hat{a}(x)$, it is not difficult to prove that (3.23) has at most one bounded solution. Note that there is $\delta > 0$ such that

$$e^{\int_s^t a(\tau,x)d\tau - \lambda(t-s)} \leq e^{-\delta(t-s)} \quad \forall\, t > s,\ x \in \bar{D}.$$

This implies that $\phi^*(t;x,g)$ is uniformly bounded in $t \in \mathbb{R}$ and $x \in \bar{D}$. Moreover, by direct computation, we have that $\phi^*(t;x,g)$ is a bounded solution of (3.23) on $\mathbb{R}$ and then $\frac{d\phi^*}{dt}(t;x,g)$ is uniformly bounded. Hence $\phi^*(t;x,g)$ is uniformly continuous in t uniformly with respect to $x \in \bar{D}$.

Next, we claim that $\phi^*(t;x,g)$ is uniformly continuous in $x \in \bar{D}$ uniformly with respect to $t \in \mathbb{R}$. In fact, if the claim is not true, then there are $\epsilon_0 > 0$, $x_n, \tilde{x}_n \in \bar{D}$, and $t_n \in \mathbb{R}$ such that

$$|x_n - \tilde{x}_n| \leq \frac{1}{n} \quad \forall\, n \geq 1$$

and

$$|\phi^*(t_n;x_n,g)-\phi^*(t_n;\tilde{x}_n,g)|\geq \epsilon_0 \quad \forall\, n\geq 1. \tag{3.24}$$

Let

$$\phi_n(t)=\phi^*(t+t_n;x_n,g), \quad \tilde{\phi}_n(t)=\phi^*(t+t_n;\tilde{x}_n,g).$$

Then $\phi_n(t)$ and $\tilde{\phi}_n(t)$ satisfy

$$\phi_n^{'}(t)=a(t+t_n,x_n)\phi_n(t)-\lambda\phi_n(t)+g(t+t_n,x_n)$$

and

$$\tilde{\phi}_n^{'}(t)=a(t+t_n,\tilde{x}_n)\phi_n(t)-\lambda\phi_n(t)+g(t+t_n,\tilde{x}_n),$$

respectively. Without loss of generality, we may assume that there are $b(t)$, $h(t)$, $\phi(t)$, and $\tilde{\phi}(t)$ such that

$$\lim_{n\to\infty} a(t+t_n,x_n)=\lim_{n\to\infty} a(t+t_n,\tilde{x}_n)=b(t), \quad \lim_{n\to\infty} g(t+t_n,x_n)=\lim_{n\to\infty} g(t+t_n,\tilde{x}_n)=h(t),$$

and

$$\lim_{n\to\infty}\phi_n(t)=\phi(t), \quad \lim_{n\to\infty}\tilde{\phi}_n(t)=\tilde{\phi}(t)$$

locally uniformly in $t\in\mathbb{R}$. It then follows that both $\phi(t)$ and $\tilde{\phi}(t)$ are bounded solutions of the following ODE

$$\psi^{'}=b(t)\psi-\lambda\psi+h(t).$$

Since $\lambda>\sup_{t\in\mathbb{R}} b(t)$, this ODE has a unique bounded solution. This implies that

$$\phi(t)\equiv\tilde{\phi}(t).$$

But by (3.24),

$$|\phi(0)-\tilde{\phi}(0)|\geq\epsilon_0,$$

which is a contradiction. Therefore, the claim holds, whence $\phi^*(t;x,g)$ is uniformly continuous in $t\in\mathbb{R}$ and $x\in\bar{D}$.

We now claim that, if $g_{\inf} := \inf_{t\in\mathbb{R},x\in\bar{D}} g(t,x) > 0$, then $\inf_{t\in\mathbb{R},x\in\bar{D}} \phi^*(t;x,g) > 0$. In fact, let $a_{\inf} = \inf_{t\in\mathbb{R},x\in\bar{D}} a(t,x)$. For any $t \in \mathbb{R}$ and $x \in \bar{D}$, we have

$$\begin{aligned}\phi^*(t;x,g) &= \int_{-\infty}^{t} e^{\int_s^t a(\tau,x)d\tau-\lambda(t-s)} g(s,x)ds \\ &\geq \int_{-\infty}^{t} e^{(a_{\inf}-\lambda)(t-s)} g_{\inf} ds \\ &= \frac{g_{\inf}}{\lambda - a_{\inf}}.\end{aligned}$$

The claim then follows and the lemma is thus proved. □

Now, we present the proof of Theorem 3.2.

Proof of Theorem 3.2(1). The proof is given in two steps.

Step 1. In this step, we prove that $\lambda'_{PE}(a) \leq \lambda_{PL}(a)$.

Note that, for any $\lambda > \lambda_{PL}(a)$, there are $M, \delta > 0$ such that

$$e^{-\lambda(t-s)}\|\Phi(t,s;a)\| \leq Me^{-\delta(t-s)} \quad \forall\, t \geq s. \tag{3.25}$$

For given $v \in \mathcal{X}$, consider

$$u_t = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u - \lambda u + v. \tag{3.26}$$

Recall that

$$\Phi_\lambda(t,s;a) = e^{-\lambda(t-s)}\Phi(t,s;a).$$

Let

$$u(t,\cdot;a,v) = \int_{-\infty}^{t} \Phi_\lambda(t,s;a)v(s,\cdot)ds. \tag{3.27}$$

By direct computation, we have that $u(t,x;a,v)$ is a solution of (3.26). By (3.25), we have that $u(t,x;a,v)$ is bounded, and then by (3.26), $u(t,x;a,v)$ is uniformly continuous in t uniformly with respect to $x \in \bar{D}$.

Let $g(t,x) = \int_D \kappa(y-x)u(t,y;a,v)dy + v(t,x)$. We have $g \in \mathcal{X}$. By Lemma 3.4, $\lambda > \sup_{x\in D} \hat{a}(x)$. Then by Lemma 3.5, $u(t,x;a,v) = \phi^*(t;x,g)$ and then $u(\cdot,\cdot;a,v) \in \mathcal{X}$. Choose $v(t,x) \equiv 1$. By Lemma 3.5 again, we have $\inf_{t\in\mathbb{R}, x\in\bar{D}} u(t,x;a,v) > 0$. Note that

$$-u_t + \int_D \kappa(y-x)u(t,y;a,v)dy + a(t,x)u(t,x;a,v) = \lambda u(t,x;a,v) - v \le \lambda u(t,x;a,v).$$

Hence $\lambda \in \Lambda'_{PE}(a)$. Therefore,

$$\lambda'_{PE}(a) \le \lambda \quad \forall \lambda > \lambda_{PL}(a).$$

This implies that

$$\lambda'_{PE}(a) \le \lambda_{PL}(a).$$

Step 2. In this step, we prove that $\lambda'_{PE}(a) \ge \lambda_{PL}(a)$.

Note that for any $\lambda > \lambda'_{PE}(a)$, there is $\phi \in \mathcal{X}$ with $\inf_{t\in\mathbb{R}, x\in\bar{D}} \phi(t,x) > 0$ such that

$$-\phi_t(t,x) + \int_D \kappa(y-x)\phi(t,y)dy + a(t,x)\phi(t,x) \le \lambda\phi(t,x) \quad a.e.\, t \in \mathbb{R},\ \forall x \in \bar{D}.$$

Let $u_0 = \inf_{t\in\mathbb{R}, x\in\bar{D}} \phi(t,x)$. By Proposition 2.4, we have

$$\Phi(t,0;a)u_0 \le e^{\lambda t}\phi(t,x) \quad \forall\ t \ge 0,\ x \in \bar{D}.$$

This implies that

$$\lambda_{PL}(a) \le \liminf_{t\to\infty} \frac{\ln \|\Phi(t,0;a)u_0\|}{t} \le \lambda.$$

Hence $\lambda_{PL}(a) \le \lambda'_{PE}(a)$ and then $\lambda'_{PE}(a) = \lambda_{PL}(a)$. □

Proof of Theorem 3.2(2). We prove Theorem 3.2(2) in three steps.

Step 1. In this first step, we prove that $\lambda_{PE}(a) \le \lambda_{PL}(a)$ for any domain D.

Choose any $\lambda \in \Lambda_{PE}$. There is $\phi \in \mathcal{X}$ with $\inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0$ and $\lambda\phi \leq L\phi$. Set $w(t,x) = e^{\lambda t}\phi(t,x)$. Then $w(t,x)$ is a subsolution of (1.1) and $w(0,x) = \phi(0,x)$. By comparison principle, we have

$$e^{\lambda t}\phi(t,\cdot) \leq \Phi(t,0;a)w(0,\cdot) \quad \forall\, t \geq 0.$$

This implies that $\lambda \leq \lambda_{PL}(a)$. Hence

$$\lambda_{PE}(a) \leq \lambda_{PL}(a). \tag{3.28}$$

Step 2. In this step, we assume that $a(t,x)$ is T-periodic in t and is also periodic in x if $D = \mathbb{R}^N$, and prove that $\lambda_{PE}(a) = \lambda_{PL}(a)$.

By Proposition 3.1, for any $\epsilon > 0$, there are $a_\epsilon(t,x), \phi_\epsilon(t,x) \in \mathcal{X}_p$ such that $\phi_\epsilon(t,x) > 0$,

$$\|a - a_\epsilon\| < \epsilon,$$

and

$$-\partial_t\phi_\epsilon(t,x) + \int_D \kappa(y-x)\phi_\epsilon(t,y)dy + a_\epsilon(t,x)\phi_\epsilon(t,x) = \lambda_{PL}(a_\epsilon)\phi_\epsilon(t,x).$$

This implies that

$$\lambda_{PL}(a_\epsilon) - \|a - a_\epsilon\| \in \Lambda_{PE}(a).$$

It then follows that

$$\lambda_{PE}(a) \geq \lambda_{PL}(a_\epsilon) - \|a - a_\epsilon\| \geq \lambda_{PL}(a) - 2\epsilon.$$

Letting $\epsilon \to 0$, we get $\lambda_{PE}(a) \geq \lambda_{PL}(a)$, which together with (3.28) implies that $\lambda_{PE}(a) = \lambda_{PL}(a)$.

Step 3. In this step, we assume that $a(t,x)$ is limiting almost periodic and prove that $\lambda_{PE}(a) = \lambda_{PL}(a)$.

Since $a(t,x)$ is limiting almost periodic, there is a sequence $\{a_n(t,x)\}$ of periodic functions such that

$$\lim_{n\to\infty} a_n(t,x) = a(t,x)$$

uniformly in $t \in \mathbb{R}$ and $x \in \bar{D}$. Then by Lemma 3.1 and the arguments in **Step 2**,

$$\lambda_{PE}(a) = \lim_{n\to\infty} \lambda_{PE}(a_n) = \lim_{n\to\infty} \lambda_{PL}(a_n) = \lambda_{PL}(a).$$

The proof of Theorem 3.2(2) is thus completed. □

Proof of Theorem 3.2(3). Assume that $a(t,x) \equiv a(t)$.

First, we prove that for any D,

$$\lambda_{PL}(a) = \hat{a} + \lambda_{PL}(0). \tag{3.29}$$

Note that

$$\Phi(t;a) = e^{\int_0^t a(s)ds}\Phi(t;0).$$

This implies that (3.29) holds.

Next, we prove that for any D,

$$\lambda_{PE}(a) = \hat{a} + \lambda_{PE}(0). \tag{3.30}$$

To this end, we first consider the case that $\int_0^t a(s)ds - \hat{a}t$ is a bounded function of t. We claim that $\Lambda_{PE}(a) = \Lambda_{PE}(\hat{a})$. In fact, for any $\lambda \in \Lambda_{PE}(a)$, let $\phi \in \mathcal{X}$ be such that $\inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0$ and

$$-\phi_t + \int_D \kappa(y-x)\phi(t,y)dy + a(t)\phi(t,x) \geq \lambda\phi(t,x).$$

Let $\psi(t,x)=e^{-(\int_0^t a(s)ds-\hat{a}t)}\phi(t,x)$. Then $\psi\in\mathcal{X}$, $\inf_{t\in\mathbb{R}}\psi(t,x)\ge\not\equiv 0$, and

$$\begin{aligned}
-\psi_t(t,x) &= (a(t)-\hat{a})\psi(t,x) - e^{-(\int_0^t a(s)ds-\hat{a}t)}\phi_t(t,x)\\
&\ge (a(t)-\hat{a})\psi(t,x)\\
&\quad + e^{-(\int_0^t a(s)ds-\hat{a}t)}\Big(-\int_D \kappa(y-x)\phi(t,y)dy - a(t)\phi(t,x)+\lambda\phi(t,x)\Big)\\
&= -\hat{a}\psi(t,x) - \int_D \kappa(y-x)\psi(t,y)dy + \lambda\psi(t,x).
\end{aligned}$$

This implies that $\lambda\in\Lambda_{PE}(\hat{a})$.

Conversely, for any $\lambda\in\Lambda_{PE}(\hat{a})$, there is $\phi\in\mathcal{X}$ with $\inf_{t\in\mathbb{R}}\phi(t,x)\ge\not\equiv 0$ such that

$$-\phi_t + \int_D \kappa(y-x)\phi(t,y)dy + \hat{a}\phi(t,x) \ge \lambda\phi(t,x).$$

Let $\psi(t,x)=e^{-(\hat{a}t-\int_0^t a(s)ds)}\phi(t,x)$. Then $\psi\in\mathcal{X}$, $\inf_{t\in\mathbb{R}}\psi(t,x)\ge\not\equiv 0$, and

$$-\psi_t(t,x) \ge -a(t,x)\phi(t,x) - \int_D \kappa(y-x)\psi(t,y)dy + \lambda\psi(t,x).$$

This implies that $\lambda\in\Lambda_{PE}(a)$. Therefore, $\Lambda_{PE}(a)=\Lambda_{PE}(\hat{a})$ and then $\lambda_{PE}(a)=\lambda_{PE}(\hat{a})=\hat{a}+\lambda_{PE}(0)$. (3.30) follows.

We now consider the general case. Let $a(t)$ be any given almost periodic function. By Proposition 2.1(3), we have that for any $\epsilon>0$, there is an almost periodic function $a_\epsilon(t)$ such that $\int_0^t a_\epsilon(s)ds-\hat{a}_\epsilon t$ is bounded and

$$\|a(\cdot)-a_\epsilon(\cdot)\|\le\epsilon.$$

By the above arguments, $\lambda_{PE}(a_\epsilon)=\hat{a}_\epsilon+\lambda_{PE}(0)$. By Lemma 3.1,

$$\hat{a}+\lambda_{PE}(0)-2\epsilon\le\lambda_{PE}(a)\le\hat{a}+\lambda_{PE}(0)+2\epsilon$$

Letting $\epsilon\to 0$, (3.30) follows.

Now, by similar arguments, we have that for any D,

$$\lambda'_{PE}(a) = \hat{a} + \lambda'_{PE}(0). \tag{3.31}$$

.

Finally, by (1) and (2), $\lambda_{PL}(0) = \lambda_{PE}(0) = \lambda'_{PE}(0)$. This together with (3.29), (3.30), and (3.31) implies (3). □

3.5 Effects of time and space variations

3.5.1 Effects of time and space variations on the generalized principal eigenvalues

Here, we discuss the effects of time and space variations on $\lambda_{PE}(a)$ and prove Theorem 3.3. We first present a lemma.

Lemma 3.6. *Consider* (1.2)

Suppose that $f(t,x,u) = u(a(x) - b(x))$*,* $a, b \in X$*, and* $\inf_{x\in D} b(x) > 0$*. If* $\lambda_{PE}(a, D_0) > 0$ *for some bounded subset* $D_0 \subset D$*, then* (1.2) *has a positive stationary solution* $\phi^*(\cdot) \in X$*.*

Proof. Let $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ be a sequence of bounded domains such that $D = \cup_{n=1}^{\infty} D_n$. Then by Theorem 3.2 and Lemma 3.3,

$$\lambda_{PE}(a, D_n) = \lambda_{PL}(a, D_n) \le \lambda_{PL}(a, D_{n+1}) = \lambda_{PE}(a, D_{n+1}) \quad \forall\, n \ge 1.$$

By Proposition 3.1,

$$\lambda_{PE}(a, D_n) = \lambda_{PL}(a, D_n) \ge \lambda_{PL}(a, D_0) > 0 \quad \forall\, n \gg 1.$$

Then by [48, Theorem E], there is a unique positive stationary solution $\phi_n^*(\cdot) \in X(D_n)$ of

$$u_t = \int_{D_n} \kappa(y - x)u(t, y)dy + u(a(x) - b(x)u), \quad x \in \bar{D}_n$$

for $n \gg 1$. By Proposition 2.4,

$$\phi_n^*(x) \le \phi_{n+1}^*(x) \quad \forall\, x \in D_n,\ n \gg 1.$$

Therefore, the limit $\phi^*(x) = \lim_{n\to\infty} \phi_n^*(x)$ exists for all $x \in \bar{D}$. Moreover, it is not difficult to see that $u = \phi^*(x)$ is a positive stationary solution of (1.2). □

We now prove Theorem 3.3.

Proof of Theorem 3.3. (1). We first prove that for any a satisfying **(H2)**, $\lambda_{PE}(a) \ge \sup_{x\in D} \hat{a}(x)$. For any $\epsilon > 0$, let $x_0 \in D$ be such that

$$\hat{a}(x_0) \ge \sup_{x\in D} \hat{a}(x) - \epsilon.$$

By Lemma 3.2, there are $\delta > 0$ and $A_0 \in W^{1,\infty}(\mathbb{R})$ such that

$$a(t, x_0) + A_0'(t) \ge \hat{a}(x_0) - \epsilon \quad \text{for} \quad a.e.\ t \in \mathbb{R} \tag{3.32}$$

and

$$a(t, x) \ge a(t, x_0) - \epsilon \quad \forall\, t \in \mathbb{R},\ x \in D_1(x_0, \delta), \tag{3.33}$$

where

$$D_1(x_0, \delta) = \{x \in D \,|\, |x - x_0| \le \delta\}.$$

By (3.32), there is $\tilde{a}(\cdot) \in X$ such that

$$\tilde{a}(x) \begin{cases} = \hat{a}(x_0) - \epsilon \quad x \in D_1(x_0, \delta/2) \\ \le a(t, x) + A_0'(t) \quad \text{for } a.e.\ t \in \mathbb{R},\ \forall\, x \in D. \end{cases}$$

For any $\lambda < \hat{a}(x_0) - \epsilon$, consider

$$\tilde{u}_t = \int_D \kappa(y - x)\tilde{u}(t, y)dy + \tilde{u}(t, x)(\tilde{a}(x) - \lambda - A_0'(t) - e^{A_0(t)}\tilde{u}), \quad x \in D. \tag{3.34}$$

Let $\tilde{v}(t,x)=e^{A_0(t)}\tilde{u}(t,x)$. Then $\tilde{v}(t,x)$ satisfies

$$\tilde{v}_t=\int_{D_1}\kappa(y-x)\tilde{v}(t,y)dy+\tilde{v}(t,x)(\tilde{a}(x)-\lambda-\tilde{v}),\quad \text{for } a.e.\, t\in\mathbb{R},\ \forall\, x\in D. \tag{3.35}$$

By Lemma 3.6, there is $\tilde{v}^*\in X$ with $\tilde{v}^*(x)>0$ such that

$$\int_D\kappa(y-x)\tilde{v}^*(y)dy+\tilde{v}^*(x)(\tilde{a}(x)-\lambda-\tilde{v}^*(x))=0\quad\forall\, x\in D.$$

Let $\tilde{u}^*(t,x)=\tilde{v}^*(x)e^{-A_0(t)}$. We have

$$-\tilde{u}_t^*+\int_D\kappa(y-x)\tilde{u}^*(t,y)dy+(\tilde{a}(x)-A_0^{'})\tilde{u}^*(t,x)\geq\lambda\tilde{u}^*(t,x)$$

for a.e. $t\in\mathbb{R}$ and all $x\in D$. This implies that

$$-\tilde{u}_t^*+\int_D\kappa(y-x)\tilde{u}^*(t,y)dy+a(t,x)\tilde{u}^*(t,x)\geq\lambda\tilde{u}^*(t,x)$$

for a.e. $t\in\mathbb{R}$ and all $x\in D$. Hence $\lambda\in\Lambda_{PE}(a)$, and

$$\lambda_{PE}(a)\geq\sup_{x\in D}\hat{a}(x)-2\epsilon.$$

Letting $\epsilon\to 0$, we obtain that $\lambda_{PE}(a)\geq\sup_{x\in D}\hat{a}(x)$.

Next, we assume that $a(t,x)$ is limiting almost periodic and show that $\lambda_{PE}(a)\geq\lambda_{PE}(\hat{a})\geq\sup_{x\in D}\hat{a}(x)$. Let $a_n(t,x)$ be a sequence of periodic functions such that $\lim_{n\to\infty}a_n(t,x)=a(t,x)$ uniformly in $t\in\mathbb{R}$ and $x\in\bar{D}$. By Theorem 3.2(2) and Proposition 3.1(1), (3), we have

$$\lambda_{PE}(a_n)\geq\lambda_{PE}(\hat{a}_n)\geq\sup_{x\in D}\hat{a}_n(x).$$

Letting $n\to\infty$, by Lemma 3.1, we obtain

$$\lambda_{PE}(a)\geq\lambda_{PE}(\hat{a})\geq\sup_{x\in D}(\hat{a}(x)).$$

(1) is thus proved.

(2) Write the eigenvalue problem

$$\int_D \kappa(y-x)\phi(y)dy + a(x)\phi(x) = \lambda\phi(x) \quad \forall\, x \in \bar{D}$$

as

$$\int_D \kappa(y-x)[\phi(y)-\phi(x)]dy + [a(x) + \int_D \kappa(y-x)dy]\phi(x) = \lambda\phi(x) \quad \forall\, x \in \bar{D}.$$

Then by the arguments of [52, Theorem 2.1(4)],

$$\lambda_{PE}(a) \geq \bar{a} + \frac{1}{|D|}\int_D\int_D \kappa(y-x)dydx.$$

(3) Let $R_n \to \infty$ and $B(0, R_n) = \{x \in \mathbb{R}^N \mid \|x\| \leq R_n\}$. Then by Theorem 3.2 and Lemma 3.3,

$$\lambda_{PE}(a, B(0, R_n)) = \lambda_{PL}(a, B(0, R_n)) \leq \lambda_{PL}(a, B(0, R_{n+1})) = \lambda_{PE}(a, B(0, R_{n+1})) \quad \forall\, n \geq 1.$$

Put

$$\lambda_\infty(a, D) = \lim_{n\to\infty} \lambda_{PE}(a, B(0, R_n)) > 0.$$

Then for any $\lambda < \lambda_\infty(a, D)$,

$$\lambda(a, B(0, R_n)) - \lambda > 0 \quad \forall\, n \gg 1.$$

By Lemma 3.6, there is $\phi \in X^+ \setminus \{0\}$ such that

$$\int_D \kappa(y-x)\phi(y)dy + a(x)\phi(x) = \lambda\phi(x) + \phi^2(x) \geq \lambda\phi(x) \quad \forall\, x \in D.$$

This implies that

$$\lambda_{PE}(a, D) \geq \lambda \quad \forall\, \lambda < \lambda_\infty(a, D).$$

Hence,

$$\lambda_{PE}(a, D) \geq \lambda_{PE}(a, B(0, R_n)) \quad \forall\, n \geq 1. \tag{3.36}$$

By (2), we have

$$\lambda_{PE}(a, B(0, R_n)) \geq \frac{1}{|B(0, R_n)|} \int_{B(0,R_n)} a(x)dx + \frac{1}{|B(0, R_n)|} \int_{B(0,R_n)} \int_{B(0,R_n)} \kappa(y-x)dydx.$$

By **(H1)**, for any $\epsilon > 0$, there is $r > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,r)} \kappa(z)dz < \epsilon.$$

This implies that

$$\begin{aligned}
\int_{B(0,R_n)} \int_{B(0,R_n)} \kappa(y-x)dydx &\geq \int_{B(0,R_n-r)} \int_{B(0,R_n)} \kappa(y-x)dydx \\
&\geq \int_{B(0,R_n-r)} [\int_{\mathbb{R}^N} \kappa(y-x)dy - \epsilon]dx \\
&= \int_{B(0,R_n-r)} (1-\epsilon)dx = |B(0, R_n - r)|(1-\epsilon).
\end{aligned}$$

Note that

$$\frac{|B(0, R_n - r)|}{|B(0, R_n)|} = \frac{(R_n - r)^N}{R_n^N} \to 1 \quad \text{as} \quad n \to \infty.$$

It then follows that

$$\lambda_{PE}(a) \geq \hat{a} + 1 - \epsilon \quad \forall\, \epsilon > 0.$$

Let $\epsilon \to 0$, we have

$$\lambda_{PE}(a) \geq \hat{a} + 1.$$

The theorem is thus proved. □

3.5.2 Effects of time and space variations on the top Lyapunov exponents

In this section, we discuss the effects of space and time variations on $\lambda_{PL}(a)$ and prove Theorem 3.4. We first present a lemma.

Lemma 3.7. *For any given $T > 0$ and compact subset $\Omega \subset \mathbb{R}^N$, let $w(t,x)$ be a positive continuous function on $[0,T] \times \Omega$. Let*

$$\theta(x,y) = \frac{1}{T}\int_0^T \frac{w(t,y)}{w(t,x)}dt.$$

Then either $w(t,x)$ is independent of x or there is $x^* \in \Omega$ such that

$$\theta(x^*,y) \geq 1 \quad \forall\, y \in \Omega$$

with strict inequality for some $y \in \Omega$.

Proof. It follows from [30, Lemma 4.3]. □

We now prove Theorem 3.4.

Proof of Theorem 3.4. (1) First we assume that D is bounded. Let $u_0^*(x) \equiv 1$. Let $u(t,\cdot;u_0^*) = \Phi(t;a)u_0^*$ and

$$v(t,\cdot;u_0^*) = e^{-\lambda_{PL}(a)t}u(t,\cdot;u_0^*).$$

Then

$$\limsup_{t\to\infty} \frac{\ln \|v(t,\cdot;u_0^*)\|}{t} = 0$$

and $v(t,x;u_0^*)$ satisfies

$$\lambda_{PL}(a)v = -v_t + \int_D \kappa(y-x)v(t,y)dy + a(t,x)v(t,x) \quad \forall\, t \geq 0,\ x \in D.$$

Hence

$$\lambda_{PL}(a) = -\frac{v_t(t,x;u_0^*)}{v(t,x;u_0^*)} + \int_D \kappa(y-x)\frac{v(t,y;u_0^*)}{v(t,x;u_0^*)}dy + a(t,x) \quad \forall\, t \geq 0,\ x \in D. \tag{3.37}$$

For any $\epsilon > 0$, by Proposition 3.1, there are $a^* \in X$ and $\phi^* \in X$ with $\phi^*(x) > 0$ such that

$$a^*(x) \leq \hat{a}(x) \leq a^*(x) + \epsilon, \tag{3.38}$$

$$\lambda_{PL}(\hat{a}) - \epsilon \le \lambda_{PL}(a^*) \le \lambda_{PL}(\hat{a}), \tag{3.39}$$

and

$$\lambda_{PL}(a^*) = \int_D \kappa(y-x)\frac{\phi^*(y)}{\phi^*(x)}dy + a^*(x) \quad \forall\, x \in D. \tag{3.40}$$

By (3.37) and (3.40), for any $T > 0$, we have

$$\begin{aligned}
&\lambda_{PL}(a^*) - \lambda_{PL}(a) \\
&= \frac{1}{T}\int_0^T \frac{v_t(t,x;u_0^*)}{v(t,x;u_0^*)}dt + \int_D \kappa(y-x)\Big(\frac{\phi^*(y)}{\phi^*(x)} - \frac{1}{T}\int_0^T \frac{v(t,y;u_0^*)}{v(t,x;u_0^*)}dt\Big)dy \\
&\quad + a^*(x) - \frac{1}{T}\int_0^T a(t,x)dt \\
&= \frac{1}{T}\big(\ln v(T,x;u_0^*) - \ln v(0,x;u_0^*)\big) + \int_D \kappa(y-x)\frac{\phi^*(y)}{\phi^*(x)}\Big(1 - \frac{1}{T}\int_0^T \frac{w(t,y)}{w(t,x)}dt\Big)dy \\
&\quad + a^*(x) - \frac{1}{T}\int_0^T a(t,x)dt \quad \forall\, x \in D,
\end{aligned} \tag{3.41}$$

where $w(t,x) = \frac{v(t,x;u_0^*)}{\phi_1^*(x)}$.

Choose $T > 0$ such that

$$\frac{1}{T}\int_0^T a(t,x)dt \ge \hat{a}(x) - \epsilon \quad \forall\, x \in D$$

and

$$\frac{1}{T}\big(\ln v(T,x;u_0^*) - \ln v(0,x;u_0^*)\big) = \frac{1}{T}\ln v(T,x;u_0^*) \le \frac{1}{T}\ln \|v(T,\cdot;u_0^*)\| \le \epsilon.$$

Fix such T. By Lemma 3.7, there is $x^* \in D$ such that

$$1 - \frac{1}{T}\int_0^T \frac{w(t,y)}{w(t,x^*)}dt \le 0 \quad \forall\, y \in D.$$

It then follows from (3.39) and (3.41) that

$$\lambda_{PL}(\hat{a}) - \epsilon - \lambda_{PL}(a) \le \lambda_{PL}(a^*) - \lambda_{PL}(a) \le a^*(x) - \hat{a}(x) + 2\epsilon \le 2\epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$\lambda_{PL}(a) \geq \lambda_{PL}(\hat{a}). \tag{3.42}$$

Next, suppose that D is unbounded. By Theorem 3.2(2) and Theorem 3.3(1), we have

$$\lambda_{PL}(a) = \lambda_{PE}(a) \geq \lambda_{PE}(\hat{a}) = \lambda_{PL}(\hat{a}).$$

It then follows that

$$\lambda_{PL}(a) \geq \lambda_{PL}(\hat{a}) \geq \sup_{x \in D} \hat{a}(x).$$

where the last inequality follows from Lemma 3.4.

(2) It follows from (1) and Theorem 3.3(2).

(3) It follows from (1) and Theorem 3.3(3). □

3.6 Characterization of the generalized principal eigenvalues

In this section, we discuss the characterization of $\lambda_{PE}(a)$ and $\lambda'_{PE}(a)$ and prove Theorem 3.5.

Proof of Theorem 3.5. (1) Assume that $a(t, x) \equiv a(x)$. Let

$$\tilde{\lambda}_{PE}(a) = \sup\{\lambda \,|\, \lambda \in \tilde{\Lambda}_{PE}(a)\} \quad \text{and} \quad \tilde{\lambda}'_{PE}(a) = \inf\{\lambda \,|\, \lambda \in \tilde{\Lambda}'_{PE}(a)\}.$$

First, by the arguments of Theorem 3.2(1), we have

$$\tilde{\lambda}'_{PE}(a) = \lambda_{PL}(a). \tag{3.43}$$

To be more precise, first, when $v(t, x) \equiv 1$ and $\lambda > \lambda_{PL}(a)$, it can be verified directly that the function $u(t, x; a, v)$ is independent of t, where $u(t, x; a, v)$ is defined in (3.27), that is,

$$u(t, x; a, v) = \int_{-\infty}^{t} \Phi_\lambda(t, s; a)v(s, \cdot)ds.$$

Then, $u(t,\cdot;a,v) \equiv u(\cdot;a,v) \in X$. By the arguments in step 1 of the proof of Theorem 3.2(1), $\lambda \in \tilde{\Lambda}'_{PE}(a)$ and

$$\tilde{\lambda}'_{PE}(a) \le \lambda_{PL}(a).$$

Secondly, it is clear that, by the arguments in step 2 of the proof of Theorem 3.2(1),

$$\tilde{\lambda}'_{PE}(a) \ge \lambda_{PL}(a).$$

(3.43) thus follows.

Next, by the arguments of Theorem 3.2(2), we have

$$\tilde{\lambda}_{PE}(a) = \lambda_{PL}(a). \tag{3.44}$$

To be more precise, first, it is clear that, by the arguments in step 1 of the proof of Theorem 3.2(2),

$$\tilde{\lambda}_{PE}(a) \le \lambda_{PL}(a).$$

Thus, by the arguments in steps 2and 3 of the proof of Theorem 3.2(2),

$$\tilde{\lambda}_{PE}(a) \ge \lambda_{PL}(a).$$

(3.44) then follows. Now by (3.43), (3.44), and Theorem 3.2,

$$\tilde{\lambda}'_{PE}(a) = \tilde{\lambda}_{PE}(a) = \lambda_{PL}(a) = \lambda_{PE}(a) = \lambda'_{PE}(a).$$

This implies (1).

(2) Assume that $a(t+T,x) \equiv a(t,x)$. Let

$$\hat{\lambda}_{PE}(a) = \sup\{\lambda \,|\, \lambda \in \hat{\Lambda}_{PE}(a)\} \quad \text{and} \quad \hat{\lambda}'_{PE}(a) = \inf\{\lambda \,|\, \lambda \in \hat{\Lambda}'_{PE}(a)\}.$$

Similarly, by the arguments of Theorem 3.2, we have

$$\hat{\lambda}'_{PE}(a) = \hat{\lambda}_{PE}(a) = \lambda_{PL}(a) = \lambda_{PE}(a) = \lambda'_{PE}(a).$$

(2) then follows. $\square$

Chapter 4

Asymptotic dynamics of Fisher-KPP equations with almost periodic dependence

In this chapter, we study the asymptotic dynamics of (1.2). Recall that the nonlinear nonlocal dispersal equation (1.2) is given by,

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + uf(t,x,u), \quad x \in \bar{D},$$

The asymptotic dynamics of (1.2) is concerned with the study of the behaviour of solutions with positive initials, especially the existence, uniqueness, and stability of a strictly positive almost periodic entire solution of (1.2). These dynamical issues have been extensively studied for population models described by reaction diffusion equations and are quite well understood in many cases. Recently there has also been extensive investigation on these dynamical issues for nonlocal dispersal population models (see [1, 2, 3, 6, 7, 8, 11, 12, 14, 21, 31, 35, 36, 37, 48, 49, 57, 56, 58, 59, 63, 64], etc.). However, the understanding of these issues for nonlocal dispersal equations is much less, and, to our knowledge they have been essentially investigated in specific situations, such as time and space periodic media or time independent and space heterogeneous media.

When D is a bounded domain and $f(t,x,u)$ is independent of t or periodic in t, the asymptotic dynamics of (1.2) has been studied in [7], [48] and [56]. When $D = \mathbb{R}^N$ and $f(t,x,u)$ are periodic in both t and x or $f(t,x,u) \equiv f(t,u)$ or $f(t,x,u) \equiv f(x,u)$, it has been studied in [6], [48]. Here, we study the existence, uniqueness, and stability of strictly positive almost periodic entire solutions of (1.2). We highlight that, in contrast to the Laplacian, the integral operator in (1.2) is not a local operator. The mathematical analysis of (1.2) appears to be difficult even though the dispersal is represented by a bounded integral operator. Unlike the case of reaction-diffusion equations, the forward flow associated with (1.2) does not have a regularizing effect.

A solution $u(t, x)$ of (1.2) defined for all $t \in \mathbb{R}$ is called an *entire solution*. An entire solution $u(t, x)$ of (1.2) is said to be *strictly positive* if $\inf_{t\in\mathbb{R}, x\in\bar{D}} u(t, x) > 0$. A strictly positive entire solution $u(t, x)$ of (1.2) is called an *almost periodic solution* if it is almost periodic in t uniformly with respect to $x \in \bar{D}$ in the case that D is bounded and is almost periodic in both t and x when $D = \mathbb{R}^N$.

4.1 Main results and remarks

In this section, we present our main results on the persistence and extintion of the population modeled by equation (1.2). We also give some remarks to highlight the contributions of these results.

4.1.1 Main results

Throughout this subsection, we assume (H1) and (H3). Observe that a function $u(t, x)$ satisfying (1.2) need not be continuous in x. However, unless otherwise specified, when we say that $u(t, x)$ is a *solution of* (1.2) *on an interval* I, it means that, for each $t \in I$, $u(t, \cdot) \in X$, and the mapping $I \ni t \mapsto u(t, \cdot) \in X$ is differentiable. Such a solution $u(t, x)$ is clearly differentiable in t and is continuous in both t and x.

Recall that, by general semigroup theory (see [47]), for any $s \in \mathbb{R}$ and $u_0 \in X$, (1.2) has a unique (local) solution $u(t, x; s, u_0)$ with $u(s, x; s, u_0) = u_0(x)$. Moreover, for any $u_0 \in X^+$, $u(t, x; s, u_0)$ exists globally, that is, $u(t, x; s, u_0)$ exists for all $t \geq s$ (see the comparison principle, Proposition 2.4 (4)).

In the rest of this chapter, $u(t, x; s, u_0)$ always denotes the solution of (1.2) with $u(s, \cdot; s, u_0) = u_0 \in X$, unless specified otherwise. Among others, we prove

Theorem 4.1.

(a) (Uniqueness) There is at most one strictly positive, bounded entire solution of (1.2).

(b) *(Almost periodicity) Any strictly positive bounded entire solution of* (1.2) *is almost periodic.*

(c) *(stability) If $u^*(t,x)$ is a strictly positive, bounded, almost periodic solution of* (1.2)*, then for any $u_0 \in X^{++}$,*

$$\lim_{t\to\infty} \|u(t,\cdot;u_0) - u^*(t,\cdot)\|_\infty = 0.$$

(d) *(Frequency module) If $u^*(t,x)$ is a strictly positive, bounded almost periodic solution of* (1.2)*, then*

$$\mathcal{M}(u^*) \subset \mathcal{M}(f),$$

where $\mathcal{M}(\cdot)$ denotes the frequency module of an almost periodic function.

Theorem 4.2.

(a) *(Existence) Equation* (1.2) *has a strictly positive bounded almost periodic entire solution if and only if $\lambda_{PE}(a,D) > 0$.*

(b) *(Nonexistence) If $\lambda_{PL}(a,D) < 0$, then the trivial solution $u \equiv 0$ of* (1.2) *is globally asymptotically stable in the sense that for any $u_0 \in \mathcal{X}^+$,*

$$\|u(t,\cdot;0,u_0)\|_X \to 0 \text{ as } t \to \infty.$$

Corollary 4.1. *(1) If $\sup_{x\in D} \hat{a}(x) > 0$, then the equation* (1.2) *has a strictly positive almost periodic solution.*

(2) *If $\kappa(\cdot)$ is symmetric, $a(t,x) \equiv a(x)$ is almost periodic in x, and $\bar{a}+1 > 0$, then the equation* (1.2) *has a strictly positive almost periodic solution.*

Proof. (1) This follows from Theorem 1.3(1) of [45] and Theorems 4.1 and 4.2.

(2) Follows from Theorem 1.3(3) of [45] and Theorems 4.1 and 4.2.

□

We also establish a new property of the generalized principal eigenvalues of the linear nonlocal equation (1.1).

Theorem 4.3. *Suppose that $a(t,x)$ satisfies (H3) with $D = \mathbb{R}^N$. For any $D_1 \subset D_2$, there holds*

$$\lambda_{PE}(a, D_1) \leq \lambda_{PE}(a, D_2), \tag{4.1}$$

where D_1 is bounded and D_2 is bounded or $D_2 = \mathbb{R}^N$.

We highlight that we consider $\lambda_{PE}(a, D)$ with D being either bounded or the whole space $\mathbb{R}^N$. Hence it is assumed that D_1 is bounded in Theorem 4.3. For otherwise, if $D_1 = \mathbb{R}^N$, then $D_2 = \mathbb{R}^N$ and nothing needs to be proved.

4.1.2 Remarks on the main results

In this subsection, we give some Remarks on the main results of this chapter.

First, we give some remarks on our results in some special cases.

Remark 4.1 (Extension of existing results in special cases)**.**

(1) For the case that the function $f(t,x,u)$ is time independent or time periodic and is periodic in x when $D = \mathbb{R}^N$, similar results on the asymptotic dynamics of (1.2) *as Theorems 4.1 and 4.2 have been obtained in [2, 7, 48, 57]). Our results recover those results in [2, 7, 48, 57].*

(2) In [50], results similar to theorems 4.1 and 4.2 for the case $D = \mathbb{R}$ were obtained for general time dependence under the condition $\liminf\limits_{t-s\to\infty} \frac{1}{t-s}\int_s^t \inf\limits_{x\in\mathbb{R}} f(\tau,x,0)d\tau > -1$. Note that when $D = \mathbb{R}$ and $f(t,x,u)$ is almost periodic in t uniformly with respect to $x \in \mathbb{R}$, $a_{\inf}(t) := \inf_{x\in\mathbb{R}} a(t,x)$ is also almost periodic in t, where $a(t,x) = f(t,x,0)$. Note also that $\lambda_{PE}(a) \geq \lambda_{PE}(a_{\inf}) = 1 + \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t a_{\inf}(\tau)d\tau$. Thus our results extend Theorem 2.1 of [50] in the case when $D = \mathbb{R}$ and $f(t,x,u)$ is almost periodic in t.

(3) It should be pointed out that, in the case that $D = \mathbb{R}^N$ and $f(t,x,u) \equiv f(x,u)$ is not almost periodic in x, the existence, uniqueness, and stability of positive solutions when

$\lambda_P(a) < 0$ was established in [6, Theorem 1.1], where $a(x) = f(x, 0)$ and

$$\lambda_p(a) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in C(\mathbb{R}^N),\ \phi > 0\ s.t.\ \int_{\mathbb{R}^N} \kappa(y-x)\phi(y)dy + a(x)\phi + \lambda\phi \le 0\}.$$

Note that the test function in the definiton $\lambda_p(a)$ may not be uniformly continuous and bounded, which are required for the test functions in the definition of $\lambda'_{PE}(a)$. Hence

$$-\lambda_p(a) \le \lambda'_{PE}(a).$$

In the case that $a(x)$ is limiting almost periodic, $\lambda_{PE}(a) = \lambda'_{PE}(a) \ge -\lambda_p(a)$. Hence, in such a case, $\lambda_p(a) < 0$ implies $\lambda_{PE}(a) > 0$ and our results improve [6, Theorem 1.1] in the sense that the positive solution we obtained is strictly positive and almost periodic.

Second, we give some remarks on the time and space variations.

Remark 4.2 (Effects of time and space variations)**.**

(1) If $a(t, x) = f(t, x, 0)$ is limiting almost periodic, then $\lambda_{PE}(a) \ge \lambda_{PE}(\hat{a})$ (see Theorem 3.3(1)), which shows that time variation does not reduce the generalized principal eigenvalue λ_{PE}. Thus Theorem 4.2(b) indicates that time variation may favor the persistence of species.

(2) If $a(t, x) = f(t, x, 0)$ is independent of t, $\kappa(\cdot)$ is symmetric, and $D = \mathbb{R}^N$, then $\lambda_{PE}(a) \ge \bar{a}+1 = \lambda_{PE}(\bar{a})$ (see Theorem 3.3(3)). Theorem 4.2(b) then indicates that space variation may favor the persistence of species.

Third, we give some remarks on the proofs of the main results.

Remark 4.3 (Difficulties in the proofs)**.** *By Theorem 4.2, $\lambda_{PE}(a) > 0$ is a necessary and sufficient condition for the existence of a unique strictly positive almost periodic solutuon of* (1.2)*, where $a(t, x) = f(t, x, 0)$. Note that $\lambda_{PE}(a) > 0$ indicates that the trivial solution $u = 0$ of* (1.2) *is unstable. It is naturally expected that the instability of the trivial solution $u = 0$ implies the existence of a positive entire solution. In fact, this has been proved for the random dispersal counterpart of* (1.2)*. However, due to the lack of the regularizing effect of the forward* *flow associated with* (1.2) *and the lack of Poincaré map in non-periodic time dependent case, it is very nontrivial to prove the existence of strictly positive almost periodic solutions of* (1.2).

Fourth, we give some remarks on the extension of the main results to more general cases.

Remark 4.4 (Extension of the main results to non-almost periodic cases)**.** *As mentioned in Remark 3.6, the definitions of $\lambda_{PL}(a), \lambda'_{PL}(a), \lambda_{PE}(a)$, and $\lambda'_{PE}(a)$ apply to general $a(t,x)$ which is bounded and uniformly continuous. When $f(t,x,u)$ is not assumed to be almost periodic in t, if $\lambda_{PE}(a) > 0$ ($a(t,x) = f(t,x,0)$), we still have a positive continuous function $u^*(t,x)$ which satisfies* (1.2) *for all $t \in \mathbb{R}$ and $x \in \bar{D}$. Moreover, if D is bounded, then $u^*(t,x)$ is a strictly positive entire solution of* (1.2) *and is asymptotically stable with respect to positive perturbations. But in general, $u^*(t,x)$ may not be strictly positive (see Remark 4.6).*

Finally, we give some remarks on the application of the main results to the study of propagation phenomena in (1.2) when $D = \mathbb{R}^N$.

Remark 4.5 (Propagation dynamics)**.** *Suppose that $D = \mathbb{R}^N$ and $\lambda_{PE}(a) > 0$, where $a(t,x) = f(t,x,0)$. Then by Theorems 4.1 and 4.2,* (1.2) *has a unique strictly positive almost periodic solution $u^*(t,x)$ that attracts all solutions with strictly positive initials uniformly, but $u^*(t,x)$ does not attract solutions with compactly supported or front-like initials uniformly. Biologically, such an initial indicates that the population initially resides in a bounded region or in one side of the whole space. Naturally, the population with such initial distribution will spread into the empty region as time evolves. It is interesting to ask how fast the population spreads. Based on the investigation in the time independent or periodic case (see [49, 57]), it is equivalent to asking how fast the region where the solution is near $u^*(t,x)$ grows. To be a little more precise, for a given compactly supported initial u_0 (i.e. $u_0(x) \geq 0$ and $\{x \in \mathbb{R}^N \,|\, u_0(x) > 0\}$ is bounded and non-empty) or front-like initial u_0 (i.e. $u_0(x) - u^*(x,0) \to 0$ as $x \cdot \xi \to -\infty$ and $u_0(x) = 0$ for $x \cdot \xi \gg 1$ for some unit vector $\xi \in \mathbb{R}^N$), and given $0 < \epsilon \ll 1$, let*

$$D(t,u_0) = \{x \in \mathbb{R}^N \,:\, |u(t,x;0,u_0) - u^*(t,x)| \leq \epsilon\}.$$

By the stability of $u^*(t,x)$ with respect to strictly positive initials, it is expected that $D(t,u_0)$ grows as t increases. It is interesting to know how fast $D(t,u_0)$ grows. We will study this problem in the next chapter.

4.2 Preliminary propositions

In this section, we present some propositions to be used in establishing the results in this chapter.

Proposition 4.1. *Let $0 < \delta_0 < 1$ and $r_0 > 0$ be given positive numbers. Suppose that* **(H1)** *holds. Then for any given positive integer k, there exist a positive number $\mu = \mu(r_0, \delta_0, k)$ and a positive integer $i = i(r_0, \delta_0, k)$ such that*

$$\inf_{x\in B_{kr_0}(0)} \sum_{j=1}^{i} \frac{(K^j u)(x)}{j!} \geq \mu \;\; \forall\, u \in L^\infty(\mathbb{R}^n),\; u \geq 0,\; \textit{with} \int_{B_{r_0}(0)} u\, dx \geq \delta_0, \tag{4.2}$$

where $Ku = \kappa * u$. In particular

$$e^K u(x) \geq \mu \quad \forall\, x \in B_{kr_0}(0).$$

Proof. From $(\mathbf{H1})$, we know that κ is continuous and $\kappa(0) > 0$ so we can find $0 < r < \frac{r_0}{2}$ such that $\kappa(x) \geq \frac{1}{2}\kappa(0)$ for every x in $\bar{B}_{2r}(0)$. Now let $u \in L^\infty(\mathbb{R}^n)$ be a nonnegative function satisfying $\int_{B_r(0)} u\, dx \geq \delta_0$. We claim that

$$\inf_{x\in \bar{B}_{(m+1)r}(0)\setminus B_{mr}(0)} (K^{m+1}u)(x) \geq \frac{[\delta_0\kappa(0)]^{m+1}}{2^{m+1}} \Pi_{i=1}^{m} \Big| B_r(ir\vec{e}_1) \cap B_r((i-1)r\vec{e}_1) \Big| \quad \forall\, m \geq 1 \tag{4.3}$$

where $\vec{e}_1$ is the unit vector in $\mathbb{R}^n$. We first observe from the definition of r that

$$(Ku)(x) \geq \int_{B_r(0)} \kappa(y-x)u(y)dy \geq \frac{1}{2}\kappa(0) \int_{B_r(0)} u(y)dy \geq \frac{1}{2}\kappa(0)\delta_0 \quad \forall\, x \in \bar{B}_r(0).$$

Hence

$$\inf_{x\in \bar{B}_r(0)} (Ku)(x) \;\; \geq \;\; \frac{1}{2}\kappa(0)\delta_0. \tag{4.4}$$

Now, we proceed by induction to show that (4.3) holds. To this end, let us first show that the claim holds for $m = 1$. Observe that for every $r \leq |x| \leq 2r$ and $y \in B_r(\frac{rx}{|x|})$, $|y - x| \leq |y - \frac{rx}{|x|}| + |x - \frac{rx}{|x|}| = |y - \frac{rx}{|x|}| + |x| - r < 2r$. Hence, by (4.4) for every $x \in \bar{B}_{2r}(0) \setminus B_r(0)$, we have

$$\begin{aligned} K^2u(x) &\geq \int_{B_r(\frac{rx}{|x|})} \kappa(y-x)(Ku)(y)dy \\ &\geq \frac{1}{2}\kappa(0) \int_{B_r(\frac{rx}{|x|})} (Ku)(y)dy \\ &\geq \frac{\kappa(0)^2}{2^2}\delta_0 \Big|B_r(\frac{rx}{|x|}) \cap B_r(0)\Big|. \end{aligned}$$

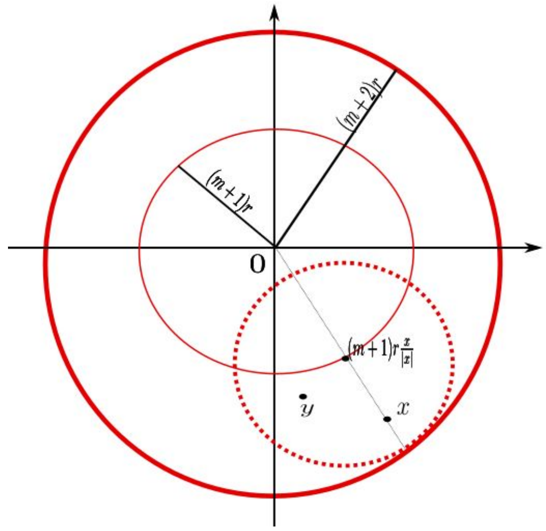
Since the Lebesgue measure is rotation invariant and $0 < \delta_0 < 1$, we conclude from the last inequality that

$$\inf_{x\in \bar{B}_{2r}(0)\setminus B_r(0)} K^2u(x) \geq \frac{\kappa(0)^2}{2^2}\delta_0\Big|B_r(r\vec{e}_1) \cap B_r(0)\Big| \geq \frac{[\delta_0\kappa(0)]^2}{2^2}\Big|B_r(r\vec{e}_1) \cap B_r(0)\Big| \qquad (4.5)$$

which proves (4.3) for $m = 1$. Next, suppose that (4.3) holds for some $m \geq 1$, we show that it also holds for $m + 1$. Indeed, as in the previous case, observe that, as shown in the schematic below, we have the following:

$$|y - x| \leq \Big|y - (m+1)r\frac{x}{|x|}\Big| + \Big|x - (m+1)r\frac{x}{|x|}\Big| < 2r$$

for $(m+1)r \leq |x| \leq (m+2)r$ and $y \in B_r(\frac{(m+1)rx}{|x|})$.



Observe also that

$$B_r(\frac{(m+1)rx}{|x|}) \cap \Big(\overline{B}_{(m+1)r}(0) \setminus B_{rm}(0)\Big) = B_r(\frac{(m+1)rx}{|x|}) \cap \overline{B}_{(m+1)r}(0) \quad \forall x \neq 0.$$

For notational convenience, let $B_{mr}(0) := B_0^m$, $B_{(m+1)r}(0) := B_0^{m+1}$ and $B_r(\frac{(m+1)rx}{|x|}) := B_1^{m+1}$. Using the induction hypothesis and recalling the choice of r, we obtain for every $x \in \bar{B}_{(m+2)r}(0) \setminus B_{(m+1)r}(0)$ that

$$\begin{aligned}
\mathcal{K}^{m+2}u(x) \geq & \int_{B_1^{m+1}} \kappa(y-x)\mathcal{K}^{m+1}u(y)dy \\
\geq & \frac{\kappa(0)}{2}\delta_0 \int_{B_1^{m+1}} \mathcal{K}^{m+1}u(y)dy \\
\geq & \frac{\kappa(0)\delta_0}{2}\Big|(B_1^{m+1}) \cap \big(\bar{B}_0^{m+1} \setminus B_0^m\big)\Big| \inf_{x \in B_1^{m+1} \cap \big(\bar{B}_0^{m+1} \setminus B_0^m\big)} \mathcal{K}^{m+1}u(x) \\
= & \frac{\kappa(0)}{2}\delta_0 \Big|B_1^{m+1} \cap \bar{B}_0^{m+1}\Big| \inf_{x \in B_1^{m+1} \cap \Big(\bar{B}_0^{m+1} \setminus B_0^m\Big)} \mathcal{K}^{m+1}u(x) \\
\geq & \frac{[\delta_0\kappa(0)]^{m+2}}{2^{m+2}}\Big|B_1^{m+1} \cap \bar{B}_0^{m+1}\Big|\Pi_{i=1}^m \Big|B_r(ire_1) \cap B_r((i-1)re_1)\Big|.
\end{aligned}$$

Again, since the Lebesgue measure is rotation invariant, then $\Big|B_1^{m+1} \cap \overline{B}_{(m+1)r}(0)\Big| = \Big|B_r((m+1)re_1) \cap \overline{B}_{(m+1)r}(0)\Big|$, which together with the last inequality show that the claim also holds for $m+1$.

Thus, we deduce that the claim holds for every $m \geq 1$. Now, by choosing $m \gg 1$ such that $B_{kr_0}(0) \subset B_{mr}(0)$, we can derive from (4.3) that (4.2) holds with $i = m$. □

For given $u, v \in X^{++}$, the part metric between u and v, denoted by $\rho(u,v)$, is defined by

$$\rho(u,v) = \inf\{\ln\alpha \,|\, \frac{1}{\alpha}u \leq v \leq \alpha u, \ \alpha \geq 1\}.$$

Proposition 4.2. *(1) For any $u_1, u_2 \in X^{++}$ and $t > s$, $\rho(u(t,\cdot;s,u_1), u(t,\cdot;s,u_2)) \leq \rho(u_1,u_2)$.*

(2) For any $\delta > 0,\ \sigma > 0,\ M > 0$ and $\tau > 0$ with $\delta < M$ and $\sigma \leq \ln\frac{M}{\delta}$, there is $\tilde{\sigma} > 0$ such that for any $u_0,\ v_0 \in X^{++}$ with $\delta \leq u_0(x) \leq M$. $\delta \leq v_0(x) \leq M$ for $x \in \mathbb{R}^n$ and

$\rho(u_0, v_0) \geq \sigma$, *there holds*

$$\rho(u(s+\tau, \cdot; s, u_0), u(s+\tau, \cdot; s, v_0)) \leq \rho(u_0, v_0) - \tilde{\sigma} \quad \forall\, s \in \mathbb{R}.$$

Proof. (1) See [48, proposition 5.1]

(2) See [33, proposition 3.4]

□

4.3 Uniqueness, stability, and frequency module of almost periodic solutions

In this section, we study the uniqueness, almost periodicity, and stability of a strictly positive bounded entire solution of (1.2) and prove Theorem 4.1.

We first prove a lemma.

Lemma 4.1. *Suppose that* $g(t, x)$ *is a uniformly continuous, bounded function in* $t \in \mathbb{R}$ *and* $x \in \bar{D}$*, with* $g(t, x) > 0$ *for all* $(t, x) \in \mathbb{R} \times D$*, and* $f(t, x, u)$ *satisfies* **(H3)**. *Then for any fixed* $x \in \bar{D}$*, the ODE*

$$u_t = g(t, x) + u f(t, x, u) \tag{4.6}$$

has at most one strictly positive bounded entire solution $u^*(t)$.

Proof. It can be proved by properly modifying the arguments in [44, Theorem 2.1]. For completeness, we provide a proof in the following.

Fix $x \in \bar{D}$. Suppose that (4.6) has two strictly positive bounded entire solutions $u_1^*(t)$ and $u_2^*(t)$, $u_1^*(t) \neq u_2^*(t)$. Without loss of generality, we may assume that $u_1^*(0) < u_2^*(0)$. Then by comparison principle for ODEs, we have

$$u_1^*(t) < u_2^*(t) \quad \forall\, t \in \mathbb{R}.$$

By **(H3)**, there is $\alpha > 0$ such that

$$\begin{aligned}\frac{d}{dt}\ln\big(\frac{u_1^*(t)}{u_2^*(t)}\big) &= \frac{u_{1t}^*}{u_1^*} - \frac{u_{2t}^*}{u_2^*} \\ &= \frac{g(t,x)}{u_1^*(t)} - \frac{g(t,x)}{u_2^*(t)} + f(t,x,u_1^*(t)) - f(t,x,u_2^*(t)) \\ &> f(t,x,u_1^*(t)) - f(t,x,u_2^*(t)) \\ &\geq \alpha(u_2^*(t) - u_1^*(t)) \quad \forall\, t \in \mathbb{R}. \end{aligned} \tag{4.7}$$

This implies that $\ln\big(\frac{u_1^*(t)}{u_2^*(t)}\big)$ increases in $\mathbb{R}$ and then there is some $0 < c < 1$ such that

$$\frac{u_1^*(t)}{u_2^*(t)} \leq \frac{u_1^*(0)}{u_2^*(0)} \leq c < 1 \quad \forall\, t \leq 0.$$

Hence

$$u_2^*(t) - u_1^*(t) = u_2^*(t)\Big(1 - \frac{u_1^*(t)}{u_2^*(t)}\Big) \geq (1-c)u_2^*(t) \quad \forall\, t \leq 0.$$

This together with (4.7) implies that there is $\beta > 0$ such that

$$\frac{d}{dt}\ln\big(\frac{u_1^*(t)}{u_2^*(t)}\big) \geq \beta \quad \forall t \leq 0.$$

Integrating the above inequality from t to 0 for $t \leq 0$, we have

$$\ln\big(\frac{u_1^*(t)}{u_2^*(t)}\big) \leq \ln\big(\frac{u_1^*(0)}{u_2^*(0)}\big) + \beta t \quad \forall\, t \leq 0$$

and then

$$\frac{u_1^*(t)}{u_2^*(t)} \leq \frac{u_1^*(0)}{u_2^*(0)} e^{\beta t} \quad \forall\, t \leq 0.$$

Letting $t \to -\infty$, we obtain

$$\lim_{t \to -\infty} \frac{u_1^*(t)}{u_2^*(t)} = 0,$$

which contradicts $u_1^*(t)$ and $u_2^*(t)$ being two strictly positive bounded entire solutions of (4.6). Hence (4.16) has at most one strictly positive bounded entire solution. □

Lemma 4.2. *Suppose that $u^*(t,x)$ is a strictly positive and bounded measurable function on $\mathbb{R}\times\bar{D}$, is differentiable in t for each $x\in\bar{D}$, and satisfies* (1.2) *for* $t\in\mathbb{R}$ *and* $x\in\bar{D}$*, that is,*

$$\frac{\partial u^*(t,x)}{\partial t}=\int_D \kappa(y-x)u^*(t,y)dy+u^*(t,x)f(t,x,u^*(t,x)),\quad t\in\mathbb{R},\ x\in\bar{D}. \tag{4.8}$$

Then $u^*(t,x)$ is uniformly continuous in $t\in\mathbb{R}$ and $x\in\bar{D}$, and $\mathbb{R}\ni t\mapsto u^*(t,\cdot)\in X$ is differentiable and hence $u^*(t,x)$ is a strictly positive bounded solution of (1.2).

Proof. We first show that $u^*(t,x)$ is uniformly continuous in t uniformly with respect to $x\in\bar{D}$ and is uniformly continuous in x uniformly with respect to $t\in\mathbb{R}$, i.e., for any $\epsilon>0$, there is $\delta>0$ such that for any $t_1,t_2\in\mathbb{R}$ and $x_1,x_2\in\bar{D}$ with $|t_1-t_2|<\delta$ and $|x_1-x_2|<\delta$, there hold

$$|u^*(t_1,x)-u^*(t_2,x)|<\epsilon\quad\forall\, x\in\bar{D}$$

and

$$|u^*(t,x_1)-u^*(t,x_2)|<\epsilon\quad\forall\, t\in\mathbb{R}.$$

Observe that $u^*_t(t,x)$ is a bounded function of $t\in\mathbb{R}$ and $x\in\bar{D}$. This implies that $u^*(t,x)$ is uniformly continuous in t uniformly with respect to $x\in\bar{D}$ and that

$$g(t,x):=\int_D\kappa(y-x)u^*(t,y)dy \tag{4.9}$$

is uniformly continuous in $t\in\mathbb{R}$ and $x\in\bar{D}$.

Assume that $u^*(t,x)$ is not uniformly continuous in $x\in\bar{D}$ uniformly with respect to $t\in\mathbb{R}$. Then there is $\epsilon_0>0$, $t_n\in\mathbb{R}$, and $x_n,\bar{x}_n\in\bar{D}$ such that

$$|x_n-\bar{x}_n|\le\frac{1}{n}\quad\forall\, n\ge 1,$$

and

$$|u^*(t_n,x_n)-u^*(t_n,\bar{x}_n)|\ge\epsilon_0\quad\forall\, n\ge 1. \tag{4.10}$$

Let $u_n(t) = u^*(t + t_n, x_n)$ and $\bar{u}_n(t) = u^*(t + t_n, \bar{x}_n)$, then

$$\frac{du_n(t)}{dt} = g(t + t_n, x_n) + u_n(t)f(t + t_n, x_n, u_n) \tag{4.11}$$

and

$$\frac{d\bar{u}_n(t)}{dt} = g(t + t_n, \bar{x}_n) + \bar{u}_n(t)f(t + t_n, \bar{x}_n, \bar{u}_n). \tag{4.12}$$

Note that $u_n(t)$ and $\bar{u}_n(t)$ are uniformly continuous in $t \in \mathbb{R}$. Since $u^*(t,x)$ is strictly positive and bounded, there are $\delta_1 > 0,\ M \gg 1$ such that

$$\delta_1 \le u^*(t,x) \le M \quad \forall\, t \in \mathbb{R}, x \in D.$$

This yields that $u_n(t)$ and $\bar{u}_n(t)$ are uniformly bounded. Furthermore, By **(H3)** and the uniform continuity of $g(t,x)$, we see that their derivatives are bounded, hence $u_n(t)$ and $\bar{u}_n(t)$ are equicontinuous. Therefore, using the usual diagonal argument and Arzela-Ascoli's theorem, without loss of generality, we may assume that there are $u_1^*(t)$, $u_2^*(t)$, $g^*(t,x)$ and $f^*(t,x,u)$ such that

$$\lim_{n\to\infty} u_n(t) = u_1^*(t), \quad \lim_{n\to\infty} \bar{u}_n(t) = u_2^*(t), \tag{4.13}$$

$$\lim_{n\to\infty} g(t + t_n, x + x_n) = g^*(t,x), \quad \lim_{n\to\infty} g(t + t_n, x + \bar{x}_n) = g^*(t,x), \tag{4.14}$$

and

$$\lim_{n\to\infty} f(t + t_n, x + x_n, u) = f^*(t,x,u), \quad \lim_{n\to\infty} f(t + t_n, x + \bar{x}_n, u) = f^*(t,x,u) \tag{4.15}$$

locally uniformly in $t \in \mathbb{R}$, $x \in \bar{D}$, and $u \in \mathbb{R}$. By (4.11)-(4.15), $\frac{du_n(t)}{dt}$ and $\frac{d\bar{u}_n(t)}{dt}$ also converge locally uniformly in $t \in \mathbb{R}$ as $n \to \infty$. It then follows that $u_1^*(t)$ and $u_2^*(t)$ are differentiable in t and are two strictly positive bounded entire solutions of

$$u_t = g^*(t,0) + uf^*(t,0,u).$$

By Lemma 4.1, $u_1^*(t) \equiv u_2^*(t)$, in particular, $u_1^*(0) = u_2^*(0)$, which contradicts (4.10). Hence $u^*(t,x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$.

Next, we prove that $\mathbb{R} \ni t \mapsto u^*(t,\cdot) \in X$ is differentiable. By the uniform continuity of $u^*(t,x)$ in $t \in \mathbb{R}$ and $x \in \bar{D}$, $\mathbb{R} \ni t \mapsto u^*(t,\cdot) \in X$ is continuous. By (4.8), for each $x \in \bar{D}$, $u^*(\cdot,x) \in L^1_{\rm loc}(\mathbb{R})$. Hence $u^*(t,x)$ is both super-solution and sub-solution of (1.2) on any interval (a,b). Then, by Proposition 2.4, for any given $t_0 \in \mathbb{R}$,

$$u^*(t,\cdot) = u(t,\cdot;t_0,u^*(t_0,\cdot)) \quad \forall\, t \geq t_0.$$

This implies that $\mathbb{R} \ni t \mapsto u^*(t,\cdot) \in X$ is differentiable, and $u^*(t,x)$ is a strictly positive bounded entire solution of (1.2). □

Next, we prove Theorem 4.1.

Proof of Theorem 4.1. (a) Suppose that there are two strictly positive bounded entire solutions u_1^* and u_2^* of (1.2). If $u_1^* \neq u_2^*$, then we can find $t_0 \in \mathbb{R}$ such that $u_1^*(t_0,\cdot) \neq u_2^*(t_0,\cdot)$. This implies that there is $\sigma > 0$ such that $\rho(u_1^*(t_0,\cdot), u_2^*(t_0,\cdot) \geq \sigma$. By Proposition 4.2(1),

$$\rho(u_1^*(t,\cdot), u_2^*(t,\cdot)) \geq \sigma \quad \forall\, t \leq t_0.$$

Then by Proposition 4.2(2), there is $\tilde{\sigma} > 0$ such that

$$\rho(u_1^*(t_0,\cdot), u_2^*(t_0,\cdot)) \leq \rho(u_1^*(t_0-k,\cdot), u_2^*(t_0-k,\cdot)) - k\tilde{\sigma} \quad \forall\, k = 1,2,\cdots. \tag{4.16}$$

Note that $\rho(u_1^*(t_0-k,\cdot), u_2^*(t_0-k,\cdot))$ is bounded for $k \in \mathbb{N}$. This together with (4.16) implies that

$$\rho(u_1^*(t_0,\cdot), u_2^*(t_0,\cdot)) \leq \rho(u_1^*(t_0-k,\cdot), u_2^*(t_0-k,\cdot)) - k\tilde{\sigma} \to -\infty$$

as $k \to \infty$, which is a contradiction. Therefore, a strictly positive bounded entire solution of (1.2) is unique.

(b) Suppose that $u^*(t,x)$ is a strictly positive bounded entire solution of (1.2). We show that $u^*(t,x)$ is almost periodic in t uniformly with respect to $x \in \bar{D}$. By Lemma 4.2, $u^*(t,x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$. It then suffices to prove that for each $x \in \bar{D}$, $u^*(t,x)$ is almost periodic in t. To this end, let $\{t_n\}$ and $\{s_n\}$ be any two sequences of $\mathbb{R}$. By **(H3)** and the uniform continuity of $u^*(t,x)$, without loss of generality, we may assume that there are $\bar{f}(t,x,u)$, $\tilde{f}(t,x,u)$, $\hat{f}(t,x,u)$ satisfying **(H3)**, and $\bar{u}^*(t,x)$, $\tilde{u}^*(t,x)$, $\hat{u}^*(t,x)$ such that

$$\lim_{n\to\infty} f(t+t_n,x,u) = \bar{f}(t,x,u), \ \lim_{m\to\infty} \bar{f}(t+s_m,x,u) = \tilde{f}(t,x,u),$$

$$\lim_{n\to\infty} f(t+t_n+s_n,x,u) = \hat{f}(t,x,u)$$

locally uniformly in $(t,x,u) \in \mathbb{R}\times\bar{D}\times\mathbb{R}$, and

$$\lim_{n\to\infty} u^*(t+t_n,x) = \bar{u}^*(t,x), \ \lim_{m\to\infty} \bar{u}^*(t+s_m,x) = \tilde{u}^*(t,x), \ \lim_{n\to\infty} u^*(t+t_n+s_n,x) = \hat{u}^*(t,x)$$

locally uniformly in $(t,x) \in \mathbb{R}\times\bar{D}$. Moreover, using (1.2), $\partial_t u^*(t+t_n,x)$ also converges locally uniformly in $(t,x) \in \mathbb{R}\times\bar{D}$ as $n\to\infty$, and then $\bar{u}^*(t,x)$ is differentiable in t and satisfies (1.2) with f replaced by $\bar{f}$ for each $t\in\mathbb{R}$ and $x\in\bar{D}$. By Lemma 4.2, $\bar{u}^*(t,x)$ is a strictly positive bounded entire solution of (1.2) with f replaced by $\bar{f}$. Similarly, $\tilde{u}^*(t,x)$ (resp. $\hat{u}^*(t,x)$) is a strictly positive bounded entire solution of (1.2) with f replaced by $\tilde{f}$ (resp. $\hat{f}$). By Proposition 2.2, $\tilde{f}(t,x,u) = \hat{f}(t,x,u)$. Then by (a), $\tilde{u}^*(t,x) = \hat{u}^*(t,x)$. By Proposition 2.2 again, $u^*(t,x)$ is almost periodic in t.

By the arguments similar to the proof of almost periodicity of $u^*(t,x)$ in t, we have that $u^*(t,x)$ is almost periodic in x when $D=\mathbb{R}^N$.

(c) Suppose that $u^*(t,x)$ is a strictly positive bounded entire solution of (1.2). We prove that $u^*(t,x)$ is asymptotically stable with respect to strictly positive perturbation. First note that there are $\delta_1>0,\ M\gg 1$ such that

$$\delta_1 \le u^*(t,x) \le M \quad \forall\, t\in\mathbb{R}, x\in D. \tag{4.17}$$

For given $u_0\in X^{++}$ and $t_0\in\mathbb{R}$, let $u(t,x;t_0,u_0)$ be the solution to (1.2) with $u(t_0,x;t_0,u_0) = u_0(x)$. Observe that, for some $0<b\ll 1,\ bu^*(t,x)$ is a subsolution of (1.2), and $u\equiv M$ is a

supersolution of (1.2) when $M \gg 1$. Therefore, we can find $0 < b \ll 1$ and $M \gg 1$ such that

$$bu^*(t_0, x) \leq u_0(x) \leq M \quad \forall\, x \in \bar{D}.$$

By Proposition 2.4,

$$bu^*(t, x) \leq u(t, x; t_0, u_0) \leq M \quad \forall\, t \geq t_0,\ x \in \bar{D}. \tag{4.18}$$

Let $\rho(t; t_0) = \rho(u(t + t_0, \cdot; u_0), u^*(t + t_0, \cdot))$ for every $t \geq 0$. We claim that

$$\limsup_{t\to\infty} \sup_{t_0\in\mathbb{R}} \rho(t; t_0) = 0. \tag{4.19}$$

Suppose on the contrary that (4.19) is false. Then we can find sequences $\{t_{0,n}\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ with $t_n \geq 1 + n$ for each $n \geq 1$ such that

$$\sigma_0 := \inf_{n\geq 1} \rho(t_n; t_{0,n}) > 0.$$

By proposition 4.2(1), we know that $\rho(t; t_{0,n}) \geq \rho(t_n; t_{0,n}) \geq \sigma_0$ for every $n \geq 1$ and $0 \leq t \leq t_n$. Thus, by (4.17), (4.18) and proposition 4.2(2), there is $\tilde{\delta} > 0$ such that

$$\rho(t + 1; t_{0,n}) \leq \rho(t; t_{0,n}) - \tilde{\delta} \quad \forall\, n \geq 1,\ 0 \leq t < t_n.$$

In particular, since $n < t_n$ for each $n \geq 1$,

$$\rho(n + 1; t_{0,n}) \leq \rho(n; t_{0,n}) - \tilde{\delta} \leq \cdots \leq \rho(0; t_{0,n}) - (n + 1)\tilde{\delta} \quad \forall\, n \geq 1.$$

Hence we have

$$0 < \sigma_0 \leq \rho(t_n; t_{0,n}) \leq \rho(n + 1; t_{0,n}) \leq \rho(0; t_{0,n}) - (n + 1)\tilde{\delta} \quad \forall\, n \geq 1. \tag{4.20}$$

This yields a contradiction since $\rho(0; t_{0,n}) = \rho(u^*(t_{0,n}, \cdot), u_0) \le \ln(\frac{M}{\delta})$ for all $n \ge 1$. Hence we conclude that (4.19) must hold. Now, (4.19) implies that

$$\lim_{t\to\infty} \sup_{t_0\in\mathbb{R}} \|u^*(t + t_0, \cdot) - u(t + t_0, \cdot; t_0, u_0)\|_\infty = 0.$$

This establishes the asymptotic stability of $u^*(t, x)$ with respect to strictly positive perturbations.

(d) Suppose that $u^*(t, x)$ is a strictly positive bounded entire solution of (1.2). We prove that $\mathcal{M}(u^*) \subset \mathcal{M}(f)$. For any given sequence $\{t_n\}$ in $\mathbb{R}$, suppose that $f(t + t_n, x, u) \to f(t, x, u)$ uniformly on bounded sets. By (a) and (b), there is a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that, $u^*(t + t_{n_k}, x) \to u^*(t, x)$ uniformly on bounded sets, as $k \to \infty$. Similarly, for any given sequence $\{x_n\}$ in $\mathbb{R}^N$, if $f(t, x + x_n, u) \to f(t, x, u)$ uniformly on bounded sets, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $u^*(t, x + x_{n_k}) \to u^*(t, x)$ as $k \to \infty$ locally uniformly. It then follows from Proposition 2.3 that $\mathcal{M}(u^*) \subset \mathcal{M}(f)$. □

4.4 Existence and nonexistence of a strictly positive entire solution

In this section, we study the existence of a strictly positive entire solution of (1.2) and prove Theorem 4.2.

Proof of Theorem 4.2. (a) First, suppose that (1.2) has a strictly positive bounded entire solution $u^*(t, x)$. By **(H3)**, $f_{\inf}(u) := \inf_{t\in\mathbb{R}, x\in\bar{D}} f_u(t, x, u)$ is continuous in $u \ge 0$ and $f_{\inf}(u) < 0$ for $u \ge 0$. Let $u^*_{\inf} = \inf_{t\in\mathbb{R}, x\in\bar{D}} u^*(t, x)$ and $u^*_{\sup} = \sup_{t\in\mathbb{R}, x\in\bar{D}} u^*(t, x)$. Then for any $0 < \lambda \le -u^*_{\inf} \cdot \sup_{u\in[0, u^*_{\sup}]} f_{\inf}(u)$, we have

$$\begin{aligned} f(t, x, u^*(t, x)) - f(t, x, 0) &= \int_0^1 \frac{d}{ds} f(t, x, su^*(t, x))ds \\ &= u^*(t, x) \int_0^1 f_u(t, x, su^*(t, x))ds \\ &\le -\lambda \quad \forall\, t \in \mathbb{R},\, x \in \bar{D}. \end{aligned} \tag{4.21}$$

This implies that

$$\begin{aligned}
u_t^* &= \int_D \kappa(y-x)u^*(t,y)dy + u^* f(t,x,u^*(t,x)) \\
&= \int_D \kappa(y-x)u^*(t,y)dy + u^*(f(t,x,0) + f(t,x,u^*(t,x)) - f(t,x,0)) \\
&\le \int_D \kappa(y-x)u^*(t,y)dy + u^*(f(t,x,0) - \lambda) \quad \forall\, t \in \mathbb{R},\, x \in \bar{D}.
\end{aligned}$$

It then follows that $\lambda_{PE}(a) \ge \lambda > 0$, where $a(t,x) = f(t,x,0)$.

Next, suppose that $\lambda_{PE}(a) > 0$. Let $M \gg 1$. Then $u(t,x) \equiv M$ is a supersolution of (1.2). By Proposition 2.4, $u(t,\cdot;-K,M) \le M$. This implies that $u(t,x;-K,M)$ decreases as K increases. Hence we can define

$$(0 \le)u^*(t,x) := \lim_{K\to\infty} u(t,x;-K,M)(\le M) \quad \forall\; t \in \mathbb{R},\;\; x \in \bar{D}. \tag{4.22}$$

It is clear that $u^*(t,x)$ is measurable in $(t,x) \in \mathbb{R} \times \bar{D}$. Moreover, note that

$$u_t(t,x;-K,M) = \int_D \kappa(y-x)u(t,y;-K,M)dy + u(t,x;-K,M)f(t,x,u(t,x;-K,M))$$

for all $t > -K$ and $x \in \bar{D}$. This together with the dominated convergence theorem implies that, for each fixed $x \in \bar{D}$,

$$u_t^*(t,x) = \int_D \kappa(y-x)u^*(t,y)dy + u^* f(t,x,u^*(t,x)) \quad \forall\, t \in \mathbb{R}, \tag{4.23}$$

and then $u_t^*(\cdot,x) \in W^{1,1}_{\rm loc}(\mathbb{R})$.

In the following, we prove that $u^*(t,x)$ is strictly positive. We do so in two steps.

Step 1. In this step, we prove that *there is* $r_x > 0$ *such that*

$$\inf_{t\in\mathbb{R}, y\in B_{r_x}(x)\cap D} u^*(t,y) > 0. \tag{4.24}$$

Let $\lambda \in \Lambda_{PE}(a)$ be such that $0 < \lambda < \lambda_{PE}$, $\lambda_{PE} - \lambda \ll 1$. Let $\phi \in \mathcal{X}^+$ satisfy $\inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0$, $\|\phi\|_{\mathcal{X}} = 1$, $\frac{\partial\phi}{\partial t} \in W^{1,1}_{\rm loc}(\mathbb{R})$ for each $x \in \bar{D}$, and

$$\lambda\phi(t,x) \leq L\phi(t,x) \quad \text{for} \quad a.e.\, t \in \mathbb{R},\ \text{all } x \in D.$$

By **(H3)**, $(f(t,x,0) - f(t,x,b\phi) - \lambda)\,\phi(t,x) \leq 0$ for $0 < b \ll 1$. Thus $u(t,x) = b\phi(t,x)$ solves

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &\leq \int_D \kappa(y-x)u(t,y)dy + a(t,x)u(t,x) - \lambda u(t,x)\\
&= \int_D \kappa(y-x)u(t,y)dy + u(t,x)f(t,x,u)\\
&\quad + (f(t,x,0) - f(t,x,u) - \lambda)\,u(t,x)\\
&\leq \int_D \kappa(y-x)u(t,y)dy + u(t,x)f(t,x,u) \quad \text{for } a.e.\, t \in \mathbb{R},\ \text{all } x \in \bar{D}.
\end{aligned}$$

Hence, $b\phi$ is a subsolution of (1.2). Therefore, by Proposition 2.4,

$$u(t,x;-K,M) \geq u(t,x;-K,b\phi(-K,x)) \geq b\phi(t,x) \quad \forall\, t \geq -K,\ x \in \bar{D}. \tag{4.25}$$

Since $\inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0$, we can find $x_0 \in D$ such that

$$\delta_1 := \inf_{t\in\mathbb{R}} b\phi(t,x_0) > 0.$$

Moreover, by the continuity of $\inf_{t\in\mathbb{R}} \phi(t,x)$ in x, we have

$$\inf_{t\in\mathbb{R}} b\phi(t,x) \geq \delta_1/2 \; for\; x \in D_0 := B_{r_0}(x_0) \cap D \ \text{ for some } \ r_0 > 0. \tag{4.26}$$

Observe that there is $m > 0$ such that $\|f(t,x,u(t,x;-K,M))\| \leq m$ for all $t \geq -K$ and $x \in D$. Thus $u(t,x;-K,M)$ solves

$$\partial_t u \geq \int_D \kappa(y-x)u(t,y)dy - mu(t,x) \quad \forall\, t > -K,\ x \in \bar{D}.$$

This together with (4.25) implies that

$$u(t+1,x;-K,M) \geq e^{-m}\big(e^{\mathcal{K}} b\phi(t,\cdot)\big)(x) \quad \forall\, t \geq -K,\ x \in D, \tag{4.27}$$

where $\mathcal{K}(u)(x) = \int_D \kappa(y-x)u(y)dy$ for $u \in X$. Hence

$$u^*(t,x) \geq e^{-m}\big(e^{\mathcal{K}} b\phi(t,\cdot)\big)(x) \quad \forall\, t \in \mathbb{R},\ x \in D. \tag{4.28}$$

By the arguments of Proposition 4.1 and (4.26), for each $x \in \bar{D}$, there are $r_x > 0$ and $\mu_x > 0$ such that

$$\big(e^{\mathcal{K}} b\phi(t,\cdot)\big)(y) \geq \mu_x \quad \forall\, t \in \mathbb{R},\ y \in B_{r_x}(x) \cap D.$$

This together with (4.28) implies (4.24).

Step 2. In this step, we prove that

$$\inf_{t\in\mathbb{R}, x\in\bar{D}} u^*(t,x) > 0. \tag{4.29}$$

In the case that D is bounded, (4.29) follows from (4.24).

In the case that $D = \mathbb{R}^N$, by the almost periodicity of $a(t,x)$ in x, for any given $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that any ball of radius r_ε contains some $\tilde{x} \in T_\varepsilon$, where

$$T_\varepsilon := \{\tilde{x} : |a(t,x) - a(t,x+\tilde{x})| < \varepsilon\ \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N\}.$$

For given $\varepsilon > 0$, we can find a sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}} \in T_\varepsilon$ such that

$$\mathbb{R}^N = \bigcup_{n\in\mathbb{N}} B_{2r_\varepsilon}(\tilde{x}_n), \tag{4.30}$$

where $B_{2r_\varepsilon}(\tilde{x}_n) := \{x \in \mathbb{R}^N : \|x - \tilde{x}_n\| < 2r_\varepsilon\}$. Let $\varepsilon = \frac{\lambda}{2}$. Then

$$\begin{aligned}\frac{\partial(\phi(t,x))}{\partial t} &\leq \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t,x+\tilde{x}_n)\phi(t,x) + (a(t,x) - a(t,x+\tilde{x}_n) - \lambda)\phi(t,x)\\ &\leq \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t,x+\tilde{x}_n)\phi(t,x) + (\varepsilon - \lambda)\phi(t,x)\\ &= \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t,x+\tilde{x}_n)\phi(t,x) - \frac{\lambda}{2}\phi(t,x).\end{aligned}$$

Hence, for some $0 < \tilde{b} < 1$, $\tilde{b}\phi$ is a subsolution of

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy + u(t,x)f(t,x+\tilde{x}_n,u(t,x)), \quad x \in \mathbb{R}^N. \tag{4.31}$$

By Proposition 2.4, we have

$$\tilde{b}\phi(t,x) \leq u(t,x+\tilde{x}_n;-K,M) \quad \text{for } t \geq -K,\ x \in \mathbb{R}^N,\ \tilde{x}_n \in T_\varepsilon.$$

By arguments similar to (4.27), we have

$$u(t+1,x+\tilde{x}_n;-K,M) \geq e^{-m}e^{\mathcal{K}}\tilde{b}\phi(t,\cdot), \quad \forall t \geq -K,\ x \in \mathbb{R}^N. \tag{4.32}$$

Without loss of generality, we may assume $x_0 = 0$ in (4.26). Then by Proposition 4.1, (4.26), and (4.27), there is $\tilde{\delta}_2 > 0$ such that

$$u(t+1,x+\tilde{x}_n;-K,M) \geq \tilde{\delta}_2 \quad \forall\, t \geq -K,\ x \in B_{2r_\epsilon}(0).$$

This together with (4.30) implies that

$$u(t+1,x;-K,M) \geq \tilde{\delta}_2 \quad \forall\, t \geq -K,\ x \in \mathbb{R}^N, \tag{4.33}$$

which implies (4.29).

By (4.22), (4.23), (4.29), and Lemma 4.2, $u^*(t,x)$ is a strictly positive bounded entire solution of (1.2).

(b) Assume that $\lambda_{PL} < 0$. For any $u_0 \ge 0$,

$$u(t, x; 0, u_0) \le \Phi(t, 0)u_0 \quad \forall\, t \ge 0,\ x \in D.$$

Note that

$$\limsup_{t\to\infty} \frac{\ln \|\Phi(t, 0)u_0\|}{t} \le \lambda_{PL} < 0.$$

Hence

$$0 \le \limsup_{t\to\infty} \|u(t, \cdot; 0, u_0)\| \le \limsup_{t\to\infty} \|\Phi(t, 0)u_0\| = 0.$$

The theorem thus follows. $\square$

Remark 4.6. *As mentioned in Remark 3.6, the definitions of $\lambda_{PL}(a), \lambda'_{PL}(a), \lambda_{PE}(a)$, and $\lambda'_{PE}(a)$ apply to general $a(t, x)$ which is bounded and uniformly continuous. When $f(t, x, u)$ is not assumed to be almost periodic in t, if $\lambda_{PE}(a) > 0$, then $u^*(t, x)$ defined in* (4.22) *is bounded on $\mathbb{R} \times \bar{D}$, differentiable in t with $\inf_{t\in\mathbb{R}} u^*(t, x) > 0$ for each $x \in \bar{D}$, and satisfies* (1.2) *for each $t \in \mathbb{R}$ and $x \in \bar{D}$. Hence $\partial_t u^*(t, x)$ is bounded on $\mathbb{R} \times \bar{D}$. We can also prove that $u^*(t, x)$ is continuous in $x \in \bar{D}$. In fact, let $g^*(t, x) = \int_D \kappa(y - x)u^*(t, y)dy$. It is clear that $g^*(t, x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$. for any $x_0 \in \bar{D}$ and $\{x_n\} \subset \bar{D}$ with $x_n \to x_0$, without loss of generality, we may assume that $u^*(t, x_n) \to \tilde{u}^*(t)$, $g(t, x_n) \to g(t, x_0)$, and $f(t, x_n, u^*(t, x_n)) \to f(t, x_0, \tilde{u}^*(t))$ as $n \to \infty$ locally uniformly in $t \in \mathbb{R}$. By* (4.23)*, we have*

$$\tilde{u}^*_t = g(t, x_0) + \tilde{u}^*(t)f(t, x_0, \tilde{u}^*(t)) \quad \forall\ t \in \mathbb{R}$$

and

$$u^*_t(t, x_0) = g(t, x_0) + u^*(t, x_0)f(t, x_0, u^*(t, x_0)) \quad \forall\ t \in \mathbb{R}.$$

By (4.24) *and Lemma 4.1, $\tilde{u}^*(t) = u^*(t, x_0)$. It then follows that $u^*(t, x)$ is also continuous in $x \in \bar{D}$. But $u^*(t, x)$ may not be strictly positive. However, if D is bounded, then $u^*(t, x)$ is a strictly positive entire solution of* (1.2) *and is asymptotically stable with respect to positive perturbations.*

4.5 Monotonicity of $\lambda_{PE}(a, D)$ in D

In this section, we investigate the monotonicity of $\lambda_{PE}(a, D)$ in D and prove Theorem 4.3.

Proof of Theorem 4.3. Let $D_1 \subset D_2$ be given. Without loss of generality, we may assume that $\lambda_{PE}(a, D_2) = 0$. For otherwise, we can replace $a(t, x)$ by $a(t, x) - \lambda_{PE}(a, D_2)$. It then suffices to prove that $\lambda_{PE}(a, D_1) \leq 0$. We prove it by contradiction.

First, assume that $\lambda_{PE}(a, D_1) > 0$. Let $\delta > 0$ be such that $\lambda_{PE}(a - \delta, D_1) > 0$. By Theorem 4.2, there is a strictly positive bounded entire solution $u_1^*(t, x)$ of

$$u_t = \int_{D_1} \kappa(y - x)u(t, y)dy + u(t, x)(a(t, x) - \delta - u(t, x)), \quad x \in \bar{D}_1. \tag{4.34}$$

For given $M > 0$, let $u_2(t, x; -K, M)$ be the solution of

$$u_t = \int_{D_2} \kappa(y - x)u(t, y)dy + u(t, x)(a(t, x) - \delta/2 - u(t, x)), \quad x \in \bar{D}_2 \tag{4.35}$$

with $u_2(-K, x; -K, M) = M$. By Propositions 2.4 and 2.5,

$$u_1^*(t, x) \leq u_2(t, x; -K, M) \quad \forall t \geq -K, \; x \in \bar{D}_1, \quad M \gg 1, \tag{4.36}$$

and

$$u_2(t, x; -K, M) \leq M \quad \forall t \geq -K, \; x \in \bar{D}_2, \quad M \gg 1. \tag{4.37}$$

Fix $M \gg 1$. By the arguments of Theorem 4.2,

$$u_2^*(t, x) := \lim_{K \to \infty} u_2(t, x; -K, M)(\leq M), \quad t \in \mathbb{R}, \; x \in \bar{D}_2 \tag{4.38}$$

is well defined, and satisfies (4.35) for all $t \in \mathbb{R}$ and $x \in \bar{D}_2$.

Next, we claim that $u_2^*(t, x)$ is strictly positive. We divide the proof of the claim into two cases.

Case 1. D_2 is bounded. Note that there is $m > 0$ such that

$$a(t,x) - \delta/2 - u_2(t,x;-K,M) \ge -m \quad \forall\, t \ge -K,\ x \in \bar{D}_2.$$

This together with (4.35) and Proposition 2.4 implies that

$$u_2(t,\cdot;-K,M) \ge e^{-m} e^{\mathcal{K}_2} u_2(t-1,\cdot;-K,M) \quad \forall\, t \ge -K+1,$$

where $\mathcal{K}_2 u = \int_{D_2} \kappa(y-x)u(y)dy$ for $u \in C^b_{\rm unif}(\bar{D}_2)$. By (4.36), there is $\delta_0 > 0$ such that

$$\int_{D_2} u_2(t-1,x;-K,M)dx \ge \delta_0 \quad \forall\ t \ge -K+1, \quad x \in \bar{D}_2.$$

This together with the arguments of Proposition 4.1 implies that there is $\tilde{\delta}_0 > 0$ such that

$$u_2(t,x;-K,M) \ge \tilde{\delta}_0 \quad \forall\, t \ge -K+1, \quad x \in \bar{D}_2.$$

It then follows that

$$u_2^*(t,x) \ge \tilde{\delta}_0 \quad \forall\, t \in \mathbb{R},\ x \in \bar{D}_2.$$

Hence the claim holds in the case that D_2 is bounded.

Case 2. $D_2 = \mathbb{R}^N$. By the almost periodicity of $a(t,x)$ in x, there are $\{x_n\} \subset \mathbb{R}^N$ and $r > 0$ such that

$$\mathbb{R}^N = \cup_{n=1}^{\infty} B_r(x_n),$$

and

$$|a(t,x+x_n) - a(t,x)| \le \delta/2 \quad \forall\, t \in \mathbb{R}, \quad x \in \mathbb{R}^N.$$

Then

$$
\begin{aligned}
\partial_t u_1^*(t,x) &= \int_{D_1} \kappa(y-x)u_1^*(t,y)dy + u_1^*(t,x)(a(t,x) - \delta - u_1^*(t,x) \\
&\le \int_{D_1} \kappa(y-x)u_1^*(t,y)dy + u_1^*(t,x)(a(t,x+x_n) \\
&- \delta/2 - u_1^*(t,x)) \quad \forall\, t \in \mathbb{R},\ x \in \bar{D}_1.
\end{aligned}
$$

This together with Propositions 2.4 and 2.5 implies that

$$
u_2(t, x + x_n; -K, M) \ge u_1^*(t,x) \quad \forall\, t \ge -K,\ x \in \bar{D}_1
$$

and then

$$
u_2^*(t,x) \ge u_1^*(t, x - x_n) \quad \forall\, t \in \mathbb{R},\ x - x_n \in \bar{D}_1.
$$

By the arguments in Case 1, there is $\tilde{\delta}_0 > 0$ such that

$$
u_2^*(t,x) \ge \tilde{\delta}_0 \quad \forall\, t \in \mathbb{R},\ x \in B_r(x_n),\ n \ge 1.
$$

Therefore, $u^*(t,x)$ is strictly positive and the claim also holds in the case $D_2 = \mathbb{R}^N$.

Now, by Lemma 4.2, $u_2^*(t,x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}_2$. Hence $u_2^*(t,x)$ can be used as a test function in the definition of $\Lambda_{PE}(a, D_2)$.

$$
-\frac{\partial u_2^*}{\partial t} + \int_{D_2} \kappa(y-x)u_2^*(t,y)dy + a(t,x)u_2^*(t,x) \ge \frac{\delta}{2} u_2^*(t,x), \quad x \in \bar{D}_2. \tag{4.39}
$$

This implies that $\lambda_{PE}(a, D_2) \ge \frac{\delta}{2} > 0$, which is a contradiction. Hence $\lambda_{PE}(a, D_1) \le 0$. The theorem is thus proved. $\square$

Chapter 5

Spreading speeds of positive solutions to nonlocal almost periodic Fisher-KPP equations

This chapter is devoted to the study of the spatial spreading speeds of positive solutions of (1.2) with front-like initials. Consider (1.2) with $D = \mathbb{R}^N$ and $f(t,x,u) = -1 + a(t,x) - b(t,x)u$ given by the following:

$$u_t = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u + u(a(t,x) - b(t,x)u), \quad x \in \mathbb{R}^N, \tag{5.1}$$

where $a(t,x)$ and $b(t,x)$ are almost periodic in t, and periodic in x.

The spatial spreading dynamics of (5.1) is of fundamental research interest. This is concerned with how fast an initial population occupying only a part of the environment invades the empty part of the environment over time. This concept has been extensively studied for the random dispersal case. Fisher [19], and Kolmogorov, Petrowsky, Piscunov [32] in their pioneering work on this version of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \quad x \in \mathbb{R} \tag{5.2}$$

established the existence of a minimal wave speed $c = 2$ which is referred to as the spreading speed of (5.2). Subsequently, several studies have investigated and established the existence of the spreading speed for the time homogenous and time periodic random dispersal cases. For the nonlocal dispersal case, [56] and [57] established the existence of spreading speed for the time homogenous and time periodic cases, respectively. However, there is not much studies on the spatial spreading dynamics of (1.2) when the reaction function is not time periodic. For the random dispersal case, [26] introduced the concept of the spreading speed interval and showed that, when the dispersal term is given by the Laplacian, the spreading speed interval is bounded. They also recovered the results in the time homogenous and periodic cases. Considering the nonlocal dispersal case, [37] investigated the case when the reaction function is time independent but space almost periodic. They showed the boundedness of the spreading speed interval in this case and established some estimates for the upper and lower bounds of the spreading speed interval. Moreover, [4] considered the space homogenous, but time almost periodic case and established the existence of the spreading speed in this case. However, to the best of our knowledge, there is no studies on the spreading speed of (1.2) for the case when the reaction function is both space heterogenous and time almost periodic. In the following, we consider this case and establish the boundedness of the spreading speed interval; providing an estimate for its upper bound in general and lower bound in specific situations.

5.1 Main results and preliminaries

This section presents the theorems on the spatial spreading speeds of positive solutions to (5.1), with some necessary notations and preliminary results needed for establishing the main results. First, we have the following notations:

Throughout this section we denote

$$X := \{u \in C(\mathbb{R}^N, \mathbb{R}) | u \ \ is \ \ uniformly \ \ continuous \ \ on \ \ \mathbb{R}^N, \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\}$$

with norm $\|u\|_X = \sup_{x\in\mathbb{R}^N} |u(x)|$ and

$$\mathcal{X}_p := \{u \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}) | u(\cdot + T, \cdot) = u(\cdot, \cdot + p_i e_i) = u(\cdot, \cdot), \ i = 1, \dots, N\},$$

where $e_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{iN}),\ \delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j,\ i, j = 1, \dots, N$. We denote $u(t, x; u_0)$ as the solution of (5.1) with $u(0, x; u_0) = u_0(x)$.

Consider also the linearization of (5.1) at $u \equiv 0$,

$$u_t = \int_{\mathbb{R}^N} k(y - x)u(t, y)dy - u + a(t, x)u, \quad x \in \mathbb{R}^N. \tag{5.3}$$

Suppose that $u(t,x)$ is a solution of (5.3). Then, for a given unit vector $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$, $v(t,x) = e^{\mu x\cdot\xi}u(t,x)$ satisfies

$$v_t = \int_{\mathbb{R}^N} k(y-x)e^{-\mu(y-x)\cdot\xi}v(t,y)dy - v + a(t,x)v, \quad x \in \mathbb{R}^N. \tag{5.4}$$

We denote the top Lyapunov exponent corresponding to (5.4) by $\lambda_{PL}(\mu,\xi;a)$. Let

$$X_c^+(\xi) = \{u \in X^+ \,|\, \liminf_{x\cdot\xi\to-\infty} u(x) > 0, \quad u(x) = 0 \quad \text{for} \quad x\cdot\xi \gg 1\},$$

$$C_{\sup}(\xi;a) = \{c \in \mathbb{R} \,|\, \lim_{t\to\infty} \sup_{x\cdot\xi\geq ct} u(t,x;u_0) = 0 \quad \forall\, u_0 \in X_c^+(\xi)\},$$

and

$$C_{\inf}(\xi;a) = \{c \in \mathbb{R} \,|\, \liminf_{t\to\infty} \inf_{x\cdot\xi\leq ct} u(t,x;u_0) > 0 \quad \forall\, u_0 \in X_c^+(\xi)\},$$

$$c^*_{\sup}(\xi;a) = \inf\{c \,|\, c \in C_{\sup}(\xi;a)\}, \quad c^*_{\inf}(\xi;a) = \sup\{c \,|\, c \in C_{\inf}(\xi;a)\}.$$

The interval $[c^*_{\inf}(\xi;a), c^*_{\sup}(\xi;a)]$ is called the *spreading speed interval* of (5.1) in the direction of ξ.

5.1.1 Main results

We have the following theorem on the boundedness of $[c^*_{\inf}(\xi;a), c^*_{\sup}(\xi;a)]$.

Theorem 5.1. *Assume that* $u \equiv 0$ *is an unstable solution of* (5.1) *in the sense that for any* u_0 *with* $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$, $\liminf_{t\to\infty} \inf_{x\in\mathbb{R}} u(t,x;u_0) > 0$.

(i)

$$-\infty < c^*_{\inf}(\xi;a) \leq c^*_{\sup}(\xi;a) \leq \inf_{\mu>0} \frac{\lambda_{PL}(\mu,\xi;a)}{\mu}. \tag{5.5}$$

(ii) If $a(t,x) \geq a_T(t,x) + a_0(t)$*, where* $a_T(t,x) \in \mathcal{X}_p$ *and* $a_0(t)$ *is almost periodic in t, then*

$$c^*_{\inf}(\xi;a) \geq \inf_{\mu>0} \frac{\lambda_{PL}(\mu,\xi;a_T+\hat{a}_0)}{\mu}. \tag{5.6}$$

5.1.2 Preliminaries

Before proving the theorem, we first present some lemmas that would be used in proving the theorem.

Lemma 5.1. *Assume that* $a(t,x) \in \mathcal{X}_p$ *and* $\lambda_{PL}(\mu,\xi;a) > 0$.

$$c^*_{\inf}(\xi;a) = c^*_{\sup}(\xi;a) = \inf_{\mu>0} \frac{\lambda_{PL}(\mu,\xi;a)}{\mu}.$$

Proof. It follows from [49, Theorem 4.1]. □

Lemma 5.2. *For any* $x \in D$ *and* $\epsilon > 0$*, there is* $A_{x,\epsilon} \in W^{1,\infty}(\mathbb{R})$ *such that*

$$a(t,x) + A^{'}_{x,\epsilon}(t) \geq \hat{a}(x) - \epsilon \quad \textit{for} \;\; a.e.\; t \in \mathbb{R}.$$

Proof. It follows from [42, Lemma 3.2]. □

5.2 Boundedness of the spreading speed interval

In this section, we establish the boundedness of the spreading speed interval, $[c^*_{\inf}, c^*_{\sup}]$, and prove Theorem 5.1.

Proof of Theorem 5.1. (i) For any given $\mu > 0$ and given $u_0 \in X^+_c(\xi)$, we can find $u_\mu \in X^+$ such that

$$u_0(x) \leq e^{-\mu x\cdot\xi} u_\mu(x) \quad \forall\, x \in \mathbb{R}^N.$$

Then by the comparison principle, we have

$$u(t,x;u_0) \leq e^{-\mu x\cdot\xi} v(t,x;u_\mu),$$

where $u(t,x;u_0)$ is the solution of (5.3) with $u(0,x;u_0) = u_0(x)$ and $v(t,x;u_\mu)$ is the solution of (5.4) with $v(0,x;u_\mu) = u_\mu(x)$. Note that

$$\limsup_{t\to\infty} \frac{\ln \|v(t,\cdot;u_\mu)\|}{t} \leq \lambda_{PL}(\mu,\xi;a).$$

Hence for any $\epsilon > 0$,

$$u(t,x;u_0) \le e^{-\mu x\cdot\xi} e^{(\lambda_{PL}(\mu,\xi;a)+\epsilon)t} = e^{-\mu(x\cdot\xi - \frac{\lambda_{PL}(\mu,\xi;a)+\epsilon}{\mu}t)} \quad \forall\, t \gg 1.$$

This implies that

$$\frac{\lambda_{PL}(\mu,\xi;a)+\epsilon}{\mu} \in C_{\sup}(\xi) \quad \forall\ \epsilon > 0.$$

It then follows that

$$c^*_{\sup}(\xi;a) \le \inf_{\mu>0} \frac{\lambda_{PL}(\mu,\xi;a)+\epsilon}{\mu} \quad \forall \epsilon > 0.$$

Letting $\epsilon \to 0$, we obtain

$$c^*_{\sup}(\xi;a) \le \inf_{\mu>0} \frac{\lambda_{PL}(\mu,\xi;a)}{\mu}.$$

To prove the case for $-\infty < c^*_{\inf}$, let us consider the space shifted equation of (5.1) given by

$$u_t = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u + u(a(t,x+z) - b(t,x+z)u), \quad x,\, z \in \mathbb{R}^N, \tag{5.7}$$

We give the proof in 2 steps.

Step 1. In this step, we show that for $\alpha_- < 0 < \alpha_+$, we can find $C_0 > 0$ such that for any $C \ge C_0$ the function

$$v^-(t,x;z) = u(t,x;\alpha_-,z)\eta(x\cdot\xi + Ct) + u(t,x;\alpha_+,z)(1-\eta(x\cdot\xi+Ct))$$

is a subsolution of (5.7) where $\eta(\cdot)$ is the function defined by $\eta(s) = \frac{1}{2}(1 + tanh(\frac{s}{2}))$ and $u(t,x;u_0,z)$ is the solution of (5.7) with $u(0,x;z) = u_0(x)$. For simplicity, we let $u(t,x)(a(t,x+z) - b(t,x+z)u(t,x)) := f(t,x+z,u)$ and for brevity in notation we may write $u(t,x;\alpha_-)$ and $u(t,x;\alpha_+)$ as u^{α_-} and u^{α_+} respectively. Without loss of generality we let $z = 0$ for the rest of this step 1.

By direct computation we have;

$$\begin{aligned}
& v_t^-(t,x) - \int_{\mathbb{R}^N} \kappa(y-x) v^-(t,y)dy + v^-(t,x) - f(t,x,v^-(t,x)) \\
= \; & C\eta^{'}(x\cdot\xi + Ct)(u(t,x;\alpha_-) - u(t,x;\alpha_+)) \\
- \; & \int_{\mathbb{R}^N} \kappa(y-x)(u^{\alpha_-} - u^{\alpha_+})(\eta(y\cdot\xi + Ct) - \eta(x\cdot\xi + Ct))dy \\
+ \; & f(t,x,u^{\alpha_-})\eta(x\cdot\xi + Ct) + f(t,x,u^{\alpha_+})(1-\eta(x\cdot\xi + Ct)) - f(t,x,v^-(t,x)).
\end{aligned}$$

Now, by applying the mean value theorem and noting that $\eta^{'}(\cdot) = \eta(\cdot)(1-\eta(\cdot))$ we have;

$$\begin{aligned}
& f(t,x,u^{\alpha_-})\eta(x\cdot\xi + Ct) + f(t,x,u^{\alpha_+})(1-\eta(x\cdot\xi + Ct)) - f(t,x,v^-(t,x)) \\
= \; & -\Big(f(t,x,u^{\alpha_+})(\eta(x\cdot\xi + Ct) - 1) - \big[(f(t,x,(u^{\alpha_-} - u^{\alpha_+}) + u^{\alpha_+})\eta(x\cdot\xi + Ct) \\
& -f(t,x,(u^{\alpha_-} - u^{\alpha_+})\eta(x\cdot\xi + Ct) + u^{\alpha_+})\big]\Big) \\
= \; & -\Big(f_u(t,x,\tilde{u}^* + u^{\alpha_+}) - f_u(t,x,\tilde{u}^*\eta(x\cdot\xi + Ct) + u^{\alpha_+}\Big)(u^{\alpha_+} - u^{\alpha_-})\eta(x\cdot\xi + Ct) \\
= \; & f_{uu}(t,x,u^{**})(u^{\alpha_+} - u^*)(u^{\alpha_-} - u^{\alpha_+})\eta^{'}(x\cdot\xi + Ct)
\end{aligned}$$

where $u^* = \tilde{u}^* + u^{\alpha_+}$ and u^*, and u^{**} are between u^{α_-} and u^{α_+}.

Therefore, we have

$$\begin{aligned}
& v_t^-(t,x) - \int_{\mathbb{R}^N} \kappa(y-x) v^-(t,y)dy + v^-(t,x) - f(t,x,v^-(t,x)) \\
= \; & C\eta^{'}(x\cdot\xi + Ct)\Bigg[(u(t,x;\alpha_-) - u(t,x;\alpha_+)) \\
& -\int_{\mathbb{R}^N} \kappa(y-x)(u(t,y;\alpha_-) - u(t,y;\alpha_+)(\frac{\eta(y\cdot\xi + Ct) - \eta(x\cdot\xi + Ct)}{\eta^{'}(x\cdot\xi + Ct)})dy \\
& f_{uu}(t,x,u^{**})(u^{\alpha_+} - u^*)(u^{\alpha_-} - u^{\alpha_+})\Bigg]
\end{aligned}$$

From the definition of $\eta(\cdot)$, we can find $M_1 > 0$ such that $|\frac{\eta(y\cdot\xi + Ct) - \eta(x\cdot\xi + Ct)}{\eta^{'}(x\cdot\xi + Ct)}| < M_1$ and from comparison principle we have $u(t,x;\alpha_-) - u(t,x;\alpha_+) \leq -M$ and for some $M > 0$ therefore, we can find $C_0 > 0$ such that

$$\begin{aligned}
& v_t^-(t,x) - \int_{\mathbb{R}^N} \kappa(y-x) v^-(t,y)dy + v^-(t,x) - f(t,x,v^-(t,x)) \\
\leq\; & C_0 \eta'(x\cdot\xi + C_0 t) \Bigg[(u(t,x;\alpha_-) - u(t,x;\alpha_+)) \\
& - \int_{\mathbb{R}^N} \kappa(y-x)(u(t,y;\alpha_-) - u(t,y;\alpha_+)) (\frac{\eta(y\cdot\xi + C_0 t) - \eta(x\cdot\xi + C_0 t)}{\eta'(x\cdot\xi + C_0 t)}) dy \\
& f_{uu}(t,x,u^{**})(u^{\alpha_+} - u^*)(u^{\alpha_-} - u^{\alpha_+}) \Bigg] \leq 0
\end{aligned}$$

Showing that $v_t^-(t,x)$ is a subsolution of (5.1).

Step 2. We show that for any $u_0 \in X_c^+(\xi)$. $\liminf_{t\to\infty} \inf_{x\cdot\xi \leq Ct} u(t,x;u_0) > 0$ for any $C \leq -C_0$

Now, since $\liminf_{x\cdot\xi\to-\infty} u_0(x) > 0$, we can find constants $\alpha_- < 0 < \alpha_+$ and $z \in \mathbb{R}^N$ such that $\alpha_+ < \liminf_{x\cdot\xi\to-\infty} u_0(x)$, for all x with $x\cdot\xi \leq z\cdot\xi$. We can take any $\alpha_- < 0$ and take $\alpha_+ = \frac{1}{2}\liminf_{x\cdot\xi\to-\infty} u_0(x)$. This implies that

$$u_0(x) \geq v^-(0,x;z) = \alpha_- \eta(x\cdot\xi) + \alpha_+(1 - \eta(x\cdot\xi))$$

therefore by step 1 and the comparison principle (Proposition 2.4) we have

$$\begin{aligned}
v^-(t,x;z) &= u(t,x;\alpha_-,z)\eta(x\cdot\xi + C_0 t) + u(t,x;\alpha_+,z)(1 - \eta(x\cdot\xi + C_0 t)) \\
&\leq u(t,x;u_0,z).
\end{aligned}$$

This implies that for $C < -C_0$ $\liminf_{t\to\infty} \inf_{x\cdot\xi\leq Ct} u(t,x;u_0,z) \geq u^*_{\inf} > 0$. Where $u^*_{\inf} = \inf_{t\geq 0,\, x\in\mathbb{R}} u^*(t,x)$ with $u^*(t,x)$ being the unique positive solution of (5.1) guaranteed by Theorem 4.2. This implies that $C_0 \leq c^*_{\inf}(\xi)$. Hence $-\infty < c^*_{\inf}(\xi;a)$

(ii) First, let $M > 0$ be such that $M > \sup_{t\in\mathbb{R},x\in\mathbb{R}^N} b(t,x)$. Consider

$$u_t = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u + u(a(t,x) - Mu), \quad x \in \mathbb{R}^N. \tag{5.8}$$

We denote $u_M(t,x;u_0)$ as the solution of (5.8) with $u_M(0,x;u_0) = u_0(x)$.

Second, by Lemma 5.2, there is $A_0 \in W^{1,\infty}(\mathbb{R})$ such that $a_0(t) + A_0'(t) \geq \hat{a}_0 - \epsilon \quad \forall t \in \mathbb{R}$.

Set $v(t,x;u_0) = e^{A_0(t)} u_M(t,x;u_0)$. Then $v(t,x;u_0)$ satisfies

$$
\begin{aligned}
v_t &\geq \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a(t,x) + A^{'}(t) - Mu) \\
&\geq \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a_T(t,x) + a_0(t) + A^{'}(t) - Mu) \\
&\geq \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a_T(t,x) + \hat{a}_0 - \epsilon - Mu) \\
&\geq \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a_T(t,x) + \hat{a}_0 - \epsilon - M_0 v)
\end{aligned}
$$

where $M_0 = M \sup_{t\in\mathbb{R}} e^{A_0(t)}$. By Lemma 5.1,

$$
c^*_{\inf}(\xi;a) \geq c^*_{\inf}(\xi; a_T + \hat{a}_0 - \epsilon).
$$

Let $\epsilon \to 0$, (ii) follows. □

Chapter 6

Concluding Remarks and Future projects

The results obtained in this dissertation opens up many significant, biologically relevant problems.

Problem 1: Establish possible criteria for the equality of the two generalized principal eigenvalues.

The spectral theory of the linear nonlocal dispersal equation has been investigated from the aspect of the top Lyapunov exponent and generalized principal eigenvalues. Consequently, it is a primary concern to establish the condition for the equality of the two generalized principal eigenvalues. This will yield a complete result on the asymptotic dynamics of the nonlinear nonlocal Fisher-KPP equations with almost periodic reaction terms.

Problem 2. Establish a precise lower bound for the spreading speed interval in the almost periodic case.

From our study of the asymptotic dynamics of the nonolcal dispersal model in the unbounded environment, we obtained a partial result on the existence of spreading speeds. Having established an upper bound for the spreading speed interval, it is also important to establish a precise lower bound. Hence, based on the study of the time periodic case, and time almost periodic but space homogenous case, we make the following conjecture:

$$c^*_{\inf}(\xi; a) \geq \inf_{\mu>0} \frac{\lambda_{PE}(\mu, \xi; a)}{\mu}.$$

We shall work at establishing the above inequality thereby completing the result on the spatial spreading speed in the nonlocal almost periodic case.

Problem 3: Establish a necessary and sufficient condition for persistence and extinction in the two species competition system with nonlocal dispersal and almost periodic dependence.

The study of the one species population model leads naturally to the consideration of two or more species in an environment. Thus, it is of biological significance to study the following system:

$$\begin{cases} u_t = d_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] \\ \qquad\qquad +u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v), \ x \in \bar{D}, \\ v_t = d_2[\int_D \kappa(y-x)v(t,y)dy - v(t,x)] \\ \qquad\qquad +v(a_2(t,x) - b_2(t,x)u - c_2(t,x)v), \ x \in \bar{D}, \end{cases} \tag{6.1}$$

where $u(t,x), v(t,x)$ represent the population density of two species at time t and location $x \in D \subset \mathbb{R}^n$ with D a bounded domain, having smooth boundary. $d_1, d_2 > 0$ stand for the dispersal rates. The functions a_i, b_i, c_i, are almost periodic in t, $\kappa(\cdot)$ is a nonnegative symmetric smooth function with compact support

Several authors have investigated the two species nonlocal competition system when the reaction term is either time homogenous or periodic. They established some sufficient conditions for the coexistence of the species and extinction of one of the species. It is of interest to investigate necessary and sufficient conditions for the persistence and extinction of the species when the environment is neither space homogenous nor time periodic.

Problem 4: Study the effects of spatial degeneracy on the coexistence and extinction in the nonlocal dispersal competition model.

The persistence and extinction dynamics of the two-species nonlocal competiton system are typically established based on the magnitude of the interaction coefficients and dispersal rates. The magnitude of the interaction coefficient determines the stronger or weaker species. It is well known that under some given conditions, the weaker species will die off eventually. However, the first natural question to ask is the following. What if the weaker species has a protection zone where it does not experience any competition from the stronger competitor; would such a protection zone save it from extinction? Hence, we shall investigate the following nonlocal competition system with a protection zone:

$$\begin{cases} u_t = \int_D \kappa(y-x)u(t,y)dy - u(t,x) + \lambda u(t,x) - u^2 - cb(x)uv, & x \in D, t > 0, \\ v_t = \int_D \kappa(y-x)v(t,y)dy - v(t,x) + \mu v(t,x) - v^2 - duv, & x \in D, t > 0. \end{cases} \tag{6.2}$$

where $D \subset \mathbb{R}^n$ is a bounded domain, with smooth boundary, We consider the case where $b(x)$ has a degeneracy (i.e., it vanishes on a nonempty proper subdomain D_0 of D and is positive on the rest of D). We are interested in the stationary solutions of (6). This amounts to finding solutions of the following equation:

$$\begin{cases} 0 = \int_D \kappa(y-x)u(y)dy - u(x) + \lambda u(x) - b(x)u^2 - cuv, & x \in D, , \\ 0 = \int_D \kappa(y-x)v(y)dy - v(x) + \mu v(x) - v^2 - duv, & x \in D. \end{cases} \tag{6.3}$$

Definition 6.1. *A solution* $(u^*, v^*) \in C(\bar{D}, \mathbb{R}) \times C(\bar{D}, \mathbb{R})$ *of (6.3) is called a coexistence state of (6) if* $u^*(x) > 0,\ v^*(x) > 0\ \ \forall\, x \in \bar{D}$. *A coexistence state is said to be globally stable if for any* $(u_0, v_0) \in C(\bar{D}, \mathbb{R}^+) \setminus \{0\} \times C(\bar{D}, \mathbb{R}^+) \setminus \{0\},\ (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \to (u^*, v^*)$ *as* $t \to \infty$.

Our aim is to study the existence and asymptotic dynamics of coexistence states of (6.2).

Problem 5: A hybrid nonlocal competition system.

As mentioned previously, the persistence and extinction of the species in the nonlocal competition system depends also on the dispersal rates. The authors in [24] established for the time homogenous case that the slower diffuser wins the competition. This generates a natural question; What if one of the species does not move, but the other disperses nonlocally, will the non-moving species win the competition? This leads to the investigation of the following model:

$$\begin{cases} u_t = \int_D \kappa(x-y)[u(t,y) - u(t,x)]dy + (a(x) - u - v)u & x \in \bar{D},\ t > 0, \\ v_t = (a(x) - u - v)v & x \in \bar{D},\ t > 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x) & x \in \bar{D}. \end{cases} \tag{6.4}$$

where D is abounded domain in $\mathbb{R}^n$ and $a(x)$ is a Hölder continuous function on $\bar{D}$.

For the random dispersal counterpart of (6.4) It was observed that

- if $a(x)$ changes sign, then u will die off in the area of the habitat where a is negative, called a sink area.

This validates the claim that the slower diffuser wins the game. Can we obtain similar dynamical scenario in the nonlocal case?

Problem 6: Multi-strain SIS epidemic model with nonlocal dispersal.

Nonlocal dispersal operators has been successfully employed in modeling the dynamics of epidemic disease models. In this regard, one can think of the Covid-19 virus which has been reported to be capable of dispersing six feet away.

Consider the multi-strain nonlocal dispersal SIS epidemic model. For $x \in \bar{D} \subset \mathbb{R}^n, t > 0$

$$\begin{cases} S_t = d_S \int_D \kappa(y-x)[S(t,y) - S(t,x)]dy + \sum\limits_{i=1}^{k} \gamma_i(x) I_i - \frac{\sum_{i=1}^{k} \beta_i(x) S I_i}{S + \sum_{i=1}^{k} I_j} \\ I_{i,t} = d_i \int_D \kappa(y-x)[I_i(t,y) - I_i(t,x)]dy - \gamma_i(x) I_i + \frac{\beta_i(x) S I_i}{S + \sum_{i=1}^{k} I_i} \\ S(0,x) = S_0(x),\ I_i(0,x) = I_{i,0}(x). \end{cases}$$

Among other things, we shall study the impact of environmental heterogeneity and movement of individuals on the persistence and extinction of a multi-strain disease.

Chapter 7

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