Color Trades on Graphs

by

John Carr

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 6, 2022

Keywords: Design Theory, Discrete Math, Edge-Colorings Graph Theory, Trades

Copyright 2022 by John Carr

Approved by

Dean Hoffman, Professor of Mathematics and Statistics
Pete Johnson, Professor of Mathematics and Statistics
Jessica McDonald, Associate Professor of Mathematics and Statistics
Chris Rodger, Professor Emeritus of Mathematics and Statistics
Abstract

Edge-colorings of graphs have a rich history and are widely studied. Trade spectra of graphs are relatively new and ripe for study. The color-trade-spectrum of a graph $G$ is defined to be the set of all $t$ for which there exist two proper edge-colorings of $G$ using $t$ colors such that each vertex of $G$ is incident to the same set of colors under each edge-coloring while each edge receives a different color under each edge-coloring. We show some general results and present various constructions which are used to determine the color-trade-spectrum of several families of graphs.
Acknowledgments

First, I want to thank Dr. Mark Greer and the rest of the mathematics faculty at the University of North Alabama. You all helped inspire my love for mathematics and gave countless hours of guidance. Your support and encouragement will never be forgotten, and I look forward to working with you all again soon.

Likewise, I would like to thank the amazing people in the Department of Mathematics and Statistics at Auburn University. I will cherish the time I spent here and the conversations I had with many of you. I am grateful to Stanley Spence, Heather Hamilton, and Paige Grice for helping me countless times with a variety of problems (often of my own creation). I would like to specifically thank Dr. Melinda Lanius, Dr. Chris Rodger, and Dr. Jessica McDonald for being wonderful mentors. I would also like to thank Dr. Jennifer Stone; although she is no longer with us, I will always remember her kind demeanor and words of wisdom.

I will forever be indebted to my advisor Dr. Dean Hoffman for all the time spent helping me in this endeavor. You were always patient with me, even when I was not patient with myself, and your confidence in me was inspiring. I hope I am able to make as positive an impact in the lives of my future students as you have made in mine. Hopefully I can take a picture of a herd of goats in party hats just for you.

Finally, I would like to thank my friends and family. It has been a long journey! In particular, I am enormously grateful to my wife Elly. Your love and support has made all of this possible, and I know I wouldn’t be at this stage in my life without you by my side. I look forward to what God has in store for us, and to celebrating our other future achievements together.
# Table of Contents

Abstract ................................................................. ii

Acknowledgments ................................................... iii

1 Introduction ......................................................... 1

2 Preliminaries ....................................................... 6

3 Theta, Wheel, and $n$-cube Graphs ............................. 17
   3.1 Theta Graphs .................................................. 17
   3.2 Wheel Graphs .................................................. 19
   3.3 $n$-cube Graphs ............................................... 20

4 Complete Bipartite Graphs ....................................... 25
   4.1 $0 \leq m \leq n \leq 3$ ......................................... 25
   4.2 $n \geq m \geq 2$ where $m$ is even .......................... 27
   4.3 $n \geq m \geq 3$ where $m$ is odd .......................... 30

5 Products of Paths ................................................ 38
   5.1 $P_2 \Box P_m$ Graphs ....................................... 38
   5.2 $P_n \Box P_m$ Graphs ....................................... 40

6 Complete Graphs ................................................ 57
   6.1 $n \equiv 0 \mod 8$ ............................................. 57
   6.2 $n \equiv 4 \mod 8$ ............................................. 60
6.3 \( n \equiv 1 \mod 8 \) ................................. 62

7 Further Work ............................................ 66

References .................................................. 67
Chapter 1

Introduction

We first give a brief history of edge-colorings and trades on graphs. Edge-colorings of graphs
can be traced back to the four-color problem, first posed in 1852 by Francis Guthrie, which asks
if every map can be colored with four colors so that adjacent countries are colored differently.
The first “proof” of this was produced by Alfred Kempe in 1879 [12]. While his arguments
were incorrect, he introduced the idea of Kempe chains, which remain a key ingredient in edge-
coloring theory. Later in the 19th century, Peter Tait attempted to improve Kempe’s arguments
and in the process introduced Tait colorings [18].

In the early 20th century, Dénes König showed that a graph is bipartite if and only if every
cycle has even length, and then showed that every $k$-regular bipartite graph can be partitioned
into 1-factors, which are sets of disjoint edges that meet all the vertices, by using Kempe chains.
In the middle of the 20th century, Claude Shannon showed that the wires of any network can
be properly colored, meaning no color appears more than once at a node in a network, with
$\left\lfloor \frac{3m}{2} \right\rfloor$ colors, where $m$ is the largest number of wires at a point [15]. An equivalent statement
is the minimum number of colors needed to properly edge-color a graph is between $\Delta$ and
$\left\lfloor \frac{3\Delta}{2} \right\rfloor$, where $\Delta$ is the maximum degree of the graph, which is the maximum number of edges
appearing at a vertex in the graph.

In 1964, Vadim Vizing proved that the minimum number of colors required to properly
edge-color a graph, the chromatic index denoted by $\chi'(G)$, is no more than $\Delta + u$ where $u$
is the maximum number of repeated (or parallel) edges between any two vertices in the graph
[19]. In particular, if $G$ is simple, meaning it has no parallel edges or loops (edges connecting
a vertex to itself), then $\chi'(G) \in \{\Delta, \Delta + 1\}$. This led to classifying simple graphs as Class 1
or Class 2, depending on whether their chromatic index is $\Delta$ or $\Delta + 1$ respectively. Classifying graphs as Class 1 or 2 is an NP-complete problem [11], but has a rich history. One possibility is to consider families of graphs whose cores have simple structure [6][7][8][10].

Work on trades in design theory originated in the 1960s [9] although the idea behind trades was used as early as 1916 [20] to construct Steiner triple systems. Trades reflect possible differences between two combinatorial objects of the same type (Steiner systems, latin squares, etc.). The original use of a trade as defined by Hedayat [9] was to avoid some undesirable blocks in an experimental design while retaining the same parameter set and variety set. Some other uses of trades include the following: solving intersection problems for two combinatorial structures, creating defining sets for block designs [16], creating critical sets for partial latin squares, creating designs with different support sizes which is used in statistical applications of designs [9], constructing $t$-designs, and constructing irreducible designs. Much of these applications are considered in the surveys by Billington [2] and Khosrovshahi [13].

The basic idea behind a trade is to partition an object into subsets satisfying some list of properties. If we can partition the object into a different set of subsets which still satisfy the same list of properties, we say the partitions form a trade. Formally, a $(k,t)$ trade of volume $m$ and foundation size $v$ (sometimes referred to as a $(v,k,t)$ trade), is a pair $\{T_1, T_2\}$ of sets of subsets of size $k$ based on a $v$-set, such that $T_1$ and $T_2$ each contain $m$ subsets of size $k$, with $T_1 \cap T_2 = \emptyset$, and so that each $t$-set chosen from the $v$-set occurs exactly the same number of times in the blocks of $T_1$ as it does in the blocks of $T_2$. A trade is called Steiner if each $t$-set occurs at most once in each $T_i$, and a trade is called simple if there are no repeated blocks in $T_1$ or in $T_2$.

While there are a variety of types of trades, four kinds appear most often in the literature; trades in designs, latin squares, graphs, and trades derived from “latin representations”. The trades described above are most often used in design theory, where trades are allowed to contain repeated subsets, often called blocks. A latin trade is a pair $\{L_1, L_2\}$ of partial latin rectangles with precisely the same $m$ filled cells such that: 1. $L_1$ and $L_2$ contain different elements in each filled cell $(i,j)$; 2. in each occupied row $i$, $L_1$ and $L_2$ contain set-wise the same symbols; 3. in each occupied column $j$, $L_1$ and $L_2$ contain set-wise the same symbols. In graph theory,
the trade-spectrum of a graph $G$ is the set of all integers $t$ for which there is a graph $H$ for which two $G$-decompositions of $H$ exist; this means the edges of $H$ can be partitioned in two unique ways into $t$ copies of $G$ with no copies of $G$ in common [3]. Some uses of trades from latin representations are addressed by Billington and Cavenagh [5][4]. Most previous work on trades has involved 2-way trades, but some work has been done on $\mu$-way trades [1]. In this dissertation, we consider a synthesis of edge-colorings and trades, called color trades.

We now give a brief summary of the contents of this dissertation. First, we cover the preliminaries in chapter 2, which includes formally defining color trades, mate colorings, and the color-trade-spectrum of a graph. In the preliminaries section we also list some general lemmas, such as determining lower and upper bounds of the color-trade-spectrum of a graph in Lemma 2.1. Additionally, we determine the color-trade-spectrum of all cycles in Lemma 2.2, and use this in Lemma 2.3 in which we use cycle decompositions to create a general method to find a subset of the color-trade-spectrum of a graph. We then define a new graph $G'$ in Lemma 2.4 based on two mate colorings of a given graph, and use $G'$ to find a subset of the color-trade-spectrum of the original graph. We conclude the preliminaries section with Lemma 2.5, which details a way to determine if two edge-colorings are mate colorings by considering a new graph $M(G, k)$. In particular, two edge-colorings of a graph $G$ are mate colorings if and only if the corresponding $M(G, k)$ is bipartite.

In chapter 3, we determine the color-trade-spectrum of three small families of graphs, namely Theta, Wheel, and $n$-cube graphs. Theta and Wheel graphs are simple modifications of cycles, so the methods for finding the color-trade-spectrum of cycles in Lemma 2.2 is used again, and explicit constructions are given for both families of graphs. The color-trade-spectrum of trivial cases of Theta graphs is determined in Lemma 3.1, and the color-trade-spectrum in full is determined in Theorem 3.2. The color-trade-spectrum for Wheel graphs is determined in Theorem 3.2. The color-trade-spectrum of trivial cases of $n$-cube graphs is determined in Lemma 3.2, and in Theorem 3.3 the full color-trade-spectrum for $n$-cube graphs is determined using a recursive construction based on the well known fact that the $(n + m)$-cube graph is isomorphic to the Cartesian product of the $n$-cube and $m$-cube graphs.
In chapter 4, we determine the color-trade-spectrum of complete bipartite graphs and list explicit constructions for attaining each value in the color-trade-spectrum. We make frequent use of Latin rectangles in the constructions, where we define and use row and column blocks of Latin rectangles. In Theorem 4.1, for a complete bipartite graph $K_{m,n}$ under a proper $\Delta$-edge-coloring with $m$ even, we list a specific construction for the corresponding Latin rectangle $L_\Delta$ which we then use to create a new Latin rectangle $\pi(L_\Delta)$ which corresponds to a mate coloring for the original edge-coloring of $K_{m,n}$. From this, we identify a 4-cycle decomposition of $K_{m,n}$ on which we use Lemma 2.3 to attain the entire color-trade-spectrum of $K_{m,n}$. We deal with $K_{3,n}$ as a special case in Lemma 4.3 and use this to create modified constructions of $L_\Delta$ and $\pi(L_\Delta)$ in Theorem 4.2. From this, we identify a cycle decomposition of $K_{m,n}$ for odd $m$ on which we again use Lemma 2.3 to attain the entire color-trade-spectrum of $K_{m,n}$.

In chapter 5, we determine a subset of the color-trade-spectrum for Cartesian products of paths $P_m \square P_n$ and fully determine the color-trade-spectrum for $P_2 \square P_m$. In Lemma 5.1, we list specific constructions for proper $\Delta$-edge-colorings of $P_2 \square P_m$ depending on the parity of $m$ along with the associated mate colorings. From this, we identify cycle decompositions of $P_2 \square P_m$ on which we again use Lemma 2.3 to achieve a subset of the color-trade-spectrum. In Theorem 5.1, we show the subset of the color-trade-spectrum from Lemma 5.1 is indeed the entire color-trade-spectrum of $P_2 \square P_m$. In Theorem 5.2, we list specific constructions for proper 4-edge-colorings and proper 5-edge-colorings of $P_n \square P_m$ depending on the parities of $n$ and $m$, along with their mate colorings. From the proper 5-edge-colorings, we identify cycle decompositions of $P_n \square P_m$ on which we use Lemma 2.3 to attain a subset of the color-trade-spectrum of $P_n \square P_m$. We conjecture that the subset from 5.2 is indeed the entire color-trade-spectrum.

In chapter 6, we fully determine the color-trade-spectrum for complete graphs $K_n$ where $n \equiv 0 \mod 8$ and $n \equiv 4 \mod 8$. In Theorem 6.1, we identify a $(K_4, C_4)$ decomposition of $K_n$ for $n \equiv 0 \mod 8$ by first finding a $K_4$ and $\frac{n}{4}K_4$ decomposition, where $\frac{n}{4}K_4$ is the complete $\frac{n}{4}$-partite graph where each part contains four vertices. From this, we consider a new graph $K_m$ which we use to find a $(K_4, C_4)$ decomposition of $\frac{n}{4}K_4$. Using this decomposition, we present a $\Delta$-edge-coloring of $K_n$ along with its mate coloring, and use Lemma 2.3 to determine the
color-trade-spectrum of $K_n$. In Theorem 6.2, we modify the process of Theorem 6.1 to find a $(K_4, C_4)$ decomposition of $K_n$ for $n \equiv 4 \mod 8$ where we consider a modified $K_m$, namely $K_{2m}$. Again, we present a $\Delta$-edge-coloring of $K_n$ along with its mate coloring, and use Lemma 2.3 to determine the color-trade-spectrum of $K_n$. In Theorem 6.3, we again modify the process of Theorem 6.1 to find a $C_4$ decomposition of $K_n$ for $n \equiv 1 \mod 8$. We present a $2n + 4$-edge-coloring of $K_n$ for $n \geq 17$ along with its mate coloring, and use Lemma 2.3 to partially determine the color-trade-spectrum of $K_n$. 
A graph is a pair $G = (V, E)$ where $V$ is a set of vertices and $E$ is a multiset of paired vertices, called edges. If $u \in V$ and $(u, u) \in E$, we say the edge joining $u$ to itself is a loop. For $u, v \in V$, we say the number of times $(u, v)$ appears in $E$ is the edge-multiplicity of $(u, v)$. We say a graph $G$ is simple if $G$ contains no loops and the maximum edge-multiplicity of $G$ is 1. We assume all graphs from this point onward are simple unless otherwise stated. If $(u, v) \in E$, we say this edge is incident to vertices $u$ and $v$. If $e_1, e_2 \in E$ are incident to a common vertex $v \in V$, we say $e_1$ and $e_2$ are adjacent edges.

A $k$-vertex-coloring of a graph $G$ is an assignment of “colors” to the vertices of $G$. More formally, $\varphi : V(G) \to C$ is a $k$-vertex-coloring of $G$ where $|C| = k$. A vertex-coloring is proper if no two adjacent vertices share a common color. Formally, for two vertices $u$ and $v$ in $V(G)$ where $u \neq v$, $\varphi(u) \neq \varphi(v)$. The minimum number of colors needed to properly vertex-color a graph $G$ is known as the chromatic number of $G$, denoted by $\chi(G)$. In studying edge-colorings, we often consider the degree of a vertex, which is the number of edges incident to a vertex in a loopless graph. The minimum degree of a graph $G$, denoted $\delta(G)$, is the degree of the vertex with the least number of edges incident to it in $G$. Likewise, the maximum degree of $G$, denoted $\Delta(G)$, is the degree of the vertex with the greatest number of edges incident to it in $G$.

Analogously, a $k$-edge-coloring of a simple graph $G$ is an assignment of “colors” to each edge of $G$. More formally, $\Phi : E(G) \to C$ is a $k$-edge-coloring of $G$ where $|C| = k$. An edge-coloring is proper if no two adjacent edges share a common color. Formally, for any vertex $v \in V(G)$ and any pair of edges incident to $v$, $u_kv$ and $u_jv$ where $k \neq j$, $\Phi(u_k, v) \neq \Phi(u_j, v)$. 
The minimum number of colors needed to properly edge-color a graph \( G \) is known as the \textit{chromatic index} of \( G \), denoted by \( \chi'(G) \).

Let \( G \) be a simple graph under a proper \( t \)-edge-coloring \( C_1 \). We say an edge-coloring \( C_2 \) of \( G \) is a \textit{mate coloring} of \( C_1 \) if and only if the following conditions are true:

1. For every \( v \in V(G) \), the set of colors assigned to edges incident to \( v \) under \( C_1 \) is the same as the set of colors assigned under \( C_2 \).
2. For every \( e \in E(G) \), the color assigned to \( e \) under \( C_1 \) is different than the color assigned under \( C_2 \).

Clearly, \( C_1 \) and \( C_2 \) must have the same cardinality, and in fact must consist of the same set of colors. We define the \textit{color-trade-spectrum} of a graph \( G \), \( CTS(G) \), to be the set of all \( t \) for which there exist two mate colorings of \( G \) using \( t \) colors.

Consider a graph \( G \) with \( n \) vertices, and where every pair of vertices is joined via an edge. Then \( G \) is the \textit{complete graph on \( n \) vertices}, denoted by \( K_n \), sometimes called the \textit{clique on \( n \) vertices}. An example of two mate colorings using three colors for \( K_4 \) is given below, showing that \( 3 \in CTS(K_4) \).

Next, we list some elementary observations about mate colorings and color trade spectra.

\textbf{Lemma 2.1.} Let \( G \) be a simple graph with chromatic index \( \chi'(G) \).

1. If \( G \) contains a vertex of degree one, then \( CTS(G) = \emptyset \).
2. In a mate coloring, each color must be assigned to at least two edges of \( G \).
3. \( \chi'(G) \leq \min CTS(G) \) and \( \max CTS(G) \leq \lfloor \frac{|E|}{2} \rfloor \).

\textit{Proof.} 1: If \( G \) contains a vertex of degree one, say \( u \), then \( CTS(G) = \emptyset \) as there is no way for two edge-colorings to both make \( u \) incident to the same color without assigning the same color to the singular edge incident to \( u \).
2: Let $C_1$ be a proper edge-coloring applied to $G$ such that only a single edge $(u, v)$ is colored $c_1$. For contradiction, suppose $C_2$ is a mate coloring of $C_1$. Then there must exist other vertices $u'$ and $v'$ such that $(u, u')$ and $(v, v')$ are colored $c_1$ under $C_2$. This means $u'$ and $v'$ must be incident to edges colored $c_1$ in $C_1$, contradicting our assumption that $u$ and $v$ are the only vertices incident to edges colored $c_1$ under $C_1$. Thus, any color in a mate coloring must be assigned to at least two edges in $G$.

3: Since mate colorings are proper edge-colorings, it immediately follows that $\chi'(G) \leq \min\{\operatorname{CTS}(G)\}$. By observation 2, each color in a mate coloring must be assigned to at least two edges in $G$, so $\max\{\operatorname{CTS}(G)\} \leq \left\lfloor \frac{|E|}{2} \right\rfloor$.

A walk is an alternating sequence of vertices and edges beginning and ending with a vertex in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it. A trail is a walk in which the edges are distinct and a path is a trail in which the vertices (and thus edges) are distinct. A cycle is a non-empty trail in which only the first and last vertices are the same. A cycle decomposition is a partitioning of a graph’s edges into cycles. We say a cycle is even or odd if it respectively contains an even or odd number of vertices. The following lemma characterizes the color trade spectra of cycles, which are often used in determining the color trade spectra of more complicated graphs.

**Lemma 2.2.** Let $G$ be a graph containing at least three vertices. If $G$ is an even cycle, then $\operatorname{CTS}(G) = \{2\}$. If $G$ is an odd cycle, then $\operatorname{CTS}(G) = \emptyset$.

**Proof.** Suppose $G$ is an even cycle containing at least four vertices, denoted by $v_0, v_1, \ldots, v_{n-1}$. Without loss of generality, suppose the edges of $G$ are of the form $(v_i, v_{i+1})$ (or $(v_{i+1}, v_i)$ since edges are unordered pairs of vertices) where $0 \leq i \leq n - 1$ and addition is done modulo $n$. Let $C_1$ be a proper edge-coloring of $G$ using two colors $c_1$ and $c_2$, and without loss of generality suppose each edge of the form $(v_i, v_{i+1})$ where $i$ is odd is colored $c_1$ while each edge of the form $(v_j, v_{j+1})$ where $j$ is even is colored $c_2$. In particular, note each vertex is incident to an edge colored $c_1$ and an edge colored $c_2$. 

8
Modify \( C_1 \) by swapping the assignment of \( c_1 \) and \( c_2 \) to create a new edge-coloring \( C_2 \). This new edge-coloring doesn’t assign the same color to any edge as \( C_1 \) by definition, and still maintains the property that each vertex is incident to an edge colored \( c_1 \) and \( c_2 \). Thus, \( C_2 \) is a mate coloring of \( C_1 \) so \( 2 \in \text{CTS}(G) \).

Now, suppose that \( C_1 \) is a proper edge-coloring of \( G \) using at least three colors, \( c_1, c_2, \ldots, c_k \). Then there must exist some sequence of three consecutive edges using three colors, say \((v_0, v_1), (v_1, v_2), \) and \((v_2, v_3)\). Without loss of generality, suppose that \( c_1 \) is assigned to \((v_0, v_1)\), while \( c_2 \) and \( c_3 \) are respectively assigned to \((v_1, v_2)\) and \((v_2, v_3)\). For any mate coloring of \( C_1 \), the edge \((v_1, v_2)\) would then need to be assigned both \( c_1 \) and \( c_3 \), but this is impossible since edges can only be assigned one color under proper edge-colorings. Thus, \( C_1 \) has no mate coloring, and we conclude that \( \text{CTS}(G) = \emptyset \). Therefore, \( \{2\} = \text{CTS}(G) \).

It is well known that for any odd cycle, a proper edge-coloring must contain at least three colors. By the previous paragraph, this shows that the color-trade-spectrum for odd cycles is empty.

\[ \square \]

We now proceed to list some general lemmas about color-trade-spectra. A graph is connected if there is a walk between every pair of vertices in the graph. We say \( H \) is a component of \( G \) if \( H \) is a connected subgraph which is not part of any larger connected subgraph of \( G \). We say a subgraph \( H \) is spanning if it contains every vertex of \( G \). Let \( C_1 \) and \( C_2 \) be two edge-colorings of \( G \) using colors \( k \) colors which are mate colorings. The following lemma gives a method to increase the number of colors used in a pair of mate colorings.

**Lemma 2.3.** Let \( C_1 \) and \( C_2 \) be mate \( k \)-edge-colorings of a graph \( G \). For each \( j \) in \( 1 \leq j \leq k \), let \( H_j \) be the subgraph of \( G \) spanned by the edges colored \( c_j \) in either \( C_1 \) or \( C_2 \). Then the components of \( H_j \) are even cycles. Let \( \alpha_j \) be the number of these cycles and \( \alpha = \sum_{j=1}^{k} \alpha_j \). Then \( \text{CTS}(G) \) contains all integers in the interval \([k, \alpha]\).

**Proof.** Suppose \( K \) is a component of \( H_j \). Since \( C_1 \) and \( C_2 \) are mate colorings, any vertex in \( K \) is incident to exactly one edge of color \( c_j \) from \( C_1 \) and exactly one (different) edge of color \( c_j \) from \( C_2 \). Thus, every vertex of \( K \) has degree two and is thus a cycle. If \( K \) is an odd cycle,
at least one vertex of \( K \) must be incident to two edges belonging to the same edge-coloring, a contradiction. Thus, \( K \) is an even cycle.

Suppose that \( H_j \) consists of disjoint even cycles. Consider two such cycles, say 
\((v_{x_1}, v_{x_2}, \ldots, v_{x_m})\) and \((v_{y_1}, v_{y_2}, \ldots, v_{y_n})\). By definition, these cycles are pairwise vertex disjoint, so we can freely assign a new color to each cycle. Since we have \( \alpha \) of these cycles, we can extend our original \( k \)-edge-coloring of \( G \) up to an \( \alpha \)-edge-coloring.

A bridge is an edge whose removal from a graph increases the graph’s number of connected components. Lemma 2.3 gives us the following corollary.

**Corollary 2.0.1.** If \( G \) is a graph which contains a bridge, then \( CTS(G) = \emptyset \).

**Proof.** By Lemma 2.3, if \( G \) had a pair of mate colorings, then each \( H_j \) would consist of even cycles. However, this can not occur if \( G \) has a bridge.

A cycle-double-cover of a graph \( G \) is a collection of cycles which together contain every edge of \( G \) exactly twice. Clearly, the \( H_j \) subgraphs from Lemma 2.3 form a cycle-double-cover, although not every cycle-double-cover correlates to the \( H_j \) subgraphs from a pair of mate colorings. Of note, this is related to the open cycle-double-cover conjecture by Seymour and Szekeres, which states that every bridgeless graph has a cycle-double-cover. [14][17]

A set of vertices is independent if no two vertices in the set are adjacent. A graph is bipartite if the vertices can be partitioned into two disjoint and independent sets. We denote the complete bipartite graph on sets of size \( m \) and \( n \) by \( K_{m,n} \). Below is a 6-edge-coloring of \( K_{4,6} \), along with its mate. Following these is an example of some of the \( H_j \) subgraphs from Lemma 2.3. In particular, note that every \( H_j \) consists of two 4-cycles. Thus, we could recolor one of the 4-cycles per color class to increase the total number of colors by one, and we could continually do this for each value of \( j \) until we have a pair of mate 12-edge-colorings of \( K_{4,6} \).
We say two colors $c_1$ and $c_2$ are incident if two edges colored $c_1$ and $c_2$ are incident.

The following lemma gives a method to decrease the number of colors used in a pair of mated colorings.

**Lemma 2.4.** Suppose that $C_1$ and $C_2$ are proper mated $k$-edge-colorings of $G$ on the same set of $k$ colors. Denote by $G'$ the graph where the $k$ vertices of $G'$ represent the colors used in both $C_1$ and $C_2$, and where vertices are adjacent if the respective colors are incident in $C_1$ and $C_2$. Then $\text{CTS}(G)$ contains all the integers in the interval $[\chi(G'), k]$.

**Proof.** Suppose $C_1$ and $C_2$ are mated $k$-edge-colorings of $G$ on the same set of colors and let $c_1$ and $c_2$ be colors which are not incident in $G$. Using notation from Lemma 2.3, we know $H_1$
and $H_2$ are edge-disjoint. Since the associated vertices of these colors in $G'$ are not adjacent, we may assign the same color to these vertices and likewise the same color to both sets of corresponding edges in $G$. Furthermore, we can freely continue this process until $G'$ is $\chi(G')$-colored.

Below is an 8-edge-coloring of $K_{4,4}$ along with its mate. Following these is an example of the $G'$ graph made from these edge-colorings, where a circular vertex represents a solid color while a square vertex represents a dashed color. Since this graph has a chromatic number of 4, we could recolor the edges of $G$ using only four colors, and still have a mate coloring.

Let $m$ be a positive integer and denote by $mG$ the graph on $V(G)$ in which two vertices are joined by $m$ edges if they are adjacent in $G$ and no edges otherwise. If we superimpose the edges from two proper edge-colorings $C_1$ and $C_2$ onto $G$, we get a copy of $2G$ where each color class is a union of even cycles by Lemma 2.3. A natural question is: when can we take some
copy of $2G$ which has been $k$-edge-colored, and decompose the graph into two separate copies of $G$, each uniquely associated with one of $C_1$ or $C_2$, where $C_1$ and $C_2$ are mate colorings. To answer this, we define a new graph, $M(G, k)$ as follows: given a $k$-edge-coloring (which is not proper) of $2G$, the vertices of $M(G, k)$ are the edges of $2G$, and two such vertices, $e_i$ and $e_j$, are adjacent in $M(G, k)$ if and only if either $e_i$ and $e_j$ are the two copies of an edge of $G$, or $e_i$ and $e_j$ are adjacent in $2G$ and belong to the same color class.

**Lemma 2.5.** Suppose $2G$ has a $k$-edge-coloring where each color class is a union of pairwise vertex disjoint cycles of length greater than two. The edge-coloring arises from a pair of mate colorings if and only if $M(G, k)$ is bipartite.

**Proof.** For the forward implication, let $C_1$ and $C_2$ be the two mate colorings giving rise to our $k$-edge-coloring of $2G$. Let $H_j$ denote the subgraph of $2G$ spanned by the edges colored $c_j$ in either $C_1$ or $C_2$. By assumption, each color class is a union of pairwise vertex disjoint cycles of length greater than two, and furthermore a pair of adjacent edges in $H_j$, $e_i$ and $e_j$, must belong to different copies of $G$ as otherwise one of $C_1$ or $C_2$ wouldn’t be proper. Suppose $\{e_i', e_i''\}$ is a multi-edge in $2G$ and without loss of generality suppose $e_i'$ is assigned its color from $C_1$ and likewise $e_i''$ is assigned its color from $C_2$. We consider two cases.

1. **Case 1:** $e_i$ and $e_j$ are vertices in $M(G, k)$ that belong to the same multi-edge in $2G$.

   By the above paragraph, $e_i$ and $e_j$ must each be associated with a different edge-coloring, and since the edge-colorings $C_1$ and $C_2$ are mates, this means the edges must receive different colors.

   2. **Case 2:** $e_i$ and $e_j$ are vertices in $M(G, k)$ that are adjacent and have the same color in $2G$.

   Since the edges share the same color, they can not belong to the same multi-edge by case 1. By the above paragraph, $e_i$ and $e_j$ receive their colors from different edge-colorings.

   Putting these cases together, we can properly color the vertices of $M(G, k)$ with two colors by assigning one color to all of the vertices associated with $C_1$ and another to all of the vertices associated with $C_2$. Therefore, $M(G, k)$ is bipartite.

For the backwards implication, let $A$ and $B$ be two independent sets of vertices of $M(G, k)$ which partition the vertices of $M(G, k)$. For contradiction, suppose $|A| > |B|$. Then there
exists vertices in $A$, say $a_i$ and $a_j$ which must correspond to the same multi-edge in $2G$, but this means $a_i$ and $a_j$ are adjacent, contradicting our assumption of $A$ being independent. Thus, $|A| = |B|$, and each set corresponds to one of the copies of $G$ which make $2G$. Suppose $C_1$ is the edge-coloring associated with $A$ and $C_2$ the edge-coloring associated with $B$. Let $e'_i$ and $e''_i$ be vertices in $M(G, k)$ whose edges belong to the same multi-edge of $2G$. Without loss of generality, suppose $e'_i = a_i \in A$ and $e''_i = b_i \in B$. Since $a_i$ belongs to the same multi-edge in $2G$ as $b_i$, this means $a_i$ and $b_i$ must have different colors, as otherwise we contradict our assumption of each color class consisting of cycles of length greater than two. Thus, for any edge $e \in E(G)$, $C_1(e) \neq C_2(e)$.

By assumption, we know $a_i$ must be adjacent to vertices which share the same color as $a_i$, say $u_i$ and $v_i$. Furthermore, if the color of $a_i$ comes from $C_1$, this means the color of both $u_i$ and $v_i$ must come from $C_2$, so we say $u_i = b_{i-1}$ and $v_i = b_{i+1}$ without loss of generality. For a given vertex $v \in V(G)$, consider the set of multi-edges incident to $v$ in $2G$. Suppose $v$ is incident to an edge colored $c_j$ in $2G$ and that this color comes from $C_1$. By assumption, there must be another edge incident to $v$ with color $c_j$ and by the previous argument, this edge must get its color from $C_2$. Therefore, the set of colors incident to $v$ under $C_1$ is the same as the set of colors under $C_2$. Thus, $C_1$ and $C_2$ are mate colorings.

Below is an example of a 3-edge-coloring of $K_4$ along with its mate. Following these is the graph $2G$ made by superimposing both edge-colorings of $K_4$ together, along with the graph $M(K_4, 3)$ associated with these edge-colorings. The circular and square vertices respectively represent edges colored under $C_1$ and $C_2$. The green edges represent that the vertices (edges in $2G$) belong to the same multi-edge, and the other colored edges show that these edges form a cycle in $2G$. 

![Example of a 3-edge-coloring of K4 and its mate](image)
Let $G = K_{3,3}$. The following is an example of $2G$ where each color class is a union of pairwise vertex disjoint cycles of even length greater than two, but $M(G, 4)$ is not bipartite. To see this, let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the partite sets of $G$. Then the 6-cycle in blue is $(a, e, c, d, b, f)$, the 4-cycle in red is $(a, d, b, e)$, the 4-cycle in black is $(b, e, c, f)$, and the 4-cycle in green is $(a, d, c, f)$. Let $xy$ denote an edge in $G$ with vertices $x$ and $y$, and denote the corresponding vertices of $M(G, 4)$ by $xy$ and $xy'$. Then $M(G, 4)$ is not bipartite since it contains the 7-cycle $(ad, ad', af', af, bf, bd, bd')$. Note this notation implies $ad$ and $ad'$ have the same color, so this forces how the labelings of $xy$ and $xy'$ are assigned. Examples of $2G$ and $M(G, 4)$ are given below to the left and right respectively.
In this chapter we determine the color-trade-spectrum of three simple families of graphs, namely Theta graphs, wheel graphs, and $n$-cube graphs.

### 3.1 Theta Graphs

Let $\Theta^k_n$ denote the Theta graph which consists of $k$ paths of length $n$, with only the first and last vertices $\infty_1$ and $\infty_2$ in common. In particular, note that $\Theta^1_n$ is a path of length $n$, $\Theta^2_n$ is a cycle of length $2n$, and $\Theta^k_1$ is a multi-edge with edge multiplicity $k$, which is not a simple graph. An example of $\Theta^3_3$ is given below.

![Theta Graph Example](image)

**Lemma 3.1.** For non-negative integers $k$ and $n$,

1. $CTS(\Theta^k_n) = \emptyset$ for $n = 0$.
2. $CTS(\Theta^k_n) = \{k\}$ for $n = 1$ and $k \geq 2$, and $CTS(\Theta^k_n) = \emptyset$ for $n = 1$ and $0 \leq k \leq 1$.

**Proof.** 1: If $n = 0$, then $\Theta^k_n$ is a graph containing no edges and two isolated vertices, regardless of the value of $k$. Thus, the color-trade-spectrum is trivially empty.
2: If \( n = 1 \), then \( \Theta_n^k \) is a multi-edge consisting of \( k \) parallel edges. When \( k = 0 \), the graph contains no edges, and by the above case, the color-trade-spectrum is empty. When \( k = 1 \), \( \Theta_n^k \) is a path of length one, which has an empty color-trade-spectrum since there is only one way to edge-color this graph. When \( k \geq 2 \), each edge is assigned a unique color from \( \{c_1, \ldots, c_k\} \). Any permutation of this set with no fixed points creates a mate coloring, so \( k \) is in the color-trade-spectrum. Since the only way to properly edge-color this graph is to use \( k \) colors, this means \( \{k\} = \text{CTS}(\Theta_n^k) \).

\[ \square \]

**Theorem 3.1.** For \( n \geq 2 \), \( \text{CTS}(\Theta_n^k) = \{k\} \) if \( k \) is even and \( \text{CTS}(\Theta_n^k) = \emptyset \) if \( k \) is odd.

**Proof.** Let the \( j \)-th path be denoted by \((v_{1,j}, v_{2,j}, \ldots, v_{n+1,j})\) where \( v_{1,j} = \infty_1 \) and \( v_{n+1,j} = \infty_2 \) for \( 1 \leq j \leq k \) (so the naming of \( \infty_1 \) and \( \infty_2 \) is not unique). If \( k = 0 \), the graph consists of two isolated vertices with no edges, so the color-trade-spectrum is trivially empty. If \( k = 1 \), the graph is a path consisting of \( n \) edges which contains two vertices of degree one, and by Lemma 2.1, the color-trade-spectrum is empty. Suppose \( k \geq 2 \) and that \( C_1 \) is a proper \( k \)-edge-coloring of \( \Theta_n^k \). Without loss of generality, suppose that edge \((v_{1,1}, v_{2,1})\) is colored \( c_1 \) and that edge \((v_{2,1}, v_{3,1})\) is colored \( c_2 \). Since \( C_1 \) is proper, no other edge incident to \( \infty_1 \) can be colored \( c_1 \) or \( c_2 \). If \((v_{3,1}, v_{4,1})\) was given a color other than \( c_1 \), say \( c_3 \), then \( C_1 \) would have no mate coloring since the edge \((v_{2,1}, v_{3,1})\) would need to be colored both \( c_1 \) and \( c_3 \) in the mate coloring. Thus, in order for \( C_1 \) to have a mate coloring, the edges along the path \( v_{1,1}, v_{2,1}, \ldots, v_{n+1,1} \) must alternate between colors \( c_1 \) and \( c_2 \). Furthermore, this assignment of alternating colors must be true for every path from \( \infty_1 \) to \( \infty_2 \) if \( C_1 \) were to have a mate coloring.

Since the maximum degree of \( \Theta_n^k \) is \( k \), no value lower than \( k \) can be in the color-trade-spectrum. By the first paragraph of the theorem, we know each path must have an alternating assignment of colors to its edges if the edge-coloring \( C_1 \) were to have a mate coloring. Furthermore, if the edges of path \( i \) alternate between \( c_1 \) and \( c_2 \), then there must be another path \( j \) where the edges alternate between \( c_2 \) and \( c_1 \) in order for \( C_1 \) to have a mate. This means that \( C_1 \) has a mate only when \( C_1 \) induces a \( 2n \)-cycle decomposition of \( \Theta_n^k \) where each cycle in the decomposition alternates between two unique colors. This only happens when \( k \) is even,
so we conclude $\emptyset = CTS(\Theta_n^k)$ if $k$ is odd. We now given an explicit construction to show $k \in CTS(\Theta_n^k)$.

Suppose that $k$ and $n$ are both even integers which are at least two, and consider the following proper $k$-edge-coloring of $\Theta_n^k$, $C_1$. Assign $c_{2j-1}$ to edges of the form $(v_{2i-1,2j-1}, v_{2i,2j-1})$ and $(v_{2i,2j}, v_{2i+1,2j})$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{k}{2}$. Then, assign $c_{2j}$ to edges of the form $(v_{2i,2j-1}, v_{2i+1,2j-1})$ and $(v_{2i-1,2j}, v_{2i,2j})$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{k}{2}$. We find a mate coloring, $C_2$, by swapping the colors between edges colored $c_{2j-1}$ and $c_{2j}$ for $1 \leq j \leq \frac{k}{2}$ under $C_1$.

If $n$ is odd, we consider a different proper $k$-edge-coloring of $\Theta_n^k$, $C_3$. Assign $c_{2j-1}$ to edges of the form $(v_{2i-1,2j-1}, v_{2i,2j-1})$, $(v_{2i,2j}, v_{2i+1,2j})$, and $(v_{n,2j-1}, v_{n+1,2j-1})$ for $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{k}{2}$. Then, assign $c_{2j}$ to edges of the form $(v_{2i,2j-1}, v_{2i+1,2j-1})$, $(v_{2i-1,2j}, v_{2i,2j})$, and $(v_{n,2j}, v_{n+1,2j})$ for $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{k}{2}$. Again, we find a mate coloring, $C_4$, by swapping the colors between edges colored $c_{2j-1}$ and $c_{2j}$. Hence, $k \in CTS(\Theta_n^k)$.

To show that nothing else is in the color-trade-spectrum, recall from the second paragraph of the theorem that $C_5$ must induce a $2n$-cycle decomposition of $\Theta_n^k$ where each cycle in the decomposition alternates between two unique colors. Since there are only $\frac{k}{2}$ such cycles in $\Theta_n^k$ (when $k$ is even), this means this condition can only be met when $C_5$ uses no more than $k$ colors. Therefore, $\{k\} = CTS(\Theta_n^k)$.

3.2 Wheel Graphs

Let $n \geq 3$ be a positive integer and let $W_n$ denote the wheel graph which consists of a single $n$-cycle and $n$ “spokes” which originate from a central vertex $v_\infty$ which is adjacent to every vertex of the $n$-cycle. An example of $W_6$ is given below.
Theorem 3.2. For \( n \geq 3 \), \( \text{CTS}(W_n) = \{n\} \).

Proof. Note that \( W_n \) consists of \( 2n \) edges, and that the central vertex \( v_\infty \) has degree \( n \). By Lemma 2.1, this means the only possible value in the color-trade-spectrum of \( W_n \) is \( n \), and we present an explicit construction to show \( n \in \text{CTS}(W_n) \).

Without loss of generality, suppose the vertices of the \( n \)-cycle are labeled in order by \( v_0, v_1, \ldots, v_{n-1} \), so that \( v_i \) is adjacent to \( v_{i-1} \) and \( v_{i+1} \) where addition is computed modulo \( n \). Denote an edge in \( W_n \) by \( (v_i, v_j) \). Consider the following proper \( n \)-edge-coloring of \( W_n \), \( C_1 \). Assign color \( c_i \) to edges \( (v_\infty, v_i) \) and \( (v_{i+1}, v_{i+2}) \), where again addition is under modulo \( n \). To find a mate coloring for \( C_1 \), consider the following proper edge-coloring \( C_2 \). Assign color \( c_i \) to edges \( (v_\infty, v_{i+2}) \) and \( (v_i, v_{i+1}) \). Geometrically, this is equivalent to rotating the spokes of the wheel two times to the right and rotating the rim of the wheel once to the left. Then \( C_1 \) and \( C_2 \) are mate colorings, so \( n \in \text{CTS}(W_n) \), and since this is the only possible value in the color-trade-spectrum, \( \{n\} = \text{CTS}(W_n) \).

\[\square\]

3.3 \( n \)-cube Graphs

The final family of graphs we will consider in this chapter is \( n \)-cubes. Denote by \( Q_n \) the \( n \)-cube graph on \( 2^n \) vertices, each labeled with an \( n \)-bit binary number, where two vertices are adjacent
if the corresponding binary numbers differ by exactly one digit, i.e. their Hamming distance is one. Note that the maximum degree of $Q_n$ is $n$, and that $|E(Q_n)| = n2^{n-1}$. By Lemma 2.1, this means the color-trade-spectrum is bounded below by $n$ and above by $n2^{n-2}$.

We say two graphs $G$ and $H$ are isomorphic, denoted $G \cong H$, if there exists a bijection $\Phi : V(G) \to V(H)$ with the property that any two vertices $u$ and $v$ in $G$ have an edge between them if and only if $\Phi(u)$ and $\Phi(v)$ have an edge between them in $H$. An equivalent construction of $Q_n$ is found by taking the Cartesian product of $n$ $K_2$ graphs, denoted $K_2 \square K_2 \square \cdots \square K_2$. Furthermore, one can show that $Q_n \square Q_m \cong Q_{n+m}$ as follows. Let the vertices of $Q_n$ and $Q_m$ be denoted by binary sequences of length $n$ and $m$ respectively. As in the previous paragraphs, vertices are adjacent when their Hamming distance is one. To construct $Q_n \square Q_m$, we take $2^m$ copies of $Q_n$. In particular, for each vertex in $Q_n$ denoted by a unique binary sequence of length $n$, we concatenate a binary sequence of length $m$ to create a new binary sequence of length $n + m$. The first $n$ digits correspond to unique vertices within a copy of $Q_n$ while the last $m$ digits correspond to unique copies of $Q_m$. For example, in $Q_2 \square Q_3$, the binary sequence 01001 would correspond to the vertex labeled 01 in $Q_2$ in the 001th copy of $Q_2$. This creates a graph with $2^{n+m}$ vertices and $(n + m)2^{n+m-1}$ edges, where vertices are adjacent only when their Hamming number is one. Therefore, $Q_n \square Q_m \cong Q_{n+m}$. In Lemma 3.2 we calculate the color trade spectra of $Q_n$ for $0 \leq n \leq 3$, and in Theorem 3.3 we calculate the remaining spectra using a recursive construction based on the fact that $Q_{n-1} \square Q_2 \cong Q_{n+1}$.

**Lemma 3.2.** Let $n$ be a non-negative integer. Then $CTS(Q_n) = \emptyset$ for $0 \leq n \leq 1$, $CTS(Q_2) = \{2\}$, and $CTS(Q_3) = \{3, 4, 5, 6\}$.

**Proof.** Trivially, $Q_0$ and $Q_1$ have no color trades since they contain less than 2 edges, and any color trade requires a graph to have at least 2 edges. Note that $Q_2$ is a 4-cycle, so by Lemma 2.2, we conclude $\{2\} = CTS(Q_2)$. However we present an alternate proof using methods that will be used for future cases of $Q_n$. By the discussion above, the only possible value in $CTS(Q_2)$ is 2. As above, denote the vertices of $Q_2$ by binary sequences of length two, $b_1b_2$, where each $b_i$ is either 0 or 1. Denote edges of $Q_2$ by $(b_1b_2, d_1d_2)$, where $b_1b_2 \neq d_1d_2$. Consider the following proper 2-edge-coloring of $Q_2$, $C_1$. Assign $c_1$ to edges $(00, 01)$ and $(10, 11)$, and assign $c_2$ to
edges \((01, 10)\) and \((00, 11)\). By swapping the colors between edges colored \(c_1\) and \(c_2\), we create a mate coloring \(C_2\). Therefore, \(\{2\} = \text{CTS}(Q_2)\). An example of these edge-colorings is given below.

Next, we show \(\{3, 4, 5, 6\} = \text{CTS}(Q_3)\). As before, denote vertices by binary sequences of length three, \(b_1b_2b_3\), and denote edges by \((b_1b_2b_3, d_1d_2d_3)\). Consider the following proper 6-edge-coloring of \(Q_3\). Assign \(c_1\) to edges \((000, 010)\) and \((100, 110)\), \(c_2\) to edges \((000, 001)\) and \((010, 011)\), \(c_3\) to edges \((000, 100)\) and \((010, 110)\), \(c_4\) to edges \((001, 011)\) and \((101, 111)\), \(c_5\) to edges \((100, 101)\) and \((110, 111)\), and \(c_6\) to edges \((001, 101)\) and \((011, 111)\). Then any permutation of the assignment of colors \(c_1\) through \(c_6\) with no fixed point creates a new edge-coloring which is a mate for \(C_1\). Thus, \(6 \in \text{CTS}(Q_3)\).

To show \(\{3, 4, 5\} \subset \text{CTS}(Q_3)\), we consider the graph \(G'\) as defined in Lemma 2.4, where the vertices of \(G'\) correspond to colors used in \(C_1\) and \(C_2\), and vertices in \(G'\) are adjacent when the respective colors are incident in \(C_1\) and \(C_2\). For \(Q_3\), \(Q'_3\) is a 3-cycle, which has chromatic number three. By Lemma 2.4, this means \(\{3, 4, 5, 6\} \subseteq \text{CTS}(Q_3)\). Since \(\frac{|E(Q_3)|}{2} = 6\) and \(Q_3\) has maximum degree 3, we conclude that \(\{3, 4, 5, 6\} = \text{CTS}(Q_3)\) by Lemma 2.1. An example of a proper 6-edge-coloring of \(Q_3\) along with its mate is given below.
**Theorem 3.3.** Let \( n \) be a nonnegative integer. Then \( CTS(Q_n) = \emptyset \) for \( 0 \leq n \leq 1 \), and \( CTS(Q_n) = \{ n, n + 1, \ldots, n2^{n-2} \} \) for \( n \geq 2 \).

**Proof.** The cases for \( 0 \leq n \leq 3 \) are covered by the above lemma, and we cover the rest of the cases using induction on \( n \), based on the following construction based on the earlier discussion that \( Q_n \square Q_m \cong Q_{n+m} \). In particular, \( Q_{n+1} \cong Q_{n-1} \square Q_2 \), and we start the induction on \( n = 2 \). By Lemma 3.2, the claim holds for \( n = 2 \), so we now suppose the claim holds for \( n \) and consider \( n + 1 \). Note that \( Q_{n-1} \square Q_2 \) yields four copies of \( Q_{n-1} \), and by induction, we can find mate colorings using any value in the color-trade-spectrum of \( Q_{n-1} \) for each copy of \( Q_{n-1} \). In particular, we can use \((n - 1)2^{n-3}\) distinct colors for each copy of \( Q_{n-1} \), for a total of \( 4(n - 1)2^{n-3} = (n - 1)2^{n-1} \) colors. Let \( v_1, v_2, \ldots, v_{2n-1} \) be a given ordering of the vertices of \( Q_{n-1} \), let \( v_{i,j} \) denote the \( i \)th vertex in the \( j \)th copy of \( Q_{n-1} \), and let \((v_{a,b}, v_{i,j})\) denote an edge between \( v_{a,b} \) and \( v_{i,j} \) in \( Q_{n-1} \square Q_2 \), assuming such an edge exists. The four copies of \( Q_{n-1} \) consist of \( 4(n - 1)2^{n-2} = (n - 1)2^n \) edges, and the remaining \((n + 1)2^n - (n - 1)2^n = 2^{n+1}\) edges form \( 2^{n-1} \) disjoint 4-cycles of the form \((v_{i,1}v_{i,2}v_{i,3}v_{i,4})\).

Coloring each of these 4 cycles with 2 distinct colors gives us an edge-coloring of \( Q_{n+1} \) using \( 4(n - 1)2^{n-3} + 2 \cdot 2^{n-1} = (n + 1)2^{n-1} \) colors. By the previous paragraph, we can find mates for this edge-coloring for each copy of \( Q_{n-1} \), and we find a mate coloring for \( Q_{n+1} \) by alternating the assignment of colors within each of the above 4-cycles. Thus, \((n + 1)2^{n-1} \in CTS(Q_{n+1})\). Since each induced copy of \( Q_{n-1} \) is disjoint, we can lower the amount of colors used in \( Q_{n+1} \) by reusing colors between copies of \( Q_{n-1} \). Since each of the 4-cycles connecting the copies of \( Q_{n-1} \) are disjoint, we can use the same 2 colors on each of these cycles. Doing this for all \( n \geq 2 \) proves the claim. An example of this construction being applied to \( Q_4 \) is given below.
Chapter 4

Complete Bipartite Graphs

Let $K_{m,n}$ be the bipartite graph consisting of vertex sets $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_n\}$ where $m \leq n$ are non-negative integers, and the edge set $E = \{(a_i, b_j) | 1 \leq i \leq m, 1 \leq j \leq n\}$.

Recall that a graph is $k$-regular if every vertex has degree $k$. A $k$-factor of a graph is a spanning $k$-regular subgraph, and a $k$-factorization of a graph partitions the graph into disjoint $k$-factors. Also, recall that a matching is a set of independent edges in a graph, meaning the edges have no common vertices. One can easily show that a 1-factorization is equivalent to a perfect matching, which is a matching which includes every vertex of the graph.

4.1 $0 \leq m \leq n \leq 3$

Before stating the main results of this section, we first determine the color-trade-spectrum of $K_{m,n}$ where at least one of $m$ or $n$ is either zero or one, $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$, which will be used in later constructions.

**Lemma 4.1.** Let $m \leq n$ for non-negative integers $m$ and $n$. Then $CTS(K_{m,n}) = \emptyset$ if at least one of $m$ or $n$ is zero or one, and the color trade spectra of $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$ are $\{2\}$, $\{3\}$, and $\{3\}$ respectively.

**Proof.** If either $m$ or $n$ is 0, we have a graph with no edges, which trivially has an empty color-trade-spectrum. If either $m$ or $n$ is one but both not zero, we have at least one vertex of degree one, and by Lemma 2.1, the color-trade-spectrum is again empty. We now consider $K_{2,2}$.

Since $|E(K_{2,2})| = 4$ and $\chi'(K_{2,2}) = 2$, the only possible value in $CTS(K_{2,2})$ is 2. Likewise, the only possible value in $CTS(K_{2,3})$ is 3. Note that $K_{2,2}$ is isomorphic to a 4-cycle.
graph, so by Lemma 2.2 \( 2 = \text{CTS}(K_{2,2}) \). To show \( 3 = \text{CTS}(K_{2,3}) \), consider the proper 3-edge-coloring \( C_1 \) given below. Assign \( c_1 \) to edges \((a_1b_1)\) and \((a_2b_2)\), \( c_2 \) to edges \((a_1b_3)\) and \((a_2b_1)\), and \( c_3 \) to edges \((a_1b_2)\) and \((a_2b_3)\). Now, consider the edge-coloring \( C_2 \) where \( c_1 \) is assigned to edges \((a_1b_2)\) and \((a_2b_1)\), \( c_2 \) is assigned to edges \((a_1b_1)\), and \((a_2b_3)\), and \( c_3 \) is assigned to edges \((a_1b_3)\) and \((a_1b_2)\). Then \( C_1 \) and \( C_2 \) are mates, so \( 3 = \text{CTS}(K_{2,3}) \). Below are examples of these edge-colorings using 2 and 3 colors for \( K_{2,2} \) and \( K_{2,3} \) respectively.

For \( K_{3,3} \), the only possible values in the color-trade-spectrum are 3 and 4, and since any proper 3-edge-coloring of \( K_{3,3} \) must be a 1-factorization, meaning that the colored edges form a perfect matching, we can show \( 3 \in \text{CTS}(K_{3,3}) \) as follows. Let \( C_1 \) be any proper 3-edge-coloring of \( K_{3,3} \), and apply any permutation with no fixed points to the colors of \( C_1 \) to create a mate coloring \( C_2 \). For example, \( \alpha := (c_1c_2c_3) \) would be such a permutation. However, 4 is not in the color-trade-spectrum, and to show this, note that any proper edge-coloring of \( K_{3,3} \) using 4 colors must contain a perfect matching, and three color classes of size two. Suppose \( c_1 \) is a perfect matching and consider assigning a color class of size 2, say \( c_2 \), to two of the remaining edges of the graph. The subgraph induced by \( c_1 \) and \( c_2 \) consists of either two components, or

![Graph](image-url)
one component. A quick exhaustive search shows that neither case leads to a mate coloring. Therefore, \( \{3\} = \text{CTS}(K_{3,3}) \).

Below are mate colorings using 3 colors for \( K_{3,3} \). Following these are examples of the two cases for \( K_{3,4} \) where the subgraph induced by \( c_1 \) and \( c_2 \) consists of either two components, or one component.

\[
\begin{array}{c}
\begin{array}{c}
\text{Case 1: Two Components} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Case 2: One Component} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Case 3: Two Components} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Case 4: One Component} \\
\end{array}
\end{array}
\]

4.2 \( n \geq m \geq 2 \) where \( m \) is even

For the following theorems, we consider \( K_{m,n} \) with a proper edge-coloring \( C_1 \). We also consider the associated (and equivalent) \( n \times m \) Latin rectangle \( L \). Given vertices in different parts, say \( a_j \) and \( b_i \), the entry of cell \( (i,j) \) of \( L \) corresponds to the color assigned to \( b_i a_j \) under \( C_1 \). Thus, finding a mate for \( K_{m,n} \) is equivalent to finding a permutation \( \pi \) of \( L \) with no fixed points, which preserves the Latin property, and where every color appearing in a row or column of \( L \)
still appears after applying \( \pi \) to \( L \). For this to happen, we note that if a set of colors appears in some column \( k \) of \( L \), then the same set must also appear in another column \( k' \) of \( L \). The following theorems give constructions for \( L \) and \( \pi (L) \).

**Theorem 4.1.** \( \{n, n+1, \ldots, \frac{mn}{2}\} = \text{CTS}(K_{m,n}) \) where \( 2 \leq m \leq n \) and \( m \) is even.

**Proof.** We construct \( L_\Delta \), the Latin rectangle corresponding to a \( \Delta \)-edge-coloring of \( K_{m,n} \). Let a **column block of size** \( s \) **on** \( t \) **colors**, denoted by \( \text{CB}(s, t) \), be a set of \( s \) columns of \( L_\Delta \) where \( t \) unique symbols appear in the column block. Partition the \( m \) columns of \( L_\Delta \) into \( \frac{m}{2} \) column blocks of size 2, and without loss of generality, suppose the 2 columns in each column block are adjacent. Denote column \( k \) by \( a_k \) and the column block containing columns \( a_{2i-1} \) and \( a_{2i} \) by \( A_i \) for \( 1 \leq i \leq \frac{m}{2} \). Fill the cells of \( a_{2i-1} \) and \( a_{2i} \) in \( L_\Delta \) in order from top to bottom with entries \( c_{2(i-1)+1}, c_{2(i-1)+2}, \ldots, c_n, c_1, \ldots, c_{2(i-1)} \) and \( c_{2(i-1)+2}, c_{2(i-1)+3}, \ldots, c_n, c_1, \ldots, c_{2(i-1)+1} \) respectively. Then each column block of \( L_\Delta \) contains the same set of \( n \) colors and we find a mate for \( L_\Delta \), \( \pi (L_\Delta) \) as follows: let \( \pi := (a_1 a_2) (a_3 a_4) \cdots (a_{m-1} a_m) \), a product of \( \frac{m}{2} \) 2-cycle permutations. Each column and row in \( \pi (L_\Delta) \) contains the same symbols as they did in \( L_\Delta \), while no cell receives the same entry. Thus, \( \pi (L_\Delta) \) and \( L_\Delta \) correspond to two mate colorings of \( K_{m,n} \).

Let \( C_1 \) and \( C_2 \) refer to the corresponding edge-colorings of \( L_\Delta \) and \( \pi (L_\Delta) \). Using notation from Lemma 2.3, note that each \( H_j \) consists of \( \frac{m}{2} \) many 4-cycles, each of which corresponds to a unique column block, so \( \alpha = \frac{mn}{2} \). By Lemma 2.3, we conclude \( \{\Delta, \Delta + 1, \ldots, \frac{mn}{2}\} \subseteq \text{CTS}(K_{m,n}) \) and since \( \Delta (K_{m,n}) = n \) and \( |E(K_{m,n})| = mn \), we conclude there are no other possible values in the color-trade-spectrum.

\[ \square \]

Below are examples of \( L_\Delta \) and \( \pi (L_\Delta) \) along with their corresponding graphs for \( K_{4,6} \).
4.3 \( n \geq m \geq 3 \) where \( m \) is odd

We now determine the color-trade-spectrum for \( K_{3,n} \), which we will use for the general case of \( K_{m,n} \) where \( m \) is odd. By Lemma 4.1, \( \text{CTS}(K_{3,3}) = \{3\} \). We first give a specific construction that shows \( \{4, 5, 6\} = \text{CTS}(K_{3,4}) \), and then use this to determine \( \text{CTS}(K_{3,n}) \) in two cases, where \( n \geq 3 \) is an integer.

**Lemma 4.2.** \( \{4, 5, 6\} = \text{CTS}(K_{3,4}) \)

**Proof.** Using the same notation from Theorem 4.1, \( L_\Delta \) contains a single column block \( A_1 \) of size three, consisting of the columns \( a_1, a_2, \) and \( a_3 \). Furthermore, we now consider a row block of size \( s \) on \( t \) colors, denoted \( \text{RB}(s, t) \), which is a set of \( s \) rows of \( L_\Delta \) where \( t \) unique symbols appear in the row block. Below is a \( \text{RB}(4, 4) \) and its corresponding 4-edge-coloring of \( K_{3,4} \).
To show that 4 is in the color-trade-spectrum, we apply the permutation $\pi := (a_1a_2a_3)$. Each column and each row in $\pi(L_\Delta)$ contains the same symbols as they did in $L$, while no cell receives the same entry. This corresponds to each vertex in $K_{3,4}$ seeing the same color set, but no edge receiving the same color, showing these two edge-colorings are mates. Note that this implies $n \in \text{CTS}(K_{3,n})$ where $n \geq 3$ is an integer by using the following method to create $L_\Delta$: from top to bottom fill the cells of $a_1$ and $a_2$ with entries $c_1, c_2, \ldots, c_n$ and $c_2, c_3, \ldots, c_n, c_1$ respectively, and the cells of $a_3$ with $c_3, c_4, \ldots, c_n, c_1, c_2$. To show that 5 is in the color-trade-spectrum, we modify $L_\Delta$ to make $L_5$, shown below along with its corresponding 5-edge-coloring of $K_{3,4}$.

Applying the permutations $\pi : (a_1a_2a_3)$ on the first 3 rows of of $L_5$ and $\sigma : (a_1a_3a_2)$ on the last row of $L_5$ gives us a mate coloring. Finally, we show 6 is in the color-trade-spectrum by further modifying $L_5$ to make $L_6$, which consists of two separate RB(2, 3), shown below along with its associated 6-edge-coloring of $K_{3,4}$.
Applying the permutations \( \pi: (a_1a_2a_3) \) on rows 2 and 4 of \( L_6 \) and \( \sigma: (a_1a_3a_2) \) on rows 1 and 3 of \( L_6 \) gives us a mate coloring. Since \( \Delta(K_{3,4}) = 4 \) and \( |E(K_{3,4})| = 12 \), we have found a mate coloring for every possible value in the color-trade-spectrum. Using these blocks as our building tools, we now determine the color-trade-spectrum of \( K_{3,n} \) by considering two cases.

**Lemma 4.3.** Let \( n \geq 4 \) be a positive integer. Then \( \{\Delta, \Delta + 1, \ldots, \frac{3n}{2}\} = CTS(K_{3,n}) \) if \( n \) is even and \( \{\Delta, \Delta + 1, \ldots, \frac{3n-1}{2}\} = CTS(K_{3,n}) \) if \( n \) is odd.

**Proof.** Case 1: \( n \) is even.

Let \( n = 2k \) where \( k \geq 2 \) is an integer. By Lemma 4.2, \( n \) is in the color-trade-spectrum by considering a single \( RB(n, n) \). To show that \( \{n + 1, \ldots, \frac{3n}{2} - 2\} \) is in the color-trade-spectrum, consider the value \( n + u \) where \( 1 \leq u \leq \frac{n}{2} - 2 \). We construct \( L_{n+u} \) by using a single \( RB(n - 2u, n - 2u) \) and \( u \) many \( RB(2, 3) \) where each block uses a different set of colors. To find a mate for this edge-coloring, we use the same permutations from Lemma 4.2, where \( \pi \) is applied to the \( RB(n - 2u, n - 2u) \) along with the second row of every \( RB(2, 3) \) and \( \sigma \) is applied to the first row of every \( RB(2, 3) \). An example of \( L_{n+u} \) is shown below along with its associated \( n + u \)-edge-coloring of \( K_{3,n} \).
Note that when \( u = \frac{n}{2} - 2 \), we have a single \( \text{RB}(4, 4) \) which we can assume is the same \( \text{RB}(4, 4) \) from the \( K_{3,4} \) case without loss of generality. This shows that \( n + u \in \text{CTS}(K_{3,n}) \) for \( 1 \leq u \leq \frac{n}{2} - 2 \). To show \( \frac{3n}{2} - 1 \) is in the color-trade-spectrum, we can modify our \( L_{n+u} \) from above by replacing the \( \text{RB}(4, 4) \) with a \( \text{RB}(4, 5) \) which again we can assume is the same \( \text{RB}(4, 5) \) from the \( K_{3,4} \). Applying \( \pi \) to the first 3 rows of the \( \text{RB}(4, 5) \) and to the second row of every \( \text{RB}(2, 3) \), and applying \( \sigma \) to the last row of \( \text{RB}(4, 5) \) and to the first row of every \( \text{RB}(2, 3) \) yields a mate. Finally, we show \( \frac{3n}{2} \) is in the color-trade-spectrum by modifying \( L_{n+u} \) again to consist of \( \frac{n}{2} \) many \( \text{RB}(2, 3) \). Applying \( \pi \) to the second row of every \( \text{RB}(2, 3) \) and \( \sigma \) to the first row of every \( \text{RB}(2, 3) \) yields a mate.

**Case 2: \( n \) is odd.**

Let \( n = 2k + 1 \) where \( k \geq 1 \) is an integer. Again, Lemma 4.2 guarantees \( n \) is in the color-trade-spectrum by considering a single \( \text{RB}(n, n) \). To show \( \{ n + 1, \ldots, \frac{3n-1}{2} \} \) is in the color-trade-spectrum, we use the same construction from the even case where we consider \( n + u \) for \( 1 \leq u \leq \frac{n-3}{2} \). To show \( \frac{3n-1}{2} \) is in the color-trade-spectrum, we consider the following
construction of $L_{3n-1/2}$.

$\begin{array}{ccc}
1 & n+1 & n-k+1 \\
2 & 1 & n-k+2 \\
3 & n+2 & 1 \\
4 & n+3 & n-k+3 \\
\vdots & \vdots & \vdots \\
k+1 & \frac{3n-1}{2} & n \\
k+2 & 2 & n+1 \\
k+3 & 3 & n+2 \\
\vdots & \vdots & \vdots \\
n-k-1 & n-k-1 & \frac{3n-1}{2} - 1 \\
n & n-k & \frac{3n-1}{2} \\
\end{array}$

To find a mate for this edge-coloring, apply the same $\pi$ from the even case to the first $k+1$ rows of $L_{3n-1/2}$ and the same $\sigma$ from the even case to the last $k$ rows of $L_{3n-1/2}$.

From Lemmas 4.1, 4.2, and 4.3, along with Theorem 4.1, we have covered every case except for $K_{m,n}$ where $m \geq 5$ is odd and $n \geq 5$. The following theorem uses Lemma 4.3 to cover this case where we consider subcases depending on the parity of $n$.

**Theorem 4.2.** Let $m \geq 5$ be an odd integer and $n \geq m$ be an integer. Then $\{\Delta, \ldots, \frac{mn}{2}\} = CTS(K_{m,n})$ if $n \geq 6$ is even, and $\{\Delta, \ldots, \frac{mn-1}{2}\} = CTS(K_{m,n})$ if $n \geq 5$ is odd.

**Proof.** We construct $L_\Delta$ by modifying the construction from Theorem 4.1. Partition the $m$ columns of $L$ into $\frac{m-3}{2}$ column blocks of size 2 where we again suppose 2 columns in a column block are adjacent without loss of generality. The final 3 columns form a column block of size 3.

As before, let $A_i$ denote the column block containing columns $a_{2i-1}$ and $a_{2i}$ where $1 \leq i \leq \frac{m-3}{2}$ and let $B$ denote the column block containing columns $a_{m-2}, a_{m-1}$, and $a_m$. Fill the cells of $a_{2i-1}$ and $a_{2i}$ in order from top to bottom with entries $c_{2(i-1)+1}, c_{2(i-1)+2}, \ldots, c_n, c_1, \ldots, c_{2(i-1)}$ and $c_{2(i-1)+2}, c_{2(i-1)+3}, \ldots, c_n, c_1, \ldots, c_{2(i-1)+1}$ respectively. Likewise, fill the cells of $a_{m-3}$ and

34
Let $\pi := (a_1a_2)(a_3a_4)\cdots(a_{m-4}a_{m-3})(a_{m-2}a_{m-1}a_m)$. Then each column and row in $\pi(L_\Delta)$ contains the same symbols as they did in $L$, while no cell receives the same entry. Thus, $\pi(L_\Delta)$ and $L_\Delta$ correspond to two mate colorings of $K_{m,n}$. Using the notation of Lemma 2.3, $C_1$ and $C_2$ refer to the corresponding edge-colorings of $L_\Delta$ and $\pi(L_\Delta)$. Note that each $H_j$ consists of $\frac{m-3}{2}$ many 4-cycles, where each cycle corresponds to some $A_i$, and one 6-cycle, which corresponds to $B$, so $\alpha = \frac{mn-n}{2}$. By Lemma 2.3, we conclude $\{\Delta, \Delta + 1, \ldots, \frac{mn-n}{2}\} \subseteq CTS(K_{m,n})$. We finish the rest of the color-trade-spectrum in cases by using Lemma 4.3.

Case 1: $n \geq 6$ is even.

To show that $\{\frac{mn-n+1}{2}, \ldots, \frac{mn}{2}\} \subseteq CTS(K_{m,n})$, note that in $L_{\frac{mn-n}{2}}$ each column block uses $n$ different colors, including the column block of size 3, $B$. Using the argument from the even case of Lemma 4.3, we conclude that we can extend the number of colors in $B$, $j$, to any value of $j$ where $n \leq j \leq \frac{3m}{2}$. Using the appropriate permutations from Lemma 4.3, we conclude that $\{\frac{mn-n+1}{2}, \ldots, \frac{mn}{2}\} \subseteq CTS(K_{m,n})$. Combining this result with the result of the previous paragraph, this shows that $CTS(K_{m,n}) = \{\Delta, \ldots, \frac{mn}{2}\}$.

Case 2: $n \geq 5$ is odd.

To show that $\{\frac{mn-n+1}{2}, \ldots, \frac{mn-1}{2}\} \subseteq CTS(K_{m,n})$, we use the same argument as in the even case, refering to the odd case of Lemma 4.3. We conclude that $CTS(K_{m,n}) = \{\Delta, \ldots, \frac{mn-1}{2}\}$.

Below are examples $L_\Delta$ and $\pi(L_\Delta)$ along with their corresponding graphs for $K_{5,5}$.  

□
\[
\begin{array}{cccccccc}
L_{\Delta} & & & & & & & \\
1 & 2 & 3 & 4 & \cdots & m - 4 & m - 3 & m - 2 & m - 1 & m \\
2 & 3 & 4 & 5 & \cdots & m - 3 & m - 2 & m - 1 & m & m + 1 \\
3 & 4 & 5 & 6 & \cdots & m - 2 & m - 1 & m & m + 1 & m + 2 \\
4 & 5 & 6 & 7 & \cdots & m - 1 & m & m + 1 & m + 2 & m + 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n - 4 & n - 3 & n - 2 & n - 1 & \cdots & m - 9 & m - 8 & m - 7 & m - 6 & m - 5 \\
n - 3 & n - 2 & n - 1 & n & \cdots & m - 8 & m - 7 & m - 6 & m - 5 & m - 4 \\
n - 2 & n - 1 & n & 1 & \cdots & m - 7 & m - 6 & m - 5 & m - 4 & m - 3 \\
n - 1 & n & 1 & 2 & \cdots & m - 6 & m - 5 & m - 4 & m - 3 & m - 2 \\
n & 1 & 2 & 3 & \cdots & m - 5 & m - 4 & m - 3 & m - 2 & m - 1 \\
\end{array}
\]

\[
\pi(L_{\Delta}) & & & & & & & \\
2 & 1 & 4 & 3 & \cdots & m - 3 & m - 4 & m & m - 2 & m - 1 \\
3 & 2 & 5 & 4 & \cdots & m - 2 & m - 3 & m + 1 & m - 1 & m \\
4 & 3 & 6 & 5 & \cdots & m - 1 & m - 2 & m + 2 & m & m + 1 \\
5 & 4 & 7 & 6 & \cdots & m & m - 1 & m + 3 & m + 1 & m + 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n - 3 & n - 4 & n - 1 & n - 2 & \cdots & m - 8 & m - 9 & m - 5 & m - 7 & m - 6 \\
n - 2 & n - 3 & n & n - 1 & \cdots & m - 7 & m - 8 & m - 4 & m - 6 & m - 5 \\
n - 1 & n - 2 & 1 & n & \cdots & m - 6 & m - 7 & m - 3 & m - 5 & m - 4 \\
n & n - 1 & 2 & 1 & \cdots & m - 5 & m - 6 & m - 2 & m - 4 & m - 3 \\
1 & n & 3 & 2 & \cdots & m - 4 & m - 5 & m - 1 & m - 3 & m - 2 \\
\end{array}
\]
Denote the Cartesian product of two graphs $G$ and $H$ by $G \square H$ which consists of the vertex set $\{(g_i, h_j) | g_i \in V(G), h_j \in V(H)\}$ where two vertices $(u, u')$ and $(v, v')$ are adjacent if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$. We will let $(i, j)$ denote the $i$th vertex along the $j$th path where $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2\}$, and let $(s, t)(u, v)$ denote the edge incident to the vertices $(s, t)$ and $(u, v)$, assuming such an edge exists. Let $2 \leq n \leq m$ be integers and denote by $P_n$ the path on $n$ vertices. For convenience, let $G = P_n \square P_m$. Then $|V(G)| = nm$ and $|E(G)| = n(m - 1) + m(n - 1)$. For $n = m = 2$, $\delta(G) = \Delta(G) = 2$. For $n = 2$ and $m \geq 3$, $\delta(G) = 2$ and $\Delta(G) = 3$. For $3 \leq n \leq m$, $\delta(G) = 2$ and $\Delta(G) = 4$. Thus, by Lemma 2.3, $\text{CTS}(G) \subseteq \{4, 5, \ldots, \left[\frac{n(m-1)+m(n-1)}{2}\right]\}$ for $3 \leq n \leq m$. When $n = m = 2$, $G \cong C_4$ so $\text{CTS}(G) = \{2\}$. When $n = 2$ and $m \geq 3$, $\text{CTS}(G) \subseteq \{3, 4, \ldots, \left[\frac{n(m-1)+m(n-1)}{2}\right]\}$. Since $P_n \square P_m$ is clearly isomorphic to $P_m \square P_n$, we will only consider $P_n \square P_m$, so this allows us to assume $n \leq m$ without loss of generality.

We begin with the case where $n = 2$ and $m \geq 3$.

### 5.1 $P_2 \square P_m$ Graphs

**Lemma 5.1.** Let $m \geq 3$ be an integer. Then $\{3, 4, \ldots, m\} \subseteq \text{CTS}(P_2 \square P_m)$.

**Proof.** As above, let $G = P_2 \square P_m$ for convenience. We consider two cases depending if $m$ is even or odd.

Case 1: $m \geq 3$ is odd.
Give $G$ a proper 3-edge-coloring as follows. Assign $c_1$ to $(1, 1)(1, 2), (2i + 2, 1)(2i + 3, 1)$, and $(2i + 2, 2)(2i + 3, 2)$ for $0 \leq i \leq \frac{m-3}{2}$. Assign $c_2$ to $(1, 1)(2, 1), (1, 2)(2, 2)$, and $(i + 3, 1)(i + 3, 2)$ for $0 \leq i \leq m - 4$. Assign $c_3$ to $(2, 1)(2, 2), (m, 1)(m, 2), (2i + 1, 1)(2i + 2, 1)$, and $(2i + 1, 2)(2i + 2, 2)$ for $1 \leq i \leq \frac{m-3}{2}$. This edge-coloring has the following mate coloring. Assign $c_1$ to $(1, 1)(2, 1), (1, 2)(2, 2)$, and $(i + 3, 1)(i + 3, 2)$ for $0 \leq i \leq m - 3$. Assign $c_2$ to $(1, 1)(1, 2), (2, 1)(2, 2), (2i + 1, 1)(2i + 2, 1)$, and $(2i + 1, 2)(2i + 2, 2)$ for $1 \leq i \leq \frac{m-3}{2}$. Assign $c_3$ to $(2i + 2, 1)(2i + 3, 1)$, and $(2i + 2, 2)(2i + 3, 2)$ for $0 \leq i \leq \frac{m-3}{2}$. Examples of these edge-colorings are given below respectively where $c_1$ is black, $c_2$ is red, and $c_3$ is blue.

![Diagram](image1)

Using the notation of Lemma 2.3, note that $H_1$ consists of $\frac{m-3}{2}$ many 4-cycles and one 6-cycle, while $H_2$ consists of $\frac{m-1}{2}$ many 4-cycles and $H_3$ consists of one $2(m - 1)$ cycle. Thus, $\alpha = m$, so by Lemma 2.3 we conclude $\{3, ..., m\} \subseteq CTS(P_2 \square P_m)$ when $m \geq 3$ is odd.

Case 2: $m \geq 3$ is even.

Give $G$ a proper 3-edge-coloring as follows. Assign $c_1$ to $(1, 1)(1, 2), (2i + 2, 1)(2i + 3, 1), (2i + 2, 2)(2i + 3, 2)$, and $(m, 1)(m, 2)$ for $0 \leq i \leq \frac{m-4}{2}$. Assign $c_2$ to $(2i + 1, 1)(2i + 2, 1)$ and $(2i + 1, 2)(2i + 2, 2)$ for $0 \leq \frac{m-2}{2}$. Assign $c_3$ to $(i + 2, 1)(i + 2, 2)$ for $0 \leq i \leq m - 3$. This edge-coloring has the following mate coloring. Assign $c_1$ to $(2i + 1, 1)(2i + 2, 1)$ and $(2i + 1, 2)(2i + 2, 2)$ for $0 \leq i \leq \frac{m-2}{2}$. Assign $c_2$ to $(i, 1)(i, 2)$ for $1 \leq i \leq m$. Assign $c_3$ to $(2i + 2, 1)(2i + 3, 1)$ and $(2i + 2, 2)(2i + 3, 2)$ for $0 \leq i \leq \frac{m-4}{2}$. Examples of these edge-colorings are given below respectively where $c_1$ is black, $c_2$ is blue, and $c_3$ is red.
Using the notation of Lemma 2.3, note that $H_1$ consists of $2m$ cycle, while $H_2$ consists of $\frac{m}{2}$ many 4-cycles and $H_3$ consists of $\frac{m-2}{2}$ many 4-cycles. Thus, $\alpha = m$, so by Lemma 2.3 we conclude \{3, ..., m\} $\subseteq$ CTS($P_2 \square P_m$) when $m \geq 3$ is even.

\[\square\]

**Theorem 5.1.** Let $m \geq 3$ be an integer. Then \{3, 4, ..., m\} = CTS($P_2 \square P_m$).

**Proof.** From Lemma 5.1, it remains to show that nothing else is contained in the color-trade-spectrum. Note that $G$ has $m$ vertical edges of the form $(i, 1)(i, 2)$ for $1 \leq i \leq m$. Suppose that we have two copies of $G$ under the mate colorings of $C_1$ and $C_2$ respectively, each on the same set of $k$ colors. Between the two copies of $G$, each color uses at least two vertical edges. Furthermore, each copy of a vertical edge receives a different color in each copy of $G$. Therefore, $k \leq m$, so the color-trade-spectrum contains no values larger than $m$. By Lemma 5.1, we conclude \{3, 4, ..., m\} = CTS($P_2 \square P_m$).

\[\square\]

### 5.2 $P_n \square P_m$ Graphs

From the beginning of this chapter, recall that $P_n \square P_m \cong P_m \square P_n$, so we assume $n \leq m$ without loss of generality.

**Theorem 5.2.** Let $m$ and $n$ be integers where $3 \leq n \leq m$. Then \{4, 5, ..., $(n-1)(m-1)+1\} \subseteq CTS(P_n \square P_m)$.

**Proof.** We consider 4 cases depending on the parities of $m$ and $n$. 

40
Case 1: $n$ and $m$ are both even.

Give $G$ a proper 4-edge-coloring as follows. Assign $c_1$ to $(2i-1, 1)(2i, 1)$, $(2i-1, n)(2i, n)$, $(2i-1, 2j)(2i-1, 2j+1)$, and $(2i, 2j)(2i, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_2$ to $(1, 2j-1)(1, 2j)$, $(m, 2j-1)(m, 2j)$, $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i-1, 2j)(2i, 2j)$ and $(2i-1, 2j+1)(2i, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i, 2j)(2i, 2j)$, and $(2i+1, 2j-1)(2i+1, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below where $c_1$ is black, $c_2$ is red, $c_3$ is blue, and $c_4$ is green.

The above edge-coloring has the following mate coloring. Assign $c_1$ to $(i, 2j-1)(i, 2j)$ for $1 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_2$ to $(2i-1, 2j-1)(2i, 2j-1)$, and $(2i-1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i-1, 2j)(2i-1, 2j+1)$, and $(2i, 2j)(2i, 2j+1)$ for...
$1 \leq i \leq \frac{m}{2}$ and $1 \leq i \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j), (2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below.

Now, give $G$ a proper 5-edge-coloring as follows. Assign $c_1$ to $(2i-1, 2j-1)(2i-1, 2j)$, and $(2i, 2j-1)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_2$ to $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i-1, 2j)(2i, 2j)$, and $(2i-1, 2j+1)(2i, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i, 2j)(2i, 2j+1)$, and $(2i+1, 2j)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_5$ to $(2i-1, 1)(2i, 1), (2i-1, n)(2i, n), (1, 2j)(1, 2j+1)$, and $(m, 2j)(m, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-4}{2}$. We use the same assignment of colors as above and include $c_5$ as purple.
The above edge-coloring has the following mate coloring. Assign $c_1$ to $(2i - 1, 2j - 1)(2i, 2j - 1)$, and $(2i - 1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i, 2j)$, and $(2i + 1, 2j - 1)(2i + 1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i - 1, 2j + 1)$, and $(2i, 2j)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i, 2j)(2i + 1, 2j)$, and $(2i, 2j + 1)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_5$ to $(2i, 1)(2i + 1, 1)$, $(2i, n)(2i + 1, n)$, $(1, 2j - 1)(1, 2j)$, and $(m, 2j - 1)(m, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below.
Using the notation of Lemma 2.3, note that $H_1$ consists of $\frac{mn}{4}$ many 4-cycles, while $H_2$, $H_3$, and $H_4$ each consist of $\frac{n(m-2)}{4}$, $\frac{m(n-2)}{4}$, and $\frac{(m-2)(n-2)}{4}$ many 4-cycles respectively. $H_5$ consists of a single $2(m-1) + 2(n-1)$ cycle, so $\alpha = (n-1)(m-1) + 1$ and by Lemma 2.3 we conclude $\{4, ..., (n-1)(m-1) + 1\} \subseteq \text{CTS}(P_n \Box P_m)$.

Case 2: $n$ is even and $m$ is odd.

Give $G$ a proper 4-edge-coloring as follows. Assign $c_1$ to $(2i-1, 1)(2i, 1), (2i-1, m)(2i, m), (2i-1, 2j)(2i-1, 2j+1)$, and $(2i, 2j)(2i, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_2$ to $(1, 2j-1)(1, 2j), (2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i-1, 2j)(2i, 2j), (2i-1, 2j+1)(2i, 2j+1)$, and $(m, 2j)(m, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_4$ to $(i, 2j-1)(i, 2j)$ for $2 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below.
The above edge-coloring has the following mate coloring. Assign $c_1$ to $(2i-1,2j-1)(2i-1,2j)$, and $(2i,2j-1)(2i,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_2$ to $(2i-1,2j-1)(2i,2j-1)$, $(2i-1,2j)(2i,2j)$, and $(m,2j-1)(m,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(1,2j)(1,2j+1)$, $(2i,2j)(2i+1,2j)$, and $(2i,2j+1)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq i \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i,1)(2i+1,1)$, $(2i,n)(2i+1,n)$, $(2i,2j)(2i,2j+1)$, and $(2i+1,2j)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. An example is given below.
Now, give $G$ a proper 5-edge-coloring as follows. Assign $c_1$ to $(2i - 1, 2j - 1)(2i - 1, 2j)$, and $(2i, 2j - 1)(2i, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i, 2j)$, and $(2i - 1, 2j + 1)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_4$ to $(2i, 2j), (2i, 2j + 1)$, and $(2i + 1, 2j)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign $c_5$ to $(2i - 1, 1)(2i, 1), (2i - 1, n)(2i, n), (1, 2j)(1, 2j + 1), (m, 2j - 1)(m, 2j)$, and $(m, n - 1)(m, n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. An example is given below.
The above edge-coloring has the following mate coloring. Assign \( c_1 \) to \((2i - 1, 2j - 1)(2i, 2j - 1)\) and \((2i - 1, 2j)(2i, 2j)\) for \(1 \leq i \leq \frac{m-1}{2}\) and \(1 \leq j \leq \frac{n}{2}\). Assign \( c_2 \) to \((2i, 2j - 1)(2i, 2j)\), and \((2i + 1, 2j - 1)(2i + 1, 2j)\) for \(1 \leq i \leq \frac{m-1}{2}\) and \(1 \leq j \leq \frac{n}{2}\). Assign \( c_3 \) to \((2i - 1, 2j)(2i - 1, 2j + 1)\), and \((2i, 2j)(2i, 2j + 1)\) for \(1 \leq i \leq \frac{m-1}{2}\) and \(1 \leq j \leq \frac{n-2}{2}\). Assign \( c_4 \) to \((2i, 2j)(2i + 1, 2j)\), and \((2i, 2j + 1)(2i + 1, 2j + 1)\) for \(1 \leq i \leq \frac{m-1}{2}\) and \(1 \leq j \leq \frac{n-2}{2}\). Assign \( c_5 \) to \((2i, 1)(2i + 1, 1), (2i, n)(2i + 1, n), (1, 2j - 1)(1, 2j), (m, 2j)(m, 2j + 1),\) and \((1, n - 1)(1, n)\) for \(1 \leq i \leq \frac{m-1}{2}\) and \(1 \leq j \leq \frac{n}{2}\). An example is given below.
Using the notation of Lemma 2.3, note that $H_1$ and $H_2$ each consist of $\frac{(m-1)n}{4}$ many 4-cycles, while $H_3$ and $H_4$ each consist of $\frac{(m-1)(n-2)}{4}$ many 4-cycles. $H_5$ consists of a single $2(m-1)+2(n-1)$ cycle, so $\alpha = (n-1)(m-1)+1$ and by Lemma 2.3 we conclude $\{4,\ldots, (n-1)(m-1)+1\} \subseteq \text{CTS}(P_n \square P_m)$.

Case 3: $n$ is odd and $m$ is even.

Give $G$ a proper 4-edge-coloring as follows. Assign $c_1$ to $(2i-1,1)(2i,1), (2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(1,2j-1)(1,2j), (m,2j-1)(m,2j), (2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i-1,2j)(2i,2j)$, and $(2i-1,2j+1), (2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i,2j-1)(2i,2j), (2i+1,2j-1)(2i+1,2j)$, and $(2i,n)(2i+1,n)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
The above edge-coloring has the following mate coloring. Assign $c_1$ to $(2i - 1, 2j - 1)(2i - 1, 2j), (2i, 2j - 1)(2i, 2j)$, and $(2i - 1, n)(2i, n)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i - 1, 2j - 1)(2i, 2j - 1)$, and $(2i - 1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(1, 2j)(1, 2j + 1), (m, 2j)(m, 2j + 1), (2i, 2j)(2i + 1, 2j)$, and $(2i, 2j + 1)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq i \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 1)(2i + 1, 1), (2i, 2j)(2i, 2j + 1)$, and $(2i + 1, 2j)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
Now, give $G$ a proper 5-edge-coloring as follows. Assign $c_1$ to $(2i - 1, 2j - 1)(2i - 1, 2j)$, and $(2i, 2j - 1)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i+1, 2j - 1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i-1, 2j)(2i, 2j)$, and $(2i - 1, 2j + 1)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 2j)(2i, 2j + 1)$, and $(2i + 1, 2j)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_5$ to $(2i - 1, 1)(2i, 1), (2i, n)(2i+1, n), (1, 2j)(1, 2j + 1)$, and $(m, 2j - 1)(m, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
The above edge-coloring admits the following mate coloring. Assign $c_1$ to $(2i - 1, 2j - 1)(2i, 2j - 1)$, and $(2i - 1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i, 2j)$, and $(2i + 1, 2j - 1)(2i + 1, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i - 1, 2j + 1)$, and $(2i, 2j)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 2j)(2i + 1, 2j)$, and $(2i, 2j + 1)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign $c_5$ to $(2i, 1)(2i + 1, 1), (2i - 1, n)(2i, n), (1, 2j - 1)(1, 2j)$, and $(m, 2j)(m, 2j + 1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
Using the notation of Lemma 2.3, note that $H_1$ and $H_3$ each consist of $\frac{m(n-1)}{4}$ many 4-cycles, while $H_2$ and $H_4$ each consist of $\frac{(m-2)(n-2)}{1}$ many 4-cycles. $H_5$ consists of a single $2(m - 1) + 2(n - 1)$ cycle, so $\alpha = (n - 1)(m - 1) + 1$ and by Lemma 2.3 we conclude \{4, ..., $(n - 1)(m - 1) + 1$\} $\subseteq$ CTS($P_n \square P_m$).

Case 4: $n$ and $m$ are both odd.

Give $G$ a proper 4-edge-coloring as follows. Assign $c_1$ to $(2i - 1, 1)(2i, 1), (2i - 1, 2j)(2i - 1, 2j + 1)$, and $(2i, 2j)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(1, 2j - 1)(1, 2j), (2i, 2j - 1)(2i + 1, 2j - 1)$, and $(2i, 2j)(2i + 1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i, 2j), (2i + 1, 2j + 1)(2i + 1, 2j)$, and $(m, 2j)(m, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 2j - 1)(2i, 2j), (2i + 1, 2j - 1)(2i + 1, 2j)$, and $(2i, n)(2i + 1, n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
The above edge-coloring admits the following mate coloring. Assign $c_1$ to $(2i - 1, 2j - 1)(2i - 1, 2j), (2i, 2j - 1)(2i, 2j)$, and $(2i - 1, n)(2i, n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i-1, 2j-1)(2i, 2j-1), (2i-1, 2j)(2i, 2j)$, and $(m, 2j-1)(m, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(1, 2j)(1, 2j + 1), (2i, 2j)(2i + 1, 2j)$, and $(2i, 2j + 1)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 1)(2i + 1, 1), (2i, 2j)(2i, 2j + 1)$, and $(2i + 1, 2j)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
Now, give $G$ a proper 5-edge-coloring as follows. Assign $c_1$ to $(2i - 1, 2j - 1)(2i - 1, 2j)$, and $(2i, 2j - 1)(2i, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i + 1, 2j - 1)$, and $(2i, 2j)(2i + 1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i, 2j)$, and $(2i - 1, 2j + 1)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 2j)(2i, 2j + 1)$, and $(2i + 1, 2j)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_5$ to $(2i - 1, 1)(2i, 1)$, $(2i - 1, n)(2i, n)$, $(1, 2j)(1, 2j + 1)$, and $(m, 2j - 1)(m, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
The above edge-coloring admits the following mate coloring. Assign $c_1$ to $(2i - 1, 2j - 1)(2i, 2j - 1)$, and $(2i - 1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_2$ to $(2i, 2j - 1)(2i, 2j)$, and $(2i + 1, 2j - 1)(2i + 1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_3$ to $(2i - 1, 2j)(2i - 1, 2j + 1)$, and $(2i, 2j)(2i, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_4$ to $(2i, 2j)(2i + 1, 2j)$, and $(2i, 2j + 1)(2i + 1, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign $c_5$ to $(2i, 1)(2i + 1, 1)$, $(2i - 1, n)(2i, n)$, $(1, 2j - 1)(1, 2j)$, and $(m, 2j)(m, 2j + 1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.
Using the notation of Lemma 2.3, note that $H_1, H_2, H_3$ and $H_4$ each consist of $\frac{(m-1)(n-1)}{4}$ many 4-cycles. $H_5$ consists of a single $2(m - 1) + 2(n - 1)$ cycle, so $\alpha = (n - 1)(m - 1) + 1$ and by Lemma 2.3 we conclude $\{4, \ldots, (n - 1)(m - 1) + 1\} \subseteq \text{CTS}(P_n \Box P_m)$.

Note there is a possible gap in these spectra, since for a given graph $G$, the maximum value of its color-trade-spectrum is $\left\lfloor \frac{|E(G)|}{2} \right\rfloor$. In particular, the gap consists of $\left\lfloor \frac{n(m-1)+m(n-1)}{2} \right\rfloor - (n - 1)(m - 1) + 1 = \left\lfloor \frac{m+n}{2} \right\rfloor - 2$ possible values. We conjecture that $\{4, \ldots, (n - 1)(m - 1) + 1\}$ is indeed the entire color-trade-spectrum of $P_n \Box P_m$. 

□
Chapter 6

Complete Graphs

Denote by $K_n$ the complete graph on $n$ vertices with $\frac{n(n-1)}{2}$ edges. By Lemma 2.1, $K_1$ and $K_2$ have empty color trade spectra since they contain vertices of degree one. Since $K_3$ is an odd cycle, it also has an empty color-trade-spectrum by Lemma 2.2. For $K \geq 4$, the color-trade-spectrum of $K_n$ is bounded below by $\chi'(K_n)$, which is $n-1$ when $n$ is even, and $n$ when $n$ is odd. The color-trade-spectrum is bounded above by $\lfloor \frac{n(n-1)}{4} \rfloor$. We first consider the case where $n \equiv 0 \mod 8$.

6.1 $n \equiv 0 \mod 8$

**Theorem 6.1.** Let $n$ be a positive integer such that $n \equiv 0 \mod 8$. Then $\{n-1, n, \ldots, n\frac{(n-1)}{4}\} = \text{CTS}(K_n)$.

**Proof.** Let $V(K_n) = \{v_{i,j} | 0 \leq i \leq \frac{n}{4} - 1, 1 \leq j \leq 4\}$ and denote by $(v_{i,j}, v_{h,k})$ the edge between vertex $v_{i,j}$ and vertex $v_{h,k}$. For each $i$, consider the vertices $v_{i,1}, v_{i,2}, v_{i,3},$ and $v_{i,4}$ and the induced $K_4$ subgraph induced by these vertices. In total, this yields $m = \frac{n}{4}$ many disjoint $K_4$ subgraphs. Let $mK_4$ denote the complete $m$-partite graph where each part contains four vertices. Then $K_n$ consists of $m$ many $K_4$s and a singular $mK_4$. We now find a $C_4$ decomposition of $mK_4$.

The induced $K_{4,4}$ on the vertices $\{v_{s,i}, v_{t,j} | s, t \in \{0, 1, \ldots, \frac{n}{4} - 1\}, s \neq t, i, j \in \{1, 2, 3, 4\}\}$ consists of the following $C_4$s: $(v_{s,1}, v_{t,1}, v_{s,2}, v_{t,2}), (v_{s,3}, v_{t,3}, v_{s,4}, v_{t,4}), (v_{s,1}, v_{t,3}, v_{s,2}, v_{t,4}),$ and $(v_{s,3}, v_{t,1}, v_{s,4}, v_{t,2})$. In particular, the first two cycles are disjoint, as are the last two cycles, which we will refer to as $T_1$ and $T_2$ cycles respectively. With the $K_4$s from above, this yields a
\((K_4, C_4)\) decomposition of \(K_n\). We give an example of the \((K_4, C_4)\) decomposition of \(K_8\) below where each \(K_4, T_1\) cycle, and \(T_2\) cycle, is colored black, red, and blue respectively.

Next, we consider a new graph, \(K_m\). In particular, let \(V(K_m) = \{v_\infty, v_i | 0 \leq i \leq m - 2\}\) where each vertex of \(K_m\) corresponds to the contraction of the \(K_4\) induced by the vertices \(\{v_{i,j} | 1 \leq j \leq 4\}\) from \(K_n\), and \(v_\infty\) corresponds to the contraction of the \(K_4\) for \(i = m - 1\). Likewise, an edge in \(K_m\) will correspond to the four \(C_4s\) between the associated \(K_4s\). Consider the following 1-factorization of \(K_m\) where addition is under modulo \(m - 1\):
\[
\{(v_\infty, v_d), (v_{d+i}, v_{d-i}) | 0 \leq d \leq m-2, 1 \leq i \leq \frac{m}{2} - 1\}
\]
In particular, each value of \(d\) yields a unique 1-factor. We give an example of the \(K_m\) associated with \(K_{16}\) below where the central vertex is \(v_\infty\) and each 1-factor is represented by one of black, red, or blue.
The 1-factorization above yields a proper \((m - 1)\)-edge-coloring of \(K_m\) where the edges in the \(d\)th 1-factor are colored \(c_d\) for \(1 \leq d \leq m - 1\). Furthermore, this yields a \(4(m - 1) = n - 4\)-edge-coloring of the edges of the \(mK_4\) in \(K_n\) by alternating between colors \(c_{4d}\) and \(c_{4d+1}\) in the associated \(T_1\) cycles and by alternating between colors \(c_{4d+2}\) and \(c_{4d+3}\) in the associated \(T_2\) cycles for each value of \(d\). Finally, we assign \(c_{n-4}\) to edges of the form \((v_{i,1}, v_{i,2}), (v_{i,3}, v_{i,4})\), \(c_{n-3}\) to \((v_{i,1}, v_{i,3}), (v_{i,2}, v_{i,4})\), and \(c_{n-2}\) to \((v_{i,1}, v_{i,4}), (v_{i,2}, v_{i,3})\). In total, this yields a \(n - 1\)-edge-coloring of \(K_n\).

To find a mate coloring, we alternate the colors in each \(C_4\), and apply a 3-cycle permutation to the color classes \(c_{n-4}, c_{n-3},\) and \(c_{n-2}\). Now, consider the subgraph \(H_i\) as mentioned in Lemma 2.3. Note that each color class consists of \(\frac{n}{4}\) many \(C_4\)s. Since we have a \(n - 1\)-edge-coloring, this yields a total of \(\frac{n(n-1)}{4}\) cycles, so \(\{n - 1, n, ..., \frac{n(n-1)}{4}\} \subseteq \text{CTS}(K_n)\) for \(n \equiv 0 \mod 8\). Since this set of values includes all possible values for the color-trade-spectrum of \(K_n\) for even \(n\), this proves the claim. We give an example of these edge-colorings for \(K_8\) below, where the top vertex is \(v_{0,1}\) and the vertices are labeled in clockwise order so the bottom vertex would be \(v_{1,1}\).
Theorem 6.2. Let $n$ be a positive integer such that $n \equiv 4 \mod 8$. Then \(\{n-1, n, \ldots, \frac{n(n-1)}{4}\} = \text{CTS}(K_n)\).

Proof. We use a modified version of the construction from the $n \equiv 0 \mod 8$ case. Let $V(K_n) = \{v_{i,j}|0 \leq i \leq \frac{n}{2} - 1, 1 \leq j \leq 2\}$ and denote by $(v_{i,j}, v_{h,k})$ the edge between vertex $v_{i,j}$ and vertex $v_{h,k}$. Let $m = \frac{n}{4}$ and consider the graph $K_{2m}$ where $V(K_{2m}) = \{v_{\infty}, v_i|0 \leq i \leq 2m - 2\}$ such that $v_{\infty}$ corresponds to the vertices $v_{\frac{n}{2} - 1,1}$ and $v_{\frac{n}{2} - 1,2}$ while $v_i$ corresponds to the vertices $v_{i,1}$ and $v_{i,2}$.

Consider the following 1-factorization of $K_{2m}$ where addition is under modulo $2m - 1$: \{(v_{\infty}, v_d), (v_{d+i}, v_{d-i})|0 \leq d \leq 2m - 2, 1 \leq i \leq m - 1\}. In particular, each value of $d$ yields a unique 1-factor, and we create a $(K_4, C_4)$ decomposition of $K_n$ from $K_{2m}$ as follows. For the 1-factor where $d = 0$, we create $K_4$s on the vertices $v_{i,1}, v_{i,2}, v_{-i,1}$, and $v_{-i,2}$ for $1 \leq i \leq m - 1$, and we also create a $K_4$ on the vertices $v_{2m-1,1}, v_{2m-1,2}, v_{0,1}$, and $v_{0,2}$. For the other 1-factors where $1 \leq d \leq 2m - 2$, we create a $C_4$ on the vertices $(v_{d+i,1}, v_{d-i,1}, v_{d+i,2}, v_{d-i,2})$ for $1 \leq i \leq m - 1$, along with the $C_4$ on the vertices $(v_{2m-1,1}, v_{d,1}, v_{2m-1,2}, v_{d,2})$. An example of the $K_{2m}$ associated with $K_{12}$ is given below where the central vertex is $v_{\infty}$ and each 1-factor is represented by one of black, red, blue, green, or purple, along with an associated partial (for readability) $(K_4, C_4)$ decomposition of $K_{12}$ where we include the $K_4$s and $C_4$s which contain the vertices $v_{5,1}$ and $v_{5,2}$.
We color $K_n$ as follows. For $1 \leq d \leq 2m - 2$, alternate between colors $c_{2d - 1}$ and $c_{2d}$ in the cycles coming from the associated 1-factor in $K_{2m}$. For $d = 0$, we assign $c_{n - 3}$ to edges of the form $(v_i, 1, v_{-i}, 1)$ and $(v_{-i}, 1, v_i, 1)$, $c_{n - 2}$ to edges of the form $(v_i, 2, v_{-i}, 2)$ and $(v_{-i}, 2, v_i, 2)$, and $c_{n - 1}$ to edges of the form $(v_i, 1, v_{-i}, 2)$ and $(v_{-i}, 2, v_i, 1)$. In total, this yields a $n - 1$-edge-coloring of $K_n$.

The same argument from the $n \equiv 0 \pmod{8}$ case shows how to find a mate, and that $\{n - 1, n, \ldots, \frac{n(n-1)}{4}\} = \text{CTS}(K_n)$ for $n \equiv 4 \pmod{8}$, proving the claim. An example of these edge-colorings is shown below applied to the partial $(K_4; C_4)$ decomposition of $K_{12}$ from above.
6.3 $n \equiv 1 \mod 8$

**Theorem 6.3.** Let $n$ be a positive integer such that $n \equiv 1 \mod 8$. Then \(\{2n, 2n+1, ..., \frac{n(n-1)}{4}\} \subseteq CTS(K_n)\) for $n \geq 17$, and \(\{14, 15, ..., 18\} \subseteq CTS(K_9)\).

**Proof.** Let $V(K_n) = \{v_i | 0 \leq i \leq n-1\}$ and consider the $K_4$ subgraphs consisting of the edges $(v_{8j+1}, v_{8j+2}), (v_{8j+1}, v_{8j+5}), (v_{8j+1}, v_{8j+6}), (v_{8j+2}, v_{8j+5}), (v_{8j+2}, v_{8j+6})$, and $(v_{8j+5}, v_{8j+6})$, along with the $K_4$ subgraphs consisting of the edges $(v_{8j+3}, v_{8j+4}), (v_{8j+3}, v_{8j+7}), (v_{8j+3}, v_{8j+8})$, $(v_{8j+4}, v_{8j+7}), (v_{8j+4}, v_{8j+8})$, and $(v_{8j+7}, v_{8j+8})$ for $j \in \{0, 1, ..., \frac{n-1}{8} - 1\}$. For any two $K_4$s with the above form on the same index $j$, along with $v_0$, we have a $C_4$ decomposition of a $K_9$ with the following nine $C_4$s: $(v_0, v_{8j+1}, v_{8j+2}, v_{8j+3}), (v_0, v_{8j+2}, v_{8j+4}, v_{8j+5}), (v_0, v_{8j+5}, v_{8j+1}, v_{8j+6}), (v_0, v_{8j+6}, v_{8j+1}, v_{8j+7}), (v_0, v_{8j+7}, v_{8j+1}, v_{8j+5}), (v_0, v_{8j+4}, v_{8j+2}, v_{8j+5}), (v_0, v_{8j+5}, v_{8j+2}, v_{8j+6}), (v_0, v_{8j+6}, v_{8j+2}, v_{8j+3}), (v_0, v_{8j+3}, v_{8j+2}, v_{8j+4}), (v_0, v_{8j+4}, v_{8j+7}, v_{8j+5}), (v_0, v_{8j+5}, v_{8j+7}, v_{8j+6})$, and $(v_0, v_{8j+7}, v_{8j+5}, v_{8j+6})$. Below is an example of $K_{17}$ under the $(v_0, K_4)$ decomposition listed above where the central vertex is $v_0$ and the other sets of vertices follow a clockwise pattern. Following this is an example of a $C_4$ decomposition of a singular $K_9$ where the leftmost vertex is $v_0$. 

![Diagram of K_17 and singular K_9 with C_4 decompositions.]

62
As in the $n \equiv 0 \mod 4$ case, we can form a $K_{4,4}$ between any two such $K_4$ subgraphs as described above (assuming $n \geq 17$), which we can partition into four pairwise disjoint $C_4$ subgraphs. If the two $K_4$s belong to the same $K_9$ as described above, the edges of the $K_{4,4}$ have already been used in the $C_4$ decomposition of the corresponding $K_9$. Otherwise, the $K_4$s belong to disjoint $K_9$s, so the $K_{4,4}$ between them decomposes into four $C_4$s. Since our construction contains two $K_4$s in each $K_9$, this gives us four different $K_{4,4}$s for each pair of $K_9$s, for a total of sixteen $C_4$s for each pair of $K_9$s as follows:

\[
(v_{8i+1}, v_{8j+1}, v_{8i+2}, v_{8j+2}), (v_{8i+1}, v_{8j+3}, v_{8i+5}, v_{8j+4}), (v_{8i+2}, v_{8j+1}, v_{8i+3}, v_{8j+2}), (v_{8i+4}, v_{8j+1}, v_{8i+3}, v_{8j+2}),
\]

\[
(v_{8i+3}, v_{8j+3}, v_{8i+4}, v_{8j+4}), (v_{8i+3}, v_{8j+3}, v_{8i+4}, v_{8j+4}), (v_{8i+5}, v_{8j+1}, v_{8i+6}, v_{8j+2}), (v_{8i+5}, v_{8j+1}, v_{8i+6}, v_{8j+2}),
\]

\[
(v_{8i+5}, v_{8j+3}, v_{8i+6}, v_{8j+4}), (v_{8i+5}, v_{8j+3}, v_{8i+6}, v_{8j+4}), (v_{8i+5}, v_{8j+3}, v_{8i+6}, v_{8j+4}), (v_{8i+5}, v_{8j+3}, v_{8i+6}, v_{8j+4}),
\]

\[
(v_{8i+7}, v_{8j+1}, v_{8i+8}, v_{8j+2}), (v_{8i+7}, v_{8j+3}, v_{8i+8}, v_{8j+4}), (v_{8i+7}, v_{8j+5}, v_{8i+8}, v_{8j+6}), (v_{8i+7}, v_{8j+5}, v_{8i+8}, v_{8j+6}),
\]

and

\[
(v_{8i+7}, v_{8j+7}, v_{8i+8}, v_{8j+8})
\]

for $i, j \in \{1, 2, \ldots, \frac{n-1}{8}\}$ where $i \neq j$. In total, this yields a $9\left(\frac{n-1}{4}\right) + 16\left(\frac{n-1}{2}\right) = \frac{n(n-1)}{8}$ $C_4$ decomposition of $K_n$. A partial example for $K_{17}$ is shown below.
Referencing the \( n \equiv 0 \mod 8 \) case, note that the sixteen \( C_4 \)s from the previous paragraph can be partitioned into four sets of disjoint cycles, which we again refer to as \( T_1, T_2, T_3, \) and \( T_4 \) type cycles respectively. In particular, we will use the following assignment of the cycles from above. \( T_1: \) 1st, 6th, 11th, 16th. \( T_2: \) 2nd, 7th, 12th, 13th. \( T_3: \) 3rd, 8th, 9th, 14th. \( T_4: \) 4th, 5th, 10th, 15th. As in the \( n \equiv 0 \mod 8 \) case, we consider \( m = \frac{n-1}{4} \) along with a new graph \( K_m \) where vertex \( v_{2i} \) in \( K_m \) corresponds to the contraction of the \( K_4 \) induced by vertices \( v_{8i+1}, v_{8i+2}, v_{8i+5}, \) and \( v_{8i+6} \) in \( K_n \) and where vertex \( v_{2i+1} \) in \( K_m \) corresponds to the contraction of the \( K_4 \) induced by the vertices \( v_{8i+3}, v_{8i+4}, v_{8i+7}, \) and \( v_{8i+8} \) in \( K_n \), for \( 0 \leq i \leq \frac{n-1}{8} - 1 \). As before, an edge between vertices in \( K_m \) corresponds to the four \( C_4 \)s between the associated \( K_4 \)s in \( K_n \). However, some of these edges were used in the \( C_4 \) decomposition of the \( K_9 \)s, and this corresponds to removing a 1-factor from our \( K_m \), denoted \( K_m - F_1 \). An example of \( K_m - F_1 \) for \( K_{17} \) is given below.

As in the \( n \equiv 0 \mod 8 \) case, a 1-factorization of \( K_m - F_1 \) yields a proper \((m - 2)\)-edge-coloring of \( K_m - F_1 \), which further corresponds to a \( 4(m - 2) = n - 9 \)-edge-coloring of the edges between the \( K_4 \)s by alternating between colors \( c_{8k+2p-2} \) and \( c_{8k+2p-1} \) in the associated \( T_p \).
cycles for $0 \leq k \leq \frac{n-1}{8} - 1$ and $1 \leq p \leq 4$, again assuming that $n \geq 17$. Next, note that within each $K_9$, the 1st and 9th cycles from the $C_4$ decomposition of $K_9$ above are disjoint, as are the 3rd and 7th cycles. Furthermore, cycles five through nine are vertex disjoint for each pair of $K_9$s in $K_n$. In total, we need eight colors for each copy of $K_9$ to account for the edges incident to $v_0$, ten colors which can be shared among all copies of $K_9$ among the cycles which are not incident to $v_0$, and $n - 9$ colors for the edges between the $K_9$s, for a total of $8\left(\frac{n-1}{8}\right) + 10 + n - 9 = 2n$ colors to properly color the edges of $K_n$ for $n \geq 17$. We find a mate for this edge-coloring by reversing the assignment of colors within each cycle as in the previous cases, and this process shows that $\{2n, 2n + 1, ..., \frac{n(n-1)}{4}\} \subseteq CTS(K_n)$ for $n \equiv 1 \mod 8$ when $n \geq 17$. For $K_9$, the $K_9$ decomposition alone shows that $\{14, 15, ..., 18\} \subseteq CTS(K_9)$. An example of the proper $m - 2$-edge-coloring of the associated $K_m - F - 1$ for $K_{25}$ is shown below.

It should be noted that our lower bound is not tight. Taking $K_{17}$ as an example, the colors assigned to the cycle $(0, 1, 2, 3)$ in the first $K_9$ copy could also be used among the cycle $(3, 11, 4, 12)$ between the $K_9$ copies. There are other examples of disjoint cycles where colors could be reused, so our lower bound can indeed be decreased. Furthermore, it remains to be determined if our construction yields the highest amount of disjoint cycles, although the answer to this question should not require extensive work.
Chapter 7

Further Work

There are many ways in which this work could be continued. Since the color-trade-spectrum of complete bipartite graphs was determined in this dissertation, a natural next step would be to determine the color-trade-spectrum of complete multipartite graphs. Perhaps the usage of Latin rectangles could be extended to higher dimensions.

The color-trade-spectrum of $P_n \square P_m$ remains incomplete. As mentioned at the end of chapter 5, we conjecture \{4, 5, ..., $(n - 1)(m - 1) + 1$\} is indeed the entire color-trade-spectrum of $P_n \square P_m$ for $3 \leq n \leq m$, but it remains to show that none of the other $\lfloor \frac{m+n}{2} \rfloor - 2$ possible values are in the color-trade-spectrum.

Likewise, the color-trade-spectrum for $K_n$ remains incomplete. As shown in chapter 6, \$\{2n, 2n + 1, ..., \frac{n(n-1)}{4}\} \subseteq CTS(K_n)$ for $n \geq 17$, but we also showed an example where the lower bound is not tight. Furthermore, the cases of $n \equiv 2, 3, 5, 6, 7 \mod 8$ are ripe for exploration.

In this dissertation, the color-trade-spectra of 2-regular graphs (cycles) and complete graphs (which are $n - 1$-regular on $n$ vertices) were explored. Further work could be done in exploring the color-trade-spectra of generic $k$-regular graphs. Extensions of this could lead to studying cages, snarks, and strongly-regular graphs.

Likewise, the color-trade-spectrum of any family of graphs is open for discovery. Some interesting graphs for which the color-trade-spectrum is unknown includes the Platonic graphs, the Heawood graph, and the Petersen graph.
References


