Color Trades on Graphs

by

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Abstract

Edge-colorings of graphs have a rich history and are widely studied. Trade spectra of graphs are relatively new and ripe for study. The color-trade-spectrum of a graph G is defined to be the set of all t for which there exist two proper edge-colorings of G using t colors such that each vertex of G is incident to the same set of colors under each edge-coloring while each edge receives a different color under each edge-coloring. We show some general results and present various constructions which are used to determine the color-trade-spectrum of several families of graphs.

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Chapter 1

Introduction

We first give a brief history of edge-colorings and trades on graphs. Edge-colorings of graphs can be traced back to the four-color problem, first posed in 1852 by Francis Guthrie, which asks if every map can be colored with four colors so that adjacent countries are colored differently. The first “proof” of this was produced by Alfred Kempe in 1879 [12]. While his arguments were incorrect, he introduced the idea of *Kempe chains*, which remain a key ingredient in edge-coloring theory. Later in the 19th century, Peter Tait attempted to improve Kempe’s arguments and in the process introduced *Tait colorings* [18].

In the early 20th century, Dénes König showed that a graph is bipartite if and only if every cycle has even length, and then showed that every k-regular bipartite graph can be partitioned into 1-factors, which are sets of disjoint edges that meet all the vertices, by using Kempe chains. In the middle of the 20th century, Claude Shannon showed that the wires of any network can be properly colored, meaning no color appears more than once at a node in a network, with $\lfloor \frac{3m}{2} \rfloor$ colors, where m is the largest number of wires at a point [15]. An equivalent statement is the minimum number of colors needed to properly edge-color a graph is between Δ and $\lfloor \frac{3\Delta}{2} \rfloor$, where Δ is the maximum degree of the graph, which is the maximum number of edges appearing at a vertex in the graph.

In 1964, Vadim Vizing proved that the minimum number of colors required to properly edge-color a graph, the chromatic index denoted by $\chi'(G)$, is no more than $\Delta + u$ where u is the maximum number of repeated (or parallel) edges between any two vertices in the graph [19]. In particular, if G is simple, meaning it has no parallel edges or loops (edges connecting a vertex to itself), then $\chi'(G) \in \{\Delta, \Delta + 1\}$. This led to classifying simple graphs as Class 1 or Class 2, depending on whether their chromatic index is Δ or $\Delta + 1$ respectively. Classifying graphs as Class 1 or 2 is an NP-complete problem [11], but has a rich history. One possibility is to consider families of graphs whose cores have simple structure [6][7][8][10].

Work on trades in design theory originated in the 1960s [9] although the idea behind trades was used as early as 1916 [20] to construct Steiner triple systems. Trades reflect possible differences between two combinatorial objects of the same type (Steiner systems, latin squares, etc.). The original use of a trade as defined by Hedayat [9] was to avoid some undesirable blocks in an experimental design while retaining the same parameter set and variety set. Some other uses of trades include the following: solving *intersection problems* for two combinatorial structures, creating *defining sets* for block designs [16], creating *critical sets* for partial latin squares, creating designs with different *support* sizes which is used in statistical applications of designs [9], constructing t-designs, and constructing *irreducible* designs. Much of these applications are considered in the surveys by Billingon [2] and Khosrovshahi [13].

The basic idea behind a trade is to partition an object into subsets satisfying some list of properties. If we can partition the object into a different set of subsets which still satisfy the same list of properties, we say the partitions form a trade. Formally, a (k, t) trade of volume m and foundation size v (sometimes referred to as a (v, k, t) trade), is a pair $\{T_1, T_2\}$ of sets of subsets of size k based on a v-set, such that T_1 and T_2 each contain m subsets of size k, with $T_1 \cap T_2 = \emptyset$, and so that each t-set chosen from the v-set occurs exactly the same number of times in the blocks of T_1 as it does in the blocks of T_2. A trade is called *Steiner* if each t-set occurs at most once in each T_i, and a trade is called *simple* if there are no repeated blocks in T_1 or in T_2.

While there are a variety of types of trades, four kinds appear most often in the literature; trades in designs, latin squares, graphs, and trades derived from "latin representations". The trades described above are most often used in design theory, where trades are allowed to contain repeated subsets, often called blocks. A *latin trade* is a pair $\{L_1, L_2\}$ of partial latin rectangles with precisely the same m filled cells such that: 1. L_1 and L_2 contain different elements in each filled cell (i, j); 2. in each occupied row i, L_1 and L_2 contain set-wise the same symbols; 3. in each occupied column j, L_1 and L_2 contain set-wise the same symbols. In graph theory, the trade-spectrum of a graph G is the set of all integers t for which there is a graph H for which two G-decompositions of H exist; this means the edges of H can be partitioned in two unique ways into t copies of G with no copies of G in common [3]. Some uses of trades from latin representations are addressed by Billington and Cavenagh [5][4]. Most previous work on trades has involved 2-way trades, but some work has been done on μ-way trades [1]. In this dissertation, we consider a synthesis of edge-colorings and trades, called *color trades*.

We now give a brief summary of the contents of this dissertation. First, we cover the preliminaries in chapter 2, which includes formally defining color trades, mate colorings, and the color-trade-spectrum of a graph. In the preliminaries section we also list some general lemmas, such as determining lower and upper bounds of the color-trade-spectrum of a graph in Lemma 2.1. Additionally, we determine the color-trade-spectrum of all cycles in Lemma 2.2, and use this in Lemma 2.3 in which we use cycle decompositions to create a general method to find a subset of the color-trade-spectrum of a graph. We then define a new graph G' in Lemma 2.4 based on two mate colorings of a given graph, and use G' to find a subset of the color-trade-spectrum of the original graph. We conclude the preliminaries section with Lemma 2.5, which details a way to determine if two edge-colorings are mate colorings by considering a new graph $M(G, k)$. In particular, two edge-colorings of a graph G are mate colorings if and only if the corresponding $M(G, k)$ is bipartite.

In chapter 3, we determine the color-trade-spectrum of three small families of graphs, namely Theta, Wheel, and n-cube graphs. Theta and Wheel graphs are simple modifications of cycles, so the methods for finding the color-trade-spectrum of cycles in Lemma 2.2 is used again, and explicit constructions are given for both families of graphs. The color-trade-spectrum of trivial cases of Theta graphs is determined in Lemma 3.1, and the color-trade-spectrum in full is determined in Theorem 3.2. The color-trade-spectrum for Wheel graphs is determined in Theorem 3.2. The color-trade-spectrum of trivial cases of n-cube graphs is determined in Lemma 3.2, and in Theorem 3.3 the full color-trade-spectrum for n-cube graphs is determined using a recursive construction based on the well known fact that the $(n + m)$-cube graph is isomorphic to the Cartesian product of the n-cube and m-cube graphs.

In chapter 4, we determine the color-trade-spectrum of complete bipartite graphs and list explicit constructions for attaining each value in the color-trade-spectrum. We make frequent use of Latin rectangles in the constructions, where we define and use row and column blocks of Latin rectangles. In Theorem 4.1, for a complete bipartite graph $K_{m,n}$ under a proper Δ-edge-coloring with m even, we list a specific construction for the corresponding Latin rectangle L_Δ which we then use to create a new Latin rectangle $\pi(L_\Delta)$ which corresponds to a mate coloring for the original edge-coloring of $K_{m,n}$. From this, we identify a 4-cycle decomposition of $K_{m,n}$ on which we use Lemma 2.3 to attain the entire color-trade-spectrum of $K_{m,n}$. We deal with $K_{3,n}$ as a special case in Lemma 4.3 and use this to create modified constructions of L_Δ and $\pi(L_\Delta)$ in Theorem 4.2. From this, we identify a cycle decomposition of $K_{m,n}$ for odd m on which we again use Lemma 2.3 to attain the entire color-trade-spectrum of $K_{m,n}$.

In chapter 5, we determine a subset of the color-trade-spectrum for Cartesian products of paths $P_m \square P_n$ and fully determine the color-trade-spectrum for $P_2 \square P_m$. In Lemma 5.1, we list specific constructions for proper Δ-edge-colorings of $P_2 \square P_m$ depending on the parity of m along with the associated mate colorings. From this, we identify cycle decompositions of $P_2 \square P_m$ on which we again use Lemma 2.3 to achieve a subset of the color-trade-spectrum. In Theorem 5.1, we show the subset of the color-trade-spectrum from Lemma 5.1 is indeed the entire color-trade-spectrum of $P_2 \square P_m$. In Theorem 5.2, we list specific constructions for proper 4-edge-colorings and proper 5-edge-colorings of $P_n \square P_m$ depending on the parities of n and m, along with their mate colorings. From the proper 5-edge-colorings, we identify cycle decompositions of $P_n \square P_m$ on which we use Lemma 2.3 to attain a subset of the color-trade-spectrum of $P_n \square P_m$. We conjecture that the subset from 5.2 is indeed the entire color-trade-spectrum.

In chapter 6, we fully determine the color-trade-spectrum for complete graphs K_n where $n \equiv 0 \mod 8$ and $n \equiv 4 \mod 8$. In Theorem 6.1, we identify a (K_4, C_4) decomposition of K_n for $n \equiv 0 \mod 8$ by first finding a K_4 and $\frac{n}{4}K_4$ decomposition, where $\frac{n}{4}K_4$ is the complete $\frac{n}{4}$-partite graph where each part contains four vertices. From this, we consider a new graph K_m which we use to find a (K_4, C_4) decomposition of $\frac{n}{4}K_4$. Using this decomposition, we present a Δ-edge-coloring of K_n along with its mate coloring, and use Lemma 2.3 to determine the color-trade-spectrum of K_n. In Theorem 6.2, we modify the process of Theorem 6.1 to find a (K_4, C_4) decomposition of K_n for $n \equiv 4 \mod 8$ where we consider a modified K_m, namely K_{2m}. Again, we present a Δ-edge-coloring of K_n along with its mate coloring, and use Lemma 2.3 to determine the color-trade-spectrum of K_n. In Theorem 6.3, we again modify the process of Theorem 6.1 to find a C_4 decomposition of K_n for $n \equiv 1 \mod 8$. We present a $2n+4$-edge-coloring of K_n for $n \geq 17$ along with its mate coloring, and use Lemma 2.3 to partially determine the color-trade-spectrum of K_n.

Chapter 2

Preliminaries

A *graph* is a pair $G = (V, E)$ where V is a set of vertices and E is a multiset of paired vertices, called edges. If $u \in V$ and $(u, u) \in E$, we say the edge joining u to itself is a loop. For $u, v \in V$, we say the number of times (u, v) appears in E is the edge-multiplicity of (u, v). We say a graph G is *simple* if G contains no loops and the maximum edge-multiplicity of G is 1. We assume all graphs from this point onward are simple unless otherwise stated. If $(u, v) \in E$, we say this edge is *incident* to vertices u and v. If $e_1, e_2 \in E$ are incident to a common vertex $v \in V$, we say e_1 and e_2 are *adjacent* edges.

A k-$vertex$-$coloring$ of a graph G is an assignment of "colors" to the vertices of G. More formally, $\varphi : V(G) \to C$ is a k-vertex-coloring of G where $|C| = k$. A vertex-coloring is *proper* if no two adjacent vertices share a common color. Formally, for two vertices u and v in $V(G)$ where $u \neq v$, $\varphi(u) \neq \varphi(v)$. The minimum number of colors needed to properly vertex-color a graph G is known as the *chromatic number* of G, denoted by $\chi(G)$. In studying edge-colorings, we often consider the degree of a vertex, which is the number of edges incident to a vertex in a loopless graph. The minimum degree of a graph G, denoted $\delta(G)$, is the degree of the vertex with the least number of edges incident to it in G. Likewise, the maximum degree of G, denoted $\Delta(G)$, is the degree of the vertex with the greatest number of edges incident to it in G.

Analogously, a k-$edge$-$coloring$ of a simple graph G is an assignment of "colors" to each edge of G. More formally, $\Phi : E(G) \to C$ is a k-edge-coloring of G where $|C| = k$. An edge-coloring is *proper* if no two adjacent edges share a common color. Formally, for any vertex $v \in V(G)$ and any pair of edges incident to v, $u_k v$ and $u_j v$ where $k \neq j$, $\Phi(u_k, v) \neq \Phi(u_j, v)$.

The minimum number of colors needed to properly edge-color a graph G is known as the *chromatic index* of G, denoted by $\chi^{'}(G)$.

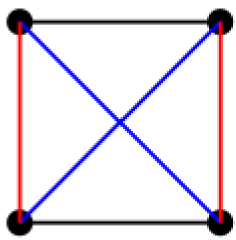
Let G be a simple graph under a proper t-edge-coloring C_1. We say an edge-coloring C_2 of G is a *mate coloring* of C_1 if and only if the following conditions are true:

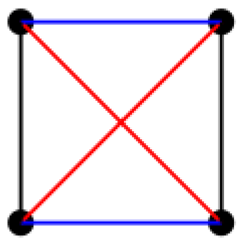
1. For every $v \in V(G)$, the set of colors assigned to edges incident to v under C_1 is the same as the set of colors assigned under C_2

2. For every $e \in E(G)$, the color assigned to e under C_1 is different than the color assigned under C_2.

Clearly, C_1 and C_2 must have the same cardinality, and in fact must consist of the same set of colors. We define the *color-trade-spectrum* of a graph $G, CTS(G)$, to be the set of all t for which there exist two mate colorings of G using t colors.

Consider a graph G with n vertices, and where every pair of vertices is joined via an edge. Then G is the *complete graph on n vertices*, denoted by K_n, sometimes called the *clique on* n *vertices*. An example of two mate colorings using three colors for K_4 is given below, showing that $3 \in \text{CTS}(K_4)$.





Next, we list some elementary observations about mate colorings and color trade spectra.

Lemma 2.1. *Let G be a simple graph with chromatic index $\chi^{'}(G)$.*

1. If G contains a vertex of degree one, then CTS$(G) = \emptyset$.

2. In a mate coloring, each color must be assigned to at least two edges of G.

3. $\chi^{'}(G) \leq$ min CTS(G) and max CTS(G) $\leq \lfloor \frac{|E|}{2} \rfloor$.

Proof. 1: If G contains a vertex of degree one, say u, then CTS$(G) = \emptyset$ as there is no way for two edge-colorings to both make u incident to the same color without assigning the same color to the singular edge incident to u.

2: Let C_1 be a proper edge-coloring applied to G such that only a single edge (u, v) is colored c_1. For contradiction, suppose C_2 is a mate coloring of C_1. Then there must exist other vertices u' and v' such that (u, u') and (v, v') are colored c_1 under C_2. This means u' and v' must be incident to edges colored c_1 in C_1, contradicting our assumption that u and v are the only vertices incident to edges colored c_1 under C_1. Thus, any color in a mate coloring must be assigned to at least two edges in G.

3: Since mate colorings are proper edge-colorings, it immediately follows that $\chi^{'}(G) \leq$ min{CTS(G)}. By observation 2, each color in a mate coloring must be assigned to at least two edges in G, so max{CTS(G)} $\leq \lfloor \frac{|E|}{2} \rfloor$.

□

A *walk* is an alternating sequence of vertices and edges beginning and ending with a vertex in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it. A *trail* is a walk in which the edges are distinct and a *path* is a trail in which the vertices (and thus edges) are distinct. A *cycle* is a non-empty trail in which only the first and last vertices are the same. A *cycle decomposition* is a partitioning of a graph's edges into cycles. We say a cycle is even or odd if it respectively contains an even or odd number of vertices. The following lemma characterizes the color trade spectra of cycles, which are often used in determining the color trade spectra of more complicated graphs.

Lemma 2.2. *Let G be a graph containing at least three vertices. If G is an even cycle, then CTS$(G) = \{2\}$. If G is an odd cycle, then CTS$(G) = \emptyset$.*

Proof. Suppose G is an even cycle containing at least four vertices, denoted by $v_0, v_1, ..., v_{n-1}$. Without loss of generality, suppose the edges of G are of the form (v_i, v_{i+1}) (or (v_{i+1}, v_i) since edges are unordered pairs of vertices) where $0 \leq i \leq n-1$ and addition is done modulo n. Let C_1 be a proper edge-coloring of G using two colors c_1 and c_2, and without loss of generality suppose each edge of the form (v_i, v_{i+1}) where i is odd is colored c_1 while each edge of the form (v_j, v_{j+1}) where j is even is colored c_2. In particular, note each vertex is incident to an edge colored c_1 and an edge colored c_2.

Modify C_1 by swapping the assignment of c_1 and c_2 to create a new edge-coloring C_2. This new edge-coloring doesn't assign the same color to any edge as C_1 by definition, and still maintains the property that each vertex is incident to an edge colored c_1 and c_2. Thus, C_2 is a mate coloring of C_1 so $2 \in \text{CTS}(G)$.

Now, suppose that C_1 is a proper edge-coloring of G using at least three colors, $c_1, c_2, ..., c_k$. Then there must exist some sequence of three consecutive edges using three colors, say (v_0, v_1), (v_1, v_2), and (v_2, v_3). Without loss of generality, suppose that c_1 is assigned to (v_0, v_1), while c_2 and c_3 are respectively assigned to (v_1, v_2) and (v_2, v_3). For any mate coloring of C_1, the edge (v_1, v_2) would then need to be assigned both c_1 and c_3, but this is impossible since edges can only be assigned one color under proper edge-colorings. Thus, C_1 has no mate coloring, and we conclude that $\text{CTS}(G) = \emptyset$. Therefore, $\{2\} = \text{CTS}(G)$.

It is well known that for any odd cycle, a proper edge-coloring must contain at least three colors. By the previous paragraph, this shows that the color-trade-spectrum for odd cycles is empty.

□

We now proceed to list some general lemmas about color-trade-spectra. A graph is *connected* if there is a walk between every pair of vertices in the graph. We say H is a *component* of G if H is a connected subgraph which is not part of any larger connected subgraph of G. We say a subgraph H is *spanning* if it contains every vertex of G. Let C_1 and C_2 be two edge-colorings of G using colors k colors which are mate colorings. The following lemma gives a method to increase the number of colors used in a pair of mate colorings.

Lemma 2.3. *LetC_1 ad C_2 be mate k-edge-colorings of a graph G. For each j in $1 \leq j \leq k$, let H_j be the subgraph of G spanned by the edges colored c_j in either C_1 or C_2. Then the components of H_j are even cycles. Let α_j be the number of these cycles and $\alpha = \sum_{j=1}^{k} \alpha_j$. Then CTS($G$) contains all integers in the interval $[k, \alpha]$.*

Proof. Suppose K is a component of H_j. Since C_1 and C_2 are mate colorings, any vertex in K is incident to exactly one edge of color c_j from C_1 and exactly one (different) edge of color c_j from C_2. Thus, every vertex of K has degree two and is thus a cycle. If K is an odd cycle, at least one vertex of K must be incident to two edges belonging to the same edge-coloring, a contradiction. Thus, K is an even cycle.

Suppose that H_j consists of disjoint even cycles. Consider two such cycles, say $(v_{x_1}, v_{x_2}, ..., v_{x_m})$ and $(v_{y_1}, v_{y_2}, ..., v_{y_n})$. By definition, these cycles are pairwise vertex disjoint, so we can freely assign a new color to each cycle. Since we have α of these cycles, we can extend our original k-edge-coloring of G up to an α-edge-coloring. □

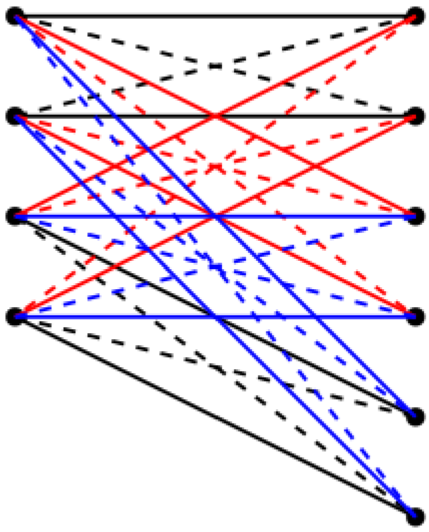
A *bridge* is an edge whose removal from a graph increases the graph's number of connected components. Lemma 2.3 gives us the following corollary.

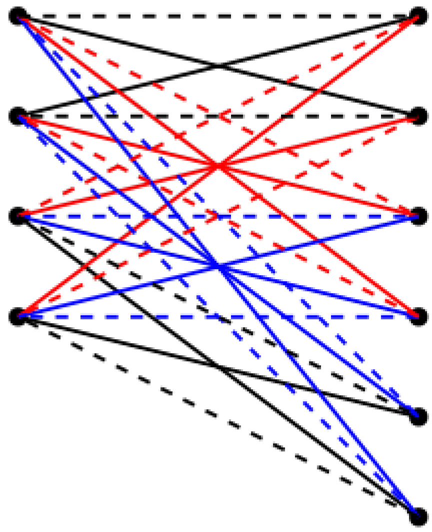
Corollary 2.0.1. *If G is a graph which contains a bridge, then CTS(G) $= \emptyset$.*

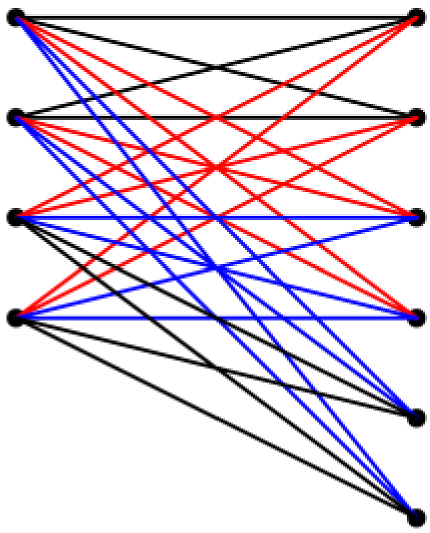
Proof. By Lemma 2.3, if G had a pair of mate colorings, then each H_j would consist of even cycles. However, this can not occur if G has a bridge. □

A *cycle-double-cover* of a graph G is a collection of cycles which together contain every edge of G exactly twice. Clearly, the H_j subgraphs from Lemma 2.3 form a cycle-double-cover, although not every cycle-double-cover correlates to the H_j subgraphs from a pair of mate colorings. Of note, this is related to the open *cycle-double-cover conjecture* by Seymour and Szekeres, which states that every bridgeless graph has a cycle-double-cover. [14][17]

A set of vertices is *independent* if no two vertices in the set are adjacent. A graph is *bipartite* if the vertices can be partitioned into two disjoint and independent sets. We denote the complete bipartite graph on sets of size m and n by $K_{m,n}$. Below is a 6-edge-coloring of $K_{4,6}$, along with its mate. Following these is an example of some of the H_j subgraphs from Lemma 2.3. In particular, note that every H_j consists of two 4-cycles. Thus, we could recolor one of the 4-cycles per color class to increase the total number of colors by one, and we could continually do this for each value of j until we have a pair of mate 12-edge-colorings of $K_{4,6}$.







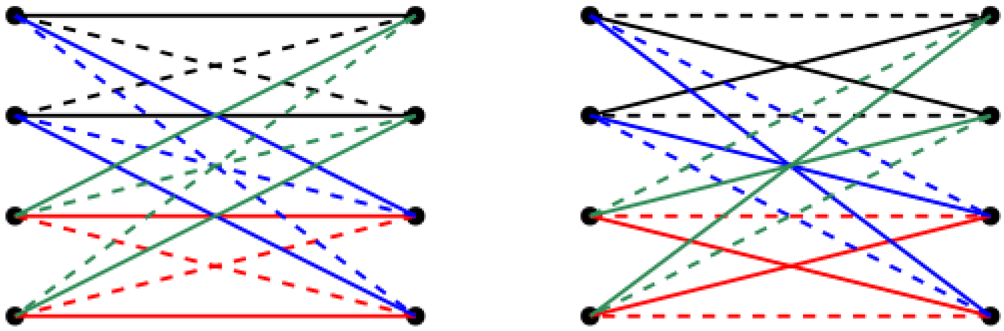
We say two colors c_1 and c_2 are incident if two edges colored c_1 and c_2 are incident. The following lemma gives a method to decrease the number of colors used in a pair of mate colorings.

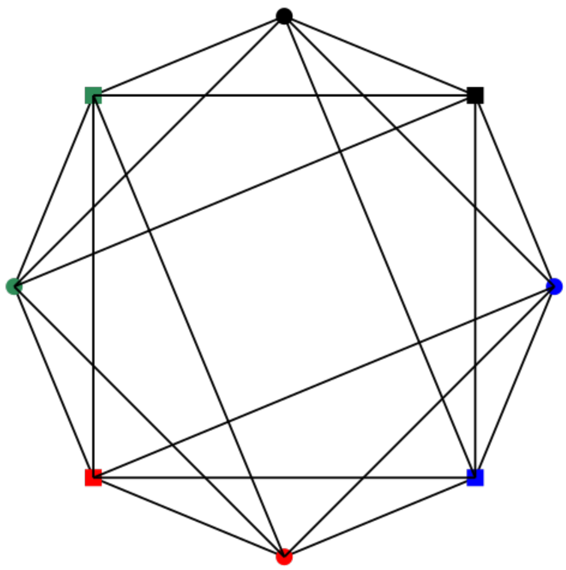
Lemma 2.4. *Suppose that C_1 and C_2 are proper mated k-edge-colorings of G on the same set of k colors. Denote by G' the graph where the k vertices of G' represent the colors used in both C_1 and C_2, and where vertices are adjacent if the respective colors are incident in C_1 and C_2. Then CTS(G) contains all the integers in the interval $[\chi(G'), k]$.*

Proof. Suppose C_1 and C_2 are mated k-edge-colorings of G on the same set of colors and let c_1 and c_2 be colors which are not incident in G. Using notation from Lemma 2.3, we know H_1

and H_2 are edge-disjoint. Since the associated vertices of these colors in G' are not adjacent, we may assign the same color to these vertices and likewise the same color to both sets of corresponding edges in G. Furthermore, we can freely continue this process until G' is $\chi(G')$-colored. $\square$

Below is an 8-edge-coloring of $K_{4,4}$ along with its mate. Following these is an example of the G' graph made from these edge-colorings, where a circular vertex represents a solid color while a square vertex represents a dashed color. Since this graph has a chromatic number of 4, we could recolor the edges of G using only four colors, and still have a mate coloring.





Let m be a positive integer and denote by mG the graph on $V(G)$ in which two vertices are joined by m edges if they are adjacent in G and no edges otherwise. If we superimpose the edges from two proper edge-colorings C_1 and C_2 onto G, we get a copy of $2G$ where each color class is a union of even cycles by Lemma 2.3. A natural question is: when can we take some copy of $2G$ which has been k-edge-colored, and decompose the graph into two separate copies of G, each uniquely associated with one of C_1 or C_2, where C_1 and C_2 are mate colorings. To answer this, we define a new graph, $M(G,k)$ as follows: given a k-edge-coloring (which is not proper) of $2G$, the vertices of $M(G,k)$ are the edges of $2G$, and two such vertices, e_i and e_j, are adjacent in $M(G,k)$ if and only if either e_i and e_j are the two copies of an edge of G, or e_i and e_j are adjacent in $2G$ and belong to the same color class.

Lemma 2.5. *Suppose $2G$ has a k-edge-coloring where each color class is a union of pairwise vertex disjoint cycles of length greater than two. The edge-coloring arises from a pair of mate colorings if and only if $M(G,k)$ is bipartite.*

Proof. For the forward implication, let C_1 and C_2 be the two mate colorings giving rise to our k-edge-coloring of $2G$. Let H_j denote the subgraph of $2G$ spanned by the edges colored c_j in either C_1 or C_2. By assumption, each color class is a union of pairwise vertex disjoint cycles of length greater than two, and furthermore a pair of adjacent edges in H_j, e_i and e_j, must belong to different copies of G as otherwise one of C_1 or C_2 wouldn't be proper. Suppose $\{e_i', e_i''\}$ is a multi-edge in $2G$ and without loss of generality suppose e_i' is assigned its color from C_1 and likewise e_i'' is assigned its color from C_2. We consider two cases.

Case 1: e_i and e_j are vertices in $M(G,k)$ that belong to the same multi-edge in $2G$.

By the above paragraph, e_i and e_j must each be associated with a different edge-coloring, and since the edge-colorings C_1 and C_2 are mates, this means the edges must receive different colors.

Case 2: e_i and e_j are vertices in $M(G,k)$ that are adjacent and have the same color in $2G$.

Since the edges share the same color, they can not belong to the same multi-edge by case 1. By the above paragraph, e_i and e_j receive their colors from different edge-colorings.

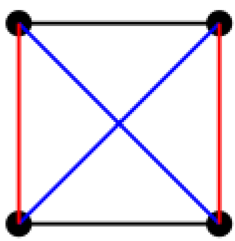
Putting these cases together, we can properly color the vertices of $M(G,k)$ with two colors by assigning one color to all of the vertices associated with C_1 and another to all of the vertices associated with C_2. Therefore, $M(G,k)$ is bipartite.

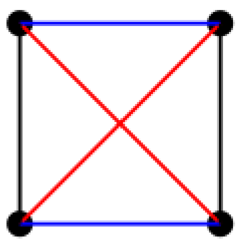
For the backwards implication, let A and B be two independent sets of vertices of $M(G,k)$ which partition the vertices of $M(G,k)$. For contradiction, suppose $|A| > |B|$. Then there exists vertices in A, say a_i and a_j which must correspond to the same multi-edge in $2G$, but this means a_i and a_j are adjacent, contradicting our assumption of A being independent. Thus, $|A| = |B|$, and each set corresponds to one of the copies of G which make $2G$. Suppose C_1 is the edge-coloring associated with A and C_2 the edge-coloring associated with B. Let e_i' and e_i'' be vertices in $M(G,k)$ whose edges belong to the same multi-edge of $2G$. Without loss of generality, suppose $e_i' = a_i \in A$ and $e_i'' = b_i \in B$. Since a_i belongs to the same multi-edge in $2G$ as b_i, this means a_i and b_i must have different colors, as otherwise we contradict our assumption of each color class consisting of cycles of length greater than two. Thus, for any edge $e \in E(G)$, $C_1(e) \neq C_2(e)$.

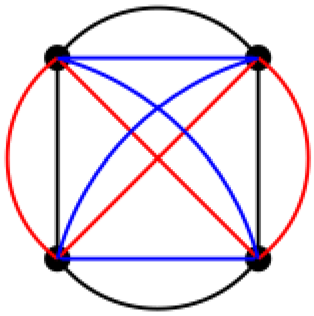
By assumption, we know a_i must be adjacent to vertices which share the same color as a_i, say u_i and v_i. Furthermore, if the color of a_i comes from C_1, this means the color of both u_i and v_i must come from C_2, so we say $u_i = b_{i-1}$ and $v_i = b_{i+1}$ without loss of generality. For a given vertex $v \in V(G)$, consider the set of multi-edges incident to v in $2G$. Suppose v is incident to an edge colored c_j in $2G$ and that this color comes from C_1. By assumption, there must be another edge incident to v with color c_j and by the previous argument, this edge must get its color from C_2. Therefore, the set of colors incident to v under C_1 is the same as the set of colors under C_2. Thus, C_1 and C_2 are mate colorings.

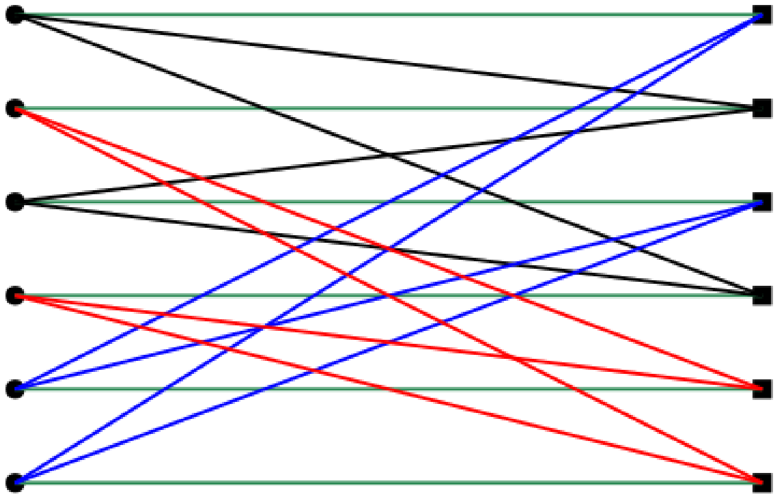
□

Below is an example of a 3-edge-coloring of K_4 along with its mate. Following these is the graph $2G$ made by superimposing both edge-colorings of K_4 together, along with the graph $M(K_4, 3)$ associated with these edge-colorings. The circular and square vertices respectively represent edges colored under C_1 and C_2. The green edges represent that the vertices (edges in $2G$) belong to the same multi-edge, and the other colored edges show that these edges form a cycle in $2G$.

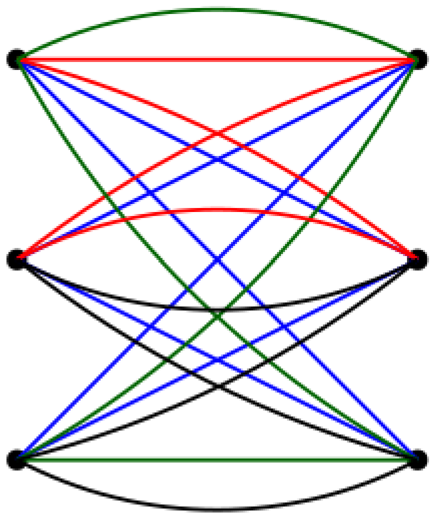


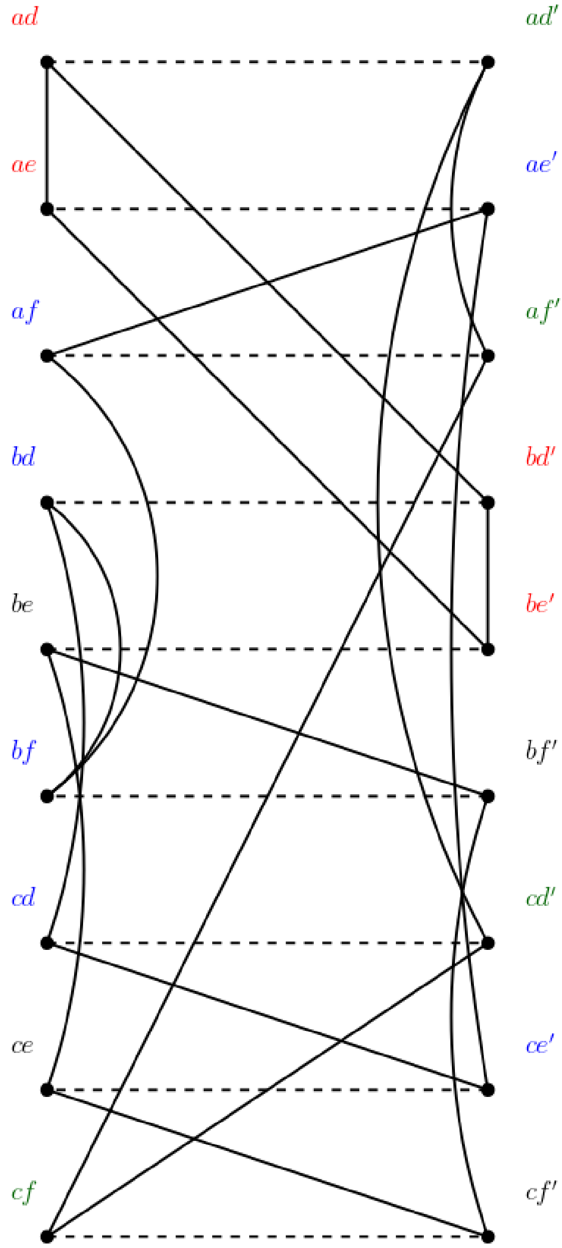






Let $G = K_{3,3}$. The following is an example of $2G$ where each color class is a union of pairwise vertex disjoint cycles of even length greater than two, but $M(G, 4)$ is not bipartite. To see this, let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the partite sets of G. Then the 6-cycle in blue is (a, e, c, d, b, f), the 4-cycle in red is (a, d, b, e), the 4-cycle in black is (b, e, c, f), and the 4-cycle in green is (a, d, c, f). Let xy denote an edge in G with vertices x and y, and denote the corresponding vertices of $M(G, 4)$ by xy and xy'. Then $M(G, 4)$ is not bipartite since it contains the 7-cycle $(ad, ad', af', af, bf, bd, bd')$. Note this notation implies ad and ad' have the same color, so this forces how the labelings of xy and xy' are assigned. Examples of $2G$ and $M(G, 4)$ are given below to the left and right respectively.




ad
ad′
ae
ae′
af
af′
bd
bd′
be
be′
bf
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cf′

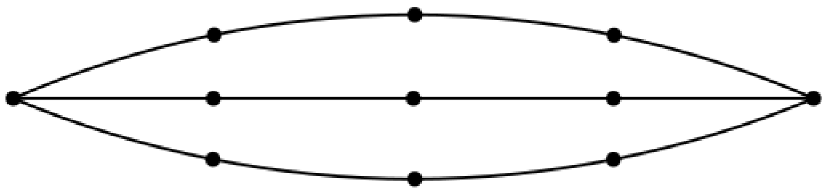
Chapter 3

Theta, Wheel, and n-cube Graphs

In this chapter we determine the color-trade-spectrum of three simple families of graphs, namely Theta graphs, wheel graphs, and n-cube graphs.

3.1 Theta Graphs

Let Θ_n^k denote the Theta graph which consists of k paths of length n, with only the first and last vertices ∞_1 and ∞_2 in common. In particular, note that Θ_n^1 is a path of length n, Θ_n^2 is a cycle of length $2n$, and Θ_1^k is a multi-edge with edge multiplicity k, which is not a simple graph. An example of Θ_4^3 is given below.



Lemma 3.1. *For non-negative integers k and n,*

1. *CTS*$(\Theta_n^k) = \emptyset$ *for* $n = 0$.
2. *CTS*$(\Theta_n^k) = \{k\}$ *for* $n = 1$ *and* $k \geq 2$, *and CTS*$(\Theta_n^k) = \emptyset$ *for* $n = 1$ *and* $0 \leq k \leq 1$.

Proof. 1: If $n = 0$, then Θ_n^k is a graph containing no edges and two isolated vertices, regardless of the value of k. Thus, the color-trade-spectrum is trivially empty.

2: If $n = 1$, then Θ_n^k is a multi-edge consisting of k parallel edges. When $k = 0$, the graph contains no edges, and by the above case, the color-trade-spectrum is empty. When $k = 1$, Θ_n^k is a path of length one, which has an empty color-trade-spectrum since there is only one way to edge-color this graph. When $k \geq 2$, each edge is assigned a unique color from $\{c_1, ..., c_k\}$. Any permutation of this set with no fixed points creates a mate coloring, so k is in the color-trade-spectrum. Since the only way to properly edge-color this graph is to use k colors, this means $\{k\} = \text{CTS}(\Theta_n^k)$.

□

Theorem 3.1. *For $n \geq 2$, CTS$(\Theta_n^k) = \{k\}$ if k is even and CTS$(\Theta_n^k) = \emptyset$ if k is odd.*

Proof. Let the jth path be denoted by $(v_{1,j}, v_{2,j}, ..., v_{n+1,j})$ where $v_{1,j} = \infty_1$ and $v_{n+1,j} = \infty_2$ for $1 \leq j \leq k$ (so the naming of ∞_1 and ∞_2 is not unique). If $k = 0$, the graph consists of two isolated vertices with no edges, so the color-trade-spectrum is trivially empty. If $k = 1$, the graph is a path consisting of n edges which contains two vertices of degree one, and by Lemma 2.1, the color-trade-spectrum is empty. Suppose $k \geq 2$ and that C_1 is a proper k-edge-coloring of Θ_n^k. Without loss of generality, suppose that edge $(v_{1,1}, v_{2,1})$ is colored c_1 and that edge $(v_{2,1}, v_{3,1})$ is colored c_2. Since C_1 is proper, no other edge incident to ∞_1 can be colored c_1 or c_2. If $(v_{3,1}, v_{4,1})$ was given a color other than c_1, say c_3, then C_1 would have no mate coloring since the edge $(v_{2,1}, v_{3,1})$ would need to be colored both c_1 and c_3 in the mate coloring. Thus, in order for C_1 to have a mate coloring, the edges along the path $v_{1,1}, v_{2,1}, ..., v_{n+1,1}$ must alternate between colors c_1 and c_2. Furthermore, this assignment of alternating colors must be true for every path from ∞_1 to ∞_2 if C_1 were to have a mate coloring.

Since the maximum degree of Θ_n^k is k, no value lower than k can be in the color-trade-spectrum. By the first paragraph of the theorem, we know each path must have an alternating assignment of colors to its edges if the edge-coloring C_1 were to have a mate coloring. Furthermore, if the edges of path i alternate between c_1 and c_2, then there must be another path j where the edges alternate between c_2 and c_1 in order for C_1 to have a mate. This means that C_1 has a mate only when C_1 induces a $2n$-cycle decomposition of Θ_n^k where each cycle in the decomposition alternates between two unique colors . This only happens when k is even, so we conclude $\emptyset = \text{CTS}(\Theta_n^k)$ if k is odd. We now given an explicit construction to show $k \in \text{CTS}(\Theta_n^k)$.

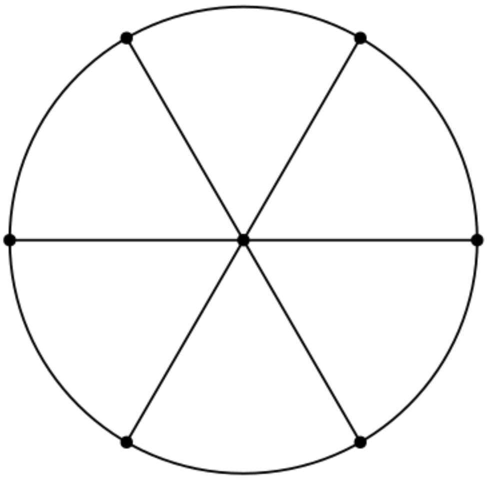
Suppose that k and n are both even integers which are at least two, and consider the following proper k-edge-coloring of Θ_n^k, C_1. Assign c_{2j-1} to edges of the form $(v_{2i-1,2j-1}, v_{2i,2j-1})$ and $(v_{2i,2j}, v_{2i+1,2j})$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{k}{2}$. Then, assign c_{2j} to edges of the form $(v_{2i,2j-1}, v_{2i+1,2j-1})$ and $(v_{2i-1,2j}, v_{2i,2j})$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{k}{2}$. We find a mate coloring, C_2, by swapping the colors between edges colored c_{2j-1} and c_{2j} for $1 \leq j \leq \frac{k}{2}$ under C_1. If n is odd, we consider a different proper k-edge-coloring of Θ_n^k, C_3. Assign c_{2j-1} to edges of the form $(v_{2i-1,2j-1}, v_{2i,2j-1})$, $(v_{2i,2j}, v_{2i+1,2j})$, and $(v_{n,2j-1}, v_{n+1,2j-1})$ for $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{k}{2}$. Then, assign c_{2j} to edges of the form $(v_{2i,2j-1}, v_{2i+1,2j-1})$, $(v_{2i-1,2j}, v_{2i,2j})$, and $(v_{n,2k}, v_{n+1,2j})$ for $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{k}{2}$. Again, we find a mate coloring, C_4, by swapping the colors between edges colored c_{2j-1} and c_{2j}. Hence, $k \in \text{CTS}(\Theta_n^k)$.

To show that nothing else is in the color-trade-spectrum, recall from the second paragraph of the theorem that C_5 must induce a $2n$-cycle decomposition of Θ_n^k where each cycle in the decomposition alternates between two unique colors. Since there are only $\frac{k}{2}$ such cycles in Θ_n^k (when k is even), this means this condition can only be met when C_5 uses no more than k colors. Therefore, $\{k\} = \text{CTS}(\Theta_n^k)$.

□

3.2 Wheel Graphs

Let $n \geq 3$ be a positive integer and let W_n denote the wheel graph which consists of a single n-cycle and n "spokes" which originate from a central vertex v_∞ which is adjacent to every vertex of the n-cycle. An example of W_6 is given below.



Theorem 3.2. *For $n \geq 3$, CTS$(W_n) = \{n\}$.*

Proof. Note that W_n consists of $2n$ edges, and that the central vertex v_∞ has degree n. By Lemma 2.1, this means the only possible value in the color-trade-spectrum of W_n is n, and we present an explicit construction to show $n \in \text{CTS}(W_n)$.

Without loss of generality, suppose the vertices of the n-cycle are labeled in order by $v_0, v_1, ..., v_{n-1}$, so that v_i is adjacent to v_{i-1} and v_{i+1} where addition is computed modulo n. Denote an edge in W_n by (v_i, v_j). Consider the following proper n-edge-coloring of W_n, C_1. Assign color c_i to edges (v_∞, v_i) and (v_{i+1}, v_{i+2}), where again addition is under modulo n. To find a mate coloring for C_1, consider the following proper edge-coloring C_2. Assign color c_i to edges (v_∞, v_{i+2}) and (v_i, v_{i+1}). Geometrically, this is equivalent to rotating the spokes of the wheel two times to the right and rotating the rim of the wheel once to the left. Then C_1 and C_2 are mate colorings, so $n \in \text{CTS}(W_n)$, and since this is the only possible value in the color-trade-spectrum, $\{n\} = \text{CTS}(W_n)$.

$\square$

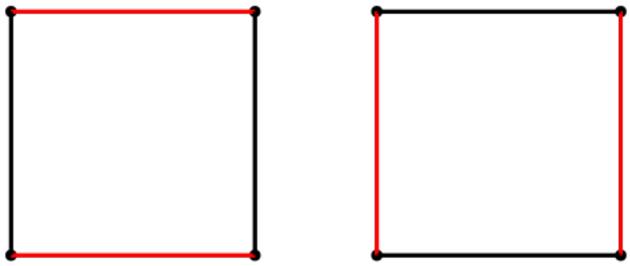
3.3 n-cube Graphs

The final family of graphs we will consider in this chapter is n-cubes. Denote by Q_n the n-cube graph on 2^n vertices, each labeled with an n-bit binary number, where two vertices are adjacent if the corresponding binary numbers differ by exactly one digit, i.e. their Hamming distance is one. Note that the maximum degree of Q_n is n, and that $|E(Q_n)| = n2^{n-1}$. By Lemma 2.1, this means the color-trade-spectrum is bounded below by n and above by $n2^{n-2}$.

We say two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\Phi : V(G) \to V(H)$ with the property that any two vertices u and v in G have an edge between them if and only if $\Phi(u)$ and $\Phi(v)$ have an edge between them in H. An equivalent construction of Q_n is found by taking the Cartesian product of n K_2 graphs, denoted $K_2 \square K_2 \square \cdots \square K_2$. Furthermore, one can show that $Q_n \square Q_m \cong Q_{n+m}$ as follows. Let the vertices of Q_n and Q_m be denoted by binary sequences of length n and m respectively. As in the previous paragraphs, vertices are adjacent when their Hamming distance is one. To construct $Q_n \square Q_m$, we take 2^m copies of Q_n. In particular, for each vertex in Q_n denoted by a unique binary sequence of length n, we concatenate a binary sequence of length m to create a new binary sequence of length $n + m$. The first n digits correspond to unique vertices within a copy of Q_n while the last m digits correspond to unique copies of Q_m. For example, in $Q_2 \square Q_3$, the binary sequence 01001 would correspond to the vertex labeled 01 in Q_2 in the 001th copy of Q_2. This creates a graph with 2^{n+m} vertices and $(n+m)2^{n+m-1}$ edges, where vertices are adjacent only when their Hamming number is one. Therefore, $Q_n \square Q_m \cong Q_{n+m}$. In Lemma 3.2 we calculate the color trade spectra of Q_n for $0 \leq n \leq 3$, and in Theorem 3.3 we calculate the remaining spectra using a recursive construction based on the fact that $Q_{n-1} \square Q_2 \cong Q_{n+1}$.

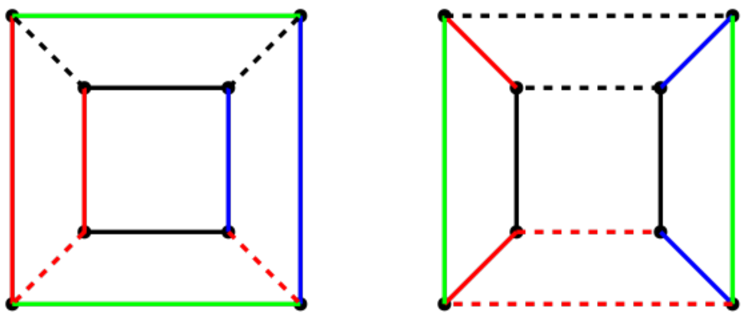
Lemma 3.2. *Let n be a non-negative integer. Then CTS$(Q_n) = \emptyset$ for $0 \leq n \leq 1$, CTS$(Q_2) = \{2\}$, and CTS$(Q_3) = \{3, 4, 5, 6\}$.*

Proof. Trivially, Q_0 and Q_1 have no color trades since they contain less than 2 edges, and any color trade requires a graph to have at least 2 edges. Note that Q_2 is a 4-cycle, so by Lemma 2.2, we conclude $\{2\} = \text{CTS}(Q_2)$. However we present an alternate proof using methods that will be used for future cases of Q_n. By the discussion above, the only possible value in CTS(Q_2) is 2. As above, denote the vertices of Q_2 by binary sequences of length two, b_1b_2, where each b_i is either 0 or 1. Denote edges of Q_2 by (b_1b_2, d_1d_2), where $b_1b_2 \neq d_1d_2$. Consider the following proper 2-edge-coloring of Q_2, C_1. Assign c_1 to edges $(00, 01)$ and $(10, 11)$, and assign c_2 to edges $(01, 10)$ and $(00, 11)$. By swapping the colors between edges colored c_1 and c_2, we create a mate coloring C_2. Therefore, $\{2\} = \text{CTS}(Q_2)$. An example of these edge-colorings is given below.



Next, we show $\{3, 4, 5, 6\} = \text{CTS}(Q_3)$. As before, denote vertices by binary sequences of length three, $b_1b_2b_3$, and denote edges by $(b_1b_2b_3, d_1d_2d_3)$. Consider the the following proper 6-edge-coloring of Q_3. Assign c_1 to edges $(000, 010)$ and $(100, 110)$, c_2 to edges $(000, 001)$ and $(010, 011)$, c_3 to edges $(000, 100)$ and $(010, 110)$, c_4 to edges $(001, 011)$ and $(101, 111)$, c_5 to edges $(100, 101)$ and $(110, 111)$, and c_6 to edges $(001, 101)$ and $(011, 111)$. Then any permutation of the assignment of colors c_1 through c_6 with no fixed point creates a new edge-coloring which is a mate for C_1. Thus, $6 \in \text{CTS}(Q_3)$.

To show $\{3, 4, 5\} \subset \text{CTS}(Q_3)$, we consider the graph G' as defined in Lemma 2.4, where the vertices of G' correspond to colors used in C_1 and C_2, and vertices in G' are adjacent when the respective colors are incident in C_1 and C_2. For Q_3, Q_3' is a 3-cycle, which has chromatic number three. By Lemma 2.4, this means $\{3, 4, 5, 6\} \subseteq \text{CTS}(Q_3)$. Since $\frac{|E(Q_3)|}{2} = 6$ and Q_3 has maximum degree 3, we conclude that $\{3, 4, 5, 6\} = \text{CTS}(Q_3)$ by Lemma 2.1. An example of a proper 6-edge-coloring of Q_3 along with its mate is given below.

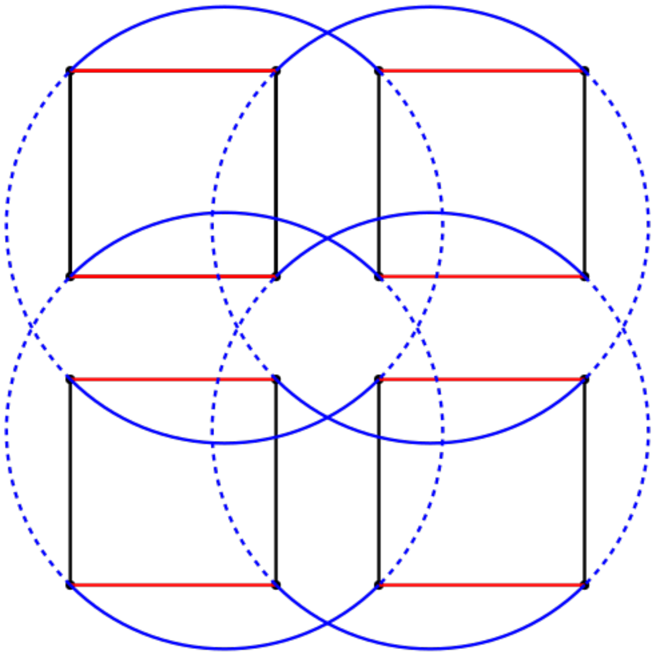


□

Theorem 3.3. *Let n be a nonnegative integer. Then CTS$(Q_n) = \emptyset$ for $0 \leq n \leq 1$, and CTS$(Q_n) = \{n, n+1, ..., n2^{n-2}\}$ for $n \geq 2$.*

Proof. The cases for $0 \leq n \leq 3$ are covered by the above lemma, and we cover the rest of the cases using induction on n, based on the following construction based on the earlier discussion that $Q_n \Box Q_m \cong Q_{n+m}$. In particular, $Q_{n+1} \cong Q_{n-1} \Box Q_2$, and we start the induction on $n = 2$. By Lemma 3.2, the claim holds for $n = 2$, so we now suppose the claim holds for n and consider $n + 1$. Note that $Q_{n-1} \Box Q_2$ yields four copies of Q_{n-1}, and by induction, we can find mate colorings using any value in the color-trade-spectrum of Q_{n-1} for each copy of Q_{n-1}. In particular, we can use $(n-1)2^{n-3}$ distinct colors for each copy of Q_{n-1}, for a total of $4(n-1)2^{n-3} = (n-1)2^{n-1}$ colors. Let $v_1, v_2, ..., v_{2^{n-1}}$ be a given ordering of the vertices of Q_{n-1}, let $v_{i,j}$ denote the ith vertex in the jth copy of Q_{n-1}, and let $(v_{a,b}, v_{i,j})$ denote an edge between $v_{a,b}$ and $v_{i,j}$ in $Q_{n-1} \Box Q_2$, assuming such an edge exists. The four copies of Q_{n-1} consist of $4(n-1)2^{n-2} = (n-1)2^n$ edges, and the remaining $(n+1)2^n - (n-1)2^n = 2^{n+1}$ edges form 2^{n-1} disjoint 4-cycles of the form $(v_{i,1}v_{i,2}v_{i,3}v_{i,4})$.

Coloring each of these 4 cycles with 2 distinct colors gives us an edge-coloring of Q_{n+1} using $4(n-1)2^{n-3} + 2 \cdot 2^{n-1} = (n+1)2^{n-1}$ colors. By the previous paragraph, we can find mates for this edge-coloring for each copy of Q_{n-1}, and we find a mate coloring for Q_{n+1} by alternating the assignment of colors within each of the above 4-cycles. Thus, $(n+1)2^{n-1} \in$ CTS(Q_{n+1}). Since each induced copy of Q_{n-1} is disjoint, we can lower the amount of colors used in Q_{n+1} by reusing colors between copies of Q_{n-1}. Since each of the 4-cycles connecting the copies of Q_{n-1} are disjoint, we can use the same 2 colors on each of these cycles. Doing this for all $n \geq 2$ proves the claim. An example of this construction being applied to Q_4 is given below.



□

Chapter 4

Complete Bipartite Graphs

Let $K_{m,n}$ be the bipartite graph consisting of vertex sets $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_n\}$ where $m \leq n$ are non-negative integers, and the edge set $E = \{(a_i b_j) | 1 \leq i \leq m, 1 \leq j \leq n\}$. Recall that a graph is $k - regular$ if every vertex has degree k. A $k - facotr$ of a graph is a spanning k-regular subgraph, and a $k - factorization$ of a graph partitions the graph into disjoint k-factors. Also, recall that a *matching* is a set of independent edges in a graph, meaning the edges have no common vertices. One can easily show that a 1-factorization is equivalent to a *perfect matching*, which is a matching which includes every vertex of the graph.

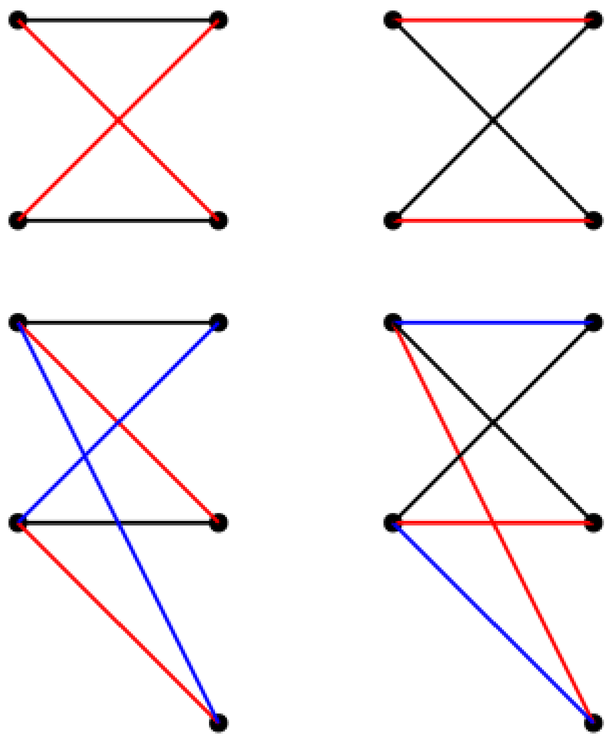
4.1 $0 \leq m \leq n \leq 3$

Before stating the main results of this section, we first determine the color-trade-spectrum of $K_{m,n}$ where at least one of m or n is either zero or one, $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$, which will be used in later constructions.

Lemma 4.1. *Let $m \leq n$ for non-negative integers m and n. Then CTS$(K_{m,n}) = \emptyset$ if at least one of m or n is zero or one, and the color trade spectra of $K_{2,2}, K_{2,3}$, and $K_{3,3}$ are $\{2\}, \{3\}$, and $\{3\}$ respectively.*

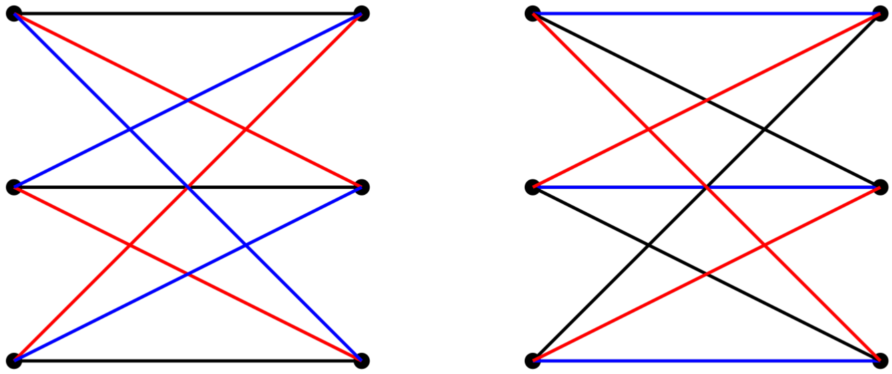
Proof. If either m or n is 0, we have a graph with no edges, which trivially has an empty color-trade-spectrum. If either m or n is one but both not zero, we have at least one vertex of degree one, and by Lemma 2.1, the color-trade-spectrum is again empty. We now consider $K_{2,2}$.

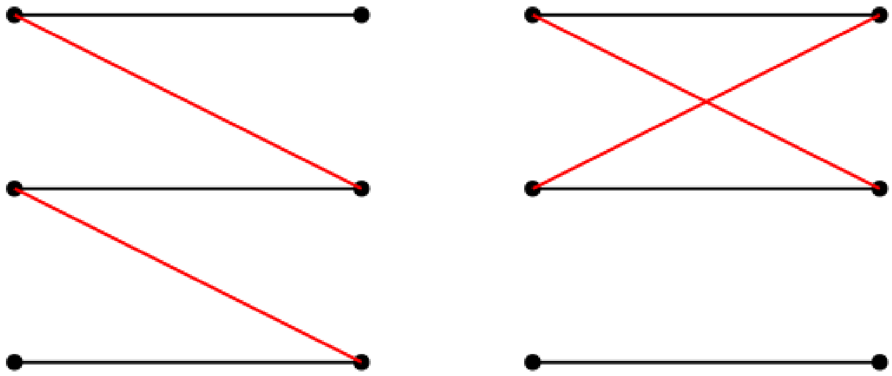
Since $|E(K_{2,2})| = 4$ and $\chi^{'}(K_{2,2}) = 2$, the only possible value in CTS($K_{2,2}$) is 2. Likewise, the only possible value in CTS($K_{2,3}$) is 3. Note that $K_{2,2}$ is isomorphic to a 4-cycle graph, so by Lemma 2.2 $2 = \text{CTS}(K_{2,2})$. To show $3 = \text{CTS}(K_{2,3})$, consider the proper 3-edge-coloring C_1 given below. Assign c_1 to edges (a_1b_1) and (a_2b_2), c_2 to edges (a_1b_3) and (a_2b_1), and c_3 to edges (a_1b_2) and (a_2b_3). Now, consider the edge-coloring C_2 where c_1 is assigned to edges (a_1b_2) and (a_2b_1), c_2 is assigned to edges (a_1b_1), and (a_2b_3), and c_3 is assigned to edges (a_1b_3) and (a_1b_2). Then C_1 and C_2 are mates, so $3 = \text{CTS}(K_{2,3})$. Below are examples of these edge-colorings using 2 and 3 colors for $K_{2,2}$ and $K_{2,3}$ respectively.



For $K_{3,3}$, the only possible values in the color-trade-spectrum are 3 and 4, and since any proper 3-edge-coloring of $K_{3,3}$ must be a 1-factorization, meaning that the colored edges form a perfect matching, we can show $3 \in \text{CTS}(K_{3,3})$ as follows. Let C_1 be any proper 3-edge-coloring of $K_{3,3}$, and apply any permutation with no fixed points to the colors of C_1 to create a mate coloring C_2. For example, $\alpha := (c_1c_2c_3)$ would be such a permutation. However, 4 is not in the color-trade-spectrum, and to show this, note that any proper edge-coloring of $K_{3,3}$ using 4 colors must contain a perfect matching, and three color classes of size two. Suppose c_1 is a perfect matching and consider assigning a color class of size 2, say c_2, to two of the remaining edges of the graph. The subgraph induced by c_1 and c_2 consists of either two components, or one component. A quick exhaustive search shows that neither case leads to a mate coloring. Therefore, $\{3\} = \text{CTS}(K_{3,3})$.

Below are mate colorings using 3 colors for $K_{3,3}$. Following these are examples of the two cases for $K_{3,4}$ where the subgraph induced by c_1 and c_2 consists of either two components, or one component.





$\square$

4.2 $n \geq m \geq 2$ where m is even

For the following theorems, we consider $K_{m,n}$ with a proper edge-coloring C_1. We also consider the associated (and equivalent) $n \times m$ Latin rectangle L. Given vertices in different parts, say a_j and b_i, the entry of cell (i, j) of L corresponds to the color assigned to $b_i a_j$ under C_1. Thus, finding a mate for $K_{m,n}$ is equivalent to finding a permutation π of L with no fixed points, which preserves the Latin property, and where every color appearing in a row or column of L

still appears after applying π to L. For this to happen, we note that if a set of colors appears in some column k of L, then the same set must also appear in another column k' of L. The following theorems give constructions for L and $\pi(L)$.

Theorem 4.1. $\{n, n+1, ..., \frac{mn}{2}\} =$ *CTS*$(K_{m,n})$ *where* $2 \leq m \leq n$ *and* m *is even.*

Proof. We construct L_Δ, the Latin rectangle corresponding to a Δ-edge-coloring of $K_{m,n}$. Let a *column block of size s on t colors*, denoted by CB(s,t), be a set of s columns of L_Δ where t unique symbols appear in the column block. Partition the m columns of L_Δ into $\frac{m}{2}$ column blocks of size 2, and without loss of generality, suppose the 2 columns in each column block are adjacent. Denote column k by a_k and the column block containing columns a_{2i-1} and a_{2i} by A_i for $1 \leq i \leq \frac{m}{2}$. Fill the cells of a_{2i-1} and a_{2i} in L_Δ in order from top to bottom with entries $c_{2(i-1)+1}, c_{2(i-1)+2}, ..., c_n, c_1, ...c_{2(i-1)}$ and $c_{2(i-1)+2}, c_{2(i-1)+3}, ..., c_n, c_1, ..., c_{2(i-1)+1}$ respectively. Then each column block of L_Δ contains the same set of n colors and we find a mate for L_Δ, $\pi(L_\Delta)$ as follows: let $\pi := (a_1a_2)(a_3a_4)\cdots(a_{m-1}a_m)$, a product of $\frac{m}{2}$ 2-cycle permutations. Each column and row in $\pi(L_\Delta)$ contains the same symbols as they did in L_Δ, while no cell receives the same entry. Thus, $\pi(L_\Delta)$ and L_Δ correspond to two mate colorings of $K_{m,n}$.

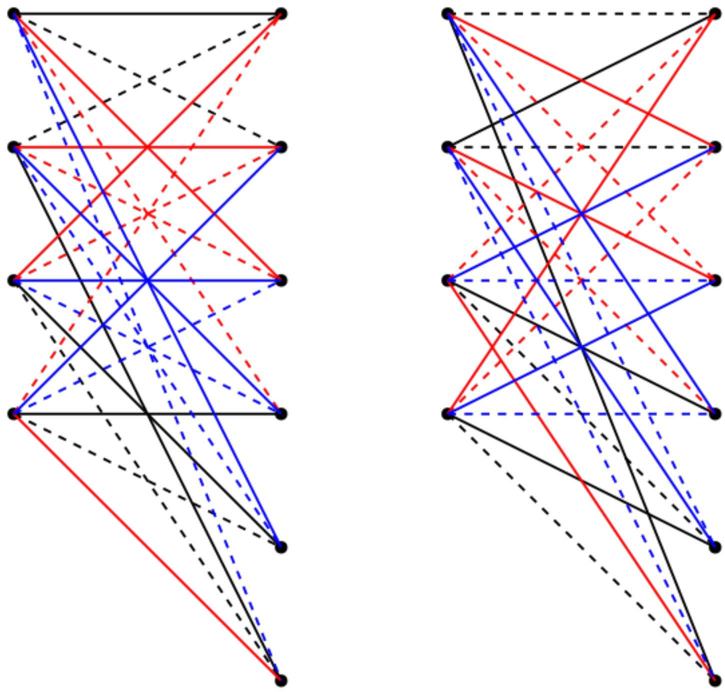
Let C_1 and C_2 refer to the corresponding edge-colorings of L_Δ and $\pi(L_\Delta)$. Using notation from Lemma 2.3, note that each H_j consists of $\frac{m}{2}$ many 4-cycles, each of which corresponds to a unique column block, so $\alpha = \frac{mn}{2}$. By Lemma 2.3, we conclude $\{\Delta, \Delta+1, ..., \frac{mn}{2}\} \subseteq$ CTS$(K_{m,n})$ and since $\Delta(K_{m,n}) = n$ and $|E(K_{m,n})| = mn$, we conclude there are no other possible values in the color-trade-spectrum.

□

Below are examples of L_Δ and $\pi(L_\Delta)$ along with their corresponding graphs for $K_{4,6}$.

L_Δ									
	1	2	3	4	$\cdots$	$m-3$	$m-2$	$m-1$	m
	2	3	4	5	$\cdots$	$m-2$	$m-1$	m	$m+1$
	3	4	5	6	$\cdots$	$m-1$	m	$m+1$	$m+2$
	4	5	6	7	$\cdots$	m	$m+1$	$m+2$	$m+3$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-3$	$n-2$	$n-1$	n	$\cdots$	$m-7$	$m-6$	$m-5$	$m-4$
	$n-2$	$n-1$	n	1	$\cdots$	$m-6$	$m-5$	$m-4$	$m-3$
	$n-1$	n	1	2	$\cdots$	$m-5$	$m-4$	$m-3$	$m-2$
	n	1	2	3	$\cdots$	$m-4$	$m-3$	$m-2$	$m-1$

$\pi(L_\Delta)$									
	2	1	4	3	$\cdots$	$m-2$	$m-3$	m	$m-1$
	3	2	5	4	$\cdots$	$m-1$	$m-2$	$m+1$	m
	4	3	6	5	$\cdots$	m	$m-1$	$m+2$	$m+1$
	5	4	7	6	$\cdots$	$m+1$	m	$m+3$	$m+2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-2$	$n-3$	n	$n-1$	$\cdots$	$m-6$	$m-7$	$m-4$	$m-5$
	$n-1$	$n-2$	1	n	$\cdots$	$m-5$	$m-6$	$m-3$	$m-4$
	n	$n-1$	2	1	$\cdots$	$m-4$	$m-5$	$m-2$	$m-3$
	1	n	3	2	$\cdots$	$m-3$	$m-4$	$m-1$	$m-2$



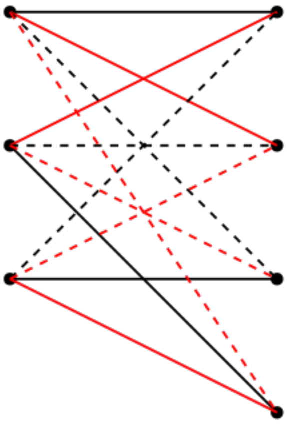
4.3 $n \geq m \geq 3$ where m is odd

We now determine the color-trade-spectrum for $K_{3,n}$, which we will use for the general case of $K_{m,n}$ where m is odd. By Lemma 4.1, CTS($K_{3,3}$) $= \{3\}$. We first give a specific construction that shows $\{4, 5, 6\}$ = CTS($K_{3,4}$), and then use this to determine CTS($K_{3,n}$) in two cases, where $n \geq 3$ is an integer.

Lemma 4.2. $\{4, 5, 6\} = $ *CTS*$(K_{3,4})$

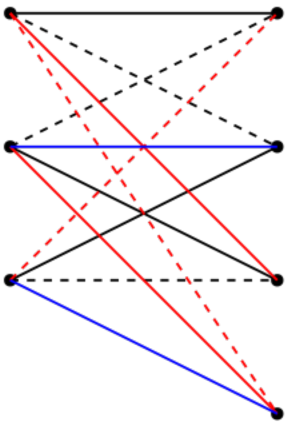
Proof. Using the same notation from Theorem 4.1, L_Δ contains a single column block A_1 of size three, consisting of the columns a_1, a_2, and a_3. Furthermore, we now consider a *row block of size s on t colors*, denoted RB(s, t), which is a set of s rows of L_Δ where t unique symbols appear in the row block. Below is a RB(4, 4) and its corresponding 4-edge-coloring of $K_{3,4}$.

1	2	3
2	3	4
3	4	1
4	1	2



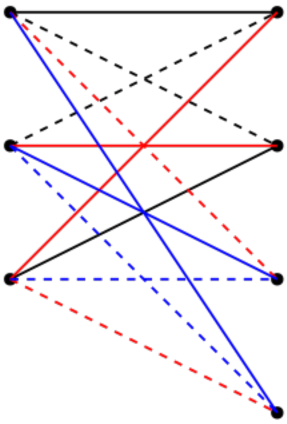
To show that 4 is in the color-trade-spectrum, we apply the permutation $\pi := (a_1a_2a_3)$. Each column and each row in $\pi(L_\Delta)$ contains the same symbols as they did in L, while no cell receives the same entry. This corresponds to each vertex in $K_{3,4}$ seeing the same color set, but no edge receiving the same color, showing these two edge-colorings are mates. Note that this implies $n \in \text{CTS}(K_{3,n})$ where $n \geq 3$ is an integer by using the following method to create L_Δ: from top to bottom fill the cells of a_1 and a_2 with entries $c_1, c_2, ...c_n$ and $c_2, c_3, ..., c_n, c_1$ respectively, and the cells of a_3 with $c_3, c_4, ..., c_n, c_1, c_2$. To show that 5 is in the color-trade-spectrum, we modify L_Δ to make L_5, shown below along with its corresponding 5-edge-coloring of $K_{3,4}$.

1	2	4
2	5	1
3	1	2
4	3	5



Applying the permutations $\pi : (a_1a_2a_3)$ on the first 3 rows of of L_5 and $\sigma : (a_1a_3a_2)$ on the last row of L_5 gives us a mate coloring. Finally, we show 6 is in the color-trade-spectrum by further modifying L_5 to make L_6, which consists of two separate $\text{RB}(2, 3)$, shown below along with its associated 6-edge-coloring of $K_{3,4}$.

1	2	3
2	3	1
4	5	6
5	6	4



Applying the permutations $\pi : (a_1a_2a_3)$ on rows 2 and 4 of L_6 and $\sigma : (a_1a_3a_2)$ on rows 1 and 3 of L_6 gives us a mate coloring. Since $\Delta(K_{3,4}) = 4$ and $|E(K_{3,4})| = 12$, we have found a mate coloring for every possible value in the color-trade-spectrum. Using these blocks as our building tools, we now determine the color-trade-spectrum of $K_{3,n}$ by considering two cases.

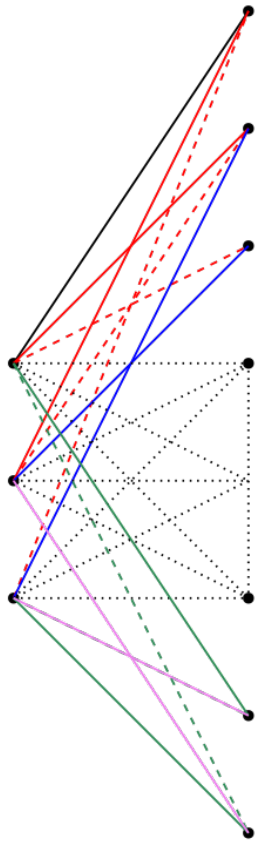
$\square$

Lemma 4.3. *Let $n \geq 4$ be a positive integer. Then $\{\Delta, \Delta+1, ..., \frac{3n}{2}\} =$ CTS$(K_{3,n})$ if n is even and $\{\Delta, \Delta+1, ..., \frac{3n-1}{2}\} =$ CTS$(K_{3,n})$ if n is odd.*

Proof. Case 1: n is even.

Let $n = 2k$ where $k \geq 2$ is an integer. By Lemma 4.2, n is in the color-trade-spectrum by considering a single RB(n, n). To show that $\{n+1, ..., \frac{3n}{2} - 2\}$ is in the color-trade-spectrum, consider the value $n+u$ where $1 \leq u \leq \frac{n}{2} - 2$. We construct L_{n+u} by using a single RB$(n - 2u, n-2u)$ and u many RB$(2, 3)$ where each block uses a different set of colors. To find a mate for this edge-coloring, we use the same permutations from Lemma 4.2, where π is applied to the RB$(n-2u, n-2u)$ along with the second row of every RB$(2, 3)$ and σ is applied to the first row of every RB$(2, 3)$. An example of L_{n+u} is shown below along with its associated $n+u$-edge-coloring of $K_{3,n}$.

1	2	3
2	3	4
$\vdots$	$\vdots$	$\vdots$
$n-2u-1$	$n-2u$	1
$n-2u$	1	2
$n-2u+1$	$n-2u+2$	$n-2u+3$
$n-2u+2$	$n-2u+3$	$n-2u+1$
$n-2u+4$	$n-2u+5$	$n-2u+6$
$n-2u+5$	$n-2u+6$	$n-2u+4$
$\vdots$	$\vdots$	$\vdots$
$n+u-5$	$n+u-4$	$n+u-3$
$n+u-4$	$n+u-3$	$n+u-5$
$n+u-2$	$n+u-1$	$n+u$
$n+u-1$	$n+u$	$n+u-2$



Note that when $u = \frac{n}{2} - 2$, we have a single RB$(4,4)$ which we can assume is the same RB$(4,4)$ from the $K_{3,4}$ case without loss of generality. This shows that $n + u \in$ CTS$(K_{3,n}$ for $1 \leq u \leq \frac{n}{2} - 2$. To show $\frac{3n}{2} - 1$ is in the color-trade-spectrum, we can modify our L_{n+u} from above by replacing the RB$(4,4)$ with a RB$(4,5)$ which again we can assume is the the same RB$(4,5)$ from the $K_{3,4}$. Applying π to the first 3 rows of the RB$(4,5)$ and to the second row of every RB$(2,3)$, and applying σ to the last row of RB$(4,5)$ and to the first row of every RB$(2,3)$ yields a mate. Finally, we show $\frac{3n}{2}$ is in the color-trade-spectrum by modifying L_{n+u} again to consist of $\frac{n}{2}$ many RB$(2,3)$. Applying π to the second row of every RB$(2,3)$ and σ to the first row of every RB$(2,3)$ yields a mate.

Case 2: n is odd.

Let $n = 2k + 1$ where $k \geq 1$ is an integer. Again, Lemma 4.2 guarantees n is in the color-trade-spectrum by considering a single RB(n,n). To show $\{n+1, ..., \frac{3n-3}{2}\}$ is in the color-trade-spectrum, we use the same construction from the even case where we consider $n+u$ for $1 \leq u \leq \frac{n-3}{2}$. To show $\frac{3n-1}{2}$ is in the color-trade-spectrum, we consider the following construction of $L_{\frac{3n-1}{2}}$.

$$\begin{array}{|ccc|}
\hline
1 & n+1 & n-k+1 \\
2 & 1 & n-k+2 \\
3 & n+2 & 1 \\
4 & n+3 & n-k+3 \\
\vdots & \vdots & \vdots \\
k+1 & \frac{3n-1}{2} & n \\
k+2 & 2 & n+1 \\
k+3 & 3 & n+2 \\
\vdots & \vdots & \vdots \\
n-1 & n-k-1 & \frac{3n-1}{2}-1 \\
n & n-k & \frac{3n-1}{2} \\
\hline
\end{array}$$

To find a mate for this edge-coloring, apply the same π from the even case to the first $k+1$ rows of $L_{\frac{3n-1}{2}}$ and the same σ from the even case to the last k rows of $L_{\frac{3n-1}{2}}$. $\square$

From Lemmas 4.1, 4.2, and 4.3, along with Theorem 4.1, we have covered every case except for $K_{m,n}$ where $m \geq 5$ is odd and $n \geq 5$. The following theorem uses Lemma 4.3 to cover this case where we consider subcases depending on the parity of n.

Theorem 4.2. *Let $m \geq 5$ be an odd integer and $n \geq m$ be an integer. Then $\{\Delta, ..., \frac{mn}{2}\} =$ CTS$(K_{m,n})$ if $n \geq 6$ is even, and $\{\Delta, ...\frac{mn-1}{2}\} =$ CTS$(K_{m,n})$ if $n \geq 5$ is odd.*

Proof. We construct L_Δ by modifying the construction from Theorem 4.1. Partition the m columns of L into $\frac{m-3}{2}$ column blocks of size 2 where we again suppose 2 columns in a column block are adjacent without loss of generality. The final 3 columns form a column block of size 3. As before, let A_i denote the column block containing columns a_{2i-1} and a_{2i} where $1 \leq i \leq \frac{m-3}{2}$ and let B denote the column block containing columns a_{m-2}, a_{m-1}, and a_m. Fill the cells of a_{2i-1} and a_{2i} in order from top to bottom with entries $c_{2(i-1)+1}, c_{2(i-1)+2}, ..., c_n, c_1, ...c_{2(i-1)}$ and $c_{2(i-1)+2}, c_{2(i-1)+3}, ..., c_n, c_1, ..., c_{2(i-1)+1}$ respectively. Likewise, fill the cells of a_{m-3} and

a_{m-2} with entries $c_{m-2}, c_{m-1}, ..., c_n, c_{n+1}, ..., c_{m-3}$ and $c_{m-1}, c_m, ..., c_n, c_{n+1}, ..., c_{m-2}$ and the cells of a_m with entries $c_m, c_{m+1}, ..., c_n, c_{n+1}, ..., c_{m-1}$.

Let $\pi := (a_1a_2)(a_3a_4)\cdots(a_{m-4}a_{m-3})(a_{m-2}a_{m-1}a_m)$. Then each column and row in $\pi(L_\Delta)$ contains the same symbols as they did in L, while no cell receives the same entry. Thus, $\pi(L_\Delta)$ and L_Δ correspond to two mate colorings of $K_{m,n}$. Using the notation of Lemma 2.3, C_1 and C_2 refer to the corresponding edge-colorings of L_Δ and $\pi(L_\Delta)$. Note that each H_j consists of $\frac{m-3}{2}$ many 4-cycles, where each cycle corresponds to some A_i, and one 6-cycle, which corresponds to B, so $\alpha = \frac{mn-n}{2}$. By Lemma 2.3, we conclude $\{\Delta, \Delta+1, ..., \frac{mn-n}{2}\} \subseteq$ CTS$(K_{m,n})$. We finish the rest of the color-trade-spectrum in cases by using Lemma 4.3.

Case 1: $n \geq 6$ is even.

To show that $\{\frac{mn-n+1}{2}, .., \frac{mn}{2}\} \subset$ CTS$(K_{m,n})$, note that in $L_{\frac{mn-n}{2}}$ each column block uses n different colors, including the column block of size 3, B. Using the argument from the even case of Lemma 4.3, we conclude that we can extend the number of colors in B, j, to any value of j where $n \leq j \leq \frac{3n}{2}$. Using the appropriate permutations from Lemma 4.3, we conclude that $\{\frac{mn-n+1}{2}, ..., \frac{mn}{2}\} \subset$ CTS$(K_{m,n})$. Combining this result with the result of the previous paragraph, this shows that CTS$(K_{m,n}) = \{\Delta, ..., \frac{mn}{2}\}$.

Case 2: $n \geq 5$ is odd.

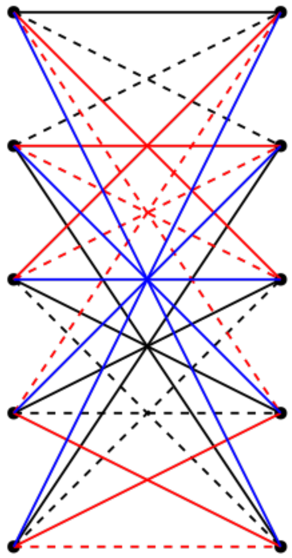
To show that $\{\frac{mn-n+1}{2}, ..., \frac{mn-1}{2}\} \subset$ CTS$(K_{m,n})$, we use the same argument as in the even case, refering to the odd case of Lemma 4.3. We conclude that CTS$(K_{m,n}) = \{\Delta, ..., \frac{mn-1}{2}\}$.

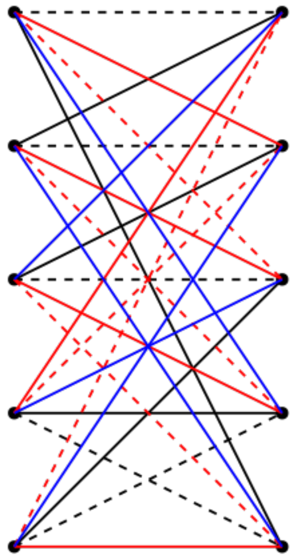
$\square$

Below are examples L_Δ and $\pi(L_\Delta)$ along with their corresponding graphs for $K_{5,5}$.

L_Δ											
	1	2	3	4	$\cdots$	$m-4$	$m-3$	$m-2$	$m-1$	m
	2	3	4	5	$\cdots$	$m-3$	$m-2$	$m-1$	m	$m+1$
	3	4	5	6	$\cdots$	$m-2$	$m-1$	m	$m+1$	$m+2$
	4	5	6	7	$\cdots$	$m-1$	m	$m+1$	$m+2$	$m+3$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-4$	$n-3$	$n-2$	$n-1$	$\cdots$	$m-9$	$m-8$	$m-7$	$m-6$	$m-5$
	$n-3$	$n-2$	$n-1$	n	$\cdots$	$m-8$	$m-7$	$m-6$	$m-5$	$m-4$
	$n-2$	$n-1$	n	1	$\cdots$	$m-7$	$m-6$	$m-5$	$m-4$	$m-3$
	$n-1$	n	1	2	$\cdots$	$m-6$	$m-5$	$m-4$	$m-3$	$m-2$
	n	1	2	3	$\cdots$	$m-5$	$m-4$	$m-3$	$m-2$	$m-1$

$\pi(L_\Delta)$										
	2	1	4	3	$\cdots$	$m-3$	$m-4$	m	$m-2$	$m-1$
	3	2	5	4	$\cdots$	$m-2$	$m-3$	$m+1$	$m-1$	m
	4	3	6	5	$\cdots$	$m-1$	$m-2$	$m+2$	m	$m+1$
	5	4	7	6	$\cdots$	m	$m-1$	$m+3$	$m+1$	$m+2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-3$	$n-4$	$n-1$	$n-2$	$\cdots$	$m-8$	$m-9$	$m-5$	$m-7$	$m-6$
	$n-2$	$n-3$	n	$n-1$	$\cdots$	$m-7$	$m-8$	$m-4$	$m-6$	$m-5$
	$n-1$	$n-2$	1	n	$\cdots$	$m-6$	$m-7$	$m-3$	$m-5$	$m-4$
	n	$n-1$	2	1	$\cdots$	$m-5$	$m-6$	$m-2$	$m-4$	$m-3$
	1	n	3	2	$\cdots$	$m-4$	$m-5$	$m-1$	$m-3$	$m-2$





Chapter 5

Products of Paths

Denote the Cartesian product of two graphs G and H by $G\Box H$ which consists of the vertex set $\{(g_i, h_j)|g_i \in V(G), h_j \in V(H)\}$ where two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. We will let (i, j) denote the ith vertex along the jth path where $i \in \{1, 2, .., m\}$ and $j \in \{1, 2\}$, and let $(s, t)(u, v)$ denote the edge incident to the vertices (s, t) and (u, v), assuming such an edge exists. Let $2 \leq n \leq m$ be integers and denote by P_n the path on n vertices. For convenience, let $G = P_n\Box P_m$. Then $|V(G)| = nm$ and $|E(G)| = n(m-1) + m(n-1)$. For $n = m = 2$, $\delta(G) = \Delta(G) = 2$. For $n = 2$ and $m \geq 3$, $\delta(G) = 2$ and $\Delta(G) = 3$. For $3 \leq n \leq m$, $\delta(G) = 2$ and $\Delta(G) = 4$. Thus, by Lemma 2.3, CTS$(G) \subseteq \{4, 5, ..., \lfloor\frac{n(m-1)+m(n-1)}{2}\rfloor\}$ for $3 \leq n \leq m$. When $n = m = 2$, $G \cong C_4$ so CTS$(G) = \{2\}$. When $n = 2$ and $m \geq 3$, CTS$(G) \subseteq \{3, 4, ..., \lfloor\frac{n(m-1)+m(n-1)}{2}\rfloor\}$. Since $P_n\Box P_m$ is clearly isomorphic to $P_m\Box P_n$, we will only consider $P_n\Box P_m$, so this allows us to assume $n \leq m$ without loss of generality.

We begin with the case where $n = 2$ and $m \geq 3$.

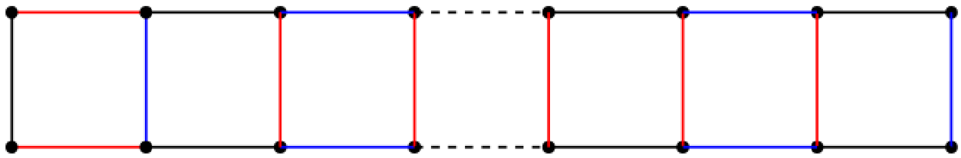
5.1 $P_2\Box P_m$ Graphs

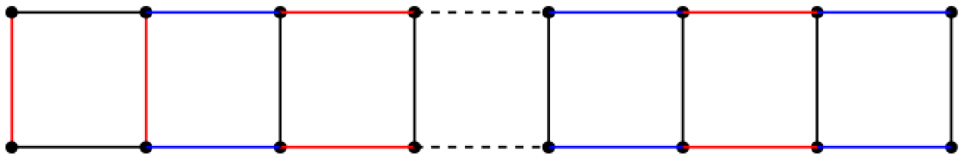
Lemma 5.1. *Let $m \geq 3$ be an integer. Then $\{3, 4, ..., m\} \subseteq CTS(P_2\Box P_m)$.*

Proof. As above, let $G = P_2\Box P_m$ for convenience. We consider two cases depending if m is even or odd.

Case 1: $m \geq 3$ is odd.

Give G a proper 3-edge-coloring as follows. Assign c_1 to $(1,1)(1,2), (2i+2,1)(2i+3,1)$, and $(2i+2,2)(2i+3,2)$ for $0 \leq i \leq \frac{m-3}{2}$. Assign c_2 to $(1,1)(2,1), (1,2)(2,2)$, and $(i+3,1)(i+3,2)$ for $0 \leq i \leq m-4$. Assign c_3 to $(2,1)(2,2), (m,1)(m,2), (2i+1,1)(2i+2,1)$, and $(2i+1,2)(2i+2,2)$ for $1 \leq i \leq \frac{m-3}{2}$. This edge-coloring has the following mate coloring. Assign c_1 to $(1,1)(2,1), (1,2)(2,2)$, and $(i+3,1)(i+3,2)$ for $0 \leq i \leq m-3$. Assign c_2 to $(1,1)(1,2), (2,1)(2,2), (2i+1,1)(2i+2,1)$, and $(2i+1,2)(2i+2,2)$ for $1 \leq i \leq \frac{m-3}{2}$. Assign c_3 to $(2i+2,1)(2i+3,1)$, and $(2i+2,2)(2i+3,2)$ for $0 \leq i \leq \frac{m-3}{2}$. Examples of these edge-colorings are given below respectively where c_1 is black, c_2 is red, and c_3 is blue.

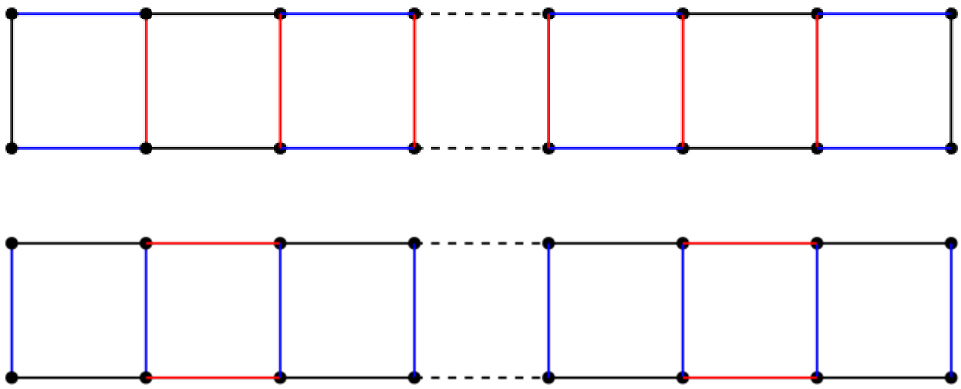




Using the notation of Lemma 2.3, note that H_1 consists of $\frac{m-3}{2}$ many 4-cycles and one 6-cycle, while H_2 consists of $\frac{m-1}{2}$ many 4-cycles and H_3 consists of one $2(m-1)$ cycle. Thus, $\alpha = m$, so by Lemma 2.3 we conclude $\{3, ..., m\} \subseteq \text{CTS}(P_2 \square P_m)$ when $m \geq 3$ is odd.

Case 2: $m \geq 3$ is even.

Give G a proper 3-edge-coloring as follows. Assign c_1 to $(1,1)(1,2), (2i+2,1)(2i+3,1), (2i+2,2)(2i+3,2)$, and $(m,1)(m,2)$ for $0 \leq i \leq \frac{m-4}{2}$. Assign c_2 to $(2i+1,1)(2i+2,1)$ and $(2i+1,2)(2i+2,2)$ for $0 \leq \frac{m-2}{2}$. Assign c_3 to $(i+2,1)(i+2,2)$ for $0 \leq i \leq m-3$. This edge-coloring has the following mate coloring. Assign c_1 to $(2i+1,1)(2i+2,1)$ and $(2i+1,2)(2i+2,2)$ for $0 \leq i \leq \frac{m-2}{2}$. Assign c_2 to $(i,1)(i,2)$ for $1 \leq i \leq m$. Assign c_3 to $(2i+2,1)(2i+3,1)$ and $(2i+2,2)(2i+3,2)$ for $0 \leq i \leq \frac{m-4}{2}$. Examples of these edge-colorings are given below respectively where c_1 is black, c_2 is blue, and c_3 is red.



Using the notation of Lemma 2.3, note that H_1 consists of 1 $2m$ cycle, while H_2 consists of $\frac{m}{2}$ many 4-cycles and H_3 consists of $\frac{m-2}{2}$ many 4-cycles. Thus, $\alpha = m$, so by Lemma 2.3 we conclude $\{3, ..., m\} \subseteq \text{CTS}(P_2 \square P_m)$ when $m \geq 3$ is even.

□

Theorem 5.1. *Let $m \geq 3$ be an integer. Then $\{3, 4, ..., m\} = CTS(P_2 \square P_m)$.*

Proof. From Lemma 5.1, it remains to show that nothing else is contained in the color-trade-spectrum. Note that G has m *vertical* edges of the form $(i, 1)(i, 2)$ for $1 \leq i \leq m$. Suppose that we have two copies of G under the mate colorings of C_1 and C_2 respectively, each on the same set of k colors. Between the two copies of G, each color uses at least two vertical edges. Furthermore, each copy of a vertical edge receives a different color in each copy of G. Therefore, $k \leq m$, so the color-trade-spectrum contains no values larger than m. By Lemma 5.1, we conclude $\{3, 4, ..., m\} = \text{CTS}(P_2 \square P_m)$.

□

5.2 $P_n \square P_m$ Graphs

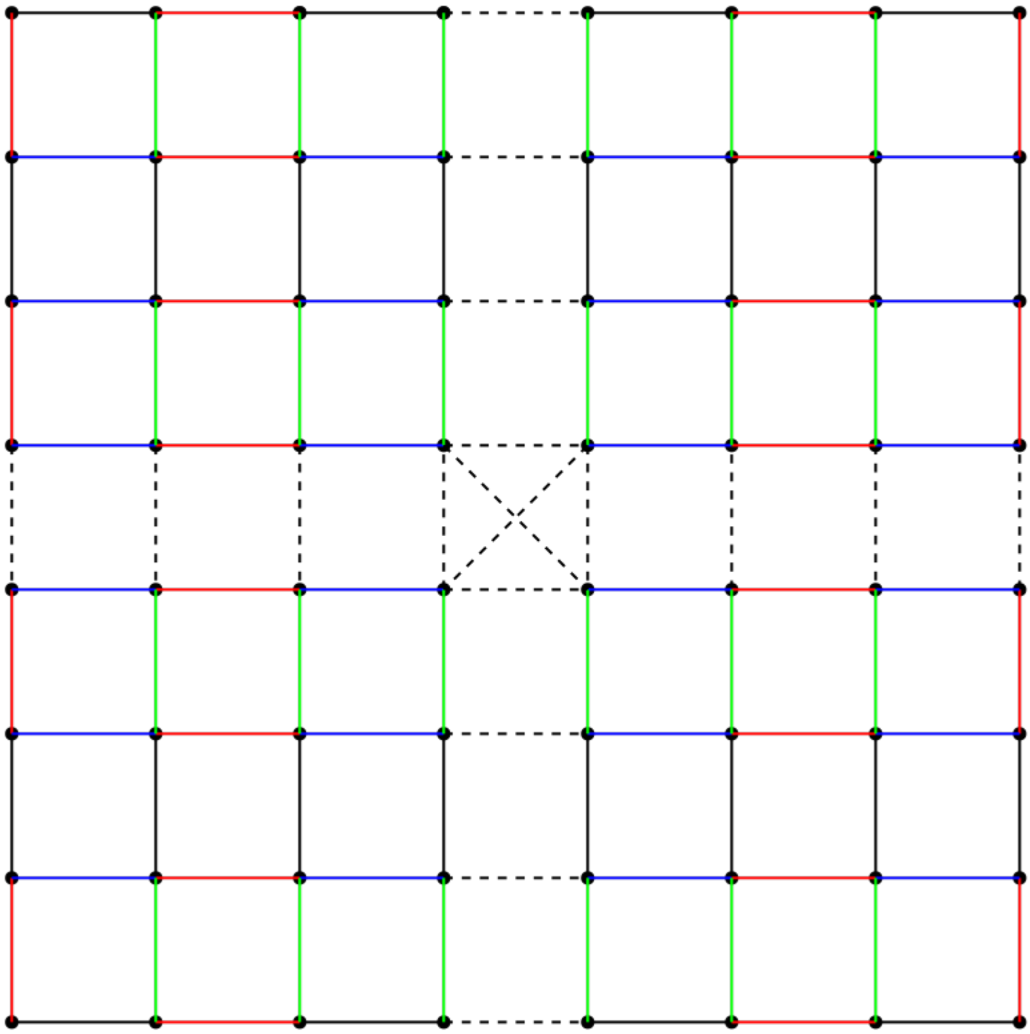
From the beginning of this chapter, recall that $P_n \square P_m \cong P_m \square P_n$ so we assume $n \leq m$ without loss of generality.

Theorem 5.2. *Let m and n be integers where $3 \leq n \leq m$. Then $\{4, 5, ..., (n-1)(m-1)+1\} \subseteq CTS(P_n \square P_m)$.*

Proof. We consider 4 cases depending on the parities of m and n.

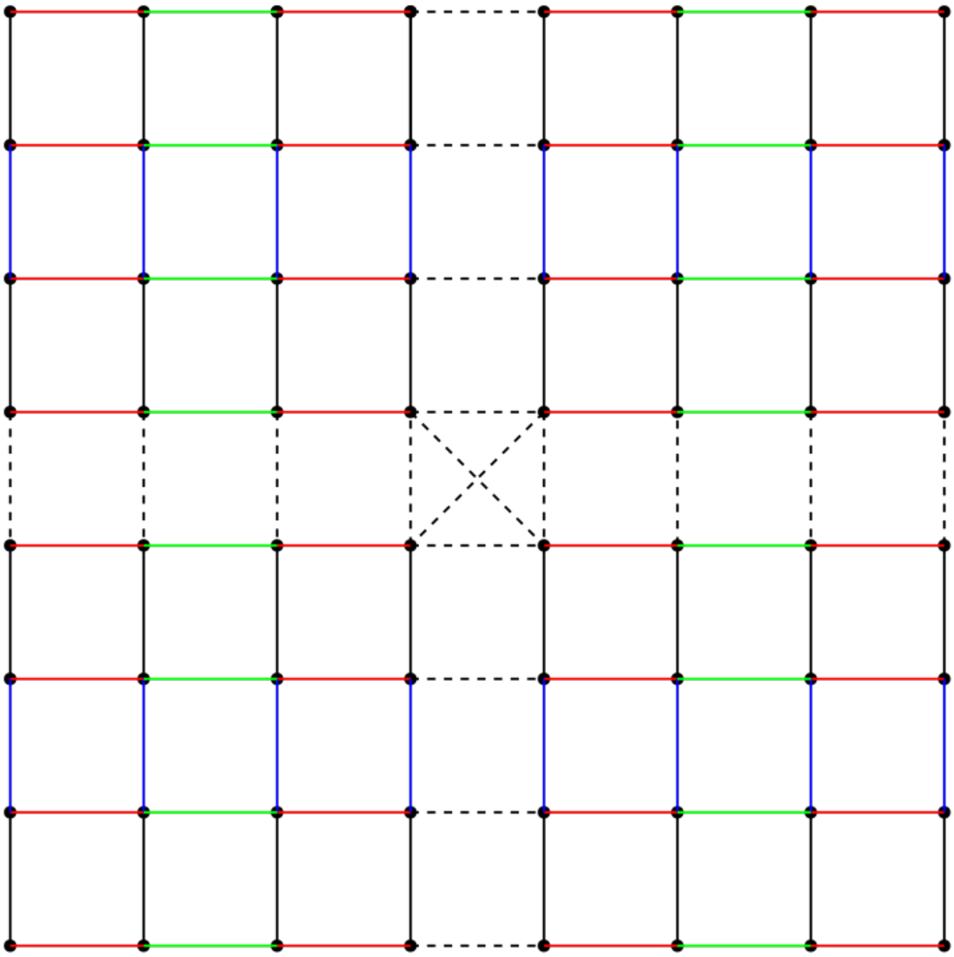
Case 1: n and m are both even.

Give G a proper 4-edge-coloring as follows. Assign c_1 to $(2i-1,1)(2i,1)$, $(2i-1,n)(2i,n)$, $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_2 to $(1,2j-1)(1,2j)$, $(m,2j-1)(m,2j)$, $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$ and $(2i-1,2j+1)(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_4 to $(2i,2j-1)(2i,2j)$, and $(2i+1,2j-1)(2i+1,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below where c_1 is black, c_2 is red, c_3 is blue, and c_4 is green.

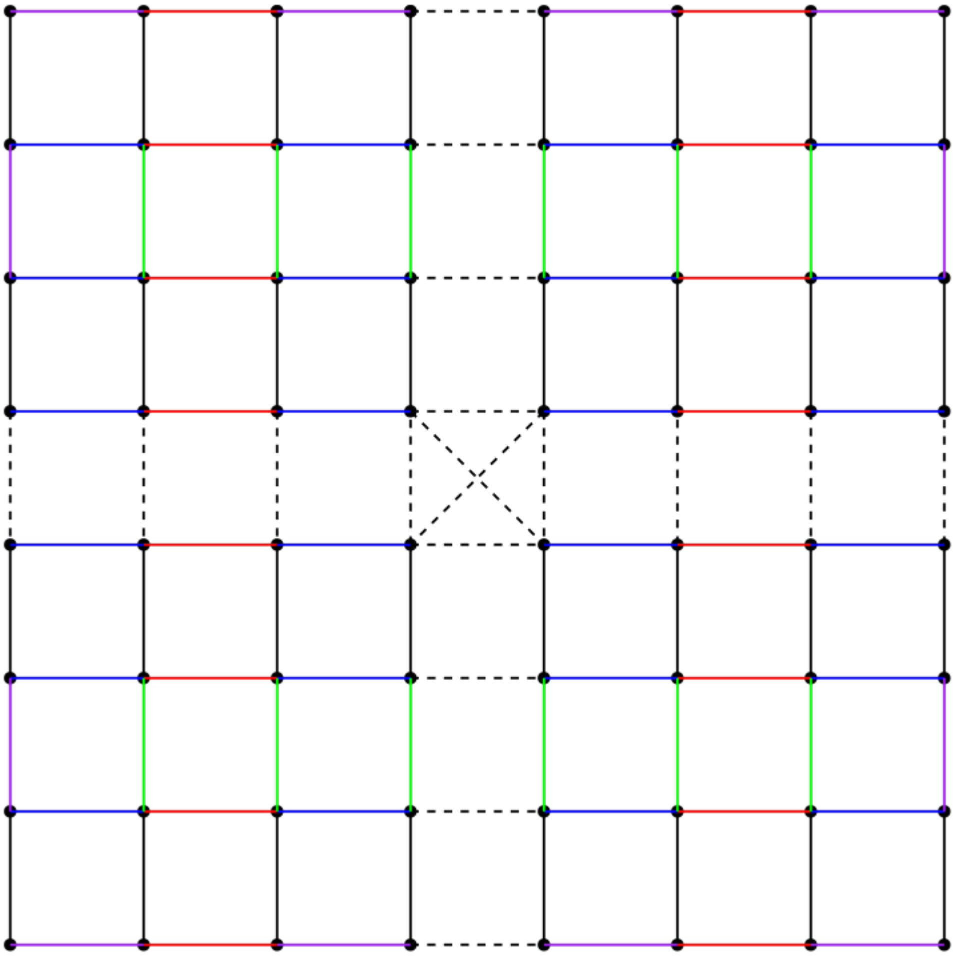


The above edge-coloring has the following mate coloring. Assign c_1 to $(i,2j-1)(i,2j)$ for $1 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2}$. Assign c_2 to $(2i-1,2j-1)(2i,2j-1)$, and $(2i-1,2j)(2i,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for

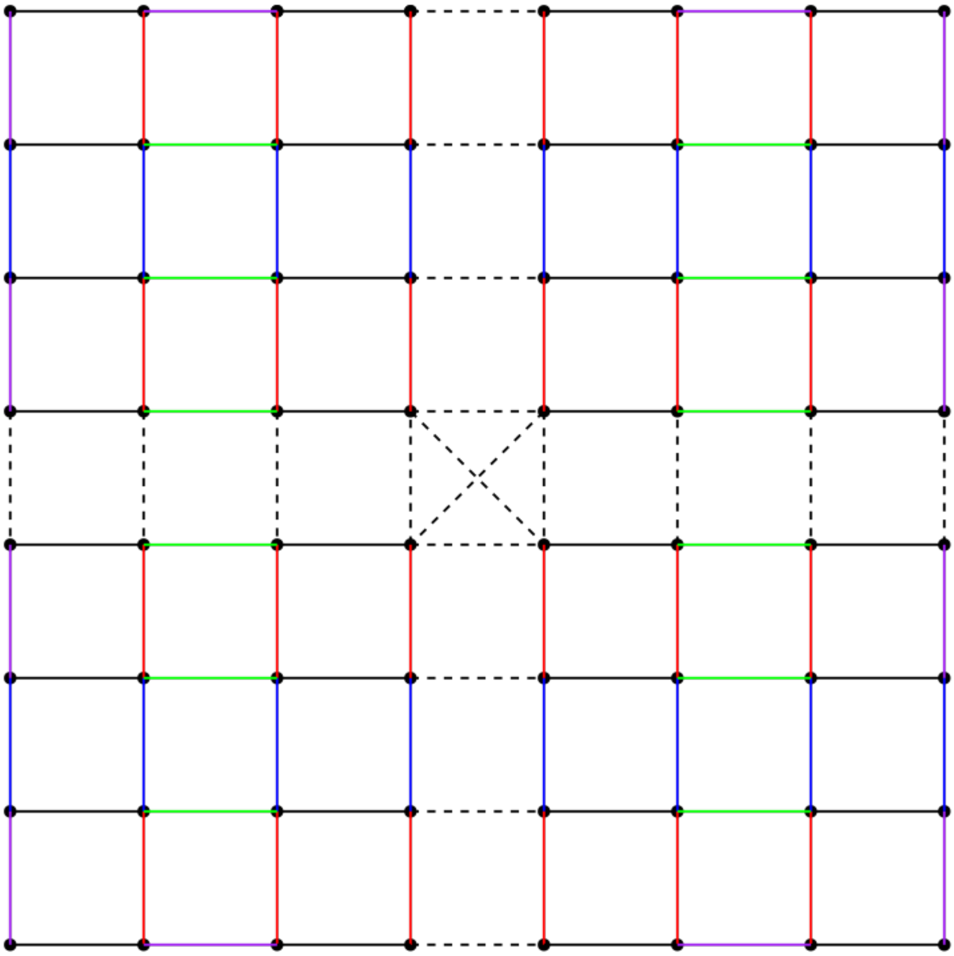
$1 \leq i \leq \frac{m}{2}$ and $1 \leq i \leq \frac{n-2}{2}$. Assign c_4 to $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)$, $(2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below.



Now, give G a proper 5-edge-coloring as follows. Assign c_1 to $(2i-1, 2j-1)(2i-1, 2j)$, and $(2i, 2j-1)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_2 to $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1, 2j)(2i, 2j)$, and $(2i-1, 2j+1)(2i, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_4 to $(2i, 2j)(2i, 2j+1)$, and $(2i+1, 2j)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_5 to $(2i-1, 1)(2i, 1)$, $(2i-1, n)(2i, n)$, $(1, 2j)(1, 2j+1)$, and $(m, 2j)(m, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-4}{2}$. We use the same assignment of colors as above and include c_5 as purple.



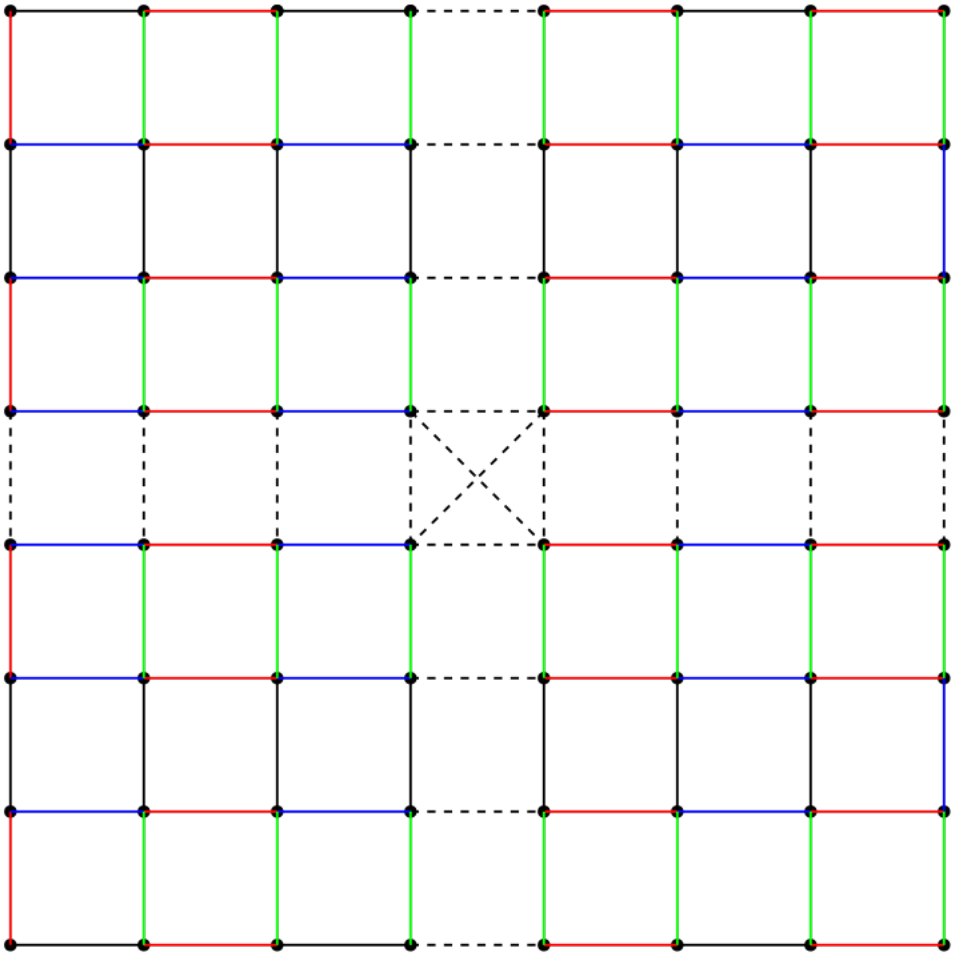
The above edge-coloring has the following mate coloring. Assign c_1 to $(2i-1,2j-1)(2i,2j-1)$, and $(2i-1,2j)(2i,2j)$ for $1 \le i \le \frac{m}{2}$ and $1 \le j \le \frac{n}{2}$. Assign c_2 to $(2i,2j-1)(2i,2j)$, and $(2i+1,2j-1)(2i+1,2j)$ for $1 \le i \le \frac{m-2}{2}$ and $1 \le j \le \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \le i \le \frac{m}{2}$ and $1 \le j \le \frac{n-2}{2}$. Assign c_4 to $(2i,2j)(2i+1,2j)$, and $(2i,2j+1)(2i+1,2j+1)$ for $1 \le i \le \frac{m-2}{2}$ and $1 \le j \le \frac{n-2}{2}$. Assign c_5 to $(2i,1)(2i+1,1)$, $(2i,n)(2i+1,n)$, $(1,2j-1)(1,2j)$, and $(m,2j-1)(m,2j)$ for $1 \le i \le \frac{m-2}{2}$ and $1 \le j \le \frac{n}{2}$. An example is shown below.



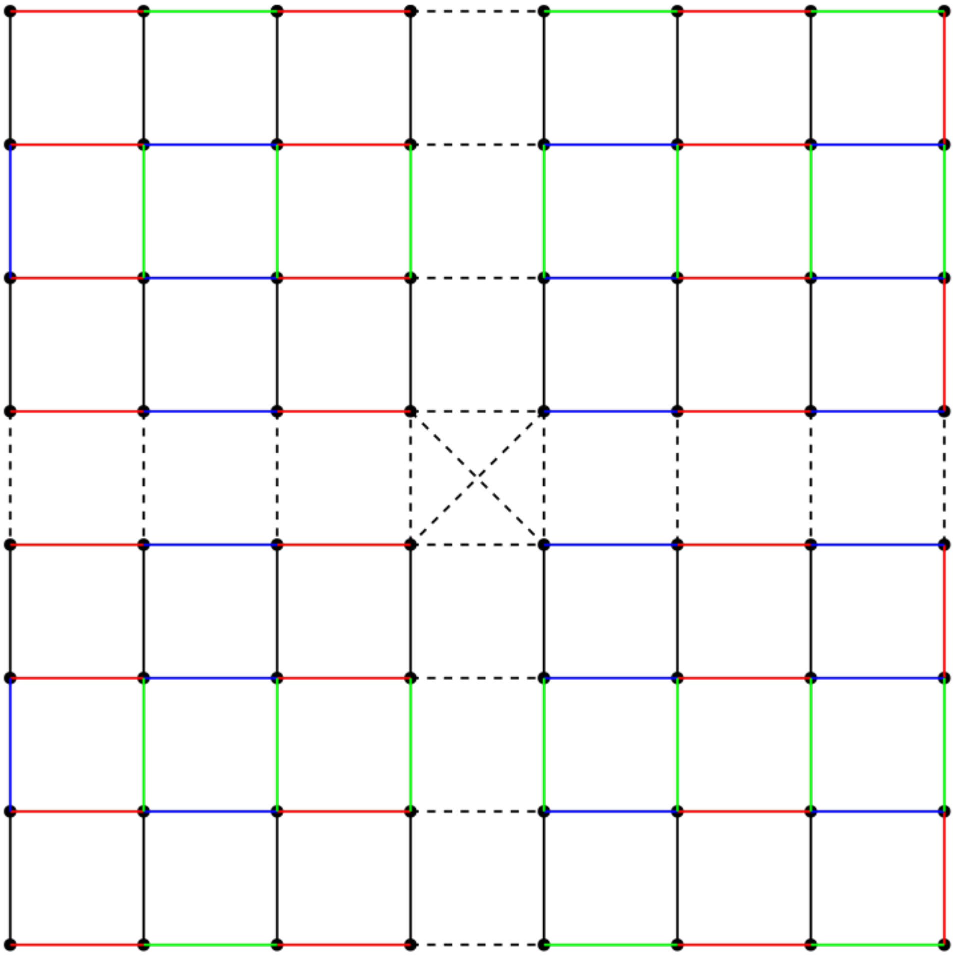
Using the notation of Lemma 2.3, note that H_1 consists of $\frac{mn}{4}$ many 4-cycles, while H_2, H_3, and H_4 each consist of $\frac{n(m-2)}{4}$, $\frac{m(n-2)}{4}$, and $\frac{(m-2)(n-2)}{4}$ many 4-cycles respectively. H_5 consists of a single $2(m-1)+2(n-1)$ cycle, so $\alpha = (n-1)(m-1)+1$ and by Lemma 2.3 we conclude $\{4, ..., (n-1)(m-1)+1\} \subseteq \text{CTS}(P_n \square P_m)$.

Case 2: n is even and m is odd.

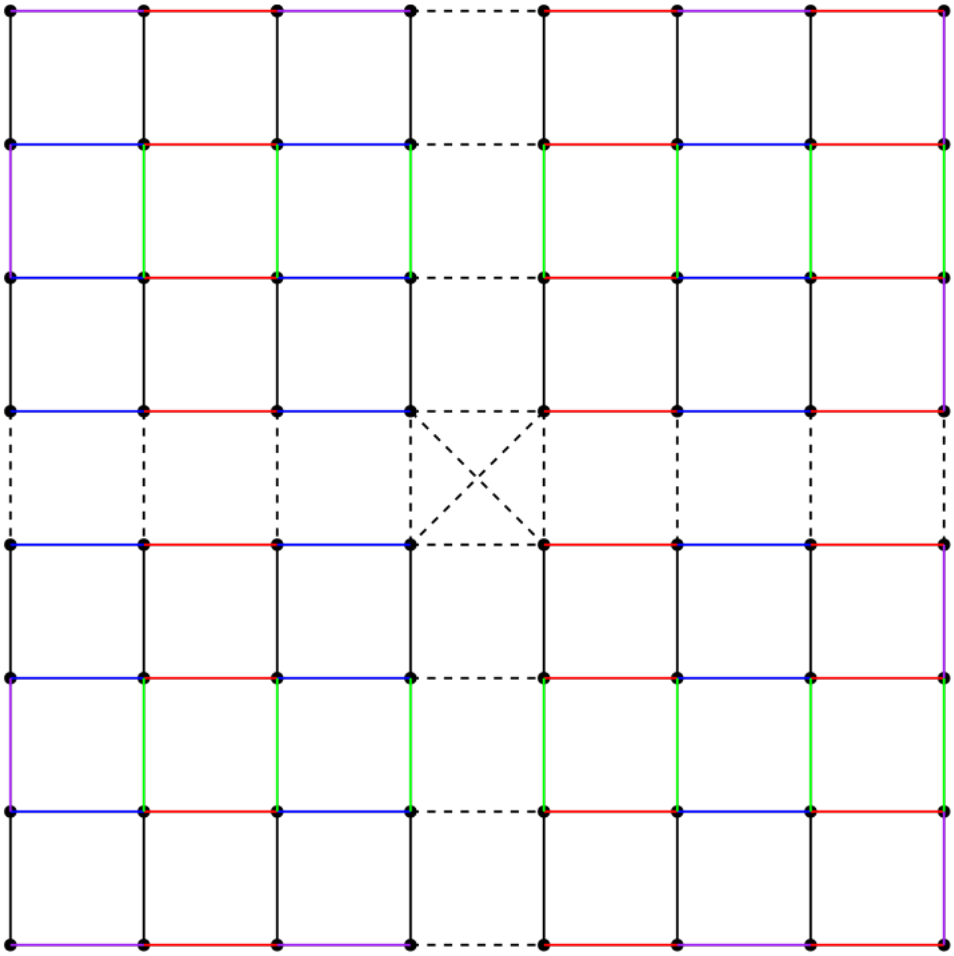
Give G a proper 4-edge-coloring as follows. Assign c_1 to $(2i-1,1)(2i,1)$, $(2i-1,m)(2i,m)$, $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_2 to $(1,2j-1)(1,2j)$, $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$, $(2i-1,2j+1)(2i,2j+1)$, and $(m,2j)(m,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_4 to $(i,2j-1)(i,2j)$ for $2 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2}$. An example is shown below.



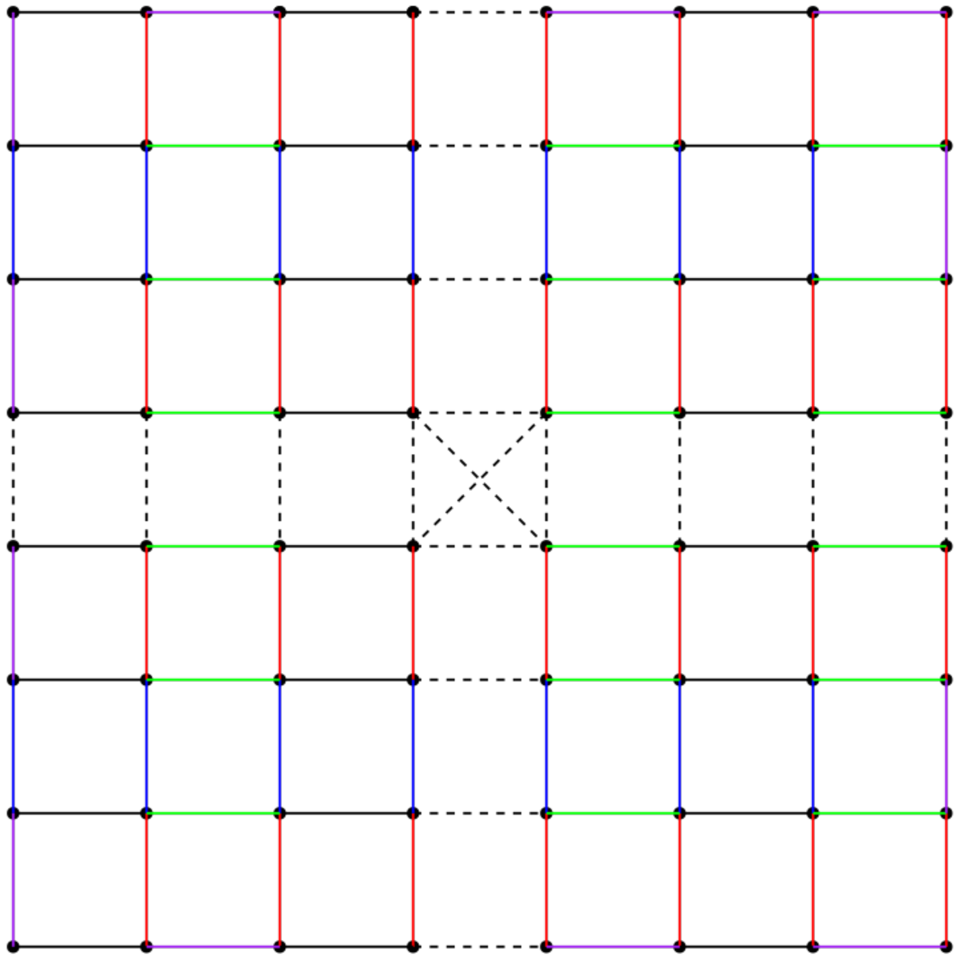
The above edge-coloring has the following mate coloring. Assign c_1 to $(2i-1,2j-1)(2i-1,2j)$, and $(2i,2j-1)(2i,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_2 to $(2i-1,2j-1)(2i,2j-1)$, $(2i-1,2j)(2i,2j)$, and $(m,2j-1)(m,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(1,2j)(1,2j+1)$, $(2i,2j)(2i+1,2j)$, and $(2i,2j+1)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq i \leq \frac{n-2}{2}$. Assign c_4 to $(2i,1)(2i+1,1)$, $(2i,n)(2i+1,n)$, $(2i,2j)(2i,2j+1)$, and $(2i+1,2j)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. An example is given below.



Now, give G a proper 5-edge-coloring as follows. Assign c_1 to $(2i-1,2j-1)(2i-1,2j)$, and $(2i,2j-1)(2i,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_2 to $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$, and $(2i-1,2j+1)(2i,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_4 to $(2i,2j),(2i,2j+1)$, and $(2i+1,2j)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_5 to $(2i-1,1)(2i,1)$, $(2i-1,n)(2i,n)$, $(1,2j)(1,2j+1)$, $(m,2j-1)(m,2j)$, and $(m,n-1)(m,n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. An example is given below.



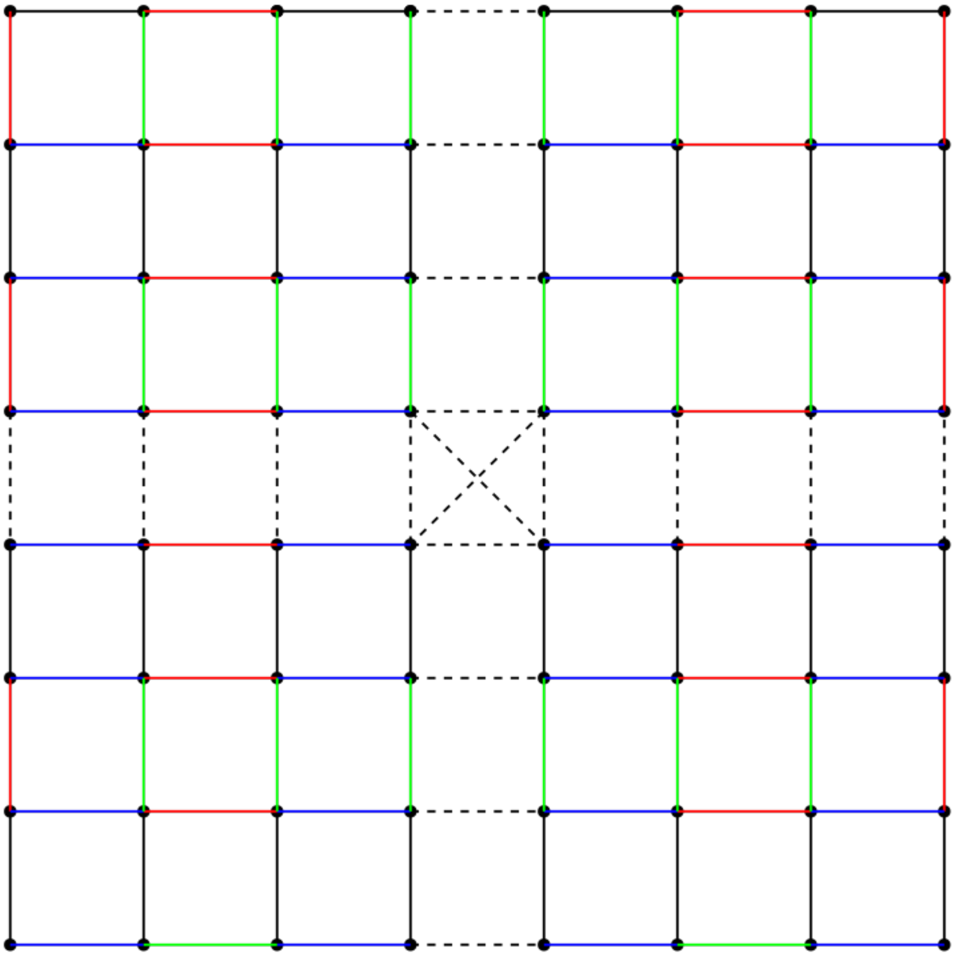
The above edge-coloring has the following mate coloring. Assign c_1 to $(2i-1,2j-1)(2i,2j-1)$, and $(2i-1,2j)(2i,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_2 to $(2i,2j-1)(2i,2j)$, and $(2i+1,2j-1)(2i+1,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_3 to $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_4 to $(2i,2j)(2i+1,2j)$, and $(2i,2j+1)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-2}{2}$. Assign c_5 to $(2i,1)(2i+1,1)$, $(2i,n)(2i+1,n)$, $(1,2j-1)(1,2j)$, $(m,2j)(m,2j+1)$, and $(1,n-1)(1,n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n}{2}$. An example is given below.



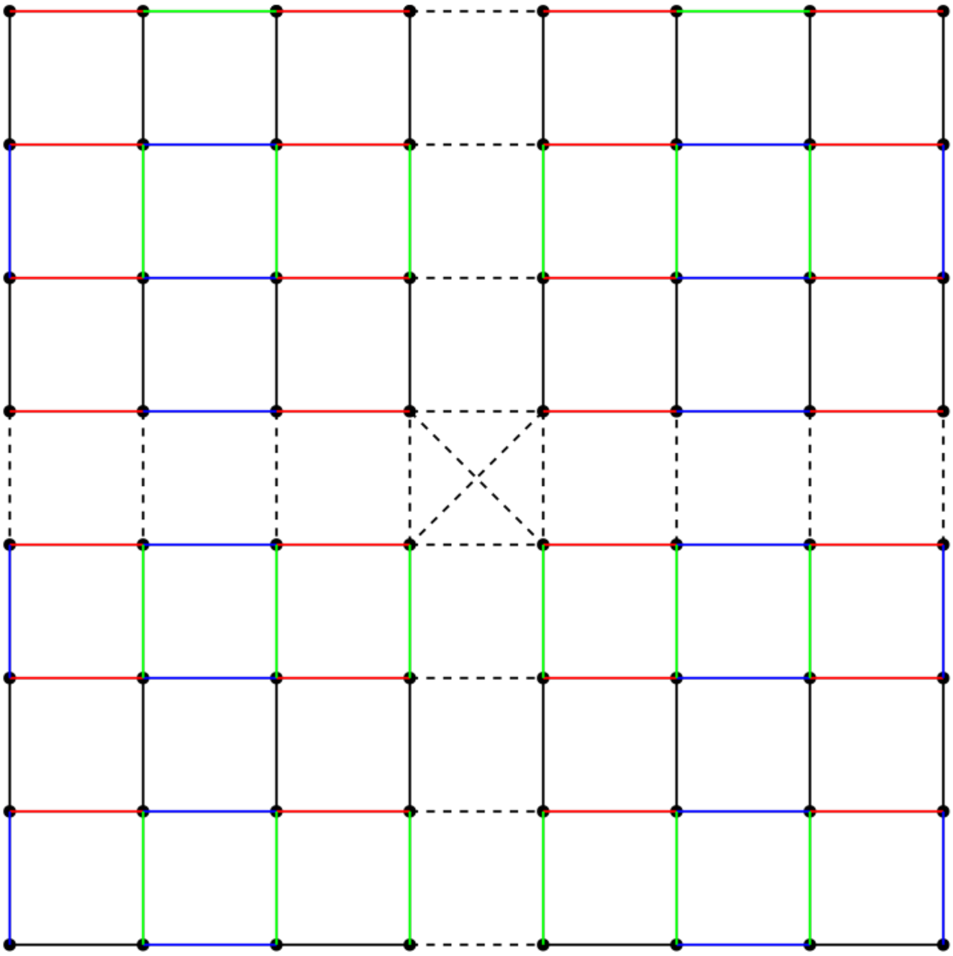
Using the notation of Lemma 2.3, note that H_1 and H_2 each consist of $\frac{(m-1)n}{4}$ many 4-cycles, while H_3 and H_4 each consist of $\frac{(m-1)(n-2)}{4}$ many 4-cycles. H_5 consists of a single $2(m-1)+2(n-1)$ cycle, so $\alpha = (n-1)(m-1)+1$ and by Lemma 2.3 we conclude $\{4, ..., (n-1)(m-1)+1\} \subseteq \text{CTS}(P_n \square P_m)$.

Case 3: n is odd and m is even.

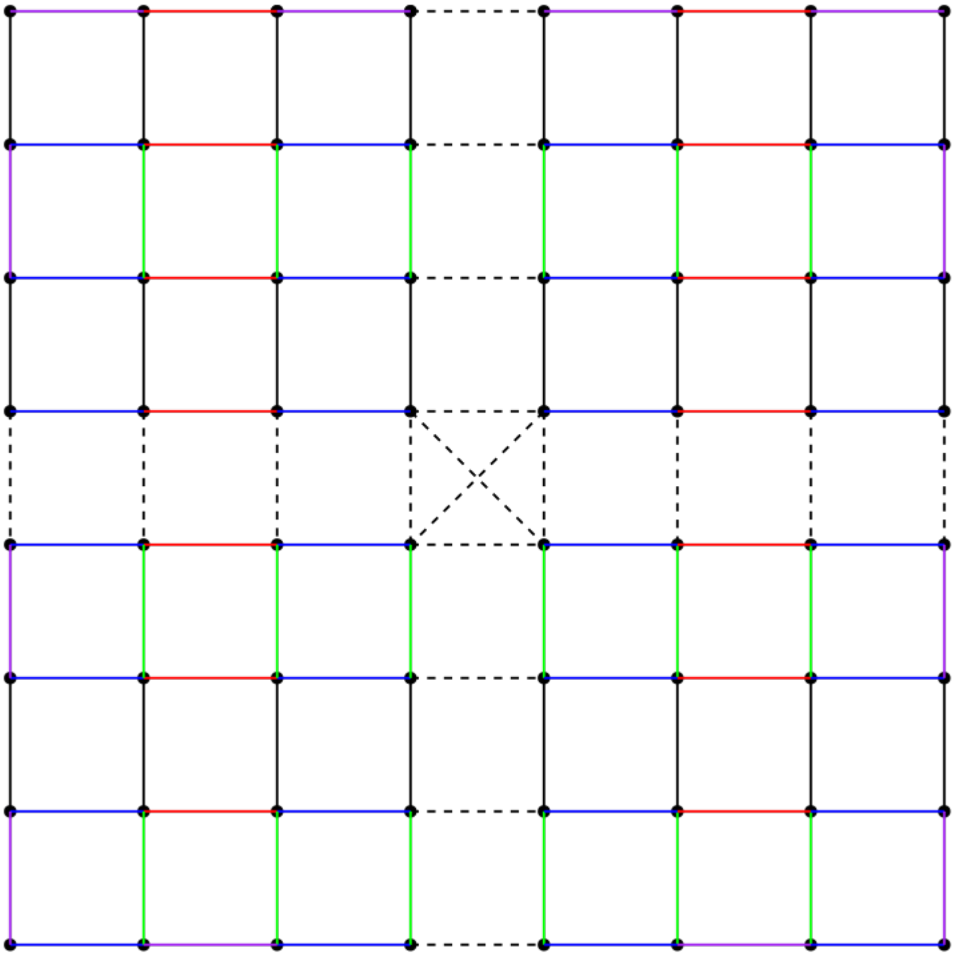
Give G a proper 4-edge-coloring as follows. Assign c_1 to $(2i-1,1)(2i,1)$, $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(1,2j-1)(1,2j)$, $(m,2j-1)(m,2j)$, $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$, and $(2i-1,2j+1)$, $(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_4 to $(2i,2j-1)(2i,2j)$, $(2i+1,2j-1)(2i+1,2j)$, and $(2i,n)(2i+1,n)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



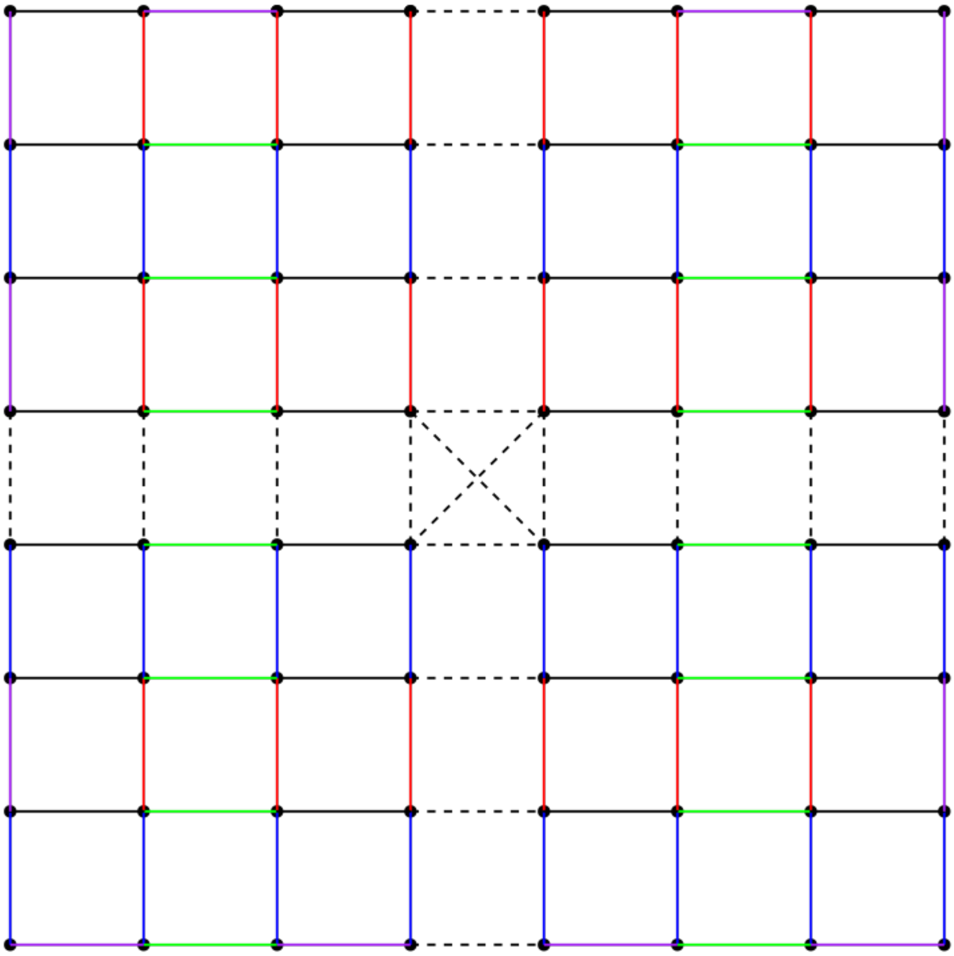
The above edge-coloring has the following mate coloring. Assign c_1 to $(2i-1, 2j-1)(2i-1, 2j)$, $(2i, 2j-1)(2i, 2j)$, and $(2i-1, n)(2i, n)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i-1, 2j-1)(2i, 2j-1)$, and $(2i-1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(1, 2j)(1, 2j+1)$, $(m, 2j)(m, 2j+1)$, $(2i, 2j)(2i+1, 2j)$, and $(2i, 2j+1)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq i \leq \frac{n-1}{2}$. Assign c_4 to $(2i, 1)(2i+1, 1)$, $(2i, 2j)(2i, 2j+1)$, and $(2i+1, 2j)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



Now, give G a proper 5-edge-coloring as follows. Assign c_1 to $(2i-1,2j-1)(2i-1,2j)$, and $(2i,2j-1)(2i,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$, and $(2i-1,2j+1)(2i,2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_4 to $(2i,2j)(2i,2j+1)$, and $(2i+1,2j)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-2}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_5 to $(2i-1,1)(2i,1)$, $(2i,n)(2i+1,n)$, $(1,2j)(1,2j+1)$, and $(m,2j-1)(m,2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



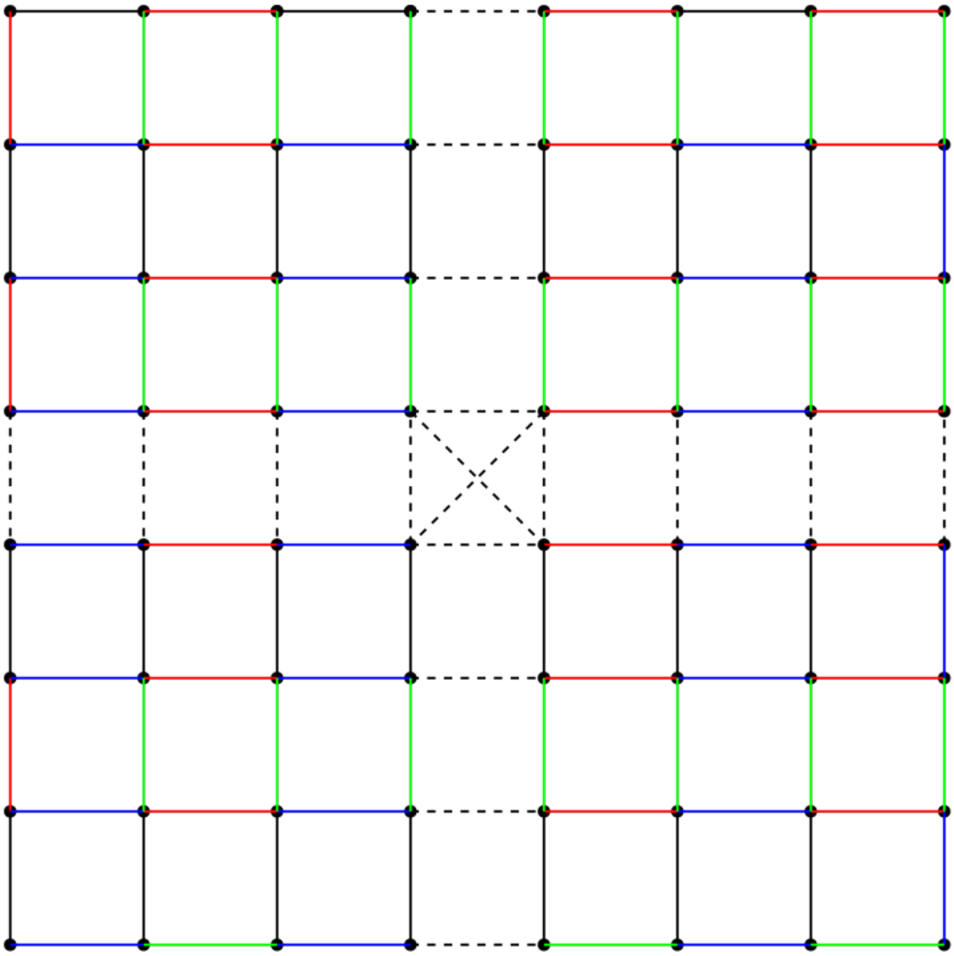
The above edge-coloring admits the following mate coloring. Assign c_1 to $(2i-1, 2j-1)(2i, 2j-1)$, and $(2i-1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i, 2j-1)(2i, 2j)$, and $(2i+1, 2j-1)(2i+1, 2j)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1, 2j)(2i-1, 2j+1)$, and $(2i, 2j)(2i, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j\frac{n-1}{2}$. Assign c_4 to $(2i, 2j)(2i+1, 2j)$, and $(2i, 2j+1)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n}{2}$. Assign c_5 to $(2i, 1)(2i+1, 1)$, $(2i-1, n)(2i, n)$, $(1, 2j-1)(1, 2j)$, and $(m, 2j)(m, 2j+1)$ for $1 \leq i \leq \frac{m}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



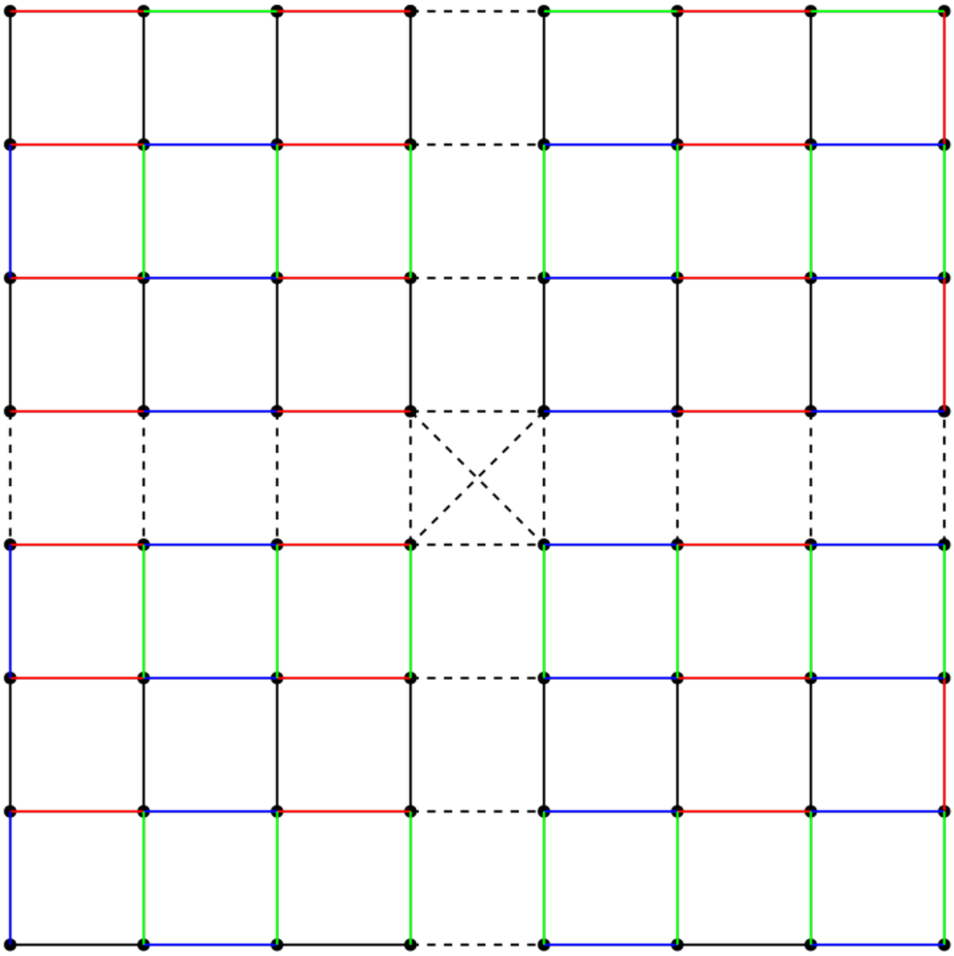
Using the notation of Lemma 2.3, note that H_1 and H_3 each consist of $\frac{m(n-1)}{4}$ many 4-cycles, while H_2 and H_4 each consist of $\frac{(m-2)(n-2)}{1}$ many 4-cycles. H_5 consists of a single $2(m-1)+2(n-1)$ cycle, so $\alpha = (n-1)(m-1)+1$ and by Lemma 2.3 we conclude $\{4, ..., (n-1)(m-1)+1\} \subseteq \mathrm{CTS}(P_n \square P_m)$.

Case 4: n and m are both odd.

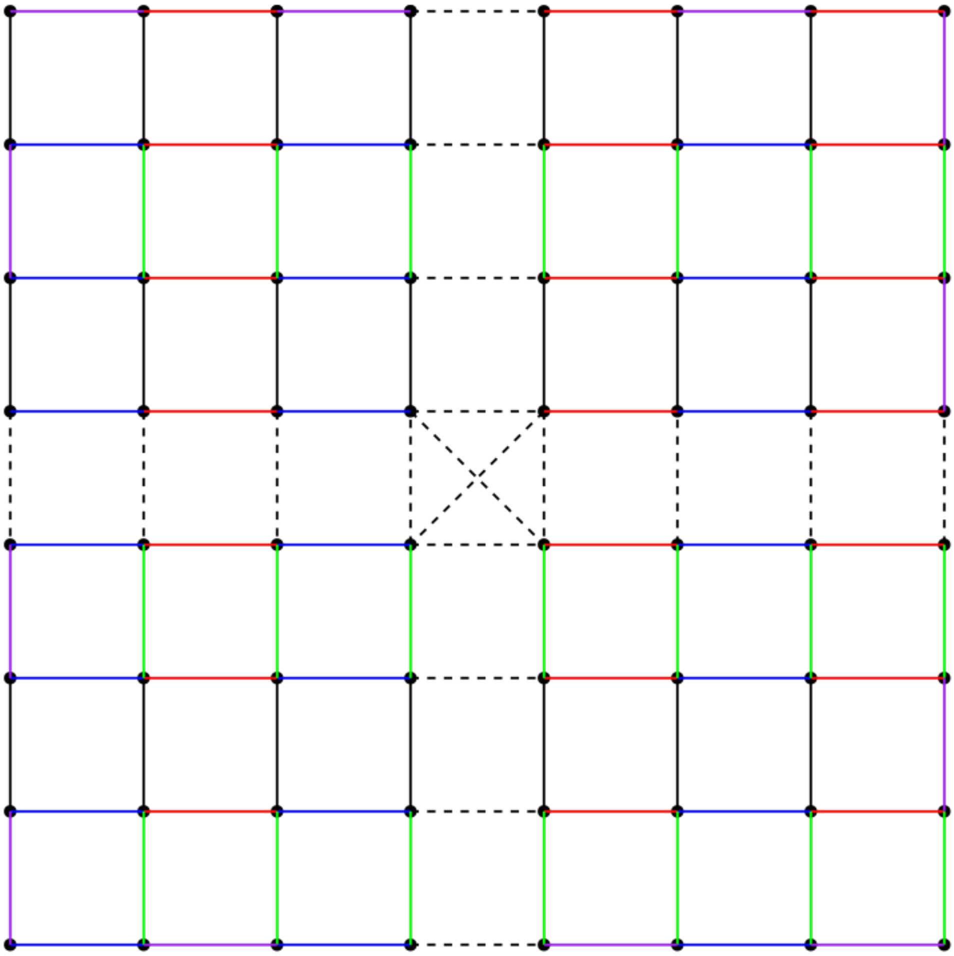
Give G a proper 4-edge-coloring as follows. Assign c_1 to $(2i-1,1)(2i,1)$, $(2i-1,2j)(2i-1,2j+1)$, and $(2i,2j)(2i,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(1,2j-1)(1,2j)$, $(2i,2j-1)(2i+1,2j-1)$, and $(2i,2j)(2i+1,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1,2j)(2i,2j)$, $(2i-1,2j+1)(2i,2j+1)$, and $(m,2j)(m,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_4 to $(2i,2j-1)(2i,2j)$, $(2i+1,2j-1)(2i+1,2j)$, and $(2i,n)(2i+1,n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



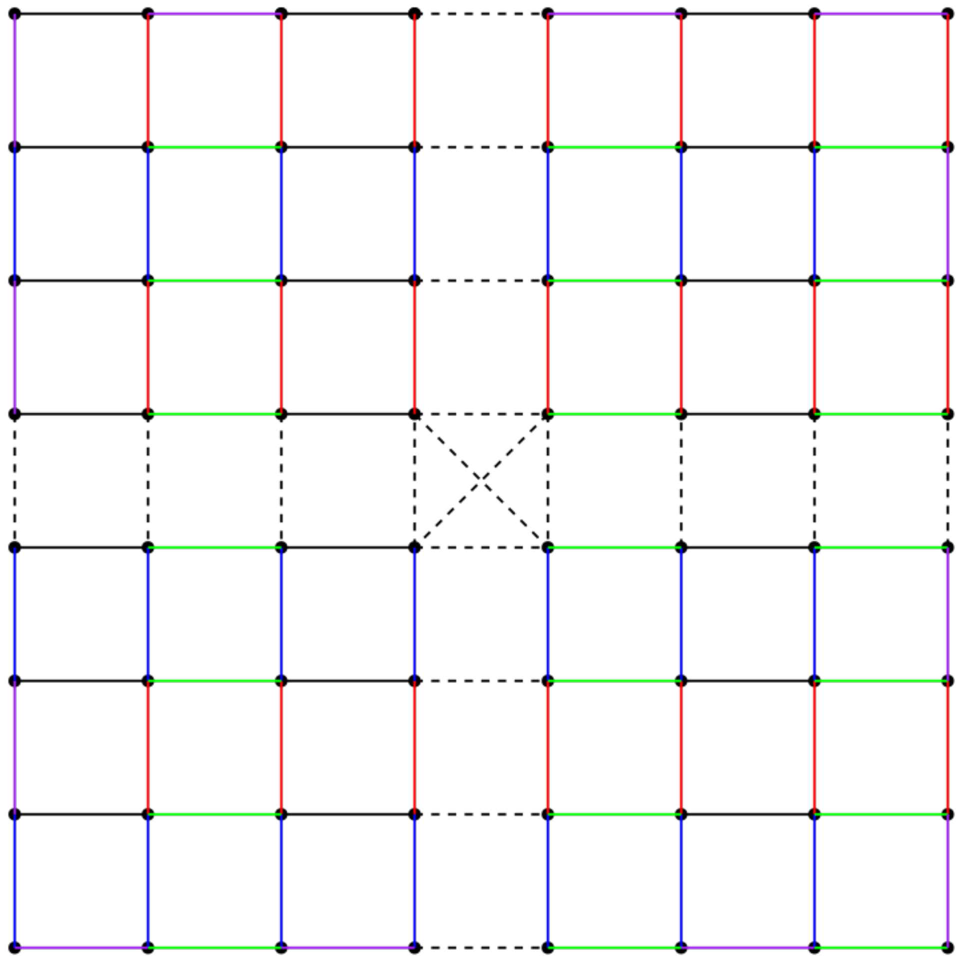
The above edge-coloring admits the following mate coloring. Assign c_1 to $(2i-1,2j-1)(2i-1,2j)$, $(2i,2j-1)(2i,2j)$, and $(2i-1,n)(2i,n)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i-1,2j-1)(2i,2j-1)$, $(2i-1,2j)(2i,2j)$, and $(m,2j-1)(m,2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(1,2j)(1,2j+1)$, $(2i,2j)(2i+1,2j)$, and $(2i,2j+1)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq i \leq \frac{n-1}{2}$. Assign c_4 to $(2i,1)(2i+1,1)$, $(2i,2j)(2i,2j+1)$, and $(2i+1,2j)(2i+1,2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



Now, give G a proper 5-edge-coloring as follows. Assign c_1 to $(2i-1, 2j-1)(2i-1, 2j)$, and $(2i, 2j-1)(2i, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i, 2j-1)(2i+1, 2j-1)$, and $(2i, 2j)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1, 2j)(2i, 2j)$, and $(2i-1, 2j+1)(2i, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_4 to $(2i, 2j)(2i, 2j+1)$, and $(2i+1, 2j)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_5 to $(2i-1, 1)(2i, 1)$, $(2i-1, n)(2i, n)$, $(1, 2j)(1, 2j+1)$, and $(m, 2j-1)(m, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



The above edge-coloring admits the following mate coloring. Assign c_1 to $(2i-1, 2j-1)(2i, 2j-1)$, and $(2i-1, 2j)(2i, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_2 to $(2i, 2j-1)(2i, 2j)$, and $(2i+1, 2j-1)(2i+1, 2j)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_3 to $(2i-1, 2j)(2i-1, 2j+1)$, and $(2i, 2j)(2i, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_4 to $(2i, 2j)(2i+1, 2j)$, and $(2i, 2j+1)(2i+1, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. Assign c_5 to $(2i, 1)(2i+1, 1)$, $(2i-1, n)(2i, n)$, $(1, 2j-1)(1, 2j)$, and $(m, 2j)(m, 2j+1)$ for $1 \leq i \leq \frac{m-1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$. An example is given below.



Using the notation of Lemma 2.3, note that H_1, H_2, H_3 and H_4 each consist of $\frac{(m-1)(n-1)}{4}$ many 4-cycles. H_5 consists of a single $2(m-1)+2(n-1)$ cycle, so $\alpha = (n-1)(m-1)+1$ and by Lemma 2.3 we conclude $\{4, ..., (n-1)(m-1)+1\} \subseteq \text{CTS}(P_n \square P_m)$.

$\square$

Note there is a possible gap in these spectra, since for a given graph G, the maximum value of its color-trade-spectrum is $\lfloor \frac{|E(G)|}{2} \rfloor$. In particular, the gap consists of $\lfloor \frac{n(m-1)+m(n-1)}{2} \rfloor - (n-1)(m-1)+1 = \lfloor \frac{m+n}{2} \rfloor - 2$ possible values. We conjecture that $\{4, ..., (n-1)(m-1)+1\}$ is indeed the entire color-trade-spectrum of $P_n \square P_m$.

Chapter 6

Complete Graphs

Denote by K_n the complete graph on n vertices with $\frac{n(n-1)}{2}$ edges. By Lemma 2.1, K_1 and K_2 have empty color trade spectra since they contain vertices of degree one. Since K_3 is an odd cycle, it also has an empty color-trade-spectrum by Lemma 2.2. For $K \geq 4$, the color-trade-spectrum of K_n is bounded below by $\chi'(K_n)$, which is $n-1$ when n is even, and n when n is odd. The color-trade-spectrum is bounded above by $\lfloor \frac{n(n-1)}{4} \rfloor$. We first consider the case where $n \equiv 0 \mod 8$.

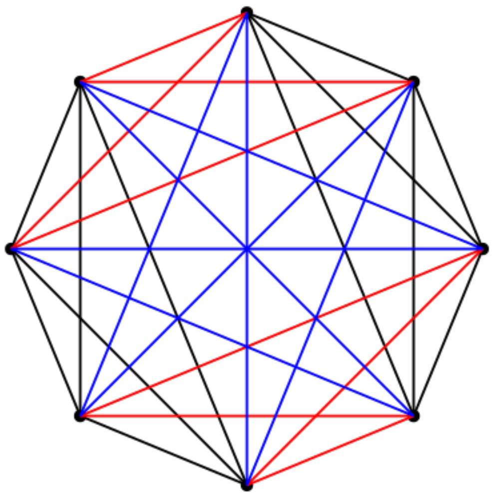
6.1 $n \equiv 0 \mod 8$

Theorem 6.1. *Let n be a positive integer such that $n \equiv 0 \mod 8$. Then $\{n-1, n, ..., n\frac{(n-1)}{4}\} =$ CTS(K_n).*

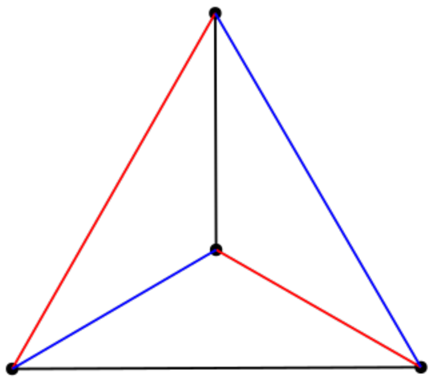
Proof. Let $V(K_n) = \{v_{i,j} | 0 \leq i \leq \frac{n}{4} - 1, 1 \leq j \leq 4\}$ and denote by $(v_{i,j}, v_{h,k})$ the edge between vertex $v_{i,j}$ and vertex $v_{h,k}$. For each i, consider the vertices $v_{i,1}, v_{i,2}, v_{i,3}$, and $v_{i,4}$ and the induced K_4 subgraph induced by these vertices. In total, this yields $m = \frac{n}{4}$ many disjoint K_4 subgraphs. Let mK_4 denote the complete m-partite graph where each part contains four vertices. Then K_n consists of m many K_4s and a singular mK_4. We now find a C_4 decomposition of mK_4.

The induced $K_{4,4}$ on the vertices $\{v_{s,i}, v_{t,j} | s, t \in \{0, 1, ..., \frac{n}{4} - 1\}, s \neq t, i, j \in \{1, 2, 3, 4\}\}$ consists of the following C_4s: $(v_{s,1}, v_{t,1}, v_{s,2}, v_{t,2})$, $(v_{s,3}, v_{t,3}, v_{s,4}, v_{t,4})$, $(v_{s,1}, v_{t,3}, v_{s,2}, v_{t,4})$, and $(v_{s,3}, v_{t,1}, v_{s,4}, v_{t,2})$. In particular, the first two cycles are disjoint, as are the last two cycles, which we will refer to as T_1 and T_2 cycles respectively. With the K_4s from above, this yields a

(K_4, C_4) decomposition of K_n. We give an example of the the (K_4, C_4) decomposition of K_8 below where each K_4, T_1 cycle, and T_2 cycle, is colored black, red, and blue respectively.

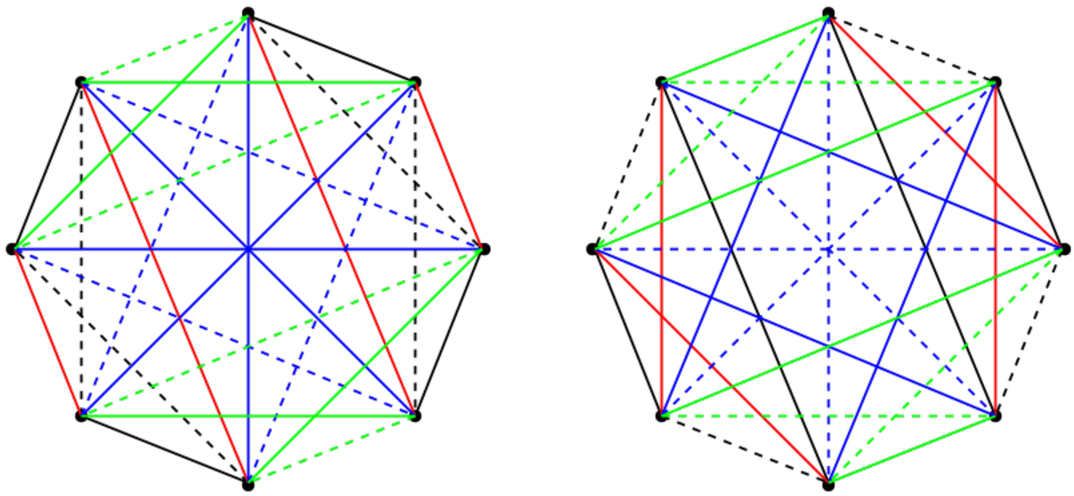


Next, we consider a new graph, K_m. In particular, let $V(K_m) = \{v_\infty, v_i | 0 \leq i \leq m-2\}$ where each vertex of K_m corresponds to the contraction of the K_4 induced by the vertices $\{v_{i,j} | 1 \leq j \leq 4\}$ from K_n, and v_∞ corresponds to the contraction of the K_4 for $i = m-1$. Likewise, an edge in K_m will correspond to the four C_4s between the associated K_4s. Consider the following 1-factorization of K_m where addition is under modulo $m-1$: $\{(v_\infty, v_d), (v_{d+i}, v_{d-i}) | 0 \leq d \leq m-2, 1 \leq i \leq \frac{m}{2} - 1\}$. In particular, each value of d yields a unique 1-factor. We give an example of the K_m associated with K_{16} below where the central vertex is v_∞ and each 1-factor is represented by one of black, red, or blue.



The 1-factorization above yields a proper $(m-1)$-edge-coloring of K_m where the edges in the dth 1-factor are colored c_d for $1 \leq d \leq m-1$. Furthermore, this yields a $4(m-1) = n-4$-edge-coloring of the edges of the mK_4 in K_n by alternating between colors c_{4d} and c_{4d+1} in the associated T_1 cycles and by alternating between colors c_{4d+2} and c_{4d+3} in the associated T_2 cycles for each value of d. Finally, we assign c_{n-4} to edges of the form $(v_{i,1}, v_{i,2}), (v_{i,3}, v_{i,4})$, c_{n-3} to $(v_{i,1}, v_{i,3}), (v_{i,2}, v_{i,4})$, and c_{n-2} to $(v_{i,1}, v_{i,4}), (v_{i,2}, v_{i,3})$. In total, this yields a $n-1$-edge-coloring of K_n.

To find a mate coloring, we alternate the colors in each C_4, and apply a 3-cycle permutation to the color classes c_{n-4}, c_{n-3}, and c_{n-2}. Now, consider the subgraph H_i as mentioned in Lemma 2.3. Note that each color class consists of $\frac{n}{4}$ many C_4s. Since we have a $n-1$-edge-coloring, this yields a total of $\frac{n(n-1)}{4}$ cycles, so $\{n-1, n, ..., \frac{n(n-1)}{4}\} \subseteq \text{CTS}(K_n)$ for $n \equiv 0 \mod 8$. Since this set of values in includes all possible values for the color-trade-spectrum of K_n for even n, this proves the claim. We give an example of these edge-colorings for K_8 below, where the top vertex is $v_{0,1}$ and the vertices are labeled in clockwise order so the bottom vertex would be $v_{1,1}$.



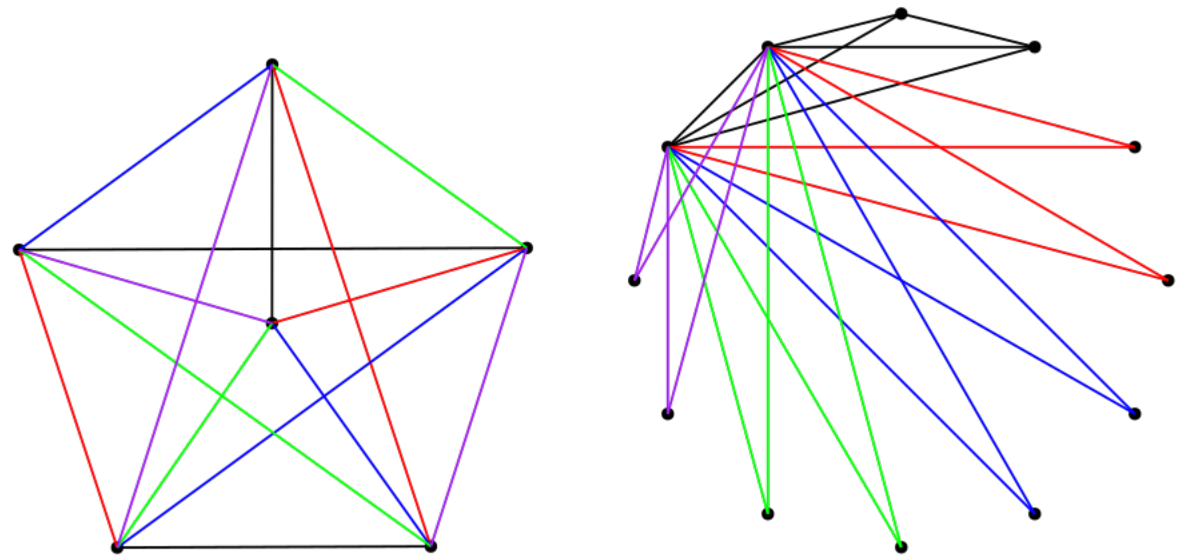
$\square$

6.2 $n \equiv 4 \mod 8$

Theorem 6.2. *Let n be a positive integer such that $n \equiv 4 \mod 8$. Then $\{n-1, n, ..., \frac{n(n-1)}{4}\} =$ CTS(K_n).*

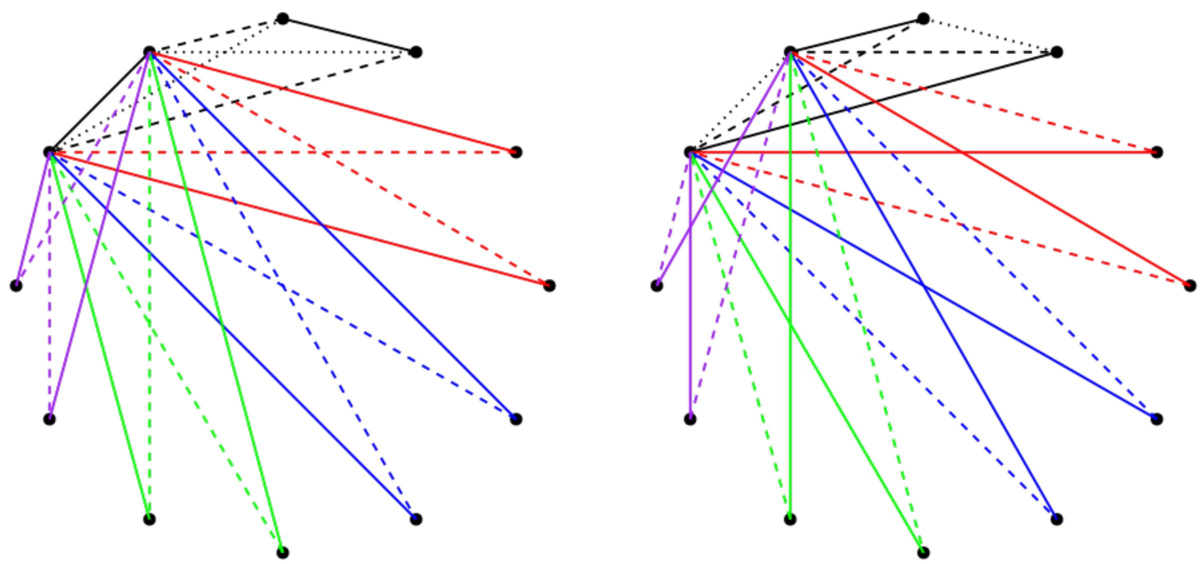
Proof. We use a modified version of the construction from the $n \equiv 0 \mod 8$ case. Let $V(K_n) = \{v_{i,j} | 0 \leq i \leq \frac{n}{2} - 1, 1 \leq j \leq 2\}$ and denote by $(v_{i,j}, v_{h,k})$ the edge between vertex $v_{i,j}$ and vertex $v_{h,k}$. Let $m = \frac{n}{4}$ and consider the graph K_{2m} where $V(K_{2m}) = \{v_\infty, v_i | 0 \leq i \leq 2m-2\}$ such that v_∞ corresponds to the vertices $v_{\frac{n}{2}-1,1}$ and $v_{\frac{n}{2}-1,2}$ while v_i corresponds to the vertices $v_{i,1}$ and $v_{i,2}$.

Consider the following 1-factorization of K_{2m} where addition is under modulo $2m-1$: $\{(v_\infty, v_d), (v_{d+i}, v_{d-i}) | 0 \leq d \leq 2m-2, 1 \leq i \leq m-1\}$. In particular, each value of d yields a unique 1-factor, and we create a (K_4, C_4) decomposition of K_n from K_{2m} as follows. For the 1-factor where $d = 0$, we create K_4s on the vertices $v_{i,1}, v_{i,2}, v_{-i,1}$, and $v_{-i,2}$ for $1 \leq i \leq m-1$, and we also create a K_4 on the vertices $v_{2m-1,1}, v_{2m-1,2}, v_{0,1}$, and $v_{0,2}$. For the other 1-factors where $1 \leq d \leq 2m-2$, we create a C_4 on the vertices $(v_{d+i,1}, v_{d-i,1}, v_{d+i,2}, v_{d-i,2})$ for $1 \leq i \leq m-1$, along with the C_4 on the vertices $(v_{2m-1,1}, v_{d,1}, v_{2m-1,2}, v_{d,2})$. An example of the K_{2m} associated with K_{12} is given below where the central vertex is v_∞ and each 1-factor is represented by one of black, red, blue, green, or purple, along with an associated partial (for readability) (K_4, C_4) decomposition of K_{12} where we include the K_4s and C_4s which contain the vertices $v_{5,1}$ and $v_{5,2}$.



We color K_n as follows. For $1 \leq d \leq 2m-2$, alternate between colors c_{2d-1} and c_{2d} in the cycles coming from the associated 1-factor in K_{2m}. For $d = 0$, we assign c_{n-3} to edges of the form $(v_{i,1}, v_{i,2})$ and $(v_{-i,1}, v_{-i,2})$, c_{n-2} to edges of the form $(v_{i,1}, v_{-i,1})$ and $(v_{i,2}, v_{-i,2})$, and c_{n-1} to edges of the form $(v_{i,1}, v_{-i,2})$ and $(v_{i,2}, v_{-i,1})$. In total, this yields a $n-1$-edge-coloring of K_n.

The same argument from the $n \equiv 0 \mod 8$ case shows how to find a mate, and that $\{n-1, n, ..., \frac{n(n-1)}{4}\} = \text{CTS}(K_n)$ for $n \equiv 4 \mod 8$, proving the claim. An example of these edge-colorings is shown below applied to the partial (K_4, C_4) decomposition of K_{12} from above.

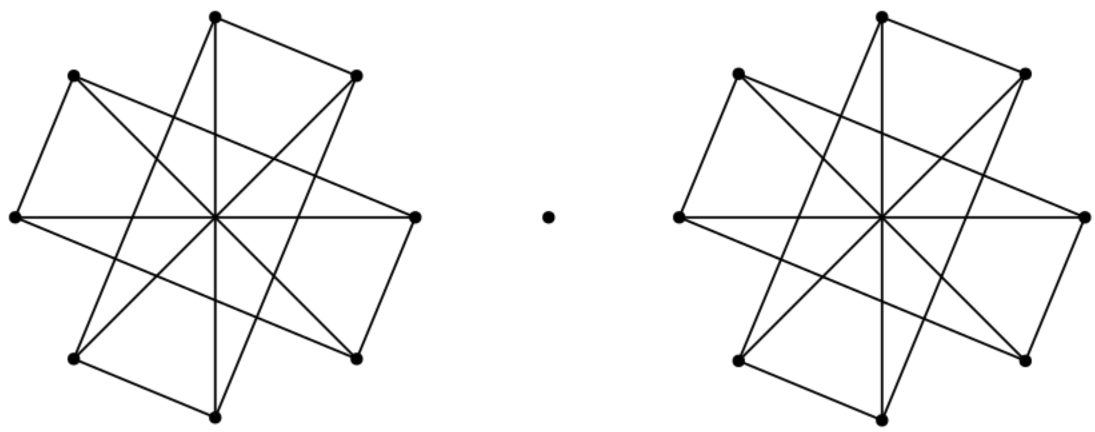


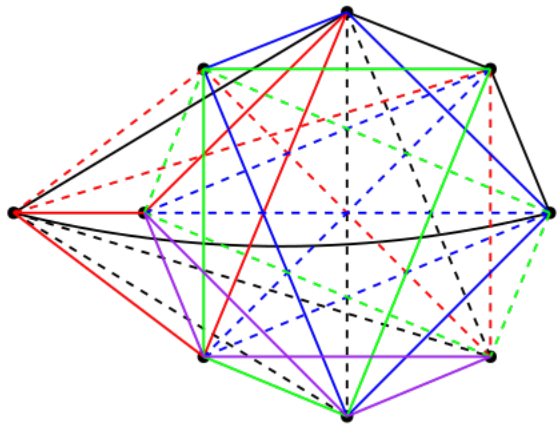
□

6.3 $n \equiv 1 \mod 8$

Theorem 6.3. *Let n be a positive integer such that $n \equiv 1 \mod 8$. Then $\{2n, 2n+1, ..., \frac{n(n-1)}{4}\} \subseteq$ CTS(K_n) for $n \geq 17$, and $\{14, 15, ..., 18\} \subseteq$ CTS(K_9).*

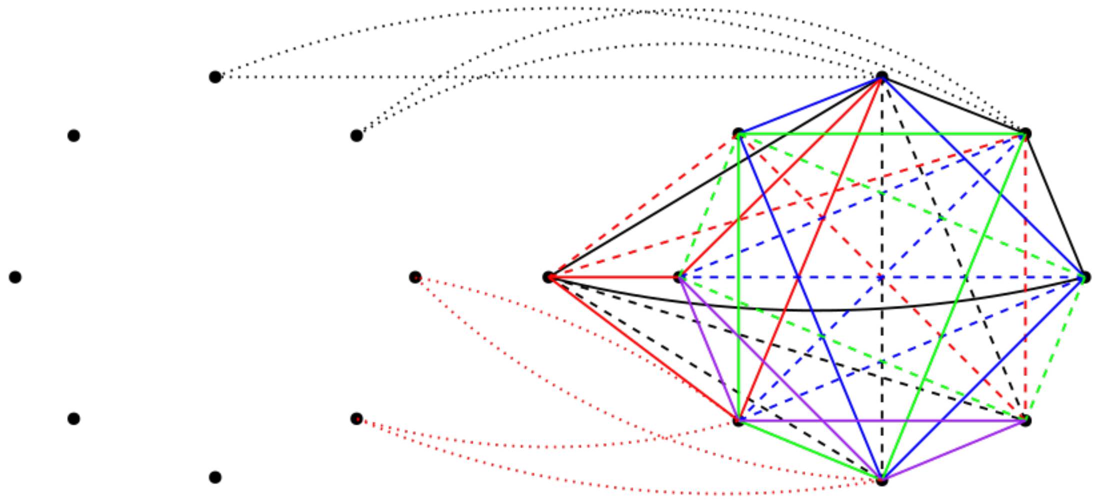
Proof. Let $V(K_n) = \{v_i | 0 \leq i \leq n-1\}$ and consider the K_4 subgraphs consisting of the edges $(v_{8j+1}, v_{8j+2}), (v_{8j+1}, v_{8j+5}), (v_{8j+1}, v_{8j+6}), (v_{8j+2}, v_{8j+5}), (v_{8j+2}, v_{8j+6})$, and (v_{8j+5}, v_{8j+6}), along with the K_4 subgraphs consisting of the edges $(v_{8j+3}, v_{8j+4}), (v_{8j+3}, v_{8j+7}), (v_{8j+3}, v_{8j+8})$, $(v_{8j+4}, v_{8j+7}), (v_{8j+4}, v_{8j+8})$, and (v_{8j+7}, v_{8j+8}) for $j \in \{0, 1, ..., \frac{n-1}{8} - 1\}$. For any two K_4s with the above form on the same index j, along with v_0, we have a C_4 decomposition of a K_9 with the following nine C_4s: $(v_0, v_{8j+1}, v_{8j+2}, v_{8j+3}), (v_0, v_{8j+2}, v_{8j+4}, v_{8j+8})$, $(v_0, v_{8j+4}, v_{8j+1}, v_{8j+5}), (v_0, v_{8j+6}, v_{8j+1}, v_{8j+7}), (v_{8j+1}, v_{8j+3}, v_{8j+5}, v_{8j+8})$, $(v_{8j+2}, v_{8j+5}, v_{8j+6}, v_{8j+8}), (v_{8j+2}, v_{8j+6}, v_{8j+3}, v_{8j+7}), (v_{8j+3}, v_{8j+4}, v_{8j+7}, v_{8j+8})$, and $(v_{8j+4}, v_{8j+5}, v_{8j+7}, v_{8j+6})$. Below is an example of K_{17} under the (v_0, K_4) decomposition listed above where the central vertex is v_0 and the other sets of vertices follow a clockwise pattern. Following this is an example of a C_4 decomposition of a singular K_9 where the leftmost vertex is v_0.



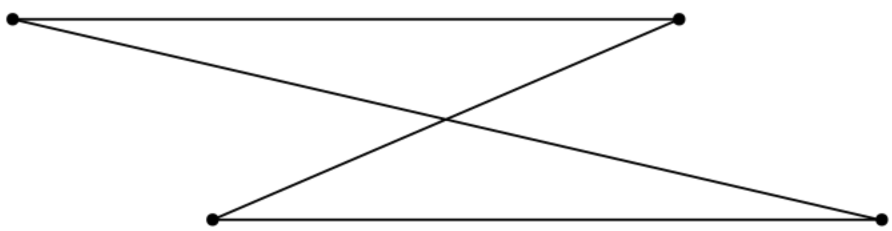


As in the $n \equiv 0 \mod 4$ case, we can form a $K_{4,4}$ between any two such K_4 subgraphs as described above (assuming $n \geq 17$), which we can partition into four pairwise disjoint C_4 subgraphs. If the two K_4s belong to the same K_9 as described above, the edges of the $K_{4,4}$ have already been used in the C_4 decomposition of the corresponding K_9. Otherwise, the K_4s belong to disjoint K_9s, so the $K_{4,4}$ between them decomposes into four C_4s. Since our construction contains two K_4s in each K_9, this gives us four different $K_{4,4}$s for each pair of K_9s, for a total of sixteen C_4s for each pair of K_9s as follows:

$(v_{8i+1}, v_{8j+1}, v_{8i+2}, v_{8j+2})$, $(v_{8i+1}, v_{8j+3}, v_{8i+2}, v_{8j+4})$, $(v_{8i+1}, v_{8j+5}, v_{8i+2}, v_{8j+6})$, $(v_{8i+1}, v_{8j+7}, v_{8i+2}, v_{8j+8})$, $(v_{8i+3}, v_{8j+1}, v_{8i+4}, v_{8j+2})$, $(v_{8i+3}, v_{8j+3}, v_{8i+4}, v_{8j+4})$, $(v_{8i+3}, v_{8j+5}, v_{8i+4}, v_{8j+6})$, $(v_{8i+3}, v_{8j+7}, v_{8i+4}, v_{8j+8})$, $(v_{8i+5}, v_{8j+1}, v_{8i+6}, v_{8j+2})$, $(v_{8i+5}, v_{8j+3}, v_{8i+6}, v_{8j+4})$, $(v_{8i+5}, v_{8j+5}, v_{8i+6}, v_{8j+6})$, $(v_{8i+5}, v_{8j+7}, v_{8i+6}, v_{8j+8})$, $(v_{8i+7}, v_{8j+1}, v_{8i+8}, v_{8j+2})$, $(v_{8i+7}, v_{8j+3}, v_{8i+8}, v_{8j+4})$, $(v_{8i+7}, v_{8j+5}, v_{8i+8}, v_{8j+6})$, and $(v_{8i+7}, v_{8j+7}, v_{8i+8}, v_{8j+8})$ for $i, j \in \{1, 2, ..., \frac{n-1}{8}\}$ where $i \neq j$. In total, this yields a $9(\frac{n-1}{4}) + 16\binom{\frac{n-1}{8}}{2} = \frac{n(n-1)}{8}$ C_4 decomposition of K_n. A partial example for K_{17} is shown below.



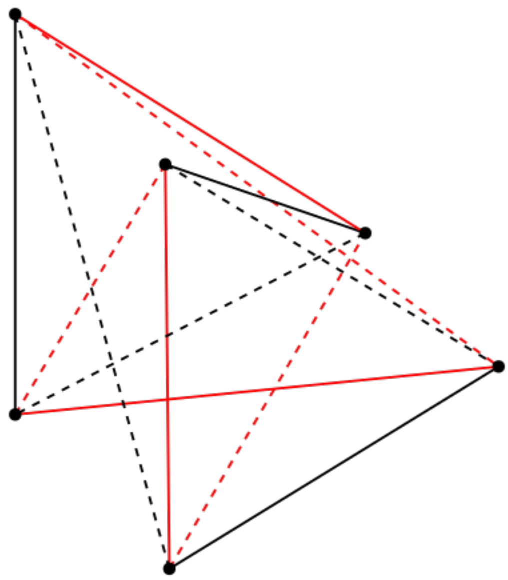
Referencing the $n \equiv 0 \mod 8$ case, note that the sixteen C_4s from the previous paragraph can be partitioned into four sets of disjoint cycles, which we again refer to as T_1, T_2, T_3, and T_4 type cycles respectively. In particular, we will use the following assignment of the cycles from above. T_1: 1st, 6th, 11th, 16th. T_2: 2nd, 7th, 12th, 13th. T_3: 3rd, 8th, 9th, 14th. T_4: 4th, 5th, 10th, 15th. As in the $n \equiv 0 \mod 8$ case, we consider $m = \frac{n-1}{4}$ along with a new graph K_m where vertex v_{2i} in K_m corresponds to the contraction of the K_4 induced by vertices $v_{8i+1}, v_{8i+2}, v_{8i+5}$, and v_{8i+6} in K_n and where vertex v_{2i+1} in K_m corresponds to the contraction of the K_4 induced by the vertices $v_{8i+3}, v_{8i+4}, v_{8i+7}$, and v_{8i+8} in K_n, for $0 \leq i \leq \frac{n-1}{8} - 1$. As before, an edge between vertices in K_m corresponds to the four C_4s between the associated K_4s in K_n. However, some of these edges were used in the C_4 decomposition of the K_9s, and this corresponds to removing a 1-factor from our K_m, denoted $K_m - F_1$. An example of $K_m - F_1$ for K_{17} is given below.



As in the $n \equiv 0 \mod 8$ case, a 1-factorization of $K_m - F_1$ yields a proper $(m-2)$-edge-coloring of $K_m - F_1$, which further corresponds to a $4(m-2) = n - 9$-edge-coloring of the edges between the K_4s by alternating between colors $c_{8k+2p-2}$ and $c_{8k+2p-1}$ in the associated T_p

cycles for $0 \leq k \leq \frac{n-1}{8} - 1$ and $1 \leq p \leq 4$, again assuming that $n \geq 17$. Next, note that within each K_9, the 1st and 9th cycles from the C_4 decomposition of K_9 above are disjoint, as are the 3rd and 7th cycles. Furthermore, cycles five through nine are vertex disjoint for each pair of K_9s in K_n. In total, we need eight colors for each copy of K_9 to account for the edges incident to v_0, ten colors which can be shared among all copies of K_9 among the cycles which are not incident to v_0, and $n - 9$ colors for the edges between the K_9s, for a total of $8(\frac{n-1}{8}) + 10 + n - 9 = 2n$ colors to properly color the edges of K_n for $n \geq 17$. We find a mate for this edge-coloring by reversing the assignment of colors within each cycle as in the previous cases, and this process shows that $\{2n, 2n+1, ..., \frac{n(n-1)}{4}\} \subseteq \text{CTS}(K_n)$ for $n \equiv 1 \mod 8$ when $n \geq 17$. For K_9, the K_9 decomposition alone shows that $\{14, 15, ..., 18\} \subseteq \text{CTS}(K_9)$. An example of the proper $m-2$-edge-coloring of the associated $K_m - F - 1$ for K_{25} is shown below.

□



It should be noted that our lower bound is not tight. Taking K_{17} as an example, the colors assigned to the cycle $(0, 1, 2, 3)$ in the first K_9 copy could also be used among the cycle $(3, 11, 4, 12)$ between the K_9 copies. There are other examples of disjoint cycles where colors could be reused, so our lower bound can indeed be decreased. Furthermore, it remains to be determined if our construction yields the highest amount of disjoint cycles, although the answer to this question should not require extensive work.

Chapter 7

Further Work

There are many ways in which this work could be continued. Since the color-trade-spectrum of complete bipartite graphs was determined in this dissertation, a natural next step would be to determine the color-trade-spectrum of complete multipartite graphs. Perhaps the usage of Latin rectangles could be extended to higher dimensions.

The color-trade-spectrum of $P_n \square P_m$ remains incomplete. As mentioned at the end of chapter 5, we conjecture $\{4, 5, ..., (n-1)(m-1)+1\}$ is indeed the entire color-trade-spectrum of $P_n \square P_m$ for $3 \leq n \leq m$, but it remains to show that none of the other $\lfloor \frac{m+n}{2} \rfloor - 2$ possible values are in the color-trade-spectrum.

Likewise, the color-trade-spectrum for K_n remains incomplete. As shown in chapter 6, $\{2n, 2n+1, ..., \frac{n(n-1)}{4}\} \subseteq \mathrm{CTS}(K_n)$ for $n \geq 17$, but we also showed an example where the lower bound is not tight. Furthermore, the cases of $n \equiv 2, 3, 5, 6, 7 \mod 8$ are ripe for exploration.

In this dissertation, the color-trade-spectra of 2-regular graphs (cycles) and complete graphs (which are $n-1$-regular on n vertices) were explored. Further work could be done in exploring the color-trade-spectra of generic k-regular graphs. Extensions of this could lead to studying cages, snarks, and strongly-regular graphs.

Likewise, the color-trade-spectrum of any family of graphs is open for discovery. Some interesting graphs for which the color-trade-spectrum is unknown includes the Platonic graphs, the Heawood graph, and the Petersen graph.

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