The Slow-Coloring Game on Path Power Graphs

by

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Abstract

The Slow-Coloring Game is a game played on a graph G by two players which we will refer to as Lister and Painter. In the ith round, Lister marks a nonempty subset $M \subseteq V(G)$ of uncolored vertices as eligible to receive color i, scoring $|M|$ points. Painter then gives color i to a subset of M that is independent in G. The game ends when all of the vertices of G are colored. Note that at each stage the resulting coloring will be a proper coloring of $V(G)$. Lister's goal is to maximize the total score while Painter seeks to minimize the total score. The *sum-color cost* of a graph G, denoted $\mathring{s}(G)$, is the best score each player can guarantee in the Slow-Coloring Game on G regardless of the play strategy of the other. [1],[2]

Puleo and West [1] showed that for every tree T on n vertices,

$$n + \sqrt{2n} \approx n + u_{n-1} = \mathring{s}(K_{1,n-1}) \leq \mathring{s}(T) \leq \mathring{s}(P_n) \leq \left\lfloor \frac{3n}{2} \right\rfloor$$

where $u_r = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor$. They also conjectured that this bound generalizes to k-trees. The k-tree generalization of the star, or k-star, is $K_k \bigoplus \overline{K}_{n-k}$, and the k-tree generalization of a path, or k-path, is P_n^k. Mahoney, Puleo, and West [2] showed that $\mathring{s}(K_s \bigoplus \overline{K}_r) = r + \binom{s+1}{2} + su_r$. We show that $\mathring{s}(P_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$, where $r \equiv n \pmod{k+1}$ and $0 \leq r < k+1$.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Background on the Slow-Coloring Game

In this thesis we study the *Slow-Coloring Game*, which was first introduced in [2]. The Slow-Coloring Game models the difficulty of producing a proper coloring of a graph G when it is not known beforehand which vertices are allowed to have which colors. The Slow-Coloring Game is a game played on a graph G by two players, which we will refer to as Lister and Painter. In the ith round, Lister marks a nonempty subset $M \subseteq V(G)$ of uncolored vertices as eligible to receive color i, scoring $|M|$ points. Painter then gives color i to a subset of M that is independent in G. The game ends when all of the vertices of G are colored. Note that the resulting coloring will be a proper coloring of $V(G)$. Lister's goal is to maximize the total score while Painter seeks to minimize the total score. The *sum-color cost* of a graph G, denoted $\mathring{s}(G)$, is the best score each player can guarantee in the Slow-Coloring Game on G regardless of the play strategy of the other.

The Slow-Coloring Game was developed over successive generalizations of a classical Graph Theory problem: proper vertex colorings of a graph. More formal definitions for the following can be found at the end of this chapter. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices receive different colors. Independently introduced by Erdos-Rubin-Taylor [3] and Vizing [4], *list coloring* is a generalization of this classical problem. In list coloring, each vertex v is assigned a set of available colors $L(v)$, called its list. A graph G is *L-colorable* if there is a proper coloring ϕ where $\phi(v) \in L(v)$ for all vertices v. Given $f : V(G) \to \mathbb{N}$ a graph G is said to be *f-choosable* if for all list assignments, L, G is L-colorable whenever $|L(v)| \geq f(v)$ for all vertices v. The *choice number* of a graph G is the least such integer k such that G is f-choosable whenever $f(v) \geq k$ for all vertices v. A variation of choice number first introduced by Isaak [5], is the *sum-choosability* denoted χ_{SC}. In sum-choosability, we seek to minimize the sum or average of the list sizes. It is the minimum $\sum f(v)$ whenever G is f-choosable.

Introducing an online variant for list coloring, where the lists of vertices are revealed a little bit at a time, produces what is called the *f-painting game*. This game is also played by two players, Lister and Painter. In each round i of the f-painting game, Lister marks a set M of vertices allowed to receive color i, which can be viewed as revealing the set of vertices having color i in their lists. Painter then chooses an independent subset of M to receive color i. Lister wins if some vertex is marked more than $f(v)$ times; Painter wins by successfully coloring all the vertices. The graph is *f-paintable* if Painter has a winning strategy. Independently introduced by Schauz [6] and by Zhu [7], and similar to choice number, the *paint number* or *paintability* of a graph G is the least k such that G is f-paintable whenever $f(v) \geq k$ for all $v \in V(G)$. The *sum paintability* of a graph G, introduced by Carraher, Mahoney, Puleo, and West [8] and written $\chi_{SP}(G)$, is the minimum of $\sum f(v)$ over all f such that G is f-paintable. Since the main concern in sum-paintability is the number of times a vertex is marked and not the number of colors used, we can view this game in a slightly different way. We instead say that $f(v)$ is a collection of tokens available to v, and whenever Lister marks v a token must be removed from its collection, and Lister wins when a vertex with no tokens is marked.

The *Slow-Coloring Game* which was first introduced by Mahoney, Puleo, and West in [2], is an online variant of the f-painting game. Just as with the online variant of List coloring, Painter is able to reveal the tokens at each vertex as they are marked rather than assigning them according to $f(v)$. Which means that we can view the sum-color cost $\mathring{s}(G)$ being the minimum number of tokens Painter needs to guarantee a proper coloring [1]. Since Painter can always act as though the tokens are assigned according to $f(v)$ we have that $\mathring{s}(G) \leq \chi_{SP}(G)$. The sum-color cost's formula can be easily described recursively, but in general its computation is not straightforward.

In [2] Mahoney, Puleo, and West gave us the sum-color cost formula which is:

Proposition 1.1.

$$\mathring{s}(G) = \max_{\varnothing \neq M \subseteq V(G)} \left(|M| + \min_{independent I \subseteq M} \mathring{s}(G - I) \right)$$

Proof. In response to the initial marked set M, Painter minimizes the additional score over colored subsets $I \subseteq M$ such that I is independent in G. Lister chooses M to maximize the resulting total score.

□

The Slow-Coloring Game is fairly new and not widely studied. Several theorems about the Slow-Coloring Game and several observations about strategies for Lister and Painter are given below. These will be useful to our results in later chapters.

Observation 1.2. *On any graph, there are optimal strategies for Lister and Painter such that Lister always marks a set M inducing a connected subgraph, and Painter always colors a maximal independent subset of M.* [2]

Proof. A move in which Lister marks a disconnected set M can be replaced with successive moves marking the vertex sets of the components of the subgraph induced by M. Also, coloring extra vertices at no extra cost cannot hurt Painter.

□

Observation 1.3. *If G_1 and G_2 are vertex disjoint subgraphs of G, then $\mathring{s}(G) \geq \mathring{s}(G_1) + \mathring{s}(G_2)$.* [2]

Proof. Lister can play an optimal strategy on G_1 while ignoring the rest and then do the same on G_2, achieving the score $\mathring{s}(G_1) + \mathring{s}(G_2)$.

□

Observation 1.4. $\mathring{s}(K_r) = \binom{r+1}{2}$. [2]

Proof. No matter the number of vertices Lister marks, Painter can only ever paint one vertex. Thus it is optimal for Lister to always mark all remaining vertices.

□

Theorem 1.5. *Among n-vertex trees, the value of $\mathring{s}$ is minimized by the star and maximized by the path. Furthermore, with $u_r = \lfloor \frac{-1+\sqrt{1+8r}}{2} \rfloor$ and T being an n-vertex tree,* [1]

$$n + \sqrt{2n} \approx n + u_{n-1} = \mathring{s}(K_{1,n-1}) \leq \mathring{s}(T) \leq \mathring{s}(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor$$

Theorem 1.5 was proved by Puleo and West in [1], where they provide a linear time algorithm to compute $\mathring{s}$ on trees, via a recursive formula.

In, [2], Mahoney, Puleo, and West posed the natural question: does this bound generalize to k-trees? While this conjecture is not proved in this thesis, we state and prove the formula for the k-path. We will define and discuss the k-path in greater detail in the next chapter.

1.2 Introductory Definitions

Definition 1.6. A *graph*, G, consists of a set $V(G)$ of objects called *vertices* and a set $E(G)$ of two element subsets of V. Each element of E is called an *edge* and can be written as $\{x, y\}$ or simply xy for vertices $x, y \in V(G)$. [9]

Definition 1.7. If $xy \in E(G)$, then x and y are said to be *adjacent* vertices; otherwise, x and y are *nonadjacent vertices* [9]

Definition 1.8. A subset of vertices of a graph G, $S \subseteq V(G)$, is *independent* if no two vertices of S are adjacent.

Definition 1.9. Any vertex adjacent to a vertex x is called a *neighbor* of x, and the set of neighbors of x is the *open neighborhood* of x, denoted by $N_G(x)$. [9]

Definition 1.10. The number of neighbors of a vertex x in a graph G is the *degree* of x, and is denoted by $\deg_G(x)$ or simply $\deg(x)$ or $\deg x$ if the graph G is clear. [9]

Definition 1.11. The maximum degree of a graph G is the maximum of the degrees of the vertices of G and is denoted by $\Delta(G)$. Similarly the *minimum degree* of a graph G is the minimum of the degrees of the vertices of G, and is denoted by $\delta(G)$. [9]

Definition 1.12. A *proper coloring* of a graph G is an assignment of colors to the vertices of G, one color to each vertex of G such that adjacent vertices of G receive different colors. [9]

Definition 1.13. The *Chromatic Number* of a graph G denoted $\chi(G)$ is the smallest number of colors needed to properly color the vertices of G. [9]

Definition 1.14. A *list assignment* for a graph, G, is a function L that assigns every vertex v a list of colors $L(v)$. $[3, 4]$

Definition 1.15. Given a graph G and list assignment L, G is *L-colorable* if it has a proper vertex coloring using the colors from the lists assigned by $L(v)$. $[3, 4]$

Definition 1.16. A graph is *k-choosable* if it it L-colorable whenever $|L(v)| \geq k$ for all vertices of the graph. $[3, 4]$

Definition 1.17. The *choice number*, or *list chromatic number*, of a graph G, denoted $\chi_l(G)$ is the least integer, k. such that G is k-choosable. $[3, 4]$

Definition 1.18. Given a function $f : V(G) \to \mathbb{N}$, we define the *f-painting game* as follows: in each round, i, Lister marks a subset M of the vertices, $M \subseteq V(G)$. Painter then selects an independent subset of M to be deleted. If Painter can guarantee that no vertex v gets marked more than $f(v)$ times, then the graph is said to be f-paintable. $[6, 7]$

Definition 1.19. The *paintability* of a graph G is the smallest positive integer k such that G is f-paintable whenever $f(v) \geq k$. $[6, 7]$

Definition 1.20. The *sum-paintability* of G, denoted $\chi_{SP}(G)$, is the least value of $\sum(f(v))$ such that G is f-paintable. $[6, 7]$

Chapter 2

k-Tree Graphs

2.1 k-Tree Definitions

Definition 2.1. A *k-tree* is a graph that can be obtained from K_k by iteratively adding a vertex whose neighborhood is a k-clique in the existing graph. [2]

Definition 2.2. The *join*, denoted $G \bigoplus H$, of graphs G and H is obtained from the disjoint union $G + H$ by making each vertex in G adjacent to each vertex in H. [2]

Definition 2.3. The *rth power* of G is the graph G^r with vertex set $V(G)$ where vertices are adjacent if and only if the distance between them in G is at most r. [2]

Definition 2.4. $K_k \bigoplus \overline{K}_{n-k}$ is the k-tree generalization of a star, or *k-star*. [2] See Figure 2.1.

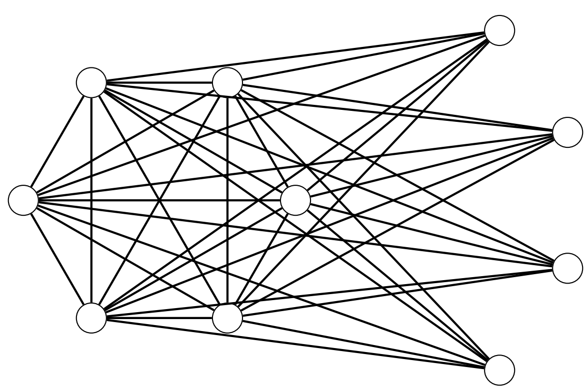

Figure 2.1: Example of a k-star: $K_6 \bigoplus \overline{K}_4$

Definition 2.5. P_n^k is the k-tree generalization of a path, or k-*path*. [2] More explicitly P_n^k is a graph whose vertices can be labeled $v_1, v_2, \ldots, v_n$ such that $v_i v_j \in E(P_n^k)$ if and only if $0 < |i - j| \leq k$. See Figure 2.2.

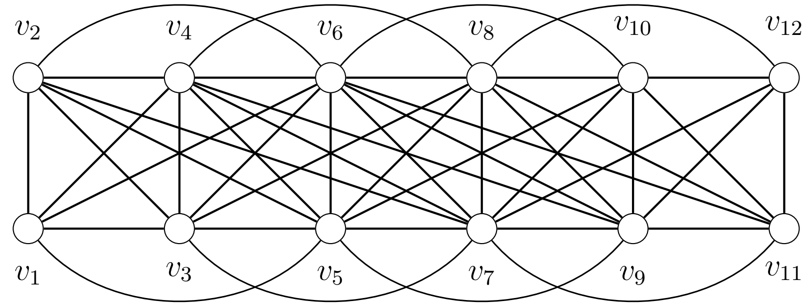

Figure 2.2: Example of a k-path: P_{12}^5

2.2 k-Path Properties

Property 2.6. $\Delta(P_n^k) \leq 2k$, *with equality only when* $n \geq 2k + 1$.

Property 2.7. *For any integer* i, *if all vertices* v_j *with* $j \equiv i$ *(mod* $k + 1$) *are removed from* P_n^k, *then the resulting graph with be a* $(k - 1)$*-path graph.*

2.3 A Strategy for Lister

A lower bound on the Slow-Coloring number for path power graphs, $\mathring{s}(P_n^k)$, follows easily from Observation 1.3. The proof of this lower bound yields a strategy for Lister to guarantee that this score is achievable. We provide both below.

Theorem 2.8. $\mathring{s}(P_n^k) \geq \lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ *where* $r \equiv n$ *(mod* $k + 1$) *and* $0 \leq r < k + 1$.

Proof. P_n^k contains $\left\lfloor \dfrac{n}{k+1} \right\rfloor$ disjoint copies of K_{k+1} and one copy of K_r. Thus if Lister plays the game on each of the disjoint K_{k+1} and K_r subgraphs, Painter can do no better than $\lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$.

□

Lister's strategy to achieve this score is to break up the P_n^k into exactly $\left\lfloor \dfrac{n}{k+1} \right\rfloor$ disjoint copies of K_{k+1} and one copy of K_r, and then play the game on each of the disjoint subgraphs and add the resulting scores together. To find these disjoint subgraphs, Lister need only go in the natural vertex ordering and greedily partition the vertices. See Figure 2.3 below.

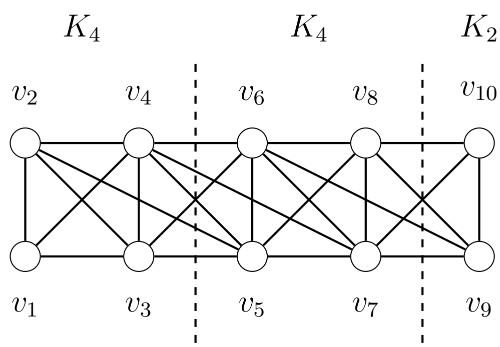

Figure 2.3: Natural Greedy Clique Partition of P_{10}^3

When $k = 1$, P_n^k is the basic path graph and the above lower bound reduces to $\lfloor \frac{3n}{2} \rfloor = \mathring{s}(P_n)$. This is the value of $\mathring{s}(P_n)$ which was proved by Puleo and West in $[1]$.

Providing a Painter strategy that results in no more than $\lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ tokens being used is not as simple, and thus it will be the main focus of Chapter 3.

Chapter 3

Main Result

We seek to show that $\left\lfloor \frac{n}{k+1} \right\rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ is also an upper bound on $\mathring{s}(P_n^k)$. In Theorem 3.14, we show that regardless of the strategy Lister adopts, Painter will never pay more than $\left\lfloor \frac{n}{k+1} \right\rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ total tokens to Lister at the end of the game.

Let $Q_1, Q_2, \ldots, Q_p$ be the natural greedy clique partition of P_n^k, with $p = \left\lceil \dfrac{n}{k+1} \right\rceil$, the same partition as described in Lister's strategy. For each clique, Q_i, we go in the natural vertex ordering assigning $1, 2, \ldots, k+1$ tokens on each vertex of the clique. See Figure 3.1 below. Note that each vertex with equivalent subscripts modulo $k+1$, will receive tokens equal to their remainder, of course except for those vertices with subscripts equivalent to $0 \pmod{k+1}$, those vertices will receive $k+1$ tokens.

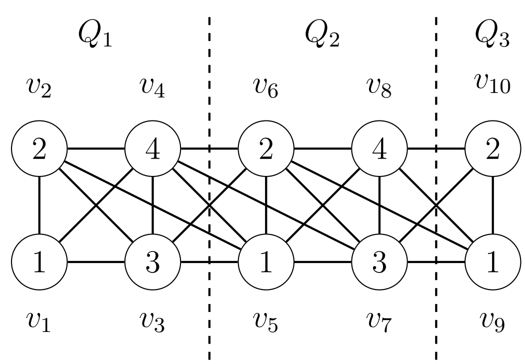

Figure 3.1: Natural Greedy Clique Token Assignment of P_{10}^3

In this assignment we assign $\binom{k+2}{2}$ tokens to each of the K_{k+1} cliques and $\binom{r+1}{2}$ tokens to the single K_r clique. Thus the total number of tokens assigned is $\left\lfloor \dfrac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2} = \left\lfloor \dfrac{n}{k+1} \right\rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$, which is exactly the upper bound value desired. See Figure 3.1.

Recall that in round i, Lister will mark a subset of vertices $M_i \subseteq V(G)$. Our goal is to show that there is a way for Painter to remove at least $|M_i|$ tokens from the graph, in such a way that (1) each vertex which is colored (deleted) has 0 tokens, (2) no uncolored (undeleted) vertex in M_i ends up with a non-positive number of tokens, and (3) each non-zero token class in G remains an independent set. We can assume that M_i is connected, otherwise, by Observation 1.3, marker does no better than if they marked each disjoint subset of M_i one at a time.

In [2] Mahoney, Puleo, and West proved the $k = 1$ case, namely that $\mathring{s}(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor$. We give an alternate proof of this case, and then use the strategies developed there to extend the result to higher powered path graphs.

Theorem 3.1. $\mathring{s}(P_n^1) = \left\lfloor \dfrac{3n}{2} \right\rfloor$.

Proof. For P_n^1, the total amount of tokens assigned is $\left\lfloor \frac{n}{1+1} \right\rfloor \binom{1+2}{2} + \binom{r+1}{2} = 3\left\lfloor \frac{n}{2} \right\rfloor + \binom{r+1}{2}$. If n is even then $r = 0$, and thus $\left\lfloor \frac{3n}{2} \right\rfloor$ total tokens have been assigned. If n is odd then $r = 1$, and $\left\lfloor \frac{n}{2} \right\rfloor \binom{3}{2} + \binom{2}{2} = 3\left(\frac{n-1}{2}\right) + 1 = \frac{3n-1}{2} = \left\lfloor \frac{3n}{2} \right\rfloor$. So in either case, the total tokens assigned is $\left\lfloor \frac{3n}{2} \right\rfloor$.

In P_n^1, there are either 1-token or 2-token vertices, and, as previously mentioned, Painter can assume that for each round, i, the marked set M_i will be connected. Thus the number of 1-token and 2-token vertices will differ by at most one, and by assignment, the vertices of each token class form an independent set. There are three possibilities for each M_i; both ends of M_i are 1-token vertices, both ends of M_i are 2-token vertices, or one end of M_i is a 1-token vertex and the other end is a 2-token vertex. See Figure 3.2.

If M_i is of the second or third type, then there are at least as many 2-token vertices as 1-token vertices in M_i. Now Painter can delete all the 2-token vertices and use their tokens to pay Lister $|M_i|$ tokens. Painter has removed all the tokens from deleted vertices, and undeleted vertices were unaffected so conditions (1) and (2) are satisfied. Painter has only deleted 2-token vertices, also

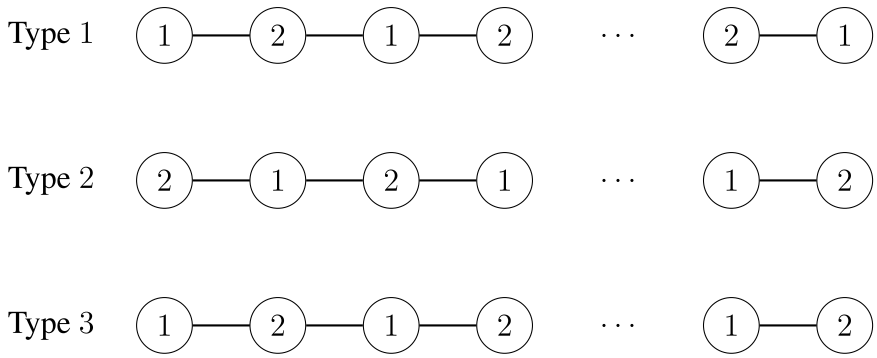

Figure 3.2: Marked Set Possibilities for P_n^1

each non-zero token class is still independent, satisfying condition (3). Thus Painter has removed at least $|M_i|$ tokens from the graph, and all conditions remain satisfied.

On the other hand, if M_i is of the first type, then there are more 1-token vertices than 2-token vertices. So if all 2-token vertices are deleted, the total number of tokens assigned to them is insufficient to pay Lister $|M_i|$ tokens. So instead Painter should remove all the tokens from and delete all the 1-token vertices, and remove one token from each of the remaining marked 2-token vertices. Since each vertex in M_i had exactly one token removed from it, we have certainly removed at least $|M_i|$ tokens from the graph. Each of the 1-token vertices, which get deleted, now have no tokens, satisfying condition (1). Each of the 2-token vertices had one token removed thus they now have 1 token, and all unmarked tokens are unaffected, thus condition (2) is satisfied. Since the 2-token vertices were reduced in value we now need to check to make sure the new 1-token class is independent. These newly added 1-token vertices were all interior vertices of M_i and thus after deletion, they are now isolated vertices. Thus the new 1-token class is still independent, satisfying condition (3). So again Painter has removed at least $|M_i|$ tokens from the graph and all conditions are still satisfied.

Painter can repeat the above process for each marked set Lister chooses, until all n vertices have been deleted. Thus, since Painter started with $\left\lfloor \frac{3n}{2} \right\rfloor$ tokens on the graph, and no vertex at any point had negative tokens, Lister scored at most $\left\lfloor \frac{3n}{2} \right\rfloor$ points.

□

To show how this proof idea extends to higher powered path graphs, we introduce some new definitions. We will also assume from this point on that $G = P_n^k$ with positive integers n and k.

3.1 Definitions

Definition 3.2. Let $v \in G$. We define $\$(v)$ to be the number of tokens assigned to v.

Definition 3.3. Let $v \in G$ and let M be a given marked set. We define $B_i(v)$ to be the highest number of tokens assigned to a vertex in $N_M(v)$, in round i, that is less than $\$(v)$. In other words, among all the marked neighbors of v with fewer tokens than $\$(v)$, we define $B_i(v)$ to be the highest number of tokens assigned to such vertices. If there are no marked neighbors with fewer tokens than $\$(v)$, then we define $B_i(v)$ to be 0.

Definition 3.4. Let $v \in G$. We define v to be a *reducible* vertex if all $w \in N_G(v)$ satisfying $\$(w) = B_i(v)$ are in the marked set M_i. Otherwise we say that v is a *non-reducible* vertex.

We will give some additional definitions later, but until they are needed.

3.2 Lemmas and Observations

Lemma 3.5. *If v is a non-reducible vertex, then there are marked and unmarked vertices in $N_G(v)$ with $B_i(v)$ tokens.*

Proof. Let $v \in G$ be a non-reducible vertex. For any integer d and any neighbor of v, either all of the d-token neighbors are marked, one is marked and one is unmarked, or none are marked. If none are marked, then $d \neq B_i(v)$, since $B_i(v)$ is defined to be the highest number of tokens assigned to a marked neighbor. So we assume that $d = B_i(v)$. If all d-token vertices in $N_G(v)$ get marked, then v would be a reducible vertex. But, since v is a non-reducible vertex, there must be marked and unmarked d-token vertices where $d = B_i(v)$.

□

Again our goal is to show that in each round, i, there is a way for Painter to remove at least $|M_{i,0}|$ tokens from the graph, in such a way that (1) each vertex which is deleted (colored with color i) has 0 tokens, (2) no undeleted vertex in $M_{i,0}$ ends up with a non-positive number of tokens, and (3) each non-zero token class in G remains an independent set. With the definitions given, each vertex will either be reducible or non-reducible. We will show that Painter has a way to reduce the size of the original marked set, creating some new marked set $M_{i,j}$, while still satisfying all of the above conditions, including removing the original $|M_{i,0}|$ tokens, in such a way that all vertices in $M_{i,j}$ will be reducible vertices.

Definition 3.6. *Let $v \in V(G)$ and define the set T_v to be the set of vertices in G which are omitted to make v a reducible vertex.*

Painter can view these omitted vertices as being unmarked, meaning Painter is no longer allowed to remove tokens from these omitted vertices. However, Painter cannot completely consider them as unmarked since Lister marked them, and thus Lister will need to receive tokens for them.

Lemma 3.7. *Let $M_{i,0}$ be the original marked set of G in round i. If $M_{i,0}$ has a non-reducible vertex, then we claim that there exist subsets $M_{i,0} \supseteq M_{i,1} \supseteq M_{i,2} \supseteq \cdots \supseteq M_{i,j}$, where in $M_{i,j}$ the vertex v is reducible, and v is not reducible in $M_{i,k}$ for any $k < j$.*

Proof. Let $M_{i,0}$ be the original marked set of G in round i, and let $v \in V(G)$ be a non-reducible vertex in $M_{i,0}$. By Lemma 3.5 there exists a marked vertex, m_i, and an unmarked vertex u_i in $N_G(v)$ such that $\$(m_i) = \$(u_i) = B_{i,0}(v)$.

Painter cannot mark the unmarked vertex, u_i, to make v a reducible vertex, so instead Painter omits the marked vertex, m_i, to obtain $M_{i,1} = M_{i,0} \setminus \{m_i\}$. With $M_{i,1}$ so defined, note that $B_{i,1}(v) \neq B_{i,0}(v)$ since Painter has omitted the only marked neighbor of v with $B_{i,0}(v)$ tokens. In particular, $B_{i,1}(v) < B_{i,0}(v)$. In $M_{i,1}$, if v is a reducible vertex then $j = 1$ and we are done. Otherwise v is still a non-reducible vertex and Painter repeats the above process obtaining subsets $M_{i,2} \supseteq M_{i,3} \supseteq \cdots \supseteq M_{i,j}$. Since $|N_{M_{i,0}}(v)|$ is finite, there are only a finite number of vertices which can be omitted, and thus this process will terminate with $M_{i,j}$ and $B_{i,j}(v)$ where v is a reducible vertex in $M_{i,j}$.

□

Observation 3.8. *If v is a non-reducible vertex, Painter could simply omit all the marked neighbors of v and this would certainly make v a reducible vertex.*

Painter, however, needs to keep in mind that they need to remove enough tokens from the graph to pay for Lister's original marked set. If Painter removed all the neighbors from every vertex that was non-reducible this could result in too many vertices being removed, and thus not enough tokens will be left on the remaining vertices to pay for Lister's original marked set. For example, in Figure 3.3, suppose Lister marks the four vertices $M_{i,0} = \{a, b, c, v\}$. Vertex v is non-reducible, and thus if Painter omits all of v's neighbors (as denoted with arrows), v is certainly a reducible vertex. However, the new marked set $M_{i,1}$ contains only v and there are only three tokens assigned to v, but Painter needs four to pay for the original marked set $M_{i,0}$.

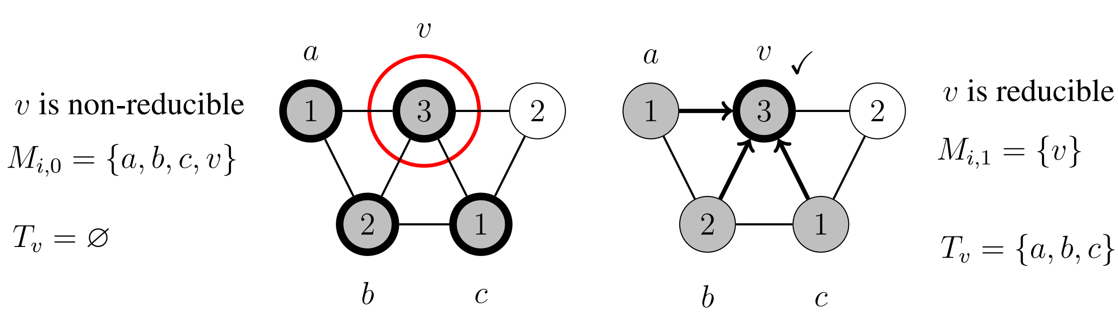

Figure 3.3: Greedy Omission Example on P_5^2

If Painter follows the omission strategy as described in Lemma 3.7, then Painter is intentionally omitting only those vertices which make a vertex non-reducible and no more. This leaves Painter with more vertices and thus more tokens to work with. Following the strategy of Lemma 3.7 and noticing that there are at most two neighbors of v with the same token assignment, thus there are at most $2\big(\$(v) - 1\big)$ neighbors of v with fewer than $\$(v)$ tokens.

Observation 3.9. *For any value, d less than $\$(v)$, if there are two d-token vertices in $N_{M_{i,0}}(v)$, then there are at most $\frac{2\left(\$(v)-1\right)-2}{2} = \$(v)-2$ vertices in T_v, and so $|T_v \cup \{v\}| = |T_v|+1 \leq \$(v)-2+1 < \$(v)$. If there are no repeated values on $N_{M_{i,0}}(v)$, then there are at most $\frac{2\left(\$(v)-1\right)}{2} = \$(v)-1$ vertices in T_v, and so $|T_v \cup \{v\}| = |T_v|+1 \leq \$(v)-1+1 = \$(v)$.*

Observation 3.9 shows that v has enough tokens assigned to it for Painter to use to pay for v and T_v. This is important because it shows that there is a vertex which has enough tokens to pay for those vertices it omitted while still also having enough tokens to pay for itself.

Observation 3.10. *Lemma 3.7 states that for any $v \in M$ Painter can alter the marked set M so that v is a reducible vertex. However, when altering more than one vertex to make all marked vertices reducible, the order in which we omit vertices matters.*

If Painter alters the non-reducible vertices with fewer tokens first, and then works up to the non-reducible vertices with the most tokens, Painter could end up in the situation where $\sum\limits_{v \in M_{i,j}} \$(v) < |M_{i,0}|$. In this case, the omissions leave Painter without enough tokens to pay for $M_{i,0}$ even though $M_{i,j}$ is a marked set in which all vertices are reducible.

For example, in Figure 3.4 suppose Lister chose as the original marked set, $M_{i,0}$ to be $\{a, b, c, d\}$. Notice that in this case vertices b and c are both non-reducible.

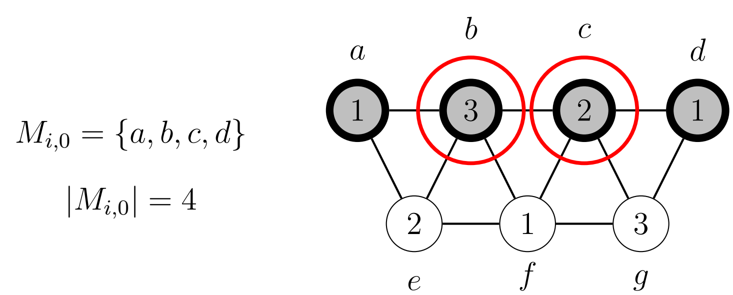

Figure 3.4: Starting Example for P_7^2

If Painter alters the marked set beginning with vertex c, which has lower value than vertex b, then to make c reducible, Painter looks at all the marked neighbors of vertex c with fewer tokens than $\$(c)$. There is only one such vertex, vertex d. Since there are marked and unmarked neighbors of vertex c with 1 token on them, then Painter omits vertex d. Now $T_c = \{d\}$, and since vertex c has no more marked neighbors with fewer tokens than $\$(c)$, vertex c is reducible. See Figure 3.5 below.

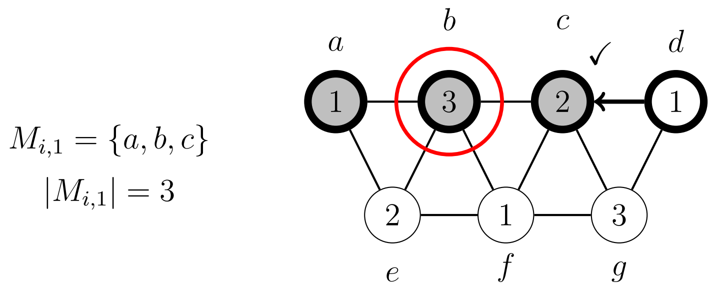

Figure 3.5: Lowest to Highest First Omission for P_7^2

Painter now moves to another non-reducible vertex. Vertex b is the only remaining non-reducible vertex. Again Painter looks at all the marked neighbors of vertex b with fewer tokens than $\$(b)$. There are two such vertices, a and c, so Painter looks at the one with more tokens, vertex c. Since there are marked and unmarked neighbors of vertex b with 2 tokens on them, then Painter omits vertex c. Now $T_b = \{c\}$, however vertex b is still non-reducible. See Figure 3.6 below.

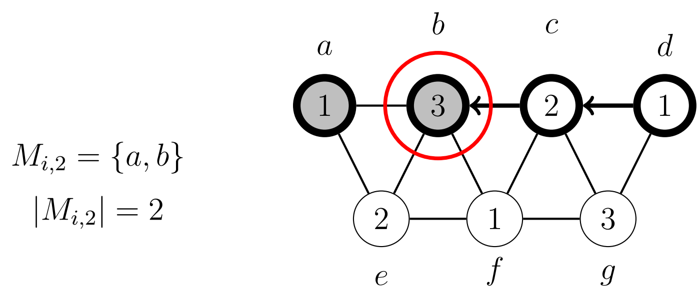

Figure 3.6: Lowest to Highest Second Omission for P_7^2

Vertex b is still the only remaining non-reducible vertex, but there is only one marked neighbor of vertex b with fewer tokens than $\$(b)$. Since there are marked and unmarked neighbors of vertex b with 1 token on them, then Painter omits vertex a. Now $T_b = \{a, c\}$ and vertex b has no more marked neighbors with fewer tokens than $\$(b)$, so vertex b is reducible. See Figure 3.7 below.

$M_{i,3} = \{b\}$
$|M_{i,3}| = 1$

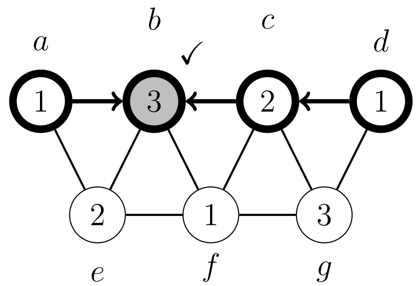

Figure 3.7: Lowest to Highest Third Omission for P_7^2

Notice that if Painter alters the original marked set $M_{i,0}$ in this order, $\sum_{v \in M_{i,3}} \$(v) = 3 < 4 = |M_{i,0}|$. In other words, Painter does not have enough remaining tokens to pay for the original marked set. Furthermore, there is a vertex which is "paying for" vertices outside their neighborhood. By Lemma 3.7 and Observation 3.9 we know that reducible vertices can "pay for" themselves and their strategically omitted neighbors, but no more. This example illustrates problems that arise when Painter makes non-reducible vertices reducible without regard to the token assignment of those non-reducible vertices. We will show that those problems do not exist if Painter makes non-reducible vertices reducible starting with non-reducible vertices with the most tokens and proceeding to subsequent non-reducible vertices $-$ always choosing one with the most tokens.

If Painter had proceeded in this manner starting with the same graph and marked set as in Figure 3.4, Painter would have begun making non-reducible vertices reducible starting with vertex b, a non-reducible vertex with the most tokens. Painter looks at all the marked neighbors of vertex b with fewer tokens than $\$(b)$. There are two such vertices, so Painter looks at the one with more tokens, vertex c. Note that $\$(c) = 2$, therefore $B_{i,0}(v) = 2$. Since there are marked and unmarked neighbors of vertex b with 2 tokens on them, Painter omits vertex c from the marked set. Now $T_b = \{c\}$, however vertex b is still non-reducible. See Figure 3.8 below.

Vertex b is still a non-reducible vertex with the most number of tokens, and there is only one marked neighbor of vertex b, vertex a, with fewer tokens than $\$(b)$. Note that $\$(a) = 1$, therefore $B_{i,1}(v) = 1$. Since there are marked and unmarked neighbors of vertex b with 1 token, then Painter

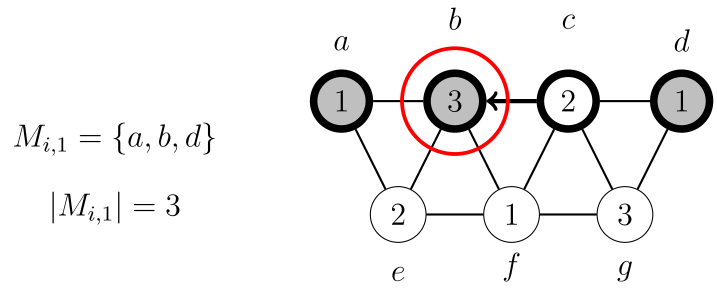

Figure 3.8: Highest First, First Omission for P_7^2

omits vertex a from the marked set. Now $T_b = \{a, c\}$ and vertex b has no more marked neighbors with fewer tokens than $\$(b)$, so vertex b is reducible. See Figure 3.9 below.

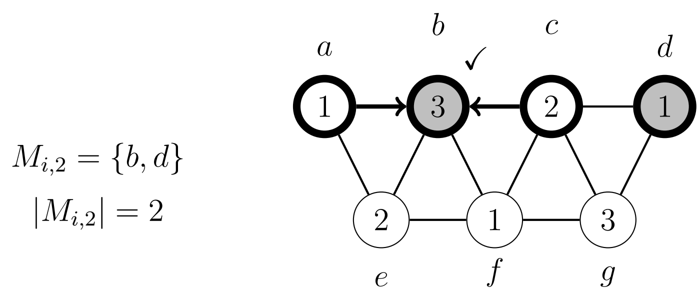

Figure 3.9: Highest First, First Omission for P_7^2

Notice that with this ordering, vertex c is omitted without being made reducible first, and thus vertex d is not omitted. Vertex d has no neighbors with fewer tokens than $\$(d)$, so vertex d is also reducible. Thus all vertices in $M_{i,2}$ are reducible and $\sum_{v \in M_{i,2}} \$(v) = 4 = |M_{i,0}|$, so Painter has exactly enough tokens to pay Lister for the original marked set $M_{i,0}$. Vertices b and d are independent of each other and thus they can both be deleted from the graph, i.e. colored with color i.

Definition 3.11. Let $M_{i,0}$ be the original marked set of G for round i. In the *Highest First Approach* Painter identifies a non-reducible vertex v with the most tokens, and omits from $M_{i,0}$ marked neighbors of v resulting in a new marked set $M_{i,1}$. Painter repeats this process on subsequent vertices $-$ always choosing a non-reducible vertex with the most tokens $-$ until all remaining marked vertices are reducible. The process will result in the final marked set for the round, $M_{i,j}$.

It is possible during the omitting process for vertex v, that the omission of a neighbor of v causes another vertex with more tokens than $\$(v)$ to become non-reducible. Using the Highest First Approach, any such newly non-reducible vertex $-$ which has more tokens than $\$(v)$ $-$ must be made reducible before proceeding any other non-reducible vertices with fewer tokens.

Lemma 3.12. *Using the Highest First Approach, if a vertex w is omitted to make vertex v reducible, then $T_w = \varnothing$.*

Proof. Let $G = P_n^k$. At the beginning of some round i with marked set $M_{i,0}$, no vertex has been omitted, thus we have that $T_v = \varnothing$ for all $v \in M_{i,0}$. If all vertices in $M_{i,0}$ are reducible then we are done. So assume that there at least one marked vertex which is non-reducible. Among all the non-reducible vertices, let h be the highest value of such a vertex, and let w_h be a non-reducible vertex with value h. Using the Highest First Approach, we now omit vertices from the neighborhood of w_h until w_h is reducible. Thus $|T_{w_h}| > 0$ and all other $T_{v \neq w_h} = \varnothing$. Since we have deleted no vertices at this step, all vertices of a given token value remain independent. Thus all the h-token vertices form an independent set and since Painter can only omit vertices in the neighborhood of a given vertex, no h-token vertex can omit another h-token vertex. So for all vertices x_w, which are omitted to make the h-token vertices reducible, $T_{x_w} = \varnothing$.

Now suppose that for some value $a < h$, all vertices of value greater than a are reducible, and let a_k be a vertex of value a which is not reducible. Vertex a_k can only omit vertices of value strictly less than a, and the only vertices with non-empty T_v sets are vertices of value strictly more than a. So any vertex which is omitted to make a_k reducible has value strictly less than a. So for all vertices x_a, which are omitted to make a_k reducible, $T_{x_a} = \varnothing$.

After Painter has omitted these x_a vertices, this may have caused higher valued vertices to become non-reducible. If this is the case, then a highest valued vertex, say vertex m, which had just been made non-reducible would be the next vertex to fix to be a reducible vertex. Since vertex

m is non-reducible, then by Lemma 3.5 there is a marked and unmarked vertex of value $B_i(m)$. Since vertex m was previously reducible and is now non-reducible, then we know that value $B_i(m)$ appears on some x_a vertex. So we also know there are no marked neighbors of vertex m of value higher than $B_i(m)$. Thus vertex m can only omit neighbors of value no greater than $B_i(m)$, and as previously stated the only vertices, v, with non-empty T_v sets are now vertex a_k, and those vertices of value strictly greater than a. However, all of these vertices have value strictly greater than $B_i(m)$, hence $T_{x_{B_i(m)}} = \varnothing$.

□

Using the Highest First approach avoids the problem shown in Figures 3.5, 3.6, and 3.7. This approach also ensures that each vertex is only ever responsible for paying for itself and possibly some its marked neighbors, but nothing outside of its closed neighborhood. Again by Observation 3.9, we know that each reducible vertex has enough tokens assigned to it to pay for itself and its set of omitted neighbors, T_v.

By Lemma 3.7 we know that Painter can make each vertex a reducible vertex, and by Lemma 3.12 we know that there is a particular order in which Painter needs to omit vertices. It remains to show that in each round i, there is a way for Painter to remove at least $|M_{i,0}|$ tokens from the graph, such that (1) each vertex which is deleted (colored with color i) has 0 tokens, (2) no undeleted (uncolored) vertex in $M_{i,0}$ ends up with a non-positive number of tokens, and (3) each non-zero token class in G remains an independent set.

Lemma 3.13. *Let $M_{i,0}$ be the initial marked set of G for round i. If x_1 and x_2 are marked vertices which are reducible in a subsequent marked set $M_{i,j}$, with $B_i(x_1) = B_i(x_2)$, then x_1 and x_2 are independent. Moreover, x_1 (and by similar argument x_2) will be independent of vertices that are unmarked, uncolored, and have $B_i(x_1)$ tokens assigned.*

Proof. Let $M_{i,0}$ be the initial marked set of G for round i, and let x_1 and x_2 be marked vertices which are reducible in a subsequent marked set $M_{i,j}$ with $B_i(x_1) = B_i(x_2) = 0$. Suppose for the sake of contradiction that x_1 and x_2 are adjacent. Since we have assumed that the token classes for round i are independent, then $\$(x_1) \neq \$(x_2)$. Without loss of generality, assume $\$(x_1) < \$(x_2)$. Note that by assumption $0 = B_i(x_2) = B_i(x_1) < \$(x_1) < \$(x_2)$. Then x_1 is a marked neighbor of x_2 with strictly fewer tokens than $\$(x_2)$, and x_1 has a non-zero number of tokens. Thus $B_i(x_2) \neq 0$, a contradiction. Therefore all vertices with $B_i(v) = 0$ are independent. Moreover, there are no vertices with 0 tokens that are unmarked and uncolored. All vertices with $B_i(v) = 0$ are independent of unmarked, uncolored vertices with 0 tokens as well.

Now assume x_1 and x_2 are reducible vertices with $B_i(x_1) = B_i(x_2) = d > 0$. Again suppose for sake of contradiction that x_1 and x_2 are adjacent. Since we have assumed that the token classes for round i are independent, then $\$(x_1) \neq \$(x_2)$. Without loss of generality, assume $\$(x_1) < \$(x_2)$. Note that by assumption, $0 < B_i(x_2) = B_i(x_1) < \$(x_1) < \$(x_2)$. So x_1 is a marked neighbor of x_2 with token value strictly between $B_i(x_2)$ and $\$(x_2)$. This cannot happen because $B_i(x_2)$ is defined to be the largest value of a marked neighbor of x_2 whose value does not exceed $\$(x_2)$.

Again since we have assumed x_1 is reducible, then by definition of reducible all neighbors of x_1 with $B_i(x_1)$ tokens assigned to them are marked. So any unmarked vertex with $B_i(x_1)$ tokens assigned to it will be independent of x_1.

$\square$

We now have all the tools needed to be able to prove the main Theorem, and provide a strategy for Painter to achieve the upper bound, $\mathring{s}(P_n^k) \leq \lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$, regardless of Lister's strategy.

3.3 Painter's Strategy and Proof of Main Theorem

For each round i, Painter assigns tokens to each vertex according to the Natural Greedy Clique Token assignment. Then Painter will apply the Highest First Approach to Lister's original marked set $M_{i,0}$. This approach yields a new set $M_{i,j}$ in which all vertices are reducible. Each surviving $v \in M_{i,j}$ will have a $B_i(v)$ value, and Painter will remove tokens from each $v \in M_{i,j}$ until the number of tokens assigned to v is $B_i(v)$, and vertices with 0 tokens assigned to them will be colored with color i.

Theorem 3.14. *Using the Painter strategy provided above, Painter will never need to give Lister more than* $\lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ *where* $r \equiv n$ *(mod* $k+1$*) tokens. Thus* $\mathring{s}(P_n^k) \leq \lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ *where* $r \equiv n$ *(mod* $k+1$*).*

Proof. Let $G = P_n^k$ and assign tokens to each vertex as described in the Natural Greedy Clique Token Assignment. For round i, let $M_{i,0}$ be the set originally marked by Lister. When Painter uses the Highest First Approach, all the vertices in the resulting marked set $M_{i,j}$ are reducible. Note that at this stage each vertex will have some $B_i(v)$ value associated with it, we will refer to this value as $B_{i,j}(v)$. By Lemma 3.12, we know that each vertex $v \in M_{i,j}$ will only be responsible for paying for itself and those vertices, T_v, which are omitted to make v reducible. Using the Painter strategy above, Painter will remove tokens from each $v \in M_{i,j}$ until the number of tokens assigned to v is $B_{i,j}(v)$. We now verify that the number of tokens removed from v are enough to pay for v and each member of T_v, that is $\$(v) - B_{i,j}(v) \geq |T_v \cup \{v\}|$. Recall that $B_{i,j}(v)$ is the highest number number of tokens assigned to a marked neighbor of v. The neighbors of v which get added to T_v are those marked vertices with fewer tokens assigned to them than $\$(v)$ but more than $B_{i,j}(v)$. Because these vertices have more tokens assigned to them than $B_{i,j}(v)$, then in some previous marked set say $M_{i,k}$ there were marked and unmarked neighbors of v with more tokens than $B_{i,j}(v)$ or there were no neighbors of v with $B_{i,j}(v)$ tokens. Thus $|T_v| \leq \big(\$(v) - 1\big) - B_{i,j}(v)$, and so $\$(v) - B_{i,j}(v) \geq |T_v| + 1 = |T_v \cup \{v\}|$. Thus we have,

$$
\begin{aligned}
|M_{i,0}| &= |M_{i,j}| + \sum_{v \in M_{i,j}} |T_v| \\
&= \sum_{v \in M_{i,j}} \Big(|T_v| + 1\Big) \\
&\leq \sum_{v \in M_{i,j}} \Big(\big[(\$(v) - 1) - B_{i,j}(v)\big] + 1\Big) \\
&= \sum_{v \in M_{i,j}} \Big(\$(v) - B_{i,j}(v)\Big).
\end{aligned}
$$

Thus following the Painter strategy provided, Painter will have enough tokens to pay for the original marked set $M_{i,0}$. To continue this for round $i+1$, we need to verify that the following three conditions hold: (1) each vertex which is deleted (colored with color i) has 0 tokens assigned to it, (2) no undeleted (uncolored) vertex in $M_{i,0}$ ends us with a non-positive number of tokens, and (3) each non-zero token class in G remains independent.

By Lemma 3.13 we know that all $v \in M_{i,j}$ with the same $B_{i,j}(v)$ are independent of each other and independent of unmarked vertices with $B_{i,j}(v)$ tokens. Thus when Painter removes tokens from the vertices in $M_{i,j}$ these vertices will have $B_{i,j}(v)$ tokens on them moving to round $i+1$. Thus at the beginning of round $i+1$ the non-zero token classes are independent, and (3) is satisfied.

At the beginning of round i each vertex had a positive number of tokens assigned to it. When Painter removes tokens from each $v \in M_{i,j}$, each v now has $B_{i,j}(v)$ tokens, which is non-negative. Painter will delete (color with color i) those vertices for which $B_{i,j}(v) = 0$, and the remaining vertices will have a positive number of tokens assigned to it. Thus (1) and (2) are also satisfied.

Now, since Painter assigned $\lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ where $r \equiv n \pmod{k+1}$ tokens to the graph and we have shown that the conditions hold. Painter removed no more than $\lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ where $r \equiv n \pmod{k+1}$ tokens to give to Lister. Thus $\mathring{s}(P_n^k) \leq \lfloor \frac{n}{k+1} \rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ where $r \equiv n \pmod{k+1}$.

□

3.4 Conclusion and Further Directions

Puleo and West [1] showed that for every tree T on n vertices,

$$n + \sqrt{2n} \approx n + u_{n-1} = \mathring{s}(K_{1,n-1}) \leq \mathring{s}(T) \leq \mathring{s}(P_n) \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

They then conjectured that this result extended to k-trees. That is, the Slow-Coloring number for k-trees is bounded below by the k-star $K_k \bigoplus \overline{K}_{n-k}$ and bounded above by the k-path P_n^k. Mahoney, Puleo, and West [2] proved that $\mathring{s}K_k \bigoplus \overline{K}_{n-k} = r + \binom{s+1}{2} + su_r$ where $u_r = \lfloor \frac{-1+\sqrt{1+8r}}{2} \rfloor$. In this

Thesis we have just proved that $\mathring{s}(P_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor \mathring{s}(K_{k+1}) + \mathring{s}(K_r)$ where $r \equiv n \pmod{k+1}$. So while the entire conjecture remains unproven, we now have an exact value for the conjectured upper bound for the Slow-Coloring number of k-trees.

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