A Study of Space-time Fractional SPDE in Bounded Domains

by

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Abstract

This dissertation considers space-time fractional stochastic heat equation on a regular bounded domain B in $\mathbb{R}^d$, $d \geqslant 1$ with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\beta u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + I_t^{1-\beta}[\lambda\sigma(u_t(x))\dot{W}(t,\, x)] \text{ for } x \in B \text{ and } t > 0 \\ u_t(x) = 0 \quad \text{for} \quad x \notin B \text{ and } t > 0, \end{cases} \tag{0.1}$$

and the initial condition $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B, where the time fractional differential operators ∂_t^β and $I_t^{1-\beta}$ respectively denote the Caputo derivative and the Riemann-Liouville fractional integral operator. The fractional Laplacian operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leqslant 2$, is the L^2-generator of a symmetric α-stable process X_t^B killed when exiting B. $\dot{W}$ denotes a space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma|x| \leqslant |\sigma(x)| \leqslant L_\sigma|x|$ where l_σ and L_σ are positive constants. The positive parameter λ is called *the level of the noise*. We studied the long time behavior of their solutions with respect to the level λ of the noise and show how the choice of the order $\beta \in (0,\, 1)$ of the fractional time derivative affects the growth and decay behavior of their solution. Further, we showed the continuity of the solution $u_t(x)$ of (0.1) with respect to the fractional parameter β. Our results extend the main results in Foondun [29] to fractional Laplacian as well as higher dimensional cases.

We also study the temporal Hölder continuity of mild random field solutions for space-time fractional stochastic heat equation driven by noise colored in space which can be obtained by constructing relevant moment bounds for increments of the stochastic convolution $(Y \circledast \sigma(u))_t(y)$. Our techniques are based on connecting the space-time fractional Green functions Y, Z to the fundamental solution of $\partial_t P_t + \frac{\nu}{2}(-\Delta)^{\alpha/2} P_t = 0, P_0 = \delta_0$ via the subordination of Wright-type function $W_{-\beta,\gamma}(-\rho)$.

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Chapter 1

Introduction

In 1855, Adolf Fick [21] developed his now-famous principle governing the transport of mass through diffusive media. In his second law, Fick showed that $u(t, x)$ obeys the classical diffusion equation $\partial_t u = D\partial_{xx} u$ in the spatial dimension, which estimates how the concentration $u(t, x)$ of a diffusive substance varies with space and time, where D is the diffusion coefficient. The space-time fractional diffusion equation which is obtained when integer-order derivative operators in space and time are replaced by fractional counterparts (in Caputo or Riemann-Liouville sense) has been recently treated by several authors (see, for example, Mijena and Nane [47], Saichev and Zaslavsky [52], and Mainardi, Luchko, and Pagnini[41]). The typical form of the space-time fractional equation is $\partial_t^\beta u = -(-\Delta)^{\alpha/2} u$, where ∂_t^β is the Caputo fractional derivative with $\beta \in (0, 1)$, $\alpha \in (0, 2)$ and $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplacian. These equations can be used to model anomalous diffusion processes or diffusion processes in non-homogeneous media with random fractal structures (see, for instance, Meerschaert *et al.* [43], Meerschaert, Nane, and Vellaisamy [45], Baeumer, Luks, and Meerschaert [3], Chen, Kim and Kim [16] and the references therein). Since these space-time fractional equations depend on the fractional parameters β and α, in [19] the importance of the continuity of their solutions with respect to these parameters is discussed. For example, if the partial derivative in time ∂_t in the classical heat equation $\partial_t u = -\Delta u$ is substituted with fractional derivatives ∂_t^β for $0 < \beta < 1$, the processes explains the sticking and trapping behavior of particle, while if the Laplacian Δ is replaced with fractional power $-(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, it describes long particle jumps (see [1]). In [47], Mijena and Nane have recently introduced time fractional SPDEs, which can be utilized to represent phenomena with random effects and thermal memory. In [47], Mijena and Nane have recently introduced time fractional SPDEs, which can be utilized to represent phenomena with random effects and thermal memory. Some other recent results in fraction time SPDEs are worked out by Xu and Caraballo [62], Lototsky and Rozovsky [39], and Thach and Tuan [55].

Let B be a regular bounded open subset of $\mathbb{R}^d$. Consider the following space-time fractional equation with Dirichlet boundary conditions (see (2.6) below for a representation of the solution).

$$\begin{cases} \partial_t^\beta u_t(x) = -(-\Delta)^{\alpha/2} u(x), & x \in B,\, t > 0, \\ u_t(x) = 0, & x \notin B,\ t > 0, \end{cases} \tag{1.1}$$

where $\alpha \in (0,\, 2)$ and $\beta \in (0,\, 1)$.

The fractional time derivative is the Caputo derivative which first appeared in [10] and is defined by

$$\partial_t^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u_r(x)}{\partial r} \frac{dr}{(t-r)^\beta}. \tag{1.2}$$

and the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ ([33]) is defined by

$$(-\Delta)^{\alpha/2} u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^\alpha (u, e^{-i\xi\cdot x}) e^{i\xi\cdot x} d\xi = \mathcal{F}^{-1} \left\{ |\xi|^\alpha \mathcal{F}\{u\}(\xi) \right\}(x), \tag{1.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively.

If $u_0(x)$ denotes the initial condition to equation (1.1), then the solution can be written as

$$u_t(x) = \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy,$$

where $G_B^{(\beta)}(t,\, x,\, y)$ is the space-time fractional heat kernel defined in (2.8).

Now consider

$$\partial_t^\beta u_t(x) = -(-\Delta)^{\alpha/2} u(x) + f(t,x), \tag{1.4}$$

with the same initial condition $u_0(x)$ and $f(t,x)$ is some nice function. To get the correct version of (1.4) we will make use of [56, 57, 58]. The fractional Duhamel principle implies that the mild solution to (1.4) for $t > 0$ is given by

$$u_t(x) = \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy + \int_0^t \int_B G_B^{(\beta)}(t-s,\, x,\, y) \partial_s^{1-\beta} f(s,y)\, dy\, ds. \tag{1.5}$$

Using the fractional order integral I_t^γ defined by

$$I_t^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau,$$

and the property

$$\partial_t^\beta I_t^\beta g(\mathrm{t}) = g(\mathrm{t}),$$

for every $\beta \in (0,\, 1)$, and $g \in L_\infty(\mathbb{R}_+)$ or $g \in C(\mathbb{R}_+)$, then by the Duhamel's principle, the mild solution to (1.4) where the force is $f(t,\, x) = I_t^{1-\beta} g(t,\, x)$, will be given by

$$\begin{aligned} u_t(x) &= \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy + \int_0^t \int_B G_B^{(\beta)}(t-s,\, x,\, y) \partial_s^{1-\beta} (I_t^{1-\beta} g(s,\, y))\, dy\, ds \\ &= \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy + \int_0^t \int_B G_B^{(\beta)}(t-s,\, x,\, y) g(s,\, y)\, dy\, ds. \end{aligned} \tag{1.6}$$

Recently, Mijena and Nane [47], Foondun, Mijena, and Nane [27], and Foondun [29] considered the following time fractional stochastic heat equation on the interval $(0,\, L)$ with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\beta u_t(x) = \frac{1}{2} \partial_{xx} u_t(x) + I_t^{1-\beta}[\lambda \sigma(u_t(x)) \dot{W}(t,\, x)] \ \text{ for } \ 0 < x < L \ \text{ and } \ t > 0 \\ u_t(0) = u_t(L) = 0 \quad \text{for} \quad t > 0, \end{cases} \tag{1.7}$$

where the initial condition $u_0 : [0, L] \to \mathbb{R}_+$ is non-random and non-negative bounded function which is strictly positive on a set of positive measures in $[0,\, L]$. $\dot{W}$ denotes a space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma |x| \leqslant |\sigma(x)| \leqslant L_\sigma |x|$ where l_σ and L_σ are positive constants. λ is a positive parameter known as *the level of the noise* and will play a significant part in this thesis.
Using Walsh [60], we define the mild solution to (1.7) as the random field $u = \{u_t(x)\}_{t>0, x\in B}$ satisfying

$$u_t(x) = (\mathcal{P}_B^\beta u_0)_t(x) + \lambda \int_0^L \int_0^t p_B^\beta(t-s,\, x,\, y) \sigma(u_s(y)) W(ds\,, dy), \tag{1.8}$$

where $p_B^\beta(t,x,y)$ denotes the probability density function of the time-changed killed Brownian motion upon exiting the domain $[0,\ L]$ associated with the fractional time operator, and

$$(\mathcal{P}_B^\beta u_0)_t(x) := \int_0^L p_B^\beta(t,\ x,\ y)u_0(y)dy.$$

In Foondun and Nualart [28] and Foondun, Guerngar, and Nane [25], the authors looked at the behavior of the solution to equation (1.7) for small and large λ when $\boldsymbol{\beta} = 1$. They showed that if λ is large enough, the second moment of the solution u_t grows exponentially fast; while if λ is small, the second moment of the solution u_t eventually decays exponentially. Nualart [49] and Xie [61] have used precise heat kernel estimates to sharpen the results in [28]. However, in [29] Foondun has shown that a more complicated situation will occur instead of such phase transition if fractional time derivative replaces the usual time derivative. That is, for any fixed $\beta \in (0,\ 1)$, the long time behavior of the supremum of the second moment of the solution to equation (1.7) behaves differently by considering the cases when $\beta \in (0,\ \frac{1}{2})$ and $\beta \in (\frac{1}{2},\ 1)$ separately, and the reasons for considering these cases are also explained. For more details about the interpretation of the results, one can refer [29]. These findings are interesting from an application standpoint since fractional time derivatives are commonly used in the modeling of various systems with memory. Therefore, it is very important to realize that the use of such derivatives can result in considerable change in the qualitative properties of the solution. The main aim of this work is to investigate the long time behavior of the solution to (3.1) with respect to the level of the noise λ. Our work extend the main results in Foondun [29] to fractional Laplacian as well as higher dimensional cases.

The Hölder continuity of the mild random field solution of SPDEs has been studied by many authors, see Balan *et. al* [4], Bezdek [6], Chen and Dalang [11], Chen *et. al* [13], Cui *et. al* [17], Foondun and Nane [26], Hu *et. al* [34], Hu *et. al* [35], Sanz-Solé and Sarrà [53], Song [54]. For example, Foondun-Khoshnevisan [23, 24] studied the the regularity of the mild solution of the following nonlinear SHE defined by (1.9) after establishing its existence and uniqueness

$$\begin{cases} \partial_t u_t(x) = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\dot{w}(t\,,x), \\ u_t(x)|_{t=0} = \varphi(x), \end{cases} \tag{1.9}$$

where $t \geqslant 0$, $x \in \mathbb{R}$, $\varphi(x)$ is a measurable and nonnegative initial function, and $\mathcal{L}$ is the $L^2(\mathbb{R})$-generator of a Lévy process $X := \{X_t\}_{t\geqslant 0}$ which acts only on the variable x, $\{\dot{w}(t\,,x)\}_{t\geqslant 0, x\in\mathbb{R}}$ is spatially colored noise, and $\sigma : \mathbb{R} \to \mathbb{R}$ be a fixed Lipschitz function.

Chen and Dalang [11] studied space–time regularity of the solution of the nonlinear stochastic heat equation in one spatial dimension driven by space–time white noise with the initial condition being a locally finite measure:

$$\begin{cases} \partial_t u_t(x) - (\partial_{xx} u_t)(x) = \rho(u_t(x))\dot{w}(t\,,x), \\ u_t(x)|_{t=0} = \mu(x), \end{cases} \tag{1.10}$$

where $t \geqslant 0$, $x \in \mathbb{R}$, $\mu(x)$ is a measurable and nonnegative initial function. They get the results that on compact sets in which $t > 0$, the classical Hölder-continuity exponents $\frac{1}{4}$-in time and $\frac{1}{2}$- in space remain valid. However, on compact sets that include $t = 0$, the Hölder-continuity exponents are $(\frac{\delta}{2} \wedge \frac{1}{4})$ in time and $(\frac{\delta}{2} \wedge \frac{1}{2})$-in space, provided μ is absolutely continuous with an δ Hölder continuous density.

In [47], Mijena and Nane considered the space-time fractional stochastic heat equations driven by white noise

$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right)\, u_t(x) = b(u_t) + \sigma(u_t(t,x))\, \dot{W}(t,x) & x \in \mathbb{R}^d, t > 0 \\ u_t(x)|_{t=0} = \varphi(x) & x \in \mathbb{R}^d, \end{cases} \tag{1.11}$$

have shown the spatially Hölder continuity

$$C_1|y^{'} - y| \leqslant (\mathbb{E}|u_t(y^{'}) - u_t(y)|^q)^{\frac{1}{q}} \leqslant C_2|y^{'} - y|^{(\frac{\alpha - N\beta}{2\beta})^{-}\wedge 1}, \tag{1.12}$$

and the and temporal Hölder continuity

$$C_1|t^{'} - t|^{\frac{1}{2} - \frac{N\beta}{2\alpha}} \leqslant (\mathbb{E}|u_{t^{'}}(y) - u_t(y)|^q)^{\frac{1}{q}} \leqslant C_2|t^{'} - t|^{\frac{1}{2} - \frac{N\beta}{2\alpha}}, \tag{1.13}$$

of the unique mild random field solution for which they have proved existence and uniqueness, where σ is globally Lipschitz continuous and $\dot{W}$ is a space-time white noise with the covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y). \tag{1.14}$$

In [7], Binh, Nguyen, and Ngoc studied the spatial–temporal Hölder continuity of mild random field solutions to the initial value problem for space-time fractional stochastic heat equation driven by colored noise given by

$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u_t(x) = b(u_t) + \sigma(u_t(t,x))\,\dot{W}(t,x) & x \in \mathbb{R}^d, t > 0 \\ u_t(x)|_{t=0} = \varphi(x) & x \in \mathbb{R}^d. \end{cases} \tag{1.15}$$

Here ∂_t^β and $(-\Delta)^{\alpha/2}$ stand for the Caputo's fractional derivative of order $\beta \in (0,\, 1)$, and the fractional Laplace operator of order $\alpha \in (0,\, 2)$ respectively, where the latter is also the generator of a strict stable process X_t of order α with $\mathbb{E}(e^{i\xi\cdot X_t}) = e^{-t|\xi|\alpha}$. The nonlinearity σ is assumed to be Lipschitz continuous.

In [12], Chen and Hu used local fractional derivatives to show the Hölder continuity of the random field solution to the following nonlinear time-fractional slow and fast diffusion equation:

$$\left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma \left[\sigma\left(u(t,x)\right)\dot{W}(t,x)\right], \quad t > 0,\ x \in \mathbb{R}^d,$$

where $\dot{W}$ is the space-time white noise, $\alpha \in (0,2]$, $\beta \in (0,2)$, $\gamma \geqslant 0$ and $\nu > 0$, under the condition that $2(\beta+\gamma) - 1 - d\beta/\alpha > 0$.

We now briefly give an outline of the dissertation. In this dissertation we used methods similar to that of Foondun [29] with crucial changes to prove our main results. These method were also used in [23] and [28] among others. Chapter 2 contains estimates and some preliminary results needed for the proof of main results. In Chapter 3 we state our main results on long time behavior of the solution for space-time fractional SPDE driven by a space-time white noise, while in Chapter 4 we state our main results for the case of space colored noise. In Chapter 5, we present preliminary results with their proof and state the main result on temporally Hölder continuity of the mild random field solution of space-time fractional stochastic heat equation driven by noise colored in space. Chapter 6 is devoted to the proof of Theorem 3.1, Theorem 3.2, and Theorem 3.3 while the proof of Theorem 3.8 is given in Section 6.2. Further, in Section 6.3 and Section 6.4 we give the proof of Theorem 4.3, Theorem 4.4, Theorem 4.5 and Theorem 4.8. Finally, the proofs of Theorem 5.9 and Theorem 5.9 are given in Section 6.5.

Chapter 2

Preliminaries

In this section we give some preliminary results, which are needed for the proofs of the main theorems. Let X_t denote a symmetric stable process of index $\alpha \in (0, 2)$ in $\mathbb{R}_d$ and B be a regular bounded open subset of $\mathbb{R}_d$. Let X_t^B denote the symmetric stable process killed upon exiting B. The following Hilbert-Schmidt expansion (see Davies [20]) holds for the probability density function p_B of X_t^B,

$$p_B(t,\, x,\, y) := \sum_{n=1}^{\infty} e^{-\mu_n t}\varphi_n(x)\varphi_n(y), \tag{2.1}$$

for all $x,\, y \in B,\, t > 0$, where $\{\varphi_n\}_{n\geqslant 0}$ is an orthonormal basis of $L^2(B)$, and $0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$ is a sequence of positive real numbers such that, for every $n \geqslant 1$, $P_t^B \varphi_n = e^{-\mu_n t}\varphi_n$,
where $\{P_t^B, t \geqslant 0\}$ denote the transition semigroup of X^B (see Chen, Meerschaert, and Nane [15]), that is

$$P_t^B f(x) = \mathbb{E}_x[f(X_t^B)] = \int_B p_B(t, x, y) f(y) dy. \tag{2.2}$$

From Theorem 2.3 of Blumenthal and Getoor [8] and the proof of Theorem 5.1 of Chen, Meerschaert, and Nane [15], we infer the following lemma:

Lemma 2.1. *[8, 15] For any bounded open subset B of $\mathbb{R}^d$, the system of eigenfunctions $\{\varphi_n\}_{n\geqslant 1}$ and the corresponding eigenvalues $\{\mu_n\}_{n\geqslant 1}$ satisfy the following:*

a) $C_1 n^{\frac{\alpha}{d}} \leqslant \mu_n \leqslant C_2 n^{\frac{\alpha}{d}}$ *for every* $n \in \mathbb{N}$,

b) $|\varphi_n(\mathrm{x})| \leqslant C \mu_n^{\frac{d}{2\alpha}}$ *for every* $\mathrm{x} \in B$,

where C, C_1, and C_2 are positive real numbers.

It is also known that

$$\begin{aligned} u_t(\mathrm{x}) &:= \mathbb{E}^x[u_0(X_t^B)] \\ &= \int_B p_B(t,\, x,\, y)\, u_0(y)\, dy, \end{aligned}$$

solves the heat equation $\partial_t u_t(\mathrm{x}) = -(-\Delta)^{\alpha/2} u_t(x)$ defined on B with Dirichlet boundary condition and initial condition u_0.

Let $D = \{D_t, t \geqslant 0\}$ denote a β-stable subordinator and $E_t = \inf\{\tau > 0 : D_\tau > t\}$ is the inverse, or first passage time of a stable subordinator D_t of index $\beta \in (0,\, 1)$. The process $X_{E_t}^B$ is just a time-changed of the killed α-symmetric stable process X_t^B and since $\beta \in (0,\, 1)$, $X_{E_t}^B$ moves more slowly than X_t^B. It is known that the density of the time changed process $X_{E_t}^B$ is given by the $G^{(\beta)}(t, x)$. By conditioning, we have

$$G^{(\beta)}(t, x) = \int_0^\infty p_B(s, x) f_t(s) ds, \tag{2.3}$$

where

$$f_t(s) = t\beta^{-1} s^{-1-1/\beta} g_\beta(t s^{-1/\beta}), \tag{2.4}$$

where $g_\beta(.)$ (Cf. Meerschaert and Straka [44]) is the density function of D_1 and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leqslant 0$; see Meerschaert and Scheffler [46] for more information about the inverse stable subordinator E_t.

The function $v_t(x) := \mathbb{E}^x[u_0(X_{E_t}^B)]$ solves the space-time fractional equation

$$\begin{cases} \partial_t^\beta v_t(x) = -(-\Delta)^{\alpha/2} v_t(x) \ \text{ for } \ x \in B \ \text{ and } \ t > 0 \\ v_t(x) = 0 \quad \text{for } \ x \notin B, \end{cases} \tag{2.5}$$

with initial condition u_0 (Chen, Meerschaert, and Nane [15]). Thus, we get the following representation of $v_t(x)$

$$
\begin{aligned}
v_t(x) &:= \mathbb{E}^x[u_0(X^B_{E_t})] \\
&= \int_B \int_0^\infty \sum_{n=1}^\infty e^{-\mu_n s} \varphi_n(x)\varphi_n(y) f_t(s)\, u_0(y)\, ds dy \\
&= \int_B \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta)\varphi_n(x)\varphi_n(y) u_0(y) dy \\
&= \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy, \qquad (2.6)
\end{aligned}
$$

this follows since the Laplace transform of $f_t(s)$ is $\hat{f}_t(\lambda) = E_\beta(-\lambda t^\beta)$ where $E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+\beta k)}$ is the Mittag-Leffler function and has the following property,

$$
\frac{1}{1+\Gamma(1-\beta)x} \leqslant E_\beta(-x) \leqslant \frac{1}{1+\Gamma(1+\beta)^{-1}x} \quad \text{for } x>0, \qquad (2.7)
$$

where $\Gamma(\cdot)$ is the gamma function. Thus, we get the following expansion,

$$
G_B^{(\beta)}(t,\, x,\, y) := \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta)\varphi_n(x)\varphi_n(y). \qquad (2.8)
$$

Using (2.4) and a change of variable from (2.8) we obtain

$$
\begin{aligned}
G_B^{(\beta)}(t,\, x,\, y) &= \int_0^\infty \sum_{n=1}^\infty e^{-\mu_n (t/u)^\beta} \varphi_n(x)\varphi_n(y) g_\beta(u) du \qquad (2.9) \\
&= \int_0^\infty p_B(\left(\frac{t}{u}\right)^\beta,\, x,\, y) g_\beta(u)\, du. \qquad (2.10)
\end{aligned}
$$

The following Lemma from Mijena and Nane [47] about the L^2-norm of the heat kernel which is a crucial ingredient for the existence and uniqueness of solution of equation (3.1) is also used in proving Theorem 3.8.

Lemma 2.2. *(Lemma 1 in [47]) Suppose that $d < 2\alpha$, then*

$$
\int_{\mathbb{R}_d} [G^{(\beta)}(t,x)]^2 dx = C^* t^{\frac{-\beta d}{\alpha}}, \qquad (2.11)
$$

where the constant C^* is given by

$$C^* = \frac{(\nu)^{-\frac{d}{\alpha}} 2\pi^{\frac{d}{2}}}{\alpha\Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty z^{\frac{d}{\alpha-1}} (\mathbb{E}_\beta(-z))^2 dz.$$

Proof. Using the Plancherel theorem and (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^d} |G^{(\beta)}(t,x)|^2 dx &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |(G^{(\beta)})^*(t,x)|^2 d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |E_\beta(-\,|\xi|^\alpha t^\beta)|^2 d\xi \\ &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty r^{d-1} (E_\beta(-\,r^\alpha t^\beta))^2 dr. \\ &= \frac{(\,t^\beta)^{-d/\alpha} 2\pi^{d/2}}{\alpha\Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty z^{d/\alpha-1} (E_\beta(-z))^2 dz.\end{aligned}$$

where $(G^{(\beta)})^*(t,x) = E_\beta(-\,|\xi|^\alpha t^\beta)$ is the Fourier transform of $G^{(\beta)}(t,x)$ in (2.3). We used the integration in polar coordinates for radially symmetric function in the last equation above. Now using equation (2.7), we get

$$\begin{aligned}\int_0^\infty \frac{z^{d/\alpha-1}}{(1+\Gamma(1-\beta)z)^2} dr &\leqslant \int_0^\infty z^{d/\alpha-1} (E_\beta(-z))^2 dz \\ &\leqslant \int_0^\infty \frac{z^{d/\alpha-1}}{(1+\Gamma(1+\beta)^{-1}z)^2} dz. \end{aligned} \tag{2.12}$$

Hence $\int_0^\infty z^{d/\alpha-1}(E_\beta(-z))^2 dz < \infty$ if and only if $d < 2\alpha$. In this case, the upper bound in equation (2.12) is

$$\int_0^\infty \frac{z^{d/\alpha-1}}{(1+\Gamma(1+\beta)^{-1}z)^2} dz = \frac{\mathrm{B}(d/\alpha, 2-d/\alpha)}{\Gamma(1+\beta)^{-d/\alpha}},$$

where $\mathrm{B}(d/\alpha, 2-d/\alpha)$ is a Beta function. □

The following three results are helpful in proving the finiteness result in Theorem 3.3, and Theorem 4.5.

Lemma 2.3. *For $p \geqslant 2$ and $\beta \in \left(\frac{1}{p}, 1\right)$, we have*

$$\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p ds \lesssim \mu_n^{-\frac{1}{\beta}}, \tag{2.13}$$

for every $t > 0$.

Proof. Using the Mittag-Leffler function bounds given in (2.7), we get

$$\begin{aligned}\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p\,ds &\leqslant C\int_0^t \left(1+\Gamma(1+\beta)^{-1}\mu_n(t-s)^\beta\right)^{-p}\,ds\\ &\leqslant C\int_0^t \left(1\wedge\left(\mu_n(t-s)^\beta\right)^{-p}\right)\,ds, \end{aligned}\tag{2.14}$$

where

$$1\wedge\left(\mu_n(t-s)^\beta\right)^{-p}=\begin{cases}1 & \text{if } \left(\mu_n(t-s)^\beta\right)^{-p}>1,\\ \left(\mu_n(t-s)^\beta\right)^{-p} & \text{if } \left(\mu_n(t-s)^\beta\right)^{-p}\leqslant 1.\end{cases}\tag{2.15}$$

Note that

$$\left(\mu_n(t-s)^\beta\right)^{-p}\leqslant 1 \implies t-\mu_n^{-\frac{1}{\beta}}\geqslant s.\tag{2.16}$$

So, from (2.15) using (2.16) we obtain

$$\begin{aligned}\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p\,ds &\leqslant C\int_0^t \left(1\wedge\left(\mu_n(t-s)^\beta\right)^{-p}\right)\,ds \qquad (2.17)\\ &= C\left(\int_0^{t-\mu_n^{-\frac{1}{\beta}}}\left(\mu_n(t-s)^\beta\right)^{-p}ds+\int_{t-\mu_n^{-\frac{1}{\beta}}}^t ds\right)\\ &= C\left(\frac{1}{\mu_n^p}\int_0^{t-\mu_n^{-\frac{1}{\beta}}}(t-s)^{-p\beta}ds+\mu_n^{-\frac{1}{\beta}}\right)\\ &= C\left(\frac{1}{\mu_n^p(1-p\beta)}\left(t^{-p\beta+1}-\left(\frac{1}{\mu_n}\right)^{-p+\frac{1}{\beta}}\right)+\mu_n^{-\frac{1}{\beta}}\right)\\ &\leqslant C\left(\frac{\mu_n^{p-\frac{1}{\beta}}}{\mu_n^p(p\beta-1)}+\mu_n^{-\frac{1}{\beta}}\right)\\ &\leqslant C\left(\frac{1}{p\beta-1}+1\right)\mu_n^{-\frac{1}{\beta}}\\ &\lesssim \mu_n^{-\frac{1}{\beta}}, \qquad (2.18)\end{aligned}$$

since $\frac{t^{-p\beta+1}}{1-p\beta}<0$.

□

Lemma 2.4. *For $p \geqslant 2$, $\beta \in \left(\frac{1}{p}, 1\right)$ and $d < \frac{\alpha}{\beta}$, we have*

$$\int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n(t-s)^\beta)^p ds < \infty, \tag{2.19}$$

for every $t > 0$.

Proof. Using Lemma 2.1(a) and Lemma 2.3, we obtain the following:

$$\begin{aligned}\int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n(t-s)^\beta)^p ds &= \sum_{n=1}^{\infty} \int_0^t E_\beta(-\mu_n(t-s)^\beta)^p ds \\ &\lesssim \sum_{n=1}^{\infty} \mu_n^{-\frac{1}{\beta}} \\ &\lesssim \sum_{n=1}^{\infty} n^{-\frac{\alpha}{\beta d}} < \infty, \end{aligned} \tag{2.20}$$

since $\frac{\alpha}{\beta d} > 1$.

□

Lemma 2.5. *For $p \geqslant 2$, $\beta \in \left(\frac{1}{p}, 1\right)$, if $d < \frac{\alpha}{2\beta}$ then we have*

$$\int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n(t-s)^\beta)^p \varphi_n^2(x)\, ds < \infty, \tag{2.21}$$

for every $t > 0$.

Proof. We prove the result using by Lemma 2.1(b) and Lemma 2.3 as follows

$$\begin{aligned}\int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n(t-s)^\beta)^p \varphi_n^2(x)\, ds &= \sum_{n=1}^{\infty} \int_0^t E_\beta(-\mu_n(t-s)^\beta)^p \varphi_n^2(x)\, ds \\ &\leqslant C \sum_{n=1}^{\infty} \mu_n^{\frac{d}{\alpha}} \mu_n^{-\frac{1}{\beta}} \\ &\lesssim \sum_{n=1}^{\infty} n^{\frac{\alpha}{d}(\frac{d}{\alpha}-\frac{1}{\beta})} \\ &\lesssim \sum_{n=1}^{\infty} n^{1-\frac{\alpha}{\beta d}} < \infty, \end{aligned} \tag{2.22}$$

since $\frac{\alpha}{\beta d} > 2$. □

The following continuity Lemma is crucial to justify the conclusion of Theorem 3.8.

Lemma 2.6. *Assume that $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$, depending on the geometrical characteristics of the domain B, i.e.,*

$$C(B) = \sup_{n\geqslant 1, x\in B} \varphi_n(x).$$

Fix $t > 0$, then for $d < \alpha$ we have

$$\lim_{\gamma\to\beta} \sup_{x\in B} \left|(\mathcal{G}_B^{(\gamma)} u_0)_t(x) - (\mathcal{G}_B^{(\beta)} u_0)_t(x)\right| = 0,$$

where

$$(\mathcal{G}_B^{(\gamma)} u_0)_t(x) := \int_B G_B^{(\gamma)}(t,\, x,\, y) u_0(y)\, dy.$$

Proof. Since the initial datum u_0 and $G_B^{(\gamma)}(t,\, x,\, y)$ are given to be bounded from above, it suffices to show that

$$\lim_{\gamma\to\beta} \sup_{x\in B} \left|G_B^{(\gamma)}(t,\, x,\, y) - G_B^{(\beta)}(t,\, x,\, y)\right| = 0.$$

Now using the Laplace transform of the Mittag-Leffler function, for any $\theta > 0$ we have

$$\lim_{\gamma\to\beta} \int_0^\infty e^{-\theta t} E_\gamma(-\mu_n t^\gamma)\, dt = \lim_{\gamma\to\beta} \frac{\theta^{\gamma-1}}{\theta^\gamma + \mu_n} = \int_0^\infty e^{-\theta t} E_\beta(-\mu_n t^\beta)\, dt.$$

This implies that

$$\lim_{\gamma\to\beta} E_\gamma(-\mu_n t^\gamma) = E_\beta(-\mu_n t^\beta). \tag{2.23}$$

Now using the expansions for the heat kernel, we have

$$|G_B^{(\beta)}(t,\, x,\, y) - G_B^{(\gamma)}(t,\, x,\, y)| \leqslant \sum_{n=1}^\infty \left[|E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma)||\varphi_n(x)||\varphi_n(y)|\right]. \tag{2.24}$$

Taking limit as $\gamma \to \beta$ on both sides of (2.24) we obtain the required result, since each terms in the summation can be bounded by a quantity independent of γ and β. This is due to the uniform boundedness of the eigenfunctions $\{\varphi_n\}_{n\geqslant 1}$ and the bounds on the Mittag-Leffler function together with Lemma 2.1(a), and summable as shown below.

For any $t > 0$, then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left[|E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma)||\varphi_n(x)||\varphi_n(y)|\right] &\leqslant [C(B)]^2 \sum_{n=1}^{\infty}\left|E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma)\right| \\ &\leqslant [C(B)]^2 \sum_{n=1}^{\infty}\left[\frac{1}{1+\Gamma(1+\beta)^{-1}\mu_n t^\beta} + \frac{1}{1+\Gamma(1+\gamma)^{-1}\mu_n t^\gamma}\right] \\ &\leqslant [C(B)]^2 \sum_{n=1}^{\infty}\frac{1}{\mu_n}\left(\frac{1}{t^\beta} + \frac{1}{t^\gamma}\right) \\ &\leqslant 2[C(B)]^2 \max(1, 1/t)\sum_{n=1}^{\infty} n^{-\frac{\alpha}{d}} < \infty, \end{aligned} \tag{2.25}$$

where C is a positive constant. □

The following result is useful to justify the next lemma.

Remark 2.7. *For any real μ, the inequality $e^{-z} \leqslant \mathscr{C}_\mu z^{-\mu}$ holds for all $z \geqslant 0$.*

Proof. From the series expansion of e^z, for any $n \geqslant 0$, we have

$$z^{-n}e^z \geqslant \frac{1}{n!}$$

This implies for any $n \geqslant 0$, we have

$$n!z^{-n} \geqslant e^{-z}.$$

Thus, the remark follows. □

The following lemma is useful in the proof of Lemma 2.9(b) and Proposition 6.3.

Lemma 2.8. *If $\beta \in (\beta_0, \beta_1)$ for some $\beta_0, \beta_1 \in (0, 1)$ and $\frac{d}{\alpha} < \lambda < \frac{1}{2\beta}$, then using the inequality $e^{-z} \leqslant \mathscr{C}_\mu z^{-\mu}$, we bound the following integral as follows:*

$$\int_0^\infty e^{-2\theta s/p}\frac{1}{s^{2\lambda\beta}}ds \lesssim p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}_{\mu_0}^2\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}_{\mu_1}^2\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right], \tag{2.26}$$

where $0 < \mu_0 < \min\left(\frac{1}{2} - \lambda,\ \frac{1-2\beta_1\lambda}{2}\right)$ and $\mu_1 > \frac{1-2\beta_0\lambda}{2}$.

Proof. Consider the integral

$$\begin{aligned}\int_0^\infty e^{-2\theta s/p}\frac{1}{s^{2\lambda\beta}}ds &= p^{1-2\lambda\beta}\int_0^\infty \frac{e^{-2\theta r}}{r^{2\lambda\beta}}dr\\ &= p^{1-2\lambda\beta}\left[\int_0^1 \frac{e^{-2\theta r}}{r^{2\lambda\beta}}dr + \int_1^\infty \frac{e^{-2\theta r}}{r^{2\lambda\beta}}dr\right]\\ &\lesssim p^{1-2\lambda\beta}\left[\mathscr{C}_{\mu_0}^2\theta^{-2\mu_0}\int_0^1 \frac{1}{r^{2\lambda\beta+2\mu_0}}dr + \mathscr{C}_{\mu_1}^2\theta^{-2\mu_1}\int_1^\infty \frac{1}{r^{2\lambda\beta+2\mu_1}}dr\right]\\ &\lesssim p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}_{\mu_0}^2\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}_{\mu_1}^2\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right],\end{aligned} \tag{2.27}$$

where $\beta\in(\beta_0,\beta_1)$ for some $\beta_0,\beta_1\in(0,1)$, $\mu_0<\min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta_1\lambda}{2}\right)<\min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta\lambda}{2}\right)$ and $\mu_1>\frac{1-2\beta_0\lambda}{2}$, which implies that the proper integral $\int_0^1\frac{1}{r^{2\beta\lambda+2\mu_0}}dr$ and $\int_1^\infty\frac{1}{r^{2\beta\lambda+2\mu_1}}dr$ are convergent. □

Lemma 2.9 (Nane and Tuan, 2022 [48])**.** *Assume that $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$, depending on the geometrical characteristics of the domain B, i.e.,*

$$C(B)=\sup_{n\geqslant 1,x\in B}\varphi_n(x).$$

a) Let $\beta\in(\frac{1}{2},1)$, and $\frac{d}{\alpha}<\lambda<\frac{1}{2\beta}$. Then there exists a constant $C=C(\lambda)$ independent of β such that

$$|G_B^{(\beta)}(t,\,x,\,y)|\leqslant Ct^{-\beta\lambda}. \tag{2.28}$$

b) Let $\beta\in(\beta_0,\beta_1)$ for some $\beta_0,\beta_1\in(0,1)$. Let also $0<\mu_0<\min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta_1\lambda}{2}\right)$ and $\mu_1>\frac{1-2\beta_0\lambda}{2}$. Then for $d<\frac{\alpha}{2\beta}$, there exists positive constant C which depends on μ_0,μ_1,p and independent of β such that

$$\left[\int_0^\infty\int_B e^{-\frac{2\theta s}{p}}|G_B^{(\beta)}(s,\,x,\,y)|^2dy\,ds\right]^{\frac{p}{2}}\leqslant Cp^{\frac{p(1-2\lambda\beta_0)}{2}}\left(\theta^{-2\mu_0}+\theta^{-2\mu_1}\right)^{p/2}. \tag{2.29}$$

Proof. (a) Using the fact that $E_\beta(-\mu_n t^\beta) \leqslant \frac{C}{(1+\mu_n t^\beta)^\lambda}$ for any $\frac{d}{\alpha} < \lambda < \frac{1}{2\beta}$ and C is independent of β, we find that

$$|G_B^{(\beta)}(t,\, x,\, y)| = \left|\sum_{n=1}^{\infty} E_\beta(-\mu_n s^\beta)\varphi_n(x)\varphi_n(y)\right| \leqslant C\sum_{n=1}^{\infty}|E_\beta(-\mu_n s^\beta)|$$
$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{\mu_n^\lambda}\frac{1}{t^{\beta\lambda}} \leqslant Ct^{-\beta\lambda}, \tag{2.30}$$

where we used the eigenfunction expansion of $G_B^{(\beta)}(t,\, x,\, y)$ and that the series $\sum_{n=1}^\infty \frac{1}{\mu_n^\lambda} \leqslant \sum_{n=1}^\infty n^{-\frac{\alpha\lambda}{d}}$ is convergent since $\lambda > \frac{d}{\alpha}$.

(b) Using part (a) of this lemma together with Lemma 2.8, we obtain

$$\int_0^\infty \int_B e^{-2\theta s/p}\big|G_B^{(\beta)}(s,\, x,\, y)\big|^2 dy\, ds \lesssim \int_0^\infty e^{-2\theta s/p}\frac{1}{s^{2\lambda\beta}}ds$$
$$\lesssim p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}_{\mu_0}^2\,\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}_{\mu_1}^2\,\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right], \tag{2.31}$$

where $\beta \in (\beta_0, \beta_1)$ for some $\beta_0, \beta_1 \in (0,1)$, $\mu_0 < \min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta_1\lambda}{2}\right) < \min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta\lambda}{2}\right)$ and $\mu_1 > \frac{1-2\beta_0\lambda}{2}$, which implies that the proper integral $\int_0^1 \frac{1}{r^{2\beta\lambda+2\mu_0}}dr$ and $\int_1^\infty \frac{1}{r^{2\beta\lambda+2\mu_1}}dr$ are convergent. It follows from (2.31) that for any $p \geqslant 0$

$$\left[\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2} \lesssim p^{\frac{p(1-2\lambda\beta_0)}{2}}\left[\frac{\mathscr{C}_{\mu_0}^2\,\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}_{\mu_1}^2\,\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right]^{p/2},$$

which allows us to deduce (2.29). That is, each terms in the summation can be bounded by a quantity independent of β.

□

Next we give an example where the sufficient condition of Lemma 2.6 is satisfied.

Example 2.10. *Let X_t denote a Brownian motion in $\mathbb{R}^2$ and X_t^B denote the Brownian motion killed upon exiting the rectangular domain $B := [0,\, L_1] \times [0,\, L_2]$. The eigenvalues of the Dirichlet Laplacian are $\mu_{m,n} = (\frac{m\pi}{L_1})^2 + (\frac{n\pi}{L_2})^2$ and $\varphi_{m,n}(x,y) := \varphi_m(x)\varphi_n(y) = \frac{2}{\sqrt{L_1L_2}}\sin(\frac{m\pi x}{L_1})\sin(\frac{n\pi y}{L_2})$ are the corresponding eigenfunction so that $|\varphi_{m,n}(x.y)| \leqslant \frac{2}{\sqrt{L_1L_2}}$ for all $(x,y) \in B$. A similar result is valid for the Brownian motion in higher dimensional rectangular boxes.*

The next theorem is the Burkholder-Davis-Gundy inequality which is one of the tool frequently used in the proof of our results.

Theorem 2.11. *([51]) For every $p \in (0,\, \infty)$ there exist constants $c_p, C_p \in (0,\, \infty)$ such that for any continuous local martingale M vanishing at 0 we have*

$$c_p \mathbb{E}[< M, M >_\infty^{p/2}] \leqslant \mathbb{E}[(M_\infty^*)^p] \leqslant C_p E[< M, M >_\infty^{p/2}], \tag{2.32}$$

*where **<M,M>** denotes the quadratic variation of a process M and $M_t^* = \sup_{s\in[0,T]} |M_s|$.*

Chapter 3

The long-time behavior of space-time fractional SPDE driven by space-time white noise

3.1 Introduction

Consider the following stochastic heat equation on a regular bounded domain B in $\mathbb{R}^d$, $d \geqslant 1$ with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\beta u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + I_t^{1-\beta}[\lambda\sigma(u_t(x))\dot{W}(t,\,x)] \ \text{ for } \ x \in B \ \text{ and } \ t > 0 \\ u_t(x) = 0 \quad \text{for} \quad x \notin B \ \text{ and } \ t > 0, \end{cases} \tag{3.1}$$

and the initial condition $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B. The operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leqslant 2$, is the L^2-generator of a symmetric α-stable process X_t^B killed when exiting B. $\dot{W}$ denotes a space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma|x| \leqslant |\sigma(x)| \leqslant L_\sigma|x|$ where l_σ and L_σ are positive constants. The positive parameter λ is called *the level of the noise*.

$\dot{W}(t,\,x)$ is a space-time white noise with $x \in B$, which is assumed to be adapted with respect to a filtered probability space $(\Omega,\,\mathcal{F},\,\mathcal{F}_t,\,\mathbb{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t,\,t \geqslant 0\}$ is right continuous.

$\dot{W}(t,\,x)$ is a generalized process with covariance given by

$$\mathbb{E}\left[\dot{W}(t,\,\mathrm{x})\dot{W}(s,\,\mathrm{y})\right] = \delta_0(t-s)\delta_0(\mathrm{x}-\mathrm{y}),$$

where δ_0 denotes the $Dirac\ delta$ distribution. That is, $W(f)$ is a random field indexed by function $f \in L^2((0,\,\infty) \times B)$ and for all $f, g \in L^2((0,\,\infty) \times B)$, we have

$$\mathbb{E}\left[W(f)W(g)\right] = \int_0^\infty \int_B f(t,x)g(t,x)\,dx\,dt.$$

Hence $W(f)$ can be represented

$$W(f) = \int_0^\infty \int_B f(t,x) W(\,dx\,dt).$$

Note that $W(f)$ is $\mathcal{F}_t$-measurable whenever f is supported on $[0,\,t] \times B$.

We need to make sense of the stochastic integral in the mild solution [18, 60]. The Walsh-Dalang Integral that is used in equation (3.3) is defined as follows. We use the Brownian filtration $\{\mathcal{F}_t\}$ and the Walsh-Dalang integral [18, 60] defined as follows:

- $(t,x) \to \Phi_t(x)$ is an elementary random field when $\exists 0 \leqslant a < b$ and an $\mathcal{F}_a$-measurable $X \in L^2(\Omega)$ and $\phi \in L^2(B)$ such that

$$\Phi_t(x) = X 1_{[a,\,b]}(t)\phi(x) \quad t > 0, x \in B.$$

- If $h = h_t(x)$ is non-random and Φ is elementary, then

$$\int h\Phi dW := X \int_{(a,\,b)\times B} h_t(x)\phi(x) W(dtdx).$$

- The stochastic integral is Wiener's, and it is well defined iff $h_t(x)\phi(x) \in L^2([a,\,b] \times B)$.

- We have Itô-Walsh isometry,

$$\mathbb{E}\left(\left| \int h\Phi dW \right|^2 \right) = \int_0^\infty ds \int_B dy [h_s(y)]^2 \mathbb{E}(|\Phi_s(y)|^2). \tag{3.2}$$

We can make sense of equation (3.1) using Walsh theory [60] again by using the integral equation below.

$$u_t(x) = (\mathcal{G}_B^{(\beta)} u_0)_t(x) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\,x,\,y)\sigma(u_s(y)) W(ds\,,dy), \tag{3.3}$$

where $G_B^{(\beta)}(t,x,y)$ denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (1.1), and

$$(\mathcal{G}_B^{(\beta)} u_0)_t(x) := \int_B G_B^{(\beta)}(t,\,x,\,y) u_0(y)\,dy.$$

If $d < (2\wedge\beta^{-1})\alpha$, the proof for the existence of a unique random-field solution of (3.1) whose second moment satisfying

$$\sup_{x\in B}\mathbb{E}|u_t(x)|^2 \leqslant c_1 e^{c_2 t\lambda^{\frac{2\alpha}{\alpha-\beta d}}}$$

for all $t > 0$ can be found in Theorem 1.1 of Foondun *et al.* [27].

Next we state our main results of this chapter.

We first note that when $\alpha = 2$ and $B = (0,\,1)$, (3.1) becomes (1.7). Thus, our new results are consistent with that obtained in Foondun [29].

3.2 Main results

Theorem 3.1. *Suppose that $d < (2\wedge\beta^{-1})\alpha$. Let u_t denote the unique solution to (3.3). Then the second moment of u_t cannot decay exponentially fast, regardless of what λ is. Indeed, if we further suppose that $\beta \in (0,\,\frac{1}{2}]$, then as t gets large, $\sup_{x\in B}\mathbb{E}|u_t(x)|^2$ grows exponentially fast for any λ.*

Theorem 3.2. *In Theorem 3.1 if $\beta \in (\frac{1}{2},\,1)$, then there exist a strictly positive real number λ_u such that for all $\lambda > \lambda_u$, $\sup_{x\in B}\mathbb{E}|u_t(x)|^2$ grows exponentially fast as time gets large.*

Since the Mittage-Leffler function $E_\beta(-t^\beta)$ (Gorenflo *et al.*[32]) behaves as a stretched exponential for $t \to 0$;

$$E_\beta(-t^\beta) \backsimeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \backsimeq e^{-t^\beta/\Gamma(\beta+1)}, \quad \text{for} \quad 0 < t \leqslant 1,$$

and as a polynomial decay for $t \to \infty$,

$$E_\beta(-t^\beta) \backsimeq \frac{\sin(\beta\pi)}{\pi}\frac{\Gamma(\beta)}{t^\beta}, \quad \text{for} \quad t \geqslant 1,$$

the polynomial decay behaviour of $E_\beta(-t^\beta)$ illustrates the need for the sharp condition of $\beta \in (0,\,\frac{1}{2}]$ in Theorem 3.1. So the representation $G_B^{(\beta)}$ in equation (2.8) which is defined interms of $E_\beta(.)$ is crucial to our results.

Theorem 3.3. *In Theorem 3.1, suppose that $\beta \in (\frac{1}{2},\,1)$. Suppose also that $d < \alpha/2\beta$ or $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$, then there exist a strictly positive real number λ_l such that for $\lambda < \lambda_l$ the quantity $\sup_{t>0}\sup_{x\in B}\mathbb{E}|u_t(x)|^2$ is finite.*

Remark 3.4. *The assumption of a uniform bound $C(B)$ on the eigenfunctions $\{\varphi_n\}_{n\geqslant 1}$ look artificial, however there is no known bound for these eigenfunctions except the cases given in Lemma 2.1 and the uniform bounds as in Example 2.10 for the Brownian motion in higher dimensional rectangular boxes.*

From the three results above in line with the interpretation given in Foondun [29], we note that whenever $\beta \in (0, \frac{1}{2}]$, the killed time changed α-stable process $X^B_{E_t}$ defined in Section 2 reaches the boundary of B slower which allows the non-linear term to grow exponentially for any λ when t becomes large. On the other hand, for $\beta \in (\frac{1}{2}, 1)$, this process does not have time to generate such growth unless λ is large enough, since it proceeds quickly enough to the boundary.

Fractional parameters play an important role in many problems involving space-time fractional equations. However, in modeling problems, these fractional parameters are unknown a priori. As a result, continuity of the solutions with respect to these parameters is essential for modeling purposes. The following continuity theorem of the unique solution $u^{\beta}_t(x)$ of equation (3.5) with respect to the parameter β is Theorem 4.3(b) of Dang $et\,al$. [19].

Theorem 3.5 (Dang, Nane and Nguyen [19])**.** *Let $u^{(\gamma)}_t$ and $u^{(\beta)}_t$ denote the solution to the following equation for parameters $\gamma, \beta \in (0,\, 1)$ with $\gamma \to \beta$. The initial condition $u_0 = f(x)$ is the same for both equations.*

$$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta} u_t(x) = \Delta u_t(x), \quad x \in B, \quad t > 0, \\ \quad u_t(x) = 0,\ x \in \partial B, \quad t > 0, \\ u(0,\, x) = f(x), \quad x \in B. \end{cases} \tag{3.4}$$

Then, we have

$$\lim_{\gamma \to \beta} ||u^{(\gamma)}_t(x) - u^{(\beta)}_t(x)||^2_H = 0,$$

where H is the Hilbert space of all functions with the bounded norm induced by

$$||f||_H = \sqrt{\sum_{k=1}^{\infty} \mu_k^2 |\langle f,\, \varphi_k\rangle|^2},$$

where φ_k is a sequence of eigenfunctions of the Laplacian operator Δ with the corresponding eigenvalues μ_k for each $k = 1, 2, \cdots$, which are specified in the preliminaries.

Remark 3.6. *Meerschaert et al. [45] established the existence of a unique solution for equation* (3.4).

The following continuity theorem of the unique solution $u_t^\beta(x)$ of equation (1.7) with respect to the parameter β is Theorem 1.3 of Foondun [29].

Theorem 3.7 (Foondun [29])**.** *Let $u_t^{(\beta)}$ and u_t denote the solution to equation* (1.7) *and the solution to equation* (1.7) *for $\beta = 1$ respectively. The initial condition u_0 is the same for both equations. Then, for any $p \geqslant 2$, we have*

$$\lim_{\beta \to 1} \sup_{x \in [0,\, L]} \mathbb{E}|u_t(x) - u_t^{(\beta)}(x)|^p = 0.$$

In the next theorem, we show the continuity of the solution $u_t(x)$ of (3.3) with respect to the fractional parameter β.

Theorem 3.8. *Assume that $\{\varphi_n\}_{n \geqslant 1}$ are uniformly bounded by a constant $C(B)$. For $d < \frac{1}{2}\min\{\frac{1}{\beta},\, \frac{1}{\gamma}\}\alpha$, let $u_t^{(\gamma)}$ and $u_t^{(\beta)}$ denote solutions to equation* (3.3) *for parameters $\beta, \gamma \in \left(\frac{1}{2},\, 1\right)$ respectively with $\gamma \to \beta$. The initial condition u_0 is the same for both equations. Then, for any $p > 0$, we have*

$$\lim_{\gamma \to \beta} \sup_{x \in B} \mathbb{E}|u_t^{(\gamma)}(\mathrm{x}) - u_t^{(\beta)}(\mathrm{x})|^p = 0.$$

Here we observe that Theorem 3.8 extends Theorem 1.3 in Foondun [29] to fractional Laplacian case, and Theorem 4.3(b) of Dang *et al.* [19] to stochastic case.

Chapter 4

The long-time behavior of space-time fractional SPDE driven by space colored noise

4.1 Introduction

The other class of equations that we consider in this thesis is equation with space colored noise stated as

$$\begin{cases} \partial_t^\beta u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + I_t^{1-\beta}[\lambda\sigma(u_t(x))\dot{F}(t,x)] \text{ for } x \in B \text{ and } t > 0 \\ u_t(x) = 0 \quad \text{for} \quad x \notin B \text{ and } t > 0, \end{cases} \tag{4.1}$$

and the initial condition $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B. The operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leqslant 2$, is the L^2-generator of a symmetric α-stable process X_t^B killed when exiting B. The function $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma|x| \leqslant |\sigma(x)| \leqslant L_\sigma|x|$ where l_σ and L_σ are positive constants. The positive parameter λ is called *the level of the noise*.

The noise $\dot{F}(t,x)$ is white in time and colored in space satisfying

$$\mathrm{Cov}(\dot{F}(t,x), \dot{F}(s,y)) = \delta_0(t-s)f(x,y),$$

where δ_0 denotes the $Dirac\ delta$ distribution, $0 < f(x,y) \leqslant g(x-y)$ and g is a locally integrable function on $\mathbb{R}^d$ with possible singularity at 0 satisfying

$$\int_{\mathbb{R}^d} \frac{\hat{g}(\xi)}{1+|\xi|^\alpha} d\xi < \infty, \tag{4.2}$$

where $\hat{g}$ denotes the Fourier transform of g.

This can be interpreted more formally as

$$Cov\Big(\int \phi \mathrm{d}F, \int \psi \mathrm{d}F\Big) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \phi_s(x)\psi_s(y) f(x-y), \tag{4.3}$$

where we use the notation $\int \phi \mathrm{d}F$ to denote the wiener integral of ϕ with respect to F, and the right-most integral converges absolutely.

We will need the following non degeneracy condition on the spatial correlation of the noise.

Assumption 4.1. *Assume there exists some positive number K_f such that*

$$\inf_{x,\, y \in B} f(x,\, y) \geqslant K_f. \tag{4.4}$$

This assumption is very mild as it is shown by the following examples.

Example 4.2. *For the following list of examples Assumption 4.2 is satisfied.*

- *Riesz Kernel:*

$$f(x, y) = \frac{1}{|x-y|^\gamma} \quad \textit{with} \quad \gamma < d \wedge \alpha.$$

- *The Exponential-type kernel:* $f(x, y) = exp[-(x \cdot y)]$.
- *The Ornstein-Uhlenbeck-type kernels:* $f(x,\, y) = exp[-|x-y|^\delta] \quad \textit{with} \quad \delta \in (0,\, 2]$.
- *Poisson Kernels:*

$$f(x, y) = \left(\frac{1}{|x-y|^2 + 1}\right)^{\frac{d+1}{2}}.$$

- *Cauchy Kernels:*

$$f(x, y) = \prod_{j=1}^d \left(\frac{1}{1 + (x_j - y_j)^2}\right).$$

Following Walsh [60], u_t is a mild solution to (4.1) if

$$u_t(x) = (\mathcal{G}_B^{(\beta)} u_0)_t(x) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)\sigma(u_s(y)) F(ds\,, dy), \tag{4.5}$$

where $G_B^{(\beta)}(t, x, y)$ denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (4.1), and

$$(\mathcal{G}_B^{(\beta)} u_0)_t(x) := \int_B G_B^{(\beta)}(t,\, x,\, y) u_0(y)\, dy.$$

We note that when the correlation function is given by the Riesz Kernel, the proof for the existence of a unique random field solution u_t of (4.1) whose second moment satisfies

$$\sup_{x \in B} \mathbb{E}|u_t(x)|^2 \leqslant c_3 e^{c_4 t \lambda^{\frac{2\alpha}{\alpha - \beta d}}}$$

for all $t > 0$ can be found in Theorem 1.3 of Foondun *et al.* [27].

Next we give the statement of the main results of this chapter.

4.2 Main results

Theorem 4.3. *Suppose that the Dalang condition (4.2) holds. Let u_t denote the unique solution to (3.3). Then no matter what λ is, the second moment of u_t cannot decay exponentially fast. In fact, if we further assume that $\beta \in (0,\, \frac{1}{2}]$ and Assumption 4.1 holds, then as t gets large, $\sup_{x \in B} \mathbb{E}|u_t(x)|^2$ grows exponentially fast for any λ.*

Theorem 4.4. *In Theorem 4.3, if $\beta \in (\frac{1}{2},\, 1)$ and Assumption 4.1 hold, then there exist a strictly positive real number λ_u such that for all $\lambda > \lambda_u$, $\sup_{x \in B} \mathbb{E}|u_t(x)|^2$ grows exponentially fast as time gets large.*

Theorem 4.5. *In Theorem 4.3, suppose that $\beta \in (\frac{1}{2},\, 1)$, $d < \frac{\alpha}{2\beta}$, and $\int_{B \times B} f(x,\, y)\, dx\, dy < \infty$. If $\{\varphi_n\}_{n \geqslant 1}$ are uniformly bounded by a constant $C(B)$, then there exist a strictly positive real number λ_l such that for $\lambda < \lambda_l$ the quantity $\sup_{t>0} \sup_{x \in B} \mathbb{E}|u_t(x)|^2$ is finite.*

Corollary 4.6. *In Theorem 4.5, if $d = 1$ and $\alpha = 2$ then the conclusion of theorem follows.*

Corollary 4.7. *In Theorem 4.5, if $d = 1$ and f is Riesz Kernel function, then the conclusion of theorem follows.*

Theorem 4.8. *Assume that* $\{\varphi_n\}_{n\geqslant 1}$ *are uniformly bounded by a constant* $C(B)$*, and* $\int_{B\times B} f(x,\, y)\, dx\, dy < \infty$*. For* $d < \frac{1}{2}\min\{\frac{1}{\beta},\, \frac{1}{\gamma}\}\alpha$*, let* $u_t^{(\beta)}$ *and* $u_t^{(\gamma)}$ *denote solutions to* (3.3) *for parameters* $\beta,\, \gamma \in \left(\frac{1}{2},\, 1\right)$ *with* $\gamma \to \beta$*. The initial condition* u_0 *is the same for both equations. Then, for any* $p > 0$*, we have*

$$\lim_{\gamma\to\beta} \sup_{x\in B} \mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p = 0.$$

Chapter 5

Temporal Hölder continuity of the mild random field solution of space-time fractional stochastic heat equation driven by colored noise

In this chapter, we consider space-time fractional stochastic heat equation driven by noise colored in space defined by

$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma \left[\sigma(u(t,x))\, \dot{W}(t,x)\right] & x \in \mathbb{R}^d, t > 0 \\ u(0,\cdot) = u_0(x) & x \in \mathbb{R}^d, \end{cases} \tag{5.1}$$

where the fractional differential operators ∂_t^β, $(-\Delta)^{\alpha/2}$ and I_t^γ respectively denote the *Caputo derivative*, the *fractional Laplacian* and the *Riemann-Liouville fractional integral operator*, the noise, $\dot{W}$, is a centered Gaussian noise that is taken to be white in time and colored in space, and study the temporal Hölder continuity of mild random field solutions, which can be obtained by constructing relevant moment bounds for increments of the stochastic convolution $(Y \circledast \sigma(u))_t(y)$. Our techniques are based on connecting the space-time fractional Green functions Y, Z to the fundamental solution of $\partial_t P_t + \frac{\nu}{2}(-\Delta)^{\alpha/2} P_t = 0, P_0 = \delta_0$ via the subordination of Wright-type function $W_{-\beta,\gamma}(-\rho)$. In other words, W is a 0-mean Gaussian processes on the space of Schwartz functions, $\mathscr{S}(\mathbb{R}^{d+1})$, with the following covariance functional:

$$\mathbb{E}\left(W(\phi)W(\psi)\right) = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}y\, \phi(x)\psi(y)\delta_0(x-y), \quad \phi,\psi \in \mathscr{S}(\mathbb{R}^{d+1}), \tag{5.2}$$

where δ_0 denotes the *Dirac delta distribution*.

5.1 Preliminary results

5.1.1 Asymptotic Behaviors of Green functions for stochastic space-fractional heat equations

Let us consider the following space-fractional SPDE

$$\begin{cases} \left(\partial_t + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) \ u(t,x) = I_t^{\gamma}\left[\sigma(u(t,x))\,\dot{W}(t,x)\right] & x\in\mathbb{R}^d, t>0 \\ u(0,\cdot) = u_0(x) & x\in\mathbb{R}^d. \end{cases} \tag{5.3}$$

Let us denote the fundamental solution of

$$\begin{cases} \partial_t P_t + \frac{\nu}{2}(-\Delta)^{\alpha/2}P_t = 0 & x\in\mathbb{R}^d, t>0 \\ P_0 = \delta_0, \end{cases} \tag{5.4}$$

by $P_t = P_t(x)$, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, which can be obtained by the inverse Fourier transform as follows

$$P_t(x) := \mathcal{F}^{-1}(e^{-|.|^{\alpha}t})(x), \quad t>0,\ x\in\mathbb{R}^d. \tag{5.5}$$

We use the Walsh theory [60] to interpret the stochastic equation (5.3) by the random integral equation as follows

$$u_t(y) = (P_t * \varphi)(y) + \int_0^t \int_{\mathbb{R}^d} P_{t-\tau}(x-y)\sigma(u_\tau(x))W(dx,d\tau). \tag{5.6}$$

The function P_t is called the space-fractional Green function, which satisfies the following asymptotic properties (Binh-Tuan-Ngoc [7], Bogdan-Jakubowski [9], Chen-Kim-Song [14], Foondun-Joseph-Li [31], and Jakubowski-Serafin [36]):

Lemma 5.1. *(Asymptotic behavior of P). For all $t>0$ and $x\in\mathbb{R}^d$,*

a) $\dfrac{c_1 t}{(t^{1/\alpha}+|x|)^{d+\alpha}} \leqslant p_t(x) \leqslant \dfrac{c_2 t}{(t^{1/\alpha}+|x|)^{d+\alpha}}$;

b) $|\nabla P_t(x)| \leqslant \dfrac{t|x|}{(t^{1/\alpha}+|x|)^{3+\alpha}}$;

c) $|(-\Delta)^{\alpha/2}P_t(x)| \leqslant \dfrac{1}{(t^{2/\alpha}+|x|^2)^{\frac{d+\alpha}{2}}}$.

where c_1 and c_2 are positive constants depending on α.

5.1.2 Space-time fractional stochastic heat equations

Here we consider the space-time fractional heat equation $\partial_t^\beta u + (-\Delta)^{\alpha/2} u = v$ with the initial condition $u_t(x)|_{t=0} = \varphi(x)$ to obtain the mild setting of the equation given in (5.1), which is the integral equation

$$u_t(y) = (Z_t * \varphi)(y) + \int_0^t \int_{\mathbb{R}^d} Y_{t-r}(x-y)v(x)dxdr. \tag{5.7}$$

by driving the equation

$$\partial_t^\beta \hat{u}_t(\xi) + |\xi|^\alpha \hat{u}_t(\xi) = \hat{v}(\xi), \qquad \hat{u}_0(\xi) = \hat{\varphi}(\xi),$$

using the Fourier transform.

In the equation (5.7), the fundamental solutions Z_t and Y_t are called the space-time fractional Green functions and can be formally expressed in terms of Mittag-Leffler functions by

$$Z_t(x) = \mathcal{F}^{-1}\Big(E_{\beta,\lceil\beta\rceil}\left(-2^{-1}\gamma t^\beta|\xi|^\alpha\right)\Big) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} E_{\beta,\lceil\beta\rceil}\left(-2^{-1}\gamma t^\beta|\xi|^\alpha\right) \mathrm{d}\xi, \tag{5.8}$$

and

$$Y_t(x) = \mathcal{F}^{-1}\Big(t^{\beta+\gamma-1}E_{\beta,\beta+\gamma}\left(-2^{-1}\gamma t^\beta|\xi|^\alpha\right)\Big) = \frac{t^{\beta+\gamma-1}}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} E_{\beta,\beta+\gamma}\left(-2^{-1}\gamma t^\beta|\xi|^\alpha\right) \mathrm{d}\xi, \tag{5.9}$$

where $E_{s,r}(z) := \sum_{j=0}^\infty \frac{z^j}{\Gamma(js+r)}$ (for $s>0$, $r\in\mathbb{R}$, and $z\in\mathbb{C}$) is the Mittag-Leffler function. For more details about Z and Y refer Binh-Tuan-Ngoc [7] and Kemppainen-Siljander-Zacher [37].

5.1.3 Absolute moment of the Wright-type function

The classical *Wright function*, that we denote by $W_{\lambda,\mu}(z)$, is defined by the series representation convergent in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^\infty \frac{z^n}{n!\Gamma(\lambda n+\mu)}, \qquad \lambda > -1, \qquad \mu\in\mathbb{C}, \tag{5.10}$$

The *integral representation* reads as:

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha_-} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu}, \qquad \lambda > -1, \qquad \mu \in \mathbb{C}, \tag{5.11}$$

where Ha_- denotes the Hankel path: this one is a loop which starts from $-\infty$ along the lower side of negative real axis, encircles with a small circle the axes origin and ends at $-\infty$ along the upper side of the negative real axis. The equivalence of the series and integral representations is easily proved using Hankel formula for the Gamma function and performing a term-by-term integration.

We now present the probability density function (pdf) known as the Wright-type function [40] $W_{-\nu,,1-\nu(-z)}$, which is defined in equation (5.12) as

$$W_{-\nu,1-\nu}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\nu n + (1-\nu)]} \tag{5.12}$$

with support in $\mathbb{R}^+$. Its *absolute moments* of order $\delta > -1$ in $\mathbb{R}^+$ are finite and turn out to be:

$$\int_0^\infty x^\delta\, W_{-\nu,1-\nu}(x)\,\mathrm{d}x = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}, \qquad \delta > -1, \qquad 0 \leqslant \nu < 1. \tag{5.13}$$

Lemma 5.2. *Let $W_{-\beta,\gamma}(-\rho)$ be the Wright-type function in $\mathbb{R}_+$, $0 < \beta < 1$ and $\theta > 1$. Then, the (finite) absolute moments of order θ are given by:*

$$\int_0^\infty \rho^\theta W_{-\beta,\gamma}(-\rho)\,\mathrm{d}\rho = \frac{\Gamma(\theta+1)}{\Gamma(\gamma+\beta+\beta\theta)}, \qquad \theta > -1, \qquad 0 < \beta < 1. \tag{5.14}$$

Proof. The proof is based on the integral representation of the Wright function $W_{-\beta,\gamma}(-\rho)$:

$$\begin{aligned}
\int_0^\infty \rho^\theta W_{-\beta,\gamma}(-\rho)\,\mathrm{d}\rho &= \int_0^\infty \rho^\theta \left[\frac{1}{2\pi i}\int_{Ha_-} e^{\sigma-\rho\sigma^\beta}\frac{d\sigma}{\sigma^\gamma}\right] d\rho \\
&= \frac{1}{2\pi i}\int_{Ha_-} e^\sigma \left[\int_0^\infty e^{-\rho\sigma^\beta}\rho^\theta d\rho\right]\frac{d\sigma}{\sigma^\gamma} \\
&= \frac{\Gamma(\theta+1)}{2\pi i}\int_{Ha_-} \frac{e^\sigma}{\sigma^{\gamma+\beta+\beta\theta}} d\sigma \\
&= \frac{\Gamma(\theta+1)}{\Gamma(\gamma+\beta+\beta\theta)}, \qquad \theta > -1, \qquad 0 < \beta < 1.
\end{aligned} \tag{5.15}$$

$\square$

Since, by integral definition of a Gamma function Γ, we have

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} \mathrm{d}s, \qquad s > 0. \tag{5.16}$$

The exchange between two integrals and the following identity contributed to the final result for Lemma 5.2:

$$\int_0^\infty e^{-\rho\sigma^\beta} \rho^\theta d\rho = \frac{\Gamma(\theta+1)}{(\sigma^\beta)^{\theta+1}}. \tag{5.17}$$

Proposition 5.3 (Theorem 1.5 [50])**.** *Suppose* $x \geqslant 0$, $y \geqslant 0$, *and* $0 < p < 1$. *Then*

$$|x^p - y^p| \leqslant |x-y|^p. \tag{5.18}$$

Proof. Assume w.lo.g $x > y$. Then the statement becomes $x^p - y^p \leqslant (x-y)^p$. Since $y > 0$, divide through out by y^p. So we need to show $(\frac{x}{y})^p - 1 \leqslant (\frac{x}{y} - 1)^p$ whenever $x > y > 0$ and $0 < p < 1$. Let $t = \frac{x}{y}$. Then $t > 0$. So we need to show $t^p - 1 \leqslant (t-1)^p$ whenever $t > 1$ and $0 < p < 1$.

Let $f(t) = (t-1)^p - (t^p - 1)$ where $0 < p < 1$.

We claim that the function $f(t)$ is increasing for $t \geqslant 1$ and when $0 < p < 1$.

So $f(t) \geqslant f(1)$ and $f(1) = 0$. So, we get the desired result.

To show $f(t)$ is increasing for $t \geqslant 1$ and when $0 < p < 1$, we need to show $0 < f'(t) = p(t-1)^{p-1} - pt^{p-1}$. Since $p > 0$, all we need to show is that $(t-1)^{p-1} > t^{p-1}, \forall t > 1$ and $0 < p < 1$.

Since $0 < p < 1$, we need to show $t^{1-p} > (t-1)^{1-p}$ which is true since $p < 1$ and $t > t - 1 > 0$. $\square$

5.1.4 More on the space-time fractional Green functions

By connecting the space-fractional Green function P with the space-time fractional Green functions Y, Z, we shall look for solutions to the problem. For $\beta \in (0, 1)$ and $x \in \mathbb{R}^d$, we can use the Fourier transforms (5.8), (5.9) and the definitions of Y_t and Z_t to form the following relationship between P and Y_t, Z_t:

$$\begin{aligned}
Y_t(x) &= \frac{t^{\beta+\gamma-1}}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} s \left(\int_0^\infty W_{-\beta,\gamma}(-\rho) e^{-2^{-1}\gamma t^\beta |\xi|^\alpha \rho} \mathrm{d}\rho \right) \mathrm{d}\xi \\
&= \frac{t^{\beta+\gamma-1}}{\sqrt{2\pi}} \int_0^\infty W_{-\beta,\gamma}(-\rho) \left(\int_{\mathbb{R}^N} e^{i\xi\cdot x} e^{-\rho 2^{-1}\gamma t^\beta |\xi|^\alpha} \mathrm{d}\xi \right) \mathrm{d}\rho \\
&= t^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(x) \mathrm{d}\rho,
\end{aligned}$$

where P is the density of the stable process and where $s = \beta + \gamma$ and

$$\begin{aligned}
Z_t(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} s \left(\int_0^\infty W_{-\beta,1-\beta}(-\rho) e^{-2^{-1}\gamma t^\beta |\xi|^\alpha \rho} \mathrm{d}\rho \right) \mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty W_{-\beta,1-\beta}(-\rho) \left(\int_{\mathbb{R}^N} e^{i\xi\cdot x} e^{-\rho 2^{-1}\gamma t^\beta |\xi|^\alpha} \mathrm{d}\xi \right) \mathrm{d}\rho \\
&= \int_0^\infty W_{-\beta,1-\beta}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(x) \mathrm{d}\rho.
\end{aligned}$$

Next we give some basic properties of the space-fractional Green function P, including its spatial and temporal continuities, which are essential to establish spatially and temporally continuities of the space-time fractional Green functions Y, and Z. These can be found in many studies, such as, Asogwa *et. al* [2], Bezdek [[6], Lemmas 2,3], and Li [[38], Lemmas 1,2]. The first part of this lemma can be proved by using Part b of Lemma 2.1. The second part can be proved due to Part c of Lemma 2.1.

Lemma 5.4 (Lemma 2 of Li [38]). *For all $t, t' > 0$, and $x \in \mathbb{R}^d$, there holds that*

a) $\int_{\mathbb{R}^d} |P_t(x-y) - P_t(x)| \,\mathrm{d}\, y \lesssim \left(\frac{|y|}{t^{1/\alpha}} \wedge 1 \right)$;

b) $\int_{\mathbb{R}^d} |P_{t'}(y) - P_t(y)| \,\mathrm{d}\, y \lesssim \left(|\log(t') - \log(t)| \wedge 1 \right)$;

Proof. (a) For all $r > 0$, define

$$\mu_d(r) = \mu_d(r,t) := \sup_{z\in\mathbb{R}^d, |z|\leqslant r} \int_{\mathbb{R}^d} |p_t(y-z) - p_t(y)| dy. \tag{5.19}$$

We first consider the case $d = 1$. Then

$$\mu_1(|x|) = \sup_{z\in(0,|x|)} \int_{-\infty}^\infty \left| \int_{y-z}^y \frac{\partial p_t(\xi)}{\partial \xi} d\xi \right| dy. \tag{5.20}$$

By (2.3) of [36] (or Lemma 5 in [9]), we have

$$\Big|\frac{\partial p_t(\xi)}{\partial \xi}\Big| \leqslant C\frac{t|\xi|}{(t^{1/\alpha}+|\xi|)^{3+\alpha}}, \tag{5.21}$$

where C only depends on α.

Taking (5.21) into (5.20) to get

$$\begin{aligned}\mu_1(|x|) &\leqslant C|x|\int_{-\infty}^{\infty}\frac{t|\xi|}{(t^{1/\alpha}+|\xi|)^{3+\alpha}}d\xi\\ &= \frac{C|x|}{t^{1/\alpha}}\int_{-\infty}^{\infty}\frac{|\nu|}{(1+|\nu|)^{3+\alpha}}d\nu\\ &\leqslant \frac{C|x|}{t^{1/\alpha}}, \qquad x\in\mathbb{R}.\end{aligned} \tag{5.22}$$

For $d>1$, integrating one coordinate at a time and then applying the triangle inequality, we have $\mu_d(|x|)\leqslant \sum_{j=1}^{d}\mu_1(|x|)\leqslant \frac{dC|x|}{t^{1/\alpha}}$, where $x=(x_1,\cdots,x_d)\in\mathbb{R}^d$. On the other hand, since $|p_t(y-x)-p_t(y)|\leqslant p_t(y-x)+p_t(y)$ and $\int_{\mathbb{R}^d}p_t(y)dy=1$, we have $\mu_d(|x|)\leqslant 2$.

(b) From (5.4) and Proposition 2.1 in [59], we show that

$$|\frac{\partial p_t(y)}{\partial t}| \leqslant \frac{Cp_t(y)}{t}. \tag{5.23}$$

Then letting $t^{'}=t+\varepsilon$, we have

$$\begin{aligned}&\int_{\mathbb{R}}|p_{t+\varepsilon}(y)-p_t(y)|dy=\int_{\mathbb{R}}\Big|\int_t^{t+\varepsilon}\frac{\partial p_s(y)}{\partial s}ds\Big|dy\leqslant C\int_{\mathbb{R}}\int_t^{t+\varepsilon}\frac{p_s(y)}{s}dsdy\\ &=C\int_t^{t+\epsilon}\frac{1}{s}ds=C(\log(t+\varepsilon)-\log(t)).\end{aligned} \tag{5.24}$$

On the other hand, we have $\int_{\mathbb{R}}|p_{t+\varepsilon}(y)-p_t(y)|dy\leqslant 2$.

□

Lemma 5.5. *For all $t,t^{'}$ such that $0<t<t^{'}$, there holds*

$$\int_{\mathbb{R}^d}|Z_{t,t^{'}}(y)|\,\mathrm{d}\,y \leqslant \beta(\log(t^{'})-\log(t))\wedge 1, \tag{5.25}$$

where $\beta \in (0,\, 1)$, $\gamma \in (0,\, 1)$, $\alpha \in (0,\, 2)$ *, and*

$$Z_{t,t'}(y) = Z_{t'}(y) - Z_t(y), \tag{5.26}$$

and

$$Z_t(y) = \int_0^\infty W_{-\beta,1-\beta}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(y)\,\mathrm{d}\,\rho. \tag{5.27}$$

Proof. For all $t, t^{'}$ such that $0 < t < t^{'}$, there holds

$$\begin{aligned}
\int_{\mathbb{R}^d} |Z_{t,t'}(y)|\,\mathrm{d}\,y &= \int_{\mathbb{R}^d} |Z_{t'}(y) - Z_{t'}(y)|\,\mathrm{d}\,y \\
&= \int_{\mathbb{R}^d} \left| \int_0^\infty W_{-\beta,1-\beta}(-\rho) P_{(\rho 2^{-1}\gamma (t')^\beta)}(y)\,\mathrm{d}\,\rho - \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(y)\,\mathrm{d}\,\rho \right| \mathrm{d}\,y \\
&= \int_0^\infty W_{-\beta,1-\beta}(-\rho) \int_{\mathbb{R}^d} |P_{(\rho 2^{-1}\gamma (t')^\beta)}(y) - P_{(\rho 2^{-1}\gamma t^\beta)}(y)|\,\mathrm{d}\,y\,\mathrm{d}\,\rho \\
&\leqslant \int_0^\infty W_{-\beta,1-\beta}(-\rho) |(\log(\rho 2^{-1}\gamma (t^{'})^\beta) - \log(\rho 2^{-1}\gamma t^\beta)| \wedge 1)\,\mathrm{d}\,\rho \\
&\leqslant \int_0^\infty W_{-\beta,1-\beta}(-\rho) |\beta((\log(t^{'})) - \log(t))| \wedge 1)\,\mathrm{d}\,\rho \\
&= (\beta(\log(t^{'}) - \log(t)) \wedge 1) \int_0^\infty W_{-\beta,1-\beta}(-\rho)\,\mathrm{d}\,\rho, \\
&\lesssim \beta(\log(t^{'}) - \log(t)) \wedge 1,
\end{aligned} \tag{5.28}$$

where $\int_0^\infty W_{-\beta,1-\beta}(-\rho)\,\mathrm{d}\,\rho = 1$, and by Lemma 5.4 (b). □

Lemma 5.6. *For all* $t, t^{'}$ *such that* $0 < t < t^{'}$*, using Proposition 5.3 there holds*

$$\int_{\mathbb{R}^d} |Y_{t,t'}(y)|\,\mathrm{d}\,y \leqslant t^{s-1}(\beta(\log(t^{'}) - \log(t)) \wedge 1) + (t^{'})^{s-1} t^{s-1} (t^{'} - t)^{1-s}, \tag{5.29}$$

where $\beta \in (0,\, 1)$, $\gamma \in (0,\, 1)$, *and* $\alpha \in (0,\, 2)$ *such that* $s = \beta + \gamma < 1$,

$$Y_{t,t'}(y) = Y_{t'}(y) - Y_t(y), \tag{5.30}$$

and

$$Y_t(y) = t^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(y)\,\mathrm{d}\,\rho. \tag{5.31}$$

Proof. For all t, t' such that $0 < t < t'$, there holds

$$\begin{aligned}
\int_{\mathbb{R}^d} |Y_{t,t'}(y)| \, \mathrm{d}\, y &= \int_{\mathbb{R}^d} |Y_{t'}(y) - Y_{t'}(y)| \, \mathrm{d}\, y \\
&= \int_{\mathbb{R}^d} |(t')^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{(\rho 2^{-1}\gamma (t')^\beta)}(y) \, \mathrm{d}\, \rho \\
&\qquad - t^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{(\rho 2^{-1}\gamma t^\beta)}(y) \, \mathrm{d}\, \rho| \, \mathrm{d}\, y \\
&\leqslant (t')^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) \int_{\mathbb{R}^d} |P_{(\rho 2^{-1}\gamma (t')^\beta)}(y) - P_{(\rho 2^{-1}\gamma t^\beta)}(y)| \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\quad + (t')^{s-1} t^{s-1} (t' - t)^{1-s} \int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho) \int_0^\infty P_{(\rho 2^{-1}\gamma t^\beta)}(y) \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\leqslant (t')^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) \int_{\mathbb{R}^d} |(\log(\rho 2^{-1}\gamma (t')^\beta) - \log(\rho 2^{-1}\gamma t^\beta)| \wedge 1) \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\quad + (t')^{s-1} t^{s-1} (t' - t)^{1-s} \int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho) \int_0^\infty P_{(\rho 2^{-1}\gamma t^\beta)}(y) \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\leqslant (t')^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) \int_{\mathbb{R}^d} |\beta((\log(t')) - \log(t))| \wedge 1) \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\quad + (t')^{s-1} t^{s-1} (t' - t)^{1-s} \int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho) \int_0^\infty P_{(\rho 2^{-1}\gamma t^\beta)}(y) \, \mathrm{d}\, y \, \mathrm{d}\, \rho \\
&\lesssim t^{s-1} (\beta(\log(t') - \log(t)) \wedge 1) + (t')^{s-1} t^{s-1} (t' - t)^{1-s},
\end{aligned} \tag{5.32}$$

where $s = \beta + \gamma$, $\int_0^\infty W_{-\beta,\gamma}(-\rho) \, \mathrm{d}\, \rho = \frac{1}{\Gamma(s)}$, and by Lemma 5.4(b). □

Lemma 5.7. *If $t, t' \in (0, T]$ are positive such that $t \vee t' \leqslant 2(t \wedge t')$, then*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{N}_{t,t'}(x, z, y, y) \, f_\lambda(x - z) \, \mathrm{d}\, z \, \mathrm{d}\, x \right| \leqslant |t' - t|^{1-s} (t' \wedge t)^{s - \frac{\beta\lambda}{\alpha} - 1} (t^{s-1} + t^{s-1}(t')^{s-1}), \tag{5.33}$$

where $\beta \in (0, 1)$, $\gamma \in (0, 1)$, and $\alpha \in (0, 2)$ such that $s = \beta + \gamma < 1$,

$$\mathcal{N}_{t,t'}(x, z, y, y) = Y_{t,t'}(x - y) Y_{t,t'}(z - y), \tag{5.34}$$

$$Y_{t,t'}(x - y) = Y_{t'}(x - y) - Y_t(x - y), \tag{5.35}$$

and

$$Y_t(x - y) = t^{s-1} \int_0^\infty W_{-\beta,\gamma}(-\rho) P_{\rho 2^{-1}\gamma t^\beta}(x - y) \, \mathrm{d}\, \rho. \tag{5.36}$$

Proof. Let us denote

$$\mathsf{S}_{\rho 2^{-1}\gamma t^{\beta},\,\rho 2^{-1}\gamma (t')^{\beta}}(x,y,z) := f_\lambda(x-z)(P_{\rho 2^{-1}\gamma (t')^{\beta}}(z-y) - P_{\rho 2^{-1}\gamma t^{\beta}}(z-y)), \tag{5.37}$$

and

$$\bar{\mathsf{S}}_{\rho 2^{-1}\gamma t^{\beta}}(x,y,z) := f_\lambda(x-z)P_{\rho 2^{-1}\gamma t^{\beta}}(z-y). \tag{5.38}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{N}_{t,t'}(x,z,y,y)\, f_\lambda(x-z)\,\mathrm{d}\,z\,\mathrm{d}\,x &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Y_{t,t'}(x-y)Y_{t,t'}(z-y)\, f_\lambda(x-z)\,\mathrm{d}\,z\,\mathrm{d}\,x\\ = (t')^{s-1}\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & W_{-\beta,\gamma}(-\rho)Y_{t,t'}(x-y)\mathsf{S}_{\rho 2^{-1}\gamma t^{\beta},\,\rho 2^{-1}\gamma (t')^{\beta}}(x,y,z)\,\mathrm{d}\,z\,\mathrm{d}\,x\,\mathrm{d}\,\rho\\ + ((t')^{s-1}-t^{s-1})\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & W_{-\beta,\gamma}(-\rho)Y_{t,t'}(x-y)\bar{\mathsf{S}}_{\rho 2^{-1}\gamma t^{\beta}}(x,y,z)\,\mathrm{d}\,z\,\mathrm{d}\,x\,\mathrm{d}\,\rho.\end{aligned} \tag{5.39}$$

We note that $|(t')^{s-1}-t^{s-1}| \leqslant (t')^{s-1}\vee t^{s-1} = (t'\wedge t)^{s-1}$, and $(t')^{s-1} \leqslant (t'\wedge t)^{s-1}$ whenever $t, t' \geqslant 0$. Upon using the triangle inequality and maximization over variables, once can get the inequality

$$\int_{\mathbb{R}^d} f_\lambda(x-z)|P_{\rho 2^{-1}\gamma t^{\beta}}(z-y') - P_{\rho 2^{-1}\gamma t^{\beta}}(z-y)|\,\mathrm{d}\,z \leqslant 2(f_\lambda * P_{\rho 2^{-1}\gamma t^{\beta}})(0). \tag{5.40}$$

See also the inequality (16) of Bezdek [6]. On the other hand, the asymptotic behavior of P (in Lemma 2.1 of Tuan et al.) ensures that $(f_\lambda * P_t)(0) \leqslant Ct^{-\frac{\lambda}{\alpha}}$ for all $t > 0$. So, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{N}_{t,t'}(x,z,y,y)\, f_\lambda(x-z)\,\mathrm{d}\,z\,\mathrm{d}\,x\right|\\ &\quad\leqslant (t'\wedge t)^{s-1}\int_0^\infty\int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho)|Y_{t,t'}(x-y)|((f_\lambda * P_{\rho 2^{-1}\gamma (t')^{\beta}})(0) + (f_\lambda * P_{\rho 2^{-1}\gamma t^{\beta}})(0))\,\mathrm{d}\,x\,\mathrm{d}\,\rho\\ &\qquad + (t'\wedge t)^{s-1}\int_0^\infty\int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho)|Y_{t,t'}(x-y)|(f_\lambda * P_{\rho 2^{-1}\gamma t^{\beta}})(0)\,\mathrm{d}\,x\,\mathrm{d}\,\rho.\\ &\quad\lesssim (t'\wedge t)^{s-1}\int_0^\infty\int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho)((\rho 2^{-1}\gamma t^{\beta})^{-\frac{\lambda}{\alpha}} + (\rho 2^{-1}\gamma (t')^{\beta})^{-\frac{\lambda}{\alpha}})|Y_{t,t'}(x-y)|\,\mathrm{d}\,x\,\mathrm{d}\,\rho\\ &\quad\lesssim (t'\wedge t)^{s-1}\int_0^\infty\int_{\mathbb{R}^d} \rho^{-\frac{\beta}{\alpha}} W_{-\beta,\gamma}(-\rho)(t^{-\frac{\beta\lambda}{\alpha}} + (t')^{-\frac{\beta\lambda}{\alpha}})|Y_{t,t'}(x-y)|\,\mathrm{d}\,x\,\mathrm{d}\,\rho\\ &\quad\lesssim (t'\wedge t)^{s-\frac{\beta\lambda}{\alpha}-1}\int_0^\infty\int_{\mathbb{R}^d} \rho^{-\frac{\beta}{\alpha}} W_{-\beta,\gamma}(-\rho)|Y_{t,t'}(x-y)|\,\mathrm{d}\,x\,\mathrm{d}\,\rho,\end{aligned} \tag{5.41}$$

since $(t^{'})^{-\frac{\beta\lambda}{\alpha}} + t^{-\frac{\beta\lambda}{\alpha}} \leqslant 2(t^{'}\wedge t)^{-\frac{\beta\lambda}{\alpha}}$ for $t,\ t^{'} > 0$. In addition, w.l.o.g. we can assume $0 < t < t^{'} < 2t$ (and so $(t^{'}-t)/t \in (0,\ 1)$). Further, we have $|\log(a^{'}) - \log(a)| \leqslant ((a^{'}-a)/a)^{\delta}$ for all $\delta \in (0,\ 1), a < a^{'} < 2a$, we imply that $|\log(t^{'}) - \log(t)| \leqslant (|t^{'}-t|/t)^{1-s}$. As a result, using Equation (5.2) and Lemma 5.6, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{N}_{t,t'}(x,z,y,y)\, f_\lambda(x-z)\,\mathrm{d}\,z\,\mathrm{d}\,x\right| \\
&\quad\lesssim (t^{'}\wedge t)^{s-\frac{\beta\lambda}{\alpha}-1}\int_0^\infty\int_{\mathbb{R}^d}\rho^{-\frac{\beta}{\alpha}}W_{-\beta,\gamma}(-\rho)|Y_{t,t'}(x-y)|\,\mathrm{d}\,x\,\mathrm{d}\,\rho \\
&\quad\lesssim (t^{'}\wedge t)^{s-\frac{\beta\lambda}{\alpha}-1}\frac{\Gamma(1-\beta/\alpha)}{\Gamma(\gamma+\beta-\beta\lambda/\alpha)}\int_{\mathbb{R}^d}|Y_{t,t'}(x-y)|\,\mathrm{d}\,x \\
&\quad\lesssim (t^{'}\wedge t)^{s-\frac{\beta\lambda}{\alpha}-1}\frac{\Gamma(1-\beta/\alpha)}{\Gamma(\gamma+\beta-\beta\lambda/\alpha)}(t^{s-1}(\beta(\log(t^{'})-\log(t))\wedge 1) + (t^{'})^{s-1}t^{s-1}(t^{'}-t)^{1-s}) \\
&\quad\lesssim |t^{'}-t|^{1-s}(t^{'}\wedge t)^{s-\frac{\beta\lambda}{\alpha}-1}(t^{s-1}+t^{s-1}(t^{'})^{s-1}).
\end{aligned} \tag{5.42}$$

□

Lemma 5.8. *For all $t > 0$, $y \in \mathbb{R}^d$, there holds that*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Y_t(x-y)Y_t(z-y)f_\lambda(x-z)\,\mathrm{d}\,z\,\mathrm{d}\,x \lesssim t^{2s-\frac{\beta\lambda}{\alpha}-2}, \tag{5.43}$$

where $s = \beta+\gamma$.

Proof. For short, we first denote

$$\mathcal{Q}_{\rho 2^{-1}\gamma t^\beta,\rho' 2^{-1}\gamma t^\beta}(x,y,z) := t^{2s-2}P_{\rho 2^{-1}\gamma t^\beta}(x-y)P_{\rho' 2^{-1}\gamma t^\beta}(z-y)f_\lambda(x-z), \tag{5.44}$$

for all positive numbers $\rho,\ t$, and $x,\ y,\ z\ \in \mathbb{R}^d$, where $s = \beta+\gamma$. By the relation (5.31) between the Green functions Y and P, we obtain the identities

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Y_t(x-y)Y_t(z-y)f_\lambda(x-z)d\,z\,\mathrm{d}\,x \\
&\quad= \int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho)W_{-\beta,\gamma}(-\rho^{'})\mathcal{Q}_{\rho 2^{-1}\gamma t^\beta,\rho' 2^{-1}\gamma t^\beta}(x,y,z)d\,z\,\mathrm{d}\,x\,\mathrm{d}\,\rho^{'}\,\mathrm{d}\,\rho \\
&\quad= t^{2s-2}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d} W_{-\beta,\gamma}(-\rho)W_{-\beta,\gamma}(-\rho^{'})P_{(\rho+\rho^{'})2^{-1}\gamma t^\beta}(x)\,f_\lambda(x)\,\mathrm{d}\,x\,\mathrm{d}\,\rho^{'}\,\mathrm{d}\,\rho,
\end{aligned} \tag{5.45}$$

where we have used the semigroup property of P_t below to get Equation (5.45)

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_{\rho 2^{-1}\gamma t^{\beta}}(x-y)P_{\rho' 2^{-1}\gamma t^{\beta}}(z-y)\, f_{\lambda}(x-z)d\,z\,\mathrm{d}\,x = \int_{\mathbb{R}^d} P_{(\rho+\rho')2^{-1}\gamma t^{\beta}}(x)\, f_{\lambda}(x)\,\mathrm{d}\,x. \tag{5.46}$$

By asymptotic behavior of P_t, we have

$$\int_{\mathbb{R}^d} P_{(\rho+\rho')2^{-1}\gamma t^{\beta}}(x)\, f_{\lambda}(x)\,\mathrm{d}\,x \lesssim \left((\rho+\rho^{'})2^{-1}\gamma t^{\beta}\right)^{-\frac{\lambda}{\alpha}}. \tag{5.47}$$

From Equation (5.45) using Equations(5.47) and Lemma 5.2 we deduce the following chain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Y_t(x-y)Y_t(z-y)f_{\lambda}(x-z)d\,z\,\mathrm{d}\,x \\
&\lesssim t^{2s-\frac{\beta\lambda}{\alpha}-2}\int_0^{\infty}\int_0^{\infty}(\rho+\rho^{'})^{-\frac{\lambda}{\alpha}}W_{-\beta,\gamma}(-\rho)W_{-\beta,\gamma}(-\rho^{'})\,\mathrm{d}\,\rho^{'}\,\mathrm{d}\,\rho \\
&\leqslant t^{2s-\frac{\beta\lambda}{\alpha}-2}\gamma^{-\frac{\lambda}{\alpha}}2^{\frac{\lambda}{\alpha}}\int_0^{\infty}\int_0^{\infty}2^{-\frac{\lambda}{\alpha}}(\rho\rho^{'})^{-\frac{\lambda}{2\alpha}}W_{-\beta,\gamma}(-\rho)W_{-\beta,\gamma}(-\rho^{'})\,\mathrm{d}\,\rho^{'}\,\mathrm{d}\,\rho \\
&= t^{2s-\frac{\beta\lambda}{\alpha}-2}\gamma^{-\frac{\lambda}{\alpha}}\left[\frac{\Gamma(1-\frac{\lambda}{2\alpha})}{\Gamma(s-\frac{\beta\lambda}{2\alpha})}\right]^2 \\
&\lesssim t^{2s-\frac{\beta\lambda}{\alpha}-2},
\end{aligned} \tag{5.48}$$

where $\rho+\rho^{'} \geqslant 2(\rho\rho^{'})^{\frac{1}{2}}$. We also used the absolute moment of Wright-type function $W_{-\beta,\gamma}$ in the last computations. □

5.2 Main result

In the next theorem, we will obtain the temporally Hölder continuity of the stochastic convolution $(Y \circledast \sigma(u))_t(y)$ by estimating the increment $E|(Y\circledast\sigma(u))_{t'}(y)-(Y\circledast\sigma(u))_t(y)|^q$. This can be proved by applying Lemma 5.8 and Lemma 5.7.

Theorem 5.9. *If $\lambda \in (0,\, d\wedge\alpha)$ and $\beta > \frac{2}{3-\frac{\lambda}{\alpha}}$, then*

$$\frac{(E|(Y\circledast\sigma(u))_{t'}(y)-(Y\circledast\sigma(u))_t(y)|^q)^{\frac{1}{q}}}{|t'-t|^{\frac{3s}{2}-\frac{\beta\lambda}{2\alpha}-1}} \lesssim \mathcal{N}_{\gamma,q,T}(u), \tag{5.49}$$

for all $t,\, t^{'} \in (0,\, T)$ *such that* $0 \neq |t^{'} - t|$ *is small enough, where* $s = \beta + \gamma$,

$$(Y \circledast \sigma(u))_t(y) = \int_0^t \int_{\mathbb{R}^d} Y_{t-s}(x-y)\sigma(u(s,x))W(ds,dx), \tag{5.50}$$

and

$$\mathscr{N}_{\gamma,\, q,\, T}(u) = \sup_{t\in[0,\, T]} \sup_{x\in\mathbb{R}^d} \left(e^{-q\gamma t} E[|u_t(x)|^q] \right)^{\frac{1}{q}}, \gamma > 1,\, q \geqslant 2. \tag{5.51}$$

Next, we shall obtain the temporally Hölder continuity of the mild random field solution u. It only requires to establish the temporally Hölder continuity of the convolution $(Z_t * u_0)(y)$ and the stochastic convolution $(Y \circledast \sigma(u))_t(y)$. The later is done in the previous theorem.

Theorem 5.10. *If* $\lambda \in (0,\, d \wedge \alpha)$ *and* $\beta > \frac{2}{3-\frac{\lambda}{\alpha}}$, *then*

$$\frac{(\mathbb{E}|u_{t'}(y) - u_t(y)|^q)^{\frac{1}{q}}}{|t' - t|^{(\frac{3s}{2} - \frac{\beta\lambda}{2\alpha} - 1)\wedge(1-\beta)^-}} \lesssim \sup_{x\in\mathbb{R}^d} |\varphi(x)| + \mathscr{N}_{\gamma,\, q,\, T}(u), \tag{5.52}$$

for all $t,\, t^{'} \in (0,\, T)$ *such that* $0 \neq |t^{'} - t|$ *is small enough and* $1 \leqslant T^\beta (t' - t)^{1-\beta}$, *where* $s = \beta + \gamma$, $\varphi(x) = u_0(x)$, *and*

$$\mathscr{N}_{\gamma,\, q,\, T}(u) = \sup_{t\in[0,\, T]} \sup_{x\in\mathbb{R}^d} \left(e^{-q\gamma t} \mathbb{E}[|u_t(x)|^q] \right)^{\frac{1}{q}}, \gamma > 1,\, q \geqslant 2. \tag{5.53}$$

Chapter 6

Proof of Main Results

6.1 The space-time white noise case

In this section we give the proof of first three main theorems of the dissertation. We denote

$$\Lambda(\theta) := \int_0^\infty e^{-\theta t} E_\beta(-\mu_1 t^\beta)^2 \, dt, \tag{6.1}$$

for $\theta > 0$. Here it is crucial to note that we can use the bounds in (2.7) to observe that $\Lambda(\theta)$ tends to infinity as θ approaches to zero if and only if $2\beta \leqslant 1$.

Lemma 6.1. *For $\beta \in (\frac{1}{2}, 1)$, the function $\Lambda(\theta)$ is bounded.*

Proof. Using the bounds in (2.7) and the fact that $e^{-\theta t} \leqslant 1$ for $\theta, t > 0$ we obtain that

$$\Lambda(\theta) := \int_0^\infty e^{-\theta t} E_\beta(-\mu_1 t^\beta)^2 \, \mathrm{d}t \leqslant \int_0^\infty \frac{1}{(1 + \Gamma(1+\beta)^{-1} \mu_1 t^\beta)^2} \, dt \leqslant 1 + \frac{1}{\mu_1^2} \int_1^\infty \frac{1}{t^{2\beta}} \, dt < \infty \tag{6.2}$$

if $2\beta > 1$ for $\theta > 0$. $\square$

Proof of Theorem 3.1. Let us first establish the second statement of the theorem. Since $\{\varphi_n\}_{n \geqslant 0}$ is an orthonormal basis of $L^2(B)$, from the mild formulation given in (3.3), using (2.7) and Stochastic Fubini theorem we have

$$\begin{aligned}
\langle u_t, \varphi_1\rangle &= \int_B u_t(x)\varphi_1(\mathrm{x})\,d\mathrm{x}\\
&= \int_B\int_B G_B^{(\beta)}(t,\,\mathrm{x},\,\mathrm{y})u_0(\mathrm{y})\,\varphi_1(\mathrm{x})\,d\mathrm{y}\,d\mathrm{x}\\
&\quad + \lambda\int_B\int_B\int_0^t E_\beta(-\mu_1(t-s)^\beta)\varphi_1^2(\mathrm{x})\varphi_1(\mathrm{y})\sigma(v_s(\mathrm{y}))W(ds\,,d\mathrm{y})\,d\mathrm{x}\\
&= E_\beta(-\mu_1 t^\beta)\int_B u_0(\mathrm{y})\,(\mathrm{x})\varphi_1(\mathrm{y})\,d\mathrm{y}\\
&\quad + \lambda\int_B\int_0^t E_\beta(-\mu_1(t-s)^\beta)\varphi_1(\mathrm{y})\sigma(v_s(\mathrm{y}))W(ds\,,d\mathrm{y})\\
&= E_\beta(-\mu_1 t^\beta)\langle u_0, \varphi_1\rangle\\
&\quad + \lambda\int_B\int_0^t E_\beta(-\mu_1(t-s)^\beta)\varphi_1(\mathrm{y})\sigma(v_s(\mathrm{y}))W(ds\,,d\mathrm{y}).
\end{aligned}\tag{6.3}$$

Taking the second moment of (6.3) and using the Itô-Walsh isometry (3.2), we get

$$\begin{aligned}
\mathbb{E}\langle u_t, \varphi_1\rangle^2 &= E_\beta(-\lambda_1 t^\beta)^2\langle u_0, \varphi_1\rangle^2\\
&\quad + \lambda^2\int_B\int_0^t E_\beta(-\lambda_1(t-s)^\beta)^2\varphi_1^2(\mathrm{y})\mathbb{E}|\sigma(u_s(\mathrm{y}))|^2 ds d\mathrm{y}.
\end{aligned}\tag{6.4}$$

Now using (6.4) and the assumption on σ gives

$$\begin{aligned}
\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2\,dt &= \Lambda(\theta)\,\langle u_0, \varphi_1\rangle^2 + \lambda^2\Lambda(\theta)\,\int_0^\infty e^{-\theta t}\int_B \mathbb{E}|\sigma(u_s(\mathrm{y}))|^2\,\varphi_1^2(\mathrm{y})d\mathrm{y}\,dt.\\
&= \int_0^\infty e^{-\theta t}E_\beta(-\mu_1 t^\beta)^2\langle u_0, \varphi_1\rangle^2\,dt\\
&\quad + \lambda^2\int_0^\infty e^{-\theta t}\int_B\int_B E_\beta(-\mu_1(t-s)^\beta)^2\varphi_1^2(\mathrm{y})\mathbb{E}|\sigma(u_s(\mathrm{y}))|^2 ds d\mathrm{y}\,dt.\\
&= \langle u_0, \varphi_1\rangle^2\int_0^\infty e^{-\theta t}E_\beta(-\mu_1 t^\beta)^2\,dt\\
&\quad + \lambda^2\int_0^\infty e^{-\theta t}\int_B\int_B E_\beta(-\mu_1(t-s)^\beta)^2\varphi_1^2(\mathrm{y})\mathbb{E}|\sigma(u_s(\mathrm{y}))|^2 ds d\mathrm{y}\,dt.\\
&= \Lambda(\theta)\,\langle u_0, \varphi_1\rangle^2 + \lambda^2\Lambda(\theta)\,\int_0^\infty e^{-\theta t}\int_B \mathbb{E}|\sigma(u_s(\mathrm{y}))|^2\,\varphi_1^2(\mathrm{y})d\mathrm{y}\,dt.\\
&\geqslant \Lambda(\theta)\,\langle u_0, \varphi_1\rangle^2 + \lambda^2 l_\sigma^2\Lambda(\theta)\,\int_0^\infty e^{-\theta t}\int_B \mathbb{E}|u_s(\mathrm{y})|^2\,\varphi_1^2(\mathrm{y})d\mathrm{y}\,dt.\\
&\gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + \lambda^2 l_\sigma^2\Lambda(\theta)\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2\,dt.
\end{aligned}\tag{6.5}$$

If $\beta \in (0, \frac{1}{2}]$, since $\Lambda(\theta)$ tends to infinity as θ goes to zero, we can choose θ small enough and from (6.5) we obtain

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt \gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + 2\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt. \tag{6.6}$$

This implies that that for sufficiently small θ, we have

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt = \infty,$$

which implies that for sufficiently large t, $\mathbb{E}\langle u_t, \varphi_1\rangle^2$ grows exponentially. Now using Cauchy-Schwarz inequality we get the following,

$$\begin{aligned}\mathbb{E}\langle u_t, \varphi_1\rangle^2 &\leqslant \mathbb{E}\left(\left(\int_B |u_t(x)|^2 \, dx\right)\left(\int_B |\varphi_1(x)|^2 \, dx\right)\right)\\ &\lesssim \sup_{\mathrm{x}\in B}\mathbb{E}|u_t(\mathrm{x})|^2.\end{aligned}$$

We can thus conclude that $\sup_{\mathrm{x}\in B}\mathbb{E}|u_t(\mathrm{x})|^2$ too grows exponentially fast for all values of λ.

Note that the first part of the conclusion of the theorem merely follows from the fact that the second term of (6.6) is positive and that the first term cannot have exponential decay. □

Proof of Theorem 3.2. Following the lines for the proof of Theorem 3.1, we get the inequality (6.5);

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt \gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + \lambda^2 l_\sigma^2 \Lambda(\theta)\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt. \tag{6.7}$$

Since $\beta \in (\frac{1}{2}, 1)$, the function $\Lambda(\theta)$ is bounded. Thus, for any fixed $\theta > 0$ we can find sufficiently large λ_u so that for all $\lambda \geqslant \lambda_u$, the above inequality (6.7) yields

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt \gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + \frac{3}{2}\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2 \, dt. \tag{6.8}$$

The result of the theorem is obtained by following the arguments of the remaining part of the proof of Theorem 3.1. □

Proof of Theorem 3.3. We establish the required result using Itô-Walsh isometry and the global Lipschitz assumption on σ. The second moment of the mild formulation given by (3.3) becomes:

$$\begin{aligned}
\mathbb{E}|u_t(x)|^2 &= |(\mathcal{G}_B^{(\beta)}u)_t(\mathrm{x})|^2 + \lambda^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})^2 \mathbb{E}|\sigma(u_s(\mathrm{y}))|^2 ds\, d\mathrm{y} \\
&\leqslant |(\mathcal{G}_B^{(\beta)}u)_t(\mathrm{x})|^2 + \lambda^2 L_\sigma^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})^2 \mathbb{E}|u_s(\mathrm{y})|^2 ds\, d\mathrm{y} \\
&:= J_1 + J_2. \qquad (6.9)
\end{aligned}$$

We observe J_1 is bounded by the square of the same constant as initial condition, since $\int_B G_B^{(\beta)}(t,\, x)\, dx \leqslant \int_{\mathbb{R}^d} G^{(\beta)}(t,\, x)\, dx = 1$ and the initial datum is assumed to be bounded above by a constant. It remains to bound J_2. Since $\{\varphi_n\}_{n\geqslant 1}$ is an orthonormal sequence, for each fixed $t > 0$ we obtain

$$\begin{aligned}
J_2 &:= \lambda^2 L_\sigma^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})^2 \mathbb{E}|u_s(\mathrm{y})|^2 ds\, d\mathrm{y} \\
&\leqslant a\lambda^2 L_\sigma^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})^2 ds\, d\mathrm{y} \\
&\leqslant a\lambda^2 L_\sigma^2 \int_B \int_0^t \left(\sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta) \varphi_n(\mathrm{x}) \varphi_n(\mathrm{y}) \right)^2 ds\, d\mathrm{y} \\
&\leqslant a\lambda^2 L_\sigma^2 \int_B \int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta)^2 \varphi_n^2(\mathrm{x}) \varphi_n^2(\mathrm{y})\, ds\, d\mathrm{y} \\
&\leqslant a_t \lambda^2 L_\sigma^2 \int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta)^2 \varphi_n^2(\mathrm{x})\, ds < \infty, \qquad (6.10)
\end{aligned}$$

by Lemma 2.5 in the case of $d < \alpha/2\beta$, where $a_t = \sup_{0<s<t} \sup_{\mathrm{x}\in B} \mathbb{E}|u_s(\mathrm{x})|^2$. In the case the eigenfunctions $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$, it follows from the above that

$$\begin{aligned}
J_2 :&\leqslant a_t \lambda^2 L_\sigma^2 \int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta)^2 \varphi_n^2(\mathrm{x})\, ds \\
&\leqslant a_t \lambda^2 L_\sigma^2 \left(C(B)\right)^2 \int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta)^2 ds \leqslant a_t \lambda^2 L_\sigma^2 \left(C(B)\right)^2 C(ML), \qquad (6.11)
\end{aligned}$$

where $a_t = \sup_{0<s<t} \sup_{\mathrm{x}\in B} \mathbb{E}|u_t(\mathrm{x})|^2$ and $C(ML) = \displaystyle\int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n (t-s)^\beta)^2 ds$. By Lemma 2.4 $C(ML) < \infty$ is a finite constant independent of t. That is, since $\beta \in (\frac{1}{2}, 1)$, we can choose λ_l sufficiently small that for all $\lambda \leqslant \lambda_l$, the above estimates shows

$$\sup_{0<s<t} \sup_{\mathrm{x} \in B} \mathbb{E}|u_t(\mathrm{x})|^2 \lesssim 1 + \frac{1}{2} \sup_{0<s<t} \sup_{\mathrm{x} \in B} \mathbb{E}|u_t(\mathrm{x})|^2.$$

This shows that $\sup_{0<t<\infty} \sup_{\mathrm{x}\in B} \mathbb{E}|u_t(\mathrm{x})|^2$ is finite and hence the conclusion of the theorem follows. $\square$

6.2 Continuity of the solution for space-time white noise case

In this section we give the proof of Theorem 3.8 and the following proposition is used in its proof. We can use the bounds of $G_B^{(\beta)}(t,\, x,\, y)$ to show the following proposition holds:

Proposition 6.2. *For $d < \frac{\alpha}{\beta}$, let $u_t^{(\beta)}$ be a solution to equation (3.3) for parameters $\beta \in \left(\frac{1}{2},\, 1\right)$. Then for some θ, the supremum on the p^{th} moment of the solution $u_s^{(\beta)}(x)$ is given by*

$$\sup_{t>0,\, x\in B} e^{-\theta t}\mathbb{E}|u_s^{(\beta)}(x)|^p, \tag{6.12}$$

is bounded above by a constant independent of β.

Proof. We use the mild formulation in equation (3.3) and follow similar computations to those used in the above proof. Now consider

$$u_t^{(\beta)}(x) = (\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x}) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})\sigma(u_s^{(\beta)}(\mathrm{y}))W(ds\,, d\mathrm{y}), \tag{6.13}$$

Now applying the inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$ for any $a, b \geqslant 0$ and using Burkholder-Davis-Gundy inequality, the p^{th} moment of (6.13) becomes

$$\mathbb{E}|u_t^{(\beta)}(x)|^p \leqslant 2^p|(\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x})|^p + (2\lambda)^p \left[\int_B \int_0^t |G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})|^2 \left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))|^p\right]^{\frac{2}{p}} ds\, d\mathrm{y}\right]^{\frac{p}{2}}, \tag{6.14}$$

$$\begin{aligned} e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p &\lesssim e^{-\theta t}|(\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x})|^p + e^{-\theta t} \left[\int_B \int_0^t |G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})|^2 \left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))|^p\right]^{\frac{2}{p}} ds\, d\mathrm{y}\right]^{\frac{p}{2}} \\ &:= J_1 + J_2. \end{aligned} \tag{6.15}$$

Using Lemma 2.2 and since σ is globally Lipschitz inequality, the second term becomes

$$
\begin{aligned}
J_2 &:= e^{-\theta t}\left[\int_B \int_0^t |G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})|^2 \left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))|^p\right]^{\frac{2}{p}} ds\, d\mathrm{y}\right]^{\frac{p}{2}} \\
&\leqslant L_\sigma^p \left[\int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})^2 e^{\frac{-2\theta t}{p}} [\mathbb{E}|u_s^{(\beta)}(\mathrm{y})|^p]^{2/p} ds\, dy\right]^{p/2} \\
&\leqslant L_\sigma^p b(\theta) \left[\int_B \int_0^t G_B^{(\beta)}(s,\, \mathrm{x},\, \mathrm{y})^2 e^{\frac{-2\theta s}{p}} ds\, dy\right]^{p/2} \\
&\lesssim L_\sigma^p b(\theta) \left[\int_0^\infty s^{-\frac{\beta d}{\alpha}} e^{\frac{-2\theta s}{p}}\, ds\right]^{p/2} \\
&\lesssim L_\sigma^p b(\theta) \left[\int_0^\infty s^{-\frac{\beta d}{\alpha}} e^{-\theta s}\, ds\right]^{p/2} \\
&\lesssim L_\sigma^p b(\theta) \left[L(s^{-\frac{\beta d}{\alpha}})\right]^{p/2} \\
&\lesssim L_\sigma^p b_t(\theta) \left[\Gamma\left(1 - \frac{\beta d}{\alpha}\right) \theta^{\frac{\beta d}{\alpha} - 1}\right]^{p/2},
\end{aligned} \tag{6.16}
$$

where $b_t(\theta) = \sup_{0<s<t} \sup_{\mathrm{z}\in B} e^{-\theta s}\mathbb{E}|u_s^{(\beta)}(\mathrm{z})|^p$. Now we fix $\theta > 0$ sufficiently large so that

$$
J_2 \lesssim \frac{1}{2} b_t(\theta). \tag{6.17}
$$

Thus, we have

$$
e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p \lesssim 1 + \frac{1}{2} b_t(\theta). \tag{6.18}
$$

Taking the supremum of the left side of (6.18) the result follows. $\square$

We are now ready to prove Theorem 3.8.

Proof of Theorem 3.8. From the mild formulation of the solutions, we have

$$
\begin{aligned}
u_t^{(\gamma)}(x) - u_t^{(\beta)}(x) &= (\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x) + \lambda \int_B \int_0^t G_B^{(\gamma)}(t-s,\, x,\, y)\sigma(u_s^{(\gamma)}(y))W(ds\, dy) \\
&\quad - \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)\sigma(u_s^{(\beta)}(y))W(ds\, dy) \\
&= (\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x) \\
&\quad + \lambda \int_B \int_0^t [G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)]\sigma(u_s^{(\gamma)}(y))W(ds\, dy) \qquad (6.19)\\
&\quad + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]W(ds\, dy).
\end{aligned}
$$

W.l.o.g., let $p > 0$. Now applying the inequality $(a+b+c)^p \leqslant 3^p(a^p + b^p + c^p)$ for any $a, b, c \geqslant 0$ and using Burkholder-Davis-Gundy inequality, the p^{th} moment of (6.19) becomes

$$
\begin{aligned}
\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p &\leqslant 3^p|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p \\
&\quad + (3\lambda)^p \left[\int_0^t \int_B |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)|^2 [\mathbb{E}|\sigma(u_s^{(\gamma)}(y))|^p]^{2/p}\, dy\, ds\right]^{p/2} \qquad (6.20)\\
&\quad + (3\lambda)^p \left[\int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)^2 [\mathbb{E}[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]^p]^{2/p} ds\, dy\right]^{p/2}.
\end{aligned}
$$

For a fixed $\theta > 0$, the inequality in (6.20) becomes

$$
\begin{aligned}
e^{-\theta t}\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p &\lesssim e^{-\theta t}|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p \\
&\quad + e^{-\theta t} \left[\int_0^t \int_B |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)|^2 [\mathbb{E}|\sigma(u_s^{(\gamma)}(y))|^p]^{2/p}\, dy\, ds\right]^{p/2} \\
&\quad + e^{-\theta t} \left[\int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)^2 [\mathbb{E}[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]^p]^{2/p} ds\, dy\right]^{p/2}. \\
&:= J_1 + J_2 + J_3.
\end{aligned}
$$

Let us first work on the third term. Using Lemma 2.2 and since σ is globally Lipschitz, we have

$$
\begin{aligned}
J_3 &:= e^{-\theta t}\left[\int_B\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)^2[\mathbb{E}[\sigma(u_s^{(\gamma)}(y))-\sigma(u_s^{(\beta)}(y))]^p]^{2/p}ds\,dy\right]^{p/2}\\
&\leqslant L_\sigma^p\left[\int_B\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})^2 e^{\frac{-2\theta t}{p}}[\mathbb{E}|u_s^{(\gamma)}(\mathrm{y})-u_s^{(\beta)}(\mathrm{y})|^p]^{2/p}ds\,dy\right]^{p/2}\\
&\leqslant L_\sigma^p b(\theta)\left[\int_B\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})^2 e^{\frac{-2\theta(t-s)}{p}}ds\,dy\right]^{p/2}\\
&\leqslant L_\sigma^p b_t(\theta)\left[\int_0^\infty\int_{\mathbb{R}^n} G^{(\beta)}(s,\,\mathrm{x},\,\mathrm{y})^2 e^{\frac{-2\theta s}{p}}dy\,ds\right]^{p/2}\\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^\infty s^{-\frac{\beta d}{\alpha}}e^{\frac{-2\theta s}{p}}\,ds\right]^{p/2}\\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^\infty s^{-\frac{\beta d}{\alpha}}e^{-\theta s}\,ds\right]^{p/2}\\
&\lesssim L_\sigma^p b(\theta)\left[\mathcal{L}_s\{s^{-\frac{\beta d}{\alpha}}\}\right]^{p/2}\\
&\lesssim L_\sigma^p b_t(\theta)\left[\Gamma\left(1-\frac{\beta d}{\alpha}\right)\theta^{\frac{\beta d}{\alpha}-1}\right]^{p/2},
\end{aligned}
\tag{6.21}
$$

where $b_t(\theta)=\sup_{0<s<t}\sup_{\mathrm{z}\in B}e^{-\theta s}\mathbb{E}|u_s^{(\gamma)}(\mathrm{z})-u_s^{(\beta)}(\mathrm{z})|^p$. Now we fix $\theta>0$ sufficiently large so that

$$
J_3\lesssim\frac{1}{2}b_t(\theta). \tag{6.22}
$$

Next we work on J_2. Since $(a+b)^2 \leqslant 4(a^2+b^2)$ for $a, b \geqslant 0$, using Proposition 6.12, Lemma 2.9(b) and the assumption on σ, we obtain

$$
\begin{aligned}
J_2 &:= e^{-\theta t}\left[\int_0^t \int_B |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)|^2 [\mathbb{E}|\sigma(u_s^{(\beta)}(y))|^p]^{2/p}\, dy\, ds\right]^{p/2} \\
&\lesssim \sup_{t>0,\, x\in B} e^{-\theta t}\mathbb{E}|u_s^{(\beta)}(x)|^p \left[\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\gamma)}(s,\, x,\, y) - G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2} \\
&\lesssim \left[\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\gamma)}(s,\, x,\, y) - G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2} \\
&\lesssim \left[\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\gamma)}(s,\, x,\, y)|^2 ds\, dy + \int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2} \\
&\lesssim \left[2\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\gamma)}(s,\, x,\, y)|^2 + |G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2} \\
&\lesssim \left[\int_0^\infty e^{-2\theta s/p}\sum_{n=1}^\infty [E_\gamma(-\mu_n s^\gamma) - E_\beta(-\mu_n s^\beta)]^2 |\phi_n(x)|^2 ds\right]^{p/2} \\
&\lesssim \left[2\int_0^\infty e^{-2\theta s/p}\sum_{n=1}^\infty \left[E_\gamma(-\mu_n s^\gamma)^2 + (E_\beta(-\mu_n s^\beta)^2\right] ds\right]^{p/2} \\
&\lesssim \left[p^{1-2\lambda'\beta_0'}\left[\frac{\mathscr{C}_{\mu_0'}^2\theta^{-2\mu_0'}}{1-2\lambda'-2\mu_0} + \frac{\mathscr{C}_{\mu_1'}^2\theta^{-2\mu_1'}}{2\lambda'\beta_0'+2\mu_1'-1}\right] + p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}_{\mu_0}^2\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}_{\mu_1}^2\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right]\right]^{p/2},
\end{aligned}
\tag{6.23}
$$

where $\beta \in (\beta_0, \beta_1)$ and $\gamma \in (\beta_0', \beta_1')$ for some $\beta_0, \beta_1, \beta_0', \beta_1' \in (0,1)$, $\frac{d}{\alpha} < \lambda < \frac{1}{2\beta}$, $\frac{d}{\alpha} < \lambda' < \frac{1}{2\gamma}$, $\mu_0 < \min\left(\frac{1}{2}-\lambda,\ \frac{1-2\beta_1\lambda}{2}\right)$, $\mu_0' < \min\left(\frac{1}{2}-\lambda',\ \frac{1-2\gamma_1\lambda'}{2}\right)$, $\mu_1 > \frac{1-2\beta_0\lambda}{2}$, and $\mu_1' > \frac{1-2\beta_0'\lambda'}{2}$, where the bound on right hand side of (6.23) doesn't depend on β and γ.

We combine the above estimates and obtain the following:

$$
\begin{aligned}
\sup_{x\in B} e^{-\theta s}\mathbb{E}|u_s^{(\gamma)}(x) - u_s^{(\beta)}(x)|^p &\lesssim e^{-\theta t}|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p + \frac{1}{2}b_t(\theta) \\
&\quad + \left[\int_0^\infty \int_B e^{-2\theta s/p}|G_B^{(\gamma)}(s,\, x,\, y) - G_B^{(\beta)}(s,\, x,\, y)|^2\, ds\, dy\right]^{p/2}.
\end{aligned}
$$

Since Lemma 2.6, and Lemma 2.9(b) allows us to use the dominated convergence theorem, taking $\gamma \to \beta$ the result of the theorem follows. □

6.3 The space colored noise case

Proof of Theorem 4.3. First, we establish the second statement of the theorem. From the mild formulation given in (3.3) and using Stochastic Fubini theorem, we set

$$
\begin{aligned}
\langle u_t, \varphi_1 \rangle &= \int_B u_t(x)\varphi_1(\mathrm{x})\, d\mathrm{x} \\
&= \int_B \left[(\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x}) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y}) \sigma(v_s(\mathrm{y})) F(ds\,, d\mathrm{y}) \right] \varphi_1(\mathrm{x})\, d\mathrm{x} \\
&= \int_B \int_B G_B^{(\beta)}(t,\, \mathrm{x},\, \mathrm{y}) u_0(\mathrm{y})\, \varphi_1(\mathrm{x})\, d\mathrm{y}\, d\mathrm{x} \\
&\qquad + \lambda \int_B \int_B \int_0^t E_\beta(-\mu_1(t-s)^\beta) \varphi_1^2(\mathrm{x}) \varphi_1(\mathrm{y}) \sigma(v_s(\mathrm{y})) F(ds\,, d\mathrm{y})\, d\mathrm{x} \\
&= \int_B \int_B E_\beta(-\mu_1 t^\beta)\, u_0(\mathrm{y})\, \varphi_1^2(\mathrm{x}) \varphi_1(\mathrm{y})\, d\mathrm{y}\, d\mathrm{x} \\
&\qquad + \lambda \int_B \int_B \int_0^t E_\beta(-\mu_1(t-s)^\beta) \varphi_1^2(\mathrm{x}) \varphi_1(\mathrm{y}) \sigma(v_s(\mathrm{y})) F(ds\,, d\mathrm{y})\, d\mathrm{x} \\
&= E_\beta(-\mu_1 t^\beta) \int_B u_0(\mathrm{y})\, (\mathrm{x}) \varphi_1(\mathrm{y})\, d\mathrm{y} \\
&\qquad + \lambda \int_B \int_0^t E_\beta(-\mu_1(t-s)^\beta) \varphi_1(\mathrm{y}) \sigma(v_s(\mathrm{y})) F(ds\,, d\mathrm{y}) \\
&= E_\beta(-\mu_1 t^\beta) \langle u_0, \varphi_1 \rangle + \lambda \int_B \int_0^t E_\beta(-\mu_1(t-s)^\beta) \varphi_1(\mathrm{y}) \sigma(v_s(\mathrm{y})) F(ds\,, d\mathrm{y}).
\end{aligned} \tag{6.24}
$$

Taking the second moment of (6.24), we get

$$
\begin{aligned}
\mathbb{E}\langle u_t, \varphi_1 \rangle^2 &= E_\beta(-\lambda_1 t^\beta)^2 \langle u_0, \varphi_1 \rangle^2 \\
&\qquad + \lambda^2 \int_{B\times B} \int_0^t E_\beta(-\lambda_1(t-s)^\beta)^2 \varphi_1(\mathrm{y}) \varphi_1(\mathrm{z}) \mathbb{E}|\sigma(u_s(\mathrm{y}))\sigma(u_s(\mathrm{z}))|\, f(y,\, z)\, ds d\mathrm{y} dz.
\end{aligned} \tag{6.25}
$$

Now using (6.25) and the assumption on σ gives

$$\begin{aligned}
\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\varphi_1\rangle^2\,dt &= \int_0^\infty e^{-\theta t}E_\beta(-\mu_1 t^\beta)^2\langle u_0,\varphi_1\rangle^2\,dt\\
&\quad+\lambda^2\int_0^\infty e^{-\theta t}\int_{B\times B}\int_0^t E_\beta(-\lambda_1(t-s)^\beta)^2\varphi_1(\mathrm{y})\varphi_1(\mathrm{z})\mathbb{E}|\sigma(u_s(\mathrm{y}))\sigma(u_s(\mathrm{z}))|f(y,\,z)\,dsd\mathrm{y}d\mathrm{z}dt.\\
&=\langle u_0,\varphi_1\rangle^2\int_0^\infty e^{-\theta t}E_\beta(-\mu_1 t^\beta)^2\,dt\\
&\quad+\lambda^2\int_0^\infty e^{-\theta t}\int_{B\times B}\int_0^t E_\beta(-\lambda_1(t-s)^\beta)^2\varphi_1(\mathrm{y})\varphi_1(\mathrm{z})\mathbb{E}|\sigma(u_s(\mathrm{y}))\sigma(u_s(\mathrm{z}))|f(y,\,z)\,dsd\mathrm{y}d\mathrm{z}dt.\\
&=\Lambda(\theta)\,\langle u_0,\varphi_1\rangle^2+\lambda^2\Lambda(\theta)\int_0^\infty e^{-\theta t}\int_{B\times B}\varphi_1(\mathrm{y})\varphi_1(\mathrm{z})\times\\
&\qquad\qquad\mathbb{E}|\sigma(u_t(\mathrm{y}))\sigma(u_t(\mathrm{z}))|f(y,\,z)d\mathrm{y}\,\mathrm{z}\,dt.\\
&\geqslant\Lambda(\theta)\,\langle u_0,\varphi_1\rangle^2+\lambda^2 l_\sigma^2K_f\Lambda(\theta)\int_0^\infty e^{-\theta t}\int_{B\times B}\varphi_1(\mathrm{y})\varphi_1(\mathrm{z})\times\\
&\qquad\qquad\mathbb{E}|u_t(\mathrm{y})u_t(\mathrm{z})|d\mathrm{y}\,d\mathrm{z}\,dt\\
&=\Lambda(\theta)\langle u_0,\varphi_1\rangle^2+\lambda^2 l_\sigma^2K_f\Lambda(\theta)\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\varphi_1\rangle^2\,dt.
\end{aligned}\tag{6.26}$$

If $\beta\in(0,\,\frac{1}{2}]$, since $\Lambda(\theta)$ tends to infinity as θ goes to zero, we can choose θ small enough and from (6.26) we obtain

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\varphi_1\rangle^2\,dt\gtrsim\Lambda(\theta)\langle u_0,\varphi_1\rangle^2+\frac{1}{2}\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\varphi_1\rangle^2\,dt.\tag{6.27}$$

This implies that for sufficiently small θ, we have

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\varphi_1\rangle^2\,dt=\infty,$$

which also implies that for sufficiently large t, $\mathbb{E}\langle u_t, \varphi_1\rangle^2$ grows exponentially. Now using Cauchy-Schwarz inequality we get the following,

$$\begin{aligned}
\mathbb{E}\langle u_t, \varphi_1\rangle^2 &= \mathbb{E}\left|\int_B u_t(x)\,\varphi_1(x)\,dx\right|^2 \\
&\leqslant \mathbb{E}\left(\left(\int_B |u_t(x)|^2\,dx\right)\left(\int_B \varphi_1^2(x)\,dx\right)\right)^2 \\
&\lesssim \sup_{\mathrm{x}\in B}\mathbb{E}|u_t(\mathrm{x})|^2.
\end{aligned}$$

We can thus conclude that $\sup_{\mathrm{x}\in B}\mathbb{E}|u_t(\mathrm{x})|^2$ too grows exponentially fast for all values of λ.

Note that the first part of the conclusion of theorem merely follows from the fact that the second term of (6.26) is positive and that the first term cannot have exponential decay. □

Proof of Theorem 4.4. Using the same notations as in the proof of Theorem 4.3, we observe that the inequality (6.26) holds;

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2\,dt \gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + \lambda^2 l_\sigma^2 K_f \Lambda(\theta)\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t,\, \varphi_1\rangle^2\,dt. \tag{6.28}$$

Since $2\beta > 1$, the function $\Lambda(\theta)$ is bounded. Thus, for any fixed $\theta > 0$ there exists sufficiently large λ_u so that for all $\lambda \geqslant \lambda_u$, from the inequality (6.28) we obtain

$$\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2\,dt \gtrsim \Lambda(\theta)\langle u_0, \varphi_1\rangle^2 + 2\int_0^\infty e^{-\theta t}\mathbb{E}\langle u_t, \varphi_1\rangle^2\,dt. \tag{6.29}$$

Again, following similar lines of arguments as in the proof of Theorem 4.3, we get result of the current theorem. □

Proof of Theorem 4.5. Using $(a+b)^2 \leqslant 4(a^2+b^2)$ for $a, b \geqslant 0$, and applying Burkholder's inequality, the second moment of the mild formulation given by (4.5) becomes:

$$\begin{aligned}\mathbb{E}|u_t(x)|^2 &\leqslant 4|(\mathcal{G}_B^{(\beta)}u)_t(\mathrm{x})|^2 + 4\lambda^2 \int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})\times \\ &\qquad \mathbb{E}|\sigma(u_s(\mathrm{y}))\sigma(u_s(\mathrm{z}))|\,f(y,\,z)\,ds\,d\mathrm{y}\,d\mathrm{z} \\ &\leqslant 4|(\mathcal{G}_B^{(\beta)}u)_t(\mathrm{x})|^2 + 4\lambda^2 L_\sigma^2 \int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})\times \\ &\qquad \mathbb{E}|u_s(\mathrm{y})u_s(\mathrm{z})|\,f(y,\,z)\,ds\,d\mathrm{y}\,dz \\ &:= J_1 + J_2. \end{aligned} \tag{6.30}$$

We observe J_1 is bounded by the square of the same constant as initial condition, since $\int_B G_B^{(\beta)}(t,\,x)\,dx \leqslant \int_{\mathbb{R}^d} G^{(\beta)}(t,\,x)\,dx = 1$ and the initial datum is assumed to be bounded above by a constant. It remains to bound J_2. Now if $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$,for each fixed $t > 0$, by Cauchy-Schwarz inequality, Lemma 2.1 and Lemma 2.3 we obtain

$$
\begin{aligned}
J_2 &:= 4\lambda^2 L_\sigma^2 \int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})\mathbb{E}|u_s(\mathrm{y})u_s(\mathrm{z})|\,f(y,\,z)\,ds\,d\mathrm{z}\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 \int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})(\mathbb{E}|u_s(\mathrm{y})|^2)^{\frac12}(\mathbb{E}|u_s(\mathrm{z})|^2)^{\frac12}\,f(y,\,z)\,ds\,d\mathrm{z}\,d\mathrm{y}\\
&\leqslant a\lambda^2 L_\sigma^2 \int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})\,f(y,\,z)\,ds\,d\mathrm{y}\,d\mathrm{z}\\
&\leqslant 4\lambda^2 L_\sigma^2 \int_0^t \sup_{x\in B}\mathbb{E}[|u_s(\mathrm{x})|^2]\int_{B\times B} G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})\,f(y,\,z)\,ds\,d\mathrm{z}\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 a_t \int_0^t \sum_{n=1}^\infty\sum_{k=1}^\infty E_\beta(-\mu_n(t-s)^\beta)E_\beta(-\mu_k(t-s)^\beta)\varphi_n(\mathrm{x})\varphi_k(\mathrm{x})\times\\
&\qquad\qquad \int_{B\times B}\varphi_n(\mathrm{y})\varphi_k(\mathrm{z})\,f(y,\,z)\,ds\,d\mathrm{z}\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 a_t \int_0^t \sum_{n=1}^\infty\sum_{k=1}^\infty E_\beta(-\mu_n(t-s)^\beta)E_\beta(-\mu_k(t-s)^\beta)\varphi_n(\mathrm{x})\varphi_k(\mathrm{x})\times\\
&\qquad\qquad \int_B \varphi_n(\mathrm{y})\left[\int_B \varphi_k(\mathrm{z})\,f(y,\,z)\,ds\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \int_0^t E_\beta(-\mu_n(t-s)^\beta)E_\beta(-\mu_k(t-s)^\beta)\varphi_n(\mathrm{x})\varphi_k(\mathrm{x})\,ds\times\\
&\qquad\qquad \int_B \varphi_n(\mathrm{y})\left[\int_B \varphi_k(\mathrm{z})\,f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \left(\int_0^t E_\beta(-\mu_n(t-s)^\beta)^2\,ds\right)^{\frac12}\left(\int_0^t E_\beta(-\mu_k(t-s)^\beta)^2\,ds\right)^{\frac12}\varphi_n(\mathrm{x})\varphi_k(\mathrm{x})\times\\
&\qquad\qquad \int_B \varphi_n(\mathrm{y})\left[\int_B \varphi_k(\mathrm{z})\,f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\leqslant 4\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \left(\mu_n^{-\frac1\beta}\right)^{\frac12}\left(\mu_k^{-\frac1\beta}\right)^{\frac12}\varphi_n(\mathrm{x})\varphi_k(\mathrm{x})\int_B \varphi_n(\mathrm{y})\left[\int_B \varphi_k(\mathrm{z})\,f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\lesssim C_1\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \left(\mu_n^{-\frac1\beta}\right)^{\frac12}\left(\mu_k^{-\frac1\beta}\right)^{\frac12}|\varphi_n(\mathrm{x})|\,|\varphi_k(\mathrm{x})|\int_B |\varphi_n(\mathrm{y})|\left[\int_B |\varphi_k(\mathrm{z})|\,f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\lesssim C_2C(B)^4\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \left(\mu_n^{-\frac1\beta}\right)^{\frac12}\left(\mu_k^{-\frac1\beta}\right)^{\frac12}\int_B\left[\int_B f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\lesssim C_3C(B)^4\lambda^2 L_\sigma^2 a_t \sum_{n=1}^\infty\sum_{k=1}^\infty \left(\mu_n^{-\frac{1}{2\beta}}\right)\left(\mu_k^{-\frac{1}{2\beta}}\right)\int_B\left[\int_B f(y,\,z)\,d\mathrm{z}\right]\,d\mathrm{y}\\
&\lesssim C_3\lambda^2 L_\sigma^2 a_t \left(\sum_{n=1}^\infty n^{-\frac{\alpha}{2d\beta}}\right)\left(\sum_{k=1}^\infty k^{-\frac{\alpha}{2d\beta}}\right)\int_{B\times B} f(y,\,z)\,d\mathrm{z}\,d\mathrm{y}\\
&\lesssim C\lambda^2 L_\sigma^2 a_t, \qquad (6.31)
\end{aligned}
$$

where $C = C_3 \left(\sum_{n=1}^{\infty} n^{-\frac{\alpha}{2d\beta}}\right) \left(\sum_{k=1}^{\infty} k^{-\frac{\alpha}{2d\beta}}\right) \int_{B\times B} f(y,\, z)\, dz\, dy < \infty$, and $a_t = \sup_{0<s<t} \sup_{x\in B} \mathbb{E}[|u_s(\mathrm{x})|^2$. Since $\beta \in (\frac{1}{2}, 1)$, we can choose λ_l sufficiently small that for all $\lambda \leqslant \lambda_l$, the above estimates shows

$$\sup_{0<s<t} \sup_{\mathrm{x}\in B} \mathbb{E}|u_t(\mathrm{x})|^2 \lesssim 1 + \frac{1}{2} \sup_{0<s<t} \sup_{\mathrm{x}\in B} \mathbb{E}|u_t(\mathrm{x})|^2.$$

This shows that $\sup_{0<t<\infty} \sup_{\mathrm{x}\in B} \mathbb{E}|u_t(\mathrm{x})|^2$ is finite and hence the conclusion of the theorem follows. $\square$

Proof of Corollary 4.6. Since $\frac{1}{\beta} > 1$, if $\alpha = 2$ and $d = 1$, (for the case of the eigenvalues $\mu_n = (\frac{n\pi}{L})^2$ and eigenvectors $\varphi_n(x) = (\frac{2}{L})^{\frac{1}{2}} \sin\left(\frac{n\pi x}{L}\right)$ of Dirichlet Laplacian corresponding to a Brownian Motion killed upon exiting the domain $[0,\, L]$ so that $|\varphi_n(x)| \leqslant (\frac{2}{L})^{\frac{1}{2}}$ for each n) from (6.10) using Lemma 2.1 we obtain the result of the corollary. $\square$

Proof of Corollary 4.7. In this case, we note that $C_4 \int_B \left[\int_B f(y,\, z)\, dz\right]\, dy < \infty$, since

$$\int_{B\times B} f(y,\, z)\, dz\, dy = 2\frac{L^{2-\gamma}}{(1-\gamma)\,(2-\gamma)} < \infty. \tag{6.32}$$

$\square$

6.4 Continuity of the solution for the white in time and colored in space noise case

In this section we give the proof of Theorem 4.8 and the following result is useful to guarantee the use of dominated convergence theorem in the proof of Theorem 4.8.

Proposition 6.3. *Assume that $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$, depending on the geometrical characteristics of the domain B, i.e.,*

$$C(B) = \sup_{n\geqslant 1, x\in B} \varphi_n(x).$$

Suppose that $\int_{B\times B} f(x,\, y)\, dx\, dy < \infty$. For $\beta \in (\frac{1}{2}, 1)$ and $\frac{d}{\alpha} < \lambda < \frac{1}{2\beta}$, let $0 < \mu_0 < \min\left(\frac{1}{2} - \lambda,\ \frac{1-2\beta_1\lambda}{2}\right) < \min\left(\frac{1}{2} - \lambda,\ \frac{1-2\beta\lambda}{2}\right)$ and $\mu_1 > \frac{1-2\beta_0\lambda}{2}$ where $\beta \in (\beta_0,\, \beta_1)$ for some $\beta_0, \beta_1 \in (0,\, 1)$. Also for $\gamma \in (\frac{1}{2}, 1)$ and $\frac{d}{\alpha} < \lambda' < \frac{1}{2\gamma}$, let $0 < \mu_0' < \min\left(\frac{1}{2} - \lambda',\ \frac{1-2\beta_1'\lambda'}{2}\right) < \min\left(\frac{1}{2} - \lambda',\ \frac{1-2\gamma\lambda'}{2}\right)$, and $\mu_1' > \frac{1-2\beta_0'\lambda'}{2}$ where

$\gamma \in (\beta_0^{'}, \beta_1^{'})$ for some $\beta_0^{'}, \beta_1^{'} \in (0, 1)$. Then for $d < \frac{1}{2}\min\{\frac{1}{\beta}, \frac{1}{\gamma}\}\alpha$, there exist positive a constant C that depends on μ_0, $\mu_0^{'}$, μ_1, $\mu_1^{'}$, and p, and independent of β and γ such that

$$\left[\int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}|G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \right.$$
$$\left. f(y,\, z)\, dy\, dz\, ds\right]^{\frac{p}{2}} \lesssim C\left[p^{1-2\lambda^{'}\beta_0^{'}}\left(\theta^{-2\mu_0^{'}} + \theta^{-2\mu_1^{'}}\right) + p^{1-2\lambda\beta_0}\left(\theta^{-2\mu_0} + \theta^{-2\mu_1}\right)\right]^{p/2}. \tag{6.33}$$

Proof. Since $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$ and $\int_{B\times B} f(x,\, y)\, dx\, dy < \infty$, for each $t > 0$, using Lemma 2.8, Lemma 2.9(a) and the fact that $(a+b)^2 \leqslant 2(a^2+b^2)$ for $a, b \geqslant 0$, we bound the integral term as follows:

$$\begin{aligned}
&\int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}|G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \\
&\qquad f(y,\, z)\, dy\, dz\, ds \\
&\quad\lesssim \int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}[(t-s)^{-2\gamma\lambda^{'}} + 2(t-s)^{-(\beta\lambda+\gamma\lambda^{'})} + (t-s)^{-2\beta\lambda}] f(y,\, z)\, dy\, dz\, ds \\
&\quad\lesssim \int_0^\infty \int_{B\times B} e^{-2\theta s/p}[s^{-\gamma\lambda^{'}} + s^{-\beta\lambda}]^2 f(y,\, z)\, dy\, dz\, ds \\
&\quad\lesssim \int_0^\infty e^{-2\theta s/p}[s^{-2\gamma\lambda^{'}} + s^{-2\beta\lambda}]\, ds \\
&\quad\lesssim p^{1-2\lambda^{'}\beta_0^{'}}\left[\frac{\mathscr{C}^2_{\mu_0^{'}}\theta^{-2\mu_0^{'}}}{1-2\lambda^{'}-2\mu_0^{'}} + \frac{\mathscr{C}^2_{\mu_1^{'}}\theta^{-2\mu_1^{'}}}{2\lambda^{'}\beta_0^{'}+2\mu_1^{'}-1}\right] + p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}^2_{\mu_0}\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}^2_{\mu_1}\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right],
\end{aligned} \tag{6.34}$$

where μ_0 and $\mu_0^{'}$ are as given in the hypothesis of the proposition. It follows from (6.34) that for any $p \geqslant 0$ we obtain

$$\begin{aligned}
&\left[\int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}|G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \right. \\
&\qquad \left. f(y,\, z)\, dy\, dz\, ds\right]^{p/2} \\
&\quad\lesssim \left[p^{1-2\lambda^{'}\beta_0^{'}}\left[\frac{\mathscr{C}^2_{\mu_0^{'}}\theta^{-2\mu_0^{'}}}{1-2\lambda^{'}-2\mu_0^{'}} + \frac{\mathscr{C}^2_{\mu_1^{'}}\theta^{-2\mu_1^{'}}}{2\lambda^{'}\beta_0^{'}+2\mu_1^{'}-1}\right] + p^{1-2\lambda\beta_0}\left[\frac{\mathscr{C}^2_{\mu_0}\theta^{-2\mu_0}}{1-2\lambda-2\mu_0} + \frac{\mathscr{C}^2_{\mu_1}\theta^{-2\mu_1}}{2\lambda\beta_0+2\mu_1-1}\right]\right]^{p/2},
\end{aligned}$$

which allows us to deduce (6.33). That is, each terms in the summation can be bounded by a quantity independent of γ and β.

$\square$

Proof of Theorem 4.8. From the mild formulation of the solutions, we have

$$\begin{aligned}
u_t^{(\gamma)}(x) - u_t^{(\beta)}(x) &= (\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x) + \lambda \int_B \int_0^t G_B^{(\gamma)}(t-s,\, x,\, y)\sigma(u_s^{(\gamma)}(y))F(ds\,dy) \\
&\quad - \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)\sigma(u_s^{(\beta)}(y))F(ds\,dy) \\
&= (\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x) \\
&\quad + \lambda \int_B \int_0^t [G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)]\sigma(u_s^{(\gamma)}(y))F(ds\,dy) \qquad (6.35) \\
&\quad + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]F(ds\,dy).
\end{aligned}$$

W.l.o.g., let $p \geqslant$. Now applying the inequality $(a+b+c)^p \leqslant 3^p(a^p\, + b^p\, + c^p)$ for any $a, b, c \geqslant 0$ and using Burkholder-Davis-Gundy inequality, the p^{th} moment of (6.35) becomes

$$\begin{aligned}
\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p &\leqslant 3^p|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p \\
&\quad + (3\lambda)^p\mathbb{E}\left|\int_B \int_0^t [G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)]\sigma(u_s^{(\gamma)}(y))F(ds\,dy)\right|^p \qquad (6.36) \\
&\quad + (3\lambda)^p\mathbb{E}\left|\int_B \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]F(ds\,dy)\right|^p.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p &\leqslant 3^p|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p \\
&\quad + (3\lambda)^p\Bigg[\int_0^t \int_{B\times B} |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \\
&\qquad [\mathbb{E}|\sigma(u_s^{(\gamma)}(y))\sigma(u_s^{(\gamma)}(z))|^{p/2}]^{2/p} f(y,\, z)\, ds\, dy\, dz\Bigg]^{p/2} \qquad (6.37) \\
&\quad + (3\lambda)^p\Bigg[\int_{B\times B} \int_0^t G_B^{(\beta)}(t-s,\, x,\, y)G_B^{(\beta)}(t-s,\, x,\, z)\times \\
&\qquad [\mathbb{E}[(\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)))(\sigma(u_s^{(\gamma)}(z)) - \sigma(u_s^{(\beta)}(z)))]^{p/2}]^{2/p} f(y,\, z)\, ds\, dy\, dz\Bigg]^{p/2}.
\end{aligned}$$

(6.38)

For a fixed $\theta > 0$ the inequality in (6.36) becomes

$$
\begin{aligned}
e^{-\theta t}\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p &\lesssim e^{-\theta t}|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p \\
&+ e^{-\theta t}\Bigg[\int_0^t \int_{B\times B} |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \\
&\qquad [\mathbb{E}|\sigma(u_s^{(\gamma)}(y))\sigma(u_s^{(\gamma)}(z))|^{p/2}]^{2/p}\, f(y,\, z)\, ds\, dy\, dz\Bigg]^{p/2} \\
&+ e^{-\theta t}\Bigg[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\, x,\, y)G_B^{(\beta)}(t-s,\, x,\, z)\times \\
&\qquad [\mathbb{E}[(\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)))(\sigma(u_s^{(\gamma)}(z)) - \sigma(u_s^{(\gamma)}(z)))]^{p/2}]^{2/p}\, f(y,\, z)\, ds\, dy\, dz\Bigg]^{p/2}. \\
&:= J_1 + J_2 + J_3. \qquad (6.39)
\end{aligned}
$$

Let us first work on the third term. Since σ is globally Lipschitz, using Lemma 2.4, and Lemma 2.9 (a), we have

$$
\begin{aligned}
J_3 &:= e^{-\theta t}\Bigg[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)\times \\
&\quad \Bigg[\mathbb{E}[(\sigma(u_s^{(\gamma)}(y))-\sigma(u_s^{(\beta)}(y)))(\sigma(u_s^{(\gamma)}(z))-\sigma(u_s^{(\gamma)}(z)))]^{p/2}]^{2/p}\,f(y,\,z)\,ds\,dy\,dz\Bigg]^{p/2}. \\
&\lesssim L_\sigma^p\Bigg[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)\times \\
&\quad e^{\frac{-2\theta t}{p}}[\mathbb{E}[(u_s^{(\gamma)}(y)-u_s^{(\beta)}(y))(u_s^{(\gamma)}(z)-u_s^{(\beta)}(z))]^{p/2}]^{2/p}\,f(y,\,z)\,ds\,dy\,dz\Bigg]^{p/2}. \\
&\lesssim L_\sigma^p\Bigg[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)\times \\
&\quad e^{\frac{-2\theta t}{p}}(\mathbb{E}(u_s^{(\gamma)}(y)-u_s^{(\beta)}(y))^p)^{\frac{1}{p}}(\mathbb{E}(u_s^{(\gamma)}(z)-u_s^{(\beta)}(z))^p)^{\frac{1}{p}}\,f(y,\,z)\,ds\,dy\,dz\Bigg]^{p/2}. \\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)e^{\frac{-2\theta(t-s)}{p}}\,f(y,\,z)\,ds\,dy\,dz\right]^{p/2} \\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^t (t-s)^{-2\beta\lambda}e^{\frac{-2\theta(t-s)}{p}}\,ds\,\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2} \\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^\infty s^{-2\beta\lambda}e^{\frac{-2\theta s}{p}}\,ds\,\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2} \\
&\lesssim L_\sigma^p p^{1-2\beta\lambda} b_t(\theta)\left[\int_0^\infty \frac{e^{-2\theta r}}{r^{2\beta\lambda}}\,dr\,\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2} \\
&\lesssim L_\sigma^p p^{\frac{p(1-2\beta\lambda)}{2}} b_t(\theta)(2\theta)^{\frac{p(2\beta\lambda-1)}{2}}\Gamma(1-2\beta\lambda)^{\frac{p}{2}}\left[\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2}, \qquad (6.40)
\end{aligned}
$$

where $b_t(\theta)=\sup_{0<s<t}\sup_{\mathrm{z}\in B} e^{-\theta s}\mathbb{E}|u_s^{(\gamma)}(\mathrm{z})-u_s^{(\beta)}(\mathrm{z})|^p$. For $t>0$, we fix $\theta>0$ sufficiently large so that

$$J_3 \lesssim \frac{1}{2}b_t(\theta). \qquad (6.41)$$

Next we work on J_2. Using Proposition 6.4 below and the assumption on σ, we obtain

$$\begin{aligned}
J_2 &:= e^{-\theta t}\bigg[\int_0^t \int_{B\times B} |G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)||G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\times \\
&\qquad \Big[[\mathbb{E}|\sigma(u_s^{(\gamma)}(y))\sigma(u_s^{(\gamma)}(z))|^{p/2}]^{2/p}\, f(y,\, z)\, dy\, dz\, ds\Big]^{p/2} \\
&\lesssim \sup_{t>0,\, x\in B} e^{-\theta t}\mathbb{E}|u_s^{(\gamma)}(x)|^p \bigg[\int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}|G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)|\times \\
&\qquad |G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\, f(y,\, z)\, dy\, dz\, ds\bigg]^{p/2}.
\end{aligned} \tag{6.42}$$

We combine the above estimates and obtain the following.

$$\begin{aligned}
\sup_{x\in B} e^{-\theta s}\mathbb{E}|u_s^{(\gamma)}(x) - u_s^{(\beta)}(x)|^p &\lesssim e^{-\theta t}|(\mathcal{G}_B^{(\gamma)}u)_t(x) - (\mathcal{G}_B^{(\beta)}u)_t(x)|^p + \frac{1}{2}b_t(\theta) + \\
&\sup_{t>0,\, x\in B} e^{-\theta t}\mathbb{E}|u_s^{(\gamma)}(x)|^p \bigg[\int_0^t \int_{B\times B} e^{-2\theta(t-s)/p}|G_B^{(\gamma)}(t-s,\, x,\, y) - G_B^{(\beta)}(t-s,\, x,\, y)|\times \\
&\qquad |G_B^{(\gamma)}(t-s,\, x,\, z) - G_B^{(\beta)}(t-s,\, x,\, z)|\, f(y,\, z)\, ds\, dy\, dz\bigg]^{p/2}.
\end{aligned}$$

Since Lemma 2.6 and Proposition 6.3 allow us to use dominated convergence theorem, taking $\gamma \to \beta$ the result of the theorem follows. □

The following proposition is used in the proof of Theorem 4.8. We use the bounds of $G_B^{(\beta)}(t,\, x,\, y)$ to show the proposition holds:

Proposition 6.4. *Assume that $\{\varphi_n\}_{n\geqslant 1}$ are uniformly bounded by a constant $C(B)$. Suppose that $d < \frac{\alpha}{2\beta}$, and $\int_{B\times B} f(x,\, y)\, dx\, dy < \infty$. For some θ, the supremum on the p^{th} moment of the solution $u_s^{(\beta)}(x)$ to equation* (4.1) *given by*

$$\sup_{0\leqslant s\leqslant t,\, x\in B} e^{-\theta s}\mathbb{E}|u_s^{(\beta)}(x)|^p \tag{6.43}$$

is bounded above by a constant independent of β.

Proof. We use the mild formulation in equation (4.5) and follow similar computations to those used in the above proof. Now consider

$$u_t^{(\beta)}(x) = (\mathcal{G}_B^{(\beta)}u_0)_t(\mathrm{x}) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y})\sigma(u_s^{(\beta)}(\mathrm{y}))F(ds\,, d\mathrm{y}), \tag{6.44}$$

Now applying the inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$ for any $a, b \geqslant 0$ and using Burkholder-Davis-Gundy inequality, the p^{th} moment of (6.44) becomes

$$\mathbb{E}|u_t^{(\beta)}(x)|^p \leqslant 2^p|(\mathcal{G}_B^{(\beta)}u_0)_t(\mathrm{x})|^p + (2\lambda)^p\left[\int_{B\times B}\int_0^t |G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})|\times \right.$$
$$\left. \left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))\sigma(u_s^{(\beta)}(\mathrm{z}))|^{p/2}\right]^{2/p} f(y,\,z)\,ds\,d\mathrm{y}\,d\mathrm{z}\right]^{\frac{p}{2}}, \tag{6.45}$$

$$e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p \lesssim e^{-\theta t}|(\mathcal{G}_B^{(\beta)}u_0)_t(\mathrm{x})|^p + e^{-\theta t}\left[\int_{B\times B}\int_0^t |G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})|\times \right.$$
$$\left. \left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))\sigma(u_s^{(\beta)}(\mathrm{z}))|^{p/2}\right]^{2/p} f(y,\,z)\,ds\,d\mathrm{y}\,d\mathrm{z}\right]^{\frac{p}{2}}$$
$$:= J_1 + J_2. \tag{6.46}$$

Since σ is globally Lipschitz, using Lemma 2.4, and Lemma 2.9(a), the second term becomes

$$\begin{aligned}
J_2 &:= e^{-\theta t}\left[\int_{B\times B}\int_0^t |G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{z})|\left[\mathbb{E}|\sigma(u_s^{(\beta)}(\mathrm{y}))\sigma(u_s^{(\beta)}(\mathrm{z}))|^{p/2}\right]^{2/p} f(y,\,z)\,ds\,d\mathrm{y}\,d\mathrm{z}\right]^{\frac{p}{2}},\\
&\lesssim L_\sigma^p\left[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)\times\right.\\
&\qquad \left. e^{\frac{-2\theta t}{p}}[(\mathbb{E}[(u_s^{(\beta)}(y))^p)^{\frac{1}{p}}(\mathbb{E}[(u_s^{(\beta)}(z))]^p)^{\frac{1}{p}} f(y,\,z)\,ds\,dy\,dz\right]^{p/2}.\\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_{B\times B}\int_0^t G_B^{(\beta)}(t-s,\,x,\,y)G_B^{(\beta)}(t-s,\,x,\,z)\,e^{\frac{-2\theta(t-s)}{p}} f(y,\,z)\,ds\,dy\,dz\right]^{p/2}\\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^t (t-s)^{-2\beta\lambda}e^{\frac{-2\theta(t-s)}{p}}\,ds\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2}\\
&\lesssim L_\sigma^p b_t(\theta)\left[\int_0^\infty s^{-2\beta\lambda}e^{\frac{-2\theta s}{p}}\,ds\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2}\\
&\lesssim L_\sigma^p p^{1-2\beta\lambda} b_t(\theta)\left[\int_0^\infty \frac{e^{-2\theta r}}{r^{2\beta\lambda}}\,ds\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2}\\
&\lesssim L_\sigma^p p^{\frac{p(1-2\beta\lambda)}{2}} b_t(\theta)(2\theta)^{\frac{p(2\beta\lambda-1)}{2}}\Gamma(1-2\beta\lambda)^{\frac{p}{2}}\left[\int_{B\times B} f(y,\,z)dy\,dz\right]^{p/2},
\end{aligned} \tag{6.47}$$

where $b_t(\theta) = \sup_{0<s<t}\sup_{\mathrm{z}\in B} e^{-\theta s}\mathbb{E}|u_s^{(\beta)}(\mathrm{z})|^p$. For $t>0$, we fix $\theta>0$ sufficiently large so that

$$J_2 \lesssim \frac{1}{2}b_t(\theta). \tag{6.48}$$

Thus, we have

$$e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p \lesssim 1 + \frac{1}{2}b_t(\theta). \tag{6.49}$$

Taking the supremum of the left side of (6.49) the result follows. □

6.5 Proof of Temporally Hölder continuity of the mild random field solution

Proof of Theorem 5.9. First set $V_\tau(x,z) := \sigma(u_\tau(x))\sigma(u_\tau(z))$. Then, by applying Hölder's inequality and the Lipschitz continuity of σ, one can observe that

$$\begin{aligned}(\mathbb{E}[|V_\tau(x,z)|^{\frac{q}{2}}])^{\frac{2}{q}} &\leqslant (E[|\sigma(u_\tau(x))|^q])^{\frac{1}{q}}(E[|\sigma(u_\tau(z))|^q])^{\frac{1}{q}} \\ &\leqslant \sup_{x\in\mathbb{R}^d}(E[|\sigma(u_\tau(x))|^q])^{\frac{2}{q}} \\ &\leqslant Lip_\sigma^2 \sup_{x\in\mathbb{R}^d}(E[u_\tau(x)|^q])^{\frac{2}{q}}. \end{aligned}\tag{6.50}$$

Now assume that $0 < t < t^{'}$, by the definition of $Y \circledast \sigma(u)$, we can directly compute the difference $(Y \circledast \sigma(u))_{t'}(y) - (Y \circledast \sigma(u))_t(y)$, where the Minkowski integral and Burkholder-Davis-Gundy inequality can be combined to show

$$\begin{aligned}&\mathbb{E}\Big|(Y\circledast\sigma(u))_{t'}(y) - (Y\circledast\sigma(u))_t(y)\Big|^q \\ &\lesssim E\Big|\int_t^{t'}\int_{\mathbb{R}^d} Y_{t'-\tau}(x-y)\sigma(u_\tau(x))W(d\tau,\mathrm{d}x)\Big|^q \\ &\quad + E\Big|\int_0^t\int_{\mathbb{R}^d} Y_{t-\tau,t'-\tau}(x,y)\sigma(u_\tau(x))\,W(d\tau,\mathrm{d}x)\Big|^q \\ &\lesssim \Big|\int_t^{t'}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Y_{t'-\tau}(x-y)Y_{t'-\tau}(z-y)f_\lambda(x-z)E[|V_\tau(x,z)|^{\frac{q}{2}}]^{\frac{2}{q}}\,\mathrm{d}z\,\mathrm{d}x\,d\tau\Big|^{\frac{q}{2}} \\ &\quad + \Big|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathcal{N}_{t-\tau,t'-\tau}(x,z,y,y)f_\lambda(x-z)E[V_\tau(x)|^{\frac{q}{2}}])^{\frac{2}{q}}\,\mathrm{d}z\,\mathrm{d}x\,d\tau\Big|^{\frac{q}{2}} \\ &=: I_1 + I_2, \end{aligned}\tag{6.51}$$

where $\mathcal{N}_{t,t'}(x,z,y,y)$ is given by Equation (5.34). Let us estimate I_1 and I_2 separately.

$$\begin{aligned} I_1 &:= \left| \int_t^{t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Y_{t'-\tau}(x-y) Y_{t'-\tau}(z-y) f_\lambda(x-z) E[|V_\tau(x,z)|^{\frac{q}{2}}]^{\frac{2}{q}} \, \mathrm{d}z \, \mathrm{d}x \, d\tau \right|^{\frac{q}{2}} \\ &\lesssim \left| \int_t^{t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Y_{t'-\tau}(x-y) Y_{t'-\tau}(z-y) f_\lambda(x-z) \sup_{x \in \mathbb{R}^d} (E[|u_\tau(x)|^q])^{\frac{2}{q}} \, \mathrm{d}z \, \mathrm{d}x \, d\tau \right|^{\frac{q}{2}}. \end{aligned} \tag{6.52}$$

By applying Lemma 5.8, we obtain

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Y_{t'-\tau}(x-y) Y_{t'-\tau}(z-y) f_\lambda(x-z) \, \mathrm{d}z \, \mathrm{d}x \right|^{\frac{q}{2}} \lesssim (t'-\tau)^{2s-\frac{\beta\lambda}{\alpha}-2}. \tag{6.53}$$

We observe that $\frac{2}{3-\frac{\lambda}{\alpha}} > \frac{1}{2-\frac{\lambda}{\alpha}}$. So, it follows that $\beta > \frac{2}{3-\frac{\lambda}{\alpha}}$ implies that $\beta > \frac{1}{2-\frac{\lambda}{\alpha}}$, and so we have $2s - \frac{\beta\lambda}{\alpha} - 2 > -1$, since $s > \beta$. Consequently, making some direct computation

$$\begin{aligned} I_1 &\lesssim \left(\int_t^{t'} (t'-\tau)^{2s-\frac{\beta\lambda}{\alpha}-2} \sup_{x \in \mathbb{R}^d} (E[|u_\tau(x)|^q])^{\frac{2}{q}} \, d\tau \right)^{\frac{q}{2}} \\ &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) \left(\int_t^{t'} (t'-\tau)^{2s-\frac{\beta\lambda}{\alpha}-2} e^{2\gamma\tau} \, d\tau \right)^{\frac{q}{2}} \\ &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) e^{\gamma T q} \frac{1}{(2s-\frac{\beta\lambda}{\alpha}-1)^{\frac{q}{2}}} |t'-t|^{\frac{2\alpha\beta q + 2\alpha r q - \beta\lambda q - \alpha q}{2\alpha}}, \end{aligned} \tag{6.54}$$

where we used the moment boundedness given below

$$\sup_{x \in \mathbb{R}^d} \left(E[|u_t(x)|^q] \right)^{\frac{1}{q}} \leqslant e^{\gamma t} \mathscr{N}_{\gamma,q,T}(u), \ \ \forall t \geqslant 0, \ \ q \geqslant 2. \tag{6.55}$$

Next we estimate I_2. Let us apply Lemma 5.7, which yields

$$\begin{aligned} I_2 &:= \left| \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{N}_{t-\tau,t'-\tau}(x,z,y,y) f_\lambda(x-z) E[V_\tau(x)|^{\frac{q}{2}}])^{\frac{2}{q}} \, \mathrm{d}z \, \mathrm{d}x \, d\tau \right|^{\frac{q}{2}} \\ &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) \left| \int_0^t |t'-t|^{1-s} (t-\tau)^{s-\frac{\beta\lambda}{\alpha}-1} (t-\tau)^{s-1} (1+(t'-\tau)^{s-1}) e^{2\gamma\tau} \, d\tau \right|^{\frac{q}{2}} \\ &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) e^{\gamma T q} |t'-t|^{\frac{q-sq}{2}} \left| \int_0^t (t-\tau)^{3s-\frac{\beta\lambda}{\alpha}-3} ((t-\tau)^{1-s}+1) \, d\tau \right|^{\frac{q}{2}}. \end{aligned} \tag{6.56}$$

We note that the power $3s - \frac{\beta\lambda}{\alpha} - 3 > -1$, since $\beta > \frac{2}{3-\frac{\lambda}{\alpha}}$. Thus, we deduce

$$\begin{aligned} I_2 &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) e^{\gamma T q}((T^{1-s}+1)^{\frac{q}{2}} |t' - t|^{\frac{q-sq}{2}} \left| \int_0^t (t-\tau)^{3s - \frac{\beta\lambda}{\alpha} - 3}\, d\tau \right|^{\frac{q}{2}} \\ &\lesssim \mathscr{N}^q_{\gamma,q,T}(u) e^{\gamma T q}((T^{1-s}+1)^{\frac{q}{2}} |t' - t|^{\frac{q-sq}{2}} \frac{1}{3s - \frac{\beta\lambda}{\alpha} - 2} t^{\frac{3s\alpha q - \beta\lambda q - 2\alpha q}{2\alpha}}. \end{aligned} \tag{6.57}$$

Finally, by combining estimates for the terms I_1 and I_2, we derive

$$\begin{aligned} &\mathbb{E}|(Y \circledast \sigma(u))_{t'}(y) - (Y \circledast \sigma(u))_t(y)|^q \\ &\quad \lesssim \mathscr{N}^q_{\gamma,q,T}(u) \left(|t' - t|^{\frac{2\alpha\beta q + 2\alpha r q - \beta\lambda q - \alpha q}{2\alpha}} + |t' - t|^{\frac{q-sq}{2}} t^{\frac{3s\alpha q - \beta\lambda q - 2\alpha q}{2\alpha}} \right). \end{aligned} \tag{6.58}$$

$$\begin{aligned} &(\mathbb{E}|(Y \circledast \sigma(u))_{t'}(y) - (Y \circledast \sigma(u))_t(y)|^q)^{\frac{1}{q}} \\ &\quad \lesssim \mathscr{N}_{\gamma,q,T}(u) \left(|t' - t|^{\frac{2\alpha\beta q + 2\alpha r q - \beta\lambda q - \alpha q}{2\alpha}} + |t' - t|^{\frac{q-sq}{2}} t^{\frac{3s\alpha q - \beta\lambda q - 2\alpha q}{2\alpha}} \right)^{\frac{1}{q}}. \end{aligned} \tag{6.59}$$

$\square$

Proof of Theorem 5.10. Suppose that $t < t'$. Similarly as in the proof of Lemma 5.7, we use the estimate $\beta|\log(t') - \log(t)| \leqslant \beta t^{\beta-1}(t'-t)^{1-\beta}$. Hence, by nothing that $1 \leqslant T^\beta (t'-t)^{1-\beta}$, we can apply Lemma 3.1 of [7] to get

$$\begin{aligned} |(Z_{t'} * u_0)(y) - (Z_t * u_0)(y)| &\leqslant \sup_{z\in\mathbb{R}^d} |u_0(x)| \int_{\mathbb{R}^d} |Z_{t,t'}(y)|\ y \\ &\lesssim \sup_{z\in\mathbb{R}^d} |u_0(x)| (\beta(\log(t') - \log(t)) \wedge 1) \\ &\lesssim \sup_{z\in\mathbb{R}^d} |u_0(x)| (\beta t^{\beta-1}(t'-t)^{1-\beta} \wedge T^\beta (t'-t)^{1-\beta}) \\ &\lesssim T^\beta (t'-t)^{1-\beta} \sup_{z\in\mathbb{R}^d} |u_0(x)| \\ &\lesssim (t'-t)^{1-\beta} \sup_{z\in\mathbb{R}^d} |u_0(x)|, \end{aligned} \tag{6.60}$$

where u_0 is nonrandom and bounded.

The desired continuity follows by combining the above estimates in Equation (6.60) with Lemma 5.9, and using the Minkowski inequality . $\square$

Chapter 7

Conclusion and Future Work

7.1 Conclusion

In this dissertation, we considered a certain class of space-time fractional SPDE with space-time white noise, and white in time and colored in space noise in bounded domains. We studied the long time behavior of their solutions with respect to the level λ of the noise. We also showed how the choice of the order $\beta \in (0, 1)$ of the fractional time derivative affects the growth and decay behavior of their solution. Further, we established continuity results in the time fractional parameter β. Finally, we showed the temporal Hölder continuity of the mild random field solution of a certain class of space-time fractional stochastic heat equation driven by space colored noise.

7.2 Future work

a) We will explore the effect of the level of the noise on the long time behavior of the second moment of the solution of the equation under consideration with Neumann boundary condition.

b) Nane and Tuan (2023+) studied the exact bound for the continuity of solutions of time fractional SPDE driven by space-time white noise in one space dimension $\mathbb{R}^1$ with respect to β parameter. We want to extend this results to higher dimensions and equations with other noises.

c) We will investigate spatial Hölder continuity of the mild random field solution u of space-time fractional stochastic heat equation (5.1) driven by space colored noise.

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