List Coloring in Graphs: Constructions Based on a Refined Scale of Choosability

by

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Abstract

A proper vertex coloring of a graph assigns colors to its vertices so that no two adjacent vertices receive the same color. List coloring is a variation of proper vertex coloring where each vertex is assigned a prescribed list of available colors. In 2020, Xuding Zhu introduced a generalization of list coloring called λ-choosability which makes use of integer partitions to categorize list assignments. λ-partitionability is another framework of list coloring that develops naturally out of λ-choosability. All λ-partitionable graphs are λ-choosable, but the converse is not true.

In this dissertation, we characterize all complete k-partite graphs which are only λ-choosable when λ is an integer partition which contains only 1's. We show several constructions of graphs which are λ-choosable but not λ-partitionable. We finally show progress towards constructing a counterexample whose purpose is to highlight a key difference between λ-choosability and λ-partitionability.

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Chapter 1

Introduction

Most attribute graph theory's beginnings to Leonhard Euler's solution of the *Seven Bridges of Königsberg* problem. The problem is whether or not there is a path through the historic city that crosses each of its seven bridges exactly once. Euler concluded that no such path existed. His solution laid the groundwork for our current study of structures called graphs.

A *graph* is a structure made up of points called *vertices* and connections between those vertices called *edges*. Two vertices are *adjacent* if they are connected by an edge. Those two vertices are *incident* with the edge that connects them. For a graph G, we call $V(G)$ its vertex set and $E(G)$ its edge set. A *simple graph* is a graph with no loops (an edge connecting a vertex to itself) nor multiple edges connecting the same pair of vertices; our focus is on simple graphs. In a simple graph, the *degree* of a vertex is the number of edges incident to it. Figure 1.1 shows two graphs; the one on the right is simple while the one on the left is not.

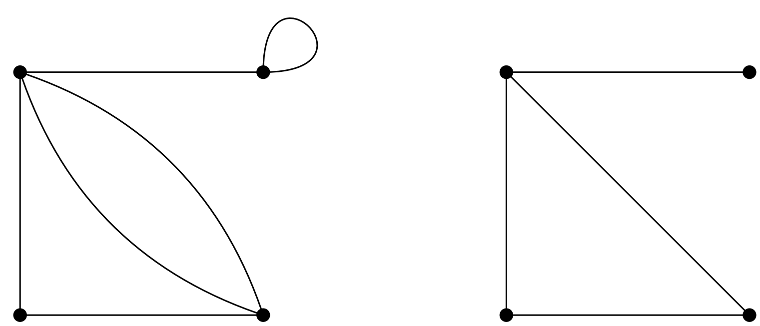


Figure 1.1: Example graphs.

1.1 Vertex coloring and choosability

The Four Color Theorem is a classic result in graph theory. It answers this question: how many colors are needed in order to color a map so that no two bordering regions receive the same color? As the theorem's name suggests, four colors are the most that will ever be needed.

To prove it, planar graphs are used as an abstraction of maps. Planar graphs are graphs which can be drawn on the plane such that there are no crossing edges. In the abstraction, vertices represent regions, and pairs of vertices corresponding to bordering regions are connected with an edge. The original question is now a matter of how many colors are needed to color the vertices of a planar graph so that no two adjacent vertices receive the same color.

Such a coloring of the vertices is called a *proper vertex coloring.* A graph G is *k-colorable* if it can be properly colored with k colors. The *chromatic number* $\chi(G)$ is the minimum k for which G is k-colorable. If $\chi(G) = k$, we say G is *k-chromatic.* The Four Color Theorem says that if G is planar, then $\chi(G) \leq 4$.

The *complete graph* K_n is a graph on n vertices such that every pair of vertices are adjacent. Naturally, $\chi(K_n) = n$.

When a graph is properly colored, its vertices are partitioned into independent sets called *color classes.* A *complete multipartite graph* G is a graph such that if the vertices of G are properly colored with $\chi(G)$ colors, then each vertex is adjacent to every vertex in a different color class. The color classes of a k-coloring of G are called *partite sets*, sometimes shortened to *parts.* If $a_1, a_2, \ldots, a_k$ are positive integers, then $K_{a_1,a_2,\ldots,a_k}$ is the complete k-partite graph where the i^{th} partite set has a_i vertices. A few examples are shown in Figure 1.2.

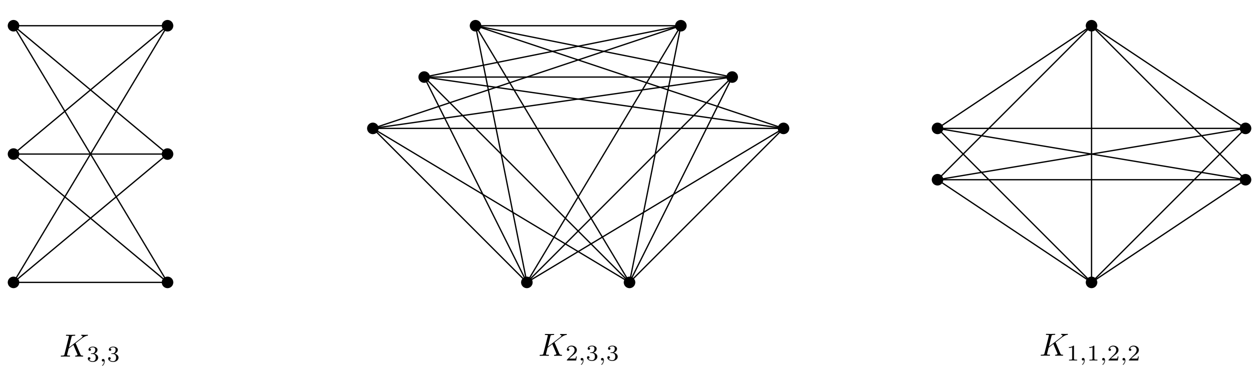


Figure 1.2: Complete multipartite graphs.

Choosability (also called *list coloring*) is a popular variation of proper vertex coloring introduced in 1976 by Vizing [16] and independently in 1979 by Erdös, Rubin, and Taylor [3] (referred to as ERT from now on). A *k-assignment* L of a graph G assigns to each vertex of G a set of k colors. G is *L-colorable* if the vertices of G can be properly colored with the colors assigned to them by L. G is *k-choosable* if it is L-colorable for all k-assignments L. The *choice number* $\mathrm{ch}(G)$ is the minimum k for which G is k-choosable.

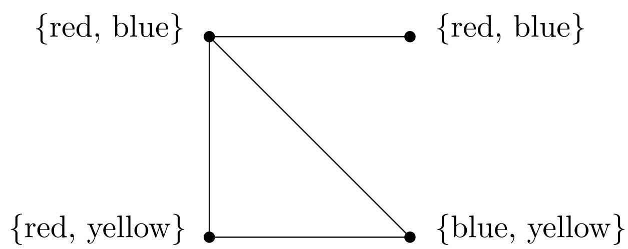


Figure 1.3: A 2-assignment of a graph.

ERT characterized all 2-choosable graphs using cycles, cores, and theta graphs. Examples of cycles are shown in Figure 1.4. The *core* of a graph is obtained by successively

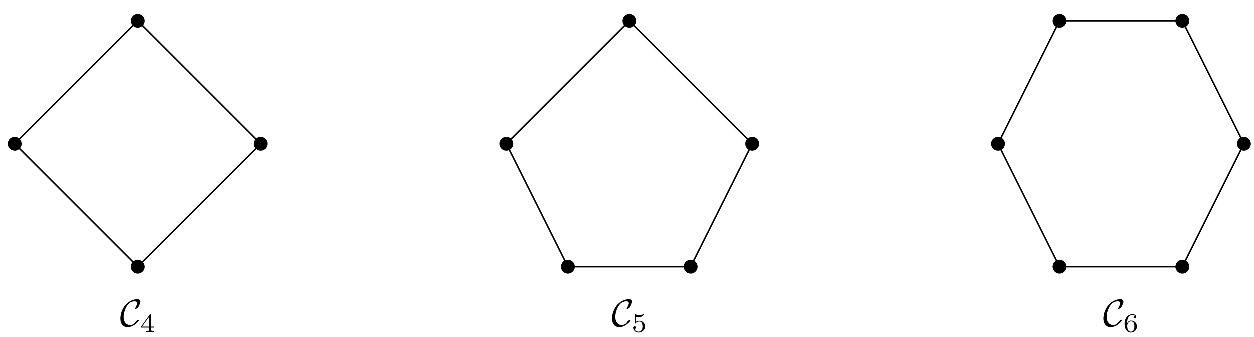
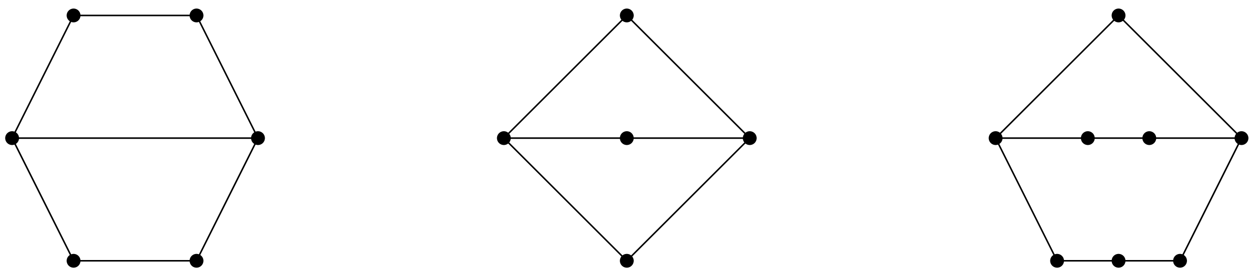


Figure 1.4: Cycles.

removing vertices of degree 1 along with its incident edge until there are no degree 1 vertices left. For example, the core of the graph in Figure 1.3 is $\mathcal{C}_3$, obtained by removing the single vertex of degree 1; the process stops after this single removal because all remaining vertices have degree 2 at that point.

The *theta graph* $\Theta_{a,b,c}$ with $a, b, c \geq 1$ is a graph made up of three disjoint paths of lengths a, b, and c where the starting vertices of each path are identified together and the ending vertices of each path are also identified together. Some sample theta graphs are shown in Figure 1.5. Theta graphs can resemble the Greek letter theta "Θ", hence their name.



$\Theta_{1,3,3}$ $\Theta_{2,2,2}$ $\Theta_{2,3,4}$

Figure 1.5: Theta graphs.

ERT showed that a graph is 2-choosable if and only if its core is one of the following: a single vertex, an even cycle, or the theta graph $\Theta_{2,2,2m}$ with $m \geq 1$.

If L assigns every vertex in G the same list, a proper coloring of L is equivalent to a normal proper vertex coloring. From this we notice $\chi(G) \leq \text{ch}(G)$ for all graphs. This inequality is often strict. For example, consider the complete bipartite graph $K_{3,3}$. Note that $\chi(K_{3,3}) = 2$. However, Figure 1.6 shows an uncolorable 2-assignment for $K_{3,3}$, meaning $\text{ch}(K_{3,3}) > 2$. It turns out $\text{ch}(K_{3,3}) = 3$.

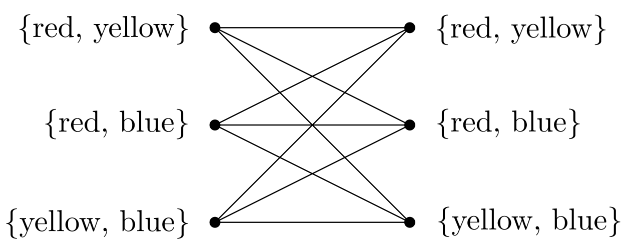


Figure 1.6: An uncolorable 2-assignment of $K_{3,3}$.

The difference between $\chi(G)$ and $\text{ch}(G)$ can be arbitrarily large. ERT [3] showed that if $m = \binom{2k-1}{k}$, then $\text{ch}(K_{m,m}) > k$. But of course, $\chi(K_{m,m}) = 2$ for all m.

For what graphs does $\chi(G) = \mathrm{ch}(G)$? Such a graph is called *chromatic-choosable*. ERT showed that complete multipartite graphs with parts of size at most 2 are chromatic-choosable. Ohba [14] gave a construction of chromatic-choosable graphs by joining together different graphs. If G_1 and G_2 are graphs, then their *join* $G_1 \vee G_2$ is obtained by requiring $V(G_1) \cap V(G_2) = \emptyset$ and then connecting every vertex of G_1 to every vertex of G_2. Ohba showed that for any graph G, there exists a non-negative integer N such that $G \vee K_n$ is chromatic-choosable for any integer $n \geq N$. In the same paper, Ohba conjectured that if $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic-choosable. This became known as the Ohba Conjecture. Noel, Reed, and Wu [13] proved the Ohba Conjecture to be true 13 years after it was posed.

Line graphs are of particular interest in studying choosability; they are constructed as follows. Start with a graph G. The vertex set of its line graph $\mathcal{L}(G)$ is the edge set of G, and two vertices in $\mathcal{L}(G)$ are adjacent if their corresponding edges in G share a vertex. All line graphs are conjectured to be chromatic-choosable.

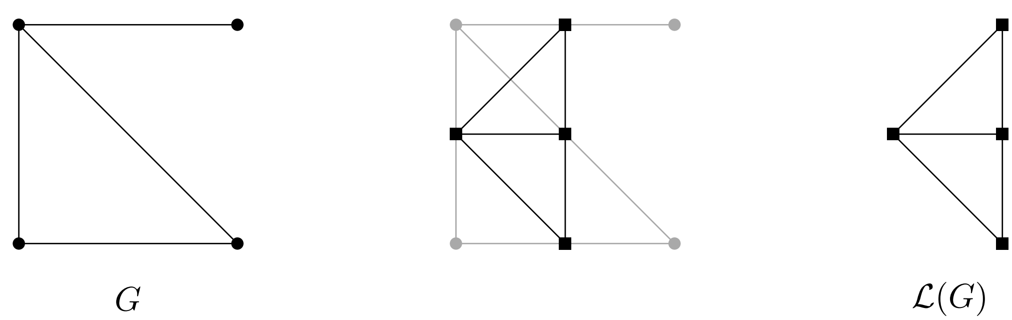


Figure 1.7: A graph G and its line graph $\mathcal{L}(G)$.

Conjecture 1.1 (The List Coloring Conjecture [1]). *If G is a graph and $\mathcal{L}$ is its line graph, then $\chi(\mathcal{L}) = \mathrm{ch}(\mathcal{L})$.*

This one of the most well known list coloring problems. It was first posed formally by Bollobás and Harris [1], and suggested independently by many others. In progress towards proving the conjecture, Galvin [4] showed that the line graphs of all bipartite graphs are chromatic-choosable. Häggkvist and Janssen [8] showed that $\mathcal{L}(K_n)$ is chromatic-choosable when n is odd. It remains unknown whether $\mathcal{L}(K_n)$ is chromatic-choosable for all even n. Borodin, Kostochka, and Woodall [2] showed that $\text{ch}(\mathcal{L}) \leq \left\lfloor \frac{3}{2}\Delta(\mathcal{L}) \right\rfloor$ for all line graphs $\mathcal{L}$ where $\Delta(\mathcal{L})$ is the maximum degree of $\mathcal{L}$. Kahn [10] showed that the conjecture holds asymptotically for all line graphs of simple graphs.

1.2 Zhu's refinement of choosability

We see that k-colorability and k-choosability are similar yet different frameworks to properly color graphs. In a recent paper, Zhu [18] introduced a refinement of choosability that unites colorability and choosability together within the same framework. This dissertation explores questions within this refinement.

Zhu's refinement categorizes list assignments using integer partitions. An *integer partition* λ of a positive integer k is a multiset of positive integers whose sum is k. For example, $\lambda = \{1, 2, 2, 2, 2, 2, 3, 3\}$ is an integer partition of 15. It's common to rewrite this example as $\lambda = \{1, 2^4, 3^2\}$, where the superscripts represent the multiplicities of each part.

For the integer partition $\lambda = \{k_1, k_2, \ldots, k_t\}$ of k, a λ-*assignment* of a graph G is a k-assignment L of G where the colors in $\bigcup_{v \in V(G)} L(v)$ can be partitioned into sets $C_1, C_2, \ldots, C_t$ so that for each $v \in V(G)$ and each $i \in \{1, 2, \ldots, t\}$, we have $|L(v) \cap C_i| = k_i$.

For example, consider the graph in Figure 1.8 with the given 5-assignment L. First,

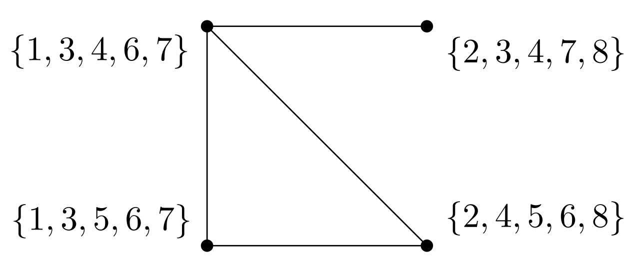


Figure 1.8: A 5-assignment L of a graph.

notice that $\bigcup_{v \in V(G)} L(v)$ (which from here on will be referred to as "the colors of L") can be partitioned into one set of colors $C_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Every list has exactly 5 colors from C_1, so L is categorized as a $\{5\}$-assignment.

Consider another partition of the colors of L. Let $C_1 = \{1, 2, 3, 4, 5\}$ and $C_2 = \{6, 7, 8\}$. Every list has exactly 3 colors from C_1 and exactly 2 colors from C_2. Thus, we call L a $\{2, 3\}$-assignment. Note that order doesn't matter in λ; we could equivalently say that L is a $\{3, 2\}$-assignment and relabel the C_i's. However, all integer partitions will be written in non-decreasing order here for consistency.

L is also a $\{1, 4\}$-assignment with color sets $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5, 6, 7, 8\}$. It's also a $\{1, 2, 2\}$-assignment with $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5\}$, $C_3 = \{6, 7, 8\}$. However, L is not a $\{1, 1, 3\}$-, $\{1, 1, 1, 2\}$-, or $\{1, 1, 1, 1, 1\}$-assignment.

As a reminder, a graph is k-choosable for some positive integer k if it is L-colorable for all k-assignments L. We now smoothly shift to integer partitions. A graph is λ-*choosable* for some integer partition λ if it is L-colorable for all λ-assignments L.

For k-colorability (and k-choosability), one can observe without much difficulty that every k-colorable (k-choosable) graph is k'-colorable (k'-choosable) if and only if $k \leq k'$. In other words, if $k \not\leq k'$, then there is a graph which is k-colorable (k-choosable) but not k'-colorable (k'-choosable).

Zhu defines a partial ordering on integer partitions that preserves this property for λ-choosablity. We say $\lambda \leq \lambda'$ if λ' can be obtained by a combination of subdividing and increasing parts of λ. We call λ' a *refinement* of λ if it is obtained only using subdivisions. The diagram in Figure 1.9 shows this partial order on the integer partitions of 4 and 5. The thick arrows represent refinements, and the skinny arrows represent part increases. Notice for example that $\{1, 3\} \leq \{1, 1, 1, 1\}$ and $\{2, 2\} \leq \{1, 1, 3\}$.

Theorem 1.2 (Zhu [18]). *Every λ-choosable graph is λ'-choosable if and only if $\lambda \leq \lambda'$.*

So if $\lambda \not\leq \lambda'$, then there is a graph which is λ-choosable but not λ'-choosable. In fact, more recently Gu and Zhu [7] strengthened this result. They showed that for $\lambda \not\leq \lambda'$ and any positive integer g, there is a graph of girth at least g which is λ-choosable but not λ'-choosable. The *girth* of a graph is the length of its smallest cycle.

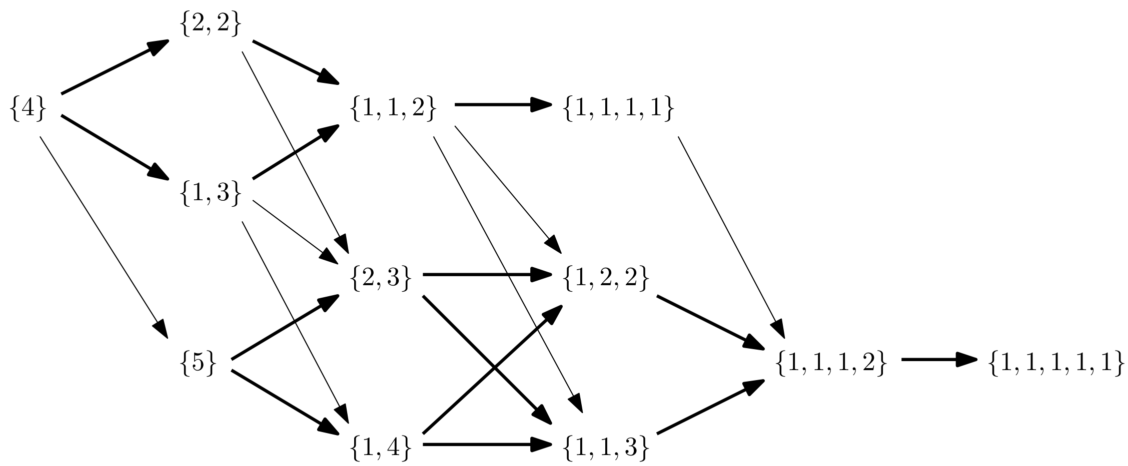


Figure 1.9: Zhu's partial ordering of integer partitions.

Recall how we showed L in Figure 1.8 is a $\{5\}$-assignment. It is no stretch to see that being $\{k\}$-choosable is equivalent to being k-choosable. Zhu proved the less obvious fact that being $\{1^k\}$-choosable is equivalent to being k-colorable. So λ-choosability conveniently houses k-colorability and k-choosability within the same framework.

There is another framework of choosability based on integer partitions that develops naturally out of Zhu's framework; we call it λ*-partitionability*. For an integer partition $\lambda = \{k_1, k_2, \ldots, k_t\}$ of k, a graph G is λ*-partitionable* if there exists a partition $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ of the vertex set $V(G)$ such that $G[V_i]$ (the induced subgraph) is k_i-choosable for $1 \le i \le t$. $\mathcal{V}$ is called a λ*-partition*. An example $\{1, 2\}$-partition is shown in Figure 1.10.

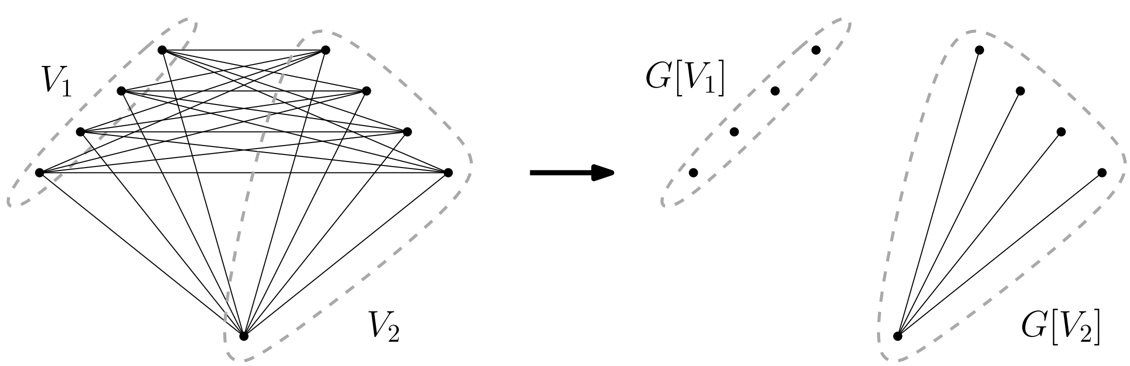


Figure 1.10: A $\{1, 2\}$-partition of $K_{1,4,4}$.

Zhu [18] proved that all planar graphs are $\{1,3\}$-choosable. The way he did it was by first showing that all planar graphs are $\{1,3\}$-partitionable, and then by showing that all $\{1,3\}$-partitionable graphs are $\{1,3\}$-choosable. The following observation generalizes the second part of his proof.

Observation 1.3. *If G is λ-partitionable, then G is λ-choosable.*

Proof. Let $\lambda = \{k_1, \ldots, k_t\}$ and let $V_1, \ldots, V_t$ be a λ-partition of $V(G)$. Let L be a λ-assignment of G with color groups $C_1, \ldots, C_t$. Because $G[V_i]$ is k_i-choosable, the vertices of V_i can be properly colored with the colors assigned to it from C_i. Because all C_i are disjoint, G is L-colorable and therefore λ-choosable since L is an arbitrary λ-assignment. $\square$

1.3 Overview of research presented

We look at three problems on λ-choosability in graphs. The first problem we explore is about k-chromatic graphs where the only integer partition λ of k for which they are λ-choosable is $\{1^k\}$. We call these *strictly k-colorable* graphs. We show some general observations about these graphs, and then characterize all strictly k-colorable complete multipartite graphs.

Next, we look at distinguishing λ-partitionability from λ-choosability. The idea was introduced to me by Greg Puleo [15] in unpublished work he did along with Dan Cranston. Observation 1.3 was already known to them prior to my introduction to the topic, and their interest was in constructing graphs which are λ-choosable but not λ-partitionable. They found two such constructions, and here we generalize these two constructions.

Finally, we explore how λ-partitionability likely does not follow the partial ordering of integer partitions that λ-choosability follows by searching for a counterexample. This study yields nine non-complete multipartite graphs which are $\{1,2\}$-choosable but not $\{1,2\}$-partitionable.

Chapter 2

Strictly k-colorable graphs

Kemnitz and Voigt [17] found a planar graph which is not L-colorable for a certain 2-common 4-assignment L; 2-*common* means L is a 4-assignment such that $\left|\bigcap_{v\in V(G)} L(v)\right| \geq 2$. Such an assignment is a $\{1,1,2\}$-assignment in Zhu's refinement language. Zhu observed that the existence of a non-$\{1,1,2\}$-choosable planar graph means that the Four Color Theorem is tight on his refined scale of choosability.

What does he mean by "tight"? The partition $\{1,1,2\}$ is a refinement of every partition of 4 except for $\{1,1,1,1\}$, as illustrated by Figure 2.1. Since Kemnitz and Voigt's planar graph is not $\{1,1,2\}$-choosable, it cannot be λ-choosable for any partition of 4 except $\lambda = \{1,1,1,1\}$ (and it is since all planar graphs are 4-colorable).

$$\begin{tikzcd}
 & \{1,3\} \arrow[dr] & & \\
\{4\} \arrow[ur] \arrow[dr] & & \{1,1,2\} \arrow[r] & \{1,1,1,1\} \\
 & \{2,2\} \arrow[ur] & &
\end{tikzcd}$$

Figure 2.1: Partitions of 4, with arrows revealing chains of refinements.

2.1 Definition and observations

This concept of tightness was not further explored by Zhu. For a partition λ of k, the idea of a graph being k-colorable and only λ-choosable strictly when $\lambda = \{1^k\}$ seems interesting. In a way, it's the "opposite" of being chromatic-choosable; a graph can be k-colorable but not even close to being k-choosable. Since these graphs exist, let's give them a name. A graph G is *strictly k-colorable* if the only integer partition λ of k for which G is λ-choosable is $\lambda = \{1^k\}$.

The following observation provides a nice alternative definition of strict k-colorability. It generalizes the idea used to explain the Four Color Theorem's tightness.

Observation 2.1. *A graph G is strictly k-colorable if and only if G is k-colorable and not $\{1^{k-2}, 2\}$-choosable.*

Proof. The forward implication follows from the definition of a strictly k-colorable graph. Let $\lambda = \{1^{k-2}, 2\}$. Suppose G is k-colorable and not λ-choosable. Let λ' be an integer partition of k such that $\lambda' \neq \{1^k\}$. Then λ is a refinement of λ'. So every λ-assignment of G is a λ'-assignment of G. Because G is not λ-choosable, G is not λ'-choosable. Therefore, G is strictly k-colorable. $\square$

So to show that G is strictly k-colorable, it suffices to show that G is k-colorable and then find a $\{1^{k-2}, 2\}$-assignment for which G is not properly colorable. This is nice because $\{1^{k-2}, 2\}$-assignments are relatively easy to deal with. Here are two more interesting observations about strictly k-colorable graphs.

Observation 2.2. *If G is strictly k-colorable, then $k = \chi(G)$.*

Proof. If $\chi(G) > k$, then G is not k-colorable, and therefore not strictly k-colorable. Suppose $\chi(G) = j < k$. Observe that $\{1^j\} \leq \{1^{k-2}, 2\}$. Because G is $\{1^j\}$-choosable, it is also $\{1^{k-2}, 2\}$-choosable, and therefore not strictly k-colorable. $\square$

So there is at most one positive integer k for which a graph G can be strictly k-colorable, that is $k = \chi(G)$.

Observation 2.3. *Suppose H is strictly k-colorable and $H \subseteq G$. Then G is strictly k-colorable if and only if $k = \chi(G)$.*

Proof. The forward is already proven. Suppose $k = \chi(G)$. Because H is not $\{1^{k-2}, 2\}$-choosable, neither is G. $\square$

So if you would like to show that some graph G is strictly k-colorable, it suffices to show that a subgraph $H \subseteq G$ with $\chi(H) = \chi(G)$ is strictly k-colorable. Put another way, if you show that some graph H is strictly k-colorable, then you get every k-chromatic graph containing H for free.

What can we say about specific values of k? If G is strictly 1-colorable, then G is an independent set. Because the only integer partition of 1 is $\{1\}$, all independent sets are strictly 1-colorable.

If G is strictly 2-colorable, then $\chi(G) = 2$ so G is bipartite. The only integer partitions of 2 are $\{1,1\}$ and $\{2\}$. So, a bipartite graph is strictly 2-colorable if and only if it is not 2-choosable. We discussed ERT's characterization of 2-choosable graphs in the introduction. Strictly 2-colorable graphs are exactly all bipartite graphs which do not fit that characterization.

The fun starts with $k = 3$; from here on, we will explore strict k-colorability only for $k \geq 3$.

2.2 Strictly k-colorable complete k-partite graphs

Complete multipartite graphs are typically of interest when studying choosability. We ask are there any strictly k-colorable complete k-partite graphs? The answer is yes, and we can describe *all* of them for every $k \geq 3$.

First, we start small. Can we find a strictly 3-colorable complete tripartite graph? Sure. Choose a sufficiently large $n \in \mathbb{N}$ such that $K_{n,n,n}$ has no hope of being $\{1,2\}$-choosable, and we have ourselves a fine example.

How large is sufficiently large? Well, $n = 2$ is too low. Recall from the introduction that ERT showed $K_{2,2,2}$ is chromatic-choosable.

What about $n = 3$? It turns out that $K_{3,3,3}$ is not $\{1,2\}$-choosable; see the uncolorable $\{1,2\}$-assignment L in Figure 2.2. To see that L is in fact a $\{1,2\}$-assignment, let $C_1 = \{0,1,2\}$ and let $C_2 = \{3\}$. To see that L is uncolorable, notice that the colors of C_1 mimic

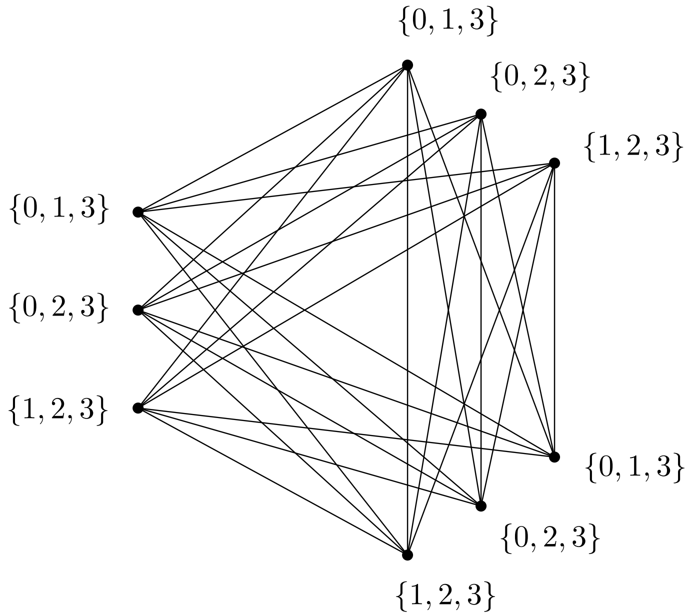


Figure 2.2: Uncolorable $\{1,2\}$-assignment of $K_{3,3,3}$.

the uncolorable 2-assignment of $K_{3,3}$ in Figure 1.6. At most one part of $K_{3,3,3}$ can be colored using only one color (namely the color 3). So at least five colors are needed in L, but only four are available. So $K_{3,3,3}$ is not L-colorable and is therefore strictly 3-colorable. By Observation 2.3, every complete tripartite graph whose smallest part size is at least 3 is strictly 3-colorable.

This result generalizes nicely for any $k \geq 3$ in the following lemma. A note on notation: it's common to use K_{a*b} as the complete b-partite graph with parts of size a. For example $K_{3*5} = K_{3,3,3,3,3}$ and $K_{5*3} = K_{5,5,5}$.

Lemma 2.4. *Let $k \geq 3$. K_{3*k} is strictly k-colorable.*

Proof. K_{3*k} is certainly k-colorable. Let $\lambda = \{1^{k-2}, 2\}$. It suffices to show that K_{3*k} is not λ-choosable. Let $V_1, V_2, \ldots, V_k$ be the partite sets of K_{3*k}. Define L to be the k-assignment shown in Figure 2.3 where $A = \{3, \ldots, k\}$.

Let $C_1 = \{0,1,2\}$ and $C_i = \{i+1\}$ for $2 \leq i \leq k-1$. Then observe $|L(v) \cap C_1| = 2$ and $|L(v) \cap C_i| = 1$ for all $v \in V(K_{3*k})$ and $2 \leq i \leq k-1$. So L is a λ-assignment to K_{3*k}.

$L(V_1)$	$L(V_2)$		$L(V_k)$
$\{0,1\} \cup A$	$\{0,1\} \cup A$		$\{0,1\} \cup A$
$\{0,2\} \cup A$	$\{0,2\} \cup A$	$\cdots$	$\{0,2\} \cup A$
$\{1,2\} \cup A$	$\{1,2\} \cup A$		$\{1,2\} \cup A$

Figure 2.3: k-assignment L of K_{3*k} where $A = \{3, \ldots, k\}$.

There are k partite sets and $k-1$ color groups. For $2 \leq i \leq k-1$, the single color in C_i can be used in a proper L-coloring on at most 1 partite set of K_{3*k}. The colors of C_1 cannot fully color 2 partite sets. Hence, all the color groups together can fully color at most $k-1$ partite sets simultaneously. So, K_{3*k} is not L-colorable, meaning it is not λ-choosable. Therefore, K_{3*k} is strictly k-colorable. □

It is worth noting that this lemma is also true for $k = 1, 2$, but since we already observed independent sets and bipartite graphs are solved, we only care for $k \geq 3$.

As mentioned before, we can describe *all* strictly k-colorable complete k-partite graphs. The way we do it is by finding a collection of subgraphs such that containing at least one as a subgraph is necessary and sufficient for a complete k-partite graph to be strictly k-colorable. "Sufficient" is simply shown by Observation 2.3; "necessary" is not so obvious.

We already encountered one family of subgraphs in Lemma 2.4. There are only two more, and we introduce them in the two lemmas that follow. First, we introduce a result by Hoffman and Johnson [9].

They showed that there is a unique uncolorable m-assignment L (up to relabeling) of $K_{m,n}$ when $n = m^m$. The lists of L in the m part are all pairwise disjoint, and the lists in the n part are exactly all m^m transversals of the lists in the m part. For $m = 2$, the unique uncolorable 2-assignment is shown in Figure 2.4. We call this the "unique bad 2-assignment of $K_{2,4}$."

Now on to the two remaining families of subgraphs. Their proofs will follow a pattern similar to Lemma 2.4.

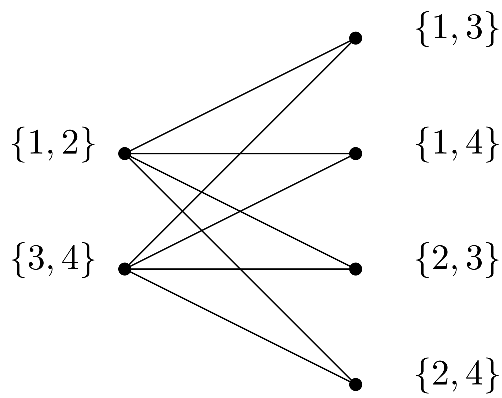


Figure 2.4: The unique bad 2-assignment of $K_{2,4}$.

Lemma 2.5. *Let $k \geq 3$. $K_{2,4,6*(k-2)}$ is strictly k-colorable.*

Proof. Let $G = K_{2,4,6*(k-2)}$. G is certainly k-colorable. Let $\lambda = \{1^{k-2}, 2\}$. It suffices to show that G is not λ-choosable. Let $V_1, V_2, \ldots, V_k$ be the partite sets of G such that $|V_1| = 2$, $|V_2| = 4$, and $|V_i| = 6$ for $3 \leq i \leq k$. Define L to be the k-assignment in Figure 2.5 on G where $A = \{5, \ldots, k+2\}$.

$L(V_1)$	$L(V_2)$	$L(V_3)$		$L(V_k)$
$\{1,2\} \cup A$	$\{1,3\} \cup A$	$\{1,3\} \cup A$		$\{1,3\} \cup A$
$\{3,4\} \cup A$	$\{1,4\} \cup A$	$\{1,4\} \cup A$		$\{1,4\} \cup A$
	$\{2,3\} \cup A$	$\{2,3\} \cup A$	$\cdots$	$\{2,3\} \cup A$
	$\{2,4\} \cup A$	$\{2,4\} \cup A$		$\{2,4\} \cup A$
		$\{1,2\} \cup A$		$\{1,2\} \cup A$
		$\{3,4\} \cup A$		$\{3,4\} \cup A$

Figure 2.5: k-assignment L of $K_{2,4,6*(k-2)}$ where $A = \{5, \ldots, k+2\}$.

Let $C_1 = \{1, 2, 3, 4\}$ and let $C_i = \{i + 3\}$ for $2 \leq i \leq k - 1$. Observe $|L(v) \cap C_1| = 2$ and $|L(v) \cap C_i| = 1$ for all $v \in V(G)$ and $2 \leq i \leq k-1$. Thus, L is a λ-assignment of G.

Notice there are k partite sets and $k-1$ color groups. In a proper L-coloring of G, each color group C_i with $2 \leq i \leq k-1$ can be seen on at most one partite set. This means in a proper L-coloring of G, at least two partite sets must only see colors from C_1. But between every pair of partite sets, their colors from C_1 contain the unique bad 2-assignment of $K_{2,4}$.

So any pair of partite sets cannot be properly colored using only C_1. So G is not L-colorable, meaning G is not λ-choosable. Therefore, G is strictly k-colorable. $\square$

Lemma 2.6. *Let $k \geq 3$. $K_{2,5*(k-1)}$ is strictly k-colorable.*

Proof. Let $G = K_{2,5*(k-1)}$. G is certainly k-colorable. Let $\lambda = \{1^{k-2}, 2\}$. It suffices to show that G is not λ-choosable. Let $V_1, V_2, \dots, V_k$ be the partite sets of G such that $|V_1| = 2$, and $|V_i| = 5$ for $2 \leq i \leq k$. Define L to be the k-assignment in Figure 2.6 on G where $A = \{5, \dots, k+2\}$.

$L(V_1)$	$L(V_2)$		$L(V_k)$
$\{1,2\} \cup A$	$\{1,3\} \cup A$		$\{1,3\} \cup A$
$\{3,4\} \cup A$	$\{1,4\} \cup A$		$\{1,4\} \cup A$
	$\{2,3\} \cup A$	$\cdots$	$\{2,3\} \cup A$
	$\{2,4\} \cup A$		$\{2,4\} \cup A$
	$\{1,2\} \cup A$		$\{1,2\} \cup A$

Figure 2.6: k-assignment L of $K_{2,5*(k-1)}$ where $A = \{5, \dots, k+2\}$.

Let $C_1 = \{1, 2, 3, 4\}$ and let $C_i = \{i + 3\}$ for $2 \leq i \leq k-1$. Observe $|L(v) \cap C_1| = 2$ and $|L(v) \cap C_i| = 1$ for all $v \in V(G)$ and $2 \leq i \leq k-1$. So L is a λ-assignment to G.

There are k partite sets and $k-1$ color groups. The color groups of size 1 can together color at most $k-2$ partite sets, leaving at least 2 partite sets left to be colored by C_1. There are 4 colors in C_1. At least two colors are needed to color any partite set with C_1.

Assume V_1 is not one of these remaining parts, otherwise the two remaining parts would contain the bad 2-assignment of $K_{2,4}$. If you completely color one of the remaining parts with only two colors from C_1, then you necessarily colored it with colors 1 and 2. Any remaining part has a vertex with only those colors remaining in its list, so that vertex will have no remaining colors. So G is not L-colorable. Therefore, G is strictly k-colorable. $\square$

Using these three subgraphs, we characterize all strictly k-colorable complete k-partite graphs.

Theorem 2.7. *Let $k \geq 3$ and let G be a complete k-partite graph. G is strictly k-colorable if and only if G contains at least one of K_{3*k}, $K_{2,4,6*(k-2)}$, or $K_{2,5*(k-1)}$ as a subgraph.*

Proof. The backwards implication follows from Observation 2.3 and Lemmas 2.4, 2.5, and 2.6. What's left to show is that containing one of the three subgraphs is necessary for G to be strictly k-colorable.

Let $G = K_{a_1,a_2,\ldots,a_k}$ such that $a_1 \leq a_2 \leq \cdots \leq a_k$. Let $V_1, \ldots, V_k$ be the corresponding partite sets of G such that $|V_i| = a_i$ for $1 \leq i \leq k$. Let $\lambda = \{1^{k-2}, 2\}$. Suppose G contains none of K_{3*k}, $K_{2,4,6*(k-2)}$, and $K_{2,5*(k-1)}$ as a subgraph. Then we have the following three cases:

Case 1: $a_1 = 1$.

In this case, the core of $G[V_1 \cup V_2]$ is a single vertex, so it is 2-choosable. So G is λ-partitionable, which means it is λ-choosable and therefore not strictly k-colorable.

Case 2: $a_1 = 2$ and $a_2 \leq 3$.

In this case, $G[V_1 \cup V_2] \subseteq K_{2,3}$, which is isomorphic to $\Theta_{2,2,2}$, which is 2-choosable. So G is λ-partitionable, which means it is λ-choosable and therefore not strictly k-colorable.

Case 3: $a_1 = 2$, $a_2 = 4$, and $a_3 \leq 5$.

It suffices to only consider $a_3 = 5$. Let L be a λ-assignment of G with color groups C_1, C_2, $\ldots$, C_{k-1} such that $|L(v) \cap C_1| = 2$ and $|L(v) \cap C_i| = 1$ for all $v \in V(G)$ and for $2 \leq i \leq (k-1)$. Color V_i with its colors from C_{i-1} for $3 \leq i \leq k$. If V_1 and V_2 can be properly colored with C_1, then we're done. If not, C_1 on V_1 and V_2 is the unique bad 2-assignment on $K_{2,4}$ as shown in Figure 2.7.

Uncolor V_3 with C_2 and color V_2 with C_2. If V_1 and V_3 can't be colored with C_1, then each transversal of the colors of C_1 in V_1 must be present on the vertices of V_3; i.e. C_1 on V_1 and V_3 contains the unique bad 2-assignment on $K_{2,4}$ as shown in Figure 2.8.

Note that a and b in Figure 2.8 are unknown colors from C_1. Uncolor V_2 with C_2 and color V_1 with C_2. If $1 \in \{a, b\}$, then we can color V_2 with $3, 4$ and V_3 with $1, 2$. We can do the same coloring if $2 \in \{a, b\}$. Otherwise, we can color V_2 with $1, 2$ and V_3 with

$L(V_1) \cap C_1$	$L(V_2) \cap C_1$
$\{1,2\}$	$\{1,3\}$
$\{3,4\}$	$\{1,4\}$
	$\{2,3\}$
	$\{2,4\}$

Figure 2.7: C_1 if V_1 and V_2 can't be colored.

$L(V_1) \cap C_1$	$L(V_2) \cap C_1$	$L(V_3) \cap C_1$
$\{1,2\}$	$\{1,3\}$	$\{1,3\}$
$\{3,4\}$	$\{1,4\}$	$\{1,4\}$
	$\{2,3\}$	$\{2,3\}$
	$\{2,4\}$	$\{2,4\}$
		$\{a,b\}$

Figure 2.8: C_1 if V_1 and V_3 also can't be colored.

$3,4,a$ since $a \neq 1,2$. Because L was arbitrary, G is λ-choosable, and therefore not strictly k-colorable. $\square$

Chapter 3

Graphs which are λ-choosable but not λ-partitionable

Chapter 1 introduced the definition of λ-partitions and we observed that every λ-partitionable graph is λ-choosable. It also mentioned that the converse is not true; i.e. there are graphs which are λ-choosable but not λ-partitionable for certain integer partitions λ. Constructing such graphs will be the exploration of this chapter.

Consider how λ-choosability and λ-partitionability compare conceptually. Both use integer partitions as a way to categorize list assignments and graphs. In λ-choosability, it is the list assignments of a graph which are directly categorized using integer partitions (e.g. a λ-assignment of the graph G). This categorization is in no way dependent upon the graphs they assign. A graph is then categorized as λ-choosable if it is properly colorable by all λ-assignments on it.

However, in λ-partitionability, graphs are directly categorized using integer partitions, not via categorized list assignments. While λ-partitionability was derived naturally out of λ-choosability, it is in no way dependent upon λ-choosability.

Dan Cranston and Greg Puleo [15] were the first (as far as I'm aware) to compare and distinguish λ-choosability and λ-partitionability. They found two constructions which are λ-choosable but not λ-partitionable in unpublished work from 2019.

We will use the shorthand λ-*cnp* to refer to graphs which are λ-$\underline{\text{c}}$hoosable but $\underline{\text{n}}$ot λ-$\underline{\text{p}}$artitionable for a given nontrivial integer partition λ, *nontrivial* meaning a partition of k which is not $\{k\}$ or $\{1^k\}$. We call these the two *trivial* partitions of k. Why? If a graph G is $\{k\}$-choosable, then you can trivially partition G into one k-choosable graph (itself). If G is $\{1^k\}$-choosable, then G is k-colorable, so color G with k colors, and each color class will induce a 1-choosable subgraph of G.

So every $\{k\}$-choosable or $\{1^k\}$-choosable graph will necessarily be $\{k\}$-partitionable or $\{1^k\}$-partitionable respectively. There are no $\{k\}$-cnp graphs and no $\{1^k\}$-cnp graphs. We only consider nontrivial integer partitions.

3.1 Examples and observations

Both constructions that Cranston and Puleo found are complete multipartite graphs. Both proofs rely on knowing that K_{m,m^m} has a unique uncolorable m-assignment (introduced and utilized in the last chapter). Here is the first λ-cnp construction that they found.

Theorem 3.1 (Cranston and Puleo [15]). *For $t \geq 2$, K_{t,t^t,t^t} is $\{1,t\}$-cnp.*

Proof. Let $G = K_{t,t^t,t^t}$. Removing any independent set from G yields a graph containing K_{t,t^t}, which is not t-choosable. So G is not $\{1,t\}$-partitionable.

Let L be a $\{1,t\}$-assignment of G with disjoint color groups C_1, C_2 such that $|L(v) \cap C_1| = 1$ and $|L(v) \cap C_2| = t$ for all $v \in V(G)$. Let V_1 be the part of size t, and let V_2, V_3 be the parts of size t^t. $G[V_1 \cup V_2]$ and $G[V_1 \cup V_3]$ are isomorphic to K_{t,t^t}, which is not t-choosable. If either of these induced subgraphs is C_2-colorable, then we're done (Note, "C_2-colorable" means the induced subgraph can be properly colored exclusively with the colors on their lists from C_2).

Suppose neither of them are C_2-colorable. Up to relabeling, there is only one bad list assignment of K_{t,t^t}. In this unique bad assignment, there are pairwise disjoint color sets $D_1, \ldots, D_t \subset C_2$ with each $|D_i| = t$ such that $D_1, \ldots, D_t$ are the C_2 parts of the lists on V_1 and all t^t transversals of $D_1, \ldots, D_t$ are the C_2 parts of the lists on V_2 and V_3. Color V_1 with C_1, color V_2 with colors from D_1, and color V_3 with colors from D_2. Because L is arbitrary, G is $\{1,t\}$-choosable. $\square$

Right away, we have an infinite family of λ-cnp graphs. Here's the other construction that they found.

Theorem 3.2 (Cranston and Puleo [15]). *$K_{3,4,4}$ and $K_{4,4,4}$ are $\{2,2\}$-cnp.*

Proof. A maximum 2-choosable induced subgraph of $K_{3,4,4}$ or $K_{4,4,4}$ has 5 vertices (either $K_{1,4}$ or $K_{2,3}$; see ERT [3]). Because both $K_{3,4,4}$ and $K_{4,4,4}$ have more than 10 vertices, neither can be partitioned into two 2-choosable graphs. Therefore, neither are $\{2, 2\}$-partitionable.

To show they are $\{2, 2\}$-choosable, it suffices to only consider $K_{4,4,4}$, so let $G = K_{4,4,4}$. Let V_1, V_2, V_3 be the parts of G. Let L be a $\{2, 2\}$-assignment of G with disjoint color groups C_1, C_2 such that $|L(v) \cap C_1| = |L(v) \cap C_2| = 2$ for all $v \in V(G)$.

Case 1: There are two nonadjacent vertices v and w such that $L(v) \cap L(w) \neq \emptyset$. WLOG, we assume $v, w \in V_1$ and $L(v) \cap L(w) \cap C_1 \neq \emptyset$. Let $X = \{v, w\}$ and let $Y = V_1 \setminus X$. Notice $G[X \cup V_2]$ and $G[X \cup V_3]$ are C_1-colorable since the unique bad 2-assignment on $K_{2,4}$ requires the lists on the part of size two to be disjoint. If either $G[Y \cup V_2]$ or $G[Y \cup V_3]$ are C_2-colorable, then we're done. Suppose they are not C_2-colorable. We may assume the lists on Y from C_2 are $\{1, 2\}$ and $\{3, 4\}$, and we may assume the lists on V_2 and V_3 from C_2 are $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{2, 4\}$ (this is the unique bad 2-assignment on $K_{2,4}$). Then $G[V_2 \cup V_3]$ is C_2-colorable by choosing $1, 2$ in V_2 and $3, 4$ in V_3. We can finish by coloring V_1 with C_1.

Case 2: $L(v) \cap L(w) = \emptyset$ for all nonadjacent vertices v and w.
Let X, Y be any partition of V_1 with $|X| = |Y| = 2$. Then $G[X \cup V_2]$ is C_1-colorable and $G[Y \cup V_3]$ is C_2-colorable because neither exhibit the unique bad 2-assignment of $K_{2,4}$.

So G is L-colorable. Because L is arbitrary, G is $\{2, 2\}$-choosable. $\square$

Note that Kierstead [12] showed $\mathrm{ch}(K_{4,4,4}) = 4$. Because $\{4\} \leq \{2, 2\}$, $K_{4,4,4}$ is necessarily $\{2, 2\}$-choosable. I included Puleo and Cranston's proof of $\{2, 2\}$-choosability for its insight.

When Puleo was my advisor, one of the first tasks he gave me was to see if I could find any generalizations of these λ-cnp graphs which he and Cranston found. I noticed a generalization of Theorem 3.1; if you join an independent set of size t^t to K_{t,t^t,t^t}, then you get K_{t,t^t*3} which is $\{1, 1, t\}$-choosable. In fact, joining any arbitrary amount of independent sets of size t^t follows this pattern: for $m, t \geq 2$, K_{t,t^t*m} is $\{1^{m-1}, t\}$-cnp (this is Corollary 3.4.2).

The idea of using graph joins had already been considered by Zhu. A note on notation: we use $\|$ to be the operator which concatenates finite multisets.

Observation 3.3 (Zhu [18]). *If G_i is λ_i-choosable for $i \in \{1, 2, \ldots, q\}$, then $\bigvee_{i=1}^{q} G_i$ is $(\|_{i=1}^{q} \lambda_i)$-choosable.*

For example, if G is $\{1, 3\}$-choosable and H is $\{1, 2\}$-choosable, then $G \vee H$ is $\{1, 1, 2, 3\}$-choosable. This makes a generalization for joining independent sets to any λ-cnp graph possible.

Lemma 3.4. *If G is λ-cnp, then $G \vee I$ is $(\lambda \,\|\, \{1\})$-cnp where I is an independent vertex set of size at least $\alpha(G)$, the size of a maximum independent set in G.*

Proof. It suffices to let I be an independent set of size $\alpha(G)$. Let $G' = G \vee I$. Let $\lambda' = \lambda \| \{1\}$. By Observation 3.3, G' is λ'-choosable. Let I' be an independent set of G'. Let $H' = G' \setminus I'$. We need to show that H' is not λ-partitionable. To do this, it suffices to show that $G \subseteq H'$ because G is not λ-partitionable.

In G', every vertex of I is adjacent to every vertex of G, so either $V(I') \subseteq V(I)$ or $V(I') \subseteq V(G)$. If the former is true, then seeing $G \subseteq H'$ is immediate.

Suppose instead that $V(I') \subseteq V(G)$. If $|E(G)| = \emptyset$, then $G \subseteq I \subseteq H'$. Suppose $|E(G)| \neq \emptyset$. To show that $G \subseteq H'$, we define an injection φ from $V(G)$ to $V(H')$ and show that if $uv \in E(G)$, then $\varphi(u)\varphi(v) \in E(H')$. Let $n = |V(G)|$ and let $\alpha = \alpha(G)$. Let $V(G) = \{v_i\}_1^n$ and let $V(I) = \{w_i\}_1^{\alpha}$. Let $|I'| = m \leq \alpha$ and WLOG let $V(I') = \{v_i\}_1^m \subseteq \{v_i\}_1^n$ (relabel the vertices of G if needed). Then $V(H') = \{v_i\}_{m+1}^n \cup \{w_i\}_1^{\alpha}$. Define $\varphi : V(G) \to V(H')$ so that

$$\varphi(v_i) = \begin{cases} w_i, & 1 \leq i \leq m \\ v_i, & m+1 \leq i \leq n \end{cases}$$

The function φ is well defined and injective because every vertex in G is mapped to one unique vertex in H'. Let $v_i v_j \in E(G)$ where $i < j$. We know $j \not\leq m$, because then $v_i, v_j \in I'$ which is an independent set. If $i \leq m$, then $\varphi(v_i) = w_i$ and $\varphi(v_j) = v_j$, and $w_i v_j \in E(H')$ because every vertex of I is adjacent to every vertex of G. If $i > m$, then $\varphi(v_i) = v_i$ and $\varphi(v_j) = v_j$, and $v_i v_j \in E(H')$ because only edges in G' with endpoints in I' were removed to obtain H'. So $G \subseteq H'$. □

Corollary 3.4.1. *If G is λ-cnp, then $G \vee \left(\bigvee_1^k I_\alpha\right)$ is $(\lambda \parallel \{1^k\})$-cnp.*

Corollary 3.4.2. *$K_{t,t^t * m}$ is $\{1^{m-1}, t\}$-cnp for $m, t \geq 2$.*

A useful analog of Observation 3.3 can be observed for λ-partitionability.

Observation 3.5. *If G_i is λ_i-partitionable for $i \in \{1, 2, \ldots, q\}$, then $\bigvee_{i=1}^q G_i$ is $\left(\|_{i=1}^q \lambda_i\right)$-partitionable.*

Proof. Let $G = \bigvee_{i=1}^q G_i$ and $\lambda = \|_{i=1}^q \lambda_i$. Let $\mathcal{V}_i$ be a λ_i-partition of G_i for $1 \leq i \leq q$. Then $\bigcup_{i=1}^k \mathcal{V}_i$ is λ-partition of G. □

It would be really nice if the inverse of Observation 3.5 was true; i.e. "If G_1 is not λ_1-partitionable and G_2 is not λ_2-partitionable, then $G_1 \vee G_2$ is not $(\lambda_1 \parallel \lambda_2)$-partitionable." This would allow cnp graphs to be constructed by naively joining smaller ones.

However, here is a counterexample.

Counterexample. $K_{2,2,2,4}$ is not $\{2, 2\}$-partitionable and $\mathcal{C}_8 \vee \mathcal{C}_8$ is not $\{1, 3\}$-partitionable, but $K_{2,2,2,4} \vee (\mathcal{C}_8 \vee \mathcal{C}_8)$ is $\{1, 2, 2, 3\}$-partitionable.

Proof. Let $G_1 = K_{2,2,2,4}$. There are three maximal 2-choosable subgraphs of G_1 we can choose from first: $K_{1,4}$, $K_{2,3}$, or $K_{2,2}$. Removing either of the first two will result in a 3-partite graph which is not 2-choosable. Removing the last could also result in a 3-partite graph, or it could leave behind $K_{2,4}$, neither of which are 2-choosable. So G_1 is not $\{2, 2\}$-partitionable.

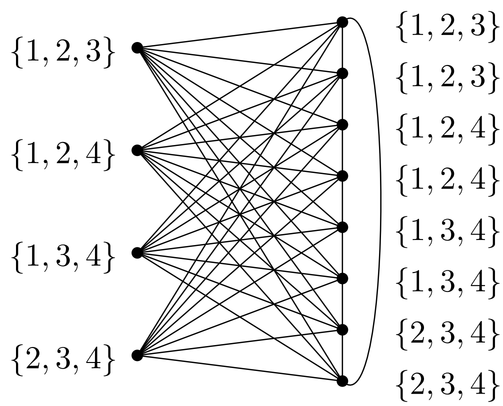


Figure 3.1: Uncolorable 3-assignment of $I_4 \vee \mathcal{C}_8$.

Let $G_2 = \mathcal{C}_8 \vee \mathcal{C}_8$. Removing any independent set from G_2 leaves a graph containing $I_4 \vee \mathcal{C}_8$. Figure 3.1 shows an uncolorable 3-assignment on $I_4 \vee \mathcal{C}_8$. So G_2 is not $\{1,3\}$-partitionable.

$V(G_1)$ can be partitioned into $\{V_1, V_2\}$ such that $G_1[V_1] = I_4$ and $G_1[V_2] = K_{2,2,2}$. Because $K_{2,2,2}$ is 3-choosable (ERT), G_1 is $\{1,3\}$-partitionable. Furthermore, $V(G_2)$ can be partitioned into $\{W_1, W_2\}$ such that $G_2[W_1] = G_2[W_2] = C_8$, which is 2-choosable. So G_2 is $\{2,2\}$-partitionable. So by Observation 3.5, $G_1 \vee G_2$ is $\{1,2,2,3\}$-partitionable. □

I was unable to find a counterexample with two complete multipartite graphs. I suspect there is no such counterexample.

Conjecture 3.6. *Let G_1 and G_2 be complete multipartite graphs. If G_1 is λ_1-cnp and G_2 is λ_2-cnp, then $G_1 \vee G_2$ is $(\lambda_1 \| \lambda_2)$-cnp.*

For now, constructing λ-cnp graphs is not automatically achieved by joining together other graphs. However, joins remain a useful tool for finding λ-cnp constructions as demonstrated in the next section.

3.2 Complete multipartite λ-cnp graphs with part size 4

Here we generalize Theorem 3.2 for any arbitrary amount of 2's in λ. This generalization characterizes all complete multipartite graphs with part size 4 which are $\{2^b\}$-cnp for $b \geq 2$.

Theorem 3.7. *For $b \geq 2$, K_{4*r} is $\{2^b\}$-cnp if and only if*

$$\frac{5b+1}{4} \leq r \leq \frac{3b}{2}.$$

To prove this, we use four lemmas that handle the choosability and partitionability portions separately.

Lemma 3.8. *For $b \geq 2$, K_{4*r} is $\{2^b\}$-choosable when $r = \lfloor 3b/2 \rfloor$.*

Proof. Let $r = \lfloor 3b/2 \rfloor$. We'll consider two cases, when b is either even or odd.

Suppose $b = 2k$ for some $k \in \mathbb{N}$. Then

$$r = \left\lfloor \frac{3(2k)}{2} \right\rfloor = 3k.$$

So

$$K_{4*r} = K_{4*3k} = \bigvee_1^k K_{4*3}.$$

By Theorem 3.2 and Observation 3.3, K_{4*r} is $\left(\Big\|_1^k \{2,2\}\right)$-choosable. Note

$$\Big\|_1^k \{2,2\} = \{2^{2k}\} = \{2^b\},$$

so K_{4*r} is $\{2^b\}$-choosable.

Now suppose $b = 2k+1$ for some $k \in \mathbb{N}$. Then

$$r = \left\lfloor \frac{3(2k+1)}{2} \right\rfloor = \left\lfloor 3k + \frac{3}{2} \right\rfloor = 3k+1.$$

So

$$K_{4*r} = K_{4*(3k+1)} = K_{4*3k} \vee I_4 = \bigvee_1^k (K_{4*3}) \vee I_4$$

where I_4 is an independent set of size 4. By Theorem 3.2 and Observation 3.3, K_{4*r} is $\{2^{2k}, 1\}$-choosable. Because $\{2^{2k}, 1\} \leq \{2^{2k}, 2\}$, K_{4*r} is $\{2^{2k}, 2\}$-choosable, and $\{2^{2k}, 2\} = \{2^{2k+1}\} = \{2^b\}$. $\square$

Lemma 3.9. *For $b \geq 2$, K_{4*r} is not $\{2^b\}$-choosable when $r = \lfloor 3b/2 \rfloor + 1$.*

Proof. Let $r = \lfloor 3b/2 \rfloor + 1$. It suffices to prove that the subgraph $H = K_{3*r}$ is not $\{2^b\}$-choosable. Let $V_1, \ldots, V_r$ be the parts of size 3 in H. Let L be a $(2b)$-assignment of H defined by

$$L(v_{i,1}) = \{x_1, y_1, x_2, y_2, \ldots, x_b, y_b\}$$

$$L(v_{i,2}) = \{x_1, z_1, x_2, z_2, \ldots, x_b, z_b\}$$

$$L(v_{i,3}) = \{y_1, z_1, y_2, z_2, \ldots, y_b, z_b\}$$

where $v_{i,1}, v_{i,2}, v_{i,3} \in V_i$ for $1 \leq i \leq r$. Let $C_j = \{x_j, y_j, z_j\}$ for $1 \leq j \leq b$. Then $|C_j \cap L(v)| = 2$ for all $v \in V(H)$ and $1 \leq j \leq b$. So L is a $\{2^b\}$-assignment of H. Notice that any proper L-coloring of H must use at least two colors on each V_i. So at least $2r$ colors are needed to properly color H since there are r parts. There are $3b$ total colors available. So if H is L-colorable, then $2r \leq 3b$.

However, I claim $2r > 3b$. Observe that $2r = 2\lfloor 3b/2 \rfloor + 2$. We'll look at two cases; either b is even or it's odd. Suppose $b = 2k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} 2r &= 2(3k) + 2, \\ &= 3(2k) + 2, \\ &= 3b + 2, \\ &> 3b. \end{aligned}$$

Now suppose $b = 2k + 1$ for some $k \in \mathbb{N}$. Then

$$
\begin{aligned}
2r &= 2\left\lfloor \frac{3(2k+1)}{2} \right\rfloor + 2, \\
&= 2\left\lfloor 3k + \frac{3}{2} \right\rfloor + 2, \\
&= 2(3k+1) + 2, \\
&= 6k + 4, \\
&= 3(2k+1) + 1, \\
&= 3b + 1, \\
&> 3b.
\end{aligned}
$$

So H is not L-colorable, and therefore not $\{2^b\}$-choosable. $\square$

Lemma 3.10. *For $b \geq 2$, K_{4*r} is not $\{2^b\}$-partitionable when $r = \lceil (5b+1)/4 \rceil$.*

Proof. Let $r = \lceil (5b+1)/4 \rceil$. A maximum 2-choosable induced subgraph of K_{4*r} has 5 vertices (either $K_{1,4}$ or $K_{2,3}$). It follows that if K_{4*r} is $\{2^b\}$-partitionable, then it must be true that $|V(K_{4*r})| \leq 5b$. However,

$$
\begin{aligned}
|V(K_{4*r})| &= 4r, \\
&= 4 \cdot \left\lceil \frac{5b+1}{4} \right\rceil, \\
&\geq 4 \cdot \frac{5b+1}{4}, \\
&> 5b.
\end{aligned}
$$

Therefore, K_{4*r} is not $\{2^b\}$-partitionable. $\square$

Lemma 3.11. *For $b \geq 2$, K_{4*r} is $\{2^b\}$-partitionable when $r = \lceil (5b+1)/4 \rceil - 1$.*

Proof. Let $r = \lceil (5b+1)/4 \rceil - 1$. Use the division algorithm to find $k, c \in \mathbb{N}$ such that $b = 4k + c$ with $0 \leq c \leq 3$. I claim $r = 5k + c$. Observe that

$$\begin{aligned} r &= \left\lceil \frac{5b+1}{4} \right\rceil - 1, \\ &= \left\lceil \frac{5(4k+c)+1}{4} \right\rceil - 1, \\ &= 5k + \left\lceil \frac{5c+1}{4} \right\rceil - 1. \end{aligned}$$

It is quickly verifiable that $\lceil (5c+1)/4 \rceil - 1 = c$ for $0 \leq c \leq 3$. So $r = 5k + c$. So,

$$K_{4*r} = K_{4*(5k+c)} = K_{4*5k} \vee K_{4*c} = \left(\bigvee_{1}^{k} K_{4*5} \right) \vee K_{4*c}.$$

Note if $c = 0$ then K_{4*0} is the null graph, and if $c = 1$ then $K_{4*1} = I_4$. Figure 3.2 shows that K_{4*5} is $\{2^4\}$-partitionable. Note that K_{4*c} is $\{2^c\}$-partitionable since it's c-colorable.

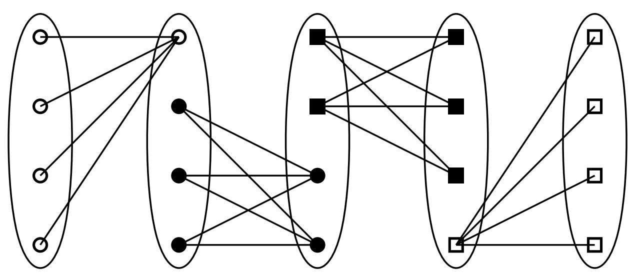


Figure 3.2: A $\{2, 2, 2, 2\}$-partition of K_{4*5}.

By Obersvation 3.5, K_{4*r} is $\left[\left(\|_{1}^{k} \{2^4\} \right) \| \{2^c\} \right]$-partitionable, and

$$\left(\Big\|_{1}^{k} \{2^4\} \right) \| \{2^c\} = \{2^{4k}\} \| \{2^c\} = \{2^{4k+c}\} = \{2^b\}.$$

Therefore, K_{4*r} is $\{2^b\}$-partitionable. $\square$

Now we tie up Theorem 3.7.

Proof of Theorem 3.7. Let $G = K_{4*r}$ and let $\lambda = \{2^b\}$. If $(5b+1)/4 \leq r \leq 3b/2$, then G is λ-cnp by Lemmas 3.8 and 3.10. If $r < (5b+1)/4$, then G is λ-partitionable by Lemma 3.11, and thus not λ-cnp. If $r > 3b/2$, then G is not λ-choosable by Lemma 3.9, and thus not λ-cnp. $\square$

Chapter 4

More on comparing λ-choosability and λ-partitionability

Each λ-cnp graph looked at so far has been complete multipartite. What about non-complete multipartite graphs? We show examples of such graphs in this chapter.

However, these examples come from attempting to solve a different problem. Puleo saw no reason for λ-partitionability to behave the way that λ-choosability does on the partial ordering of integer partitions that Zhu gives. Recall the partial ordering: if λ and λ' are integer partitions of k and k' respectively where $k \leq k'$, then we say $\lambda \leq \lambda'$ if λ' can be obtained by a combination of subdividing and increasing parts of λ. Every λ-choosable graph is λ'-choosable if and only if $\lambda \leq \lambda'$ [18].

Puleo suspected that λ-partitionability does not behave this way. The goal is to find a counterexample to show it. The first two sections of this chapter show the work done in my master's project. The work is included here because it was not previously written up and is important background for the work shown in the third section. The third section shows progress made beyond my master's project.

4.1 Constructing a counterexample

As far as we could tell, a simplest counterexample would be a graph which is $\{3\}$-partitionable but not $\{1,2\}$-partitionable, noting that $\{3\} \leq \{1,2\}$. Being $\{3\}$-partitionable is the same as being 3-choosable. It seems plausible that a counterexample should exist because otherwise every 3-choosable graph can be partitioned into an independent set and a 2-choosable graph, which seems unlikely.

We began our search in the familiar territory of complete multipartite graphs. However, Puleo discovered that there is no counterexample among complete multipartite graphs.

Lemma 4.1 (Puleo [15]). *Every 3-choosable complete multipartite graph is $\{1,2\}$-partitionable.*

Proof. Let G be a 3-choosable complete multipartite graph. G must have no more than three parts, otherwise G is not 3-colorable and certainly not 3-choosable. If G has less than three parts, then it's either an independent set or a complete bipartite graph, both of which are trivially $\{1, 2\}$-partitionable.

So assume G has exactly three parts. Say $G = K_{p,q,r}$ with $p \leq q \leq r$. We know either $p = 1$ or $p = 2$ because if $p \geq 3$, then $K_{3,3,3} \subseteq G$. Kierstead [11] showed that $\mathrm{ch}(K_{3,3,3}) = 4$, so G would not be 3-choosable.

If $p = 1$, then $K_{p,q}$ is a star which is 2-choosable. Therefore, G is $\{1, 2\}$-partitionable.

If $p = 2$ and $q \leq 3$, then $K_{p,q} \subseteq K_{2,3}$ which is 2-choosable. Therefore, G is $\{1, 2\}$-partitionable.

We cannot have $p = 2$ and $q \geq 4$ because then $K_{2,4,4} \subseteq G$ and $K_{2,4,4}$ is not 3-choosable. To see this, it suffices to show that $K_{1,4,4}$ is not 3-choosable. Let V_1, V_2, V_3 be the parts of size 1, 4, and 4 respectively of $K_{1,4,4}$. Let L be the 3-assignment of $K_{1,4,4}$ defined such that

$$L(V_1) = \{\{0, 1, 2\}\},$$
$$L(V_2) = L(V_3) = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}.$$

A proper L coloring of $K_{1,4,4}$ would require all four colors to be used for V_2 and V_3, leaving no available colors for V_1. Hence $K_{1,4,4}$ is not 3-choosable, so neither is $K_{2,4,4}$. Therefore, every 3-choosable complete multipartite graphs is also $\{1, 2\}$-partitionable. □

So any counterexample for our choice of $\lambda = \{1, 2\}$ will not be complete multipartite. When Puleo emailed me the initial proof of this, he also shared an idea to get around this problem. He suggested that perhaps a complete multipartite graph with an edge removed could be a counterexample. $K_{3,3,3}$ was one of the important graphs in his proof above, so we started there.

$K_{3,3,3}$ is neither 3-choosable nor $\{1, 2\}$-partitionable. My strategy to find a subgraph of $K_{3,3,3}$ which is 3-choosable but not $\{1, 2\}$-partitionable was to remove as many edges from

$K_{3,3,3}$ as possible while maintaining its property of not being $\{1,2\}$-partitionable. The hope was that this would be enough to make it 3-choosable.

How many edges can you remove from $K_{3,3,3}$ without it becoming $\{1,2\}$-partitionable? First, notice that removing any independent set from $K_{3,3,3}$ leaves a graph containing $K_{3,3}$ as a subgraph, which is not 2-choosable. The answer to the question depends on how many edges can we remove from $K_{3,3}$ without it becoming 2-choosable.

Recall that ERT showed that a graph is 2-choosable if and only if its core is a single vertex, an even cycle, or the theta graph $\Theta_{2,2,2m}$ with $m \geq 1$. We'll use this to figure out how many edges we can remove.

If we remove one edge from $K_{3,3}$, then it contains $\Theta_{1,3,3}$ which is not 2-choosable. If we remove a second edge which is adjacent to the first edge we removed, then the core is $\Theta_{2,2,2}$, so it *is* 2-choosable.

If we remove a second edge *independent* to the first, then we are left with $\Theta_{1,3,3}$ which is not 2-choosable (see Figure 4.1). If we remove any third edge, the core of the resulting

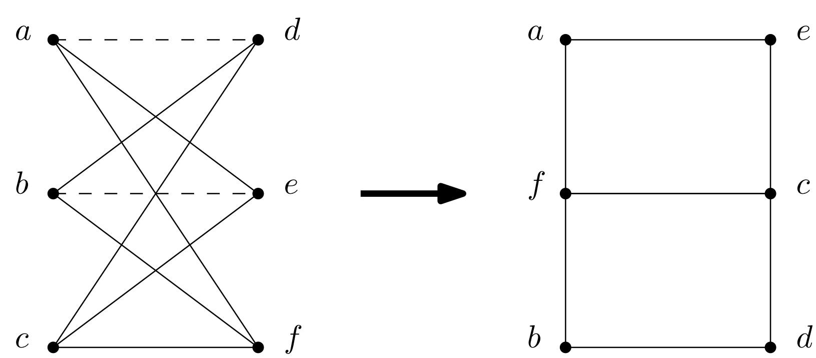


Figure 4.1: Removing two independent edges from $K_{3,3}$ yields $\Theta_{1,3,3}$.

graph will be an even cycle, making it 2-choosable. So, the most edges we can remove from $K_{3,3}$ without making it 2-choosable is two independent edges.

This means that for $K_{3,3,3}$, removing at most two independent edges between any pair of its parts is the best we can hope to do. If we remove a pair of independent edges between each pair of parts of $K_{3,3,3}$, then that's six removed edges (two edges which have been removed from different pairs of parts aren't necessarily independent). Let $\mathcal{G}$ be the class of graphs obtained by removing six such edges from $K_{3,3,3}$. Figure 4.2 shows examples of graphs in $\mathcal{G}$, where dashed lines represent edges removed from $K_{3,3,3}$.

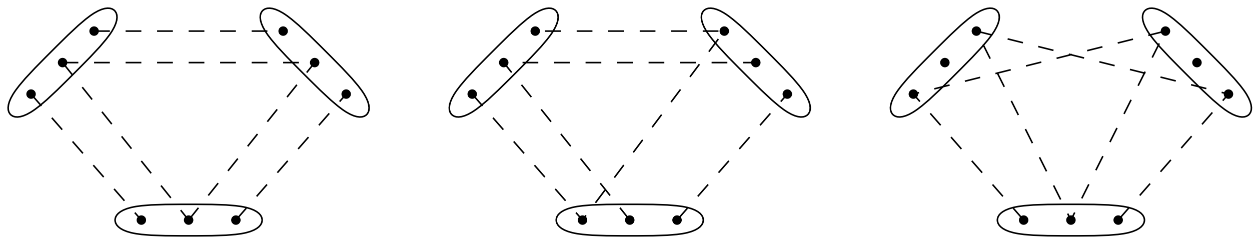


Figure 4.2: Three graphs in $\mathcal{G}$.

Lemma 4.2. *Let $G \in \mathcal{G}$. G is not $\{1,2\}$-partitionable.*

Proof. Let V_1, V_2, V_3 be the parts of G which correspond to the three partite sets of $K_{3,3,3}$. Let I be an independent set of G. Either $I \subseteq V_i$ for $i \in \{1,2,3\}$ or it is not. If it is, then $\Theta_{1,3,3} \subseteq G \setminus I$ which is not 2-choosable.

If $I \not\subseteq V_i$ for all i, then I has at most one vertex from each part. This is because adjacent edges were not removed from $K_{3,3,3}$ in constructing G. It must be that $|I| \in \{2,3\}$. It suffices to show that $G \setminus I$ contains a triangle, because then it will not be 2-colorable and therefore not 2-choosable.

If $|I| = 3$, then $G \setminus I$ is $K_{2,2,2}$ with three edges removed. There are $2^3 = 8$ total triangles in $K_{2,2,2}$. Removing an edge from $K_{2,2,2}$ removes at most 2 triangles. If 3 edges are removed, then at most $3 \cdot 2 = 6$ of the 8 triangles are removed. So $G \setminus I$ contains at least 2 triangles.

If $|I| = 2$, then $G \setminus I$ is $K_{2,2,3}$ with at most 5 edges removed where exactly one edge is removed between the two parts of size 2. There are $2 \cdot 2 \cdot 3 = 12$ total triangles in $K_{2,2,3}$. Removing an edge between the parts of size 2 removes at most 3 triangles, and removing any other edge removes at most 2 triangles. If 5 edges are removed (where exactly one of those is between the parts of size 2), then at most $3 + 4 \cdot 2 = 11$ of the 12 total triangles are removed. So $G \setminus I$ contains at least one triangle. $\square$

Showing that at least one of the graphs in $\mathcal{G}$ is 3-choosable is enough to show that λ-partitionability does not follow the partial ordering of integer partitions that λ-choosability follows. This turns out to be a challenge.

Note that if a graph in $\mathcal{G}$ is 3-choosable, then it would necessarily also be $\{1, 2\}$-choosable. Showing that graphs in $\mathcal{G}$ are $\{1, 2\}$-choosable is a good first step to proving 3-choosability. In the next section, we show that all but three of the graphs in $\mathcal{G}$ are $\{1, 2\}$-choosable. This makes them non-complete multipartite $\{1, 2\}$-cnp graphs.. The section following the next shows progress towards proving the original goal of 3-choosability in these graphs.

4.2 $\{1, 2\}$-cnp graphs in $\mathcal{G}$

We first study $\Theta_{1,3,3}$ closer. We know $\Theta_{1,3,3}$ is not 2-choosable, and Puleo determined that up to relabeling there are exactly two unique 2-assignments of $\Theta_{1,3,3}$ which are not properly colorable.

Lemma 4.3 (Puleo [15])**.** *The only uncolorable* 2*-assignments of* $\Theta_{1,3,3}$ *are shown in Figure 4.3 where* $a = b$ *or* $a \neq b$*.*

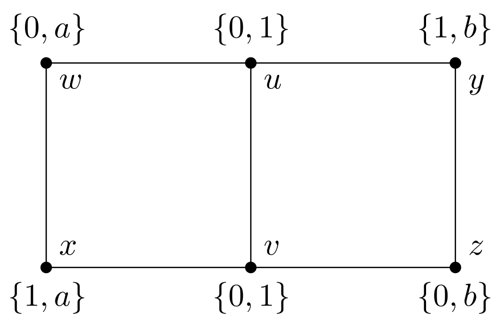


Figure 4.3: Unique uncolorable 2-assignments of $\Theta_{1,3,3}$.

Proof. Let $G = \Theta_{1,3,3}$ with vertices labeled as pictured above. Let L be a 2-assignment of G. Let (p, q) represent the partial L-coloring of G that colors vertex u with color p and colors vertex v with color q. We say (p, q) is "bad" for list assignment L if we can't extend (p, q)

to an L-coloring. Let $W_1 = \{w, x\}$ and $W_2 = \{y, z\}$ be the "wings" of G. If (p, q) is bad, then both vertices of at least one of the wings W_i only have one color available to it in L with (p, q). We say "W_i blocks (p, q)" in this case.

If W_i blocks (p_1, q_1), then the lists of L on W_i are of the form $\{p_1, a\}, \{q_1, a\}$ for some color a. If W_i also blocks p_2, q_2, then the lists of L on W_i are of the form $\{p_2, b\}, \{q_2, b\}$ for some color b. If $a = b$, then this means $p_1 = p_2$ and $q_1 = q_2$. If $a \neq b$, then $p_1 = q_1 = a$, which is a contradiction because $(p_1, q_1) = (a, a)$ is not a partial coloring. So each wing blocks at most one partial coloring.

If $|L(u) \cap L(v)| = 0$, then there are four partial colorings. Since the wings only block up to two partial colorings, G is L-colorable. If $|L(u) \cap L(v)| = 1$, then there are three partial colorings. Again, the wings block up to two partial colorings, so G is L-colorable.

So, if G is not L-colorable, then WLOG it must be that $L(u) = L(v) = \{0, 1\}$, W_1 blocks $(0, 1)$ with lists $\{0, a\}, \{1, a\}$, and W_2 blocks $(1, 0)$ with lists $\{1, b\}, \{0, b\}$. Because $a = b$ and $a \neq b$ are both valid possibilities, there are exactly two unique uncolorable assignments of $\Theta_{1,3,3}$. □

Furthermore, if any list L on $\Theta_{1,3,3}$ assigns more than 2 colors to a vertex while all others have at least two colors, then $\Theta_{1,3,3}$ is L-colorable. This is due to the following lemma.

Lemma 4.4 (Gravier and Maffray [6]). *Let L be a list assignment of $K_{3,3}$. If each vertex has at least two colors on their lists and at least one vertex has at least three colors, then $K_{3,3}$ is L-colorable.*

Corollary 4.4.1. *$\Theta_{1,3,3}$ is L-colorable.*

Proof. $\Theta_{1,3,3} \subseteq K_{3,3}$. □

We use the uncolorable 2-assignments described in Lemma 4.3 to show which graphs in $\mathcal{G}$ are $\{1, 2\}$-choosable. There are exactly three graphs in $\mathcal{G}$ which aren't $\{1, 2\}$-choosable, pictured in Figure 4.4.

Let $\mathcal{G}^*$ be all the graphs of $\mathcal{G}$ excluding the three pictured in Figure 4.4.

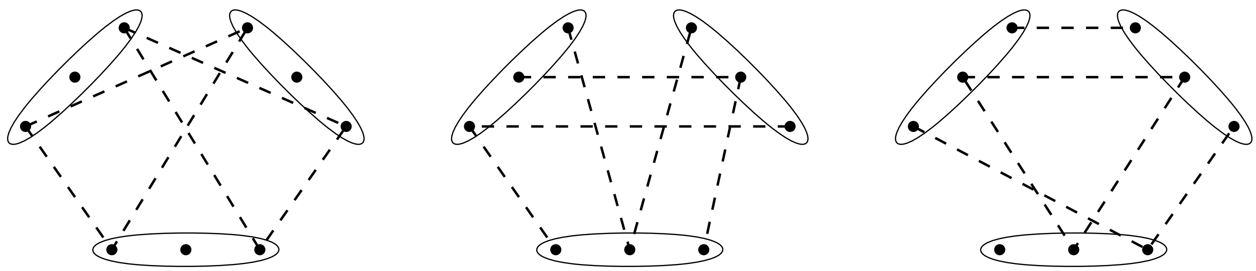


Figure 4.4: The three graphs in $\mathcal{G}$ which are not $\{1,2\}$-choosable.

Theorem 4.5. *Every graph in* $\mathcal{G}^*$ *is* $\{1,2\}$*-choosable.*

Proof. Let $G \in \mathcal{G}^*$. Let L be a $\{1,2\}$-assignment of G with C_1 and C_2 the color groups of L such that $|L(v) \cap C_1| = 1$ and $|L(v) \cap C_2| = 2$ for all $v \in V(G)$. Let V_1, V_2, V_3 be the parts of G corresponding to the partite sets of $K_{3,3,3}$. Color each verticex in V_1 with its color from C_1. Note that $G[V_2 \cup V_3] \cong \Theta_{1,3,3}$. If $V_2 \cup V_3$ cannot be colored with their assigned colors from C_2, then by Lemma 4.3 there are two cases for what their lists look like shown in Figure 4.5; note that each "$*$" is a placeholder for the colors from C_1.

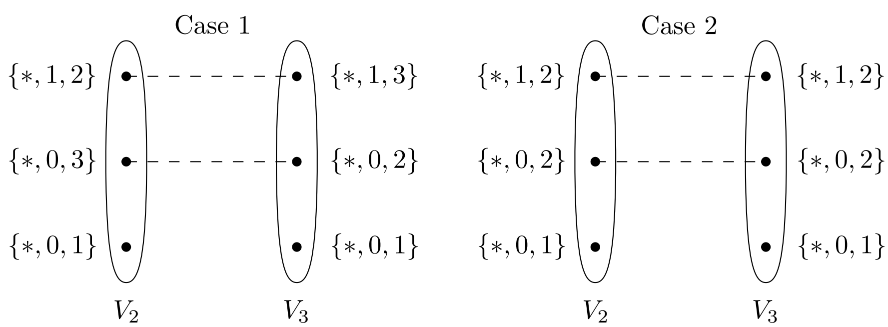


Figure 4.5: Two cases for lists on V_2 and V_3.

Case 1: Uncolor the vertices of V_1 and color the vertices of V_2 each with their color from C_1. If the coloring cannot be completed, then each vertex on V_1 and V_3 must have only two colors remaining on their lists by Corollary 4.4.1. By Lemma 4.3, those remaining two colors must correspond to Figure 4.6. Note that the removed edges from between V_1 and V_2 are not shown because they don't affect anything so far.

Uncolor V_2 and color the vertices of V_3 each with their color from C_1. Color V_1 with colors $0, 2$, and color V_2 with colors $1, 3$. G is L-colorable.

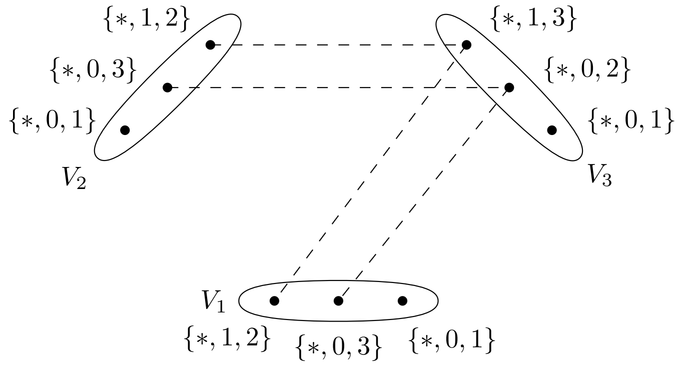


Figure 4.6: L and G if coloring V_2 with C_1 cannot be extended.

Case 2: As before, uncolor V_1 and color the vertices of V_2 each with their color from C_1. If the coloring cannot be completed, then each vertex on V_1 and V_3 must have only have two colors remaining on their lists. The graph and the remaining two colors on each list must now correspond to one of three cases shown in Figure 4.7.

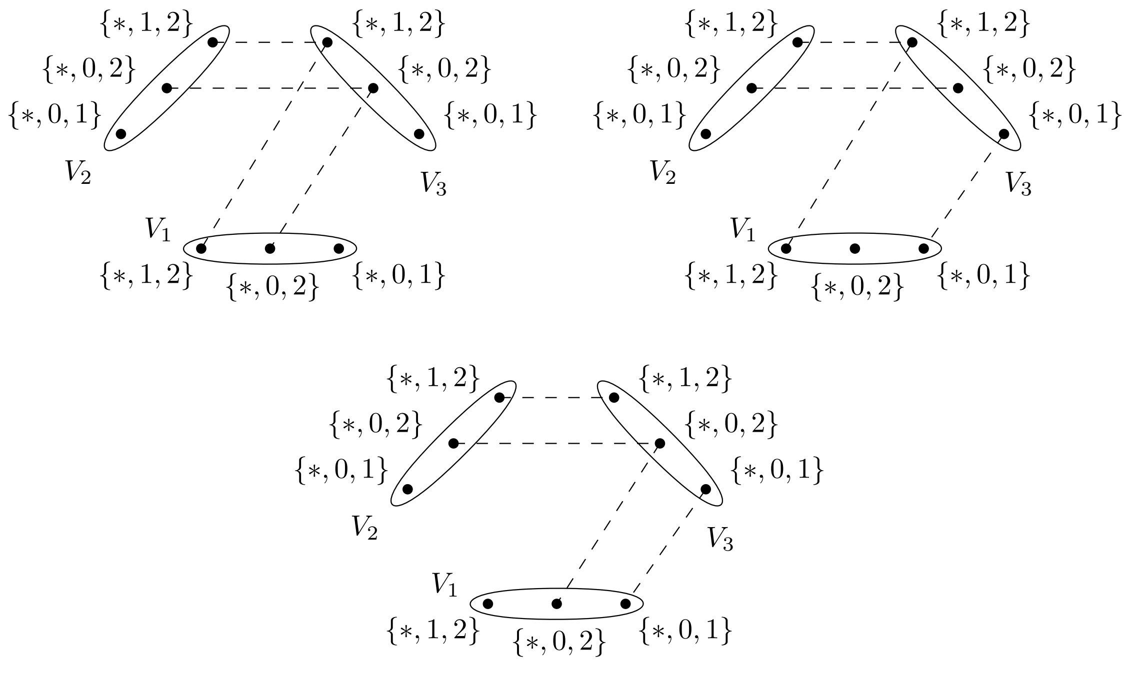


Figure 4.7: Case 2.1 (top left), Case 2.2 (top right), and Case 2.3 (bottom).

In all three sub-cases, uncolor V_2. If V_1 and V_2 cannot be colored using C_2 after coloring V_3 with C_1, then we have three uncolorable sub-cases for each sub-case in Figure 4.7. The sub-cases are shown in Figures 4.8, 4.9, and 4.10.

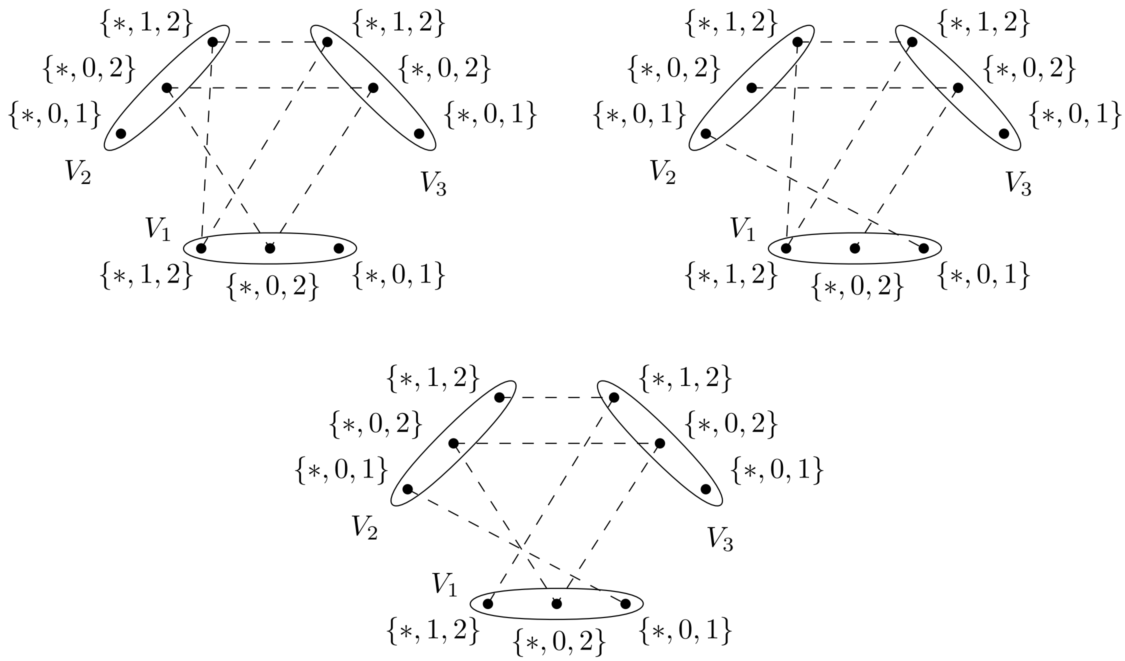


Figure 4.8: Sub-cases of 2.1.

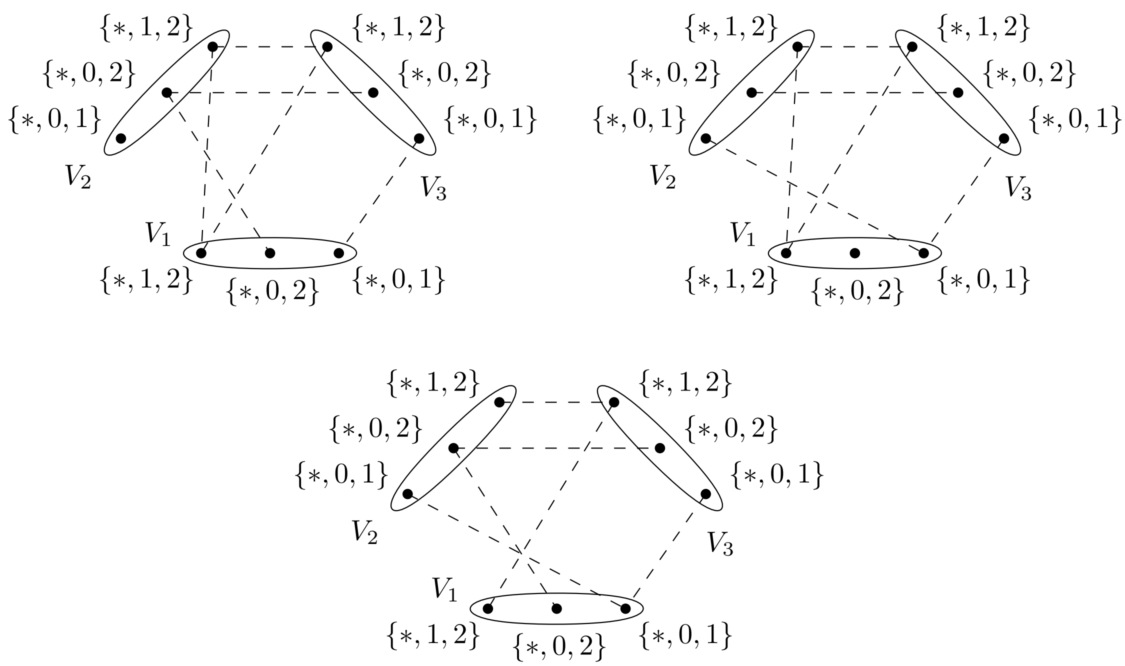


Figure 4.9: Sub-cases of 2.2.

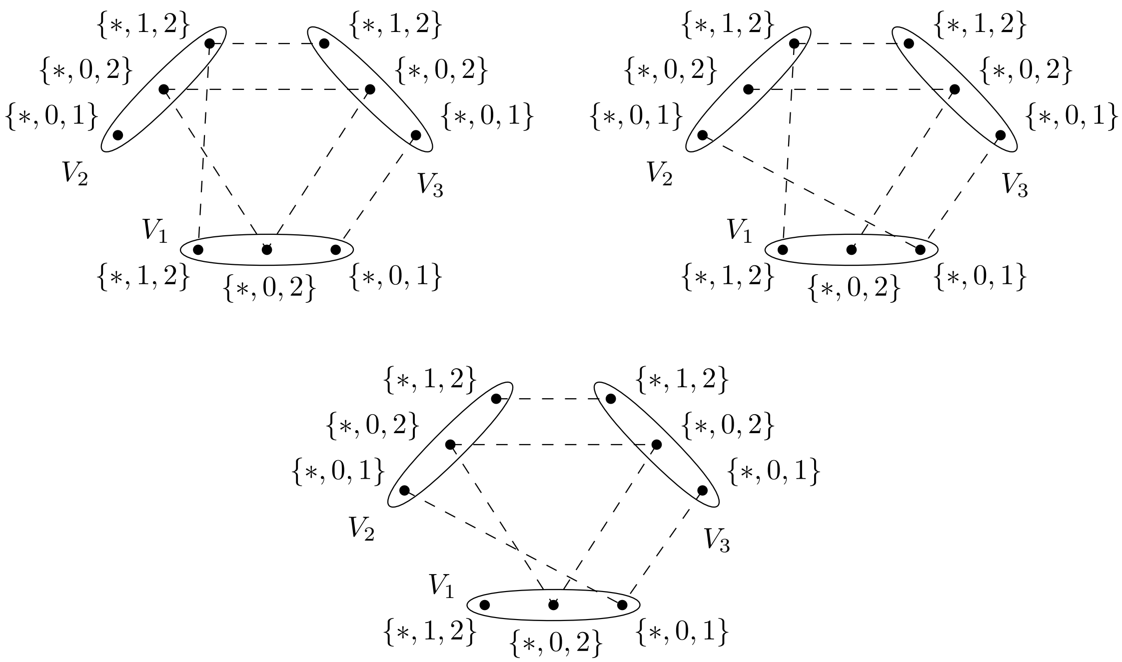


Figure 4.10: Sub-cases of 2.3.

Every graph in Figures 4.8, 4.9, and 4.10 is isomorphic to one of the three graphs in Figure 4.4, meaning they are not in $\mathcal{G}^*$. To see this, notice each of these graphs is $K_{3,3,3}$ with one of the following three edge configurations removed:

- Two disjoint triangles,
- Three disjoint paths of length two, or
- A triangle, a path of length two, and a single edge, all disjoint.

It turns out that each of these graphs is uncolorable if you let $*$ be a single color.

Regardless, G is not any of the graphs in Figures 4.8, 4.9, and 4.10 since $G \in \mathcal{G}^*$. This means V_1 and V_2 can be properly colored with C_2. So G is L-colorable. Because G and L are arbitrary, every graph in $\mathcal{G}^*$ is $\{1,2\}$-choosable. $\square$

Corollary 4.5.1. *Every graph in $\mathcal{G}^*$ is $\{1,2\}$-cnp.*

4.3 Progress towards 3-choosability in $\mathcal{G}^*$

There are exactly nine graphs up to isomorphism in $\mathcal{G}^*$. I first determined this through brute force by hand. I then confirmed it by using Macaulay2 to generate every possible graph obtained by removing 2 independent edges from between each pair of partite sets in $K_{3,3,3}$ (as graphs in $\mathcal{G}$ are defined). I then used the NautyGraphs package to remove any isomorphic graphs. This left the twelve graphs in $\mathcal{G}$, so removing the three from Figure 4.4 leaves the nine in $\mathcal{G}^*$. The details of the code are in Appendix A.

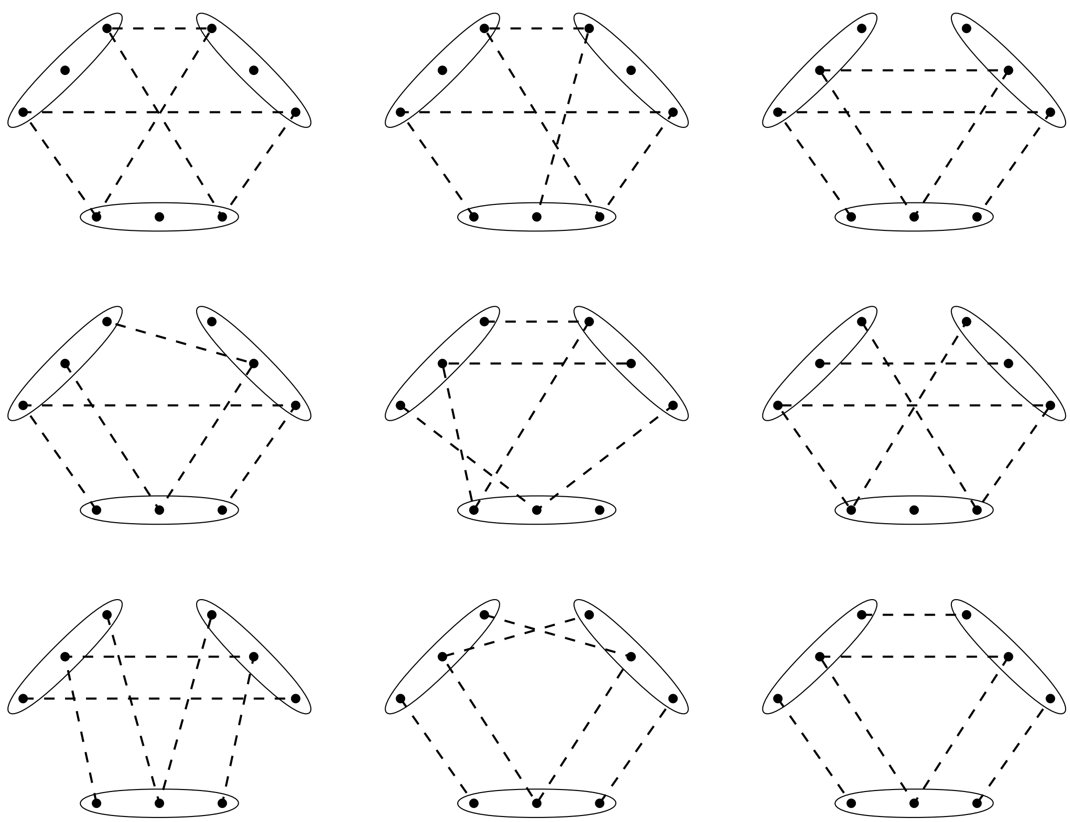


Figure 4.11: The nine graphs in $\mathcal{G}^*$.

I believe that all nine of these graphs are 3-choosable.

Conjecture 4.6. *Every graph in $\mathcal{G}^*$ is 3-choosable.*

However, my attempts to prove this often devolve into many cases. One approach I tried was to study 3-assignments on $K_{3,3,3}$. I wanted to see if I could find something helpful to say about which 3-assignments of $K_{3,3,3}$ are properly colorable. The only uncolorable 3-assignments of $K_{3,3,3}$ I have found so far contain either 4 or 5 colors in total. I have not yet found an uncolorable 3-assignment on $K_{3,3,3}$ which has 6 or more total colors. Either such a 3-assignment exists and I have not come across it yet, or the following is true.

Conjecture 4.7. *Let $V = V(K_{3,3,3})$. If L is a 3-assignment of $K_{3,3,3}$ such that $\left|\bigcup_{v\in V} L(v)\right| \geq 6$, then G is L-colorable.*

Not only would this be a very interesting result in its own right, but it would also do much of the heavy lifting to prove if a graph in $\mathcal{G}^*$ is 3-choosable. In Corollary 4.10.1, we show that $G \in \mathcal{G}^*$ is L-colorable if L has at most 4 colors. If Conjecture 4.7 is true, that takes care of L having at least 6 colors because $G \subseteq K_{3,3,3}$. All that would be left to check in G would be the 3-assignments with exactly 5 colors.

Unfortunately, this approach sweeps one hard problem under another hard problem. Ganjali et al. [5] showed that there is a 3-assignment on $K_{3,3,3}$ with exactly 6 colors which admits only one possible proper coloring. This lack of flexibility suggests that affirming Conjecture 4.7 would not be easy.

Given this, I abandoned that approach for another which has so far yielded good progress. This other approach is to study different configurations of colors in 3-assignments on graphs in $\mathcal{G}^*$.

Here's a theorem we'll be making use of.

Theorem 4.8 (Gravier and Maffray [6])**.** *$K_{2*k,3,3}$ is chromatic-choosable for $k \geq 1$.*

In particular, $K_{2,3,3}$ is 3-choosable. From this, we make a nice observation about colors in a 3-assignment which appear only on independent vertices for a subgraph of $K_{3,3,3}$.

Observation 4.9. *Let $H \subseteq K_{3,3,3}$ and let L be a 3-assignment of H. If there is a color in L which only appears on an independent set of H, then H is L-colorable.*

Proof. Let c be the color that only appears on an independent set, and call that independent set I. Now color the vertices of I with c. All other vertices of H still have three choices left.

Note that $H \setminus I \subseteq K_{2,3,3}$, which is 3-choosable by Theorem 4.8. Hence all the remaining vertices of H have a choice. $\square$

Since every graph in $\mathcal{G}^*$ is a subgraph of $K_{3,3,3}$, this observation applies to them all. We can say a few more things about all the graphs in $\mathcal{G}^*$.

Lemma 4.10. *Let L be a 3-assignment of $G \in \mathcal{G}^*$. If there is a color in L which appears on all three vertices of a partite set of G corresponding to those of $K_{3,3,3}$, then G is L-colorable.*

Proof. Let V_1, V_2, V_3 be the partite sets of G corresponding to those of $K_{3,3,3}$. WLOG, suppose the vertices of V_1 share a common color in there lists. Call that color c. If $c \in L(v)$ for all $v \in V(G)$, then L is a $\{1,2\}$-assignment and G is L-colorable by Theorem 4.5. If c is only on the lists of V_1, then c appears only on an independent set of G. So G is L-colorable by Observation 4.9. Otherwise, c appears on some but not all of the vertices of $V_2 \cup V_3$. Choose c for all the vertices of V_1. At least one vertex in $V_2 \cup V_3$ will have three colors remaining while the rest have at least two, so we can finish according to Corollary 4.4.1 because $G[V_2 \cup V_3] \cong \Theta_{1,3,3}$. $\square$

Corollary 4.10.1. *If $\left|\bigcup_{v \in V_i} L(v)\right| \leq 4$ for any $i \in \{1,2,3\}$, then G is L-colorable.*

Proof. Any 3-assignment on 3 vertices with at most 4 colors will assign at least one common color to all three vertices. To see this, we use the pigeonhole principle. Let the 4 colors each represent a container. There are nine possible positions in a 3-assignment on 3 vertices. Let those 9 positions each correspond to a pigeon. The pigeonhole principle says one of those containers will have at least three pigeons. This means one of the colors will appear at least three times in L. Because a color cannot appear on a vertex's list more than once, this color must appear on all three vertices. $\square$

Lemma 4.11. *Let L be a 3-assignment of $G \in \mathcal{G}^*$. If there is a color in L which appears on exactly three vertices and two of them are in the same partite set, then G is L-colorable.*

Proof. Let V_1, V_2, V_3 be the partite sets of G corresponding to those of $K_{3,3,3}$. We call the three vertices with the unique common color v_1, v_2, v_3 such that WLOG $v_1, v_2 \in V_1$ and $v_3 \in V_2$. Call their unique common color c_1. Choose c_1 for v_1 and v_2. Now v_3 has two choices left and all other uncolored vertices still have three choices. Let v_4 be the third vertex in V_1. There is a color in $L(v_4)$ for which choosing it still leaves two choices for v_3. Call it c_2 and choose it on v_4. Figure 4.12 illustrates the partial coloring so far. The edges removed from $K_{3,3,3}$ in G are not shown to keep it general.

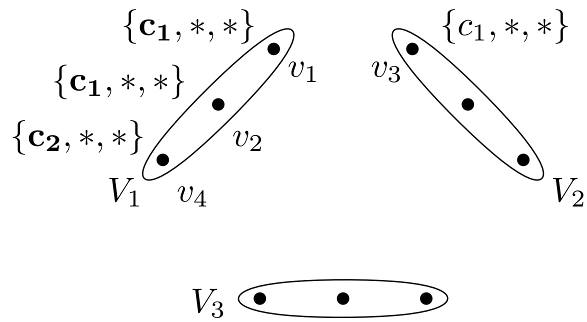


Figure 4.12: Partial coloring of G.

The remaining uncolored vertices are all the vertices of V_2 and V_3. They induce $\Theta_{1,3,3}$, and they all have at least two colors left. If they all have *exactly* two colors left, then c_2 appears on all three vertices V_3 (shown in Figure 4.13) because c_1 only appears in $L(v_1), L(v_2), L(v_3)$. So G is L-colorable by Lemma 4.10. Otherwise, if there is at least one vertex that still has three choices, then we can finish the coloring according to Corollary 4.4.1. □

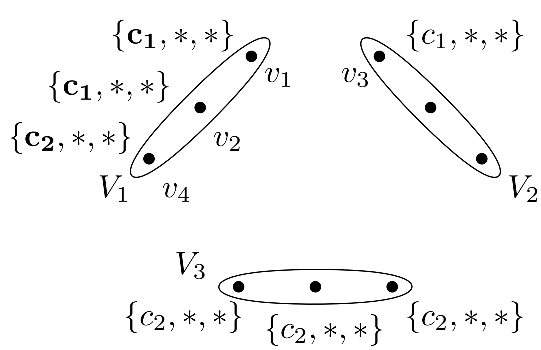


Figure 4.13: Partial coloring of G if all other vertices have two colors left.

Next, we focus on a specific specific graph G^* in $\mathcal{G}^*$ shown in Figure 4.14. In progress

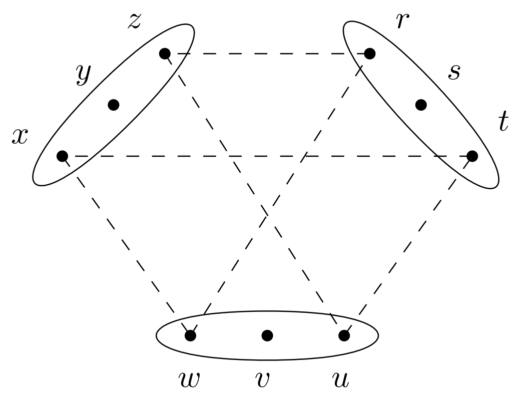


Figure 4.14: G^*.

towards 3-choosability, we show several color configurations whose presence in a 3-assignment L of G^* makes G^* L-colorable.

Lemma 4.12. *Let L be a 3-assignment of G^*. If there is a color in L which appears on exactly four lists in one of the three configurations shown in Figure 4.15, then G^* is L-colorable. Square vertices are where the color appears.*

Proof. Let 0 be the color that appears only on the four square vertices.

For the configuration on the top left of Figure 4.15, color x and y with 0. The vertices t and u will have two choices left and the rest will still have three choices. There is a choice on z which is not on t, so choose it. Then r will still have all three of its choices and the rest will have at least 2 choices left. What remains is $\Theta_{1,3,3}$ with the list assignment in Corollary 4.4.1. Therefore G^* is L-colorable.

The configuration on the top right of Figure 4.15 follows the same protocol. Color x and y with 0. Vertices s and u will have two choices left while the rest still have three. There is a choice on z which is not on s, so choose it. Again, r will still have all three of its choices and the rest will have at least 2 choices left. Apply Corollary 4.4.1, and G^* is L-colorable.

The configuration on the bottom of Figure 4.15 is trickier but reasonable. We start as before by coloring x and y with 0. If $L(z) \neq L(i)$ for any $i \in \{s, t, v, w\}$, then there is a

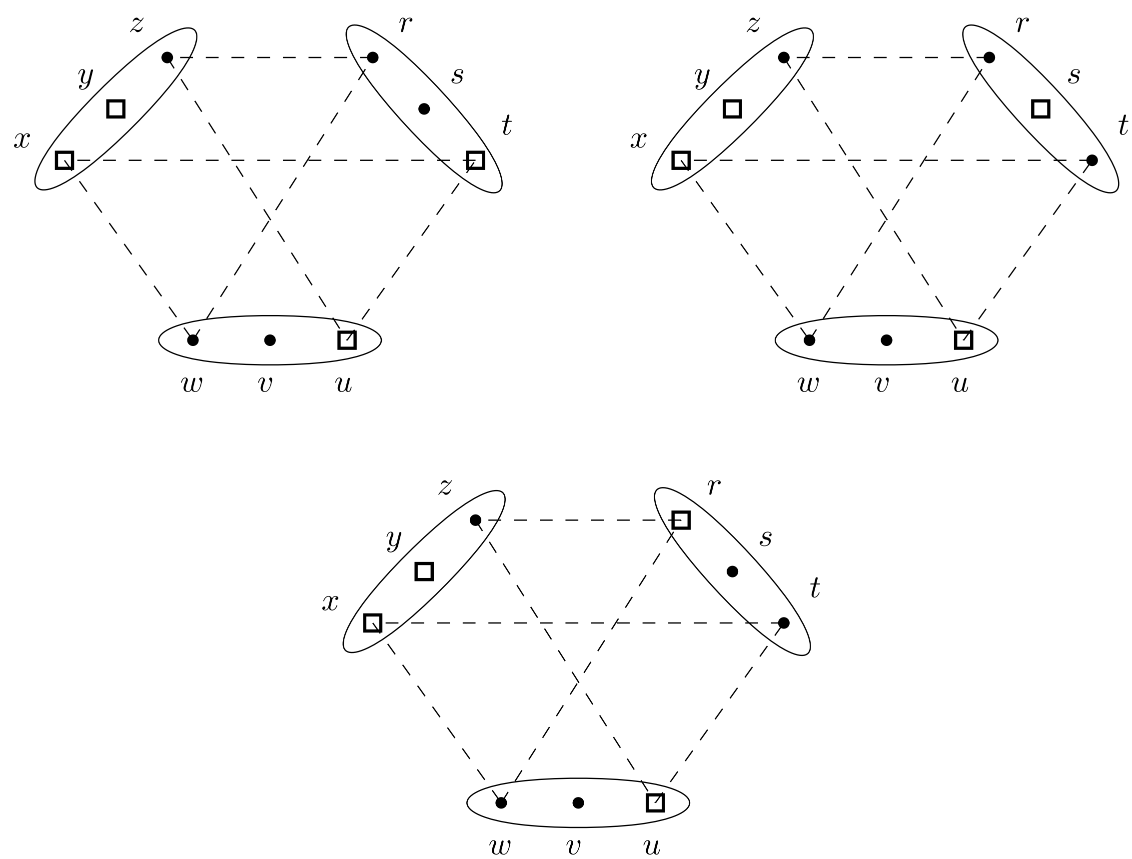


Figure 4.15: Configurations of a color on 4 vertices in L of G^*.

choice on z that leaves one of s, t, v, w with three choices and all the rest of the vertices with at least two choices. Finish with Corollary 4.4.1.

Suppose $L(z) = L(i)$ for all $i \in \{s, t, v, w\}$, and let $L(z) = \{1, 2, 3\}$. If either $L(r)$ or $L(u)$ contain any one of $1, 2, 3$, then all three vertices of a partite set in G^* will have a common color, and we can color G^* according to Lemma 4.10.

Otherwise, choose 1 for z, choose 2 for s and t, and choose 3 for v and w, shown in Figure 4.16. Then both r and u still have two choices because their lists contain none of

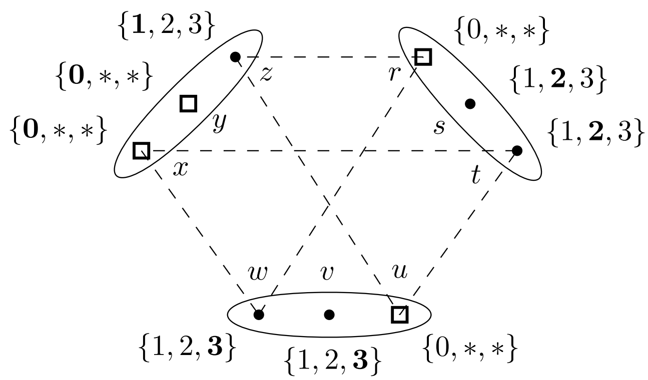


Figure 4.16: Partial coloring of G^*.

$1, 2, 3$. We can finish this coloring, so G^* is L-colorable. $\square$

Lemma 4.13. *Let L be a 3-assignment of G^*. If there is a color in L which appears on exactly five lists in one of the two configurations shown in Figure 4.17, then G^* is L-colorable. Square vertices are where the color appears.*

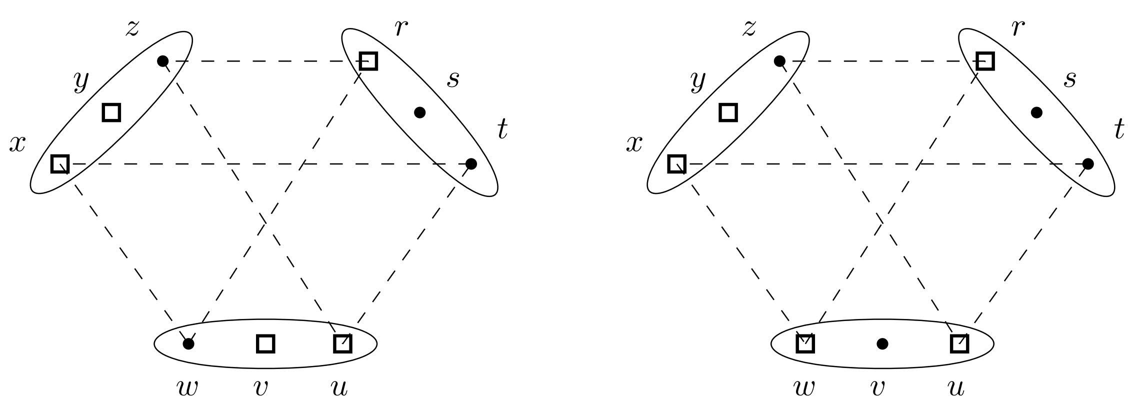


Figure 4.17: Configurations of a color on 5 vertices in L of G^*.

Proof. Call the color that appears only on the five square vertices c_1.

For the configuration on the left, start by choosing c_1 on x and y. Vertices r, u, and v will have two choices remaining while z, s, t, w have three. There is a choice in $L(z)$, call it c_2, that is not in $L(v)$ Choose c_2 for z. All remaining vertices have at least two choices remaining. If at least one of them has three choices, finish with Corollary 4.4.1.

Suppose all remaining vertices have two choices left. If the coloring can be finished at this point, then by all means do it. However, since $\Theta_{1,3,3}$ is not 2-choosable, we might not be able to finish the coloring as we have started it. But since we know exactly what the uncolorable 2-assignments of $\Theta_{1,3,3}$ are from Lemma 4.3, all is not lost.

According to Lemma 4.3, if the coloring can't be finished, then $\{0, a\} \subseteq L(r)$, $\{0, 1\} \subseteq L(s)$, $\{1, b\} \subseteq L(t)$, $\{1, a\} \subseteq L(u)$, $\{0, 1\} \subseteq L(v)$, and $\{0, b\} \subseteq L(w)$ where either $a = b$ or $a \neq b$. We also know that c_2, must be assigned to s, t, and w. So we know $L(s) = \{c_2, 0, 1\}$, $L(t) = \{c_2, 1, b\}$, and $L(w) = \{c_2, 0, b\}$. The partial coloring so far is shown in Figure 4.18.

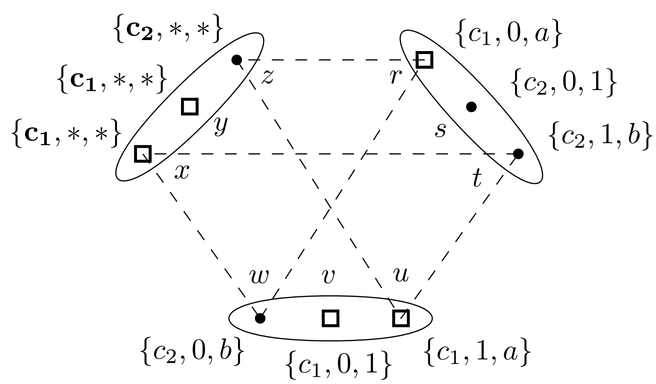


Figure 4.18: Partial coloring of G^*.

Remove your choice of c_2 on z. Choose a for r, choose c_2 for both s and t, choose 1 for u, and choose 0 for both v and w as shown in Figure 4.19.

If any one of $1, a, b$ are in $L(z)$, regardless of whether $a = b$ or not, then we can choose it for z to complete the coloring. If none of $1, a, b$ are in $L(z)$, then there is a color in $L(z)$ distinct from $c_1, c_2, 0, 1, a, b$, call it c_3. This is because $c_1 \notin L(z)$ and even if $0 \in L(z)$, there's

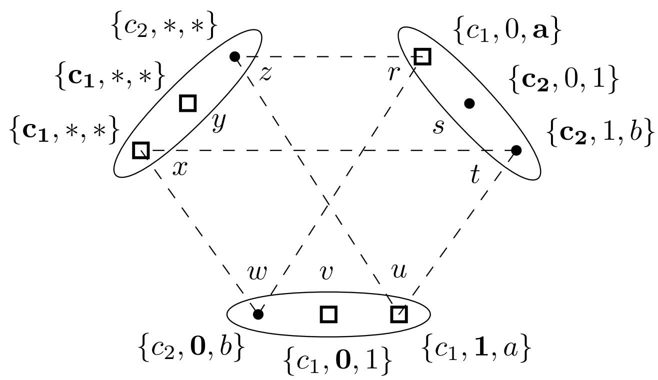


Figure 4.19: Partial coloring of G^*.

still one more color unaccounted for. We can finish the coloring with c_3 on z. Therefore G^* is L-colorable.

Now, the configuration on the right in Figure 4.17 can be colored in the same fashion with a slightly different setup. We will walk through the steps so that there is no doubt.

Choose c_1 on x and y as before, now leaving r, u, and w with two choices and the rest with three. There is a choice in $L(z)$, call it c_2 again, that leaves w with both of its remaining colors and all other vertices with at least two colors. If at least one of them has three choices, finish with Corollary 4.4.1.

Suppose all remaining vertices have two choices left. If the coloring cannot be finished, then we know by Lemma 4.3 that $\{0, a\} \subseteq L(r)$, $\{0, 1\} \subseteq L(s)$, $\{1, b\} \subseteq L(t)$, $\{1, a\} \subseteq L(u)$, $\{0, 1\} \subseteq L(v)$, and $\{0, b\} \subseteq L(w)$ where either $a = b$ or $a \neq b$. We also know that c_2, must be assigned to s, t, and v. So we know $L(s) = \{c_2, 0, 1\}$, $L(t) = \{c_2, 1, b\}$, and $L(v) = \{c_2, 0, 1\}$. The partial coloring so far is shown in Figure 4.20.

The rest follows exactly as in the first configuration. Remove your choice of c_2 on z. Choose a for r, choose c_2 for both s and t, choose 1 for u, and choose 0 for both v and w as shown in Figure 4.21.

If any one of $1, a, b$ are in $L(z)$, regardless of whether $a = b$ or not, then we can choose it for z to complete the coloring. If none of $1, a, b$ are in $L(z)$, then there is a color in $L(z)$

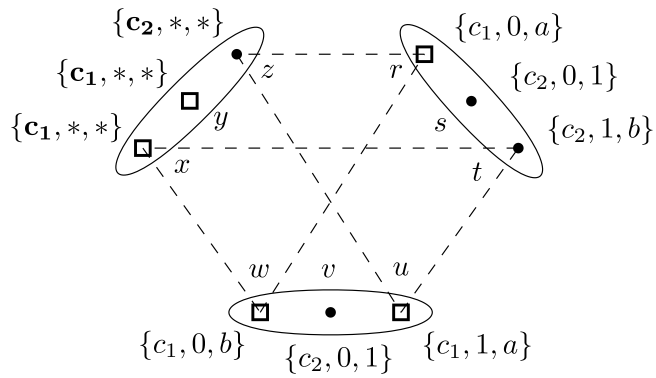


Figure 4.20: Partial coloring of G^*.

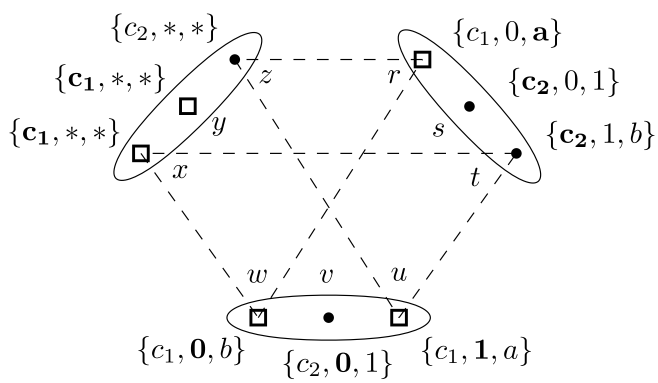


Figure 4.21: Partial coloring of G^*.

distinct from $c_1, c_2, 0, 1, a, b$, call it c_3. We can finish the coloring with c_3 on z. Therefore G^* is L-colorable. $\square$

Chapter 5

Conclusion and future work

Zhu's refinement of choosability acts as a fine comb for those who study list coloring. It offers stepping stones for partial results to harder problems in list coloring. The work in this dissertation explores this refinement to offer more insight into its structure and to discover graphs which demonstrate interesting consequences of the refinement.

There are still many unanswered questions stemming from what was presented. First, can anything be said about non-complete multipartite strictly colorable graphs? The inspiration to study strictly colorable of graphs came from the strictly 4-colorable planar graph that Kemnitz and Voigt [17] found, so it would be natural to return to planar graphs for further study.

Can further generalizations or characterizations of strictly colorable graphs be found? The characterization of strictly colorable complete multipartite graphs in Theorem 2.7 was done using three classes of subgraphs. Is there a definitive list of strictly colorable subgraphs which completely characterize *all* strictly colorable graphs? A good place to start exploring this would be to search for strictly 3-colorable graphs on n vertices and work from there.

More work can be done to find more λ-cnp graphs. As of right now, it cannot be definitively said whether or not there is a λ-cnp graph for every nontrivial integer partition λ. It is probably true, and a constructive proof would be very nice. However, finding one has been difficult. A good place to start trying would be to prove Conjecture 3.6.

As a side note, it would very interesting if there turns out to be a nontrivial λ for which λ-choosability and λ-partitionability are equivalent. But such a counterexample likely does not exist.

It would be very nice to finish proving that G^* from Chapter 4 is 3-choosable. This will either require more clever case-by-case work, or perhaps some computer code to check the remaining list assignments not yet covered by the cases I've already shown. One idea is to look at cases based on the total number colors in a given list assignment. Once that is done, a next step could be to find k-choosable graphs which are not $\{1, k-1\}$-partitionable for $k > 3$. Perhaps a generalized version of G^* would do the trick, or maybe a simpler construction exists.

Hoffman and Johnson's [9] unique bad 2-assignment of $K_{2,4}$ was used a lot in this dissertation. They end their proceeding with an open problem. They showed that

$$\mathrm{ch}(K_{m,n}) = \begin{cases} m+1, & m \geq 1 \ \text{ and } \ n \geq m^m, \\ m, & m \geq 3 \ \text{ and } \ (m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m. \end{cases}$$

The next natural step is to find for what n is $\mathrm{ch}(K_{m,n}) = m - 1$. We've made attempts at this problem, and it is hard. Our current idea is for an improved upper bound on n.

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Appendices

Appendix A

Macaulay2 code to generate all graphs in $\mathcal{G}$

Here each input line `i#` and output line `o#` is shown preceded by explanations. First, the Graphs and NautyGraphs packages are loaded.

```
i1 : needsPackage "Graphs"
-- storing configuration for package Graphs in /home/m2user/.Macaulay2/init-
   Graphs.m2
o1 = Graphs
o1 : Package
i2 : needsPackage "NautyGraphs"
-- storing configuration for package NautyGraphs in /home/m2user/.Macaulay2/
   init-NautyGraphs.m2
o2 = NautyGraphs
o2 : Package
```

Next, $K_{3,3,3}$ is stored in `K` using the Graphs package.

```
i3 : K = completeMultipartiteGraph {3,3,3}
o3 : Graph
```

Next, let V_1, V_2, V_3 be the partite sets of $K_{3,3,3}$. The vertices of `K` in `i3` are named with the numbers $0, 1, \ldots, 8$. In particular, $V_1 = \{0, 1, 2\}$, $V_2 = \{3, 4, 5\}$, and $V_3 = \{6, 7, 8\}$. The Cartesian product of each pair of partite sets is used to generate three lists of edges between them. For example, `E12` is the list of edges between partite sets V_1 and V_2.

```
i4 : E12 = {0,1,2} ** {3,4,5}
o4 = {(0,3),(0,4),(0,5),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)}
o4 : List
i5 : E13 = {0,1,2} ** {6,7,8}
o5 = {(0,6),(0,7),(0,8),(1,6),(1,7),(1,8),(2,6),(2,7),(2,8)}
```

```
o5 : List

i6 : E23 = {3,4,5} ** {6,7,8}

o6 = {(3,6),(3,7),(3,8),(4,6),(4,7),(4,8),(5,6),(5,7),(5,8)}

o6 : List
```

Next, a list of every independent pair of edges between partite sets V_1 and V_2 is made. To do this, all two-element subsets of `E12` are generated in a list `P`, and independent pairs from `P` are filtered into another list `M12`

```
i7 : P = subsets(E12,2)

o7 = {{(0,3),(0,4)},{(0,3),(0,5)},{(0,4),(0,5)},{(0,3),(1,3)},{(0,4),(1,3)
   },{(0,5),(1,3)},{(0,3),(1,4)},{(0,4),(1,4)},{(0,5),(1,4)},{(1,3),(1,4)
   },{(0,3),(1,5)},{(0,4),(1,5)},{(0,5),(1,5)},{(1,3),(1,5)},{(1,4),(1,5)
   },{(0,3),(2,3)},{(0,4),(2,3)},{(0,5),(2,3)},{(1,3),(2,3)},{(1,4),(2,3)
   },{(1,5),(2,3)},{(0,3),(2,4)},{(0,4),(2,4)},{(0,5),(2,4)},{(1,3),(2,4)
   },{(1,4),(2,4)},{(1,5),(2,4)},{(2,3),(2,4)},{(0,3),(2,5)},{(0,4),(2,5)
   },{(0,5),(2,5)},{(1,3),(2,5)},{(1,4),(2,5)},{(1,5),(2,5)},{(2,3),(2,5)
   },{(2,4),(2,5)}}

o7 : List

i8 : M12={}

o8 = {}

o8 : List

i9 : for p in P do if #(set p#0 * set p#1) == 0 then M12=append(M12,p)
```

Next, these steps are repeated to get `M13` and `M23`. `M13` is the list of all independent pairs of edges between V_1 and V_3, and `M23` is the list of all independent pairs of edges between V_2 and V_3.

```
i10 : P = subsets(E13,2)
```

```
o10 = {{(0,6),(0,7)},{(0,6),(0,8)},{(0,7),(0,8)},{(0,6),(1,6)},{(0,7),(1,6)
   },{(0,8),(1,6)},{(0,6),(1,7)},{(0,7),(1,7)},{(0,8),(1,7)},{(1,6),(1,7)
   },{(0,6),(1,8)},{(0,7),(1,8)},{(0,8),(1,8)},{(1,6),(1,8)},{(1,7),(1,8)
   },{(0,6),(2,6)},{(0,7),(2,6)},{(0,8),(2,6)},{(1,6),(2,6)},{(1,7),(2,6)
   },{(1,8),(2,6)},{(0,6),(2,7)},{(0,7),(2,7)},{(0,8),(2,7)},{(1,6),(2,7)
   },{(1,7),(2,7)},{(1,8),(2,7)},{(2,6),(2,7)},{(0,6),(2,8)},{(0,7),(2,8)
   },{(0,8),(2,8)},{(1,6),(2,8)},{(1,7),(2,8)},{(1,8),(2,8)},{(2,6),(2,8)
   },{(2,7),(2,8)}}
o10 : List
i11 : M13={}
o11 = {}
o11 : List
i12 : for p in P do if #(set p#0 * set p#1) == 0 then M13=append(M13,p)
i13 : P = subsets(E23,2)
o13 = {{(3,6),(3,7)},{(3,6),(3,8)},{(3,7),(3,8)},{(3,6),(4,6)},{(3,7),(4,6)
   },{(3,8),(4,6)},{(3,6),(4,7)},{(3,7),(4,7)},{(3,8),(4,7)},{(4,6),(4,7)
   },{(3,6),(4,8)},{(3,7),(4,8)},{(3,8),(4,8)},{(4,6),(4,8)},{(4,7),(4,8)
   },{(3,6),(5,6)},{(3,7),(5,6)},{(3,8),(5,6)},{(4,6),(5,6)},{(4,7),(5,6)
   },{(4,8),(5,6)},{(3,6),(5,7)},{(3,7),(5,7)},{(3,8),(5,7)},{(4,6),(5,7)
   },{(4,7),(5,7)},{(4,8),(5,7)},{(5,6),(5,7)},{(3,6),(5,8)},{(3,7),(5,8)
   },{(3,8),(5,8)},{(4,6),(5,8)},{(4,7),(5,8)},{(4,8),(5,8)},{(5,6),(5,8)
   },{(5,7),(5,8)}}
o13 : List
i14 : M23={}
o14 = {}
o14 : List
i15 : for p in P do if #(set p#0 * set p#1) == 0 then M23=append(M23,p)
```

Next, `M12`, `M13`, and `M23` are used to make the list `M`, which is a collection of every possible configuration of 6 edges removed from $K_{3,3,3}$ to produce a graph in $\mathcal{G}$.

```
i16 : M={}

o16 = {}

o16 : List

i17 : for m12 in M12 do for m13 in M13 do for m23 in M23 do M=append(M,m12|m13
    |m23)
```

Next, we make a list of graphs `G` by removing each list of edges in `M` from $K_{3,3,3}$ and adding the graph to `G`. This produces every graph in $\mathcal{G}$. Each graph is converted to Graph6 string format so that the NautyGraphs package can be used next.

```
i18 : G={}

o18 = {}

o18 : List

i19 : for m in M do G=append(G, graphToString deleteEdges(K,m))
```

Finally, *NautyGraphs* is used to remove all isomorphisms from `G`.

```
i20 : G = removeIsomorphs G

o20 ={Hl‘Mjy{,Hl‘Kzzs,Hl‘M^q{,Hl‘M~Q{,Hl‘Mxzs,Hl‘MnR{,Hl‘Mnrk,Hl‘M|rs,Hl‘M^b{,
    Hl‘M~bk,Hl‘M|rk,Hl‘Mxzk}

o20 : List
```

What remains in `G` are the Graph6 string representations of the twelve graphs in $\mathcal{G}$ up to isomorphism.