Rainbow Connectivity and Proper Rainbow Connectivity

by

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Abstract

A connected graph G is rainbow connected with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path—a path with no color appearing on more than one edge of the path. G is strongly rainbow connected if each pair of distinct vertices of G are joined by a rainbow geodesic, a shortest path in G between the vertices. The (strong) rainbow connection number of G, denoted (s)rc(G), is the smallest number of colors in an edge coloring of G with respect to which G is (strongly) rainbow connected.

Two more recently introduced parameters, prc and psrc, are defined as rc and src were, with the additional requirement that the edge colorings be proper. Some relations among the four parameters are mentioned and they are evaluated for some classes of graphs, including some of the theta graphs and some graphs constructed by joining arbitrarily many cycles at a cut vertex. The impact of several types of graph modifications on the values of parameters is also considered.

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Chapter 1

On the Rainbow Connectivity Parameters

Rainbow connectivity and strong rainbow connectivity were introduced by Gary Chartrand, Garry L. Johns, Kathleen A. McKeon, and Ping Zhang in 2006 [2]. In recent years, the rainbow connectivity and strong rainbow connectivity parameters (defined below) have been heavily researched [4]. Two other parameters – proper rainbow connectivity and proper strong rainbow connectivity – were introduced more recently [1]. We will begin by extending some early results for the initial parameters to the more recent proper versions.

Definitions. A connected graph G is *rainbow connected* with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path – a path with no color appearing on more than one edge of the path.

A connected graph G is *strongly rainbow connected* with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow geodesic – a shortest rainbow path.

The *rainbow connection number of* G, denoted $\text{rc}(G)$, is the smallest number of colors required to obtain an edge coloring with respect to which G is rainbow connected.

The *strong rainbow connection number of* G, denoted $\text{src}(G)$, is the smallest number of colors required to obtain an edge coloring with respect to which G is strongly rainbow connected.

Of more recent interest are the proper versions of rainbow connectivity and strong rainbow connectivity, in which we require the edge coloring of the graph to be proper.

Definitions. A connected graph G is *proper rainbow connected* with respect to a proper edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path.

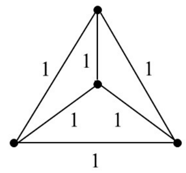


Figure 1.1: rc(K_4) = src(K_4) = 1

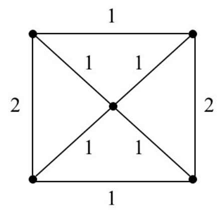


Figure 1.2: rc(W_4) = src(W_4) = 2

A connected graph G is *proper strongly rainbow connected* with respect to a proper edge coloring of G if each pair of distinct vertices of G are joined by a rainbow geodesic.

The *proper rainbow connection number of* G, denoted prc(G), is the smallest number of colors required to obtain an edge coloring with respect to which G is proper rainbow connected.

The *proper strong rainbow connection number of* G, denoted psrc(G) is the smallest number of colors required to obtain an edge coloring with respect to which G is proper strongly rainbow connected.

Throughout, G will be a finite connected simple graph. The chromatic index, or edge chromatic number, of G is denoted $\chi'(G)$ and the diameter by diam(G). The inequalities in the following theorem are straightforward to see from the definitions. The first appears in [2].

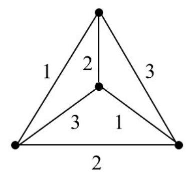


Figure 1.3: prc(K_4)= psrc(K_4) = 3

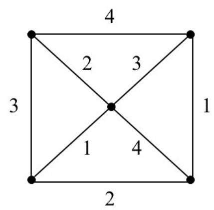


Figure 1.4: prc(W_4) = psrc(W_4) = 4

Theorem 1.1. *diam(G) $\leq$ rc(G) $\leq$ src(G) $\leq$ |E(G)|,*
χ'(G) $\leq$ prc(G) $\leq$ psrc(G) $\leq$ |E(G)|, rc(G) $\leq$ prc(G),
and src(G) $\leq$ psrc(G).

Theorem 1.2. *The following are equivalent.*

(a) rc(G) = |E(G)|;

(b) src(G) = |E(G)|;

(c) G is a tree.

The equivalence of (a) and (c), and thus (c) $\Rightarrow$ (b), is proven in [2]. A proof that (b) implies (c) is given in [1].

Corollary 1.2.1. *If G is a tree then prc(G) = psrc(G) = |E(G)|.*

Neither converse of Corollary 1.2.1 holds, since prc(K_3) = psrc(K_3) = 3 = |E(K_3)|, which raises two extremal questions:

1. For which G is it true that prc(G) = |E(G)|?

2. For which G is it true that psrc(G) = |E(G)|?

The following corollary proves both are true precisely when G is a tree or G is K_3:

Corollary 1.2.2. *K_3 is the only connected non-tree for which prc$(G) = |E(G)|$ or psrc$(G) =$ $|E(G)|$.*

Proof. Let G be a connected graph that is not K_3 and is not a tree. Since G is not a tree, G must contain at least one cycle as a subgraph. Let C_n, $n \geq 3$, be such a subgraph of minimal length. Because C_n is of minimal length, no chord of C_n is in $E(G)$.

Case 1: $G = C_n$. Then $n \geq 4$.

Color $E(G)$ with $n-1$ colors, with two edges e, f of G a maximum distance from each other ($\frac{n-2}{2}$ if n is even, $\frac{n-3}{2}$ if n is odd) bearing the same color. It is then straightforward to see that every geodesic path of $G = C_n$ contains at most one of e, f.

Case 2: $G \neq C_n$.

Because no chord of C_n is in $E(G)$, and because G is connected, there must be some edge $e = uv \in E(G)$ such that $u \in V(C_n)$ and $v \notin V(C_n)$. Because C_n is a cycle of least order length in G, any geodesic path on C_n is also a geodesic path in G. Let $f = xy \in E(C_n)$ be as far away from u as possible ($\frac{n-1}{2}$ if n is odd, $\frac{n-2}{2}$ if n is even) on C_n. Color $E(G)$ with $|E(G)| - 1$ colors by letting e and f be colored with the same color and then the remaining $|E(G)| - 2$ colors be assigned to the remaining $|E(G)| - 2$ edges.

In both cases, the colorings are clearly proper and provide strong rainbow connection in G. Thus psrc(G) $\leq |E(G)| - 1$. □

Proposition 1.1. *If diam(G) ≤ 2 then prc(G) = psrc(G) = χ'(G).*

Proof. We have χ'(G) $\leq$ prc(G) $\leq$ psrc(G). If G is properly edge-colored then every path in G of length 1 or 2 is rainbow, so, if diam(G) ≤ 2 then every geodesic is rainbow in a proper coloring. Thus, diam(G) $\leq 2 \Rightarrow$ psrc(G) $\leq \chi'$(G). □

Let $\mathcal{P}$ denote the Petersen graph, and C_n the cycle on n vertices.

Theorem 1.3. *rc($\mathcal{P}$) = 3, src($\mathcal{P}$) = prc($\mathcal{P}$) = psrc($\mathcal{P}$) = 4, and, for $n > 3$, rc(C_n) = src(C_n) = prc(C_n) = psrc(C_n) = $\lceil \frac{n}{2} \rceil$.*

Proof. It is shown in [2] that rc($\mathcal{P}$) = 3, src($\mathcal{P}$) = 4, and rc(C_n) = src(C_n) = $\lceil \frac{n}{2} \rceil$ if $n > 3$. Since χ'($\mathcal{P}$) = 4 and $\mathcal{P}$ has diameter 2, prc($\mathcal{P}$) = psrc($\mathcal{P}$) = 4 follows from Proposition 1.1.

To see that prc(C_n) = psrc(C_n) = $\lceil \frac{n}{2} \rceil$ if $n > 3$, it suffices (since rc $\leq$ prc $\leq$ psrc) to see that C_n has a proper edge coloring with $\lceil \frac{n}{2} \rceil$ colors such that C_n is strongly rainbow connected with respect to this coloring. If $n > 3$ is even, color around the cycle with $1, 2, \ldots, \frac{n}{2}, 1, 2, \ldots, \frac{n}{2}$. If $n > 3$ is odd, color around the cycle with $1, 2, \ldots, \lceil \frac{n}{2} \rceil, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. $\square$

Two facile conjectures exploded

It might seem reasonable to conjecture that if rc(G) $\geq \chi'(G)$ then prc(G) = rc(G). In [1] this conjecture was disposed of by the example of a (large) clique with a (long) path attached. By making the clique large and the path yet longer, we can arrange to have prc(G) - rc(G) arbitrarily large while rc(G) $\geq \chi'(G)$. (In fact, rc(G) - $\chi'(G)$ can be arbitrarily large simultaneously). Here we note that the same class of examples dismisses another plausible conjecture, that rc(G) $\leq \chi'(G)$ might imply that prc(G) = $\chi'(G)$. By making the order of the clique much larger than the (large) order of the path we can arrange that $\chi'(G)$ - rc(G) and prc(G) - $\chi'(G)$ are simultaneously arbitrarily large.

However, in these examples, in the first case (rc(G) $\geq \chi'(G)$) we always have $\frac{\text{prc}(G)}{\text{rc}(G)} < 2$, and in the second (rc($G$) $\leq \chi'(G)$) we always have $\frac{\text{prc}(G)}{\chi'(G)} < 2$. Verification is left to the reader. This observation raises two more extremal questions of a different sort from 1 and 2, above.

3. Is $\left\{ \frac{\text{prc}(G)}{\text{rc}(G)} \;\middle|\; G \text{ is finite, simple, connected, } |E(G)| > 0 \text{, and rc}(G) \geq \chi'(G) \right\}$ unbounded, and, if not, what is its least upper bound?

4. Is $\left\{ \frac{\text{prc}(G)}{\chi'(G)} \;\middle|\; G \text{ is finite, simple, connected, } |E(G)| > 0, \text{ and rc}(G) \leq \chi'(G) \right\}$ unbounded, and, if not, what is its least upper bound?

Chapter 2

Rainbow Connectivity Parameters for Theta Graphs

Definition. A *theta graph* $\theta(m_1, m_2, \ldots, m_k)$ is a set of two vertices u and v with $k \geq 3$ internally disjoint (simple) paths between them such that the paths have lengths $m_1, m_2, \ldots, m_k$, and at most one of the path lengths is 1.

Theorem 2.1. *Let $G = \theta(m_1, m_2, \ldots, m_k)$ such that $m_1 \leq m_2 \leq \cdots \leq m_k$, $m_1 = s$, and $m_k = t$. Then rc$(G) \leq s + t$.*

Proof. Let the $u - v$ paths with lengths $m_1, m_2, \ldots, m_k$ be $P_1, P_2, \ldots, P_k$, respectively. Color the edges of P_1 with distinct colors $c_1, c_2, \ldots, c_s$. For all remaining $u - v$ paths P_i, color the edges of P_i with the distinct colors $c_{s+1}, c_{s+2}, \ldots, c_{s+t}$ consecutively from u to v. That is, if $1 < i \leq k$, the edges of P_i from u to v are colored $c_{s+1}, \ldots, c_{s+m_i}$.

Then every $u - v$ path is rainbow and, since the edges of P_1 are colored distinctly from all other edges of G, every geodesic between a vertex of P_1 and a vertex of P_i, $i \in \{2, \ldots, k\}$, is rainbow.

Now suppose that $x, y \in V(G)$ are internal vertices on P_i and P_j, respectively, $i \neq j$, $i, j > 1$. Without loss of generality, suppose that the distance in P_i from x to u is no greater than the distance in P_j from y to u. Then the path from x to y described by: Go from x to u along P_i, then from u to v along P_1, and then from v to y along P_j, is rainbow.

Thus there is a rainbow path between each pair of vertices of G when G is equipped with this coloring, so rc$(G) \leq s + t$. □

This raises the question of when equality holds for this upper bound of rc(G). Note that this bound is quite helpful in some cases, as for theta graphs with many paths of short length.

However, $(s+t) - \text{rc}(G)$ can be made arbitrarily large. This is the case in the following class of examples: Let G be a theta graph of the form $G = \theta(1, 2, t)$. It was proven in [3] that $\text{rc}(G) = \text{diam}(G) = \left\lfloor \frac{2+t}{2} \right\rfloor$. In this case, then, where s is the length of the shortest path in G and t is the length of the longest path, $(s+t) - \text{rc}(G) = 1 + t - \left\lfloor \frac{2+t}{2} \right\rfloor = t - \left\lfloor \frac{t}{2} \right\rfloor$, the value of which can be made arbitrarily large by choosing sufficiently large maximum path length t.

Theorem 2.2. *Let $G = \theta(m_1, m_2, \ldots, m_k)$, $m_1 = m_2 = \cdots = m_k = 2$. Then*

I. For $3 \leq k \leq 16$, rc(G) = $\lceil \sqrt{k} \rceil$

II. For all $k \geq 10$, rc(G) = 4

III. For all $k \geq 3$, src(G) = $\lceil \sqrt{k} \rceil$

IV. For all $k \geq 3$, prc(G) = psrc(G) = k

Proof. Since $\text{diam}(G) = 2$ and $\chi'(G) = k$, IV follows immediately from Proposition 1.1.

For $k \geq 3$, let $t = \lceil \sqrt{k} \rceil$. Let the edges of G incident to u be $e_1, e_2, \ldots, e_k$ and those incident to v be $f_1, f_2, \ldots, f_k$. Let the midpoints of the paths of length 2 joining u to v be $c_1, c_2, \ldots, c_k$, named so that c_i is incident to e_i and f_i, $1 \leq i \leq k$.

To prove III, let us assume by contradiction that $\text{src}(G) < t$. Then there exists an edge coloring $\alpha : E(G) \longrightarrow \{1, 2, \ldots, t-1\}$ with respect to which G is strongly rainbow connected.

Since there are k internally disjoint paths between u and v, any such edge coloring α must apply some color $p \in \{1, 2, \ldots, t-1\}$ to no fewer than t edges of the form e_i. Without loss of generality, suppose $\alpha(e_1) = \alpha(e_2) = \cdots = \alpha(e_t) = p$. Now, consider $f_1, f_2, \ldots, f_t \in E(G)$. For some $f_i, f_j \in \{f_1, f_2, \ldots, f_t\}$ with $i \neq j$, $\alpha(f_i) = \alpha(f_j) = q$ for some $q \in \{1, 2, \ldots, t-1\}$. Then neither of the two geodesics $e_i e_j$ and $f_i f_j$ joining c_i and c_j are rainbow; therefore $\text{src}(G) > t - 1$.

As r and s each vary over $\{1, \ldots, t\}$, $(r-1)t + s$ varies over $\{1, \ldots, t^2\}$. Since $k \leq t^2$, we can define $\beta : E(G) \longrightarrow \{1, 2, \ldots, t\}$ as follows: If $i \in \{1, \ldots, k\}$, let $r, s \in \{1, \ldots, t\}$ be the unique pair such that $i = (r-1)t + s$, and set $\beta(e_i) = r$, $\beta(f_i) = s$.

Since $\beta(e_2) = 1 \neq 2 = \beta(f_2)$, there is a rainbow geodesic joining u and v.

Let $i, j \in \{1, 2, \ldots, k\}$, $i < j$, and suppose $\beta(f_i) = \beta(f_j) = s_0$ for some $s_0 \in \{1, 2, \ldots, t\}$. Then for some $r_i, r_j \in \{1, 2, \ldots, t\}$ we have $i = (r_i - 1)t + s_0$ and $j = (r_j - 1)t + s_0$, implying $j - i = [(r_j - 1)t + s_0] - [(r_i - 1)t + s_0] = (r_j - r_i)t$. Since $j - i \neq 0$, we have $r_i \neq r_j$ and thus $r_i = \beta(e_i) \neq \beta(e_j) = r_j$.

Thus, for every c_i, c_j, $i \neq j$, either $\beta(e_i) \neq \beta(e_j)$ or $\beta(f_i) \neq \beta(f_j)$, so there is either an e_ie_j geodesic or an f_if_j geodesic, respectively, joining c_i and c_j. Thus, $\text{src}(G) = t$.

Returning to I, suppose by contradiction that $q = \text{rc}(G) < \lceil\sqrt{k}\rceil = \text{src}(G)$. Then there is a coloring of $E(G)$ with q colors with respect to which there is a rainbow path in G joining any two vertices – but for two of these vertices there is no rainbow geodesic between them.

These vertices, then, must be among the c_i, and any rainbow path between them must be of length 4. Since such a path exists, we have $4 \leq q < \text{src}(G) = \lceil\sqrt{k}\rceil$, which yields a contradiction when $3 \leq k \leq 16$. Thus, $\text{rc}(G) = \text{src}(G) = \lceil\sqrt{k}\rceil$ for $3 \leq k \leq 16$.

Finally, let $k \geq 10$. By the result just above, $\text{rc}(G) = \text{src}(G) = \lceil\sqrt{k}\rceil = 4$ for $10 \leq k \leq 16$. By Theorem 2.1, $\text{rc}(G) \leq 2 + 2 = 4$ for all $k \geq 3$. Since $\text{rc}(G)$ is non-decreasing as k increases, and $\text{rc}(G) = 4$ when $k = 16$, it follows that $\text{rc}(G) = 4$ for all $k \geq 16$, and thus for all $k \geq 10$. □

Corollary 2.2.1. *Let $H = \theta(m_1, m_2, \ldots, m_{k+1})$, $k \geq 2$, $m_1 = m_2 = \cdots = m_k = 2$, $m_{k+1} = 1$. The values of the (strong) rainbow connectivity and proper (strong) rainbow connectivity parameters are as follows:*

I. For $2 \leq k \leq 4$, rc(H) = 2

II. For all $k \geq 5$, rc(H) = 3

III. For all $k \geq 2$, src(H) = $\lceil\sqrt{k}\rceil$

IV. For all $k \geq 2$, prc(H) = psrc(H) = $k + 1$

Proof. Since $\text{diam}(H) = 2$ and $\chi'(H) = k + 1$, we immediately obtain that prc(H) = psrc(H) = $k + 1$ for all $k \geq 2$ by Proposition 1.1.

To prove that src(H) = $\lceil\sqrt{k}\rceil$, let us first dispose of the smallest case. By Theorem 2.4 yet to come, or by reference [3], $\text{rc}(H) = \text{src}(H) = \text{diam}(H) = 2$ when $k = 2$.

For the remaining cases, when $k \geq 3$, let $t = \lceil \sqrt{k} \rceil$. Let $G = \theta(m_1, m_2, \ldots, m_k)$ and $H = \theta(m_1, m_2, \ldots, m_{k+1})$ such that $m_1 = m_2 = \cdots = m_k = 2$ and $m_{k+1} = 1$.

Note that H is obtained from G by adding an edge with endpoints u and v to $E(G)$. That is, $V(H) = V(G)$ and $E(H) = E(G) \cup \{uv\}$. Let the edges and vertices of H be defined as they were for G in the proof of Theorem 2.2. Then the $u - v$ geodesic of H is a path of length 1, while all $c_i - c_j$ geodesics remain paths of length 2. Since we have shown that t colors are needed in any edge coloring of G to obtain an edge coloring with respect to which G is strongly rainbow connected, and since the $c_i - c_j$ geodesics remain unchanged in H, we must use at least t colors on any edge coloring of H to obtain an edge coloring with respect to which H is strongly rainbow connected as well. That is, $\text{src}(H) \geq \text{src}(G)$.

Define $\beta : E(G) \longrightarrow \{1, 2, \ldots, t\}$ as in the proof of Theorem 2.2. Define an edge coloring $\beta' : E(H) \longrightarrow \{1, 2, \ldots, t\}$ such that $\beta(e_i) = \beta'(e_i)$, $\beta(f_i) = \beta'(f_i)$, and $\beta'(uv) = 1$. Note that the $u - v$ path of length 1 is trivially rainbow, and there exists a rainbow geodesic between all pairs of vertices of the form c_i and c_j, with $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$ as this was proven to be true of β and the edges of the paths of length 2 are colored the same under β'. Thus, H is strongly rainbow connected with respect to β', an edge coloring of H using t colors. Since we have shown that we must use at least t colors to obtain strong rainbow connectivity and have now provided an edge coloring with t colors with respect to which H is strongly rainbow connected, $\text{src}(H) = t$.

To prove I and II, note that $2 = \text{diam}(H) \leq \text{rc}(H) \leq \text{src}(H)$. When $2 \leq k \leq 4$, then, by the above proof, $\text{rc}(H) = \text{src}(H) = 2$.

For $k \geq 5$, note that the only paths joining c_i and c_j for any $i \neq j$ with $i, j \in \{1, 2, \ldots, k\}$ are paths of length 2, 3, or 4. Thus, for any edge coloring of H containing fewer than 3 colors, a rainbow $c_i - c_j$ geodesic is required to obtain a $c_i - c_j$ path. Thus, when $k \geq 5$, $\text{rc}(H) \geq 3$ since $\text{src}(H) \geq 3$.

By Theorem 2.1, $\text{rc}(H) \leq \min\limits_{1 \leq i \leq k+1} \{m_i\} + \max\limits_{1 \leq i \leq k+1} \{m_i\} = 1 + 2 = 3$. Thus, $\text{rc}(H) = 3$ for all $k \geq 5$. $\square$

Theorem 2.3. *Let $G = \theta(m_1, m_2, \ldots, m_k)$ be a theta graph. Then src$(G) \geq \sum\limits_{\substack{1 \leq i \leq k \\ m_i \geq 3}} \left\lfloor \frac{m_i - 1}{2} \right\rfloor$.*

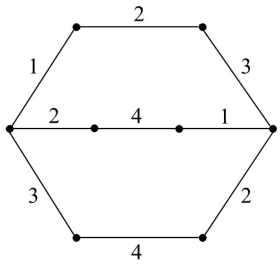


Figure 2.1: $\theta(3,3,3)$ with a proper strongly rainbow connective edge coloring with 4 colors

Proof. Let $G = \theta(m_1, m_2, \ldots, m_k)$ be a theta graph. Let the $u-v$ paths with lengths $m_1, m_2, \ldots, m_k$ be denoted $P_1, P_2, \ldots, P_k$, respectively. Let the vertices of P_i, $1 \leq i \leq k$, be denoted as $u = v_{(i,0)}, v_{(i,1)}, \ldots, v_{(i,m_i)} = v$. Let $l_i = \lfloor \frac{m_i-1}{2} \rfloor$. Then for each pair of $v_{(i,l_i)}, v_{(j,l_j)} \in V(G)$ such that $1 \leq i,j \leq k$, $i \neq j$, $m_i \geq 3$, there is a unique $v_{(i,l_i)} - v_{(j,l_j)}$ geodesic, which passes through u. Since each of these geodesics are a unique geodesic between a pair of vertices of G, then, we require distinct colors on all edges of these $u - v_{(i,l_i)}$ paths, again for all $1 \leq i \leq k$, $m_i \geq 3$, to obtain an edge coloring of G with respect to which the graph is strongly rainbow connected. Thus, src(G) $\geq \sum_{\substack{1 \leq i \leq k \\ m_i \geq 3}} \lfloor \frac{m_i-1}{2} \rfloor$. □

Chartrand et al. identified the rainbow connection number and strong rainbow connection number of $G = \theta(3,3,3)$: rc(G) = src(G) = 4 [2]. The proof states that diam(G) = 3 and shows it is not possible to edge color G with 3 colors such that G is rainbow connected. It goes on to provide an edge coloring of G using 4 colors with respect to which the graph is rainbow connected and strongly rainbow connected. As this edge coloring, shown in Figure 2.1, is proper, it follows that prc(G) = psrc(G) = 4.

In [3] we identified the rainbow connection numbers and proper rainbow connection numbers of theta graphs with three paths in which $k = 3$, $m_1 = 1$, $m_2 = s = |E(S)|$, and $m_3 = t = |E(T)|$ for paths S and T. The common ends of the 3 paths will be u and v; they will also bear other names in the proofs.

Theorem 2.4. *Let $G = \theta(1, s, t)$, $s, t > 1$.*

I. If s and t are both odd, then rc(G) = src(G) = prc(G) = psrc(G) = diam(G).

II. If s and t are both even, then rc(G) = src(G) = diam(G).

(a) If $s = t = 2$, then prc(G) = psrc(G) = $\chi'(G)$ = 3 = diam(G)+1.

(b) If $s \geq 2$ and $t \geq 4$, then prc(G) = psrc(G) = diam(G).

III. Suppose s is even and t is odd.

(a) If $s = 2$ or $t = 3$, then rc(G) = src(G) = diam(G) and prc(G) = psrc(G) = diam(G)+1.

(b) If $s = 4$ and $t \geq 5$ then rc(G) = diam(G) and src(G) = prc(G) = psrc(G) = diam(G)+1.

(c) If $s \geq 6$ and $t \geq 5$, then rc(G) = src(G) = prc(G) = psrc(G) = diam(G)+1.

Proof. The edge uv will also be denoted e. In all cases, diam(G) = $\lfloor \frac{s+t}{2} \rfloor$. If $s = t = 2$ or $s = 2$, $t = 3$ then diam(G) = 2, so, by Proposition 1.1, prc(G) = psrc(G) = $\chi'(G)$ = 3. In these cases it is easy to see that rc(G) = src(G) = 2. Thus, the claim in II(a) and one claim in III(a) are established. We will deal with the remaining claims below.

Before proceeding to the hard parts of the proof, we note that the claims for rc(G) and prc(G) have easy proofs for many pairs (s, t). Observe that $G - e = C_{s+t}$, so by Theorem 1.3, $A(G - e) = \lceil \frac{s+t}{2} \rceil$ for all $A \in \{$rc, src, prc, psrc$\}$.

If $s + t$ is even, we have rc($G - e$) = $\frac{s+t}{2}$ = diam(G) = rc(G); the last equality follows because if $G - e$ is rainbow connected with respect to an edge coloring with $\frac{s+t}{2}$ colors, then so is G if the coloring is extended to G by coloring e with any of the $\frac{s+t}{2}$ colors already appearing.

The same argument shows that prc(G) = rc(G) = $\frac{s+t}{2}$ when $s + t$ is even and $\frac{s+t}{2} \geq 5$; this last inequality would guarantee that if $G - e$ is properly edge-colored with $\frac{s+t}{2}$ colors so that $G - e$ is rainbow connected, then some color among the $\frac{s+t}{2}$ can be found which does not appear on any of the 4 edges of $G - e$ incident to either u or v. Then assigning such a color to e results in a proper edge coloring of G with respect to which G is rainbow connected. (However, our proofs below of the theorem's claims about prc(G) when $s + t$ is even apply even when $s + t \leq 8$.)

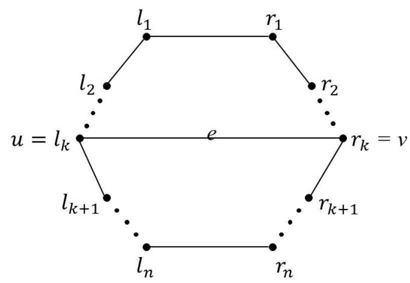


Figure 2.2: $\theta(1, s, t)$, s, t both odd

When $s+t$ is odd, consideration of $G-e$ gives us rc(G) $\leq \lceil \frac{s+t}{2} \rceil$ = diam(G)+1 and, when $s+t \geq 9$, prc(G) $\leq \lceil \frac{s+t}{2} \rceil$ = diam(G)+1. Therefore, if we show that rc$(G) > \lfloor \frac{s+t}{2} \rfloor =$ diam(G) when $s+t$ is odd, then the claims of III(c) for rc(G) and prc(G) will be established.

Consideration of $G-e$ is not much help in determining src(G) and psrc(G) because, for many pairs of vertices, the geodesics between them in G are quite different from the geodesics between them in $G-e$.

Now we will start the serious portion of this proof, with the proof of I.

Suppose that s and t are both odd, and $s \leq t$. Then $s, t \geq 3$. Let $k = \frac{s+1}{2}$ and n = diam(G) = $\frac{s+t}{2}$. Let the vertices of G be as in Figure 2.2.

Color the edges of G by the following instructions:

1. The diametral geodesic path $l_1, l_2, \ldots, l_n, r_n$ has its edges colored $1, 2, \ldots, n$. That is, $l_i l_{i+1}$ is colored i, $i = 1, \ldots, n-1$, and $l_n r_n$ is colored n. Note that $l_{k-1} l_k = l_{k-1} u$ is colored $k-1$ and $l_k l_{k+1}$ is colored k; note that $n - k = \frac{s+t}{2} - \frac{s+1}{2} = \frac{t-1}{2} \geq 1$.

2. Color the edges of the path $r_n, \ldots, r_k$ with $k, \ldots, n-1$, in that order. That is, $r_j r_{j-1}$ is colored $k + n - j$, $j = k+1, \ldots, n$.

3. Color the edges of the path $r_1, \ldots, r_k$ with $k-1, \ldots, 1$, in that order. That is, $r_i r_{i+1}$ is colored $k - i$, $i = 1, \ldots, k-1$.

4. Color the edges e and $l_1 r_1$ with n.

This edge coloring is illustrated in Figure 2.3 for the cases $s = t = 3$, $s = 3$, $t = 5$, and $s = 5$, $t = 7$.

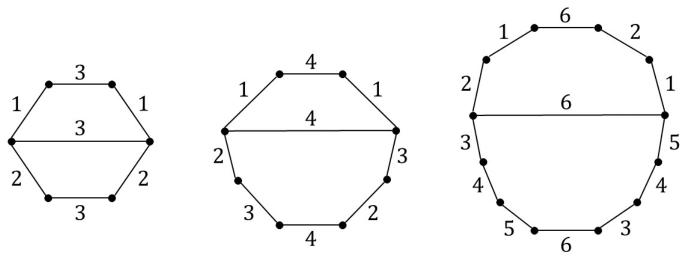


Figure 2.3: Proper strongly connecting edge coloring of $G = \theta(1, s, t)$ in the cases $(s, t) \in \{(3, 3), (3, 5), (5, 7)\}$

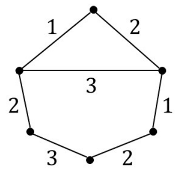


Figure 2.4: A proper strong rainbow connecting edge coloring of $\theta(1, 2, 4)$ with 3 colors

Clearly this coloring is proper. We leave it to the reader to verify that G is strongly rainbow connected by this coloring.

This concludes the proof of I.

Next: The proof of II.

Suppose that $2 \leq s \leq t$ and that both s and t are even. The diameter of $G = \theta(1, s, t)$ is $\frac{s+t}{2} = n$. We have already disposed of the case $s = t = 2$. In all other cases, it suffices to show that psrc(G) $\leq$ diam(G) = n.

Case 1: $s = 2$, $t = 4$. Then $n = 3$.

A proper edge 3-coloring of G with respect to which G is strongly rainbow connected is given in Figure 2.4.

Case 2: $s = 2$, $t \geq 6$. Then $n = \frac{t}{2} + 1 \geq 4$.

Let the vertices of $G = \theta(1, 2, t)$ be named as in Figure 2.5.

Included in Figure 2.5. is an indication of a proper edge coloring of G with n = diam(G) colors with respect to which G is strongly rainbow connected. In case the color assignment is unclear, here it is in words.

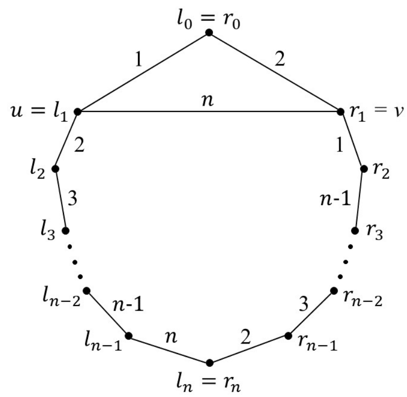


Figure 2.5: $\theta(1, 2, t)$, $t \geq 6$, even, with a proper strong rainbow connecting edge coloring with $n = \frac{t}{2} + 1$ colors

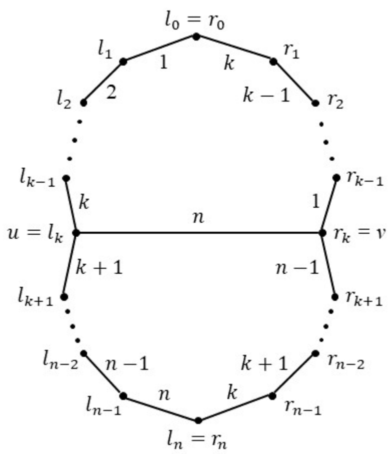


Figure 2.6: A proper coloring of $\theta(1, s, t)$, $4 \leq s \leq t$, s, t even, with $n = \frac{s+t}{2}$ colors, with respect to which the graph is strongly rainbow connected

For $i = 1, \ldots, n$, the edge $l_{i-1}l_i$ is colored with i; that is, the edges of the "left" diametral geodesic from l_0 to l_n are colored $1, \ldots, n$, in that order. For $j = 3, \ldots, n$, the edge $r_{j-1}r_j$ is colored $n - j + 2$; that is, the edges of the path on the vertices $r_n, r_{n-1}, \ldots, r_2$ are colored $2, 3, \ldots, n-1$, in that order. Then r_1r_2 is colored 1, r_0r_1 is colored 2, and $l_1r_1 = uv$ is colored n.

Note that the coloring in the case $s = 2$, $t = 4$ was a degenerate instance of this coloring. We judged that it would be clearer if dealt with as a separate case.

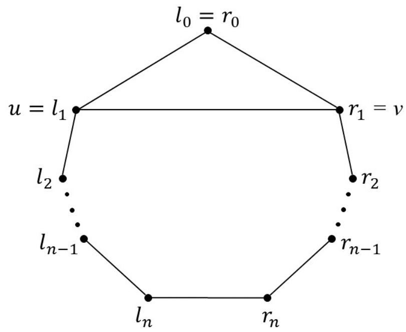
Case 3. $4 \leq s \leq t$. Then $n = \frac{s+t}{2} \geq 4$. Let $k = \frac{s}{2}$. We will indicate below a naming of the vertices and a proper edge coloring of $G = \theta(1, s, t)$ with n colors with respect to which G is strongly rainbow connected, and that will finish the proof of II.

In the coloring above, $l_{i-1}l_i$ is colored i for $i = 1, \ldots, n$; $r_{j-1}r_j$ is colored $n - j + k$, $j = k+1, \ldots, n$; $r_{j-1}r_j$ is colored $k - j + 1$, $j = 1, \ldots, k$; and $l_k r_k = uv$ is colored n.

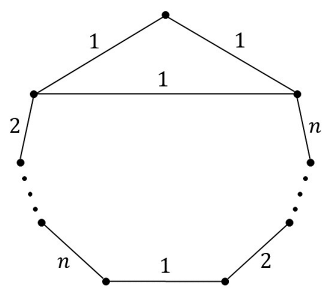
Finally, we undertake the proof of III. Throughout, $s \geq 2$ is even, $t \geq 3$ is odd, so n = diam(G) = $\lfloor \frac{s+t}{2} \rfloor = \frac{s+t-1}{2}$. The case $s = 2, t = 3$ has already been dealt with.

Case 1. $s = 2, t \geq 5$. In this case, $n = \frac{t+1}{2}$.

The vertices of $G = \theta(1, 2, t)$ will be named as follows:



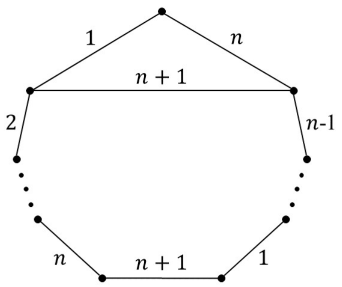
First we will show that rc(G) = src(G) = n by giving an edge coloring of G with n colors appearing such that G is strongly rainbow connected with respect to this coloring:



We hope that this coloring is clear. In case it is not: The diametral geodesic $l_0, l_1, \ldots, l_n$ is edge-colored $1, \ldots, n$, and this coloring is repeated on $l_n, r_n, \ldots, r_1$. Then $r_0 r_1$ and $r_1 l_1 = uv$ are colored 1. This edge coloring is not proper, but the graph is strongly rainbow connected with respect to this edge coloring.

To finish the proof in this case we must do two things: (1) show that there is a proper strongly rainbow connective edge coloring of G with $n + 1$ colors, and (2) show that there is no proper rainbow connective edge coloring of G with n colors.

The picture following disposes of (1).

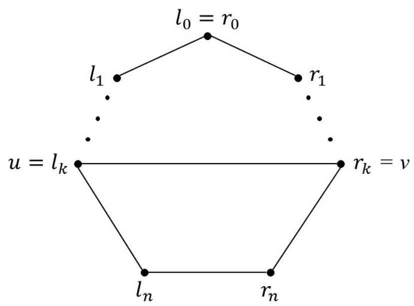


As for (2): Suppose that ϕ: E$(G) \longrightarrow \{1, \ldots, n\}$ is a proper rainbow connective coloring. There is exactly one geodesic joining l_0 and l_n. Therefore we may assume that the edges of the path $l_0, l_1, \ldots, l_n$ are colored $1, 2, \ldots, n$, in that order.

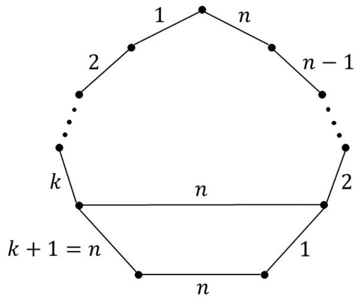
The unique geodesic in G from r_0 to r_n, which is $r_0, r_1, \ldots, r_n$ must be rainbow, and either the path $l_1, \ldots, l_n, r_n$ or the path $l_1, r_1, \ldots, r_n$ must be rainbow. But if the latter path is rainbow then the rainbowness of $r_0, \ldots, r_n$ and the assumption that there are only n colors appearing forces $\phi(r_0r_1) = \phi(l_1r_1)$, which would contradict the presumed properness of ϕ. Therefore the path $l_1, \ldots, l_n, r_n$ is rainbow. Therefore $\phi(l_nr_n) = 1$.

But applying the mirror-image argument to the rainbow geodesic path $r_0, \ldots, r_n$ forces us to the conclusion that $\phi(l_nr_n) = \phi(r_0r_1)$, which again contradicts the assumption that ϕ is proper. This completes the proof of III in this case.

Case 2. $s \geq 4$ and $t = 3$. Then $n = \frac{s}{2} + 1 = k + 1$. The vertices of $G = \theta(1, s, 3)$ will be as follows:



Here is an improper edge coloring of G with n colors with respect to which G is strongly rainbow connected.



Recoloring uv and $l_n r_n$ with $n+1$ gives a proper strongly rainbow connective edge coloring with $n+1$ colors appearing.

It remains to be seen that there is no proper edge coloring of G with n colors with respect to which G is rainbow connected. Suppose, to the contrary, that ϕ: E(G) $\longrightarrow \{1, \ldots, n\}$ is a proper edge coloring of G with respect to which G is rainbow connected.

Both diametral geodesics $l_0, l_1, \ldots, l_n$ and $l_0 = r_0, r_1, \ldots, r_n$ must be rainbow. Let the colors be named so that $\phi(l_{i-1}l_i) = i$ and $\phi(r_{i-1}r_i) = i'$, $i = 1, \ldots, n$. Because ϕ is proper, $1 \neq 1'$.

At least one of the two $l_1 - r_n$ geodesics, either $l_1, \ldots, l_k, l_n, r_n$ or $l_1, \ldots, l_k, r_k, r_n$, must be rainbow. In either case, since the colors on $l_1, \ldots, l_k$ are $2, \ldots, k$, the colors on the last two edges of the path must be 1 and n. Because $\phi(l_k l_n) = n$, $\phi(l_k r_k) \neq n \neq \phi(l_n r_n)$. From these considerations we conclude that if $\phi(l_n r_n) \neq 1$ then $\phi(l_k r_k) = 1$ and $n' = \phi(r_k r_n) = n$.

Applying the same reasoning with l_1, r_n replaced by r_1, l_n, and the colors i replaced by the colors i', $i = 1, \ldots, n$, we conclude that if $\phi(l_n r_n) \neq 1'$ then $\phi(l_k r_k) = 1'$ and $n = \phi(l_k l_n) = n'$.

Since $\phi(l_n r_n)$ cannot equal both 1 and 1', we conclude that $n = n'$ and $\phi(l_k r_k) \in \{1, 1'\}$. Without loss of generality, we assume that $\phi(l_k r_k) = 1$.

There are two paths of length $\leq n$ with end vertices r_1 and l_k, namely, $r_1, r_2, \ldots, r_k, l_k$ and $r_1, r_0 = l_0, l_1, \ldots, l_k$. At least one of these must be rainbow. It cannot be the first, because $1 \notin \{1', n\}$ and $r_0, r_1, \ldots, r_n$ is rainbow, so 1 appears as a color somewhere on the path $r_1, \ldots, r_k$ as well as on $r_k l_k$.

Therefore the path $r_1, l_0, l_1, \ldots, l_k$ is rainbow. Therefore, because the path $l_0, \ldots, l_k, l_n$ is rainbow and there are only n colors, $1' = \phi(l_0 r_1) = n$. But then the path $l_0 = r_0, r_1, \ldots, r_n$ is not rainbow. This contradiction finishes the proof in Case 2.

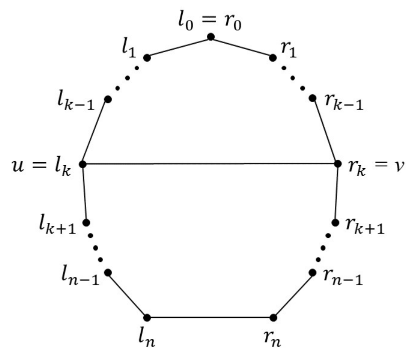


Figure 2.7: $\theta(1, s, t), s \geq 4, t \geq 5$ with s even and t odd

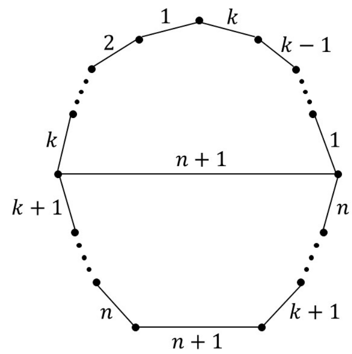


Figure 2.8: A proper strongly rainbow connective edge coloring of G with $n + 1$ colors

Case 3. $s \geq 4, t \geq 5$.

As before, let $k = \frac{s}{2}$ and n = diam(G) = $\frac{s+t-1}{2}$. The vertices of $G = \theta(1, s, t)$ will be named as in Figure 2.7.

The proper coloring shown in Figure 2.8 with $n + 1$ colors shows that psrc(G) $\leq n + 1$. It then remains to show that, when $s \geq 6$, rc$(G) > n$ and when $s = 4$, rc(G) = n while src(G), prc$(G) > n$.

Assuming only that $s \geq 4$, even, and $t \geq 5$, odd, suppose that ϕ: E(G) $\longrightarrow \{1, \ldots, n\}$ is a coloring with respect to which G is rainbow connected. Our argument will show that this is impossible when $s > 4$, and, while possible when $s = 4$, is only achievable by improper edge colorings with respect to which G is *not* strongly rainbow connected.

Both diametral geodesics $l_0, l_1, \ldots, l_n$ and $l_0 = r_0, r_1, \ldots, r_n$ must be rainbow. Let the colors along the first be $1, 2, \ldots, n$, (i.e. $\phi(l_{j-1}l_j) = j$, $j = 1, \ldots, n$), and let the colors along the second be $1', \ldots, n'$ (i.e., $\phi(r_{j-1}r_j) = j'$, $j = 1, \ldots, n$).

At least one of the paths $P_1 = [r_1, r_2, \ldots, r_n, l_n]$ and $P_2 = [r_1, r_2, \ldots, r_k, l_k, \ldots, l_n]$ is rainbow, since these are the only paths in G of length $\leq n$ from r_1 to l_n, and, for a similar reason, at least one of $P_3 = [l_1, \ldots, l_n, r_n]$ and $P_4 = [l_1, l_2, \ldots, l_k, r_k, \ldots, r_n]$ must be rainbow.

Suppose that P_2 is rainbow. Then, because $[l_0, \ldots, l_n]$ is rainbow and the colors on $[l_k, \ldots, l_n]$ are $k+1, \ldots, n$, it must be that $\{2', \ldots, k', \phi(l_k r_k)\} = \{1, \ldots, k\}$. We claim that this forces the conclusion that $\phi(l_k r_k) = 1$. Otherwise, $1 = \phi(r_{j-1} r_j) = j'$ for some $j \in \{2, \ldots, k\}$. Then neither path around $S \cup l_k r_k \simeq C_{s+1}$ from r_j to l_1 is rainbow: the color 1 appears twice on the "top" path, and the color $\phi(l_k r_k)$ appears twice on the "bottom" path. The only other path in G with end vertices r_j, l_1, the one with T as a subpath, is longer than n, and is therefore also not rainbow.

Therefore $1 = \phi(uv) = \phi(l_k r_k)$. Therefore $\{2', \ldots, k'\} = \{2, \ldots, k\}$. Since $[r_0, r_1, \ldots, r_n]$ is rainbow, it follows that $\{1', (k+1)', \ldots, n'\} = \{1, k+1, \ldots, n\}$.

Now, it cannot be that $1 = 1' = \phi(r_0 r_1)$, for, if it were, there would be no rainbow path in G from r_1 to l_1. The path of length 2, r_1, r_0, l_1, would have two 1's appearing, the other path around $S \cup uv$ would have $\phi(r_1 r_2) = 2' \in \{2, \ldots, k\}$ appearing twice, and the only other path in G from r_1 to l_1 is too long to be rainbow.

Therefore, $1 = j' = \phi(r_{j-1} r_j)$ for some $j \in \{k+1, \ldots, n\}$. If $j < n$ then there is no rainbow path in G from r_j to l_1. Therefore, $\phi(r_{n-1} r_n) = 1$ and the path $P_3 = [l_1, l_2, \ldots, l_n, r_n]$ must be rainbow. Because $[l_0, l_n]$ is rainbow, it follows that $\phi(l_n r_n) = \phi(l_0 l_1) = 1$.

From $\{1', (k+1)', \ldots, n'\} = \{1, k+1, \ldots, n\}$ and, as we now know, $n' = 1$, we conclude that $\{1', (k+1)', \ldots, (n-1)'\} = \{k+1, \ldots, n\}$. But now it follows that there is no rainbow path in G from r_{n-1} to l_n. One path from r_{n-1} to l_n around the cycle $T \cup uv$ has the color 1 appearing twice; the other has $\phi(r_{n-2} r_{n-1}) = (n-1)' \in \{k+1, \ldots, n\}$ appearing twice, and the only other path in G joining r_{n-1} to l_n is too long to be rainbow.

This shows, finally, that P_2 being rainbow means that G cannot be rainbow connected with respect to ϕ after all.

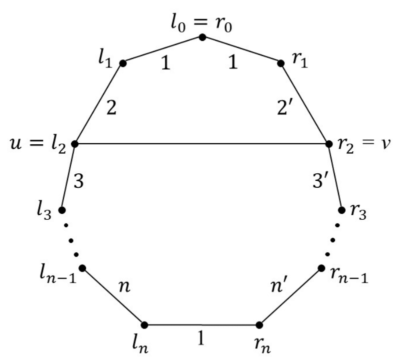
Therefore we may assume that neither P_2 nor P_4 are rainbow. Therefore both P_1 and P_3 are rainbow. By an inference previously applied, it follows that $1 = 1' = \phi(l_n r_n) = \phi(l_0 l_1) = \phi(r_0 r_1)$.

The path $P_2 - l_n = [r_1, \ldots, r_k, l_k, \ldots, l_{n-1}]$ must be rainbow, because of the other two paths in G with end vertices r_1 and l_{n-1}, one is too long to be rainbow and the other has the color 1 appearing twice. From the rainbowness of this path we conclude that $\{2', \ldots, k', \phi(l_k r_k)\} \subseteq \{1, \ldots, k, n\}$.

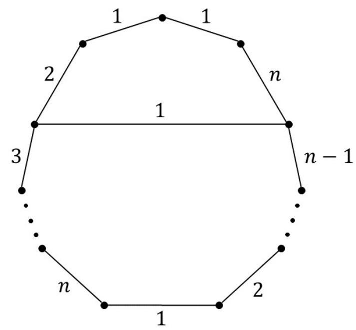
On the other hand, $\{2', \ldots, k', \phi(l_k r_k)\} \cap \{2, \ldots, k\} = \emptyset$, because, otherwise, $\phi(r_j w) = q \in \{2, \ldots, k\}$ for some $j \in \{1, \ldots, k\}$ and $w = r_{j+1}$ if $j < k$, $w = l_k$ if $j = k$. But then there is no rainbow path in G from r_j to l_1; on one of the possible paths the color q appears twice, on another 1 appears twice, and the third is too long to be rainbow.

Therefore, if $s \geq 6 \Rightarrow k = \frac{s}{2} \geq 3$, we have a proof-ending contradiction.

So assume that $s = 4$. The situation at present is depicted as follows.



The presence of a monochromatic path of length 2 in G, or in any edge-colored triangle-free graph, implies both that the coloring is not proper and that the graph is not strongly rainbow connected with respect to the coloring. Since we were forced to $l_1 l_0$ and $r_0 r_1$ having the same color by the assumption that ϕ is a rainbow connective coloring using only n colors, this proves, even in the case $s = 4$, that prc(G), src$(G) > n$. That rc(G) = n when $s = 4$ is proven by the following coloring:



□

Chapter 3

Rainbow Connectivity Parameters for Cycles with a Cut Vertex

Theorem 3.1. *Let G be the graph constructed by joining k cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$, with $n_i \geq 3$ for all $1 \leq i \leq k$, at a single cut vertex v_0. Then rc$(G) \leq \max\limits_{1 \leq i \leq k} \{n_i\}$.*

Proof. Let G be as defined above, suppose $n_1 \leq n_2 \leq \cdots \leq n_k$, and let the vertices of C_{n_i} be defined as $v_0 = v_{(i,0)}, v_{(i,1)}, \ldots, v_{(i,n_i-1)}$ for all $1 \leq i \leq k$. Define an edge coloring $\phi : E(G) \longrightarrow \{1, 2, \ldots, n_k\}$ such that, for all $1 \leq i \leq k$ and $1 \leq s \leq n_i - 1$, $\phi(v_{(i,s-1)}v_{(i,s)}) = s$ and $\phi(v_{(i,n_i-1)}v_0) = n_k$. Then ϕ is an edge coloring of G with n_k colors with respect to which no color is repeated within a given cycle C_{n_i}.

That is, for every pair of vertices of the form $v_{(i,s)}, v_{(i,t)} \in V(C_{n_i})$, with $s \neq t$ and $s, t \in \{0, 1, \ldots, n_i - 1\}$ there are two rainbow paths on C_{n_i} joining $v_{(i,s)}$ and $v_{(i,t)}$.

Now, suppose $v_{(i,s)} \in C_{n_i}$ and $v_{(j,t)} \in C_{n_j}$ for some $i \neq j$, with $i, j \in \{1, 2, \ldots, k\}$. Furthermore, suppose without loss of generality that $s \leq t$. Then the path of the form

$v_{(i,s)}, v_{(i,s-1)}, \ldots, v_{(i,1)}, v_0, v_{(j,n_j-1)}, v_{(j,n_j-2)}, \ldots, v_{(j,t)}$ is a rainbow path joining $v_{(i,s)}$ and $v_{(j,t)}$.

Thus, there is a rainbow path between all pairs of vertices of G with respect to ϕ, so rc$(G) \leq \max\limits_{1 \leq i \leq k} \{n_i\}$. □

Theorem 3.2. *Let G be the graph constructed by joining k cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$, $k \geq 2$, $n_i \geq 3$ for all $1 \leq i \leq k$, at a cut vertex v_0. Then src$(G) = \max \left\{ \text{diam}(G), \sum\limits_{1 \leq i \leq k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor \right\}$.*

Proof. Let G be constructed as above, and let the vertices of C_{n_i} be denoted as $v_0 = v_{(i,0)} = v_{(i,n_i)}, v_{(i,1)}, \ldots, v_{(i,n_i-1)}$. Let $\left\lfloor \frac{n_i-1}{2} \right\rfloor = m_i$ for all $1 \leq i \leq k$. For every pair of vertices $v_{(i,m_i)}, v_{(j,m_j)} \in V(G)$ with $i \neq j$, there is a unique geodesic joining $v_{(i,m_i)}$ and $v_{(j,m_j)}$ that passes through the cut vertex v_0.

Thus, for all $v_{(i,m_i)}, v_{(j,m_j)} \in V(G)$, every edge within the unique $v_{(i,m_i)} - v_{(j,m_j)}$ geodesic must be colored with a distinct color. Since this is true of all pairs $v_{(i,m_i)}, v_{(j,m_j)} \in V(G)$, then, we must use at least $\sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$ colors in any edge coloring of G to obtain an edge coloring with respect to which G is strongly rainbow connected. That is, src$(G) \geq \sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$. Furthermore, by Theorem 1.1, src$(G) \geq$ diam(G). Thus, both diam(G) and $\sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$ are lower bounds for src(G).

Let us begin, then, with a comparison of these values. Note that diam$(G) = \left\lfloor \frac{n_k}{2} \right\rfloor + \left\lfloor \frac{n_{k-1}}{2} \right\rfloor$. Then diam$(G) > \sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor = \left\lfloor \frac{n_k - 1}{2} \right\rfloor + \left\lfloor \frac{n_{k-1} - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_1 - 1}{2} \right\rfloor$ in the following cases:

I. $k = 2$ and at least one of the cycle lengths n_1 and n_2 is even.

II. $k = 3$, $n_1 = 3$ or $n_1 = 4$, and n_2, n_3 are both even.

In all other cases, diam$(G) \leq \sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$. Thus, providing an edge coloring with diam(G) colors with respect to which G is strongly rainbow connected for the above cases, and an edge coloring with $\sum\limits_{1 \le i \le k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$ colors for all remaining cases, suffices to prove the desired result.

In Case I, when $k = 2$, suppose n_1 is even and color the edges of C_1 around the cycle in order from $v_0, v_{(1,1)}, v_{(1,2)}, \ldots, v_{(1,n_1)}$ as follows: $1, 2, \ldots, \frac{n_1}{2}, 1, 2, \ldots, \frac{n_1}{2}$. If n_2 is also even, color the edges of C_2 around the cycle in order from $v_0, v_{(2,1)}, v_{(2,2)}, \ldots, v_{(2,n_1)}$ as follows: $\frac{n_1}{2} + 1, \frac{n_1}{2} + 2, \ldots, \frac{n_1}{2} + \frac{n_2}{2}, \frac{n_1}{2} + 1, \frac{n_1}{2} + 2, \ldots, \frac{n_1}{2} + \frac{n_2}{2}$. On the other hand, if n_2 is odd, color the edges of C_2 around the cycle in order from $v_0, v_{(2,1)}, v_{(2,2)}, \ldots, v_{(2,n_1)}$ as follows: $\frac{n_1}{2}+1, \frac{n_1}{2}+2, \ldots, \frac{n_1}{2}+\left\lfloor \frac{n_2}{2} \right\rfloor, 1, \frac{n_1}{2}+1, \frac{n_1}{2}+2, \ldots, \frac{n_1}{2}+\left\lfloor \frac{n_2}{2} \right\rfloor$. This provides an edge coloring of G with diam$(G) = \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor$ colors with respect to which G is strongly rainbow connected.

For the remainder of the proof, suppose $n_1 \leq n_2 \leq \cdots \leq n_k$.

For case II, when $k = 3$, color the edges of G as follows:

1. Color the $v_{\left(3, \frac{n_3}{2}\right)} - v_{\left(2, \frac{n_2}{2}\right)}$ geodesic along the edges $v_{\left(3, \frac{n_3}{2}\right)} v_{\left(3, \frac{n_3}{2} - 1\right)}, \ldots, v_{(3,1)} v_0$, $v_0 v_{(2,1)}, \ldots, v_{\left(2, \frac{n_2}{2} - 1\right)} v_{\left(2, \frac{n_2}{2}\right)}$ in order with the colors $1, 2, \ldots, \frac{n_3}{2} + \frac{n_2}{2} =$ diam(G).

2. Color the remaining edges of C_{n_2}, that is, $v_{\left(2,\frac{n_2}{2}\right)}v_{\left(2,\frac{n_2}{2}+1\right)}, \ldots, v_{(2,n_2-1)}v_{(2,n_2)}$, in order with the colors $1, \frac{n_3}{2}+1, \frac{n_3}{2}+2, \ldots, \frac{n_3}{2}+\frac{n_2}{2}-1$ until all remaining edges of C_{n_2} are colored.

3. Color the remaining edges of C_{n_3}, that is, $v_{(3,n_3)}v_{(3,n_3-1)}, \ldots, v_{\left(3,\frac{n_3}{2}+1\right)}v_{\left(3,\frac{n_3}{2}\right)}$, in order with the colors $2, 3, \ldots, \frac{n_3}{2}, \frac{n_3}{2}+\frac{n_2}{2}$ until all remaining edges of C_{n_3} are colored.

4. Color the edges of C_{n_1} as follows:

   (a) Color $v_0v_{(1,1)}$ and $v_0v_{(1,n_1-1)}$ with color 1.

   (b) If n_1 is odd, color $v_{(1,1)}v_{(1,2)}$ with the color 1 as well.

   (c) If n_1 is even, color $v_{(1,1)}v_{(1,2)}$ with the color used on $v_0v_{(2,1)}$ and color $v_{(1,2)}v_{(1,3)}$ with the color used on $v_0v_{(3,1)}$. That is, we color the two remaining edges of this cycle C_4 with distinct colors, each used on an a neighbor of v_0 within either C_{n_2} or C_{n_3}, and with exactly one color borrowed from each of the cycles C_{n_2} and C_{n_3}.

Then G is strongly rainbow connected with respect to this edge coloring of $\frac{n_2}{2}+\frac{n_3}{2} = \text{diam}(G)$ colors, and we have shown that $\text{src}(G) = \text{diam}(G)$ for all graphs of the form described in Case II.

Finally, it remains to show that $\text{src}(G) = \sum\limits_{1\le i\le k} \left\lfloor \frac{n_i-1}{2} \right\rfloor$ for all remaining cases by providing an edge coloring with $\sum\limits_{1\le i\le k} \left\lfloor \frac{n_i-1}{2} \right\rfloor$ colors with respect to which G is strongly rainbow connected.

Define an edge coloring $\phi : E(G) \longrightarrow \left\{1, 2, \ldots, \sum\limits_{1\le i\le k} \left\lfloor \frac{n_i-1}{2} \right\rfloor\right\}$ by coloring the edges of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ as follows:

1. For $1 \le s \le \left\lfloor \frac{n_i-1}{2} \right\rfloor$, let $\phi\left(v_{(i,s-1)}v_{(i,s)}\right) = s + \sum\limits_{1\le j<i} \left\lfloor \frac{n_j-1}{2} \right\rfloor$

2. For $\left\lceil \frac{n_i+1}{2} \right\rceil + 1 \le s \le n_i$, let $\phi\left(v_{(i,s-1)}v_{(i,s)}\right) = s - \left\lceil \frac{n_i+1}{2} \right\rceil + \sum\limits_{1\le j<i} \left\lfloor \frac{n_j-1}{2} \right\rfloor$

3. For every cycle C_{n_i} such that n_i is odd,

   (a) For all $2 \le i \le k$, let $\phi\left(v_{\left(i,\left\lfloor \frac{n_i-1}{2} \right\rfloor\right)}v_{\left(i,\left\lfloor \frac{n_i-1}{2} \right\rfloor+1\right)}\right) = \phi\left(v_{(i-1,0)}v_{(i-1,1)}\right)$.

   (b) For $i = 1$, let $\phi\left(v_{\left(1,\left\lfloor \frac{n_1-1}{2} \right\rfloor\right)}v_{\left(1,\left\lfloor \frac{n_1-1}{2} \right\rfloor\right)+1}\right) = \phi\left(v_{(k,0)}v_{(k,1)}\right)$.

4. For every cycle C_{n_i} such that n_i is even,

   (a) For all $1 < i \leq k$, let $\phi\left(v_{\left(i,\left\lfloor\frac{n_i-1}{2}\right\rfloor\right)}v_{\left(i,\frac{n_i}{2}\right)}\right) = \phi\left(v_{(i-1,0)}v_{(i-1,1)}\right)$.

   (b) For $i = 1$, let $\phi\left(v_{\left(1,\left\lfloor\frac{n_i-1}{2}\right\rfloor\right)}v_{\left(1,\frac{n_i}{2}\right)}\right) = \phi\left(v_{(k,0)}v_{(k,1)}\right)$.

   (c) For all $1 \leq i < k$, let $\phi\left(v_{\left(i,\frac{n_i}{2}\right)}v_{\left(i,\left\lceil\frac{n_i+1}{2}\right\rceil\right)}\right) = \phi\left(v_{(i+1,0)}v_{(i+1,1)}\right)$.

   (d) For $i = k$, let $\phi\left(v_{\left(k,\frac{n_i}{2}\right)}v_{\left(k,\left\lceil\frac{n_i+1}{2}\right\rceil\right)}\right) = \phi\left(v_{(1,0)}v_{(1,1)}\right)$.

In case this edge coloring is unclear, the instructions below can be followed to obtain this edge coloring of G. Note that while Steps 3 and 4, below, allow a bit more freedom when selecting which edge of an adjacent cycle to borrow from, any edge coloring achieved by the below instructions will suffice to provide an edge coloring of $\sum\limits_{1\leq i\leq k}\left\lfloor\frac{n_i-1}{2}\right\rfloor$ colors with respect to which G is strongly rainbow connected; the specific edge coloring defined by ϕ above is among these possible colorings.

1. Color the cycles $C_{n_1}, \ldots, C_{n_k}$ in order such that the path $v_{(i,0)}, \ldots, v_{\left(i,\left\lfloor\frac{n_i-1}{2}\right\rfloor\right)}$ in C_{n_i} is colored with $\left\lfloor\frac{n_i-1}{2}\right\rfloor$ distinct colors, each also distinct from the colors used on the edges of each previously colored cycle C_{n_j}, $1 \leq j < i \leq k$.

2. For each cycle C_{n_i}, skip one edge for odd cycles, and two edges for even cycles. Then wrap the $\left\lfloor\frac{n_i-1}{2}\right\rfloor$ distinct colors just applied to the path $v_{(i,0)}, \ldots, v_{\left(i,\left\lfloor\frac{n_i-1}{2}\right\rfloor\right)}$ along the edges of the path $v_{\left(i,\left\lceil\frac{n_i+1}{2}\right\rceil\right)}, \ldots, v_{(i,n_i)}$ in the same order.

3. For each odd cycle, color the remaining edge with a color "borrowed" from an edge incident with v_0 in the "previous" cycle. (If C_{n_1} is an odd cycle, borrow a color from an edge incident with v_0 in C_{n_k}.)

4. For each even cycle, color the two remaining edges such that each edge borrows a color from an edge incident with v_0 in an adjacent cycle, with each of the remaining two edges borrowing from a different adjacent cycle.

□

Corollary 3.2.1. *Let G be the graph constructed by joining k cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$, $k \geq 2$, $n_i \geq 3$ for all $1 \leq i \leq k$, at a cut vertex v_0. Then psrc$(G) = \sum\limits_{1\leq i\leq k}\left\lfloor\frac{n_i-1}{2}\right\rfloor + \sum\limits_{n_i=3,4} 1$.*

Proof. Let G be the graph constructed as above. If $n_i \geq 5$ for all $1 \leq i \leq k$, the edge coloring ϕ provided in the proof of Theorem 3.2 is a proper edge coloring of G, so $\sum_{n_i=3,4} 1 = 0$ and $\text{psrc}(G) = \sum_{1\leq i\leq k} \left\lfloor \frac{n_i-1}{2} \right\rfloor + \sum_{n_i=3,4} 1$.

To resolve the remaining cases, then, suppose $n_i = 3$ or $n_i = 4$ for some $1 \leq i \leq k$.

We showed in Theorem 3.2 that, for all $v_{(i,m_i)}v_{(j,m_j)} \in V(G)$ such that $i \neq j$, $m_i = \left\lfloor \frac{n_i-1}{2} \right\rfloor$ for all $1 \leq i \leq k$, every edge within the unique $v_{(i,m_i)} - v_{(j,m_j)}$ geodesic must be colored with a distinct color, which requires $\sum_{1\leq i\leq k} \left\lfloor \frac{n_i-1}{2} \right\rfloor$ colors when we consider all such geodesics. Without loss of generality, color the $v_{(i,m_i)} - v_{(j,m_j)}$ geodesics as defined in steps 1 and 2 of the edge coloring ϕ in the proof of Theorem 3.2.

Now, consider C_{n_i} for some $1 \leq i \leq k$ such that $n_i = 3$ or $n_i = 4$. We must place some color on the edge $v_0v_{(i,n_i-1)}$. We cannot use the same color used on the edge $v_0v_{(i,1)}$, else the edge coloring is not proper. If we use any of the remaining $\left(\sum_{1\leq i\leq k} \left\lfloor \frac{n_i-1}{2} \right\rfloor\right) - 1$ colors on the edge $v_0v_{(i,n_i-1)}$, however, the color must have already been used on some edge contained within a $v_0v_{\left(j,\left\lfloor \frac{n_j-1}{2} \right\rfloor\right)}$ or $v_0v_{\left(j,\left\lceil \frac{n_j+1}{2} \right\rceil\right)}$ geodesic for some cycle C_{n_j}, $i \neq j$, in which case the unique $v_{(i,n_i-1)} - v_{\left(j,\left\lfloor \frac{n_j-1}{2} \right\rfloor\right)}$ or $v_{(i,n_i-1)} - v_{\left(j,\left\lceil \frac{n_j+1}{2} \right\rceil\right)}$ geodesic is not rainbow. Thus, we require an additional color, distinct from the $\sum_{1\leq i\leq k} \left\lfloor \frac{n_i-1}{2} \right\rfloor$ colors placed on the edges of G so far, to obtain a proper edge coloring with respect to which G is strongly rainbow connected. Since this same argument can be subsequently applied to each remaining cycle of length 3 or 4, then, we require an additional distinct color for each cycle of length 3 or 4, and $\text{psrc}(G) = \sum_{1\leq i\leq k} \left\lfloor \frac{n_i-1}{2} \right\rfloor + \sum_{n_i=3,4} 1$. $\square$

Theorem 3.3. *Let $H_1, H_2, \ldots, H_k$ be finite, simple, connected graphs. Construct a graph G by joining the graphs $H_1, H_2, \ldots, H_k$ at a single cut vertex v_0. Then $src(G) \leq \sum_{1\leq i\leq k} src(H_i)$.*

Proof. Suppose $\text{src}(H_i) = t_i$, and let $\phi_i : E(H_i) \longrightarrow \{1, \ldots, t_i\}$ be an edge coloring of H_i with respect to which H_i is strongly rainbow connected. Define $\phi : E(G) \longrightarrow \{1, \ldots, \sum_{1\leq i\leq k} \text{src}(H_i)\}$ such that $\phi(e) = \left(\sum_{1\leq j<i} t_i\right) + \phi_i(e)$.

Since H_i is strongly rainbow connected with respect to ϕ_i for all $1 \leq i \leq k$, every pair of vertices $x, y \in V(H_i)$ has an $x - y$ rainbow geodesic in H_i with respect to ϕ_i. By our definition of ϕ, this $x - y$ rainbow geodesic in H_i is also a rainbow geodesic joining x and y

in G with respect to ϕ. Now, suppose $x \in H_i, y \in H_j$ for some $1 \leq i < j \leq k$. Then there is a rainbow geodesic joining x and v_0 in H_i under ϕ_i, and a rainbow geodesic joining y and v_0 in H_j under ϕ_j. Joining these two paths results in a rainbow geodesic joining x and y under ϕ. Thus, there exists a rainbow geodesic between every pair of vertices of G with respect to ϕ, and $\mathrm{src}(G) \leq \sum_{1 \leq i \leq k} \mathrm{src}(H_i)$. $\square$

Chapter 4

Impact of Graph Modifications on Rainbow Connectivity Parameters

4.1 Adding an Edge of the Complement of a Graph

Theorem 4.1. *If G is a connected graph and e is an edge of the complement of G, then rc$(G \cup e)$ is no greater than rc(G).*

Proof. Let G be a connected graph that is not a clique, e an edge of the complement of G, and suppose $\text{rc}(G) = t$. Furthermore, suppose G is rainbow connected with respect to an edge coloring $\phi : E(G) \longrightarrow \{1, \ldots, t\}$. That is, under the edge coloring ϕ, there exists a rainbow path joining x and y for all $x, y \in V(G)$.

Now, consider $G \cup e$. Let ϕ' be an edge coloring of $G \cup e$, defined such that $\phi'(f) = \phi(f)$ for all $f \in E(G)$ and $\phi'(e) = 1$. Then every $x - y$ rainbow path of G under ϕ is an $x - y$ rainbow path of $G \cup e$ under ϕ', hence $G \cup e$ is rainbow connected with respect to ϕ'. Thus, every edge coloring of G with respect to which G is rainbow connected can be extended to an edge coloring of $G \cup e$ with respect to which $G \cup e$ is rainbow connected by re-using any color of ϕ on e; thus $\text{rc}(G \cup e) \leq \text{rc}(G)$. □

Equality holds for $\text{rc}(G \cup e)$ and $\text{rc}(G)$ in the following class of examples: Let $G = C_{s+t}$, $G \cup e = \theta(1, s, t)$, and suppose s and t are either both even or both odd, with $s, t \geq 2$. By Theorem 1.3, then, $\text{diam}(G) = \text{diam}(C_{s+t}) = \left\lceil \frac{s+t}{2} \right\rceil$ and $\text{rc}(G \cup e) = \text{rc}\left(\theta(1, s, t)\right) = \text{diam}(\theta(1, s, t)) = \left\lceil \frac{s+t}{2} \right\rceil$ by Theorem 2.4. Thus $\text{rc}(G \cup e) = \left\lceil \frac{s+t}{2} \right\rceil = \text{rc}(G)$ for this class of examples.

By the same class of examples, the following equivalences are also possible: $\text{src}(G \cup e) = \text{src}(G)$, $\text{prc}(G \cup e) = \text{prc}(G)$, and $\text{psrc}(G \cup e) = \text{psrc}(G)$, with the latter two results requiring either $s > 2$ or $t > 2$.

On the other hand, while we have proved that $\text{rc}(G \cup e)$ can be no greater than $\text{rc}(G)$, it is possible for $\text{prc}(G \cup e) > \text{prc}(G)$ and for $\text{psrc}(G \cup e) > \text{psrc}(G)$ by the following example: Let $G = C_4$ and let e be either edge of the complement of C_4. Then $\text{prc}(G \cup e) = \text{psrc}(G \cup e) = 3 > 2 = \text{prc}(G) = \text{psrc}(G)$.

Corollary 4.1.1. *If G is a connected graph and e is an edge of the complement of G, then prc$(G \cup e)$ is no greater than prc$(G) + 1$.*

Proof. Let G, e, and $G \cup e$ be as in the proof of Theorem 4.1. Let $\text{prc}(G) = s$ and let $\phi : E(G) \longrightarrow \{1, \ldots, s\}$ be a proper edge coloring of G with respect to which G is rainbow connected. Define $\phi' : E(G) \longrightarrow \{1, \ldots, s, s+1\}$ such that $\phi'(f) = \phi(f)$ for all $f \in E(G)$ and $\phi'(e) = s + 1$. Then ϕ' is a proper edge coloring of $G \cup e$ with respect to which $G \cup e$ is rainbow connected, and $\text{prc}(G \cup e) \leq \text{prc}(G) + 1$. □

It is possible for the values of the (proper) rainbow connection number and (proper) strong rainbow connection number of G to be arbitrarily much larger than the same values for $G \cup e$. That is, adding an edge can significantly decrease the values of these parameters. Consider, for example, $G = P_n$, the path with n vertices, for $n \geq 4$. Let $G \cup e = C_n$, the cycle with n vertices, obtained by adding an edge between the endpoints of P_n. Then $\text{rc}(G) = \text{rc}(P_n) = n - 1$, and $\text{rc}(G \cup e) = \text{rc}(C_n) = \lceil \frac{n}{2} \rceil$. Thus, $\text{rc}(G) - \text{rc}(G \cup e) = n - 1 - \lceil \frac{n}{2} \rceil$, which becomes arbitrarily large as n increases. By the same class of examples and reasoning, $\text{src}(G) - \text{src}(G \cup e)$, $\text{prc}(G) - \text{prc}(G \cup e)$, and $\text{psrc}(G) - \text{psrc}(G \cup e)$ can also be made arbitrarily large.

Note that, in the above examples, $\frac{\text{rc}(G)}{\text{rc}(G \cup e)}, \frac{\text{src}(G)}{\text{src}(G \cup e)}, \frac{\text{prc}(G)}{\text{prc}(G \cup e)}, \frac{\text{psrc}(G)}{\text{psrc}(G \cup e)} < 2$. This raises the extremal question: For a finite, simple, connected graph G, are the maximum values of these ratios unbounded and, if not, what is the least upper bound for each?

It remains to determine whether $\text{src}(G + e)$ can possibly be larger than $\text{src}(G)$ and, if so, by how much. The "by how much" portion of this also remains open for psrc.

4.2 Deleting a Non-Cut Edge of a Graph

Let G be a connected graph, e a non-cut edge of G, and let $G-e$ denote the graph obtained by deleting e from $E(G)$.

Corollary 4.1.2. *If G is a connected graph and e is a non-cut edge of G, then rc$(G-e)$ is no less than rc(G).*

Proof. Since G can be obtained from $G-e$ by adding an edge of its complement, this follows immediately from Theorem 4.1. □

Similar to the consideration of equality of parameters for G and $G \cup e$, equality holds for rc(G) and rc$(G-e)$ when $G = \theta(1, s, t)$, $G - e = C_{s+t}$, and s and t are either both odd or both even with $s, t \geq 2$. For this class of examples, $\text{rc}(G) = \text{src}(G) = \text{diam}(\theta(1, s, t)) = \frac{s+t}{2} = \lceil \frac{s+t}{2} \rceil = \text{rc}(G-e) = \text{src}(G-e)$. When it is also true that either $s > 2$ or $t > 2$, it is also the case that $\text{prc}(G) = \text{psrc}(G) = \frac{s+t}{2} = \text{prc}(G-e) = \text{psrc}(G-e)$.

Borrowing another example from our consideration of how adding an edge impacts the rainbow connectivity parameters of a graph, let $G = C_n$ and $G - e = P_{n-1}$ for $n \geq 4$. Then as n becomes arbitrarily large, so do the values of $\text{rc}(G-e) - \text{rc}(G)$, $\text{src}(G-e) - \text{src}(G)$, $\text{prc}(G-e) - \text{prc}(G)$, and $\text{psrc}(G-e) - \text{psrc}(G)$. Furthermore, for this class of examples, $\frac{\text{rc}(G-e)}{\text{rc}(G)}, \frac{\text{src}(G-e)}{\text{src}(G)}, \frac{\text{prc}(G-e)}{\text{prc}(G)}, \frac{\text{psrc}(G-e)}{\text{psrc}(G)} < 2$.

Finally, again similar to our consideration of G and $G \cup e$, we can show it is possible for the proper (strong) rainbow connection number of G to be larger than the proper (strong) rainbow connection number of $G-e$ by considering the example in which G is constructed by joining C_4 and an edge e of the complement of C_4, in which case $\text{prc}(G) = \text{psrc}(G) = 3 > 2 = \text{prc}(G-e) = \text{psrc}(G-e)$.

4.3 Contracting an Edge of a Graph

Let G be a graph and denote the graph with an edge of G contracted as G'. There are many questions remaining regarding how edge contraction affects the values of the (strong) rainbow connectivity and proper (strong) rainbow connectivity parameters. A few early examples follow.

Consider $G = C_n$, $n \geq 4$. If n is even, then $\mathrm{rc}(G) = \mathrm{src}(G) = \mathrm{prc}(G) = \mathrm{psrc}(G) = \mathrm{rc}(G') = \mathrm{src}(G') = \mathrm{prc}(G') = \mathrm{psrc}(G')$. If n is odd, then $\mathrm{rc}(G) - \mathrm{rc}(G') = \mathrm{src}(G) - \mathrm{src}(G') = \mathrm{prc}(G) - \mathrm{prc}(G') = \mathrm{psrc}(G) - \mathrm{psrc}(G') = 1$. Thus, it is possible for contracting an edge of a graph to result in equal values of all rainbow connectivity parameters, or to result in a decrease in these values.

The difference $\mathrm{prc}(G') - \mathrm{prc}(G)$ can be made arbitrarily large by the following example: Let G be the graph consisting of two complete graphs K_m and K_n, $m, n \geq 4$, with a single edge e between them. By contracting e, we can make the maximum degree of G' arbitrarily large when compared to the maximum degree of G, thus increasing the number of colors needed to obtain a proper edge coloring of the modified graph and allowing for $\mathrm{prc}(G') - \mathrm{prc}(G)$ to be made arbitrarily large.

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