SOME NECESSARY CONDITIONS FOR LIST COLORABILITY OF GRAPHS AND A CONJECTURE ON COMPLETING PARTIAL LATIN SQUARES

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Benkam Benedict Bobga

Certificate of Approval:

Dean G. Hoffman,
Professor
Mathematics and Statistics

Peter D.Johnson Jr., Chair
Professor
Mathematics and Statistics

Overtoun M. Jenda,
Professor
Mathematics and Statistics

George T. Flowers,
Interim Dean
Graduate School

SOME NECESSARY CONDITIONS FOR LIST COLORABILITY OF GRAPHS AND A CONJECTURE ON COMPLETING PARTIAL LATIN SQUARES

Benkam Benedict Bobga

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Date of Graduation

Dissertation Abstract

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Benkam Benedict Bobga

Doctor of Philosophy, December 19, 2008
(M.S., East Tennessee State University,–U.S.A 2005)
(M.Sc.,Buea University–Cameroon, 1998)
(B.Sc (Honors)., Buea University–Cameroon, 1996)

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Let $\mathcal{C}$ be an infinite set of symbols, $\mathfrak{F}$ the collection of finite subsets of $\mathcal{C}$. A function $\mathcal{L}$ is a list assignment to a graph G if $\mathcal{L}$ assigns to each vertex of G a finite subset of $\mathcal{C}$. A proper $\mathcal{L}$-coloring occurs when adjacent vertices are colored with different colors from their corresponding lists. Interpreted as a theorem about proper list colorings of complete graphs, P. Hall's theorem on systems of distinct representatives inspires a generalization, a necessary condition for proper colorings, known as Hall's Condition (HC). The problem of completing an $n \times n$ partial latin square (PLS) is a list coloring problem in which G is the cartesian product of two n-cliques and $\mathcal{L}$ is determined in an obvious way.

Cropper's problem is the question: does a completion exist whenever the associated list coloring problem satisfies Hall's Condition? One will show that HC is sufficient for completion of a PLS in several circumstances addressed in well established theorems, including Ryser's theorem. Generalizations of Cropper's problem and refinements of HC for colorings recently developed by Hilton, Johnson, Lehel et al, completes this dissertation.

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My first appreciation go to all the members of the committee for their time and energy reading through this work. The vital corrections from them are greatly appreciated. I would also like to acknowledge the help of Professors A.J.W. Hilton, Chris Rodger and D. Leonard for their separate contribution to this work. Their advice and critique on both the work on latin squares and Hall's condition are greatly appreciated

Needless to say the maximum input from Professor Johnson, my advisor and supervisor, deserves very special appreciation. Professor Johnson was always there to point out the flaws or missing pieces in my proofs. Most of the material used for this research was from his personal collection of topics and problems.

Finally, I will like to thank my family both here and abroad (Cameroon) for their special prayers and advice. I very much appreciate my family's understanding during those occasions that they had to make-do without a full time FATHER, HUSBAND or son that I became due to this work. These 3 years in Auburn have been very challenging but my family was always there either physically, by phone or by emails, to make me smile and to know that all will be well.

This work is dedicated as a special birthday present to all my lovely children.

Thank you for every thing. Yours BOBGA, Benkam B.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

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Chapter 1

INTRODUCTION AND BACKGROUND KNOWLEDGE

Graph List Coloring was introduced in the late 1970s independently by Vizing [20] with the intention to study total colorings, and Erdös, Rubin and Taylor [9] with motivation from Dinitz's conjecture on n x n matrices. Many questions have been asked and answered in this subject. Many questions remain unasked and some questions, though innocent looking, are still open. In this work, one will give some answers and ask some questions concerning a fairly recent departure in the world of list-colorings.

To better understand the concepts, let us start with some elementary definitions in graph theory. The reader should refer to [8].

For a start, all our graphs are assumed to be both finite and simple. Allow a possibility for a graph G to be the *empty graph* on n vertices, if it has no edges; this is not to be confused with the *"nothing graph"* Γ in which both the vertex and edge sets are empty.

A simple graph $H = (V(H), E(H))$ is called a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$, where E is the edge set and V the vertex set.

H is a *proper* subgraph if at least one of the containments above is proper. A subgraph H of graph G is called *spanning subgraph* whenever $V(H) = V(G)$.

Let $U \subset V(G)$ and let $E(U)$ denote the subset of $E(G)$ of all edges with both ends in U. Then $H = (U, E(U))$ is called the subgraph of G *induced* by U, denoted at times by $G[U]$. In simple terms, an induced subgraph on a set of vertices $\mathcal{U} = \{u_1, u_2, \ldots, u_m\}$ of a graph G is the subgraph on vertex set U that contains every edge of G whose end points are in U.

A graph can also be induced by an edge set, $E \subset E(G)$, in which case the vertex set of the subgraph is just the set of ends of edges in E, with no additional vertices.

The *line graph*, $L(G)$ of a graph G has the edges of G as its vertices; any two vertices of of the line graph $L\ (G)$ are adjacent if the edges in G to which they correspond have a common vertex. Also, a graph H is said to be a line graph if there exists a graph G such that H is *isomorphic* to $L\ (G)$.

An *independent set* or a stable set of a graph $G=\ (V\ ,\ E)$ is a set of vertices $I \subset V$ in G no two of which are adjacent.

A *maximal independent set* will be an independent set of vertices such that adding another vertex of V into I will force an edge. Thus for a maximal independent set, any vertex from G not in the set is adjacent to at least one vertex in the maximal set.

A *maximum independent set* is a largest independent set of G. It is the largest amongst all the maximal independent sets. A *clique* in a graph is a set of pairwise adjacent vertices.

I will now turn your attention to the concepts of *Graph Colorings* in general and *List Colorings* in particular.

A *vertex coloring* (or *proper vertex coloring*, when emphasis is necessary), for a simple graph $G=\ (V\ ,\ E)$, is a function $\phi : V(G) \to \mathcal{C}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$, where the collection $\mathcal{C}$ is a set of colors or symbols.

The *independence number* of a graph G is the maximum size of an independent set of vertices.

A *list assignment* to a simple graph G is a function $\mathcal{L} :\ V(G) \to \mathfrak{F}$,

where $\mathfrak{F}$ is the collection of all finite subsets of $\mathcal{C}$. By a *proper $\mathcal{L}$-coloring* of G, I will mean a function $\varphi : V(G) \to \mathcal{C}$ satisfying, for all $u, v \in V$, *i*). $\varphi\ (u) \in \mathcal{L}\ (u)$

ii). if u, v is an edge of G, then $\varphi(u) \neq \varphi(v)$ (i.e., φ is a proper vertex coloring of G).

Condition (*ii*) is equivalent to the requirement that for each symbol $\sigma \in \mathcal{C}$, the *"support"* of σ with respect to φ, defined by

$$\varphi^{-1}(\sigma) = \{v \in V | \varphi(v) = \sigma\}$$

is an independent set of vertices.

Example 1.1

The following graph has a proper $\mathcal{L}$-coloring, where $\mathcal{L}$ is indicated in the picture. G can be

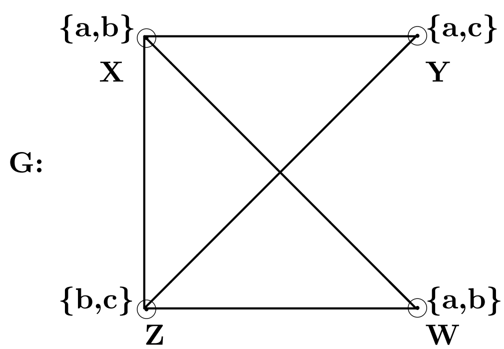


Figure 1.1: K_4 - minus - an edge with assigned list

list colored as follows: $\varphi(\mathbf{X}) = \mathbf{b}$, $\varphi(\mathbf{Y}) = \mathbf{a}$, $\varphi(\mathbf{Z}) = \mathbf{c}$, $\varphi(\mathbf{W}) = \mathbf{a}$

The *choice number* of a graph G, denoted $ch(G)$ is the smallest integer j such that there is a proper $\mathcal{L}$-coloring of G whenever $|\mathcal{L}\ (v)| \geqslant j$, for every vertex $v \in V(G)$.

Since the *chromatic number* $\mathcal{X}(G)$ is defined similarly with the list assignment constrained to be constant, clearly $\mathcal{X}(G) \leqslant ch(G)$ for every G. If $ch(G) \leqslant k$ I will some times say that G is $k-$ choosable, or k list colorable.

The *hall number* of a graph G is the smallest positive integer m such that whenever each list has size greater or equal to m and HC is satisfied, a proper $\mathcal{L}$-coloring of G exists.

The next graph is the graph $K_{3,3}$ minus two independent edges, which is the smallest graph whose choice number (the smallest list length that guarantees a proper coloring) is greater than its chromatic number.

Example 1.2

G:

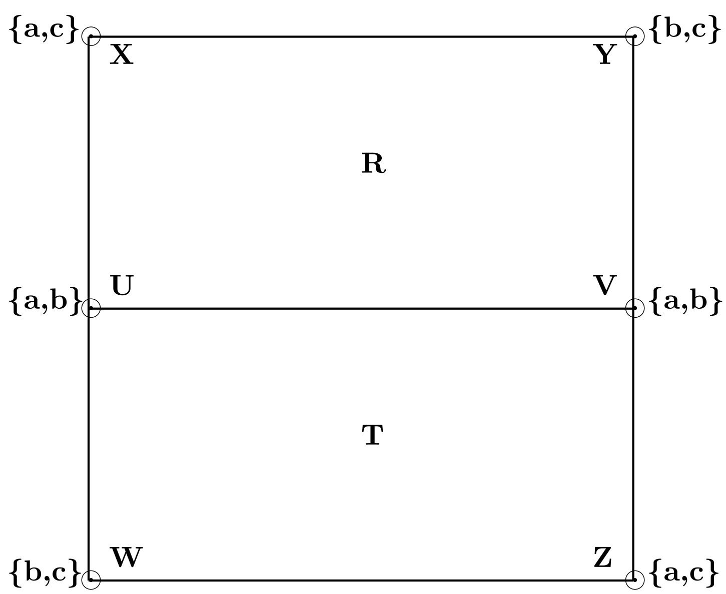


Figure 1.2: $K_{3,3}$minus two independent edges with a list assignment with lists of length 2 from which no proper coloring is possible.

G is bipartite yet is not 2 list-colorable. Indeed, if vertex **U** is colored with color **a**, consider the subgraph induced by the vertices $\{\mathbf{X,Y,U,V}\}= \mathbf{R}$, then the following coloring will be forced: $\varphi(\mathbf{X})= \mathbf{c}$ and $\varphi(\mathbf{V})= \mathbf{b}$. This leaves vertex **Y** not properly list colorable from the given list because the only two colors on $\mathcal{L}(\mathbf{Y})$ have been taken up by its neighbors. Similarly, if I let **U** to be colored with **b** and consider the subgraph induced by the vertices $\{\mathbf{U,V,W,Z}\}= \mathbf{T}$, then the following coloring will also be forced: $\varphi(\mathbf{W})= \mathbf{c}$ and $\varphi(\mathbf{V})= \mathbf{a}$. Whence vertex **Z** can not be properly list colored from this list because the only two colors on $\mathcal{L}(\mathbf{Z})$ have both been taken up by its neighbors. Thus this graph cannot be colored from the given lists.

Theorem 1.1 *P.Hall [12]*

Suppose $A_1, A_2, \ldots, A_n$ are finite sets. There exist distinct representatives: $a_1, a_2, \ldots, a_n$ such that $a_i \in A_i, i = 1, 2, \ldots, n$, if and only if for each $J \subset \{1, 2, \ldots, n\}$, $|\bigcup_{j\in J} A_j| \geqslant |J|$.

It was noticed in [14] that Hall's theorem can be viewed as a theorem about list colorings of cliques. Let V be the vertex set of the complete graph on n vertices, K_n, and let $\mathcal{L}$ be a list assignment to V. Then because any two vertices of K_n are adjacent, a proper $\mathcal{L}$-coloring of K_n is a selection of distinct representatives from the sets $\mathcal{L}(v)$, $v \in V$, and vice versa. Therefore, with V replacing the index set $\{1, 2, \ldots, n\}$, and with the sets $\mathcal{L}(v)$ replacing the sets A_i, Hall's theorem may be restated as follows:

Hall's Theorem, restated: Suppose $\mathcal{L}$ is a list assignment to a complete graph K with vertex set V. Then there is a proper $\mathcal{L}$-coloring of K if and only if for each $U \subset V$,

$$|\bigcup_{u\in U} \mathcal{L}(u)| \geqslant |U|.$$

This leads to the so called **Hall's Condition (HC)** for the existence of a proper $\mathcal{L}$-coloring in an arbitrary finite simple graph G, to which $\mathcal{L}$ might be a list assignment.

Suppose that $\mathcal{L}$ is a list assignment to G and let H be a subgraph of G. Let σ be a color in $\mathcal{C}$. Denote by H_σ (or $H(\sigma, \mathcal{L})$) the subgraph of H induced by the support set

$$\{v \in V(H) \quad | \quad \sigma \in \mathcal{L}(v)\}.$$

Hall's Condition is the following:

Hall's Condition (HC) on G,L: For each subgraph H of graph G,

$$\sum_{\sigma \in \mathcal{C}} \alpha(H_\sigma) \quad \geqslant \quad |V(H)|, \tag{1.1}$$

(where α in equation (1.1) is the vertex independence number).

Remark 1.1

Since removing edges does not lower the vertex independence number α, **HC** will be satisfied if the inequality (1.1) holds for every induced subgraph H of G.

Hall's Condition is a necessary condition for a proper $\mathcal{L}$-coloring of G because in any proper $\mathcal{L}$-coloring of any subgraph H of G, $\alpha(H(\sigma, \mathcal{L}))$ is greater or equal to the number of occurrences of σ in the coloring because the set of vertices on which σ appears in a proper coloring is an independent set.

Remark 1.2

To show that a graph G and a list assignment $\mathcal{L}$ satisfy **Hall's Condition (HC)**, it suffices to show that $G-v$ is properly $\mathcal{L}$-colorable for every vertex $v \in V(G)$ and that the inequality (1.1) in Hall's condition holds for G itself.

Indeed, every induced subgraph H of G other than G itself is an induced subgraph of $G - v$ for some vertex $v \in V(G)$. If $G - v$ is properly $\mathcal{L}$-colorable, then since **HC** is a necessary condition for proper $\mathcal{L}$-coloring, it must be that the inequality in **HC** holds for every subgraph of $G - v$.

Remark 1.3

As a follow up of the restatement of Hall's theorem, one should note that if U is a subset of $V(G)$, then the subgraph H of the complete graph K induced by U forms a clique. So for each symbol $\sigma \in \mathcal{C}$, $\alpha(H(\sigma, \mathcal{L})) = 1$ if $\sigma \in \bigcup_{u \in U} \mathcal{L}(u)$, otherwise it is zero. That is, as a function of σ, $\alpha(H(\sigma, \mathcal{L}))$ is the characteristic function of $\cup_{u\in U}\mathcal{L}(u)$. Consequently, (1.1) becomes

$$| \cup_{u\in U} \mathcal{L}(u)| \quad = \quad \sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \quad \geqslant \quad |V(H)| = |U|. \tag{1.2}$$

Thus Hall's Condition is equivalent to the condition in Hall's theorem on the Systems of Distinct Representatives (SDR) in the case of complete graphs. Therefore Hall's theorem can be restated: if K is a complete graph and $\mathcal{L}$ is a list assignment to K, then there is a proper $\mathcal{L}$-coloring of K if and only if K and $\mathcal{L}$ satisfy Hall's Condition.

Example 1.3

Consider the four cycle, C_4 with the given list assignment as shown below. First note

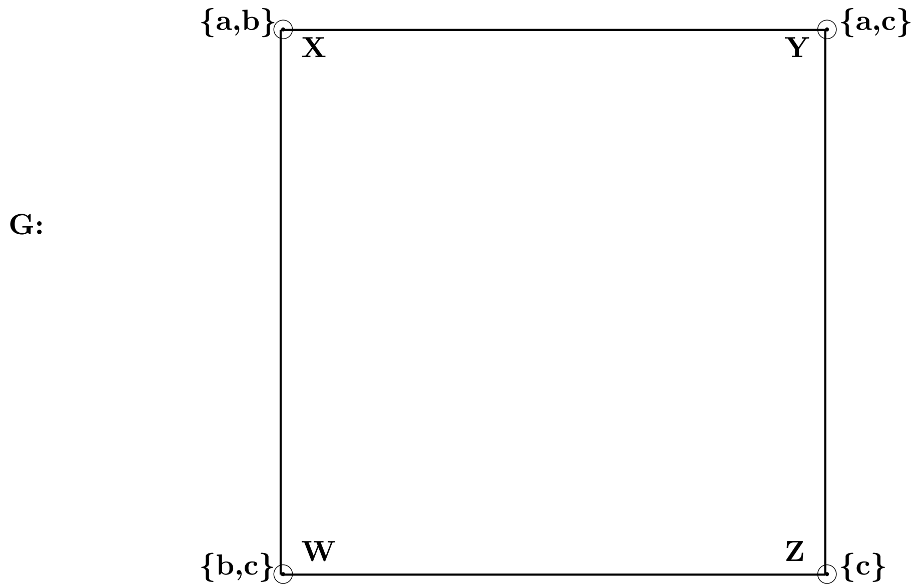


Figure 1.3: Four Cycle

that graph G is not properly list colorable from the given list because once I color the vertex **Z** with **c**, both **Y** and **W** are forced to be colored **a** and **b** respectively. This leaves vertex **X** without any available colors since we require adjacent vertices to be colored differently. To show that **HC** holds, note that for every vertex $v \in V(G)$, the graph $G - v$ is properly list colorable from the lists (which implies the desired inequality for every proper induced subgraph H of G) and $\alpha(G_a) = \alpha(G_b) = 1$, $\alpha(G_c) = 2$, hence the alpha sums equals the order of G, 4. Thus Hall's Condition is satisfied.

Theorem 1.2 ***Hilton and Johnson:*** *[14] A graph G has the property that for all $\mathcal{L}$, if G, $\mathcal{L}$ satisfy Hall's Condition then there is a proper $\mathcal{L}$-coloring of G, if and only if every block of G is a clique.*

Recall that a block of a graph G is a subgraph maximal with respect to being connected and containing no cut-vertex.

Lemma 1.1 ***Hall's Condition*** *holds for G and $\mathcal{L}$ if and only if the inequality in* ***Hall's Condition*** *(equation (1.1)) holds for each connected induced subgraph of G.*

Cropper's Problem

Let $G = K_n \square K_n \equiv$ the line graph of $K_{n,n}$, normally represented as an n x n array of cells. Let some cells of G be filled in with symbols from the set $\mathcal{C}_n = \{1, 2, \ldots, n\}$, so that no symbol appears more than once in any row or column. So I have a partial latin square. Let the list for an unfilled cell $v(i, j)$ be defined in an obvious way as follows:

$\mathcal{C}_n$ - {symbols appearing in the filled cells in row i, column j}.

The list for a filled cell (of size one) will simply be the symbol in that cell.

Cropper posed the following question: is Hall's Condition on such a graph G with such a list assignment $\mathcal{L}$ sufficient for a proper $\mathcal{L}$ - coloring?

(Briefly, is Hall's Condition on G and $\mathcal{L}$ sufficient for a completion of a partial latin square?).

A theorem of Ryser's answers in the affirmative Cropper's question when the filled-in parts of the partial latin square form a subrectangle.

Theorem 1.3 ***Ryser's Theorem:*** *[19]*

An $r \times s$ latin rectangle R on n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be completed to a latin square of order n if and only if

N_R $(\sigma_i) \geqslant r+s-n$ for each $i = 1, 2, \ldots, n$, where N_R (σ_i) is the number of occurrences of the symbol σ_i in R.

Theorem 1.4 ***Bobga, Hilton and Johnson:*** *[2]*

A latin rectangle on n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be completed to a latin square of order n if and only if inequality (1.1) holds for $H = G = K_n \square K_n$.

This statement turns out to be "equivalent" to Ryser's Theorem, in the sense that each theorem is easily derivable from the other. As a corollary, when the filled-in parts form a subrectangle, Hall's Condition on G and $\mathcal{L}$ implies completability.

Theorem 1.5 ***Bobga and Johnson:*** *[2]*

If the filled-in part of a partial latin square form a subrectangle minus one cell, then Hall's Condition on G and $\mathcal{L}$ is sufficient for completion of the partial latin square.

A proof of theorem (1.5) and a variant of theorem(1.4) will be given in chapter 2.

Part of what I will try to do is to recast results by Andersen and Hilton [1], Buchanan and Ferencak [3], Rodger [18], and Hoffman [17] in the form: when the filled-in parts belong to a certain class of configurations, Hall's Condition on G and $\mathcal{L}$ suffices for completion. (This is not exactly right, as will be explained; Hoffman's result is an exception). In all the other cases so far, the theorems say that not only is Hall's Condition on G and $\mathcal{L}$ sufficient, but that only a few instances of the inequality in Hall's Condition suffice for proper $\mathcal{L}$-coloring of G (i.e. for a completion).

Inspired by Cropper's problem are the following concepts, whose application to Cropper's problem have not yet been found but are of independent interest.

Let G and $\mathcal{L}$ satisfy Hall's Condition and let H be a subgraph of G. One will say that H is an $\mathcal{L}$-*tight* subgraph of G if

$$\sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, \mathcal{L})) = |V(H)| \tag{1.3}$$

If H is non-induced but is $\mathcal{L}$-tight, then the subgraph generated by the vertices of H will also be $\mathcal{L}$-tight. If H is $\mathcal{L}$-tight, then in every proper $\mathcal{L}$-coloring of H, every color σ has to appear $\alpha(H(\sigma, \mathcal{L}))$ times; i.e. a maximum independent subset of $V(H_\sigma)$ is colored σ for each σ.

The next set of definitions will consist of different modifications to Hall's Condition on G and $\mathcal{L}$. They are actually stronger than Hall's Condition for an arbitrary graph G and list assignment $\mathcal{L}$. However, when the underlining graph is complete, then they all coincide; each is equivalent to the existence of a proper $\mathcal{L}$-coloring of the graph. Each definition demands first that Hall's Condition on G and $\mathcal{L}$ be satisfied (although this is not explicit in the case of **HC***, below).

A simple graph G is said to satisfy **Hall's Condition plus** on G and $\mathcal{L}$, denoted **HC+**, if

(i) G and $\mathcal{L}$ satisfy **HC**, and

(ii) there is an indexed family $[S_\sigma : \sigma \in \mathcal{C}]$ of independent subsets of $V(G)$ satisfying:

(a) S_σ lies in $V(G(\sigma, \mathcal{L}))$ for all $\sigma \in \mathcal{C}$ (i.e., for all vertices $v \in S_\sigma$, $\sigma \in \mathcal{L}(v)$)

(b) for each $\mathcal{L}$-tight subgraph H of G, $|S_\sigma \bigcap V(H)| = \alpha\ (H(\sigma, \mathcal{L}))$.

I shall call such a family $[S_\sigma : \sigma \in \mathcal{C}]$ an **HC+** *satisfying family*.

A simple graph G is said to satisfy **Hall's Condition plus plus** on G and $\mathcal{L}$, denoted **HC++**, if G and $\mathcal{L}$ satisfy **HC+**, and in some **HC+** satisfying family $[S_\sigma : \sigma \in \mathcal{C}]$ of independent subsets of $V(G)$, the S_σ's are pairwise disjoint. (That is, $S_\sigma \bigcap S_\tau = \emptyset$ for $\sigma \neq \tau$).

If φ is a proper $\mathcal{L}$-coloring of G, then the supports of $\sigma \in \mathcal{C}$,

$$S_\sigma = \varphi^{-1}(\sigma) = \{v \in V | \varphi(v) = \sigma\}, \quad \sigma \in \mathcal{C} \tag{1.4}$$

form an **HC++** satisfying family for G and $\mathcal{L}$.

So **HC++** (and hence **HC+**) is a necessary condition for a proper $\mathcal{L}$-coloring of G.

The next definition is due to Jeno Lehel. It deals with the conditional independence number of a graph relative to a subgraph.

For an induced subgraph H of G, the *conditional independence number* of graph G with respect to H, denoted $\alpha(G|H)$, is the maximum cardinality of an independent set I of $V(G)$ such that $|I \bigcap V(H)| = \alpha(H)$.

This says that $\alpha(G|\Gamma) = \alpha(G)$, where Γ is the "NOTHING" graph.

A simple graph G with list assignment $\mathcal{L}$ is said to satisfy **Hall's Condition star** , denoted **HC***, if

$$\sum_{\sigma \in \mathcal{C}} \alpha(H_\sigma | T_\sigma) \quad \geqslant \quad |V(H)|, \tag{1.5}$$

holds for every induced subgraph H of G and every induced $\mathcal{L}$-tight subgraph T of H.

Consider the "NOTHING" graph Γ to be an $\mathcal{L}$-tight subgraph of any induced subgraph H of G, and $\Gamma_\sigma = \Gamma$ for each $\sigma \in \mathcal{C}$.

Therefore if a simple graph G and list assignment $\mathcal{L}$ satisfy **HC***, then they satisfy **Halls Condition**.

A simple graph G with list assignment $\mathcal{L}$ is said to satisfy **Hall's Condition star star** (denoted **HC****) if

$$\sum_{\sigma \in \mathcal{C}} \min_{T \lhd H} \alpha(H_\sigma | T_\sigma) \quad \geqslant \quad |V(H)|, \tag{1.6}$$

holds for every induced subgraph H of G, where the minimum is taken over all $\mathcal{L}$-tight induced subgraphs T of H.

Because Γ is allowed as an induced tight subgraph of every graph for any list assignment, we clearly have **HC**** $\Longrightarrow$ **HC*** $\Longrightarrow$ **HC**.

Certainly, since **HC** is a part of or implied by the requirements for **HC**$\diamondsuit$ for $\diamondsuit \in \{+, ++, *, **\}$, it must be that each **HC**$\diamondsuit \Longrightarrow$ **HC** for $\diamondsuit \in \{+, ++, *, **\}$.

I will now show that **HC**** is a necessary condition for the existence of a proper $\mathcal{L}$-coloring of a graph G.

Theorem 1.6 ***HC***** *is a necessary condition for the existence of a proper $\mathcal{L}$-coloring of a graph G.*

Proof: Suppose there exists such a proper $\mathcal{L}$-coloring φ of a graph G; we show that **HC**** holds. Let H be an induced subgraph of G, I will show that the inequality in equation (1.6) holds.

Suppose T is an $\mathcal{L}$-tight induced subgraph of H and suppose $\sigma \in \mathcal{C}$. Then

$$\sum_{\sigma \in \mathcal{C}} \alpha(T_\sigma) \quad = \quad |V(T)|, \tag{1.7}$$

and the number of times that σ appears as a color assigned by the function φ on T,

i.e., $|\varphi^{-1}(\sigma) \bigcap V(T)|$, is actually $\alpha(T_\sigma)$.

Therefore the number of times σ (for an arbitrary color σ) appears as a color in H must satisfy the inequality

$$|\varphi^{-1}(\sigma) \bigcap V(H)| \quad \leqslant \quad \alpha(H_\sigma | T_\sigma) \tag{1.8}$$

because $\varphi^{-1}(\sigma) \bigcap V(H) \subseteq V(H_\sigma)$ is an independent set of vertices of H_σ which extends (by T being $\mathcal{L}$-tight), a maximum independent set of vertices of T_σ.

(Recall that the vertices in $\varphi^{-1}(\sigma)\bigcap V(H)$ have σ on their $\mathcal{L}$ list and $\alpha(H_\sigma|T_\sigma)$ is the conditional independence number).

Since inequality (1.8) holds for every $\mathcal{L}$-tight induced subgraph T of H, it must be that

$$|\varphi^{-1}(\sigma)\bigcap V(H)| \leqslant \min_{T\lhd H}\alpha(H_\sigma|T_\sigma) \tag{1.9}$$

$$\Longrightarrow \sum_{\sigma\in\mathcal{C}}|\varphi^{-1}(\sigma)\bigcap V(H)| \leqslant \sum_{\sigma\in\mathcal{C}}\min_{T\lhd H}\alpha(H_\sigma|T_\sigma) \tag{1.10}$$

$$\Longrightarrow \sum_{\sigma\in\mathcal{C}}|V(H)| \leqslant \sum_{\sigma\in\mathcal{C}}\min_{T\lhd H}\alpha(H_\sigma|T_\sigma) \tag{1.11}$$

which establishes the inequality (1.6) and hence the claim of the theorem.

□

G is a **Hall** graph if whenever G, $\mathcal{L}$ satisfy HC, then there is a proper $\mathcal{L}$-coloring of G.

Theorem 1.7 ***Hilton and Johnson:*** *[14] G is **Hall** if and only if every block of G is a clique.*

(This is just a restatement of theorem 1.2).

Definition 1.1 *G is a **Hall+** graph if whenever G, $\mathcal{L}$ satisfy **HC+**, there is a proper $\mathcal{L}$-coloring of G.*

Definition 1.2 *G is a **Hall++** graph if whenever G, $\mathcal{L}$ satisfy **HC++**, there is a proper $\mathcal{L}$-coloring of G.*

Definition 1.3 *G is a **Hall*** graph if whenever G, $\mathcal{L}$ satisfy **HC***, there is a proper $\mathcal{L}$-coloring of G.*

Definition 1.4 *Finally, G is a* ***Hall***** *graph if whenever G, $\mathcal{L}$ satisfy* ***HC*****, *there is a proper $\mathcal{L}$-coloring of G.*

I shall investigate these graph properties with a view to finding analogues of theorem 1.7, and possibly connections between the properties, but it is early days in this area. I shall, for example, prove that all cycles are **Hall+**. Also, it will be shown that if a clique were to be attached to a **Hall**$\diamondsuit$ graph at a cut-vertex, then the resulting graph will still belong to **Hall**$\diamondsuit$. This result holds for all $\diamondsuit \in$ { empty string, +, ++, *, ** }.

As to what happens when one joins two **Hall**$\diamondsuit$ graphs with a cut-edge and whether or not cycles were **Hall*** remain unanswered.

Lemma 1.2 *Suppose that $\mathcal{L}$ is a list assignment to G, $\tau \in \mathcal{C}$ and $\tilde{\mathcal{L}}$ is obtained from $\mathcal{L}$ by replacing τ in the lists on each component of $G(\tau, \mathcal{L})$ by a new color, for that component, that does not appear in $\bigcup \mathcal{L}(v)$, where the union is taken over all $v \in V(G)$.*

Then:

(a) G and $\mathcal{L}$ satisfy ***HC*** $\Longleftrightarrow$ *G and $\tilde{\mathcal{L}}$ satisfy* ***HC****;*

(b) G and $\mathcal{L}$ satisfy ***HC+*** $\Longleftrightarrow$ *G and $\tilde{\mathcal{L}}$ satisfy* ***HC+****;*

(c) G and $\mathcal{L}$ satisfy ***HC++*** $\Longleftrightarrow$ *G and $\tilde{\mathcal{L}}$ satisfy* ***HC++****;*

(d) G and $\mathcal{L}$ satisfy ***HC**** $\Longleftrightarrow$ *G and $\tilde{\mathcal{L}}$ satisfy* ***HC*****;*

(e) G and $\mathcal{L}$ satisfy ***HC***** $\Longleftrightarrow$ *G and $\tilde{\mathcal{L}}$ satisfy* ***HC******;*

(f) There is a proper $\mathcal{L}$-coloring of G $\Longleftrightarrow$ there is a proper $\tilde{\mathcal{L}}$-coloring of G.

Proof: Parts (a), (b), (c), (d) and (e) depart from the observation that the independence number of any graph is the sum of the independence numbers of its components. From this it follows that if Hall's Condition is satisfied, then a subgraph is tight if and only if each component of it is tight, and also T is $\mathcal{L}$-tight if and only if it is $\tilde{\mathcal{L}}$-tight. From there the arguments become quite laborious but straight forward, and I will omit them.

The proof of part (f) is as follows:

If there exists a proper $\mathcal{L}$-coloring of G, then replace each occurrence of τ with its replacement on the vertex's list in the formation of $\tilde{\mathcal{L}}$. This will also be a proper $\tilde{\mathcal{L}}$-coloring of G.

Conversely, if there is a proper $\tilde{\mathcal{L}}$-coloring of G, we proceed as above and replace each color τ-clone by τ.

$\square$

Corollary 1.1 *$G \in$ **Hall** $\diamondsuit$, where $\diamondsuit \in \{$ empty string, $+$, $++$, $*$, $**$ $\}$ $\Longleftrightarrow$ G is $\mathcal{L}$-colorable for every $\mathcal{L}$ satisfying **HC** $\diamondsuit$ with G and such that for each color $\sigma \in \mathcal{C}$, $G(\sigma, \mathcal{L})$ is connected.*

Proof:

The forward part is clear by definition.

Conversely, suppose G satisfies the proposed less stringent requirement. Suppose $\mathcal{L}$ is a list assignment for G that satisfies **HC** $\diamondsuit$ but for some colors, the support is not connected. Then for one of those colors, one can modify $\mathcal{L}$ to $\tilde{\mathcal{L}}_1$ and if there is another different bad color, one can repeat the procedure on $\tilde{\mathcal{L}}_1$ and get a new list say $\tilde{\mathcal{L}}_2$. This procedure can be repeated as necessary. One then ends up with a new list assignment $\tilde{\mathcal{L}}$ of the desired properties (i.e. every color has connected support). By assumption, G is $\tilde{\mathcal{L}}$-colorable. Therefore, by repeated application of the backwards implication in (f) of lemma 1.2, G is $\mathcal{L}$-colorable. Therefore $G \in$ **Hall** $\diamondsuit$.

$\square$

Chapter 2

CROPPER'S PROBLEM AND RECASTED RESULTS

2.1 Introduction to Cropper's Problem

The problem of completing a partial n by n latin square is a *List Coloring Problem* in which the graph is the cartesian product of two n-cliques and the lists are determined in an obvious way by the filled-in cells. Hall's Condition (fairly well known) is a necessary condition on a graph with a list assignment for the existence of a proper coloring. Matt Cropper some years ago asked whether Hall's Condition is sufficient for the completion of a PLS. I will show in this chapter that the answer is a "yes" when the filled-in cells form

-a sub-rectangle, or

-a sub-rectangle minus one cell.

In the former case, Hall's Condition implies Ryser's Condition.

A **partial latin square** of order n is an n x n array of n^2 cells in which the cells may contain either 0 or 1 symbols, the symbols all lying in a set $\mathcal{C}_n = \{1, 2, \ldots, n\}$ of n elements. Furthermore, no symbol can occur in more than one cell in any row or column. If there are no empty cells, then the partial latin square is a latin square.

Over the years there has been quite a considerable amount of interest in the question of completing partial latin squares, ranging from, but not limited to, Ryser's theorem [19], Marshall Hall's theorem [11], and the Evans' Conjecture [10], to the result of C. Colbourn [5] that it is an $NP-complete$ problem to decide whether a partial latin square can be completed.

Before delving into the results for this section, I will state the results mentioned above due to their importance in the studies of latin square completions. Marshall Hall used Phillip Hall's theorem (theorem 1.1) on SDR to prove the following important theorem on latin square completion.

Theorem 2.1 ***Marshall Hall, 1945*** *[11] Every r by n latin rectangle, $0 \leqslant r \leqslant n$, on n symbols, can be completed to a latin square of order n.*

Proof: [7]

The cases $r = 0$ and $r = n$ are both trivial. So it will be assumed that P is an r x n latin rectangle, $0 < r < n$. It will be shown that P can be extended to an $(r+1)$ x n latin rectangle, and hence ultimately to a latin square of order n.

For $0 \leqslant j \leqslant n$, let S_j denote the set of all $x \in$ {1,2,. . . ,n} such that x does not occur in the column j of P. Note that $\mid Sj \mid = n - r$, and since P is a latin rectangle each $x \in$ {1,2,. . . ,n} belongs to exactly $n - r$ of the sets $S_1, S_2, \ldots, S_n$. It will be shown that there exists a SDR $\langle\ s_1, s_2, \ldots, s_n\rangle$ for the collection $S_1, S_2, \ldots, S_n$ and consequently that the $(r+1)$ x n latin rectangle $P\ \cup\{(r+1, 1, s_1), (r+1, 2, s_2),\ldots, (r+1, n, s_n)\}$ is an extension of P.

Assume that such a System of Distinct Representatives does not exist. Then there exists some $k \in$ {1,2,. . . ,n} and some choice of $S_{j1}, S_{j2}, \ldots, S_{jk}$ for which $\mid S_{j1} \cup S_{j2} \cup \ldots \cup S_{jk} \mid < k$. But then $\mid S_{j1} \mid + \mid S_{j2} \mid + \ldots + \mid S_{jk} \mid = k(n-r)$ implies that there must exist some symbol $x \in$ {1,2,. . . ,n} occurring in more than $n - r$ of the sets $S_{j1}, S_{j2}, \ldots, S_{jk}$. However this is a contradiction as each $x \in$ {1,2,. . . ,n} belongs to exactly $n - r$ of the sets $S_1, S_2, \ldots, S_n$. Hence a SDR must exist and so the r x n latin rectangle P can be extended to $P\ \cup\{(r+1, 1, s_1), (r+1, 2, s_2),\ldots, (r+1, n, s_n)\}$. If $P\ \cup\{(r+1, 1, s_1), (r+1, 2, s_2),\ldots, (r+1, n, s_n)\}$ is a latin square of order n the argument is complete, otherwise the above process is repeated and $P \cup \{(r+1, 1, s_1), (r+1, 2, s_2), \ldots, (r+1, n, s_n)\}$ is extended to an $(r+2)$ x n latin rectangle, and so on until a latin square is obtained. □

The next two results are important in discussions relating to latin squares.

Conjecture 2.1 ***T.Evans' Conjecture****, 1960 [10] Every partial latin square of order n containing at most $n-1$ filled cells is completable.*

The theorem of Ryser given in the introductory chapter is also important. Later in this chapter, I shall restate it and use a lemma to establish it.

Theorem 2.2 ***Colbourn*** *[5] Deciding whether a latin square can be completed is NP-complete.*

Example 2.1

1	3	4	8	2	**{57}**	**{56}**	**{567}**
4	2	1	3	8	**{57}**	**{56}**	**{567}**
5	6	3	1	7			**{24}**
7	5	2	4	6			**{13}**
6	7	8	2	5			**{134}**
		{57}	**{57}**		6		**{123457}**
		{56}	**{56}**			7	**{123456}**
{23}	**{14}**	**{567}**	**{567}**	**{134}**	**{123457}**	**{123456}**	8

Figure 2.1: An Incomplete PLS with some lists indicated

This PLS due to L. D. Anderson is incompletable and HC fails (and so it is not a counter example to the conjecture that the answer in Cropper's problem is yes!). Indeed, look at the four cells $(1,6),(1,7),(2,6),(2,7)$. Note that 6 and 7 can each be used exactly once and so 5 must be used twice. Thus each of rows 1, 2, columns 6, 7 will have a 5 on it. A careful look at cells (6,3), (6,4), (7,3), (7,4) indicate that 6 and 7 will each be used exactly once, leaving the 5 to be used twice to complete the 4 cells. Hence each of rows 6, 7, columns 3, 4 will also have a 5. Thus in any attempted completion row 8 and column 8 cannot have a 5. However, to fill up the 8 cells in row 8, every number has to be used once. Hence one cannot complete the PLS.

To see that HC fails, consider the subgraph H induced by all the preassigned cells together with the cells:

(1,6), (1,7), (2,6), (2,7), (6,3), (6,4), (7,3), (7,4) and $(i,8)$, $(8,i)$ for i= $1,2,\cdots,7$.

Then $\mid V(H) \mid = 50$. Moreover, $\alpha(H_1)$=5, $\alpha(H_2)$=6, $\alpha(H_3)$=5, $\alpha(H_4)$=5, $\alpha(H_5)$=8, $\alpha(H_6)$=8, $\alpha(H_7)$=8, $\alpha(H_8)$=4.

$$\Longrightarrow \sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \quad = \quad 49 \quad < \quad 50.$$

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \quad < \quad |V(H)|$$

means that the inequality in HC fails and so HC is not satisfied for G.

□

Remark 2.1

If the inequality for HC does not hold for some set of cells, it must also not hold for the subset of that set of cells consisting of the empty, unprescribed, cells. This is because adding a prescribed cell to a set of cells increases both sides of the inequality 1.1 by exactly

1. Thus we could simply have considered verifying HC only for the 22 cells within braces (excluding the preassigned cells). The conclusion would have been the same.

2.2 Cropper's Problem

Let $G = K_n \Box K_n \equiv$ the line graph of $K_{n,n}$, normally represented as an $n \times n$ array of cells.

Let some cells of G be filled in with symbols from the set $\{1, 2, 3, \ldots, n\} = C_n$, with the understanding that no symbol appears more than once in any row or column. So I have a partial latin square. Below is an example with $n = 7$.

		7	3	5		
				6	1	5
	6					3
6						
4						
2		1				
3	4	2				

Figure 2.2: 7×7 partial latin square.

This one was provided to us by A. J. W. Hilton, and is an example due to Ron Aharoni, the significance of which A. J. W. H. has now forgotten. But this particular one was thought to possibly have great significance in the work on Cropper's problem. It is not completable. We did spend quite some time in studying this example with the hope of using it as a counter example to the problem. However, it turned out that there is at least one "bad" set of cells (one that does not satisfies Hall's Condition, see below) and so cannot serve as a counter example. A. J. W. Hilton recently emailed us an example due to J. Goldwasser believed to be a counter example to Cropper,s problem. If that were to be true, then Cropper's problem will be solved in the negative and the question will then become: *when* does HC imply completability?

As stated in chapter 1, let the list for an unfilled cell $v(i,j)$ be $C_n\setminus$ {symbols appearing in the filled cells in row i, column $j\}$. The list for a filled cell (of size one) will simply be the symbol in that cell. The list assignment corresponding to each cell in figure (2.2) is as follows:

{1}	{12}	7	3	5	{246}	{1246}
{7}	{237}	{34}	{24}	6	1	5
{157}	6	{45}	{12457}	{1247}	{2457}	3
6	{12357}	{345}	{12457}	{12347}	{23457}	{1247}
4	{12357}	{356}	{12567}	{1237}	{23567}	{1267}
2	{357}	1	{4567}	{347}	{34567}	{467}
3	4	2	{1567}	{17}	{567}	{167}

Figure 2.3: A partial latin square with its accompanying list assignment.

To show that this cannot serve as a counter example to Cropper's problem, I shall show that there is a set of cells that satisfy the inequality in Hall's condition. Such a set will be referred to as a *bad set*. Note that any bad set remains bad if any of the preassigned cells are added (or removed) from it. One such bad set is the following:

{**1**}	{1**2**}	////	////	////	////	////
{**7**}	{2**3**7}	{3**4**}	////	////	////	////
////	////	{4**5**}	////	////	////	////
////	{**1**235**7**}	{**3**45}	{12**45**7}	{12347}	{**2**3457}	{1247}
////	{123**5**7}	{35**6**}	{**1**256**7**}	{1**2**37}	{2**3**567}	{1267}
////	{357}	////	{45**67**}	{**3**4**7**}	{34567}	{**4**67}
////	////	////	{1567}	{**1**7}	{**5**67}	{1**67**}

Figure 2.4: Bad set indicated by the lists from figure (2.2).

Hall's inequality is not satisfied because letting ($\alpha(i)$ stand for $\alpha(H_i)$), $\alpha(1) = 4$, $\alpha(2) = 3$, $\alpha(3) = 4$, $\alpha(4) = 3$, $\alpha(5) = 4$, $\alpha(6) = 3$, $\alpha(7) = 5$. Hence

$$\sum_{\sigma=1}^{7} \alpha(\sigma) \quad = \quad 26 \quad < \quad 27 \quad = \quad |\, V(H)\,| \,.$$

Theorem 2.3 *The answer to the question in Cropper's problem is 'yes' when the filled-in cells form either*

(i) a subrectangle, or

(ii) a subrectangle minus one cell.

We will see that the first case is "equivalent" to Ryser's theorem [19]: An $r \times s$ latin rectangle R on n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be extended to a latin square of order n if and only if $N_R(\sigma_i) \geqslant r + s - n$ for each $i = 1, \ldots, n$, where $N_R(\sigma_i)$ is the number of occurrences of the symbol σ_i in rectangle R.

Proof:

First I show that HC implies completability for an $n \times n$ partial latin square, when the filled-in parts form a sub-rectangle. (The proof is due to Hilton, 1988 [13], but this statement of what was proved is new). WOLOG, the filled-in sub-rectangle R is in the upper left hand corner of the partial latin square. Let $G = K_n \square K_n$ consist of four portions as in Figure 2.5, with the filled-in part R of dimension $r \times s$ as in Ryser's theorem. Assume HC holds.

For $\sigma \in C_n$, $\alpha(G_\sigma) \leqslant \alpha(G) = n$. Therefore, Hall's Condition applied to the graph G implies the following: $n^2 \geqslant \sum_{\sigma \in C_n} \alpha(G_\sigma) \geqslant |V(G)| = n^2$, where the second inequality results from HC with $H = G$.

Thus for each symbol $\sigma \in C_n$, $\alpha(G_\sigma) = n$. A maximum independent set of vertices (cells) in G_σ must have a representative from every row and column, and therefore must contain a total of $r - N_R(\sigma)$ cells from A as well as $s - N_R(\sigma)$ cells from B. Therefore, for all $\sigma \in C_n$,

$$\begin{aligned} n = \quad & \alpha(G_\sigma) \geqslant N_R(\sigma) + r - N_R(\sigma) + s - N_R(\sigma) \\ = \quad & r + s - N_R(\sigma). \end{aligned}$$

Hence, $N_R(\sigma) \geqslant r + s - n$. Whence, G is completable by Ryser's theorem.

It is worth noting that in case (i), completability was implied by the inequality in Hall's condition for only one choice of H, namely, the whole graph. We shall see a similar phenomenon in the proof for Case (ii).

Before proceeding to that case, I will give another proof of the sub-rectangle case, not essentially different from the proof just given, but which will be convenient to refer to

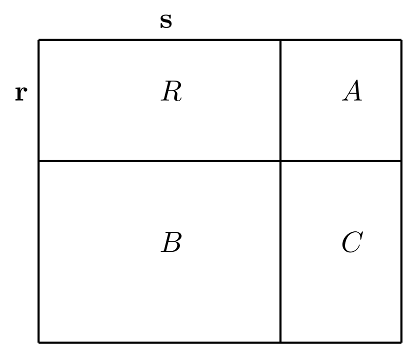


Figure 2.5: Rectangular blocks

later. This time, take H to be the copy of $K_r \Box K_n$ represented by the first r rows of the partial latin square. Since, for each $i \in C_n$, $\alpha(H_i) \leqslant \alpha(H) = r$, by the assumption that HC is satisfied, $|V(H)| = rn \leqslant \sum_{i=1}^{n} \alpha(H_i) \leqslant rn$; therefore, $\alpha(H_i) = r$ for each $i \in C_n$. Therefore a maximum independent set of cells in H_i must have a representative from each of the r rows of H, and therefore i must occur on the lists of at least $r - N_R(i)$ cells in different columns, in the last $n - s$ columns of H. It follows that for each $i \in \{1, \ldots, n\}$, $r - N_R(i) \leqslant n - s$, which implies the condition in Ryser's theorem and hence the claim (i). Again the completability arises from the application of the inequality in Hall's condition to a single induced subgraph of the underlying graph $G = K_n \Box K_n$.

Now suppose that the filled-in parts of the partial latin square form a sub-rectangle minus one cell. Permuting rows and columns, which amounts to merely renaming vertices in the factors K_n of $G = K_n \Box K_n$, I can assume that the filled-in cells occupy R', which is an $r \times s$ rectangle in the upper left hand corner of the square, but with the $(1, s)$ cell removed. Let v_{ij} stand for the (i, j) cell, $1 \leqslant i,\ j \leqslant n$, and let $R = \{v_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$, so $R' = R \backslash \{v_{1s}\}$. Let $\mathcal{L}$ denote the list assignment induced by the filled-in cells.

First observe that $\mathcal{L}(v_{1s}) \neq \emptyset$, for, if $\mathcal{L}(v_{1s}) = \emptyset$, then taking $H = \{v_{1s}\}$ we would have $\sum_{\alpha \in C_n} \alpha(H(\sigma, \mathcal{L})) = 0 < 1 = |V(H)|$, contradicting the assumption that Hall's condition is satisfied. Next, I may as well suppose that for each $\sigma \in \mathcal{L}(v_{1s})$, the latin square cannot be completed with σ in cell v_{1s}. In view of the second proof of part (i) of the theorem, given above, this means that if I fill v_{1s} with σ and then define list assignment $\mathcal{L}'$ by

$\mathcal{L}'(v_{1k}) = \mathcal{L}(v_{1k}) \setminus \{\sigma\}, s < k \leqslant n, \mathcal{L}'(v_{1s}) = \{\sigma\}, \mathcal{L}'(v_{is}) = \mathcal{L}(v_{is}) \setminus \{\sigma\}, r < i \leqslant n$, and $\mathcal{L}' = \mathcal{L}$ otherwise (since $\sigma \in \mathcal{L}(v_{1s})$, σ does not appear in any of the filled cells $v_{2s}, \ldots, v_{rs}$), it must be that the inequality in Hall's condition does not hold for $\mathcal{L}'$ and the subgraph $H = K_r \square K_n$ of G induced by the vertices (cells) v_{ij}, $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$. That is, $\sum_{\tau \in C_n} \alpha(H(\tau, \mathcal{L}')) < rn = |V(H)|$. It must be, therefore, that for some $\tau \in \mathcal{C}_n$, $\alpha(H(\tau, \mathcal{L}')) < r = \alpha(H)$. Since $\mathcal{L}$ satisfies Hall's condition, by assumption, so $\sum_{i \in \mathcal{C}_n} \alpha(H(i, \mathcal{L})) \geqslant rn$, so $\alpha(H(i, \mathcal{L})) = r$ for all $i \in \mathcal{C}_n$, and since, in H, $\mathcal{L}'$ differs from $\mathcal{L}$ only on the cells $v_{1s}, v_{1,s+1}, \ldots, v_{1n}$, where $\mathcal{L}'(v_{1s}) = \{\sigma\}$ and $\mathcal{L}'(v_{1k}) = \mathcal{L}(v_{1k}) \setminus \{\sigma\}$, $s < k \leqslant n$, it must be that $\tau \in \mathcal{L}(v_{1s})$.

The fact that $\alpha(H(\sigma, \mathcal{L})) = r$ means that there is a choice of r cells, one from each row of H, no two in the same column, which have σ on their $\mathcal{L}$-list. Since σ does not occur in the filled-in cells in column s, below v_{1s}, in H (and I know that it does not because $\sigma \in \mathcal{L}(v_{1s})$), I can let v_{1s} be one of those r cells. It follows that $\alpha(H(\sigma, \mathcal{L}')) = r$, so $\tau \in \mathcal{L}(v_{1s}) \setminus \{\sigma\}$. Further, it must be that for every choice of r independent cells from H bearing τ on their $\mathcal{L}$-list, v_{1s} is the cell chosen from row 1. Consequently, letting $H' = H - v_{1s}$, I have that $\alpha(H'(\tau, \mathcal{L})) = r - 1$.

Now, σ was an arbitrary element of $\mathcal{L}(v_{1s})$; letting τ now play the role just played by σ, we deduce the existence of $\eta \in \mathcal{L}(v_{1s})$, $\eta \neq \tau$, such that $\alpha(H'(\eta, \mathcal{L})) = r - 1$. Since

$\alpha(H'(i,\mathcal{L})) \leqslant \alpha(H') = r$ for every $i \in \mathcal{C}_n$, I have

$$\begin{aligned}\textstyle\sum_{i\in\mathcal{C}_n}\alpha(H'(i,\mathcal{L})) &= \alpha(H'(\tau,\mathcal{L})) + \alpha(H'(\eta,\mathcal{L})) + \textstyle\sum_{i\in\mathcal{C}_n\setminus\{\tau,\eta\}}\alpha(H'(i,\mathcal{L})) \\ &\leqslant r-1+r-1+r(n-2) \\ &= rn-2 < rn-1 = |V(H')|,\end{aligned}$$

contradicting the assumption that G and $\mathcal{L}$ satisfy Hall's condition. So it must be that the latin square can be completed by filling in v_{1s} with a color from $\mathcal{L}(v_{1s})$ after all.

It is worth noting that while the completability in case (i) was implied by the inequality in Hall's condition for a single choice of H, either $H = G$ or $H = K_r \Box K_n$, in case (ii) the completability was implied by three instances of the inequality in Hall's condition: $H = \{v_{1s}\}$, $H = K_r \Box K_n$, and $H = K_r \Box K_n - v_{1s}$. (In all of this, by $K_r \Box K_n$, we mean of course a particular choice of $K_r \Box K_n$ in G, the one constituted by the first r rows of the array).

□

2.3 Recasted Results

In this section, I will examine several theorems on completing PLS, as mentioned in the introductory chapter, due to Buchanan and Ferencak [3], Andersen and Hilton [1], Rodger [18] and a result of Hoffman [17] which deals with commutative latin squares. We shall discover that each of them except Hoffman's, possibly, can be restated in the form:

if the prescribed cells form such - and - such a configuration, then the satisfaction of the inequality in Hall's Condition for just a few special choices of induced subgraph(s) suffices for the existence of a completion.

2.3.1 Recasting

Is HC sufficient for the completion of a partial latin square? Most think no!

What I have found: in 6 different theorems giving necessary and sufficient conditions for a partial latin square of a certain sort (i.e. with the prescribed cells filling a certain configuration) to be completable, not only do the theorems confirm that HC suffices in those cases, but in each case, a small number of the $2^{n^2} - 1$ inequalities constituting HC suffices for completion.

I now present, in the form of a list, the re-casted results. I shall indicate the prescribed cells and the H's such that the inequality in HC for those H's suffices for completability.

The proofs of the claims in 1 and 2 are given in section 2.2 above. The proofs of the claims in parts 3,4,5 and 6 will use the well-known ancestor of Ryser's Theorem, due to Marshall Hall [11], theorem 2.1.

1. H.J. Ryser (1951 [19])

Prescribed cells or configuration: top left $r \times s$ sub-rectangle.

Cell set such that HC inequality implies completability: In section 2.2 it is shown that either the full array of n^2 cells, or the cells of the first r rows, are such that the HC inequality (1.1) for the subgraph induced by those cells implies completability. There are numerous other choices; for instance, the upper right $r \times (n - s)$ sub-rectangle will do the job. This can be inferred from the proof in section 2.2 and the fact that in the case of the graph and list assignment involved in a partial latin square, if inequality (1.1) holds for some H then it will also hold for any subgraph obtained from H by adding or deleting prescribed cells. Thus the inequality (1.1) for the upper right $r \times (n - s)$ sub-rectangle in a PLS in which the prescribed cells occupy the upper left $r \times s$ sub-rectangle implies inequality (1.1) for

$H \equiv K_r \square K_n$, the first r rows, and by the proof of theorem 2.3 that implies that the PLS is completable.

Here is a direct demonstration that (1.1) for $H \equiv K_r \square K_{n-s}$, the upper right $r \times (n-s)$ rectangle in a PLS with the cells of the upper left $K_r \square K_s$ prescribed, implies completability. As in the proof of theorem 2.3, Ryser's theorem will be the main engine of the proof.

Each $i \in \mathcal{C}_n$ appears on the lists of all the cells of $r - N_R(i)$ rows of H, and on no other lists in H. (We appeal to the terminology in the proof of theorem 2.3). Therefore, H_i is an $(r - N_R(i)) \times (n-s)$ sub-rectangle of H, so $\alpha(H_i) = min[(r - N_R(i)), (n-s)]$. Therefore, assuming (1.1) holds for H, I have

$$|V(H)| = r(n-s) \leqslant \sum_{i \in \mathcal{C}_n} \alpha(H_i) \leqslant \sum_{i \in \mathcal{C}_n} (r - N_R(i))$$

$$= \sum_{i \in \mathcal{C}_n} r - \sum_{i \in \mathcal{C}_n} N_R(i) = nr - rs = r(n-s).$$

Therefore, $r - N_R(i) = min[(r - N_R(i)), (n-s)]$ for each $i \in \mathcal{C}_n$. Thus, for each i, $r - N_R(i) \leqslant n - s$, which implies completability, by Ryser's theorem. $\square$

2. B. B. Bobga and P. D. Johnson (2007 [2])

Prescribed cells or configuration: the upper left r by s sub-rectangle minus one cell. Without loss of generality, the missing cell will be $v_{1,s}$ in the first row and the s^{th} column.

Cell sets or subgraphs induced by them such that HC inequality implies completability: $H \in \Big\{ \{v_{1,s}\}, K_r \square K_n, (K_r \square K_n) - v_{1,s} \Big\}$, where $K_r \square K_n$ is in the first r rows.

See theorem 2.3 and its proof.

3. H.L. Buchanan and M. N. Ferencak (2000 [3])

Prescribed cells or configuration: first s rows together with the first d cells of row $s+1$.

Cell sets such that HC inequality implies completability:

The $H's$ here are precisely the $2^{(n-d)}-1$ subgraphs induced by the non-empty subsets of the set of the last $n-d$ cells in row $s+1$.

Proof:

What is to be proven is that the inequalities in Hall's Condition for the described choices of H suffice for the given PLS to have a completion.

By theorem 2.1, for completion, it suffices that the last $n-d$ cells in row $s+1$ be properly filled from their lists, so that the result is an $(s+1)\times n$ latin rectangle. Because of the way the lists are formed, it therefore suffices that the clique on $n-d$ vertices induced by those $n-d$ cells be properly colorable from their lists. But then the original theorem of Phillip Hall [12], reconstrued to be about list-colorings of cliques, says that $2^{(n-d)}-1$ inequalities constituting Hall's condition in this case suffices for a completion.

□

4. H.L. Buchanan and M. N. Ferencak (2000 [3])

Prescribed cells: top right $s\times(n-d)$ sub-rectangle together with the first d cells of row $s+1$.

Cell sets such that HC inequality implies completability: The $H's$ here are precisely the $2^{(n-d)}-1$ sets referred to in 3 above together with the upper left $s\times d$ sub-rectangle.

Proof:

Buchanan and Ferencak prove in [3] that one can complete this PLS if and only if the following three conditions are satisfied:

1. There are no collections X of columns and $\sum$ of symbols such that

   (i) the d prescribed symbols in row $s+1$ lie in $\sum$;

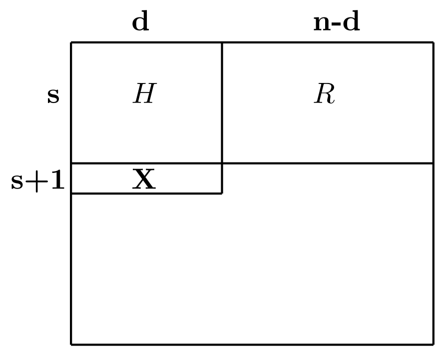


Figure 2.6: Rectangular blocks with R and C filled-in

(ii) X is contained in the set of columns $d+1$ to n;

(iii) Every symbol σ in the set of all the symbols $\mathcal{C}_n \setminus \sum$ appears in each column in X in the first s rows

(iv) $|X| \quad > \quad |\sum| - d.$

2. Each symbol occurs (prescribed) at least $s-d$ times in the first s rows.

3. No symbol occurring (prescribed) exactly $s-d$ times in the first s rows also occurs in the first d cells of row $s+1$.

Condition 1 above was given by Buchanan and Ferencak as necessary and sufficient for completion in case 3, above, and it is easy to see why. If there existed a set X of columns and a set $\sum \subseteq \mathcal{C}_n$ satisfying $(i)-(iv)$ under 1, just above, then the cells in row $s+1$ in the columns of X would have their lists contained in $\sum \setminus$ {symbols in the first d cells of row $s+1$}, so that $|X| \quad > \quad |\sum| - d$ constitutes a contradiction of one of the HC inequalities associated with one of the $2^{(n-1)}-1$ non-empty subsets of the last $n-d$ cells in row $s+1$. Therefore, if all of those inequalities hold, then there are no such X and $\sum$, and the first condition is satisfied.

Now let H be the subgraph of $K_n \Box K_n$ consisting of the $K_s \Box K_d$ depicted in the upper left of the array. For each symbol $\sigma \in \mathcal{C}_n$, let $N(\sigma)$ be the number of appearances of σ in the $s \times (n-d)$ rectangle R. Clearly, σ appears on the list of a cell in H if and only if the cell is in one of the $s - N(\sigma)$ rows in which σ does not appear in R, and σ does not appear as the prescribed entry of row $s+1$ in the column of the cell.

Therefore, H_σ is induced by the cells of a subrectangle of dimensions

$(s - N(\sigma)) \times d$ if $\sigma \notin D = \{$symbols appearing in the first d cells of row $s+1\}$, and of dimensions $(s - N(\sigma)) \times (d-1)$ if $\sigma \in D$. Thus $\alpha(H_\sigma) = min(s - N(\sigma), d)$ if $\sigma \notin D$ and $\alpha(H_\sigma) = min(s - N(\sigma), d-1)$ if $\sigma \in D$. Inequality (1.1) for H implies

$$sd = |V(H)| \leqslant \sum_{\sigma \in \mathcal{C}_n} \alpha(H_\sigma) \leqslant \sum_{\sigma \in \mathcal{C}_n} (s - N(\sigma)) = ns - \sum_{\sigma \in \mathcal{C}_n} N(\sigma) = ns - s(n-d) = sd. \quad (2.1)$$

Since the ends of equation (2.1) are equal, the middles must also be equal. Hence all the inequalities are equalities.

Hence $s - N(\sigma) \leqslant d$ if $\sigma \notin D$ and $s - N(\sigma) \leqslant d-1$ if $\sigma \in D$, which imply 2 and 3 above.

□

5. L. D. Andersen and A. J. W. Hilton (1980 [1])

Prescribed cells or configuration: upper left r by s sub-rectangle, S, of a t by t array, together with some or all of the cells $(r+i, s+i)$, $i = 1, 2, \ldots, t-s$, $0 < r < s < t$; if $0 < r = s < t$, the prescribed diagonal elements outside of R are in some or all of the cells $(r+i, s+i)$, $i = 1, 2, \ldots, t-s-1$. [Taking into account the equivalences afforded by simultaneous row and column permutations, the last requirement really says that if

$0 < r = s < t$, the prescription on the diagonal elements outside of R misses at least one cell.]

Take H to be the r by $(t-s)$ rectangle in the upper right of the array.

Proof:

For $\sigma \in \mathcal{C}_t$, let $N_S(\sigma)$ denote the number of occurrences of σ in S and let $f^*(\sigma)$ denote the

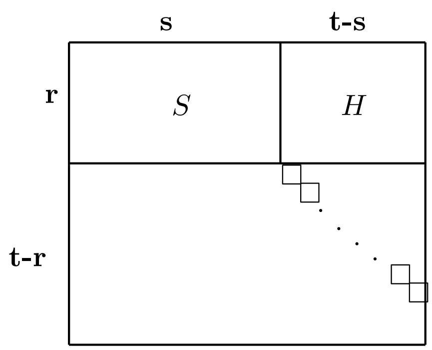


Figure 2.7: Rectangular blocks with S and local diagonals filled-in

number of occurrences of σ in the prescribed cells among those on the diagonal $(r+i, s+i)$, $1 \leqslant i \leqslant t-s$, if $r < s$, or $1 \leqslant i \leqslant t-s-1$, if $r = s$.

Note that

$$\sum_{\sigma \in \mathcal{C}_t} f^*(\sigma) \quad \leqslant \quad \begin{cases} t-s-1, & \text{if r = s ;} \\ t-s, & \text{if r < s .} \end{cases}$$

Anderson and Hilton prove that the PLS with prescribed entries filling the rectangle S and part of the diagonal outside of S, as described, can be completed to a Latin Square of order t if and only if $N_S(\sigma_i) \geqslant r+s-t+f^*(\sigma_i)$ for each $\sigma \in \mathcal{C}_t$.

For any symbol $\sigma \in \mathcal{C}_t$, H_σ is induced by the cells in the $(r - N_S(\sigma)) \times (t-s-f^*(\sigma))$ subrectangle formed by the intersection of the $r - N_S(\sigma)$ rows, among the first r rows, in which σ does not appear in that row in S, and the $t-s-f^*(\sigma)$ columns, among the last $t-s$

columns, such that σ does not appear in that column in a prescribed cell on the diagonal outside of S.

Therefore $\quad \alpha(H_\sigma) = min(r - N_S(\sigma), t - s - f^*(\sigma))$ for each σ.

Then inequality (1.1) implies

$$r(t-s) \quad = \quad |V(H)| \quad \leqslant \quad \textstyle\sum_{\sigma \in \mathcal{C}} \alpha(H_\sigma) \quad \leqslant \quad \textstyle\sum_{\sigma \in \mathcal{C}} (r - N_S(\sigma))$$

$$= \quad \textstyle\sum_{\sigma \in \mathcal{C}} r \quad - \quad \textstyle\sum_{\sigma \in \mathcal{C}} N_S(\sigma) \quad = \quad tr - rs \quad = \quad r(t-s).$$

Ends equal means middles must also be equal. We conclude that for each σ,

$r - N_S(\sigma) \quad \leqslant \quad t - s - f^*(\sigma) \quad \Longleftrightarrow \quad N_S(\sigma) \quad \geqslant \quad r + s - t + f^*(\sigma)$ as required.

$\square$

6. C. A. Rodger (1984 [18])

Prescribed cells or configuration: the top left n by n subrectangle, R of a t by t array, $t \geqslant 2n+1$, together with all the main diagonal elements, (i,i) for $i = n+1, n+2, \ldots, t$.

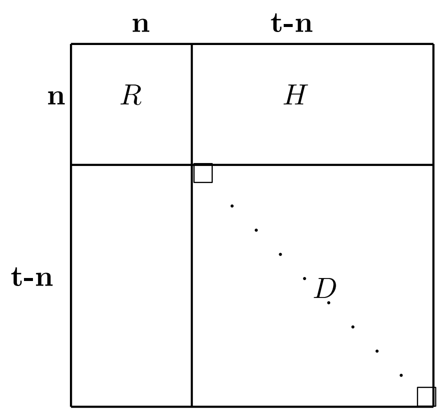


Figure 2.8: $t \times t$ square with block R and main diagonal D filled-in

For $\sigma \in \mathcal{C}_t$, let $N_R(\sigma)$ denote the number of occurrences of σ in R and let $f(\sigma)$ denote the number of occurrences of σ on the diagonal outside of R, in the cells (i,j), $n+1 \leqslant i \leqslant t$.

Rodger proves that a PLS with such a prescription configuration can be completed to a LS of order t if and only if: [Andersen and Hilton's requirement in part 5 above]

(i) $N_R(\sigma) \geqslant 2n - t + f(\sigma)$ for each $\sigma \in \mathcal{C}_t$;

(ii) for each $\sigma \in \mathcal{C}_t$, if $N_R(\sigma) = n$ then $f(\sigma) \neq t - n - 1$; and

(iii) if R is a LS on the symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $t = 2n + 1$, then $\sum_{i=1}^{n} f(\sigma_i) \neq 1$.

Consider $V(H) = \{v(i,j) : 1 \leqslant i \leqslant n, \quad n < j \leqslant t\}$.

Then $\quad |V(H)| \quad = \quad n(t-n)$.

For this choice of H, the inequality in Hall's Condition will imply condition (i) of this theorem; the proof is just like that in item 5, above, but I give it any way.

As in 5, for each symbol $\sigma \in \mathcal{C}_t$, H_σ is an $(n - N_R(\sigma)) \times (t - n - f(\sigma))$ subrectangle of H.

Then for each $\sigma \in \mathcal{C}_t$,

$\alpha(H_\sigma) \quad = \quad min(t - n - f(\sigma), n - N_R(\sigma))$.

If the inequality in 1.1 holds for H, then

$$
\begin{aligned}
n(t-n) \quad &\leqslant \quad \sum_{\sigma \in \mathcal{C}_t} \alpha(H_\sigma) \quad = \quad \sum_{\sigma \in \mathcal{C}_t} min(t - n - f(\sigma), n - N_R(\sigma)) \\
&\leqslant \quad \sum_{\sigma \in \mathcal{C}_t} (n - N_R(\sigma)) \\
&= \quad \sum_{\sigma \in \mathcal{C}_t} n \quad - \quad \sum_{\sigma \in \mathcal{C}_t} N_R(\sigma) \quad = \quad nt - n^2 \quad = \quad n(t-n).
\end{aligned}
$$

It then follows that $min(t - n - f(\sigma), n - N_R(\sigma)) = n - N_R(\sigma))$ for each σ; hence

$n - N_R(\sigma) \quad \leqslant \quad t - n - f(\sigma) \quad$ for each σ, which is condition (i).

Next I consider condition (*ii*). It is my claim that if the inequality (1.1) holds for each $H \in \{\text{column } n+1, \text{ column } n+2, \dots, \text{ column } t\}$, then condition (*ii*) also holds.

Proof:

Suppose (*ii*) does not hold; then for some σ, $N_R(\sigma) = n$ and $f(\sigma) = t - n - 1$. Then σ appears at all but one place on D, the part of the main diagonal outside R. Let H be the column of the cell on D where σ does not appear. Then σ appears on no list on H and H is a clique, hence $\alpha(H_\sigma) = 0$ and $\alpha(H_\tau) \leqslant 1$ for all $\tau \in \mathcal{C}_t$.

So

$$\sum_{\tau \in \mathcal{C}_t} \alpha(H_\tau) \quad \leqslant \quad t-1 \quad < \quad t \quad = \quad |V(H)|\,.$$

This contradicts HC and so it must be that if HC holds, then condition (*ii*) holds.

Finally, consider condition (*iii*).

When $t = 2n+1$, (*iii*) is guaranteed by the inequality (1.1) for the following choices of H (notice there will be $t-n$ of them in all):

$$H^{(k)} = \{v(i,j) : 1 \leqslant j \leqslant n, \quad n+1 \leqslant i \leqslant t\} \cup \{v(i,j) : n+1 \leqslant i \leqslant t, \quad 1 \leqslant j \leqslant n\}$$
$$\cup \{k^{th} \text{ row}\} \cup \{k^{th} \text{ column}\}, \text{ for } k = n+1, n+2, \dots, t.$$

you have $\left|V(H^{(k)})\right| = 2n(t{-}n) + 2(t{-}n) - 1 = 2n(n{+}1) + 2n + 1$ because $t = 2n+1 \Leftrightarrow t - n = n+1$. I will show that if condition (*iii*) fails, then inequality (1.1) for one of the $H^{(k)}$ will fail.

Suppose $t = 2n+1$, and R is a latin square on the symbols $\sigma_1, \sigma_2, \dots, \sigma_n$, but $\sum_{i=1}^{n} f(\sigma_i) = 1$. Then exactly one symbol σ_i, $i \in \{1, 2, \dots, n\}$, appears on D, and it appears exactly once, say in the k^{th} row and k^{th} column as shown in figure 2.9 above.

Because R is a LS on $\sigma_1, \sigma_2, \dots, \sigma_n$, for $1 \leqslant j \leqslant n$, σ_j does not appear on the list of any cell $v(i,j)$, $n+1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant n$ or $1 \leqslant i \leqslant n$, $n+1 \leqslant j \leqslant t$.

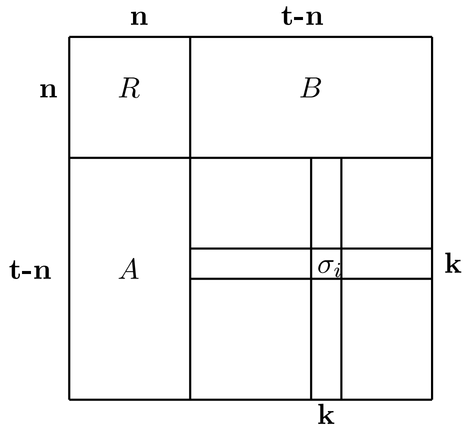


Figure 2.9: $t \times t$ square with indicated $H^{(k)}$ filled-in

If $j \neq i$ then σ_j appears on the lists of the cells of row k in columns $n+1, n+2, \ldots, t$, except row k, and the same for column k, so $\alpha(H_{\sigma_j}) = 2$. Clearly, $\alpha(H_{\sigma_i}) = 1$. Letting $\sigma_{n+1}, \ldots, \sigma_t$ denote the symbols in $\mathcal{C}_t \setminus \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, for $n+1 \leqslant j \leqslant t$ it is easily seen that H_{σ_j} consists of a $(t-n-f(\sigma_j)) \times n$ sub-rectangle of the lower left $(t-n) \times n$ rectangle A, together with an $n \times (t-n-f(\sigma_j))$ subrectangle of the upper right $n \times (t-n)$ rectangle B, together with $t-n-f(\sigma_j)-1$ cells in each of row and column k, outside of A and B.

To see how to estimate $\alpha(H_{\sigma_j})$ in this case, assume that $k = n+1$ (which could be actually achieved by permuting rows and columns, an isomorphism of the graph). Then $\alpha(H_{\sigma_j})$ is contained in the union of two rectangles, one $(n+1-f(\sigma_j)) \times (n+1)$ and the other $(n+1) \times (n+1-f(\sigma_j))$. (Since $t = 2n+1, t-n = n+1$).

Therefore

$\alpha(H_{\sigma_j}) \quad \leqslant \quad 2min[n+1, n+1-f(\sigma_j)] \quad = \quad 2(n+1-f(\sigma_j))$, for each $j = n+1, \ldots, 2n+1$. But if $f(\sigma_j) = 0$, $\alpha(H_{\sigma_j}) \quad \leqslant \quad 2n+1 \quad = \quad 2(n+1-f(\sigma_j))-1$, because it is not possible to find $2n+2$ independent cells in the union of A, B, row $n+1$ and column $n+1$.

And $f(\sigma_j) = 0$ for some $j \in \{n+1, \ldots, 2n+1\}$, because

$$\sum_{j=n+1}^{2n+1} f(\sigma_j) \quad = \quad n+1-1 \quad = \quad n,$$

and there are $n+1$ values of j summed over.

Therefore,

$$\begin{aligned}
\sum_{j=1}^{2n+1} \alpha(H_{\sigma_j}) \quad &= \quad \sum_{j=1}^{n} \alpha(H_{\sigma_j}) \quad + \quad \sum_{j=n+1}^{2n+1} \alpha(H_{\sigma_j}) \\
&\leqslant \quad 2(n-1)+1+2\sum_{j=n+1}^{2n+1} [(n+1) - f(\sigma_j)] - 1 \\
&= \quad 2n-1+2(n+1)^2 - 2\sum_{j=n+1}^{2n+1} f(\sigma_j) - 1 \\
&= \quad 2n-1+2(n^2+2n+1) - 2n - 1 \\
&= \quad 2n^2+4n \quad < \quad 2n^2+4n+1 \quad = \quad \left|V(H^{(k)})\right|
\end{aligned}$$

Thus $\quad \sum_{j=1}^{t} \alpha(H_{\sigma_j}^{(k)}) \quad < \quad \left|V(H^{(k)})\right| \quad$ which contradicts HC.

Thus for this choice of $H^{(k)}$, Hall's Condition fails. By contraposition, one can then conclude that if HC holds, then so also will condition (*iii*).

□

There is one other result related to Cropper's Problem that I wish to inspect.

7. D. G. Hoffman (1983 [17])

This theorem gives two necessary and sufficient conditions for a partial (incomplete) commutative LS to be completable to a commutative LS. One is equivalent to the satisfaction of an HC inequality for a choice of one H, but the other does not seem to arise from any combination of the inequalities in HC.

Referring to figure 2.10, let $c(i)$ be the number of appearances of the symbol i in all of A, $d(i)$ be the number of occurrences of the symbol i on the main diagonal of A, and $t(i)$ be the number of appearances of the symbol i along the tail of A. We have the following theorem by Hoffman:

Theorem 2.4 ***D.G. Hoffman 1983****:([17]) Let A be a commutative incomplete latin square of size r and order n, with content c and diagonal d. Let $t : \boldsymbol{N} \rightarrow \mathbb{N}$, with $\sum_{i \in N} t(i) = n - r$. Then A can be embedded in a commutative latin square B with tail t if and only if, for each $i \in \boldsymbol{N}$, (1). $d(i) + t(i) \equiv n (\mathrm{mod}\ 2)$, and (2). $2r + t(i) \leqslant n + c(i)$.*

Since this theorem is about completing partial commutative latin squares, it would not finish off Cropper's Problem if it turned out that Hall's Condition is not sufficient for completability in such circumstances, but such an example would bear on a more general question:

which graphs G have the property that for every partial proper $\mathcal{L}$-coloring of G with $\mathcal{X}(G)$ colors, $\{1, 2, \ldots, \mathcal{X}(G)\}$, there is an extension to a proper coloring of G with $\mathcal{X}(G)$ colors whenever G and the list assignment $\mathcal{L}$ defined by the partial coloring in the obvious way, (for an uncolored vertex v, I have: $\mathcal{L}(v) = \{1, 2, \ldots, \mathcal{X}(G)\}$-{colors on $\mathcal{N}(v)$}) satisfy Hall's Condition?

I will now show that the second condition of Hoffman's result is equivalent to HC for some prescribed set of vertices.

Proof:

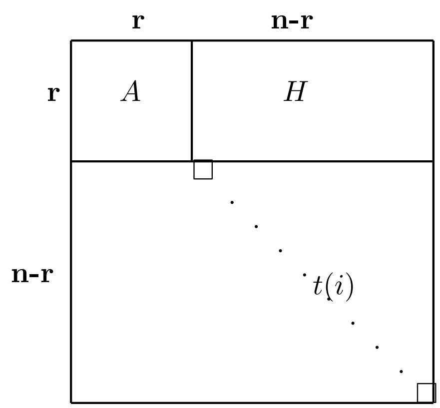


Figure 2.10: $n \times n$ commutative square with block A and main diagonal filled-in

Referring to figure 2.10, given A and its tail (main diagonal outside A) prescribed, let H be defined by

$V(H) = \{V_{ij} : 1 \leqslant i \leqslant r, \quad r+1 \leqslant j \leqslant n\}$

I claim that the inequality in HC for this H will imply condition (2) : $2r + t(i) \leqslant n + c(i)$ for each symbol $i = 1, 2, \ldots, n$.

For each symbol i, we see that i appears on the lists in cells of $r - c(i)$ rows of H where they intersect $n - r - t(i)$ columns.

So $H(i, \mathcal{L}) \equiv H_i$ is an $(r - c(i)) \times (n - r - t(i))$ rectangle.

$\Longrightarrow \quad \alpha(H_i) \quad = \quad min(r - c(i), n - r - t(i))$ and so by HC, I have

$$|V(H)| \quad = \quad r(n-r) \quad \leqslant \quad \sum_{i=1}^{n} \alpha(H_i) \quad = \quad \sum_{i=1}^{n} min[r - c(i), n - r - t(i)]$$

$$\leqslant \sum_{i=1}^{n}(r-c(i)) = nr - \sum_{i=1}^{n}c(i) = nr-r^2 = r(n-r) = |V(H)|.$$

Since the ends are equal, it must be that all the inequalities between were actually equalities. By termwise equality, it means that $min(r-c(i), n-r-t(i))= (r-c(i)$ and so it must be that $r-c(i) \leqslant n-r-t(i))$ for all symbol $i = 1, 2, \ldots, n$. Thus $(2) : 2r+t(i) \leqslant n+c(i)$ for each symbol i is satisfied if the inequality for HC is satisfied for this choice of H.

$\square$

The problem of completing a partial commutative latin square to a commutative latin square is indeed a list-coloring problem in the same way that completing a partial latin square to a latin square is, but the underlying graph is not $K_n \square K_n$, nor are the lists generated from the prescribed cells just as they are in the P.L.S case. However, the underlying graph in partial commutative latin square case is obtainable from $K_n \square K_n$ by adding edges, and therefore the lists dictated by a partial prescription are subsets of the lists dictated in $K_n \square K_n$ by the same prescription. Therefore, HC in the true partial commutative latin square case implies HC in the $K_n \square K_n$, or P.L.S., setting. Therefore, the preceding shows that HC in the true partial commutative latin square case implies condition (2) in Hoffman's theorem. we doubt that it implies (1), which does not seem the sort of thing that could be implied by a collection of inequalities, but the matter requires further investigation.

2.3.2 Hoffman's Lemma

I will state and proof a result that will enable us give another proof of Ryser's theorem. This result is due to D.G Hoffman.

Lemma 2.1 ***Hoffman's Lemma***: *Let $G = K_m \square K_n \equiv$ the line graph of $K_{m,n}$ be an m by n rectangular array of cells such that each cell holds exactly one symbol.*

Then one can permute within the rows (columns) to make the array G to become column latin (respectively row latin) if and only if no symbol appears more than n times (respectively, m times) in the m by n array.

G can be rearranged by permutations within lines to be both row and column latin if and only if no symbol appears more than $min(m, n)$ times.

Before giving a proof of this lemma, I will first give a remark, an example of the use of the lemma, a corollary and a result of König.

By the *pigeon hole principle*,we remark that if there were $n + 1$ (respectively $m + 1$) copies of a symbol σ in the m by n array G, then two of them will be forced to appear on the same column (respectively, row). This will contradict latinness.

An application of this lemma is the following example.

Example 2.2

2	2	1	3	4	7
1	2	3	4	5	6
2	3	4	5	6	7
1	1	2	4	3	3
2	4	6	8	8	8

Figure 2.11: Randomly filled-in 5×6

Since each symbol appears not more than $n = 6$ times, one can make the array to be column latin by permuting within each row.

2	2	1	3	4	7
6	3	2	4	5	1
3	4	5	6	7	2
1	1	4	2	3	3
4	6	8	8	2	8

Figure 2.12: Column latin 5×6

However, there are more than $m = 5$ copies of the symbol 2. Hence the table cannot be made row latin because one cannot place 6 copies of 2 in exactly 5 rows without duplicating in some row.

If we modify this table to a new table in which no symbol appears more than $5 = min(5, 6)$ times in the whole array, we see by Hoffman's Lemma that the new array can be made to become a latin rectangle as shown in our next example.

Example 2.3

Consider the array given by

1	2	1	3	4	7
1	2	3	4	5	6
2	3	4	5	6	7
1	1	2	4	3	3
2	4	6	8	8	8

Figure 2.13: Randomly filled-in 5×6 Rectangle

By permuting within the rows, I get the following array which is column latin

2	7	1	3	1	4
6	3	2	4	5	1
3	4	5	6	7	2
1	1	4	2	3	3
4	6	8	8	2	8

2	7	8	3	1	4
6	3	2	4	5	1
3	4	1	6	7	2
1	6	4	2	3	8
4	1	5	8	2	3

Figure 2.14: 5×6 Latin Rectangle

The final latin rectangle results from permutations within the columns.

Theorem 2.5 ***König 1916***:*[8] If G is a bipartite multigraph with chromatic index $\mathcal{X}'(\mathcal{G})$ and maximum degree $\triangle(G)$, then* $\quad \mathcal{X}'(\mathcal{G}) = \triangle(G)$.

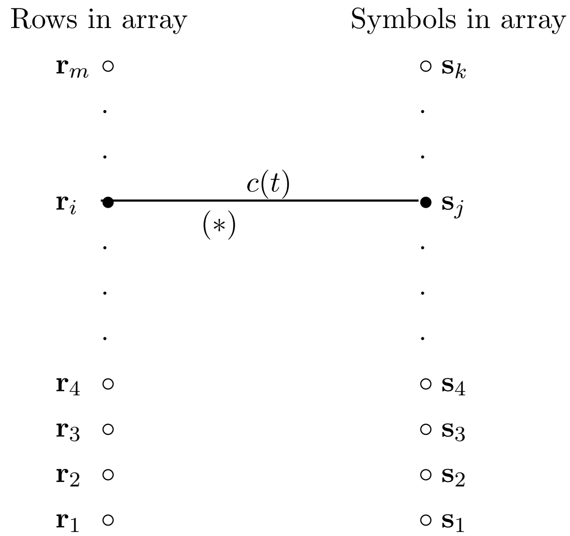
Proof of Hoffman's Lemma:

The "only if" claim has been proven already.

Suppose each symbol appears no more than n times in the array. I aim to show that one can permute within the rows to make the array column latin. Make a bipartite graph as follows.

Each row has n entries, so n symbols appear in each row. Hence I get $deg(r_i) = n$, $i = 1, 2, \ldots, m$. $deg(s_j)$ = number of appearances of symbol s_j in array, for $j = 1, 2, \ldots, k$.

But each symbol appears no more than n times by hypothesis. Hence $deg(s_j) \leqslant n$, for $j = 1, 2, \ldots, k$.



where $(*)$ stands for as many edges as appearances of symbol $\mathbf{s}_j$ in row $\mathbf{r}_i$.

Figure 2.15: Rows $\mathbf{r}_i$ and symbols $\mathbf{s}_j$ in the array.

By theorem 2.5, $\mathcal{X}'(\mathcal{G}) = \triangle(G) = n$ as G is a bipartite multigraph. So I can properly color the edges of this bipartite graph with n colors: $c_1, c_2 \ldots c_n$ which in turn represent the columns.

Now whenever an edge from row r_i to symbol s_j is colored c_t, put one of the s_j's in row i in column t. Since the coloring is proper, only one symbol will appear in row i, column t after this arrangement and, by the way the bipartite graph is defined, each row will still contain the same numbers of the different symbols that it did before. That is, there has been rearrangement only within each row.

□

2.3.3 Another look at Ryser's Theorem

I now present another proof of the theorem of Ryser's, a result that has been the backbone of the section on recasted results.

Theorem 2.6 ***Ryser's Theorem:*** *[19]*

An $r \times s$ latin rectangle R on n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be completed to a latin square of order n if and only if $N_R\ (\sigma_i) \geqslant r + s - n$ for each $i\ = 1, 2, \ldots, n$,

where $N_R\ (\sigma_i)$ is the number of occurrences of the symbol σ_i in R.

Proof: I need to complete the partial latin rectangle R to an r by n latin rectangle on $\sigma_1, \sigma_2, \ldots, \sigma_n$. By an old result of M.Hall, (Theorem 2.1) which can also be proved using Hoffman's Lemma 2.1, any r by n latin rectangle on n symbols can be completed to a latin square. Complete each row to be of length n so that each row contains all n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$. One can then complete to an r by n latin rectangle if and only if one can rearrange within the rows of the r by $n - s$ block made by the first r rows, last $n - s$ columns, so that the result is (also) column latin. By Hoffman's Lemma 2.1, one can do this rearrangement if and only if for each $i = 1, 2, \ldots, n,$, σ_i appears at most $n - s$ times in the array.

But σ_i appears in each row, so in all, it appears r times in the r by n array. In R, σ_i appears $N_R(\sigma_i)$ times and so in the r by $n - s$ block, it appears $r - N_R(\sigma_i)$ times. In the r by $n - s$ block, the $'n'$ in Hoffman's Lemma is the number of columns, $n - s$. Thus

the completion to an $r \times n$ latin rectangle is possible if and only if

$r - N_R(\sigma_i) \quad \leqslant \quad n - s \quad \Longleftrightarrow \quad N_R\ (\sigma_i) \geqslant r + s - n$, where $n - s$ is the number of times the symbol σ_i appears in the r by $n - s$ rectangle.

$\square$

Chapter 3

HALL+ AND HALL++ GRAPHS

3.1 Introduction

In this chapter, I shall explore the relations that exist between **HALL+** and **HALL++** graphs and attempt answers to questions about obtaining new **HALL+** and **HALL++** graphs from old ones. I shall also look at a few graph theoretic operations that either preserve or modify the property of a graph being **HALL+** or **HALL++**.

The main result of this chapter is that the operation of attaching cliques to vertices of **HALL+** graphs yield a **HALL+** graph, and the same for **HALL++**. I shall also prove any induced subgraph of a **HALL+** graph is **HALL+**, and the same holds for **HALL++**. Hence the classes of **HALL+** and **HALL++** graphs have *forbidden subgraph characterizations.*

3.2 Inclusion Property

I shall now try to build up a theory for **HALL+** graphs that resembles that for **HALL** graphs.

Recall that a simple graph $G \in$ **HALL+** if and only if for every list assignment $\mathcal{L}$ of G, if G and $\mathcal{L}$ satisfy **HC+**, then there is a proper $\mathcal{L}$-coloring of G.

Theorem 3.1 *If $G \in$ **HALL**, then $G \in$ **HALL+** and the converse is false. Thus **HALL** $\subsetneq$ **HALL+** $\subseteq$ **HALL++**.*

Proof:

$G \in$ **HALL** means whenever G and $\mathcal{L}$ satisfy **HC**, then there is a proper $\mathcal{L}$-coloring of G. Now suppose **HC+** is satisfied for G and $\mathcal{L}$. I wish to show that a proper $\mathcal{L}$-coloring for G exists. Since **HC+** is satisfied, it follows that **HC** also holds for G and $\mathcal{L}$. Hence $G \in$ **HALL** $\Longrightarrow$ there is a proper $\mathcal{L}$-coloring for G. The same logic works for **HALL+** $\subseteq$ **HALL++** because if G and $\mathcal{L}$ satisfy **HC++**, then G and $\mathcal{L}$ satisfy **HC+** and so there is a proper $\mathcal{L}$-coloring of G as $G \in$ **HALL+**. Hence $G \in$ **HALL++**.

To complete the proof, I will give an example of a graph which is **HALL+** but not **HALL**. Consider $G = C_4$. Then by a later result, proposition 3.3, G is **HALL+**. However, by the characterization of Hall graphs in theorem 1.7, I know that C_4 is not **HALL**.

$\square$

Lemma 3.1 *Let H be an induced subgraph of G with list assignment $\mathcal{L}$. Suppose that H and $\mathcal{L}$ satisfy HC+. Let $\widetilde{\mathcal{L}}$ be an extension of $\mathcal{L}$ to $V(G)$ such that G and τ satisfy HC and any $\widetilde{\mathcal{L}}$-tight subgraph of G is also an $\mathcal{L}$-tight subgraph of H. Then any* ***HC+*** *- satisfying family [$S_\sigma : \sigma \in \mathcal{C}$] for H and $\mathcal{L}$ is also an* ***HC+*** *- satisfying family for G and $\widetilde{\mathcal{L}}$; in particular, G and $\widetilde{\mathcal{L}}$ satisfy HC+.*

Proof:

Since $S_\sigma \subseteq V(H(\sigma, \mathcal{L}\ )) \subseteq V(G(\sigma, \mathcal{L})$ for each symbol $\sigma \in \mathcal{C}$, we see that S_σ must belong to $V(G(\sigma, \mathcal{L})$.

By assumption, any $\widetilde{\mathcal{L}}$-tight subgraph T of G is also an $\mathcal{L}$-tight subgraph of H. So $\forall \sigma \in \mathcal{C}$, $T(\sigma, \widetilde{\mathcal{L}}) = T(\sigma, \mathcal{L})$ and so I can conclude that $|S_\sigma \bigcap V(T)| = \alpha\ (T(\sigma, \mathcal{L})) = \alpha\ (T(\sigma, \widetilde{\mathcal{L}}))$.

$\square$

Proposition 3.1 *If $G \in$ **HALL+**, (respectively **HALL++**) and H is an induced subgraph of G, then $H \in$ **HALL+** (respectively **HALL++**).*

Proof:

I will do the proof for **HALL+** only. The proof for **HALL++** is very similar.

Let $G \in$ **HALL+**. Then by definition, if G and $\mathcal{L}$ satisfy **HC+**, then there is a proper $\mathcal{L}$-coloring. Now take H induced in G as postulated and suppose that $\mathcal{L}$ is a list assignment to H such that H and $\mathcal{L}$ satisfy **HC+**. The proof will be complete once I can show that there is an $\mathcal{L}$-coloring for H. Extend $\mathcal{L}$ to a list assignment $\widetilde{\mathcal{L}}$ of G by assigning huge lists to the part of G outside of H, i.e. to the vertices in $V(G)\backslash V(H)$, huge enough so that G and $\widetilde{\mathcal{L}}$ satisfy HC and any $\widetilde{\mathcal{L}}$-tight subgraph T of G will also be an $\mathcal{L}$-tight subgraph of H. Then by the result above, lemma 3.1, whatever the **HC+**-satisfying family $[S_\sigma : \sigma \in \mathcal{C}]$ was for H and $\mathcal{L}$, it is also an **HC+**-satisfying family for G and $\widetilde{\mathcal{L}}$. Therefore G has a proper $\widetilde{\mathcal{L}}$-coloring, and therefore H has a proper $\mathcal{L}$-coloring. □

Example 3.1

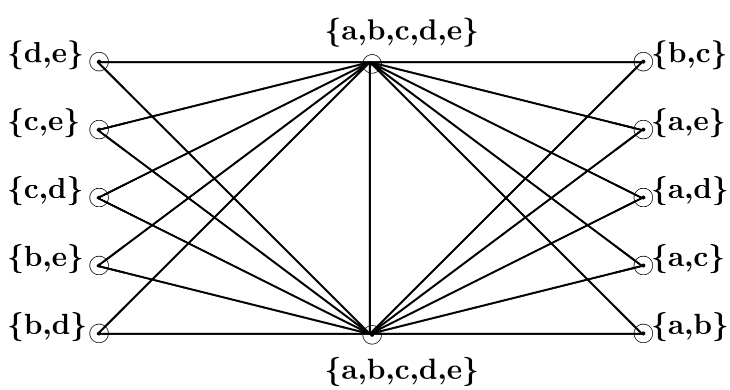


Figure 3.1: $G = K_2 \vee \overline{K_{10}}$ with assigned list.

This example is due to Matt Cropper.

The graph G has no tight subgraphs and one cannot properly color. However, deleting any vertex leaves a properly $\mathcal{L}$-colorable graph. One can therefore check HC for G alone.

Since $\alpha(G_\sigma) = 4$ for each symbol, we see that $\sum_{\sigma\in\mathcal{C}} \alpha(G_\sigma) = 20 > 12 = |V(G)|$, and so HC is satisfied. Since there are no tight subgraphs, one can take $S_\sigma = \emptyset, \forall\sigma$. Then $[S_\sigma; \sigma \in \mathcal{C}]$ is an HC++ - satisfying family.

Conclude that $G \notin Hall\diamondsuit$ for $\diamondsuit \in$ { empty string, +, ++, *, **}.

3.3 Forbidden Induced Subgraph Characterization

Let $\mathcal{P}$ be a property of graphs (such as being a **Hall**$\diamondsuit$, where $\diamondsuit \in$ { empty string, +, ++, *, ** }).

Definition 3.1 *I will say that $\mathcal{P}$ has a forbidden induced subgraph characterization if and only if there is a collection $\mathbb{Q}$ of graphs such that a graph G has property $\mathcal{P}$ if and only if G has no induced subgraph in the set $\mathbb{Q}$.*

In general, to see that a property $\mathcal{P}$ has an induced subgraph characterization, we check whether or not property $\mathcal{P}$ is closed under the property of taking induced subgraphs.

Proposition 3.2 *A property $\mathcal{P}$ of graphs (which some graphs actually have) has a forbidden induced subgraph characterization if and only if the collection $\mathfrak{G}(\mathcal{P})$ of graphs, defined by $\mathfrak{G}(\mathcal{P}) = \{G$: G has property $\mathcal{P}\}$ is closed under the operation of taking induced subgraphs.*

Proof: The "only if" part is true because if I is an induced subgraph of an induced subgraph H of G, then I is also induced in G. Thus the collection is closed under taking induced subgraphs, if $\mathcal{P}$ has forbidden induced subgraph characterization, for if G can contain no induced subgraph belonging to $\mathbb{Q}$, then neither can H.

To see that the other implication is also true, suppose that $\mathfrak{G}(\mathcal{P})$ is closed under taking induced subgraphs and define the collection $\mathbb{Q}$ to be:

$\mathbb{Q} = \{H \mid H$ is a graph that does not have property $\mathcal{P}$, but $\forall\ v \in V(H),\ H - v \in \mathfrak{G}(\mathcal{P})\ \}$.

I will show that property $\mathcal{P}$ has a forbidden induced subgraph characterization, with $\mathbb{Q}$ as the collection of forbidden induced subgraphs, whenever $\mathfrak{G}(\mathcal{P})$ is closed under the operation of taking induced subgraphs.

If the graph G has an induced subgraph $H \in \mathbb{Q}$, then by hypothesis, G does not have property $\mathcal{P}$ because G has an induced subgraph without property $\mathcal{P}$. Suppose that G does not have property $\mathcal{P}$. I want to see that G has an induced subgraph in $\mathbb{Q}$. If $G - v$ has property $\mathcal{P}$ for every vertex $v \in V(G)$, then G itself is in $\mathbb{Q}$ and we are done. Otherwise, $G_1 = G - v_1$ does not have property $\mathcal{P}$, for some $v_1 \in V(G)$. If $G_1 - v$ has property $\mathcal{P}$ for every $v \in V(G_1)$ then $G_1 \in \mathbb{Q}$ and we are done. Otherwise, for some $v_2 \in V(G_1)$, $G_1 - v_2 = G - v_1 - v_2$ does not have property $\mathcal{P}$.

Continuing in this way, we have a sequence $G_1, G_2, \ldots$ of graphs, $G_k = G_{k-1} - v_k$ for some $v_k \in V(G_{k-1})$, each without property $\mathcal{P}$. The sequence much terminate at some graph $H = G_k$, an induced subgraph of G with at least two vertices which does not have property $\mathcal{P}$ but such that $H - v$ does have property $\mathcal{P}$ for every $v \in V(H)$; i.e., $H \in \mathbb{Q}$. H must have at least two vertices because by assumption $\mathfrak{G}(\mathcal{P})$ is non-empty and closed under taking induced subgraphs, so K_1 has property $\mathcal{P}$.

□

Of course, forbidden subgraphs exist, however, completely characterizing the minimal set $\mathbb{Q}$, can be a little more involving. This is left for further research.

Corollary 3.1 *[6] The class of Hall$\diamondsuit$ graphs, for $\diamondsuit \in \{$ empty string, +, ++, *, ** $\}$ has a forbidden induced subgraph characterization.*

This result says that if $\mathcal{P}$ is the property of being **Hall**, then there is a collection $\mathbb{Q}$ of graphs such that a graph G is **Hall** if and only if G has no induced subgraph in the set $\mathbb{Q}$.

Following is a characterization of **HALL** graphs. I do not yet have forbidden induced subgraph characterizations for **HALL+** and **HALL++**, and I do not know if the **HALL*** or **HALL**** graphs have such a characterization.

For a start, let us restate the following equivalent results of Hilton and Johnson [15].

Theorem 3.2 *[15] $G \in$ **Hall** if and only if G has no induced $C_n, n \geqslant 4$ nor K_4 -minus-an-edge.*

Question : What characterization similar to theorem 3.2 above can be stated for the class of **HALL+**, **HALL++** as well as **HALL*** and **HALL**** graphs?

3.4 Closure Property

We now have enough background knowledge of **HALL+** graphs to tackle one of the main results in this work.

Theorem 3.3 *If a graph $H \in$ **Hall+**, and a graph G is obtained from H by attaching a clique to an arbitrary vertex v of H, then $G \in$ **HALL+**. Thus the class of **HALL+** graphs is closed under attachment of cliques.*

Lemma 3.2 *Suppose K is a clique, $\mathcal{L}$ a list assignment to the graph K, and K and $\mathcal{L}$ satisfy Hall's Condition. Suppose further that H_1, H_2 are $\mathcal{L}$-tight sub-cliques of K. Then*

1). $H_0 = H_1 \cap H_2$ is also $\mathcal{L}$-tight.

2). $H_3 = < H_1 \cup H_2 >$ is also $\mathcal{L}$-tight.

It is clear that both H_0 and H_3 are cliques.

Proof of Lemma:

Since H_3 is a clique, $\alpha(H_3(\sigma, \mathcal{L}))$ as a function of the symbol σ is the characteristic function of $\bigcup_{v\in V(H_3)} \mathcal{L}(v)$. This is true since the α value will either be 1 or 0 depending on whether or not $\sigma \in \cup\mathcal{L}(v)$. That is,

$$\alpha(H_3(\sigma,\mathcal{L})) = \begin{cases} 1, & \text{if } \sigma \in \bigcup_{v\in V(H_3)} \mathcal{L}(v); \\ 0, & \text{otherwise.} \end{cases}$$

By Hall's Condition on H_3 and $\mathcal{L}$, and the fact that $V(H_3) = V(H_1) \cup V(H_2)$, you have the following:

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_3(\sigma,\mathcal{L})) \quad \geqslant \quad |V(H_3)| \quad = \quad |V(H_1) \cup V(H_2)|\,. \tag{3.1}$$

Also, $\alpha(H_3(\sigma, \mathcal{L}))$ being a characteristic function

$\Rightarrow \quad \sum_{\sigma\in\mathcal{C}} \alpha(H_3(\sigma,\mathcal{L})) \quad = \quad \left|\bigcup_{v\in V(H_3)} \mathcal{L}(v)\right| \quad = \quad \left|\bigcup_{v\in V(H_1)\cup V(H_2)} \mathcal{L}(v)\right|$

$= \quad \left|\{\bigcup_{v\in V(H_1)} \mathcal{L}(v)\} \quad \bigcup \quad \{\bigcup_{v\in V(H_2)} \mathcal{L}(v)\}\right|$

$= \quad \left|\bigcup_{v\in V(H_1)} \mathcal{L}(v)\right| \quad + \quad \left|\bigcup_{v\in V(H_2)} \mathcal{L}(v)\right| \quad - \quad \left|\{\bigcup_{v\in V(H_1)} \mathcal{L}(v)\} \quad \bigcap \quad \{\bigcup_{v\in V(H_2)} \mathcal{L}(v)\}\right|$

(*) $\leqslant \quad |V(H_1)| \quad + \quad |V(H_2)| \quad - \quad \left|\{\bigcup_{v\in V(H_1)\cap V(H_2)} \mathcal{L}(v)\}\right|$

$$\textbf{(**)} \leqslant \quad |V(H_1)| + |V(H_2)| - \left|V(H_1) \bigcap V(H_2)\right| = \left|V(H_1) \bigcup V(H_2)\right| = |V(H_3)|\,. \tag{3.2}$$

[The inequality (*) is true since $\left|\bigcup_{v\in V(H_i)} \mathcal{L}(v))\right| = |V(H_i)|$, $i = 1, 2$ by the assumption of tightness and $\{\bigcup_{v\in V(H_1)} \mathcal{L}(v)\} \quad \bigcap \quad \{\bigcup_{v\in V(H_2)} \mathcal{L}(v)\} \quad \supseteq \quad \bigcup_{v\in V(H_1)\cap V(H_2)} \mathcal{L}(v)$.

The inequality (**) is an application of HC to $H_0 = H_1 \cap H_2 = < V(H_1) \cap V(H_2) >$: $\left|\bigcup_{v \in V(H_1)\cap V(H_2)} \mathcal{L}(v)\right| \geqslant |V(H_1) \bigcap V(H_2)|]$.

From inequalities (3.1) and (3.2), we see that the ends are equal and so it must be that the middles were also equal. Hence the inequalities in (*) and (**) are actually equalities. Thus H_3 is $\mathcal{L}$-tight and H_0 as well, implied by equality at (**).

□

By observing that $S_\sigma \cap V(H) \subseteq S_\sigma$ is an independent set of vertices, I have the following lemma.

Lemma 3.3 *If [$S_\sigma : \sigma \in \mathcal{C}$] is an **HC+** (or **HC++**) satisfying family for G and $\mathcal{L}$, then for any induced subgraph H of G, [$S_\sigma \cap V(H) : \sigma \in \mathcal{C}$] is also an **HC+** (respectively **HC++**) satisfying family for G and $\mathcal{L}$.*

Lemma 3.4 *Suppose K is a clique with a list assignment $\mathcal{L}$, and K and $\mathcal{L}$ satisfy HC. Suppose that for some color τ, removing τ from K wherever it appears results in a list assignment which does not satisfy HC with K. Then some subclique K_τ of K is $\mathcal{L}$-tight. Further, $\tau \in \mathcal{L}(K_\tau)$.*

Proof of theorem 3.3:

Suppose G and $\mathcal{L}$ satisfy **HC+**, and show that the new graph G has a proper $\mathcal{L}$-coloring. Let [$S_\sigma : \sigma \in \mathcal{C}$] be an **HC+** - satisfying family for G and $\mathcal{L}$; H is **HC+** and $\widehat{K}$ is a clique, i.e. $\widehat{K} = K_n$ for some $n \geqslant 2$. Let $K = \widehat{K} - v \equiv K_{n-1}$. Since H is an induced subgraph of G, **HC+** on G and $\mathcal{L}$ means **HC+** on H and $\mathcal{L}$. Thus there is a proper $\mathcal{L}$-coloring of H.

Let $\mathcal{P}$ be the set of all colors that appear on the vertex of attachment, say the specific vertex v, in the proper $\mathcal{L}$-coloring of H.

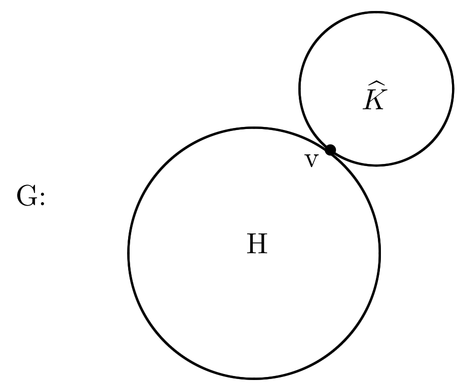


Figure 3.2: Hall+ graph H, with an attached clique $\widehat{K}$

Define $\mathcal{P}$ as follows: $\mathcal{P} = \{\tau \in \mathcal{C} \mid \text{for some proper } \mathcal{L}\text{-coloring } \varphi \text{ of } H,\ \varphi(v) = \tau\ \}$.

For $\tau \in \mathcal{P}$ arbitrary, if I remove τ from all the lists on K, then I may as well assume that the resulting list assignment to K does not satisfy Hall's Condition. This is so because if the new list did satisfy Hall's Condition, then K being a clique means I can color. If one puts a coloring of K without using τ together with a proper $\mathcal{L}$-coloring of H using τ at v, you get an $\mathcal{L}$-coloring of G. We therefore see from lemma 3.4 that for each $\tau \in \mathcal{P}$, there is an $\mathcal{L}$ - tight subclique say K_τ with τ on its lists.

By lemma 3.2 above applied $\mid \mathcal{P} \mid$-1 times successively, I see that $K_p = < \bigcup_{\tau\in\mathcal{P}} K_\tau >$ is $\mathcal{L}$-tight and since the α value for each color σ is 1 (for cliques), we must have some vertex of K_p in $\mathcal{S}_\tau$ for each $\tau \in \mathcal{P}$ (for all $\mathcal{L}$-tight subgraphs). Thus $\forall\tau \in \mathcal{P}$, $v \notin \mathcal{S}_\tau$.

Now define a new list assignment $\mathcal{L}'$ on the graph H as follows:

$\mathcal{L}'(v) = \mathcal{L}(v)\backslash\mathcal{P}$, and $\mathcal{L}'(u) = \mathcal{L}(u)$ for $u \neq v$.

Now H and $\mathcal{L}'$ can not satisfy **HC+** because if they did, then by $H \in$ **Hall+**, there exists a proper $\mathcal{L}'$-coloring of H. But such a proper $\mathcal{L}'$-coloring will also be a proper $\mathcal{L}$-coloring for H given that $\mathcal{L}$ and $\mathcal{L}'$ coincide except at the vertex v and $\mathcal{L}'(v) \subseteq \mathcal{L}(v)$. But I have eliminated $\mathcal{P}$ on $\mathcal{L}'$ i.e. the colors on $\mathcal{L}(v)$ that are in the proper $\mathcal{L}$-colorings of H. Hence there is no proper $\mathcal{L}'$-coloring of H. Thus conclude that H and $\mathcal{L}'$ do not satisfy **HC+**.

Now consider the definition of **HC+** and note that it has two parts: the existence of **HC** on one hand and an **HC+** satisfying family as the other part. Thus the failure of **HC+** for H and $\mathcal{L}'$ means that at least one of the two parts in the definition must fail.

Case 1: If H and $\mathcal{L}'$ do not satisfy **HC**, then for some induced subgraph H_1 of H, the following must be true:

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma,\mathcal{L}')) \quad < \quad |V(H_1)|\,. \tag{3.3}$$

But $\mathcal{L}$ and $\mathcal{L}'$ are the same except at v. Also, H and $\mathcal{L}$ did satisfy **HC** and so the problem must be at the vertex v itself. Thus, surely, H_1 contains vertex v.

Let $H_2 = < H_1 \cup K_p >$ (see figure). This is a disjoint union because $H_1 \subseteq H$ and $K_p \subseteq K$.

$$|V(H_2)| \quad = \quad |V(H_1)| \quad + \quad |V(K_p)| \qquad \leqslant \qquad \sum_{\sigma\in\mathcal{C}} \alpha(H_2(\sigma,\mathcal{L})) \tag{3.4}$$

$$\leqslant \qquad \sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma,\mathcal{L}')) \quad + \quad \sum_{\sigma\in\mathcal{C}} \alpha(K_p(\sigma,\mathcal{L})) \tag{3.5}$$

$$= \qquad \sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma,\mathcal{L}')) \quad + \quad \left|\cup_{u\in K_p)}\mathcal{L}(u)\right| \tag{3.6}$$

$$= \qquad \sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma,\mathcal{L}')) \quad + \quad |V(K_p)| \tag{3.7}$$

$$< \quad |V(H_1)| \quad + \quad |V(K_p)| \tag{3.8}$$

Thus $|V(H_1)| + |V(K_p)| < |V(H_1)| + |V(K_p)|$ which is a contradiction. Thus H and $\mathcal{L}'$ do satisfy **HC** but not **HC+**.

The inequality in (3.4) follows from the fact that G and $\mathcal{L}$ satisfy **HC**. The inequality in (3.5) follows from the fact that the vertex independence number of a union is less than or equal to that of its separate parts. Indeed, taking term by term sum, if a color $\sigma \notin \mathcal{P}$, then its independence number in H_2 is less or equal to the sum of its independence number from H_1 and that from the clique K_p . However, if color $\sigma \in \mathcal{P}$, then on its $\mathcal{L}$-list, any maximum independent set in $H_2(\sigma, \mathcal{L})$, if it contains the vertex v, it will not contain any vertex of K_p. Thus I can trade v for one vertex from K_p to obtain a maximum independent set in $H_2(\sigma, \mathcal{L})$ consisting of an independent set in $H_1(\sigma, \mathcal{L}')$ and a vertex of K_p. On the other hand, a maximum independent set in $H_2(\sigma, \mathcal{L})$ not containing v is already of that form. The details for (3.5) can be explained as follows: suppose U is a maximum independent set in $H_2(\sigma, \mathcal{L})$. Let $U_1 = U \cap V(H_1)$, and $U_2 = U \cap V(K_p)$. Then U_2 has a maximum of one vertex. If $\sigma \notin \mathcal{P}$, then from above, $H_2(\sigma, \mathcal{L}) = H_2(\sigma, \mathcal{L}')$ and since by definition $|U_1| \leqslant \alpha(H_1(\sigma, \mathcal{L}'))$ and $|U_2| \leqslant \alpha(K_p(\sigma, \mathcal{L}))$,

$$\Longrightarrow \quad \alpha(H_2(\sigma, \mathcal{L})) = |U_1| + |U_2| \leqslant \alpha(H_1(\sigma, \mathcal{L}')) + \alpha(K_p(\sigma, \mathcal{L})).$$

Now if $\sigma \in \mathcal{P}$, then either $v \notin U$ so that the inequality follows or $v \in U$ in which case $U_2 = \emptyset$; I can replace vertex v with some vertex of K_p to get a new maximum independent set.

The equality in (3.6) holds because K_p is an $\mathcal{L}$-tight clique.

The equality in (3.7) follows because K_p is $\mathcal{L}$-tight. Finally the strict equality in (3.8) follows from (3.3).

As a recap, we have proven that H and $\mathcal{L}'$ do satisfy **HC** but not **HC+**. Hence the problem must come from the **HC+** satisfying family. I had $K_p = < \bigcup_{\tau \in \mathcal{P}} K_\tau >$ to be an $\mathcal{L}$-tight sub-clique of $K = \widehat{K} - v \equiv K_{n-1}$, for $n \geqslant 2$, with every color in $\mathcal{P}$ appearing in K_p.

For every color σ appearing in K_p, $\quad |\mathcal{S}_\sigma \bigcap V(K_p)| = 1$. So if $\sigma \in \mathcal{P}$, then $v \notin \mathcal{S}_\sigma$ because some $u \in V(K_p) \cap S_\sigma$ and u is adjacent to v. Hence $v \notin \mathcal{S}_\sigma$.

There must be some $\mathcal{L}'$ - tight subgraph, say $\widehat{H}_1$ of H, such that for some color $\tau \in \mathcal{C}$, $\left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau, \mathcal{L}'))\right| < \alpha(\widehat{H}_1(\tau, \mathcal{L}'))$. Thus it must follow that $v \in V(\widehat{H}_1)$ because other than on the vertex v, $\mathcal{L}$ and $\mathcal{L}'$ coincide.

Now define $G_1 = < \widehat{H}_1 \bigcup K_p >$. Our aim at this stage is to show that G_1 is $\mathcal{L}$- tight. This is done by letting G_1 play H_2 in the first part of our proof above.

Claim: G_1 is $\mathcal{L}$ - tight.

Proof:

G:

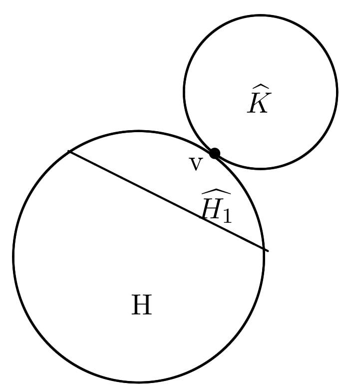


Figure 3.3: Hall+ graph H $\supset \widehat{H}_1$ and clique $\widehat{K}$

$$|V(\widehat{H}_1)| \quad + \quad |V(K_p)| \quad = \quad |V(G_1)| \quad \leqslant \quad \sum_{\sigma \in \mathcal{C}} \alpha(G_1(\sigma, \mathcal{L})) \tag{3.9}$$

$$= \quad \sum_{\sigma \in \mathcal{C} \setminus \mathcal{P}} \alpha(G_1(\sigma, \mathcal{L})) \quad + \quad \sum_{\sigma \in \mathcal{P}} \alpha(G_1(\sigma, \mathcal{L})) \tag{3.10}$$

$$\leqslant \sum_{\sigma\in\mathcal{C}\setminus\mathcal{P}} [\alpha(\widehat{H}_1(\sigma,\mathcal{L}')) + \alpha(K_p(\sigma,\mathcal{L}))] \quad + \quad \sum_{\sigma\in\mathcal{P}} [\alpha(\widehat{H}_1(\sigma,\mathcal{L}')) + \alpha(K_p(\sigma,\mathcal{L}))] \tag{3.11}$$

$$= \quad \sum_{\sigma\in\mathcal{C}} \alpha(\widehat{H}_1(\sigma,\mathcal{L}')) \quad + \quad \sum_{\sigma\in\mathcal{C}} \alpha(K_p(\sigma,\mathcal{L})) \quad = \quad |V(\widehat{H}_1)| \quad + \quad |V(K_p)|\,, \tag{3.12}$$

where the inequality in (3.9) follows from Hall's Condition on G_1 and $\mathcal{L}$, and (3.10) follows by breaking set $\mathcal{C}$ into two disjoint sets. Finally, for (3.11), one note first that the independence number of a union of two graphs is at most the sum of the separate independence numbers. Also, the replacement of $\mathcal{L}$ by $\mathcal{L}'$ in the first sum is a consequence of the fact that if $\sigma \notin \mathcal{P}$, then $\alpha(\widehat{H}_1(\sigma,\mathcal{L}')) = \alpha(\widehat{H}_1(\sigma,\mathcal{L}))$. For the second sum, if $\sigma \in \mathcal{P}$ and you take a maximum independent set in $G_1(\sigma,\mathcal{L})$; if v belongs to that set, then we can delete v and replace it with some vertex of K_p (i.e. trade places). The second equality in (3.12) is as a result of $\widehat{H}_1$ being $\mathcal{L}'$ - tight and K_p being $\mathcal{L}$ - tight.

Thus we see that the ends of the set of inequalities above are equal. Consequently, the middle inequalities are forced to become equalities. Thus the inequality in (3.9) will become an equality, giving $|V(G_1)| \quad = \quad \sum_{\sigma\in\mathcal{C}} \alpha(G_1(\sigma,\mathcal{L}))$ and so G_1 is $\mathcal{L}$ - tight. The claim is thus established.

As a further consequence,

$$|\mathcal{S}_\sigma \bigcap V(G_1)| \quad = \quad \alpha(G_1(\sigma,\mathcal{L})) \quad = \quad \alpha(\widehat{H}_1(\sigma,\mathcal{L}')) \quad + \quad \alpha(K_p(\sigma,\mathcal{L})), \quad \forall \sigma \in \mathcal{C}.$$

Recall that $\tau \in \mathcal{C}$ is such that $\left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}'))\right| \quad < \quad \alpha(\widehat{H}_1(\tau,\mathcal{L}'))$. We will see that the assumption of the existence of τ leads to a contradiction, which will finish the proof.

First suppose that $\tau \notin \mathcal{P} \quad \Longrightarrow \quad \widehat{H}_1(\tau,\mathcal{L}') = \widehat{H}_1(\tau,\mathcal{L})$. We have

$$\left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}'))\right| \quad + \quad \alpha(K_p(\tau,\mathcal{L})) \quad = \quad \left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}))\right| \quad + \quad |\mathcal{S}_\tau \bigcap V(K_p(\tau,\mathcal{L}))|$$
$$= \quad |\mathcal{S}_\tau \bigcap V(G_1(\tau,\mathcal{L}))| \quad = \quad \alpha(G_1(\tau,\mathcal{L})) \quad = \quad \alpha(\widehat{H}_1(\tau,\mathcal{L}')) \quad + \quad \alpha(K_p(\tau,\mathcal{L}))$$

$\Longrightarrow \quad \left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}'))\right| \quad = \quad \alpha(\widehat{H}_1(\tau,\mathcal{L}')),$ contradicting the original statement that $\left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}'))\right| \quad < \quad \alpha(\widehat{H}_1(\tau,\mathcal{L}')).$

Now suppose that $\tau \in \mathcal{P} \quad \Longrightarrow \quad v \notin \mathcal{S}_\tau$.

$\Longrightarrow \quad \left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}'))\right| \quad = \quad \left|\mathcal{S}_\tau \bigcap V(\widehat{H}_1(\tau,\mathcal{L}))\right|,$ so the sequence of equalities just above holds, and again I have a contradiction.

$\square$

I will now state and prove a lemma that was stated in the introductory chapter.

Lemma 3.5 ***Hall's Condition*** *holds for G and $\mathcal{L}$ if and only if the inequality in* ***Hall's Condition*** *(equation (1.1)) holds for each connected induced subgraph of G.*

Proof: If G and $\mathcal{L}$ satisfies **Hall's Condition**, then the inequality holds for every subgraph of G, and thus for every connected induced subgraph of G.

Now suppose the inequality in equation (1.1) holds for every induced subgraph of G. If H is a subgraph of G, then H is a spanning subgraph of an induced subgraph $\widehat{H}$ and for each symbol $\sigma \in \mathcal{C}$, $H(\sigma,\mathcal{L})$ is a spanning subgraph of $\widehat{H}(\sigma,\mathcal{L})$. Since $H(\sigma,\mathcal{L})$ is either equal to $\widehat{H}(\sigma,\mathcal{L})$ or obtained from it by deleting edges, it follows that $\alpha(H(\sigma,\mathcal{L})) \geqslant \alpha(\widehat{H}(\sigma,\mathcal{L}))$. Therefore, using the assumption that the inequality defining **HC** holds for $\widehat{H}$, I have

$$\sum_{\sigma\in\mathcal{C}} \alpha(H(\sigma,\mathcal{L})) \quad \geqslant \quad \sum_{\sigma\in\mathcal{C}} \alpha(\widehat{H}(\sigma,\mathcal{L})) \quad \geqslant \quad |V(\widehat{H})| \quad = \quad |V(H)|\,. \tag{3.13}$$

Now suppose that the inequality defining **HC** holds for every connected induced subgraph of G. To show that it holds for every subgraph of G, it suffices to show that it holds for every induced subgraph of G from above. Suppose that H is an induced subgraph of G with components $H_1, H_2, \ldots, H_k$. Then $H_1, H_2, \ldots, H_k$ are induced subgraphs of G and for each symbol $\sigma \in \mathcal{C}$, $H(\sigma, \mathcal{L})$ is the disjoint union of $H_1(\sigma, \mathcal{L}), H_2(\sigma, \mathcal{L}), \ldots, H_k(\sigma, \mathcal{L})$. Therefore, by the assumption,

$$\sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, \mathcal{L})) \quad = \quad \sum_{\sigma \in \mathcal{C}} \sum_{i=1}^{k} \alpha(H_i(\sigma, \mathcal{L})) \quad = \quad \sum_{i=1}^{k} \sum_{\sigma \in \mathcal{C}} \alpha(H_i(\sigma, \mathcal{L})) \tag{3.14}$$

$$\geqslant \quad \sum_{i=1}^{k} |V(H_i)| \quad = \quad |V(H)| \, . \tag{3.15}$$

Hence, I have inequality (1.1) and hence the lemma.

□

Our next two results identifies two families of graphs that are always in the family of **Hall+** graphs.

First note that by a result of Changiz Eslahchi et al [4] and the definition of the *hall number* given at the introduction, cycles with 4 or more vertices have hall number 2. This means that if $\mathcal{L}$ is a list assignment to such a graph satisfying HC and $|\mathcal{L}(v)| \geqslant 2$ for all vertices v, then there is a proper $\mathcal{L}$-coloring of the cycle.

Proposition 3.3 *All cycles with 4 or more vertices are **Hall+***

Proof:

$C_n = G$:

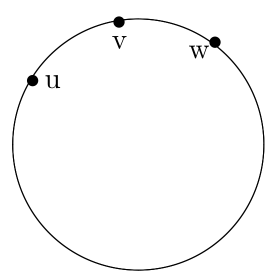


Figure 3.4: Cycle with n vertices, C_n

Our aim is to show that if C_n, for $n \geqslant 4$ and $\mathcal{L}$ satisfy **HC+**, then there exists a proper $\mathcal{L}$-coloring.

Suppose C_n, for $n \geqslant 4$ and some list assignment $\mathcal{L}$ satisfy **HC+** and let $[S_\sigma; \sigma \in \mathcal{C}]$ be an HC+ - satisfying family for C_n and $\mathcal{L}$. Then since the hall number of C_n is 2, for $n \geqslant 4$, I may as well assume that some list is a singleton, say $\mathcal{L}(v) = \{\sigma\}$. Then $v \in S_\sigma$. Let u, w be the vertices on either side of v on C_n. Consider $C_n - uv$ i.e. C_n-minus the edge between vertex u and v. Then $P = C_n - uv$ is a path, and so every block is a clique. Thus P satisfies Hall's Condition with $\mathcal{L}$. So P has a proper $\mathcal{L}$-coloring in which the vertex v is colored σ. One may as well suppose, without loss of generality that in every proper $\mathcal{L}$-coloring of P, the vertex u is always colored σ. Indeed, if there is some proper $\mathcal{L}$-coloring of P in which the vertex u is not colored with σ, then bring back the edge uv and get a proper $\mathcal{L}$-coloring of C_n and we will be done.

So if a new list assignment $\mathcal{L}'$ is defined by $\mathcal{L}'(u) = \mathcal{L}(u) \backslash \{\sigma\}$ and $\mathcal{L} = \mathcal{L}'$ otherwise, then P and $\mathcal{L}'$ do not satisfy Hall's Condition. So for some (connected) induced subgraph H of P, I have $\quad \sum_{\tau \in \mathcal{C}} \alpha(H(\tau, \mathcal{L}')) \quad < \quad |V(H)|$.

Conclude that

(i) $u \in V(H)$ and H is $\mathcal{L}$-tight (since bringing back σ at u increases the independence number $\alpha(H_\sigma)$ by at most one).

Furthermore,(ii) u is in every maximum independent set $H(\sigma, \mathcal{L})$.

So $u \in S_\sigma$ and this is a contradiction because by assumption, $\mathcal{L}(v) = \{\sigma\}$ thus forcing $v \in S_\sigma$. But then $u \in S_\sigma$ and $v \in S_\sigma$ is a contradiction as $uv \in E(C_n)$. This forces us to conclude that $C_n \in$ **Hall+**. $\square$

Proposition 3.4 *K_4 - minus - an edge is **Hall+**.*

Proof:

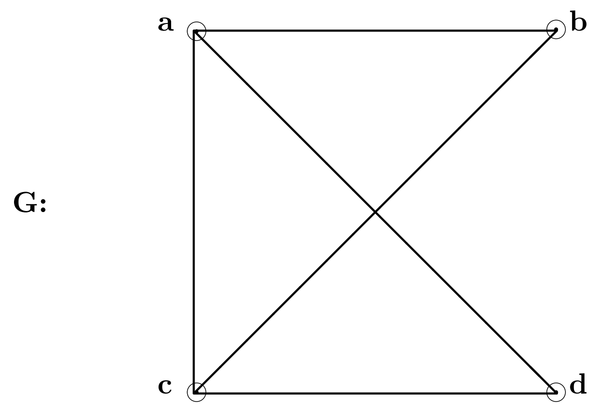


Figure 3.5: K_4 - minus - an edge

Let $\mathcal{L}$ be a list assignment to $G = K_4$ - minus - an edge, such that G and $\mathcal{L}$ satisfy **HC+**. Then in particular, G and $\mathcal{L}$ satisfy **HC**. Since the hall number of G is 2, if I had $|\mathcal{L}(v)| \geqslant 2, \quad \forall v \in V(G),$ then there must be a proper $\mathcal{L}$-coloring of G. Let us suppose that $\mathcal{L}(v) = \{\sigma\}$ for some vertex $v \in V(G)$ and let the collection $[\mathcal{S}_\lambda : \lambda \in \mathcal{C}]$ be an **HC+** - satisfying family for G and $\mathcal{L}$. From here the proof is almost word for word like that of Proposition 3.3; v has a neighbor u (in fact, there are two choices for u) such that $G - uv$ is **Hall**. Let $\mathcal{L}'$ be defined as in the proof preceding, and everything goes through.

□

The next example shows that attaching two **Hall+** graphs at one vertex does not necessarily result to a **Hall+** graph.

Example 3.2

Notice that G satisfies HC+ but cannot be properly colored.

Obviously, a graph G' obtained by contracting an edge e of a clique G is still a clique. Since **Hall** graphs are graphs with blocks being cliques, we see that contracting an edge of a **Hall** graph results in a **Hall** graph.

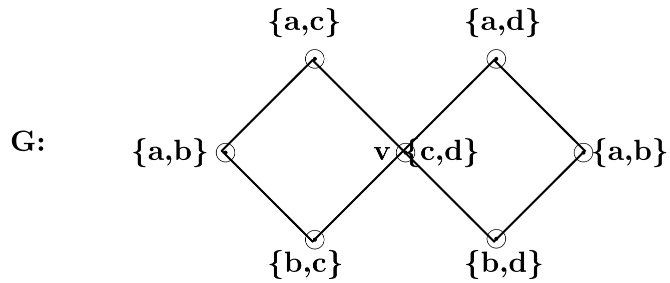


Figure 3.6: Attached Hall+ graphs at vertex **v** that does not yield a Hall+ graph

Question: Does the contraction of an edge of a **Hall+** or **Hall++** graph leave it in the same category, as was the case of **Hall** graphs?

Chapter 4

HALL* AND HALL** GRAPHS

4.1 Introduction

The main result in this chapter will be one similar to theorem 3.3 of the previous chapter. Here, I shall similarly prove that if a graph $H \in$ **Hall*** and a clique $\widehat{K}$ is attached to a vertex $v \in V(H)$, then the resulting graph G will also be **Hall***. A deduction from this will be a similar conclusion for **Hall**** graphs. For completeness, I will recall some definitions stated earlier.

Definition 4.1 *G is a **Hall*** graph if whenever G, $\mathcal{L}$ satisfy **HC***, there is a proper $\mathcal{L}$-coloring of G.*

Definition 4.2 *G is a **Hall**** graph if whenever G, $\mathcal{L}$ satisfy **HC****, there is a proper $\mathcal{L}$-coloring of G.*

Example 4.1

Consider H to be the six 3 by 2 sub squares induced by the coordinates:
$< (1,1),(1,2),(2,1),(2,2),(3,1),(3,2) >$ of the Aharoni-Hilton example, figure(2.2).

H has 19 $\mathcal{L}$-tight induced subgraphs.

Indeed, let the list assignment on H be as in chapter 2. H and its list assignments are shown in figure(4.1).

{1}	**{1,2}**
7	**{2,3}**
{1,5}	**6**

Figure 4.1: $3\times$ 2 Rectangle of cells

The 19 $\mathcal{L}$-tight induced subgraphs of H are as follows:

$< (1,1) >$, $< (2,1) >$, $< (3,2) >$, $< (1,1),(1,2) >$, $< (1,1),(2,1) >$, $< (1,1),(3,2) >$, $< (1,1),(3,1) >$, $< (2,1),(3,2) >$, $< (1,1),(1,2),(2,1) >$, $< (1,1),(2,1),(3,1) >$, $< (1,1),(1,2),(2,2) >$, $< (1,1),(1,2),(3,2) >$, $< (1,1),(2,1),(3,2) >$, $< (1,1),(3,1),(3,2) >$, $< (1,1),(1,2),(2,1),(2,2) >$, $< (1,1),(1,2),(2,1),(3,2) >$, $< (1,1),(2,1),(3,1),(3,2) >$, $< (1,1),(1,2),(2,2),(3,2) >$, $< (1,1),(1,2),(2,1),(2,2),(3,2) >$.

For instance, when $T = < (1,1),(1,2),(2,1),(2,2),(3,2) >$, then $\alpha(T_4) = 0 = \alpha(T_5)$, while $\alpha(T_\sigma) = 1$ for $\sigma \in \{1,2,3,6,7\}$. Thus

$$\sum_{\sigma=1}^{7} \alpha(T_\sigma) \quad = \quad 1+1+1+0+0+1+1 \quad = \quad 5 \quad = \quad |\, V(T) \,| \, .$$

This shows that T is $\mathcal{L}$-tight.

H is properly $\mathcal{L}$-colorable because I can actually fill each cell using say, $\varphi((1,1)) = 1$, $\varphi((1,2)) = 2$, $\varphi((2,1)) = 7$, $\varphi((2,2)) = 3$, $\varphi((3,1)) = 5$, $\varphi((3,2)) = 6$.

The fact that H is properly $\mathcal{L}$-colorable implies that H and $\mathcal{L}$ satisfy HC**.

However, H is not **HALL****, (and therefore not **HALL***), as shown by the following list assignment:

{1,3}	**{2,3}**
{1,2}	**{1,2}**
{2,3}	**{1,3}**

Figure 4.2: 3× 2 Rectangle of cells with no completion

This example demonstrates how tedious it can be to verify HC* or HC**. (In our example, six cells have as many as 19 tight subgraphs. For the complete 7×7 in figure(2.2), the number of tight subgraphs to be considered can be very large, and to verify HC* by brute force, each $\mathcal{L}$-tight subgraph must be considered together with every induced subgraph containing it.) Thus HC* and HC** are of theoretical interest alone, so far.

4.2 Closure Property

I now state and prove the main result in this chapter.

Theorem 4.1 *If a graph $H \in$ **Hall***, and a graph G is obtained from H by attaching a clique to an arbitrary vertex v of H, then $G \in$ **HALL***. Thus the class of **HALL*** graphs is closed under attachment of cliques. The same is true if* **Hall*** *is replaced by* **Hall****.

Notice that the statement is very similar to that of theorem(3.3) of the previous chapter. I will build the proof with this in mind.

Proof:

Let $K = \widehat{K} - v \equiv K_{n-1}$, for $n \geqslant 2$ and suppose $\mathcal{L}$ is a list assignment on G such that G and $\mathcal{L}$ satisfy **HC***. I must show that there is a proper $\mathcal{L}$-coloring of G.

Assume the contrary. By lemma 3.1, every induced subgraph H of G with $\mathcal{L}$ restricted on $V(H)$ will satisfy **HC***. Now suppose there is no proper $\mathcal{L}$-coloring of G.

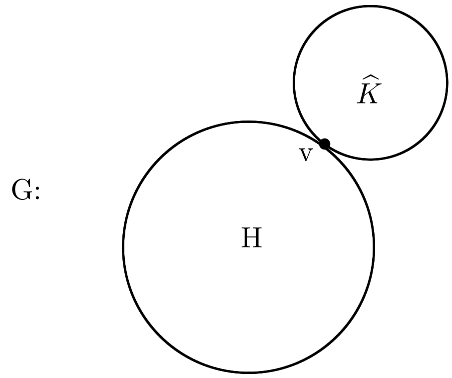


Figure 4.3: Hall* graph H, with an attached clique $\widehat{K}$

Since $H \subseteq G$ and G and $\mathcal{L}$ satisfy **HC***, it must be that H and $\mathcal{L}$ restricted on $V(H)$ will satisfy **HC***. Since $H \in$ **Hall***, it follows that there is a proper $\mathcal{L}$ coloring of H. However, because there is no proper $\mathcal{L}$-coloring of G for every proper $\mathcal{L}$-coloring of H, if the color on the vertex v is removed from the lists on $V(K)$, then K and the new list assignment do not satisfy **HC**.

Define the set of symbols $\mathcal{P} = \{\tau \in \mathcal{C} \mid \text{for some proper } \mathcal{L}\text{-coloring } \varphi \text{ of } H, \varphi(v) = \tau\}$

We therefore see from lemma 3.4 that for every symbol $\sigma \in \mathcal{P}$, there is an $\mathcal{L}$-tight subclique of the clique K with σ appearing on its lists. Hence there is a tight complete subgraph K_p of K with all colors of $\mathcal{P}$ on its lists. Now define a new list assignment $\mathcal{L}'$ on the graph H as follows:

$\mathcal{L}'(v) = \mathcal{L}(v) \backslash \mathcal{L}(K_p)$, and $\mathcal{L}'(u) = \mathcal{L}(u)$ for all $u \in V(H - v)$.

Then it must be that H and $\mathcal{L}'$ do not satisfy **HC*** because there is no proper $\mathcal{L}'$-coloring of H, but they do satisfy Hall's condition by the argument for the analogous claim in the proof of theorem (3.3) in chapter 3. I can therefore conclude this far that H and $\mathcal{L}'$ so defined do satisfy **HC** but do not satisfy **HC***.

HC* failing means there is some induced subgraph H' of H and some $\mathcal{L}'$-tight subgraph T' of H' such that the inequality in **HC*** fails. Thus I have

$$\sum_{\sigma \in \mathcal{C}} \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) \quad < \quad \left|V(H')\right|. \tag{4.1}$$

Thus the vertex v must belong to H' since $\mathcal{L}' = \mathcal{L}$ on the graph $H - v$ and H and $\mathcal{L}$ satisfy **HC***.

With $v \in V(H')$ and $T' \subseteq H'$, two cases arise: $v \in V(T')$ and $v \notin V(T')$.

G:

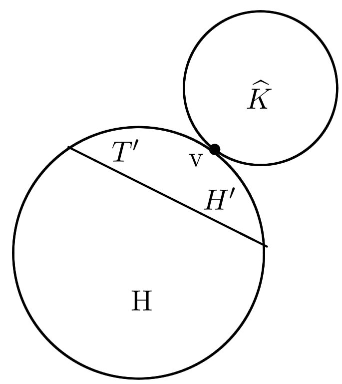


Figure 4.4: Hall* graph H $\supset H' \supset T'$, and clique $\widehat{K}$

Let $H'' = < H' \cup K_p >$ be the subgraph induced by the disjoint union of H' and K_p and let $T'' = < T' \cup K_p >$ be the subgraph induced by the disjoint union of T' and K_p. Recalling that T' is an $\mathcal{L}'$-tight subgraph of H', an induced subgraph of H containing the vertex v, I have the following two claims:

Claim 1: $T'' = < T' \cup K_p >$ is $\mathcal{L}$-tight;

Claim 2: $\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) \quad < \quad |V(H'')|$.

Note that if claim 2 holds, then it will contradict the hypothesis that G and $\mathcal{L}$ satisfy **HC***; hence I can conclude that theorem 4.1 is true.

Remark 4.1

Before getting into the proof, it is worth remarking that it is a fairly known result that if each component of a graph G and a list assignment $\mathcal{L}$ satisfy **HC**, then G and $\mathcal{L}$ will satisfy **HC**. Moreover, if each component is $\mathcal{L}$-tight, then so too is the whole graph.

Proof of Claim 1:

If $v \notin V(T')$, then $\mathcal{L}' = \mathcal{L}$ on T'' and so T'' must also be $\mathcal{L}$-tight as the disjoint union of two $\mathcal{L}$-tight subgraphs. Now suppose $v \in V(T')$. If the symbol $\sigma \notin \mathcal{L}(K_p)$, then

$\alpha(T''(\sigma, \mathcal{L})) \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L})) \quad + \quad \alpha(K_p(\sigma, \mathcal{L})) \quad = \quad \alpha(T'(\sigma, \mathcal{L}')) \quad + \quad \alpha(K_p(\sigma, \mathcal{L}))$.

On the other hand, if $\sigma \in \mathcal{L}(K_p)$, then I will prove the following claim:

$\alpha(T''(\sigma, \mathcal{L})) \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L}')) \quad + \quad \alpha(K_p(\sigma, \mathcal{L}))$.

Noting that K_p is a clique in which case $\alpha(K_p(\sigma, \mathcal{L})) = 1$, one can equivalently prove that

$$\alpha(T''(\sigma, \mathcal{L})) \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L}')) \quad + \quad 1. \tag{4.2}$$

Indeed, let W be a maximum independent set of vertices in $T''(\sigma, \mathcal{L})$, then

Case (i): if $v \in W$, we see that W contains no vertices of the clique K_p and so $W \setminus \{v\}$ is an independent set in $T''(\sigma, \mathcal{L}')$ so $|W \setminus \{v\}| \quad = \quad |W| \quad - \quad 1 \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L}')$ $\iff \quad |W| \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L}') \quad + \quad 1$

and by the definition of W being a maximum independent set in $T''(\sigma, \mathcal{L})$, it follows that

$\alpha(T''(\sigma, \mathcal{L})) \quad \leqslant \quad \alpha(T'(\sigma, \mathcal{L}')) \quad + \quad 1,$ which is equation (4.2).

Case (ii): if $v \notin W$, then W consists of exactly one vertex of K_p together with an independent set in $T'(\sigma, \mathcal{L}')$. This immediately gives $|W| \leqslant \alpha(T'(\sigma, \mathcal{L}') + 1$, hence equation (4.2).

Finally, by Hall's Condition and the definition of T'',

$$\left|V(T'')\right| \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(T''(\sigma, \mathcal{L})) \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(T'(\sigma, \mathcal{L}')) + \sum_{\sigma \in \mathcal{C}} \alpha(K_p(\sigma, \mathcal{L})) \tag{4.3}$$

$$= \left|V(T')\right| + |V(K_p)| = \left|V(T'')\right| \tag{4.4}$$

where the first equality in (4.4) follows because T' is $\mathcal{L}'$-tight and K_p is $\mathcal{L}$-tight and the last equality follows from the fact that $T'' = < T' \cup K_p >$. Thus T'' is $\mathcal{L}$-tight.

Proof of Claim 2:

I will aim at showing that

$$\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) < \left|V(H'')\right|, \tag{4.5}$$

which will contradict the assumption that G and $\mathcal{L}$ satisfy **HC***.

$$\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) = \sum_{\sigma \in \mathcal{C} \setminus \mathcal{L}(K_p)} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) + \sum_{\sigma \in \mathcal{L}(K_p)} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) \tag{4.6}$$

$$= \sum_{\sigma \in \mathcal{C} \setminus \mathcal{L}(K_p)} \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) + \sum_{\sigma \in \mathcal{L}(K_p)} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) \tag{4.7}$$

$$\leqslant \sum_{\sigma \in \mathcal{C} \setminus \mathcal{L}(K_p)} \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) + \sum_{\sigma \in \mathcal{L}(K_p)} \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) + |\mathcal{L}(K_p)| \tag{4.8}$$

where the equality in (4.7) follows because if the symbol σ is not in $\mathcal{L}(K_p)$, then $H''(\sigma, \mathcal{L}) = H'(\sigma, \mathcal{L}')$ and $T''(\sigma, \mathcal{L}) = T'(\sigma, \mathcal{L}')$.

The inequality in (4.8) is a consequence of the following claim:

Claim:

If $\sigma \in \mathcal{L}(K_p)$, then

$$\alpha(H''(\sigma,\mathcal{L}) \mid T''(\sigma,\mathcal{L})) \quad \leqslant \quad \alpha(H'(\sigma,\mathcal{L}') \mid T'(\sigma,\mathcal{L}')) \quad + \quad 1. \tag{4.9}$$

Proof of Claim:

Suppose U is a maximum independent set of vertices in $T''(\sigma,\mathcal{L})$ and W is an independent set of vertices in $H''(\sigma,\mathcal{L})$ containing the set U and suppose $|W| = \alpha(H''(\sigma,\mathcal{L}) \mid T''(\sigma,\mathcal{L}))$. If U contains a vertex x of K_p, then W does not contain the vertex v nor any other vertex of K_p; same for U. So $U' \; = \; U \setminus \{x\}$ is an independent set of vertices in $T'(\sigma,\mathcal{L}')$, contained in $W \setminus \{x\}$, an independent set of vertices in $H'(\sigma,\mathcal{L}')$. I claim that U' is a maximum independent set of vertices in $T'(\sigma,\mathcal{L}')$. For, if $\widetilde{U}$ is any independent set of vertices in $T'(\sigma,\mathcal{L}')$ then, since $\sigma \notin \mathcal{L}'(v)$, whence $v \notin \widetilde{U}$, it follows that $\widetilde{U} \cup \{x\}$ is an independent set of vertices in $T''(\sigma,\mathcal{L})$, whence

$$\left|\widetilde{U} \cup \{x\}\right| \quad = \quad \left|\widetilde{U}\right| \quad + \quad 1 \quad \leqslant \quad |U| \quad = \quad |U' \cup \{x\}| \quad = \quad |U'| \quad + \quad 1.$$

From the fact that $U' \; = \; U \setminus \{x\}$ is an independent set of vertices of $T'(\sigma,\mathcal{L}')$ contained in $W \setminus \{x\}$, an independent set of vertices in $H'(\sigma,\mathcal{L}')$, it follows that

$$|W \setminus \{x\}| \quad = \quad |W| - 1 \quad = \quad \alpha(H''(\sigma,\mathcal{L}) \mid T''(\sigma,\mathcal{L})) - 1 \quad \leqslant \quad \alpha(H'(\sigma,\mathcal{L}') \mid T'(\sigma,\mathcal{L}'))$$

$$\Longleftrightarrow \quad |W| \quad \leqslant \quad \alpha(H'(\sigma,\mathcal{L}') \mid T'(\sigma,\mathcal{L}')) \quad + \quad 1.$$

Now if U does not contain a vertex of K_p, then since $\sigma \in \mathcal{L}(K_p)$, it must be that $v \in U$ because if $v \notin U$ then adding a vertex of K_p to U would give a larger independent set of vertices of $T''(\sigma,\mathcal{L})$. Therefore $v \in V(T'(\sigma,\mathcal{L}))$ and since W is an independent set containing U, it must be that W contains no vertices of K_p. Thus $U \backslash \{v\}$ is a maximum independent set of vertices in $T'(\sigma, \mathcal{L}')$ because if not, take say $\widetilde{U}$ an independent set of vertices in $T'(\sigma, \mathcal{L}')$ such that $|U| > |U \backslash \{v\}|$. Then for any $y \in K_p(\sigma, \mathcal{L})$, $\widetilde{U} \cup \{y\}$ is an independent set of vertices in $T''(\sigma, \mathcal{L})$ bigger than U, contrary to the assumption that U is a maximum independent set of vertices in $T''(\sigma, \mathcal{L})$.

Recalling that $W \setminus \{v\}$ contains $U \setminus \{v\}$ one can complete the prove of the claim as follows:

$$|W \setminus \{v\}| \quad = \quad |W| - 1 \quad \leqslant \quad \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}'))$$

$$\Longleftrightarrow |W| \quad \leqslant \quad \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) \quad + \quad 1 \quad \text{which is (4.9).}$$

To complete the proof, note that the left hand side of (4.6) and the right hand side of inequality (4.8) gives

$$\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, \mathcal{L}) \mid T''(\sigma, \mathcal{L})) \quad \leqslant \quad \sum_{\sigma \in \mathcal{C}} \alpha(H'(\sigma, \mathcal{L}') \mid T'(\sigma, \mathcal{L}')) \quad + \quad |\mathcal{L}(K_p)| \tag{4.10}$$

$$< \quad \left|V(H')\right| \quad + \quad |V(K_p)| \quad = \quad \left|V(H'')\right|. \tag{4.11}$$

from (4.1) above coupled with the definition of H'' and K_p being a tight clique.

To get a corresponding proof for the **Hall**** case, we assume the notation of the preceding proof and proceed as follows:

suppose $\mathcal{L}$ is a list assignment on G such that G and $\mathcal{L}$ satisfy **HC****. One must show that there is a proper $\mathcal{L}$-coloring of G.

Assume the contrary. By lemma 3.1, every induced subgraph H of G with $\mathcal{L}$ restricted on $V(H)$ will satisfy **HC****. Thus H and $\mathcal{L}$ satisfy **HC**** as G, $\mathcal{L}$ does.

Since $H \in$ **Hall****, it follows that there is a proper $\mathcal{L}$ coloring of H. Now suppose there is no proper $\mathcal{L}$-coloring of G. Then for every proper $\mathcal{L}$-coloring of H, if the color on v is removed from the lists on $V(K)$, then K and the new list assignment do not satisfy **HC**.

As before, define $\mathcal{P} = \{\tau \in \mathcal{C} \mid \text{for some proper } \mathcal{L}\text{-coloring } \varphi \text{ of } H,\ \varphi(v) = \tau\ \}$.

We therefore see from lemma 3.4 that for every symbol $\sigma \in \mathcal{P}$, there is an $\mathcal{L}$-tight subclique of the clique K with σ appearing on its lists. Hence there is a tight complete subgraph K_p of K with all colors of $\mathcal{P}$ on its lists.

Now define a new list assignment $\mathcal{L}'$ on the graph H as follows:

$\mathcal{L}'(v) = \mathcal{L}(v) \backslash \mathcal{L}(K_p)$, and $\mathcal{L}'(u) = \mathcal{L}(u)$ for all $u \in V(H - v)$. Then as in the claim in the proof of theorem (3.3) I can conclude that H and $\mathcal{L}'$ satisfy **HC** but do not satisfy **HC****.

Hence there must be an induced subgraph H' of H and some $\mathcal{L}'$-tight subgraph T' of H' such that the inequality in **HC**** fails, i.e. such that

$$\sum_{\sigma \in \mathcal{C}} \min_{T' \lhd H'} \alpha(H'(\sigma, \mathcal{L}') | T'(\sigma, \mathcal{L}')) \quad \geqslant \quad |V(H')|, \tag{4.12}$$

where the minimum is taken over each $\mathcal{L}'$-tight subgraph T' of H'.

Thus $v \in V(H')$ since $\mathcal{L}' = \mathcal{L}$ on the graph $H - v$ and H and $\mathcal{L}$ satisfy **HC****.

With $v \in V(H')$ and $T' \subseteq H'$, either $v \in V(T')$ and $v \notin V(T')$.

As in the preceding proof, let $H'' = < H' \cup K_p >$; and for each T' an $\mathcal{L}'$-tight subgraph of H', let $T'' = < T' \cup K_p >$. Then by an argument similar to that in the preceding proof, it must be that T'' is $\mathcal{L}$-tight in H''.

Claim:

$$\sum_{\sigma \in \mathcal{C}} \min_{\widetilde{T} \lhd H''} \alpha(H''(\sigma, \mathcal{L}) | \widetilde{T}(\sigma, \mathcal{L})) \quad < \quad |V(H'')| \quad = \quad |V(H')| + |V(K_p)|, \tag{4.13}$$

where the minimum is taken over each $\mathcal{L}$-tight subgraph $\widetilde{T}$ of H''.

This claim will contradict the hypothesis that G and $\mathcal{L}$ satisfy **HC****; hence we can conclude.

Indeed, by the preceding proof for equation 4.9, one conclude that for each $\sigma \in \mathcal{L}(K_p)$, and each $\mathcal{L}'$-tight subgraph T' of H',

$$\alpha(H''(\sigma,\mathcal{L}) \mid T''(\sigma,\mathcal{L})) \quad \leqslant \quad \alpha(H'(\sigma,\mathcal{L}') \mid T'(\sigma,\mathcal{L}')) \quad + \quad 1. \tag{4.14}$$

Taking minimum respectively over $\widetilde{T}$ an $\mathcal{L}$-tight subgraph of H'' or over T' an $\mathcal{L}'$-tight subgraph of H' for each symbol σ, and noting that $T'' =< T' \cup K_p >$ one obtain

$$\sum_{\sigma\in\mathcal{C}} \min_{\widetilde{T}\lhd H''} \alpha(H''(\sigma,\mathcal{L})|\widetilde{T}(\sigma,\mathcal{L})) \quad \leqslant \quad \sum_{\sigma\in\mathcal{C}} \min_{T'\lhd H'} \alpha(H''(\sigma,\mathcal{L})|T''(\sigma,\mathcal{L})) \tag{4.15}$$

$$= \sum_{\sigma\in\mathcal{C}\setminus\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H''(\sigma,\mathcal{L})|T''(\sigma,\mathcal{L})) + \sum_{\sigma\in\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H''(\sigma,\mathcal{L})|T''(\sigma,\mathcal{L})) \tag{4.16}$$

$$= \sum_{\sigma\in\mathcal{C}\setminus\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H'(\sigma,\mathcal{L}')|T'(\sigma,\mathcal{L}')) + \sum_{\sigma\in\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H''(\sigma,\mathcal{L})|T''(\sigma,\mathcal{L})) \tag{4.17}$$

$$\leqslant \sum_{\sigma\in\mathcal{C}\setminus\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H''(\sigma,\mathcal{L})|T''(\sigma,\mathcal{L})) + \sum_{\sigma\in\mathcal{L}(K_p)} \min_{T'\lhd H'} \alpha(H'(\sigma,\mathcal{L}')|T'(\sigma,\mathcal{L}')) + |\mathcal{L}(K_p)| \tag{4.18}$$

$$\leqslant \quad \sum_{\sigma\in\mathcal{C}} \min_{T'\lhd H'} \alpha(H'(\sigma,\mathcal{L}')|T'(\sigma,\mathcal{L}')) \quad + \quad |\mathcal{L}(K_p)| \tag{4.19}$$

$$< \quad \left|V(H')\right| \quad + \quad |V(K_p)| \quad = \quad \left|V(H'')\right|. \tag{4.20}$$

Where the inequality in (4.18) follows from (4.14) while inequalities (4.19) and (4.20) follows from (4.1) given earlier coupled with the definition of H'' and K_p being a tight clique.

□

Remark 4.2

One could rephrase the proof architecture by supposing that G, $\mathcal{L}$ satisfy **HC** and H, $\mathcal{L}$ satisfy **HC*** (or respectively **HC****) but there is no proper $\mathcal{L}$-coloring of G. One then show that G, $\mathcal{L}$ do not satisfy **HC*** (or respectively **HC****).

Bibliography

[1] L. D. Andersen and A. J. W. Hilton, Thank Evans!. *Proc. London Math. Soc.* 47 (3) (1983), 507-522.

[2] B. B. Bobga and P. D. Johnson Jr., Completing Partial Latin Squares: Cropper's Problem, *Congressus Numerantium*, 188 (2007), 211-216.

[3] H. L. Buchanan and M. N. Ferencak, On completing Latin squares, *J. Comb. Math. and Comb. Comp.*, 34 (2000), 129-132.

[4] Changiz Eslachi and M. Johnson, Characterization of graphs with Hall number 2, *J. Graph theory* 45 (2004), no. 2 81-100.

[5] C. Colbourn, The complexity of completing partial latin squares, *Discrete Appl. Math.*, 8 (1984), 151158.

[6] M. M. Cropper, J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, P. D. Johnson Jr. Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs, *Journal of Graph Theory*, 33 no.4 (2000), 199-219.

[7] D. Donovan, The completion of partial latin squares, *Australasian Journal of Combinatorics*, 22 (2000), 247-264.

[8] Douglas B. West, Introduction to Graph Theory, 2nd Ed., Pearson Edu., 2005.

[9] Erdös, Rubin and Taylor, Choosability in Graphs. *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, 26 (1979), 125-157.

[10] T. Evans, Embedding incomplete latin rectangles, *Amer. Math. Monthly*, 67 (1960), 958-961.

[11] M. Hall, An Existence Theorem for latin squares, *Bull. Amer. Math. Soc.* 51 (1945), 387-388.

[12] P. Hall, On representatives of subsets, *J. London Math. Soc.* 10 (1935), 26-30.

[13] A. J. W. Hilton and P. D. Johnson Jr., A variation of Ryser's theorem and a necessary condition for the list-colouring problem, Chap. 10, pp. 135-143, *Graph Colourings*, Ray Nelson and Robin Wilson, Editors; Pitman Research Notes in Mathematics Series 218, Longman Scientific and Technical with John Wiley and Sons, Harlow, Essex, and New York, 1990.

[14] A. J. W. Hilton and P. D. Johnson Jr., Extending Hall's theorem, *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, Physica-Verlag, Heidelberg, 1990, 359-371.

[15] A. J. W. Hilton and P. D. Johnson Jr., and E. B. Wantland, The Hall number of a simple graph, *Congressus Numerantium*, 121 (1996), 161-182.

[16] A. J. W. Hilton and P. D. Johnson Jr., List multicolorings of graphs with measurable sets, *Journal of Graph Theory*, 54 no. 3 (2007) 179-193.

[17] D. G. Hoffman, Completing commutative latin squares with prescribed diagonals, *European J. Combinatorics*, 4 (1983), 33-35.

[18] C. A. Rodger, Embedding an incomplete latin square in a latin square with a prescribed diagonal, *Discrete Math.*, 51 (1984), 73-89.

[19] H. J. Ryser, A combinatorial theorem with an application to Latin squares, *Proc. Amer. Math. Soc.,* 2 (1951), 550-552.

[20] V. G. Vizing, Coloring the Vertices of a Graph in Prescribed Colors. (Russian) *Diskret. Analiz. No. 29 Metody Diskret. Anal. V. Teorii Kodov i Shem*,29 (1976), 3-10, 101(MR5816371).