Palm Measure Invariance and Exchangeability for Marked Point Processes

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Palm Measure Invariance and Exchangeability for Marked Point Processes

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Dissertation Abstract

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A random measure ξ on a real interval I is known to be exchangeable iff suitably reduced versions of the Palm distributions Q_t are independent of $t \in I$. In this dissertation we prove a corresponding result where ξ is a point process on I with marks in some Borel space. For this case, the Palm distributions $Q_{s,t}$ depend on parameters $s \in S$ and $t \in I$, and we show that ξ is exchangeable iff the reduced versions of $Q'_{s,t}$ are independent of t.

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Chapter 1

Introduction

1.1 Motivation and History

This dissertation deals with the relation between Palm measures and symmetry properties of marked point processes. More precisely, we characterize the property of exchangeability in terms of an invariance property of the Palm measures. In this introduction, we will only discuss some of the basic notions and ideas in intuitive terms. The precise definitions will be given in later sections.

The first studies of Palm measures date back to the work of Palm [25], Khinchin ([19], 1955), Kaplan (1955), Ryll-Nardzewski [28], Slivnyak [29], Matthes [22], and Mecke [23]. Palm's monograph ([25], 1943) deals with telephone traffic intensity variations. This marks the beginning of queuing theory, which was later to be developed more systematically by Khinchin [19] and others. Palm theory was originally used to develop tools for formulating and studying the basic relationships between time and event averages in queuing theory, but its applications have been explored in other subjects in recent work. Most studies in the literature deal with Palm distributions in the stationary case, but very few treat the exchangeable case.

The research on exchangeability started with de Finetti ([8], 1930). The characterization of infinite exchangeable sequences of random variables was established by de Finetti ([9], 1937), and such sequences are also contractable, as noted by Ryll-Nardzewski ([27], 1957). Hewitt and Savage ([12], 1955) extended the result to random elements in a compact Hausdorff space, and they also proved the celebrated Hewitt-Savage zero-one law in that paper. Bühlmann ([3], 1960) studied exchangeable processes and showed that a process on $\mathbb{R}_+$ is exchangeable iff it has conditional i.i.d. increments. Exchangeable random measures on $[0, 1]$ were characterized by Kallenberg ([13], 1975), and the corresponding results for random measures on product spaces $S \times \mathbb{R}_+$ and $S \times [0, 1]$ were given by Kallenberg (1990). The fact that any exchangeable, simple point process on [0, 1] is a mixed binomial process was noted independently by Kallenberg (1973), Davidson ([7], 1974) and Matthes, Kerstan, and Mecke (1974–82). The characterization of mixed Poisson processes by exchangeability and stationarity with exchangeable spacing variables ([17], Proposition 1.28) was proved by Nawrotzki (1962), Kallenberg (1975), Matthes (1978) and Freedman. For more details on exchangeability, the reader may refer to the monographs of Aldous ([1], 1983) and Kallenberg ([17], 2005). The reader may also find more complete historical and bibliographical remarks on exchangeability from Kallenberg ([17], 2005).

The adopted version of the celebrated result of Slivnyak ([29], 1962) states that a point process is Poisson iff the associated reduced Palm distributions agree with the original distribution of the process. Slivnyak's work linked the notions of exchangeability with Palm measures. Later, some people proved extensions in various direction. Papangelou ([26], 1974) characterized the mixed Poisson processes through the invariance of their reduced Palm distributions. The characterization of mixed binomial processes appeared in Kallenberg (1972), and the version for general random measures was proved by Kallenberg ([13], 1975). The extension amounts to characterizing exchangeability of a random measure by the invariance of the associated Palm measures. More precisely, a random measure ξ is exchangeable iff the associated reduced Palm measures Q'_s can be chosen to be independent of s. Inspired by these previous characterizations, especially the result in Kallenberg ([13], 1975), we characterize exchangeable marked point processes in terms of suitably defined Palm distributions. In the main result of this dissertation, we prove that a marked point process ξ, with σ-finite intensity measure $E\xi$ admitting a factorization $\nu \otimes \lambda$, is exchangeable iff the associated reduced Palm measures $Q'_{s,t}$ can be chosen to depend only on $s \in S$. The factorization of the intensity measure $E\xi$ is also necessary, due to the translation invariance of $E\xi$ when ξ is exchangeable.

1.2 Organization

The basic notions used in this dissertation are defined in Chapter 2. Here, we begin in Section 2.1 with definitions of different types of random measures, such as general random measures, point processes, marked point processes. The dissertation relates the notions of Palm measures and exchangeability, which are the main topics of Sections 2.2 and 2.3, respectively. In those sections we will also make some bibliographical comments on Palm measures. This is followed by some other crucial definitions concerning Palm measures in Section 2.2, such as Campbell measures, reduced Palm distributions. In Section 2.3, a brief summary of the classical theory of exchangeable point processes and random measures is presented. In particular, Section 2.3 discusses the relations between different symmetries for random measures and gives a unqiue representation of an exchangeable random measure. This section also collects some characterization results for exchangeable simple point processes and marked point processes.

In Chapter 3, we begin with a discussion of previously known results relevant to this dissertation. Former results characterize the exchangeability of a few special types of random measure and general random measures in terms of Palm measures, where Palm measures Q_s for general random measures on S are assumed to depend only on $s \in S$. The discussion in Section 3.2 in the setting without marks provides a motivation for the main result of this dissertation, Theorem 3.4, dealing with marked point processes, where the Palm measures depend on two parameters $s \in S$ and $t \in I$.

In Chapter 4 we prove some previously known results on this subject, in order to provide some more details to the terse and technical proofs given in the references. This makes this dissertation self-contained, and may help to explain some of the ideas underlying the proof of our main result, Theorem 3.4, in Chapter 5.

Finally, the proof of our main result is given in Chapter 5. Since the proof is quite complicated, we divide it into several steps, as follows: First we show that the proof of Theorem 3.4 can be reduced to the case where the "time scale" I is bounded, the total mass of the marked point process ξ is finite, and the projection of ξ onto the mark space is nonrandom. Then we provide an equivalence condition for the exchangeability of ξ. We complete the proof by using a connection between regular condition probability and conditional reduced Palm distributions.

1.3 Notation

For convenience, we assume all random elements appearing in this dissertation to be defined on some abstract probability space $(\Omega, \mathcal{F}, P)$ with associated expectation operator E. For ease of reading, we list some symbols that will be used throughout the dissertation:

$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{R}_+$	set of non-negative real numbers
$\mathbb{Z}, \mathbb{Z}_+$	set of integers in $\mathbb{R}$ or $\mathbb{R}_+$
$\mathbb{Q}, \mathbb{Q}_+$	set of rational numbers in $\mathbb{R}$ or $\mathbb{R}_+$
S	σ-finite state space of a point process or random measure ξ, often $S = \mathbb{R}^d$
I	interval $[0,1]$ or $\mathbb{R}_+$
p	permutation $(p_1, p_2, ...)$ of integers $(1, 2, ...)$
$\xi \circ p$	sequence $(\xi_{p_1}, \xi_{p_2}, ...)$, where ξ is a sequence of random elements and p is a permutation
$A \equiv B$	A is defined by B
$\mathcal{B}(\mathbb{R})$	Borel σ-field on $\mathbb{R}$
$\mathcal{M}(S)$	class of σ-finite measures on a Borel space S
$\mathcal{N}(S)$	class of locally finite counting measures on a Borel space S
$\mathcal{B}_{\mathcal{M}(S)}$	Borel σ-field on $\mathcal{M}(S)$
$\xi \perp\!\!\!\perp_\tau \eta$	ξ and η are conditionally independent given τ
$T_{a,b}(\xi)$	transposition of a random measure ξ
$1_B\xi$	restriction of random measure ξ to a set B, i.e. $1_B\xi(A) = \xi(B \cap A)$
λ	intensity of a random measure or Lebesgue measure on $\mathbb{R}$
δ_t	$1_B(t) = \delta_t(\mathrm{B})$
ξ_B	projection of a random measure ξ on $S \times I$ onto a set B
id	$id(x) = x$

Chapter 2

Basic Notions

2.1 Random Measures and Point Processes

A *random measure* ξ on a Borel state space S is defined as a σ-finite kernel from the basic probability space Ω to $\mathbb{R}^d$, i.e. $\xi(\omega, \cdot)$ is a measure on $\mathbb{R}^d$ for any $\omega \in \Omega$. Here the underlying probability space for the random measure is $(\Omega, \mathcal{F}, P)$.

A *point process* is a random measure ξ such that ξB is integer-valued for every bounded Borel set $B \in \mathcal{S}$, in which case, we have a representation for the point process $\xi = \sum_i \delta_{\tau_i}$, where $\delta_\tau B = 1\{\tau \in B\}$. In particular, if $\xi\{s\} \leq 1$ for every $s \in \mathbb{R}^d$ outside a P-null kernel, then the point process ξ is said to be *simple.*

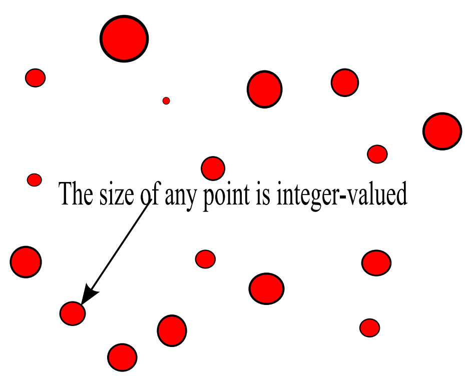


Figure 2.1: Point Process

A *Poisson process* on S with intensity measure $\mu \in \mathcal{M}(\mathbb{R}_+)$ is a point process ξ on $\mathbb{R}_+$ with independent increments such that ξB is Poisson with mean μB whenever

$\mu B < \infty$. A *Cox process* ξ on space S is the point process whose distribution is Poisson with a random underlying intensity measure. A point process ξ on S is a Cox process directed by some random measure η on S when, conditionally on η, realizations of ξ are those of a Poisson process $\xi(\cdot|\eta)$ on S with parameter measure η. A Cox process, also known as a doubly stochastic Poisson process, is a generalization of a Poisson process. It was originally introduced by Cox ([5], 1955). For a Cox process on $\mathbb{R}_+$, taking $\eta = \rho\lambda$ for some random variable $\rho \geq 0$ and σ-finite measure λ turns the Cox process to a *mixed Poisson process*. We say that ξ is a *binomial process* on $[0,1]$ based on $k \in \mathbb{N}$ if ξ can be written as $\sum_{j\leq k} \delta_{\tau_j}$ a.s. for some i.i.d. random variables $\tau_1,\ldots,\tau_k$ with distribution $U(0,1)$. A *mixed binomial process* on $[0,1]$ is obtained by replacing integer k with an integer-valued random variable κ independent of the τ_j.

By a *marked point process* ξ (or, MPP for short) on I with marks in S we mean a simple point process ξ such that $\xi\{S \times \{t\}\} \leq 1$ for any $t \in I$. If $\beta = \sum_j \delta_{\beta_j}$ is an arbitrary point process on S, then a *uniform* or λ*-randomization* of β is defined as a point process on $S \times [0,1]$ of the form $\xi = \sum_j \delta_{\beta_j,\tau_j}$, where the τ_j are i.i.d. $U(0,1)$ and independent of β. An instance of such an exchangeable marked point process is given in Figure 2.2, where the projections of marks onto $[0,1]$ are i.i.d. $U(0,1)$. Since the τ_j are also independent of β, for any positive integer $k \leq \xi(S \times [0,1])$, the τ_j sharing the same β_k are also i.i.d. $U(0,1)$.

Here, we may present some results based on stronger conditions on the probability space. Readers may refer to David Vere-Jones [6] for more details. For a probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{X}$ be a locally compact, second countable Hausdorff

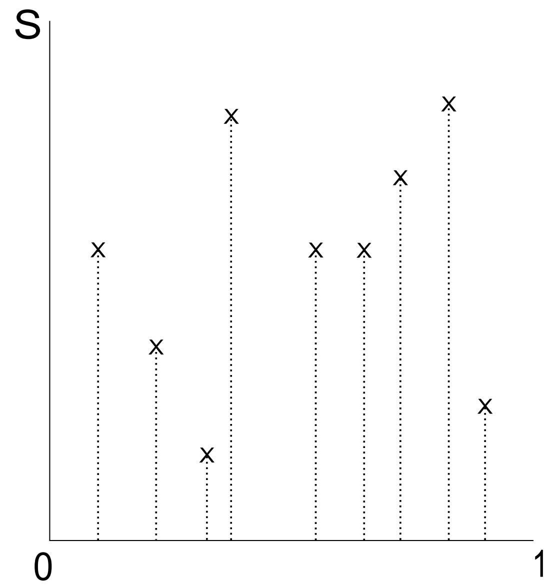


Figure 2.2: Exchangeable Point Process on $S \times [0,1]$

space (abbreviated as $lcscH$), and let d be the metric such that the space $(\mathcal{X}, d)$ is Polish. We denote by $\mathcal{B}$ the ring of relatively compact subsets of $\mathcal{X}$. Let $\mathcal{M}_{\mathcal{X}}$ be the space of Radon measures endowed with the vague topology, in other words $\mathcal{M}_{\mathcal{X}}$ is the space of Borel nonnegative measures that are finite on $\mathcal{B}$. The vague topology is the topology induced by functions π_f, where $\pi_f(\mu) = \mu f$ for any $\mu \in \mathcal{M}_{\mathcal{X}}$ and $f \in C_K(\mathcal{X})$, the space of real-valued functions on $\mathcal{X}$ with compact support. Denote by $\mathcal{M}(\mathcal{X})$ the space of random measures on $(\Omega, \mathcal{F}, P)$ taking values on $\mathcal{M}_{\mathcal{X}}$.

The finite-dimensional distributions of a random measure ξ are the family of proper distribution functions

$$P[\xi(A_i) \in B_i; i = 1, ..., k] \tag{2.1}$$

for all finite families of bounded Borel sets $A_1, ..., A_k$, and Borel sets B_i are chosen from $\mathbb{R}_+$, and $k = 1, 2, ...$

Proposition 2.1. *The distribution of a random measure ξ on an lcscH space $\mathcal{X}$ is totally determined by its finite-dimensional distributions.*

2.2 Palm Measures

The definition of Palm measures requires a random measure ξ on a measurable space $(S, \mathcal{S})$, along with a random element η in a measurable space $(T, \mathcal{T})$, where ξ is also defined as a σ-finite kernel from the basic probability space $(\Omega, \mathcal{F}, P)$ to the space S. Define the set of *Palm distributions* Q_s of η with respect to ξ as Radon-Nikodým densities, given by

$$Q_s(A) = \frac{E[\xi(ds); \eta \in A]}{E\xi(ds)}, \qquad s \in S,\ A \in \mathcal{T}, \tag{2.2}$$

and regular version of $Q_s(A)$ is a measure in S for each s. In order to make sense of this definition for every A, we need the intensity measure $E\xi$ to be σ-finite. In order to ensure the existence of Q as a probability kernel from S to T, we may also assume the space T to be Borel. Rewriting the above equation of Palm distributions as a disintegration, we get

$$Cf \equiv E \int f(s, \eta)\xi(ds) = \int E\xi(ds) \int f(s,t) Q_s(dt), \qquad f \geq 0\ , \tag{2.3}$$

where f is understood to be an arbitrary non-negative, measurable function on $S \times T$. Measure C is the corresponding *Campbell measure* of the pair (ξ, η) on $S \times T$ admitting factorization $C = E\xi \otimes Q$, where this product measure is defined in the sense of (2.3).

When ξ is a simple point process, the Palm distribution Q_s of η with respect to ξ is the conditional distribution of η, given that ξ has a unit mass (or point) at s. In particular, if $\xi = \delta_\sigma$ for some random element σ in S, then Q_σ reduces to a regular conditional distribution $P[\eta \in \cdot|\sigma]$. If $E\xi$ is not σ-finite, then the denominator $E\xi$ in (2.2) needs to be replaced by a σ-finite *supporting measure* ν of ξ on S such that $\nu B = 0$ iff $\xi B = 0$ a.s. for every $B \in \mathcal{S}$. The *supporting measure* ν is unique up to an equivalence, in the sense of mutual absolute continuity. The Q_s are ν-a.e. bounded iff the intensity measure $E\xi$ is σ-finite, in which case we may choose $\nu = E\xi$ and normalize the Q_s to be probability measures on T if we compare (2.2) with (2.3). If Ω itself is Borel, we may choose η to be the identity mapping on Ω (of course, $\Omega = T$ in this case), which makes Q a kernel from S to Ω. Our main emphasis is on the case when $\eta \equiv \xi$, S is Borel, and $E\xi$ is σ-finite. In such a setting, Q becomes a kernel from S to $\mathcal{M}(S)$, in which case the Q_s are called the Palm measures of ξ.

Similarly, we may also define the Palm measures $Q_{s,t}$ of the random measure ξ on a measurable space $S \times I$ by changing the S in (2.3) to $S \times I$, where $s \in S$, $t \in I$.

When ξ is a point process on S and $\eta = \xi$, the Palm measures Q_s of ξ are $E\xi$-a.e. confined to the set of measures $\mu \in \mathcal{N}(S)$ with $\mu\{s\} \geq 1$ for $s \in S$ a.e. ν. The *reduced Palm measures* Q'_s on $\mathcal{N}(S)$ are obtained by subtracting a trivial unit mass (or point) at s from the point process ξ, in which case, the formula of Q'_s is given explicitly by

$$Q'_s(B) = \int_{\mu - \delta_s \in B} Q_s(d\mu) = Q_s\{\mu; \mu - \delta_s \in B\}, \quad s \in S. \tag{2.4}$$

To justify this definition, we may introduce the *reduced Campbell measure* $C' = C'_\xi$ on $S \times \mathcal{N}(S)$, given by

$$C'f = E\int f(s, \xi - \delta_s)\xi(ds), \quad f \geq 0,$$

where f is an arbitrary non-negative, measurable function on $S \times \mathcal{N}(S)$, and $\delta_s A = 1_A(s)$. A reduced Palm distribution Q'_s of the simple point process ξ with σ-finite intensity measure $E\xi$ is the conditional probability distribution of the simple point process obtained by removing the point s from ξ given that ξ has a unit mass at $s \in S$. Again, if C' is σ-finite, then C' also admits a disintegration $C' = \nu \otimes Q'$, where ν is the supporting measure of ξ, and the product measure is also defined in the sense of (2.3). The measures Q_s and Q'_s are ν-a.e. related by (2.4).

A comprehensive introduction to Palm measures is given by Daley [6], and basic facts and results on Palm measures associated with stationary point processes are offered in Chapter 1 of Baccelli [2]. Palm probabilities give us a way of calculating probabilities of events conditioning on sets of measure zero.

Next result (David Vere-Jones [6], section 12.1) states that the Campbell Measure of random measure ξ uniquely determines the distribution of ξ, obviously, the converse is also true.

Lemma 2.2. *Let $\xi \in \mathcal{M}(\mathcal{X})$ be a random measure on an lcscH space $\mathcal{X}$, and define*

$$\nu\left(A \times B\right) = E[\xi A; \xi \in B] \tag{2.5}$$

for any measurable $A \subset \mathcal{X}$ and $B \subset \mathcal{M}(\mathcal{X})$. Then, ν is a σ-finite measure on $\mathcal{X} \times \mathcal{B}_{\mathcal{M}(\mathcal{X})}$ and uniquely determines $\mathcal{L}(\xi)$.

The following result shows how Palm measures can be described in terms of ordinary conditional probabilities by introducing some auxiliary random element τ. The result was first given by Kallenberg (2007) ([18], Proposition 4.1). Here we may fill some details in the short proof given in this paper. For convenience, we write $1_B\xi$ for the restriction of ξ to the set B, i.e. $1_B\xi(A) = \xi(B \cap A)$.

Proposition 2.3. *Let ξ be a random measure on a Borel state space S. For any set $B \in \mathcal{S}$ with $\xi B < \infty$ a.s. Consider a random element η in T such that the Campbell measure C of (ξ, η) is σ-finite. If a random element τ in S with $\tau \notin B$ whenever $\xi B = 0$, and*

$$P[\tau \in \cdot|\xi, \eta] = \frac{1_B\xi}{\xi B} \text{ a.s. on } \{\xi B > 0\} \ . \tag{2.6}$$

Then, the Palm measures Q_s of η with respect to ξ having supporting measure $\nu = \mathcal{L}(\tau)$ on B are given by

$$Q_\tau(A) = E[\xi B; \eta \in A|\tau] \quad \text{a.s. on } \{\tau \in B\}$$

for any $A \in \mathcal{S}$

Proof: Taking expectation on both sides of (2.6), we have that on the set $\{\xi B > 0\}$

$$E\left[P[\tau \in \cdot|\xi, \eta]\right] = P(\tau \in \cdot) = E\left[\xi/\xi B\right].$$

Write $\nu = \mathcal{L}(\tau) \equiv P\{\tau \in \cdot\}$. Let us now check $\nu \sim E\xi$. Assume that $E\xi B = 0$ for some $B \in \mathcal{S}$, then $\xi B = 0$ a.s., so $P(\tau \in B) = 0$ by the assumption of proposition. Similarly, if $\nu B = 0$, then

$$\begin{aligned}
0 &= P\,(\tau \in B) = P(\tau \in B, \xi B = 0) + P(\tau \in B, \xi B > 0) \\
&= P(\tau \in B|\xi B = 0)P(\xi B = 0) + P(\tau \in B|\xi B > 0)P(\xi B > 0) \\
&= P(\tau \in B|\xi B = 0)P(\xi B = 0) + E\left[\xi B/\xi B|\xi B > 0\right]P(\xi B > 0) \\
&= P(\tau \in B|\xi B = 0)P(\xi B = 0) + P(\xi B > 0).
\end{aligned}$$

So, $P(\xi B = 0) = 1 - P(\xi B > 0) = 1$, which gives $E\xi B = 0$. Hence $\nu \sim E\xi$.

From the introduction part of Palm measure, we know that $E\xi$ is a supporting measure on B since $\xi B < \infty$ a.s., therefore ν is also a supporting measure of ξ on B. To see $C = \nu \otimes Q$ on B with ν, Q as stated in the proposition, we shall apply the disintegration theorem twice and use the definition of Campbell measure in the calculations as follows. Let $f \geq 0$ be an arbitrary $\mathcal{S} \otimes \mathcal{T}$-measurable function, then on the set $\{\xi B > 0\}$

$$\begin{aligned} C(1_B f) &= E\int_B f(s,\eta)\xi(ds) = E\left[\xi B \cdot \int_B f(s,\eta)\frac{\xi(ds)}{\xi B}\right] \\ &= E\left[\xi B \cdot \int_B f(s,\eta) P[\tau \in ds|\xi,\eta]\right] = E\left[\xi B \cdot E\left[f(\tau,\eta)|\xi,\eta\right]\right] \\ &= E\left[E\left[\xi B \cdot f(\tau,\eta)|\xi,\eta\right]\right] = E\left[\xi B \cdot f(\tau,\eta)\right] \\ &= E\left[E\left[\xi B \cdot f(\tau,\eta)|\tau\right]; \tau \in B\right] \qquad \text{for } \tau \notin B \text{ when } \xi B = 0 \\ &= \int_B \nu(ds) \int f(s,t) Q_s(dt), \end{aligned}$$

which shows that the Q_s are Palm measures (not necessary to be probability distributions) of η associated with the supporting measure ν of ξ. The last step implies $Q_\tau(dt) = E[\xi B; \eta \in dt|\tau]$ a.s. on $\{\tau \in B\}$. $\square$

2.3 Symmetries

We turn to the brief summary of classical theory of exchangeable random measures, in particular, the exchangeable point processes. The detailed discussions of different types of symmetries including exchangeability are offered in the book of Kallenberg [17].

A finite or infinite sequence of random elements $\xi = (\xi_1, \xi_2, ...)$ in a measurable space $(S, \mathcal{S})$ is said to be *exchangeable* if

$$(\xi_{k_1}, ..., \xi_{k_m}) \stackrel{d}{=} (\xi_1, ..., \xi_m) \tag{2.7}$$

for any sequence k_1,..., k_m of distinct elements in the index set of ξ. We also say that ξ is *contractable* if (2.7) holds whenever $k_1 < \cdots < k_m$. Note that any exchangeable sequence is also contractable.

For a random measure ξ on $(0, \infty)$ or $(0, 1]$, it is exchangeable if the infinite or finite sequence of random elements

$$(\xi(0, 1/n], \xi(1/n, 2/n], ...)$$

is exchangeable for any $n \in \mathbb{N}$.

Assume that $\xi\{0\} = 0$ a.s. or $\xi\,(S \times \{0\}) = 0$ a.s. if ξ is random measure on I or $S \times I$, where $I = [0, \infty)$ or $[0, 1]$. For convenience, we write $\xi_B\,(\cdot) \equiv \xi(\cdot \times B)$ as the projection of ξ onto $B \in \mathcal{B}(I)$ if the random measure ξ is on $S \times I$. Let p denote a permutation $(p_1, p_2, ...)$ of the sequence $(1, 2, ...)$ such that only finitely many integers of sequence $(1, 2, ...)$ are rearranged, and $L = (I_1, I_2, ...)$ be a sequence of disjoint equal-length subintervals of I. Write $L \circ p \equiv (I_{p_1}, I_{p_2}, ...)$ and $\xi \circ L \equiv (\xi_{I_1}, \xi_{I_2}, ...)$. A random measure ξ on $S \times I$, where S is Borel and $I = \mathbb{R}_+$ or $[0, 1]$, is exchangeable if $\xi \circ L \overset{d}{=} \xi \circ (L \circ p)$ for any such permutation p and sequence L.

The celebrated de Finetti's theorem ([9], 1937) states that the distribution of an infinite exchangeable sequence of random variables is a mixture of distributions of i.i.d. sequences. A further extension ([17], Theorem 1.1) of his result shows that the distribution of an infinite sequence ξ of random elements in a Borel space S is exchangeable if ξ is conditionally i.i.d. given some σ-field $\mathcal{F}$, i.e.

$$P[\xi \in \cdot|\mathcal{F}] = \nu^\infty \text{ a.s.} \tag{2.8}$$

for some random probability measure ν on S ([17], section 1.1), which is a stronger result than de Finetti's. The counterpart of this extension result that a random sequence $\xi = (\xi_1, ..., \xi_n)$ in a Borel space S is exchangeable iff ξ is *conditionally "urn"*, i.e.

$$P[\xi \in B|\mathcal{F}] = \frac{1}{n!} \sum_p 1_B(\xi \circ p), \quad B \in \mathcal{S}^n, \tag{2.9}$$

where $\mathcal{F}$ can be taken to be the σ-field generated by $\beta = \sum_{k=1}^n \delta_{\xi_k}$, and the summation extends over all permutations $p = (p_1, ..., p_n)$ of $\{1, ..., n\}$, and the $\xi \circ p \equiv (\xi_{p_1}, ..., \xi_{p_n})$ ([17], section 1.2). Ryll-Nardzewski ([27]) proves that the exchangeability of an infinite sequence ξ of random elements in a Borel measurable space S is equivalent to the contractibility of ξ, and also equivalent to the condition that ξ is conditionally i.i.d in the sense of (2.8).

There are three common notions of symmetries considered in this section, and we gave a short introduction to one type of symmetry for random sequence, the exchangeability. Formulas (2.8) and (2.9) characterize the exchangeabilities for infinite and finite random sequences. We turn to the other two symmetries for random measures by introducing some simple transformations on $I = \mathbb{R}_+$ or $[0,1]$, and give their equivalence for random measures on I and product space $S \times I$.

For any $a \geq 0$, the *reflection* R_a on I is defined by

$$R_a(t) = \begin{cases} a - t, & t \leq a \\ t, & t > a \end{cases}.$$

A random measure ξ on I is said to be *reflectable* if $\xi \circ R_a^{-1} \stackrel{d}{=} \xi$ for every $a \in I$. In the case when ξ is a simple point process. For any fixed $0 \leq a \leq b$, the *contraction* $C_{a,b}$ on I is defined by

$$C_{a,b}(t) = \begin{cases} t, & t \leq a \\ \infty, & t \in (a,b] \\ t - b + a, & t > b \end{cases}.$$

Note that the effect of $C_{a,b}$ is to remove the interval $(a, b]$ and join the remaining two disconnected intervals together. Similarly, we say a random measure ξ on $\mathbb{R}_+$ is *contractable* if $\xi \circ C_{a,b}^{-1} \overset{d}{=} \xi$ for any $[a,b] \subset \mathbb{R}_+$, where $\xi \circ C_{a,b}^{-1}$ is obtained by sticking $1_{[0,a]}\xi$ and $1_{(b,1]}\xi$ (or $1_{(b,\infty)}\xi$) together, but a random measure ξ on $[0,1]$ is said to be contractable if $\xi \circ C_{a,b}^{-1} \overset{d}{=} \xi \circ C_{c,d}^{-1}$ whenever the lengths of intervals $[a,b]$ and $[c,d]$ are equal. The *transposition* $T_{a,b}$ on I is defined for any $0 \leq a \leq b$ by

$$T_{a,b}(t) = \begin{cases} t+b-a, & t \leq a \\ t-a, & t \in (a,b] \\ b, & t > b \end{cases} .$$

In other words, $T_{a,b}$ switches the intervals $[0,a]$ and $(a,b]$, but the interval after b remains the same. An example of such transposition is shown in the Figure 2.3.

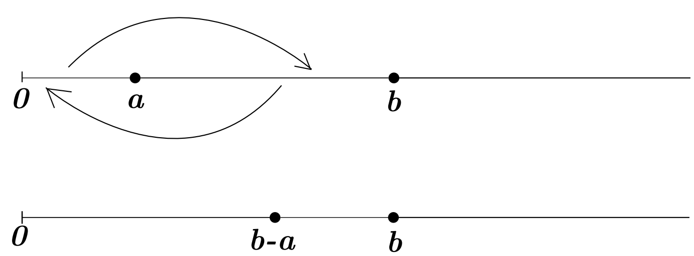


Figure 2.3: Transposition $T_{a,b}$ operated on $\mathbb{R}_+$

Informally, for any random measure ξ on I, $\xi \circ T_{a,b}^{-1}$ is a random measure on I by switching two parts $1_{[0,a]}\xi$ and $1_{(a,b]}\xi$. Intuitively, we may see from the definition of $T_{a,b}$ that a marked point process $\xi \circ T_{a,b}^{-1}$ is obtained by switching two "slices" of ξ on $S \times [0,a]$, $S \times (a,b]$.

According to Theorem 1.15 in [17], the above three notions of symmetry defined by reflection, contraction and transposition are equivalent for a random measure ξ on

$\mathbb{R}_+$. Indeed, this theorem still holds for random measures on $S \times \mathbb{R}_+$, where S is a Borel space, but again, the notations $\xi \circ R_a^{-1}$, $\xi \circ C_{a,b}^{-1}$ and $\xi \circ T_{a,b}^{-1}$ are defined under the condition $\xi(S \times \{0\}) = 0$ a.s. We say f is a λ-preserving function on I if $\lambda = \lambda \circ f^{-1}$. The next result summarizes the relations between the different symmetries quoted from sections 1.4 and 1.5 in [17] for a random measure.

Lemma 2.4. *(exchangeable random measures) Let ξ be a random measure on I or $S \times I$, where $I = \mathbb{R}_+$ and S is Borel. Then these conditions are equivalent:*

(i) ξ is contractable,

(ii) ξ is exchangeable,

(iii) ξ is reflectable,

(iv) $\xi \circ f^{-1} \overset{d}{=} \xi$ for any λ-preserving function f on I,

If instead $I = [0,1]$, then (i)$\Leftarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv), and the exchangeability of ξ is equivalent to the condition

(v) ξ has a representation

$$\xi = \alpha \otimes \lambda + \sum_j \beta_j \otimes \delta_{\tau_j} \quad a.s.$$

for some i.i.d. $U(0,1)$ random variables τ_1,τ_2,... and an independent collection of random measures α and β_1,β_2,... on S.

Furthermore, if ξ is a simple point process on $\mathbb{R}_+$ with infinitely many points $\tau_1 < \tau_2 < \cdots$, then (i)–(iv) are also equivalent to

(vi) ξ is stationary with exchangeable spacing variables $\tau_k - \tau_{k-1}$, $k \in \mathbb{N}$.

Of course, Lemma 2.4 also holds for marked point processes on $S \times \mathbb{R}_+$ or $S \times [0,1]$ and simple point processes on $\mathbb{R}_+$ or $[0,1]$. The following Lemma outlines the exchangeability results in sections 1.4, 1.5 and 1.6 of [17] for simple point processes.

Lemma 2.5. *(characterization of exchangeable simple and marked point processes)*

(i) Let ξ be a simple point process on $\mathbb{R}_+$ or $[0,1]$. Then ξ is exchangeable iff it is a mixed Poisson or mixed binomial process.

(ii) Let ξ be a marked point process on $\mathbb{R}_+$ with marks in Borel space S. Then ξ is exchangeable iff it is a Cox process directed by $\nu \otimes \lambda$ for some random measure ν on S, where λ is Lebesgue measure.

(iii) Let ξ be a marked point process on $S \times [0,1]$, where mark space S is Borel. Then ξ is exchangeable iff it is a λ-randomization of $\beta \equiv \xi(\cdot \times [0,1])$.

Chapter 3

Previous and New Results

3.1 Introduction

Section 3.2 starts with Slivnyak's theorem stating that the only point process whose associated Palm distributions are the same as the distribution of this point process is Poisson, which characterizes the Poisson process in terms of Palm measures. Kallenberg and Papangelou ([17], section 2.7) provide the condition with explicit formula to characterize the exchangeable point processes, in which case, the reduced Palm distributions turn out to be invariant, so it naturally gives arise to conjecturing if there is also an invariance property of reduced Palm distributions for the exchangeable marked point processes. This is the origin of the ideas about the main results of this dissertation presented in Section 3.3.

3.2 Previously Known Results

Pure Poisson processes were first characterized in terms of the Palm distributions by Slivnyak ([29], 1962). The following Theorem 3.2 ([17], section 2.7) extends Slivnyak's classical result and shows that the distribution of a Poisson process with a fixed point added to s is the Palm distribution of this Poisson process at s. The converse is also true.

Theorem 3.1. *(Poisson criterion, Slivnyak 1962) Let ξ be a point process on a Borel space S with reduced Palm distributions Q'_s, $s \in S$. Then ξ is a Poisson process iff $Q'_s = \mathcal{L}(\xi)$ for $E\xi$-a.e. s.*

Theorem 3.1 is a simple special case of the following Theorem 3.2 that completely characterizes the exchangeable point processes through reduced Palm measures, and also gives the explicit formula for the reduced Palm distributions of such exchangeable simple point processes. There are several different approaches of the proof in literature, but here we may take the approach in [17], Section 2.7. In Section 4.1 we also add a few more details to that proof.

Recall from Lemma 2.5 that the only exchangeable simple point processes on $\mathbb{R}_+$ are mixed Poisson, whereas the only exchangeable simple point processes on $[0, 1]$ are mixed binomial processes. However this is not true for point processes.

Consider a mixed Poisson process ξ on S directed by $\rho\lambda$ for some random variable $\rho \geq 0$ and σ-finite measure λ on a measurable space $(S, \mathcal{S})$. Letting $\varphi(t) = Ee^{-t\rho}$ denote the Laplace transform of ρ, we have

$$P\{\xi B = 0\} = Ee^{-\rho\lambda B} = \varphi(\lambda B), \qquad B \in \mathcal{S}.$$

Next let the measure λ on S be such that $0 < \lambda S < \infty$. If ξ is a mixed binomial process based on the probability measure $\lambda/\lambda S$ and a $\mathbb{Z}_+$-valued random variable κ, then the following formula holds.

$$P\{\xi B = 0\} = E\left(1 - \frac{\lambda B}{\lambda S}\right)^{\kappa} = \varphi(\lambda B),$$

where $\varphi(t) = E\left[1 - (t/\lambda S)\right]^{\kappa}$. The following result was obtained by Papangelou (1974), Kallenberg (1975) ([17], section 2.7).

Theorem 3.2. *(reduced Palm measures) Let ξ be a point process on a Borel space S with reduced Palm measures Q'_s, $s \in S$. Then, ξ is a mixed Poisson or binomial process iff Q'_s are independent of s. Moreover, $\mathcal{L}(\xi) \equiv P \circ \xi^{-1} = M(\lambda, \varphi)$ iff $Q'_s = M(\lambda, -\varphi')$ for λ-a.e. s.*

A random measure ξ on S may be decomposed into diffuse and atomic components. Define $\xi \in S_1(\lambda, \alpha, \beta)$ if ξ is *symmetrically distributed* associated with the positive bounded measure λ on S. Then the diffuse component equals $\xi_d = \alpha\lambda/\lambda S$, and the atomic component of ξ is given by $\sum_j \beta_j \delta_{\tau_j}$, where the β_j are atom sizes at positions τ_j, where the τ_j are i.i.d. with the common distribution $\lambda/\lambda S$ and $\beta \equiv \sum_j \delta_{\beta_j}$. Note that since $\xi_d/\alpha = \lambda/\lambda S$, λ is diffuse when $\alpha > 0$.

Similarly, define $\xi \in S_\infty(\lambda, \alpha, \nu)$ if ξ is λ-*symmetric* and has conditionally independent increments directed by the pair (α, ν), where the diffuse component of ξ is $\xi_d = \alpha\lambda$ and atoms are given by a Cox process $\eta \perp\!\!\!\perp_\nu \alpha$ on $S \times (0, \infty)$ directed by $\lambda \otimes \nu$. The following Palm measure invariance result was extended to the general random measures by Kallenberg ([13], 1975).

Theorem 3.3. *(invariant Palm measures) Let ξ be a random measure on a Borel space S with supporting measure λ and associated reduced Palm measures Q'_s. Then $Q'_s = Q'$ is a.e. independent of s iff ξ is λ-symmetric, in which case even the reduced Palm measure Q' is λ-symmetric, and λ is diffuse unless ξ is a.s. degenerate.*

A detailed proof of this result appears in [17], Section 2.7. See also Section 4.2 below.

3.3 Main Result

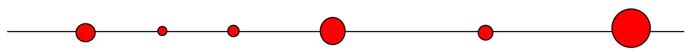
We are now ready to present the main result of this dissertation. Theorem 3.3 suggests the following result for characterizing the exchangeable marked point process by using Palm measure invariance. Here λ denotes the Lebesgue measure, and the Palm measures $Q_{s,t}$ for marked point processes are defined on page 12.

Theorem 3.4. *Let ξ be a marked point process on Borel space $S \times I$, where $I = \mathbb{R}_+$ or $[0,1]$. Assume $E\xi = \nu \otimes \lambda$ for some σ-finite measure ν on S. The following two conditions are equivalent:*

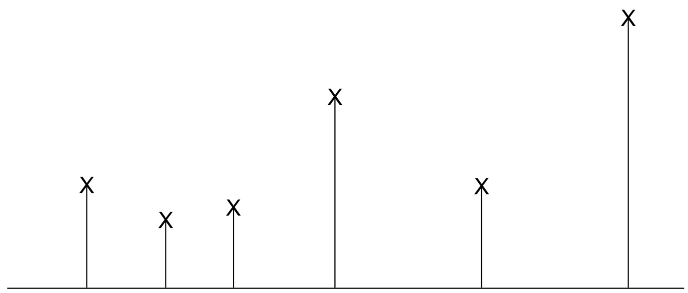
(i) ξ is exchangeable.

(ii) $Q'_{s,t}$ has a version that is independent of t.

Note that the first statement of Theorem 3.2 is the special case where the mark space S is replaced by a single point. The following figure is displayed here for elaborating the connection between Theorem 3.3 and our main result.



A random measure without the diffuse component



A marked point process with the same times as positions of atoms of the above random measure. The "heights" of marks are chosen to be equal to the sizes of atoms of the random measure

The atomic part of a random measure may be coded by a point process in a product space $S \times I$, where the first component gives the location, and the second component gives the size of an atom, which make it seemingly reasonable to give an invariance result for Palm measures of an exchangeable marked point process. As the above figure shows, for a random measure without the diffuse component, we may think of its atom sizes as the marks in the mark space if this random measure turns out to be a marked point process. The atom positions of such a random measure play a role similar to the times of marks on time scale I in the present case. The first statement of Theorem 3.3 shows that the exchangeable random measure has associated reduced Palm measures that are invariant only with respect to the atom positions, which is the motivation behind Theorem 3.4. It naturally gives rise to the extension of these results for random measures to marked point processes. The intuition from Theorem 3.3 suggests that there is a strong connection between the exchangeability of a marked point process and the invariance of its associated reduced Palm measures with respect to the times of marks. However, the definitions of Palm measures and reduced Palm measures for random measures are different from those for marked point processes since, unlike random measures, the Palm measures depend on both mark space S and time scale I in the case of marked point processes. This major difference explains why we need to impose some further conditions, such as requiring $E\xi$ to admit the stated factorization. Recall from Lemma 2.5 that a marked point process on $S \times [0, 1]$ is exchangeable if and only if it is a λ-randomization of point process β on the mark space. Apparently this suggests the need of factoring the intensity measure of an exchangeable marked point process on $S \times [0, 1]$ into a product of measures on the mark space and the time scale.

Chapter 4

Proofs of Previously Known Results

In this chapter, we show some technical proofs of the classical results shown in section 3.3.2 basically following the ideas of proofs given by Kallenberg ([17], section 2.7). The main reasons for including the following technical proofs are to make this dissertation self-contained and to explain some basic ideas that will be used again in the proofs of the main results for this dissertation. Meanwhile, the proofs presented in this chapter provide more details that will be helpful in the next chapter for proving the main result, and also unify notations used later on.

4.1 Proof of Theorem 3.2

Here we may state Theorem 3.2 again to save time of turning pages back and forth for readers.

Theorem 3.2 *Let ξ be a point process on a Borel space S with reduced Palm measures Q'_s, $s \in S$. Then, ξ is a mixed Poisson or binomial process iff Q'_s are independent of s. Moreover, $L(\xi) \equiv P \circ \xi^{-1} = M(\lambda, \varphi)$ iff $Q'_s = M(\lambda, -\varphi')$ for λ-a.e. s.*

First we need the following two auxiliary lemmas to prove Proposition 4.3 that is important for proving equivalence between two equations in this Theorem. For elementary proofs of those lemmas and references, see Lemmas 12.2 and 12.4 in Kallenberg ([16], 2002). We start with the Lemma that characterizes Poisson processes by their unique Laplace functionals.

Lemma 4.1. *ξ is Poisson with intensity measure $E\xi = \lambda$ iff it has Laplace functional*

$$Ee^{-\xi f} = \exp\left\{-\lambda(1 - e^{-f})\right\}, \qquad f \geq 0 \text{ measurable.}$$

Next Lemma provides the Laplace functional for a mixed binomial process.

Lemma 4.2. *If ξ is a mixed binomial process based on the probability measure $\lambda/\lambda S$ with $0 < \lambda S < \infty$ and an integer-valued random variable κ, then ξ has Laplace functional*

$$Ee^{-\xi f} = E\left(\lambda e^{-f}/\lambda S\right)^{\kappa}, \qquad f \geq 0 \text{ measurable.}$$

The following proposition gives the Laplace functional for an exchangeable simple point process using the function φ defined in Section 3.2. (refer to [17], section 2.7)

Proposition 4.3. *A point process ξ is either mixed Poisson or mixed binomial iff it has Laplace functional*

$$Ee^{-\xi f} = \varphi\left(\lambda\left(1 - e^{-f}\right)\right), \qquad f \geq 0 \text{ measurable,} \tag{4.1}$$

where φ, λ in both mixed Poisson and mixed binomial cases are defined as in Section 3.2. Hence, by the uniqueness of Laplace functional, we may write $\mathcal{L}(\xi) \equiv P \circ \xi^{-1} = M(\lambda, \varphi)$, a function of φ and λ.

Proof of Proposition 4.3: When ξ is a mixed Poisson process directed by $\rho\lambda$ for some random variable $\rho \geq 0$ and σ-finite measure λ on the measurable space $(S, \mathcal{S})$, then by Lemma 4.1, we have

$$Ee^{-\xi f} = E\left[E\left[e^{-\xi f}|\rho\right]\right] = E\left[E\left[\exp\left\{-\rho\lambda(1-e^{-f})\right\}|\rho\right]\right] = \varphi\left(\lambda(1-e^{-f})\right) \ .$$

If ξ is a mixed binomial process based on the probability measure $\lambda/\lambda S$ with $0 < \lambda S < \infty$ and a random variable κ, then by Lemma 4.2, we obtain

$$Ee^{-\xi f} = E\left(\lambda e^{-f}/\lambda S\right)^{\kappa} = E\left(1 - \lambda(1-e^{-f})/\lambda S\right)^{\kappa} = \varphi\left(\lambda(1-e^{-f})\right) \ .$$

The distribution of the point process ξ is uniquely determined by the Laplace functional as in (4.1), so we may write the distribution of ξ as a function of λ, φ, in other words, $\mathcal{L}(\xi) = M(\lambda, \varphi)$ for some function M. $\square$

Now we are ready to prove Theorem *3.2*. This is a more detailed version of the proof in Kallenberg (2005) (refer to [17], section 2.7).

Proof of Theorem 3.2: Assume that ξ is a mixed Poisson or binomial process. Proposition 4.3 shows that $\mathcal{L}(\xi) = M(\lambda, \varphi)$ for some function M, in which case $Ee^{-\xi f} = \varphi\left(\lambda\left(1-e^{-f}\right)\right)$ for any measurable $f \geq 0$. If ξ is a mixed Poisson process directed by directed $\rho\lambda$ for some random variable $\rho \geq 0$ and σ-finite measure λ on the measurable space $(S, \mathcal{S})$, then λ is a supporting measure of ξ for $\lambda \sim E\xi$ as an observation from $E\xi = E\rho \cdot \lambda$.

Fix a measurable function $f \geq 0$ on S with $\lambda f > 0$ and a set $B \in \mathcal{S}$ with $\lambda B < \infty$. Then

$$Ee^{-\xi f - t\xi B} = \varphi\left(\lambda\left(1 - e^{-f - t1_B}\right)\right), \quad t \geq 0.$$

Taking derivatives with respect to t on both sides, and by dominated convergence theorem together with the equation

$$\frac{d}{dt}\int_B g(s,t)\lambda(ds) = \int_B \frac{\partial g(s,t)}{\partial t}\lambda(ds) \quad \text{if } |g| \leq 2, \lambda B < \infty, \tag{4.2}$$

we have

$$E\left[-\xi B \cdot e^{-\xi f - t\xi B}\right] = \varphi'\left(\lambda\left(1 - e^{-f - t1_B}\right)\right) \cdot \lambda\left(1_B e^{-f - t1_B}\right).$$

Let $t = 0$, then

$$E\left[\xi B \cdot e^{-\xi f}\right] = -\varphi'\left(\lambda\left(1 - e^{-f}\right)\right) \cdot \lambda\left(1_B e^{-f}\right).$$

Let Q_s denote the Palm measures of ξ corresponding to the supporting measure λ. Note that

$$E\left[\xi B \cdot e^{-\xi f}\right] = E\int_B e^{-\xi f}\xi(ds) = C(1_B e^{-\xi f}) = \int_B \lambda(ds)\int e^{-\mu f}Q_s(d\mu).$$

Therefore, we get

$$\int e^{-\mu f} Q_s(d\mu) \int_B \lambda(ds) = -\varphi'\left(\lambda\left(1-e^{-f}\right)\right) \cdot \int_B e^{-f(s)} \lambda\left(ds\right).$$

For any λ-a.e. $s \in S$, we have

$$\int e^{-\mu f} Q_s(d\mu) = -\varphi'\left(\lambda\left(1-e^{-f}\right)\right) \cdot e^{-f(s)},$$

because B is arbitrary, which implies

$$\begin{aligned}
\int e^{-\mu f} e^{f(s)} Q_s(d\mu) &= \int e^{-(\mu-\delta_s) f} Q_s(d\mu) \\
&= \int e^{-\mu f} Q_s'(d\mu) \\
&= -\varphi'\left(\lambda\left(1-e^{-f}\right)\right),
\end{aligned}$$

where $\int e^{-\mu f} Q_s'(d\mu)$ is the Laplace functional of ξ corresponding to the reduced Palm measures. To extend the result to an arbitrary measurable function $f \geq 0$, we simply take non-negative measurable functions $0 \leq f_n \uparrow f$. Note that $-e^{-\mu f_n} \uparrow -e^{-\mu f}$, hence

$$\varphi'\left(\lambda\left(1-e^{-f_n}\right)\right) \uparrow \varphi'\left(\lambda\left(1-e^{-f}\right)\right)$$

by monotone convergence theorem.

Comparing the following equation

$$\int e^{-\mu f} Q'_s(d\mu) = -\varphi'\left(\lambda\left(1 - e^{-f}\right)\right)$$

with (4.1), we conclude that $Q'_s = M(\lambda, -\varphi')$ for λ-a.e. s. In particular, $Q'_s = Q'$ is independent of $s \in S$ a.e. λ.

Let us now prove the theorem in the converse direction. Choose the supporting measure λ and the associated reduced Palm measures Q'_s such that $Q'_s = Q'$ is independent of $s \in S$.

First, assume that ξS is bounded a.s. and $P\{\xi \neq 0\} > 0$. Introduce a random element τ in S satisfying

$$P[\tau \in \cdot|\xi] = \xi/\xi S \quad \text{a.s. on } \{\xi S > 0\}. \tag{4.3}$$

By the definition of Palm measures, for an arbitrary set $B \in \mathcal{S}$, we have

$$\begin{aligned} P\left[\tau \in B|\xi S = n\right] &= E\left[\frac{\xi B}{\xi S}\Big|\xi S = n\right] && (4.4)\\ &= \frac{E[\xi B; \xi S = n]}{nP\{\xi S = n\}} = \frac{\int_B Q_s\left\{\mu S = n\right\}\lambda(ds)}{nP\{\xi S = n\}} && (4.5)\\ &= \frac{\int_B Q_s\left\{(\mu - \delta_s)S = n-1\right\}\lambda(ds)}{nP\{\xi S = n\}}\\ &= \frac{\int_B Q'_s\left\{\mu S = n-1\right\}\lambda(ds)}{nP\{\xi S = n\}}\\ &= \frac{Q'\left\{\mu S = n-1\right\}\lambda B}{nP\{\xi S = n\}}. \end{aligned}$$

In particular, taking $B = S$ implies $0 < \lambda S < \infty$ since ξS is a.s. bounded by assumption.

If $P\{\xi S = n\} > 0$ for some $n \in \mathbb{Z}_+$, then by Proposition 2.3

$$
\begin{aligned}
&P\left[\xi - \delta_\tau \in B | \xi S = n, \tau \in ds\right] \\
&= E\left[1\left\{\xi S = n\right\} \cdot 1_B(\xi - \delta_\tau) | \tau \in ds\right] / P\{\xi S = n\} \\
&= \int 1\left\{\xi S = n\right\} \cdot 1_B(\mu - \delta_s) Q_s(d\mu) / P\{\xi S = n\} \\
&= Q_s[\mu - \delta_s \in B | \mu S = n],
\end{aligned}
$$

which implies $\tau \perp\!\!\!\perp_{\xi S} (\xi - \delta_\tau)$ on the set $\{\xi S > 0\}$ by the invariance of reduced Palm measures. Since (4.4) shows that $P\left[\tau \in B | \xi S = n\right]$ is proportional to λB for a fixed $n \in \mathbb{N}$, we get that

$$
P[\tau \in B | \xi S; \xi - \delta_\tau] = P[\tau \in B | \xi S] = \lambda B / \lambda S \quad \text{a.s. on } \{\xi S > 0\}. \tag{4.6}
$$

Since ξ is a point process and S is Borel, we may write $\xi = \sum_{j \le \kappa} \delta_{\gamma_j}$, where $\gamma_1, ..., \gamma_\kappa$ are listed by a suitable ordering and $\kappa \equiv \xi S$. In order to generate an exchangeable sequence $\tau_1, ..., \tau_\kappa$ by permuting the sequence $\gamma_1, ..., \gamma_\kappa$, we may introduce a sequence of independent integer-valued random variables $\pi_1, ..., \pi_\kappa$ with $(\pi_i) \perp\!\!\!\perp (\xi, \kappa)$ and $1 \le \pi_i \le i$. The distributions of π_i are given by

$$
P\{\pi_i = j\} = j^{-1}, \quad 1 \le j \le i, \, j \in \mathbb{Z}_+, \, 1 \le i \le \kappa. \tag{4.7}
$$

Define $\tau_1 \equiv \gamma_{\pi_\kappa}$, and rearrange the remains $\{\gamma_1, ..., \gamma_\kappa\} \setminus \{\gamma_{\pi_\kappa}\}$ by the same ordering as we used for ordering the set $\{\gamma_1, ..., \gamma_\kappa\}$. Pick the π_{k-1}-th element as τ_2. Continue to pick the τ_i recursively until the whole sequence $\tau_1, ..., \tau_\kappa$ is constructed. It is easy to see that the sequence $\tau_1, ..., \tau_\kappa$ is exchangeable as $\kappa = \xi S$ is given (a short proof of this conclusion is given on page 49), which shows that (4.3) follows with τ replaced by τ_1.

Note that $\xi = \sum_{j \leq \kappa} \delta_{\gamma_j} = \sum_{j \leq \kappa} \delta_{\tau_j}$. Then by (4.6) and independence of (π_i), we get

$$P[\tau_1 \in B | \kappa; \tau_2, ..., \tau_\kappa] = \lambda B / \lambda S \quad \text{a.s. on } \{\kappa > 0\}, \tag{4.8}$$

which extends by the exchangeability of the sequence $\tau_1, ..., \tau_\kappa$ for a given κ to

$$P[\tau_i \in B | \kappa; \{\tau_1, ..., \tau_\kappa\} \backslash \{\tau_i\}] = \lambda B / \lambda S \quad \text{a.s. on } \{\kappa > 0\}, 1 \leq i \leq \kappa. \tag{4.9}$$

Equation (4.9) shows that the random elements (τ_i) are i.i.d. with distribution $\lambda / \lambda S$ a.s. on $\{\xi S > 0\}$, which means that $\xi = \sum_{j \leq \kappa} \delta_{\tau_j}$ is a mixed binomial process based on the probability measure $\lambda / \lambda S$. It follows by Proposition 4.3 that $\mathcal{L}(\xi) = M(\lambda, \varphi)$ for some completely monotone function $\varphi(t) = E\left[1 - (t/\lambda S)\right]^\kappa$ on the bounded interval $[0, \lambda S]$.

Next, we consider the case when ξS is unbounded by borrowing the result obtained from the previous argument for the bounded case. Choose some sequence of sets $B_n \uparrow S$ with $\xi B_n < \infty$ a.s. for every n. By the previous argument, we see that

$$\mathcal{L}(1_{B_n}\xi) = M(1_{B_n}\lambda, \varphi_n), \quad n \in \mathbb{N}$$

for some completely monotone functions $\varphi_n(t) = E\left[1 - (t/\lambda S)\right]^{\xi B_n}$ on bounded intervals $[0, \lambda B_n]$. Fix an arbitrary measurable function $f \geq 0$ on S and put $f_n \equiv 1_{B_n} f$. Then $f_n \uparrow f$. Note that

$$Ee^{-\xi f_n} = \varphi(\lambda(1 - e^{-f_n})), \quad n \in \mathbb{N},$$

where φ on $[0, \lambda S)$ is obtained by the uniqueness of φ_n on the bounded intervals $[0, \lambda B_n]$. Therefore, φ is unique since the extension of φ_n on the increasing intervals $[0, \lambda B_n]$ to $[0, \lambda S)$ is unique.

Note that $e^{-\xi f_n}$ and $e^{-\xi f}$ are bounded by 1, and φ is decreasing and continuous, we get

$$Ee^{-\xi f} = \lim_{n\to\infty} Ee^{-\xi f_n} = \lim_{n\to\infty} \varphi(\lambda(1 - e^{-f_n})) = \varphi(\lambda(1 - e^{-f})),$$

where the first convergence is by dominated convergence and the second convergence is due to monotone convergence and the continuity of φ.

Summarizing the previous arguments together with Theorem 12.4 in [16] (2002), we conclude that ξ is a mixed binomial process when ξS is a.s. bounded or a mixed Poisson process when ξS is a.s. unbounded. $\square$

4.2 Proof of Theorem 3.3

For convenience, we may again state Theorem 3.3 in the previous chapter.

Theorem 3.3 *Let ξ be a random measure on a Borel space S with supporting measure λ and associated reduced Palm measures Q'_s. Then $Q'_s = Q'$ is a.e. independent of s iff ξ is λ-symmetric, in which case even the reduced Palm measure Q' is λ-symmetric, and λ is diffuse unless ξ is a.s. degenerate.*

The following lemmas are required to reduce the proof of Theorem 3.3.

Lemma 4.4. *Let ξ be a random measure on a Borel space S with $0 < \xi S < \infty$ a.s. If the associated reduced Palm measures Q'_s are a.e. independent of s, then for any fixed $x > 0$, the measures*

$$Q_s[(\mu\{s\}, \mu - \mu\{s\}) \in \cdot | \mu S \in dx] \tag{4.10}$$

are also a.e. independent of s.

Proof: Fix any measurable set $B \subset S \times \mathcal{M}(S)$ and $C \in \mathcal{B}(\mathbb{R}_+)$, then

$$Q_s[(\mu\{s\}, \mu - \mu\{s\}) \in B; \mu S \in C] = Q'_s[(y, \nu) \in B; \nu S + y \in C]. \tag{4.11}$$

We note that the right-hand side of (4.11) are a.e. independent of s since reduced Palm measures Q'_s are a.e. independent of s. Write (4.10) as the Radon-Nikodým derivatives, i.e.

$$Q_s\left[(\mu\{s\},\mu-\mu\{s\})\in\cdot|\mu S\in dx\right]=\frac{Q_s[(\mu\{s\},\mu-\mu\{s\})\in\cdot;\mu S\in dx]}{Q_s\{\mu S\in dx\}}. \tag{4.12}$$

Then from (4.11) we see that (4.10) are a.e. independent of s if we take $C=\{x\}$ and $B=S\times\mathcal{M}(S)$. □

Next proposition shows that some properties of a random measure ξ such as exchangeability hold locally so as to reduce the proof of Theorem 3.3 by assuming the total mass of the random measure ξ to be finite.

Proposition 4.5. *Let ξ be a random measure on a Borel space S with σ-finite intensity measure $E\xi$ and Palm distributions Q_s. If $B_n \uparrow S$, then*
(i) ξ is exchangeable iff $1_{B_n}\xi$ is exchangeable for every $n\in\mathbb{N}$.
(ii) $\lim_{n\to\infty} Q_s^{(n)}=Q_s$ for a.e. s, where $Q_s^{(n)}$ are the Palm distribution associated with the random measure $1_{B_n}\xi$.

Proof: The proof is very similar to that of Lemma 5.6. Readers may refer to page 64 of this dissertation for details. □

Let us now prove Theorem 3.3 by some reduction.

Proof of Theorem 3.3: To prove the first assertion, it suffices to just consider the restrictions of ξ to the sets $B\in\mathcal{S}$ with $\xi B<\infty$ a.s. by Proposition 4.5. So, we may henceforth assume $\xi S<\infty$ a.s.

Assume that $Q'_s = Q'$ are a.e. independent of s. Let τ be the random variable with conditional distribution

$$P[\tau \in \cdot|\xi] = \xi/\xi S \ \text{ on } \{\xi S > 0\}, \tag{4.13}$$

and set $(\eta, \zeta) \equiv (\xi\{\tau\}, \xi - \xi\{\tau\}\delta_\tau)$. By Proposition 2.3, we get that any $x > 0$ and $s \in S$

$$P\left[(\eta, \zeta) \in \cdot|\xi S \in dx, \tau \in ds\right] \tag{4.14}$$

$$= P[(\xi\{\tau\}, \xi - \xi\{\tau\}\delta_\tau) \in \cdot|\xi S \in dx, \tau \in ds] \tag{4.15}$$

$$= Q_s[(\mu\{s\}, \mu - \mu\{s\}\delta_s) \in \cdot|\mu S \in dx].$$

Since $Q'_s = Q'$ are a.e. independent of s, the right-hand side of (4.14) is independent of s by Lemma 4.4, which gives the conditional independence

$$\tau \perp\!\!\!\perp_{\xi S} (\eta, \zeta) \quad \text{on } \{\xi S > 0\}. \tag{4.16}$$

Therefore, we have

$$P\left[\tau \in \cdot|\xi S, \eta, \zeta\right] = P[\tau \in \cdot|\xi S] = E[\xi/\xi S|\xi S] \tag{4.17}$$

$$= E[\xi|\xi S]/\xi S \ \text{ a.s. on } \{\xi S > 0\}.$$

Recall that the atom sizes β_j of random measure ξ can be given by the point process $\beta = \sum_j \delta_{\beta_j}$. Also note that β is a measurable function of (η, ζ). So, combining the following equation

$$P[\tau \in \cdot|\xi, \beta] = P[\tau \in \cdot|\xi]$$

with

$$P[\tau \in \cdot|\xi S, \eta, \zeta, \beta] = P[\tau \in \cdot|\xi S],$$

we may henceforth assume β to be non-random. A detailed proof of the sufficiency of similar reduction is given on page 51.

Let λ be the supporting measure of ξ. To see that ξ is λ-symmetric, we may note that for any $B \in \mathcal{S}$ and $A \in \mathcal{B}(\mathbb{R}_+)$,

$$\begin{aligned} E\left[\xi B; \xi S \in A\right] &= \int_B Q_s\{\mu S \in A\}\lambda(ds) \\ &= \int_B Q'_s\{(x,\mu); x + \mu S \in A\}\lambda(ds) \\ &= Q'\{(x,\mu); x + \mu S \in A\}\lambda B \end{aligned}$$

by the assumption that $Q'_s = Q'$ are a.e. independent of $s \in S$. In particular, taking $B = S$. gives

$$E[\xi B; \xi S \in A]/E[\xi S; \xi S \in A] = \lambda B/\lambda S.$$

So,

$$E[E[\xi B|\xi S]; \xi S \in A] = E[\xi B; \xi S \in A] = E[\xi S; \xi S \in A]\lambda B/\lambda S \tag{4.18}$$

shows that $E[\xi B|\xi S] = \xi S \cdot \lambda B/\lambda S$ a.s. since A is arbitrary. Comparing it with (4.17), we get

$$P[\tau \in \cdot|\xi S, \xi\{\tau\}, \xi - \xi\{\tau\}\,\delta_\tau] = P[\tau \in \cdot|\xi S] = \lambda/\lambda S \quad \text{a.s. on } \{\xi S > 0\}. \tag{4.19}$$

By the same means of generating a random permutation as the previous proof on page 35 shows, positions of the atoms are enumerated in exchangeable order when atom sizes at those positions are the same, therefore, we obtain a sequence of positions τ_1,τ_2,... of the atoms of sizes β_1,β_2,...

Let n be the number of distinct atom sizes and write β^i as the i-th smallest atom size. Construct a random variable ϖ independent of ξ, τ such that

$$P\left(\varpi = i\right) = \frac{\beta^i \cdot \beta(\beta^i)}{\xi S}.$$

Write $j_\varpi = \min\{j : \xi\{\tau_j\} = \beta^\varpi\}$. Define $\tau \equiv \tau_{j_\varpi}$. Let us now show that such τ satisfies the condition $P[\tau \in \cdot|\xi] = \xi/\xi S$ a.s. on $\{\xi S > 0\}$ by the following calculations. By the exchangeability of atom positions corresponding to the same atom size, we have a.s.

$$\begin{aligned}
P\left[\tau \in A, \varpi = i|\xi\right] &= \frac{1}{\beta(\beta^i)} \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} P\left[\tau_j \in A, \varpi = i|\xi\right] \\
&= \frac{1}{\beta(\beta^i)} \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} E\left[1\{\tau_j \in A\}\cdot 1\{\varpi = i\}|\xi\right] \\
&= \frac{1}{\beta(\beta^i)} \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} 1\{\tau_j \in A\}\cdot P[\varpi = i|\xi] \\
&= \frac{1}{\beta(\beta^i)} \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} 1\{\tau_j \in A\}\cdot P\{\varpi = i\} \\
&= \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} \beta^i \cdot 1\{\tau_j \in A\}/\xi S,
\end{aligned}$$

hence,

$$\begin{aligned}
P\left[\tau \in A|\xi\right] &= \sum_{i=1}^{n} P\left[\tau \in A, \varpi = i|\xi\right] \\
&= \sum_{i=1}^{n} \sum_{j\in\{k:\xi\{\tau_k\}=\beta^i\}} \beta^i \cdot 1\{\tau_j \in A\}/\xi S \\
&= \xi A/\xi S,
\end{aligned}$$

which shows that the constructed τ satisfies the condition $P[\tau \in \cdot|\xi] = \xi/\xi S$ a.s. on $\{\xi S > 0\}$.

On the set $\{\eta = \beta_k\}$, (4.19) together with the exchangeability of $\{\tau_i : \xi\{\tau_i\} = \beta_k, i \in \mathbb{N}\}$ implies

$$P\left[\tau_k \in \cdot|\xi\left\{\tau_k\right\}, \xi - \xi\left\{\tau_k\right\}\delta_{\tau_k}; \tau_j, j \neq k\right] = \lambda/\lambda S \quad \text{a.s.}$$

by a similar argument as in the previous proof on page 35, which shows that τ_1,τ_2,... are i.i.d. with distribution $\lambda/\lambda S$. It proves that ξ is λ-symmetric. Here, λ is diffuse unless ξ is a.s. degenerate, since otherwise, say $\lambda\{t\} > 0$ for some t, then we have

$$P\left(\tau_1 = \tau_2\right) > P(\tau_1 = t, \tau_2 = t) = \left(\lambda\{t\}\right)^2 > 0,$$

a contradiction to the fact that the τ_i are distinct.

Conversely, suppose that ξ is λ-symmetric, and λ is diffuse unless ξ is a.s. degenerate. Again by Proposition 4.5, it is enough to consider the restrictions of ξ on the sets $B \in \mathcal{S}$ with $0 < \lambda B < \infty$, so we may assume that $\xi \in S_1(\lambda, \alpha, \beta)$ with $0 < \lambda S < \infty$. As before, we may also assume that $\beta = \sum_i \delta_{\beta_i}$ and α are non-random. Note that the λ-symmetry implies $E[\xi|\xi S] = (\xi S) \cdot \lambda/\lambda S$ a.s. Fix an arbitrary measurable set $B \in \mathcal{S}$, by calculations similar to (4.18) we obtain that

$$\begin{aligned}
&\int_B Q_s\left\{\mu S \in \cdot\right\}\lambda(ds) \\
&= E[\xi B; \xi S \in \cdot] = E[E[\xi B|\xi S]; \xi S \in \cdot] \\
&= E[\xi S; \xi S \in \cdot]\lambda B/\lambda S,
\end{aligned}$$

which shows that for a.e. $s \in S$,

$$Q_s\left\{\mu S \in \cdot\right\} = E[\xi S; \xi S \in \cdot]/\lambda S. \tag{4.20}$$

Hence, $Q_s\{\mu S \in \cdot\}$ are a.e. independent of s.

Next, we may construct a random variable τ satisfying $P[\tau \in \cdot|\xi] = \xi/\xi S$ a.s. and $\tau \perp\!\!\!\perp_{\xi S} (\eta, \zeta)$ so that (4.14) holds and gives the invariance property for Palm measures, here $\eta \equiv \xi\{\tau\}$ and $\zeta \equiv \xi - \xi\{\tau\}\delta_\tau$ as before.

By the assumption that ξ is λ-symmetric, ξ has an a.s. representation

$$\xi = \alpha \frac{\lambda}{\lambda S} + \sum_j \beta_j \delta_{\tau_j}, \tag{4.21}$$

for some i.i.d. random elements τ_1,τ_2,... with distribution $\lambda/\lambda S$ and independent of α,β_1,β_2,....

Choose η to be independent of τ_1,τ_2,... with distribution

$$P\{\eta = 0\} = \frac{\alpha}{\xi S},$$

and

$$P\{\eta = \beta_k\} = \frac{\beta(\beta_k) \cdot \beta_k}{\xi S}, \quad k \geq 1.$$

Let $j_k = \min\{j : \xi\{\tau_j\} = \beta_k\}$ for $k \geq 1$. Choose $\tau = \tau_{j_k}$ if $\eta = \beta_k$ as $k \geq 1$, and set τ to be an independent random variable with distribution $\lambda/\lambda S$ if $\eta = 0$. Let $\tau_{j_{k,1}}$,$\tau_{j_{k,2}}$,... denote the positions of atoms with $j_k \equiv j_{k,1} \leq j_{k,2} \leq \cdots$ such that

$$\xi\{\tau_{j_{k,1}}\} = \xi\{\tau_{j_{k,2}}\} = \cdots = \beta_k.$$

For any measurable $A \in \mathcal{S}$ and $k \geq 1$, letting $k' = j_k$ for convenience, we have

$$
\begin{aligned}
&P\left[\tau \in A, \eta = \beta_k | \xi\right] \\
&\quad = P\left[\tau \in A, \eta = \beta_{k'} | \xi\right] = P\left[\tau_{j_{k',1}} \in A, \eta = \beta_{k'} | \xi\right] \\
&\quad = P\left[\tau_{j_{k',2}} \in A, \eta = \beta_{k'} | \xi\right] = \cdots\cdots \\
&\quad = \frac{1}{\beta(\beta_k)} \sum_{n=1} P\left[\tau_{j_{k',n}} \in A, \eta = \beta_{k'} | \xi\right] \\
&\quad = \frac{1}{\beta(\beta_k)} \sum_{n=1} 1\{\tau_{j_{k',n}} \in A\} P\left[\eta = \beta_{k'} | \xi\right] \\
&\quad = \frac{\beta_k}{\xi S} \sum_{n=1} 1\{\tau_{j_{k',n}} \in A\} \\
&\quad = \frac{\beta_k}{\xi S} \sum_{i=1} 1\left\{\beta_i = \beta_k\right\} 1\{\tau_i \in A\}
\end{aligned}
$$

by λ-symmetry of ξ.

The calculations for verifying $P[\tau \in \cdot|\xi] = \xi/\xi S$ a.s. are as follows: For any measurable $A \in \mathcal{S}$, by λ-symmetry of ξ we have

$$
\begin{aligned}
P\left[\tau \in A | \xi\right] &= P\left[\tau \in A, \eta = 0 | \xi\right] + \sum_{j=1} P\left[\tau \in A, \eta = \beta^j | \xi\right] \\
&= \frac{\alpha}{\xi S} \frac{\lambda A}{\lambda S} + \sum_{j=1} \frac{\beta^j}{\xi S} \sum_{i=1} 1\{\tau_i \in A\} 1\{\beta_i = \beta^j\} \\
&= \frac{1}{\xi S}\left[\alpha \frac{\lambda A}{\lambda S} + \sum_{k=1} \beta_k \cdot \delta_{\tau_k}(A)\right] = \xi A / \xi S.
\end{aligned}
$$

Note that $\Omega = \cup_{k=1}(\eta = \beta_k) \cup (\eta = 0)$ and recall that ξS is fixed at the beginning of the proof. Then, on the set $(\eta = \beta_k)$ for some $k \geq 1$, by the independence of η and

τ_1,τ_2,... we have

$$\begin{aligned} P\left[\tau \in \cdot|\eta, \zeta\right] &= P\left[\tau \in \cdot|\zeta, \eta = \beta_k\right] \\ &= P\left[\tau_{j_k} \in \cdot|\tau_i, i \neq j_k; \eta = \beta_k\right] \\ &= P\left\{\tau_{j_k} \in \cdot\right\} = \lambda/\lambda S \text{ a.s.} \end{aligned}$$

Similarly, since τ is defined to be independent of ξ with distribution $\lambda/\lambda S$ and $\zeta = \xi$ on the set $(\eta = 0)$, we have

$$P\left[\tau \in \cdot|\eta, \zeta\right] = P\left[\tau \in \cdot|\xi, \eta = 0\right] = P\left[\tau \in \cdot|\eta = 0\right] = \lambda/\lambda S \text{ a.s.}$$

on the set $(\eta = 0)$. Therefore, $\tau \perp\!\!\!\perp_{\xi S} (\eta, \zeta)$ with τ satisfying (4.13), so (4.14) shows that the conditional distributions

$$Q_s[(\mu\{s\}, \mu - \mu\{s\}\delta_s)|\mu S], \quad s \in S \tag{4.22}$$

are a.e. independent of s. Combining it with (4.20), we have $Q'_s = Q'$ are a.e. independent of s.

The η constructed above is assumed to be independent of ξ, so on the set $(\eta = 0)$ we have

$$P\left[\zeta \in \cdot|\eta\right] = P[\xi \in \cdot|\eta = 0] = P\left\{\xi \in \cdot\right\}. \tag{4.23}$$

Similarly, on the set $(\eta = \beta_k)$ for some $k \geq 1$, we get

$$P\left[\zeta \in \cdot|\eta\right] = P[\xi - \beta_k \delta_{j_k} \in \cdot|\eta = \beta_k] = P\left\{\xi - \beta_k \delta_{j_k} \in \cdot\right\}. \tag{4.24}$$

Note that the λ-symmetry of ξ also implies that $\xi - \beta_k \delta_{j_k}$ is λ-symmetric because of (4.21), the representation of λ-symmetric random measures. This together with (4.23) and (4.24) implies ζ is conditionally λ-symmetric given η, hence (η, ζ) is also λ-symmetric. Then we see from (4.14) that the conditional distributions in (4.22) are also λ-symmetric. This together with (4.20) implies that Q' is λ-symmetric. $\square$

Chapter 5

Proof of Main Theorem

5.1 Some Auxiliary Results

The most important results for proving Theorem 3.4 are Propositions 5.4 and 5.5. In Proposition 5.4 we will establish an equivalence result for connecting the exchangeability of a marked point process on $S \times [0, 1]$ with some independence result by introducing a so-called average random variable τ in $[0, 1]$. One assumption of Proposition 5.5 is the average property of this τ introduced in Proposition 5.4. This equivalence result together with Proposition 5.5 are very handy when dealing with the Palm distributions through a simple calculation of regular conditional probabilities.

The following Lemma provides a construction of an exchangeable sequence of random variables by picking integers $1, ..., n$ one by one at random. This exchangeable sequence is used in Proposition 5.4 to generate an exchangeable sequence of times with the same mark for a marked point process.

Lemma 5.1. *Let $\pi_1, ..., \pi_n$ be independent random elements such that $P(\pi_i = j) = i^{-1}$, where $1 \leq j \leq i \leq n$. Define $\tau_1 \equiv \pi_n$,, and let τ_2 be the π_{n-1}-st smallest integer in the remaining set $\{1,...,n\} \backslash \{\tau_1\}$. Continue recursively to construct the sequence $(\tau_1, ..., \tau_n)$. Then $(\tau_1, ..., \tau_n)$ is an exchangeable permutation of $(1, ..., n)$.*

Proof: We use induction. Fix a positive integer $k \leq n - 1$ and assume that the sequence $(\tau_1, ..., \tau_k)$ is exchangeable. For any distinct integers $a_1, ..., a_k \in [1, n]$,

$2 \leq k \leq n-1$, we define $A \equiv \{\omega : (\tau_1(\omega) = a_1, ..., \tau_k(\omega) = a_k)\} \subset \Omega$. Note that $\pi_1(A)$,...,$\pi_k(A)$ are fixed integers due to the construction of the sequence $(\tau_1, ..., \tau_k)$, i.e. $(\tau_1 = a_1, ..., \tau_k = a_k)$ is equivalent to $B \equiv (\pi_1 = n_1, ..., \pi_k = n_k)$ for some integers n_1,...,n_k.

Define b_1,...,b_{n-k} to be the integers in the set $\{1, ..., n\}\backslash\{a_1, ..., a_k\}$ listed in increasing order. Since π_{k+1} is independent of random elements π_1,...,π_k, for any positive integer $i \leq n-k$, we see from the exchangeability of the sequence $(\tau_1, ..., \tau_n)$ that

$$\begin{aligned} &P\left[\tau_{k+1} = b_i, \tau_1 = a_1, ..., \tau_k = a_k\right] = P\left[\tau_{k+1} = b_i | A\right] P(A) \\ &\quad = P\left[\pi_{k+1} = i | B\right] \frac{(n-k)!}{n!} = P(\pi_{k+1} = i) \frac{(n-k)!}{n!} = \frac{[n-(k+1)]!}{n!}, \end{aligned}$$

It follows by induction that $(\tau_1, ..., \tau_n)$ is an exchangeable sequence. □

We continue with an elementary result about the existence of probability kernels.

Lemma 5.2. *Let ξ be a random element in a Borel space $(S, \mathcal{S})$, and β be a random element in a Borel space T. Then there exists a regular conditional distribution of ξ with respect to some σ-field $\mathcal{F}$ generated by β. Furthermore, if μ is such a probability kernel from T to S satisfying $\mu(\beta, \cdot) = P[\xi \in \cdot | \beta]$ a.s., then $\mu \circ f^{-1}$ is also a regular conditional distribution of $f(\xi)$ with respect to $\mathcal{F}$ for any measurable function $f : S \to S$.*

Proof: The existence of μ is clear from Theorem 6.3 in [16]. For any measurable set $A \in \mathcal{S}$, we have

$$P[f(\xi) \in A|\beta] = P[\xi \in f^{-1}(A)|\beta] = \mu_\beta(f^{-1}A) \text{ a.s.}$$

This completes the proof. $\square$

If two random elements ξ, β are related by $\beta = h(\xi)$ for some measurable function h, then by saying β is invariant under f we mean $h = h \circ f$ for some transformation function f on ξ. The following Lemma 5.3 helps reduce the proof of Proposition 5.4.

Lemma 5.3. *Fix two measurable Borel spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$. Let ξ be a random measure on S, and η be random elements in T. Let β be a ξ-measurable random element in a Borel space $(U, \mathcal{U})$. Let μ be a probability kernel satisfying $\mu_\beta\left[(\xi, \eta) \in \cdot\right] = P[(\xi, \eta) \in \cdot|\beta]$ a.s. Let f be a measurable transformation on S such that β remains invariant under f. Then,*

(i) Fix two measurable sets $A \in \mathcal{S}$, $B \in \mathcal{T}$. For an arbitrary $\mathcal{L}(\beta)$-a.e. b, we have

$$\mu_b\left[\eta \in B, \xi \in A\right] = E\left[\tilde{\xi}_B; \xi \in A|\beta = b\right] \;\; a.s.,$$

where the random variable $\tilde{\xi}_B = P\left[\eta \in B|\xi\right]$ a.s.

(ii) ξ and η are conditionally independent given β iff

$$\mu_b \circ (\xi, \eta)^{-1} = \left(\mu_b \circ \eta^{-1}\right) \otimes \left(\mu_b \circ \xi^{-1}\right)$$

for an arbitrary $\mathcal{L}(\beta)$-a.e. b, where $\mu_b \circ (\xi, \eta)^{-1}$ denotes the joint probability distribution of (ξ, η) under probability measure μ_b.

(iii) f is $\mathcal{L}(\xi)$-preserving iff f is $\mu_b \circ \xi^{-1}$-preserving for $\mathcal{L}(\beta)$-a.e. b.

Proof: (i) Since β is ξ-measurable, by the assumption $P[\eta \in B|\xi] = \tilde{\xi}_B$ a.s., we have, for any measurable $C \in \mathcal{U}$,

$$\begin{aligned}
&\int_{b \in C} \mu_b [\eta \in B; \xi \in A] P\{\beta \in db\} \\
&E[\mu_\beta [\eta \in B; \xi \in A]; \beta \in C] \\
&= P[\eta \in B; \xi \in A, \beta \in C] \\
&= E\left[\tilde{\xi}_B; \xi \in A, \beta \in C\right] \\
&= \int_{b \in C} E\left[\tilde{\xi}_B; \xi \in A|\beta = b\right] P\{\beta \in db\} \\
&= \int_{b \in C} E_{\mu_b}\left[\tilde{\xi}_B; \xi \in A\right] P\{\beta \in db\},
\end{aligned}$$

which shows that, for an $\mathcal{L}(\beta)$-a.e. b,

$$\mu_b [\eta \in B; \xi \in A] = E\left[\tilde{\xi}_B; \xi \in A|\beta = b\right].$$

(ii) Assume that ξ and η are conditionally independent given β, then we have

$$\mu_\beta \{(\xi, \eta) \in A \times B\} = P[(\xi, \eta) \in A \times B|\beta] = \mu_\beta \{\xi \in A\} \cdot \mu_\beta \{\eta \in B\} \quad \text{a.s.}$$

for any measurable sets $A \in \mathcal{S}$, $B \in \mathcal{T}$. Since S and T are Borel, there exits a measure-determing class $\{A_i \times B_j\}_{i=1}^{\infty}$ of $(S \times T, \mathcal{S} \times \mathcal{T})$ satisfying

$$P \bigcap_{i,j=1}^{\infty} [\mu_\beta \{(\xi, \eta) \in A_i \times B_j\} = \mu_\beta \{\xi \in A_i\} \cdot \mu_\beta \{\eta \in B_j\}] = 1. \tag{5.1}$$

Write $C = \{b \in U; \mu_b \circ (\xi, \eta)^{-1} = (\mu_b \circ \eta^{-1}) \otimes (\mu_b \circ \xi^{-1})\}$, and note that $P(\beta \in C) = 1$ from (5.1). Then, for $\mathcal{L}(\beta)$-a.e. b, we have

$$\mu_b \circ (\xi, \eta)^{-1} = \left(\mu_b \circ \eta^{-1}\right) \otimes (\mu_b \circ \xi^{-1}). \tag{5.2}$$

Conversely, if (5.2) holds for $\mathcal{L}(\beta)$-a.e. b, then

$$\mu_b \circ (\xi, \eta)^{-1} = \left(\mu_b \circ \eta^{-1}\right) \otimes (\mu_b \circ \xi^{-1}) \quad \text{a.s.}$$

which shows that ξ and η are conditionally independent given β.

(iii) Assume f is $\mathcal{L}(\xi)$-preserving. Write $\beta = h(\xi)$ since β is a measurable function of ξ. Since β is invariant under f, we get

$$\xi \stackrel{d}{=} f(\xi) \Rightarrow (\xi, h(\xi)) \stackrel{d}{=} (f(\xi), h(f(\xi)) = (f(\xi), h(\xi)),$$

and it implies

$$P[\xi \in A|\beta] = P[f(\xi) \in A|(h \circ f)(\xi)] = P\left[\xi \in f^{-1}A|\beta\right] \text{ a.s.}, \tag{5.3}$$

Therefore, f is $\mu_b \circ \xi^{-1}$-preserving for $\mathcal{L}(\beta)$-a.e. b by a measure-determing class argument as in the proof of (ii).

Conversely, if f is $\mu_b \circ \xi^{-1}$-preserving for $\mathcal{L}(\beta)$-a.e. b, then (5.3) holds by reversing the previous argument. Hence, f is $\mathcal{L}(\xi)$-preserving by taking expectation on both sides of (5.3). $\square$

5.2 Exchangeability of MPP

Let us first state the following Proposition that plays a crucial role in the proof of our main result. In particular, it provides a condition equivalent to exchangeability for marked point processes on $S \times [0,1]$ in terms of a pair of random elements. This pair connects regular conditional probabilities with conditional Palm distributions in the sense of Proposition 5.5.

Proposition 5.4. *Let ξ be a marked point process on a Borel space $S \times I$ with $\xi(S \times I) < \infty$ a.s., where $I = [0,1]$. Let (σ, τ) be a random element in $S \times I$ such that*

$$P[(\sigma,\tau) \in \cdot|\xi] = \xi/\xi(S \times I) \quad \text{a.s. on } \{\xi \neq 0\}. \tag{5.4}$$

Then, ξ is λ-symmetric iff $\tau \perp\!\!\!\perp_{\xi \neq 0} (\sigma, \xi - \delta_{\sigma,\tau})$ with distribution λ.

To make this long proof easier to read, we prove the proposition in three steps. The proof is organized as follows:

Proof: Step 1: Here we consider the special case when β is non-random and prove that λ-symmetry of ξ implies $\tau \perp\!\!\!\perp_{\xi \neq 0} (\sigma, \xi - \delta_{\sigma,\tau})$.

We may assume that $\beta = \sum_{j\leq N} \delta_{\beta_j}$. Here $N = \beta(S)$ is a constant since β is non-random. Since $\{\xi \neq 0\} = \{N > 0\}$, and $\xi(S \times I) < \infty$ a.s. by assumption, we henceforth assume that $N > 0$ and ξ is bounded. Recall that a λ-symmetric ξ has an a.s. representation $\xi = \sum_{j\leq N} \delta_{\beta_j,\tau_j}$ for some i.i.d. random variables $\tau_1, \tau_2, ...$ with distribution λ.

To construct a pair (σ', τ') satisfying the condition

$$P[(\sigma', \tau') \in \cdot|\xi] = \xi/\xi(S \times I) \quad \text{a.s.},$$

we choose a random variable σ' independent of τ_1,τ_2,... with distribution β/N. Note that $\xi = \sum_i \delta_{\beta_i,\tau_i}$ is a function of $(\tau_1, \tau_2, ...)$ since the β_i are non-random. Then $\sigma' \perp\!\!\!\perp \xi$.

Define $i_{\sigma',j}$ to be the j-th smallest number in the index set $\{i \geq 1; \beta_i = \sigma'\}$. Similarly, let $i_{k,j}$ be the j-th smallest number in the set $\{i \geq 1; \beta_i = \beta_k\}$. For convenience, write $\tau_{i_{k,1}} \equiv \tau_{k,1}$ and $m_k \equiv \beta(\beta_k)$. Define $\tau' \equiv \tau_{\sigma',1}$.

The exchangeability of $\tau_1, \tau_2, ...$ implies

$$(\tau_{p_1}, ..., \tau_{p_N}) \overset{d}{=} (\tau_1, ..., \tau_N), \tag{5.5}$$

for any permutation $(p_1, ..., p_N)$ of $1, ..., N$. For any sets $A \in [0, 1]^N$, $B \subset \mathcal{N}(S)$ and $C \subset \mathcal{N}(S \times I)$, since the β_i are non-random, (5.5) implies

$$\begin{aligned} &P\left\{(\tau_{p_1}, ..., \tau_{p_N}) \in A; \sum\nolimits_{1\leq i\leq N} \delta_{\tau_i} \in B\right\} \\ &\quad = P\left\{(\tau_1, ..., \tau_N) \in A; \sum\nolimits_{1\leq i\leq N} \delta_{\tau_i} \in B\right\}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&P[(\tau_{p_1}, ..., \tau_{p_N}) \in A; \xi \in C] \\
&\quad = P[(\tau_{p_1}, ..., \tau_{p_N}) \in A; \sum\nolimits_{1 \le i \le N} \delta_{\beta_{p_i}, \tau_{p_i}} \in C] \\
&\quad = P[(\tau_1, ..., \tau_n) \in A; \sum\nolimits_{1 \le i \le N} \delta_{\beta_i, \tau_i} \in C] \\
&\quad = P[(\tau_1, ..., \tau_n) \in A, \xi \in C],
\end{aligned}
$$

which gives

$$P[(\tau_{p_1}, ..., \tau_{p_N}) \in \cdot | \xi] = P[(\tau_1, ..., \tau_N) \in \cdot | \xi] \quad \text{a.s.}$$

Thus, for any positive integer $k \le N$ and an arbitrary interval $A \subset [0, 1]$, we have

$$P[\tau_{k,1} \in A | \xi] = P[\tau_{k,2} \in A | \xi].$$

Note that $\sigma' \perp\!\!\!\perp \xi$ implies $\sigma' \perp\!\!\!\perp (\tau_{k,1}, \tau_{k,2})$. Therefore, for any $C \subset \mathcal{N}(S \times I)$, we get

$$P[\tau_{k,1} \in A, \sigma' = \beta_k | \xi] = P[\tau_{k,2} \in A, \sigma' = \beta_k | \xi] \quad \text{a.s.}$$

Hence, we get

$$\begin{aligned}
&P[\tau' \in A, \sigma' = \beta_k|\xi] \\
&\qquad = P[\tau_{k,1} \in A, \sigma' = \beta_k|\xi] \\
&\qquad = P[\tau_{k,2} \in A, \sigma' = \beta_k|\xi] = \cdots \\
&\qquad = P[\tau_{k,m_k} \in A, \sigma' = \beta_k|\xi] \\
&\qquad = m_k^{-1}\sum\nolimits_{i\le m_k} P[\tau_{k,i} \in A, \sigma' = \beta_k|\xi] \\
&\qquad = m_k^{-1}E\left[\sum\nolimits_{i\le m_k} 1_A(\tau_{k,i}) \cdot 1\{\sigma' = \beta_k\}\Big|\xi\right] \\
&\qquad = m_k^{-1} \cdot \sum\nolimits_{i\le m_k} 1_A(\tau_{k,i}) \cdot P\left[\sigma' = \beta_k|\xi\right] \\
&\qquad = m_k^{-1} \cdot \sum\nolimits_{i\le m_k} 1_A(\tau_{k,i}) \cdot P\{\sigma' = \beta_k\} \\
&\qquad = m_k^{-1} \cdot \xi(\{\beta_k\} \times A) \cdot m_k/N = \frac{\xi(\{\beta_k\} \times A)}{N}.
\end{aligned}$$

Then for any measurable $B \times C \subset S \times I$, we have

$$P[(\sigma', \tau') \in B \times C|\xi] = \xi(B \times C)/\xi(S \times I) \text{ a.s.}$$

By a monotone-class argument, we get

$$P[(\sigma', \tau') \in \cdot|\xi] = \xi/\xi(S \times I) \text{ a.s.,} \tag{5.6}$$

which proves that the constructed pair (σ', τ') satisfies condition (5.4).

By the exchangeability of the i.i.d. sequence $(\tau_{k,1}, \tau_{k,2}, ..., \tau_{k,m_k})$ and the condition $\sigma' \perp\!\!\!\perp \tau_{k,1}$, for any measurable $M \subset \mathcal{M}(S \times I)$, $A \subset I$, we have

$$P\{\tau' \in A\} = \sum\nolimits_k P\{\sigma' = \beta_k, \tau_{k,1} \in A\} = \lambda A,$$

and so

$$\begin{aligned}
&P\{\xi - \delta_{\sigma',\tau'} \in M, \tau' \in A\} \\
&\quad = \sum\nolimits_k P\left\{\xi - \delta_{\beta_k,\tau_{k,1}} \in M, \tau_{k,1} \in A, \right\} \\
&\quad = P\{\tau_{1,1} \in A\} \sum\nolimits_k P\left\{\xi - \delta_{\beta_k,\tau_{k,1}} \in M\right\} \\
&\quad = P\{\xi - \delta_{\sigma',\tau'} \in M\} P\{\tau' \in A\},
\end{aligned}$$

where the first step is due to the fact that $\xi - \delta_{\beta_k,\tau_{k,1}}$ and $\tau_{k,1}$ are independent. Since σ' is a measurable function of β and $\xi - \delta_{\sigma',\tau'}$, where β is non-random, we have

$$\begin{aligned}
P\left[\tau' \in \cdot | \sigma', \xi - \delta_{\sigma',\tau'}\right] &= P[\tau' \in \cdot | \beta, \xi - \delta_{\sigma',\tau'}] \\
&= P[\tau' \in \cdot | \xi - \delta_{\sigma',\tau'}] \\
&= \lambda.
\end{aligned}$$

Therefore $\tau' \perp\!\!\!\perp (\sigma', \xi - \delta_{\sigma',\tau'})$ with distribution λ. Note that the joint distribution of (ξ, σ', τ') is determined by (5.6). Likewise, (5.4) determines the joint distribution of

(ξ, σ, τ), and so $(\xi, \sigma', \tau') \stackrel{d}{=} (\xi, \sigma, \tau)$. Hence, the independence $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ also holds since the independence only depends on the joint distribution of (ξ, σ, τ).

Step 2: Here we prove the converse assertion. Assume that $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ with distribution λ conditionally on $\{\xi \neq 0\}$. We may again assume that ξ is bounded and non-zero and β is non-random as before.

Fix a positive integer $k \leq N$. Let $\tau_1^k < \cdots < \tau_{m_k}^k$ be the times with a common mark β_k. Consider a sequence of independent random elements π_1^k,..., $\pi_{m_k}^k$ that are independent of ξ with distributions

$$P\{\pi_i^k = j\} = i^{-1}, \quad 1 \leq j \leq i \leq m_k \ ; i, j \in \mathbb{N}. \tag{5.7}$$

Choose $\tau_{k,1}$ to be the $\pi_{m_k}^k$-th smallest time among $\tau_1^k, \tau_2^k, ..., \tau_{m_k}^k$. Then let $\tau_{k,2}$ be the $\pi_{m_k-1}^k$-st smallest time in the remaining set $\{\tau_i^k; i \neq \pi_{m_k}^k\}$. Continue to construct the sequence $(\tau_{k,i})_{i=1}^{m_k}$ recursively. By Lemma 5.1 the constructed sequence $(\tau_{k,i})_{i \leq m_k}$ is exchangeable. Construct sequences $(\tau_{k,i})$ for all other k in the same way. Since the space $S \times I$ is Borel, we may write $\xi = \sum_{i \leq N} \delta_{\sigma_i, \tau_i}$ by choosing

$$\sigma_i = \beta_k \quad \text{if} \quad \sum\nolimits_{n \leq k-1} m_n + 1 \leq i \leq \sum\nolimits_{n \leq k} m_n,$$

and

$$\tau_i = \tau_{k,j} \quad \text{if} \quad \sigma_i = \beta_k \text{ and } i = j + \sum\nolimits_{n \leq k-1} m_n.$$

Define σ' to be a random variable independent of ξ and all the random elements (π_j^k) with distribution β/N as before. Let $\tau' \equiv \tau_{\sigma',1}$. By computations similar to those proving (5.6), we see that the pair (σ', τ') satisfies the same relation. Write $\tau^k \circ \tilde{\pi} \equiv \tau_{k,1}$ for convenience. Then $\tau^k \circ \tilde{\pi}$ and $\pi_{m_k}^k$ are uniquely determined by each other.

Fix an arbitrary measurable set $A \subset I$ and an integer $k \leq N$. By Lemma 6.2 in Kallenberg [16], we see that the assumption $\tau' \perp\!\!\!\perp (\sigma', \xi - \delta_{\sigma',\tau'})$ with distribution λ implies

$$\begin{aligned}
\lambda A &= P[\tau' \in A | \sigma', \xi - \delta_{\sigma',\tau'}] \\
&= P[\tau^k \circ \tilde{\pi} \in A | \sigma' = \beta_k, \xi - \delta_{\beta_k, \tau^k \circ \tilde{\pi}}] \\
&= P[\tau^k \circ \tilde{\pi} \in A | \xi - \delta_{\beta_k, \tau^k \circ \tilde{\pi}}] \quad \text{a.s. on } (\sigma' = \beta_k)\,,
\end{aligned}$$

where the last step is due to the fact that σ' is a measurable function of $\xi - \delta_{\sigma',\tau'}$ and the non-random β. Note that $(\tau_i)_{i \leq N} \backslash (\tau^k \circ \tilde{\pi})$ and $\left(\pi_j^i\right) \backslash \pi_{m_k}^k$ are enough to determine $\xi - \delta_{\beta_k, \tau^k \circ \tilde{\pi}}$, i.e., they contain complete information about $\xi - \delta_{\beta_k, \tau^k \circ \tilde{\pi}}$, by the construction of the sequences (τ_i) and (π_j^i). Therefore, by the independence of the random elements π_j^i, we have

$$\begin{aligned}
\lambda &= P[\tau^k \circ \tilde{\pi} \in \cdot | \xi - \delta_{\beta_k, \tau^k \circ \tilde{\pi}}] \\
&= P[\tau^k \circ \tilde{\pi} \in \cdot | (\tau_i)_{i \leq N} \backslash (\tau^k \circ \tilde{\pi}); \left(\pi_j^i\right) \backslash \pi_{m_k}^k] \\
&= P[\tau_{k,1} \in \cdot | (\tau_i) \backslash \tau_{k,1}] \quad \text{a.s. on } (\sigma' = \beta_k)\,.
\end{aligned}$$

This, together with the exchangeability of the sequence $(\tau_{k,i})$ with a common mark position β_k, implies that the times $(\tau_{k,i})_{k,i}$ are i.i.d. with the common distribution λ a.s. Hence, if any pair (σ, τ) satisfies both (5.4) and $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$, then the λ-symmetry of ξ follows. This completes the proof for the case of non-random β.

Step 3: We now consider the case for a general β. By Lemma 5.3 (i), we get

$$E_{\mu_b}(\xi/\xi S) = \mu_b\left[(\sigma, \tau) \in \cdot | \xi\right] \quad \text{a.s.}$$

for $\mathcal{L}(\beta)$-a.e. b, where η in Lemma 5.3 is replaced by the pair (σ, τ). Combining Lemma 5.2 with Lemma 5.3 (i)-(ii), we get $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ with distribution λ if and only if

$$\mu_b \circ (\tau, \sigma, \xi - \delta_{\sigma,\tau})^{-1} = \left(\mu_b \circ \tau^{-1}\right) \otimes \left(\mu_b \circ (\sigma, \xi - \delta_{\sigma,\tau})^{-1}\right).$$

for $\mathcal{L}(\beta)$-a.e. b. By Proposition 5.3 (iii), we see that ξ is λ-symmetric iff $\mu_b \circ \xi^{-1}$ is invariant under an arbitrary transposition $T_{c,d}$ for $\mathcal{L}(\beta)$-a.e. b, where $c, d \in \mathbb{Q}_+ \cap I$. Therefore, for $\mathcal{L}(\beta)$-a.e. b, the assumption $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ with distribution λ, the λ-symmetry of ξ, and equation (5.4) all remain true simultaneously under probability measures P and μ_b. This shows that the proof for non-random β, using μ_b, is enough to prove the general case by a change of probability measures. $\square$

In many cases, the direct calculations of Palm distributions can be difficult. The following result gives a way of using regular conditional probabilities to do calculations involving Palm distributions.

Proposition 5.5. *Let ξ be a marked point process on a Borel space $S \times I$ with Palm distribution $Q_{s,t}$, where $(s,t) \in S \times I$ and $\kappa = \xi(S \times I) < \infty$ a.s. Let the random element (σ, τ) in $S \times I$ have the conditional distribution*

$$P[(\sigma,\tau) \in \cdot|\xi] = \xi/\xi(S \times I) \text{ a.s. on } \{\kappa > 0\}.$$

Then for any measurable function $f \geq 0$, $n \in \mathbb{N}$, we have a.s. on $\{\kappa > 0\}$

$$E\left[f(\xi,\sigma,\tau)|\xi(S \times I) = n\right] = \int f(\mu,\sigma,\tau) Q_{\sigma,\tau}[d\mu|\mu(S \times I) = n] \ .$$

Proof: We may assume $\xi(S \times I) > 0$ without loss of generality. Let $\nu \equiv E\xi$ be the supporting measure associated with the probability kernel Q from $S \times I$ to $\mathcal{N}(S \times I)$. By Proposition 7.26 in FMP[16], the following formula

$$\frac{Q_{s,t}\left[d\mu; \mu(S \times I) = n\right]}{Q_{s,t}\left\{\mu(S \times I) = n\right\}}$$

has a version of probability kernel, say q, from $(S \times I) \times \mathbb{N}$ to $\mathcal{N}(S \times I)$. Then, for any non-negative product measurable function f, we have

$$\begin{aligned}
E\left[f(\xi, \sigma, \tau)\right] &= EE\left[f(\xi, \sigma, \tau)|\xi\right] \\
&= E\int f(\xi, s, t)P[(\sigma, \tau) \in dsdt|\xi] \\
&= E\int f(\xi, s, t) \cdot \xi(dsdt)/\xi(S \times I) \\
&= \int \nu(dsdt) \int_{\mathcal{N}(S\times I)} f(\mu, s, t) \cdot Q_{s,t}(d\mu)/\mu(S \times I) \\
&= \int \nu(dsdt) \sum_{k\in\mathbb{N}} k^{-1} Q_{s,t}\left[\mu(S \times I) = k\right] \int f(\mu, s, t) Q_{s,t}\left[d\mu|\mu(S \times I) = k\right].
\end{aligned}$$

Writing $r_{s,t}(k) \equiv k^{-1}Q_{s,t}\left[\mu(S \times I) = k\right]$, $q_{s,t,k}(d\mu) \equiv Q_{s,t}[d\mu|\mu(S \times I) = k]$, we get

$$\mathcal{L}\{(\sigma, \tau), \xi(S \times I), \xi\} = \nu \otimes r_{s,t} \otimes q_{s,t,k},$$

and so,

$$\mathcal{L}\{(\sigma, \tau), \xi(S \times I)\} = \nu \otimes r_{s,t}$$

since q is a probability kernel. Thus, it follows that

$$\begin{aligned}
&E\left[f(\xi, \sigma, \tau); \xi(S \times I) = n, (\sigma, \tau) \in dsdt\right] \\
&\quad = \nu(dsdt) \cdot r_{s,t}(n) \int f(\mu, s, t) q_{s,t,n}(d\mu) \\
&\quad = \int f(\mu, s, t) Q_{s,t}[d\mu|\mu(S \times I) = n] \cdot P\left[\xi(S \times I) = n, (\sigma, \tau) \in dsdt\right],
\end{aligned}$$

which shows

$$E[f(\xi,\sigma,\tau)|\xi(S\times I)=n]=\int f(\mu,\sigma,\tau)Q_{\sigma,\tau}[d\mu|\mu(S\times I)=n] \quad \text{a.s.}$$

$\square$

5.3 Proof of Theorem 3.4

Let us recall our main result of this dissertation, Theorem 3.4 on page 26. Here, λ denotes Lebesgue measure as before.

Theorem 3.4 *Let ξ be a marked point process on Borel space $S\times I$, where $I=\mathbb{R}_+$ or $[0,1]$. Assume $E\xi=\nu\otimes\lambda$ for some σ-finite measure ν on S. The following two conditions are equivalent:*

(i) ξ is exchangeable.

(ii) $Q'_{s,t}$ has a version that is independent of t.

Our plan is to prove the statement in the special case when $I=[0,1]$, and ξ is a.s. bounded and non-zero. We may need the following auxiliary results to show that it is enough to reduce the proof of Theorem 3.4 in this special case.

Lemma 5.6. *It is enough to prove Theorem 3.4 in the case when $I=[0,1]$.*

Proof: Write $I_n=[0,n]$. Assume that Theorem 3.4 holds when $I=I_1$. Then, for every $n\in\mathbb{N}$, Theorem 3.4 also holds for I_n by changing every 1 in $[0,1]$ to n in the proof for the case when $I=I_1$.

Let $I = \mathbb{R}_+$. Write ξ_n for the restriction of ξ on $S \times I_n$, i.e. $\xi_n = 1_{S\times I_n}\xi$.

Assume $E\xi = \nu \otimes \lambda$. Then, for any $n \in \mathbb{N}$, $E\xi_n$ also admits such factorization since $\xi = \xi_n$ on $S \times I_n$. Likewise, if $E\xi_n = \nu \otimes \lambda$ for every n, then $E\xi = \nu \otimes \lambda$ since, for any $A \times B \subset S \times \mathbb{R}_+$, we get $E\xi\,(A \times B) = E\xi_n\,(A \times B) = \nu A \cdot \lambda B$ for some sufficiently large n.

Assume that ξ is exchangeable (or λ-symmetric). Define φ_n by

$$\varphi_n \mu = 1_{S\times I_n}\mu, \quad \mu \in \mathcal{N}\,(S \times I)\,.$$

Fix an arbitrary $n \in \mathbb{N}$. For any $a, b \in I_n$, we have $T_{a,b}\varphi_n = \varphi_n T_{a,b}$. Thus

$$T_{a,b}\xi_n = T_{a,b}\,(\varphi_n \xi) = \varphi_n\,(T_{a,b}\xi) \overset{d}{=} \varphi_n \xi = \xi_n,$$

which shows that ξ_n is also exchangeable.

Assume that ξ_n is exchangeable for each n. Let $\mathcal{D}$ be the class of measurable sets $A \subset \mathcal{N}(S \times I)$ satisfying $P\{\xi \in A\} = P\{T_{a,b}\xi \in A\}$ for any $[a, b] \subset \mathbb{Q}_+$. Let B_1, B_2,... be a semi-ring generating σ-algebra $\mathcal{B}_{\mathcal{N}(S\times I)}$. Also let $\mathcal{C}$ be the class of consisting of sets

$$C_i^n = B_i \cap \mathcal{N}(S \times I_n), \quad i, n \in \mathbb{N}.$$

Clearly $\Omega \in \mathcal{D}$. For any $[a,b] \subset \mathbb{Q}_+$, if $A, B \in \mathcal{D}$ with $A \subset B$, then

$$\begin{aligned} P\{\xi \in B \backslash A\} &= P\{\xi \in B\} - P\{\xi \in A\} \\ &= P\{T_{a,b}\xi \in B\} - P\{T_{a,b}\xi \in A\} \\ &= P\{T_{a,b}\xi \in B \backslash A\}. \end{aligned}$$

If $A_1, A_2, ... \in \mathcal{D}$ with $A_n \uparrow A$, then $A \in \mathcal{D}$ by the continuity of probability measures. This shows that $\mathcal{D}$ is a λ-system. If $C_i^{n_1}$, $C_j^{n_2} \in \mathcal{C}$, then

$$C_i^{n_1} \cap C_j^{n_2} = (B_i \cap B_j) \cap \mathcal{N}(S \times I_{n_1 \wedge n_2}) \in \mathcal{C},$$

which shows that $\mathcal{C}$ is a π-system. For any $C_i^n \in \mathcal{C}$, the assumption $T_{a,b}\xi_n \stackrel{d}{=} \xi$ implies

$$P\{\xi \in C_i^n\} = P\{\xi_n \in C_i^n\} = P\{T_{a,b}\xi_n \in C_i^n\} = P\{T_{a,b}\xi \in C_i^n\},$$

which shows that $\mathcal{C} \subset \mathcal{D}$. Therefore, we see that $\mathcal{B}_{\mathcal{N}(S\times I)} = \sigma(\mathcal{C}) = \mathcal{D}$ by a standard monotone-class theorem. Together with the previous result, this shows that the λ-symmetry of ξ_n for any n implies the λ-symmetry of ξ. This proves that ξ is λ-symmetric iff ξ_n is λ-symmetric for each n.

Write $Q_{s,t}$ and $Q_{s,t}^{(n)}$ for the Palm distributions of the marked point processes ξ and ξ_n, respectively. Let C', C_n' be the reduced Campbell measures associated with ξ and ξ_n, respectively. Assume that the reduced Palm distribution $Q_{s,t}^{(n)'}$ has a version independent of t for any $n \in \mathbb{N}$. Note that $Q_{s,t} = Q_{s,t}^{(n)}$ on $\mathcal{N}(S \times I_n)$. For any bounded measurable function $f \geq 0$ on $S \times \mathbb{R}_+ \times \mathcal{N}(S \times \mathbb{R}_+)$, we have

$$
\begin{aligned}
C'\left(1_{S\times I_n\times\mathcal{N}(S\times I_n)}f\right) &= \int_{S\times I_n} E\xi\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,t})\,Q_{s,t}\,(d\mu) \\
&= \int_{S\times I_n} E\xi_n\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,t})\,Q^{(n)}_{s,t}\,(\varphi_n d\mu) \\
&= \int_{S\times \mathbb{R}_+} E\xi_n\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,0})\,Q_{s,0}\,(d\mu)\,.
\end{aligned}
$$

By dominated convergence, we get

$$
\begin{aligned}
& \int_{S\times \mathbb{R}_+} E\xi\,(ds\,dt)\int_{\mathcal{N}(S\times \mathbb{R}_+)} f\,(s,t,\mu-\delta_{s,t})\,Q_{s,t}\,(d\mu) \\
= \;& C'f \leftarrow C'\left(1_{S\times I_n\times\mathcal{N}(S\times I_n)}f\right) \\
= \;& \int_{S\times \mathbb{R}_+} E\xi_n\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,0})\,Q_{s,0}\,(d\mu) \\
\to \;& \int_{S\times \mathbb{R}_+} E\xi\,(ds\,dt)\int_{\mathcal{N}(S\times \mathbb{R}_+)} f\,(s,t,\mu-\delta_{s,0})\,Q_{s,0}\,(d\mu)\,.
\end{aligned}
$$

We conclude that $Q'_{s,t}$ has a version independent of t since f is arbitrary.

Conversely, assume that $Q'_{s,t}$ has a version independent of t. For any measurable $f \geq 0$ on $S \times I_n \times \mathcal{N}(S \times I_n)$, we obtain

$$
\begin{aligned}
& \int_{S\times I_n} E\xi_n\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,t})\,Q^{(n)}_{s,t}\,(d\mu) \\
= \;& \int_{S\times I_n} E\xi\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\varphi_n\mu-\delta_{s,t})\,Q_{s,t}\,(d\mu) \\
= \;& \int_{S\times I_n} E\xi\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\varphi_n\mu-\delta_{s,0})\,Q_{s,0}\,(d\mu) \\
= \;& \int_{S\times I_n} E\xi_n\,(ds\,dt)\int_{\mathcal{N}(S\times I_n)} f\,(s,t,\mu-\delta_{s,0})\,Q^{(n)}_{s,0}\,(d\mu)\,,
\end{aligned}
$$

which shows that $Q_{s,t}^{(n)'}$ has a version independent of t for any $n \in \mathbb{N}$. This proves that the Palm-measure invariance property holds for ξ and all ξ_n simultaneously.

Write $(\bigstar)$ for the statement $E\xi = \nu \otimes \lambda$. Statements (i) and (ii) refer to those in Theorem 3.4. For an arbitrary $n \in \mathbb{N}$, also write $(\bigstar)_n$, $(i)_n$ and $(ii)_n$ for the versions of statements $(\bigstar)$, (i) and (ii) with ξ replaced by ξ_n. By previous discussions, if statements $(\bigstar)$ and (i) hold for any $n \in \mathbb{N}$, then $(\bigstar)_n$ and $(i)_n$ also hold, which implies that $(ii)_n$ holds. So does (ii). Likewise, statements $(\bigstar)$ and (ii) imply (i). This shows that it is enough to prove Theorem 3.4 in the case when $I = [0, 1]$. $\square$

The next lemma shows another reduction of the proof of our main theorem.

Lemma 5.7. *Let $I = [0, 1]$. Assume that Theorem 3.4 holds in the special case when $\xi(S \times I) < \infty$ a.s. Then it is enough to prove Theorem 3.4 in this special case.*

Proof: Since S is Borel, we can choose $S_n \uparrow S$ with $\xi(S_n \times I) < \infty$ a.s. For convenience, let $1_{S_n \times I}\xi$ denote the restriction of ξ to the set $S_n \times I$.

Since $\xi = 1_{S_n \times I}\xi$ on $S_n \times I$, the intensity measure $E(1_{S_n \times I}\xi)$ admits the same factorization $\nu \otimes \lambda$ on $S_n \times I$. If $E(1_{S_n \times I}\xi) = \nu \otimes \lambda$ on any $S_n \times I$, then for any measurable $A \times B \subset S \times I$, there exists an $n \in \mathbb{N}$ such that $A \subset S_n$, and so $E\xi(A \times B) = E(1_{S_n \times I}\xi)(A \times B) = \nu A \cdot \lambda B$. Thus, $E\xi = \nu \otimes \lambda$ iff $E(1_{S_n \times I}\xi)$ also admits such factorization on $S_n \times I$ for any $n \in \mathbb{N}$.

Next, we show that ξ is λ-symmetric if and only if $1_{S_n \times I}\xi$ is λ-symmetric for any $n \in \mathbb{N}$.

Assume that ξ is λ-symmetric. Then $T_{a,b}\xi \stackrel{d}{=} \xi$ for any $[a,b] \subset I$. For any $n \in \mathbb{N}$, define a surjective mapping p_n by

$$p_n\mu = 1_{S_n \times I}\mu \ , \quad \mu \in \mathcal{N}(S \times I).$$

Note that $p_n T_{a,b} = T_{a,b} p_n$. Hence,

$$p_n\xi \stackrel{d}{=} p_n T_{a,b}\xi = T_{a,b} p_n \xi,$$

which gives the λ-symmetry of $1_{S_n \times I}\xi$ since $[a,b]$ is an arbitrary interval.

Assume that $p_n\xi$ is λ-symmetric for any $n \in \mathbb{N}$. By a similar argument as in the proof of Lemma 5.6, we conclude that the λ-symmetry of $p_n\xi$ for any n implies the λ-symmetry of ξ. This proves that ξ is λ-symmetric iff $1_{S_n \times I}\xi$ is λ-symmetric for any $n \in \mathbb{N}$.

Write ($\sharp$) for the following equivalence statement

$$\begin{aligned} & Q'_{s,t} \text{ has a version independent of } t \\ \Leftrightarrow \quad & Q^{(n)\prime}_{s,t} \text{ has a version independent of } t. \end{aligned}$$

By changing φ_n and $S \times I_n$ in the proof of the statement ($\sharp$) in Lemma 5.6 to p_n and $S_n \times I$, respectively, we see that the statement ($\sharp$) in the current lemma is also true. Therefore, it is enough to prove Theorem 3.4 in the case when $\xi\,(S \times I) < \infty$ a.s. by the same argument as in the last paragraph of the proof for Lemma 5.6. $\square$

We also need the following result to reduce the proof of Theorem 3.4 to the case when ξ is non-zero.

Lemma 5.8. *Let $I = [0, 1]$. Assume that Theorem 3.4 holds in the special case when $\xi(S \times I) \neq 0$ a.s. Then it is enough to prove Theorem 3.4 in this special case.*

Proof: If $P(\xi = 0) = p \in [0, 1)$, then by transfer theorem we may find a marked point process η on $S \times I$ satisfying $P(\eta \in \cdot) = P[\xi \in \cdot | \xi \neq 0]$. Since $E\eta = (1-p)^{-1} E\xi = (1-p)^{-1} \nu \otimes \lambda$, even $E\eta$ admits such factorization.

Assume that ξ is λ-symmetric. Then for any transposition $T_{a,b}$ and $A \subset \mathcal{N}(S \times I)$,

$$\begin{aligned} P\left\{\eta \in T_{a,b}^{-1} A\right\} &= P[\xi \in T_{a,b}^{-1} A | \xi \neq 0] = P[T_{a,b}\xi \in A | T_{a,b}\xi \neq 0] \\ &= P[\xi \in A | \xi \neq 0] = P\left\{\eta \in A\right\}, \end{aligned}$$

where the second equality holds since $\{T_{a,b}\xi \neq 0\} = \{\xi \neq 0\}$. So, the λ-symmetry of ξ implies the λ-symmetry of η. Furthermore, if η is λ-symmetric, then

$$\begin{aligned} P\left\{T_{a,b}\xi \in \cdot\right\} &= (1-p)P\left\{T_{a,b}\eta \in \cdot\right\} + p\,1\{0 \in \cdot\} \\ &= (1-p)P\left\{\eta \in \cdot\right\} + p\,1\{0 \in \cdot\} \\ &= P\left\{\xi \in \cdot\right\}. \end{aligned}$$

Therefore, ξ is λ-symmetric iff η is λ-symmetric.

Write $(s,t) = u$. By simple calculations, we get

$$\begin{aligned} Q_{s,t}(A) &= \frac{E[\xi(ds \times dt); \xi \in A]}{E\xi(ds \times dt)} = \frac{E[\xi(du); \xi \in A]}{E\xi(du)} \\ &= \frac{E[\xi(du); \xi \in A; \xi \neq 0]}{E[\xi(du), \xi \neq 0]} \\ &= \frac{E[\xi(du); \xi \in A | \xi \neq 0]}{E[\xi(du) | \xi \neq 0]} \\ &= \frac{E[\eta(du); \eta \in A]}{E\eta(du)}. \end{aligned}$$

This shows that the Palm distributions are the same for ξ and η. Thus, it is enough to further assume ξ is non-zero a.s. $\square$

Let us now prove Theorem 3.4 in some special case.

Proof of Theorem 3.4: By Lemmas 5.6, 5.7 and 5.8, we may henceforth assume that $I = [0,1]$ and $0 < \xi(S \times I) < \infty$ a.s.

Let Ω be a canonical space, i.e. $\Omega = S \times I \times \mathcal{N}(S \times I)$. Assume that ξ is λ-symmetric. Define (σ, τ) to be the pair of random variables satisfying $P[(\sigma,\tau) \in \cdot | \xi] = \tilde{\xi}$ a.s., where $\tilde{\xi} = \xi / \xi(S \times I)$. By Proposition 5.4 we have $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ due to the λ-symmetry of ξ. Hence, $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau}, \xi(S \times I))$ with distribution λ since $\xi(S \times I)$ is a measurable function of $\xi - \delta_{\sigma,\tau}$. Fix an arbitrary measurable set $M \subset \mathcal{N}(S \times I)$ and let f be a measurable function on Ω satisfying $f(x,y,z) \equiv 1_{z - \delta_{x,y}}(M)$. By Proposition 5.5 we get, for any $(s,t) \in S \times I$

$$P[\xi - \delta_{\sigma,\tau} \in M | \xi(S \times I) = n, (\sigma,\tau) \in ds\, dt] \tag{5.8}$$

$$= Q_{s,t}[\mu - \delta_{s,t} \in M | \mu(S \times I) = n] \text{ a.s.} \tag{5.9}$$

Combining (5.8) with the condition $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau}, \xi(S \times I))$, we have

$$
\begin{aligned}
&P[\xi - \delta_{\sigma,\tau} \in M | \xi(S \times I) = n, (\sigma, \tau) \in ds\, dt] \\
&\quad = P[\xi - \delta_{\sigma,\tau} \in M | \xi(S \times I) = n, \sigma \in ds],
\end{aligned}
$$

which is a.e. independent of t. Then the right-hand side of (5.8) is also a.e. independent of t for any measurable set M. Therefore, the expressions $Q_{s,t}[\mu - \delta_{s,t} \in \cdot | \mu(S \times I) = n]$ are a.e. independent of t.

Since ξ is a λ-symmetric marked point process on the Borel space $S \times I$, we may write $\xi = \sum_{i=1}^{\kappa} \delta_{\beta_i, \tau_i}$ and $\beta = \xi(\cdot \times I) = \sum_{i=1}^{\kappa} \delta_{\beta_i}$, where the τ_i are i.i.d. $U(0,1)$ and independent of marks β_1,β_2,... For any measurable set $A \in \mathcal{S}$ and $B \subset [0,1]$, by the independence between τ_i and β_i, we have

$$
\begin{aligned}
E\xi(A \times B) &= E\sum\nolimits_{i \le \kappa} 1\{\beta_i \in A, \tau_i \in B\} = \sum\nolimits_{i \le \kappa} P(\beta_i \in A) P(\tau_i \in B) \\
&= \lambda B \cdot \sum\nolimits_{i \le \kappa} P(\beta_i \in A) = \lambda B \cdot E\beta(A) = \nu A \cdot \lambda B,
\end{aligned}
$$

which shows that the intensity measure of the λ-symmetric marked point process ξ on $S \times I$ has a version of factorization $\nu \otimes \lambda$.

Fix an arbitrary $n \in \mathbb{N}$. For any measurable sets $A \in \mathcal{S}$, $B \subset [0,1]$, by the definition of Palm distributions and the λ-symmetry of ξ, we have

$$\begin{aligned}
\int_{A\times B} & Q_{s,t}\left\{\mu(A\times I)=n\right\}\nu\otimes\lambda(dsdt) \\
&= E[\xi(A\times B);\xi(A\times I)=n] \\
&= E[E[\xi(A\times B)|\xi(A\times I)];\xi(A\times I)=n] \\
&= E[\lambda B\cdot\xi(A\times I);\xi(A\times I)=n] \\
&= \lambda B\cdot nP[\xi(A\times I)=n].
\end{aligned}$$

Since B is arbitrary, for a λ-a.e. t, we have

$$\int_A Q_{s,t}\{\mu(A\times I)=n\}\nu(ds) = nP\{\xi(A\times I)=n\}. \tag{5.10}$$

Hence, $Q_{s,t}\{\mu(A\times I)=n\}$ is a.e. independent of t for any n. Since $Q_{s,t}[\mu-\delta_{s,t}\in\cdot|\mu(S\times I)=n]$ is a.e. independent of t, we conclude that $Q'_{s,t}\equiv Q_{s,t}\left\{\mu-\delta_{s,t}\in\cdot\right\}$ is a.e. independent of t by Lemma 4.4, i.e., $Q'_{s,t}$ is independent of t a.e. λ.

Conversely, suppose that $Q'_{s,t}$ is independent of t a.e. λ, and $E\xi=\nu\otimes\lambda$ for some measure ν. Define (σ,τ) as in Proposition 5.4 (see page 59) satisfying

$$P\left[(\sigma,\tau)\in\cdot|\xi\right]=\xi/\xi S \quad \text{a.s.}$$

Then, Proposition 5.5 shows that, for any $(s,t) \in S \times I$, (5.8) also holds for an arbitrary measurable set $M \subset \mathcal{N}(S \times I)$, and so,

$$P\left[\xi - \delta_{\sigma,\tau} \in \cdot | \xi(S \times I) = n, (\sigma,\tau) \in ds\, dt\right] \tag{5.11}$$

$$= Q'_{s,t}[\mu \in \cdot | \mu(S \times I) = n-1] \text{ a.s.} \tag{5.12}$$

Clearly the right-hand side of (5.11) is independent of t by assumption. So is the left-hand side, which implies

$$\tau \perp\!\!\!\perp_{\xi(S\times I),\sigma} \xi - \delta_{\sigma,\tau}.$$

To show that ξ is λ-symmetric, we just need to establish the independence $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ with distribution λ, in order to apply Proposition 5.4. Note that $\xi\,(S \times I)$ is a measurable function of $\xi - \delta_{\sigma,\tau}$, it is enough to prove $\tau \perp\!\!\!\perp (\sigma, \xi(S \times I))$ with distribution λ, and so, the independence $\tau \perp\!\!\!\perp (\sigma, \xi - \delta_{\sigma,\tau})$ follows by chain rule (Proposition 6.6, [16]). By the definition of Palm distributions and the assumption

$E\xi = \nu \otimes \lambda$, for any measurable sets $A \in \mathcal{S}$ and $B \subset [0,1]$, and any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&P\left[\tau \in B | \sigma \in A, \xi(S \times I) = n\right] \\
&\quad = \frac{P[(\sigma,\tau) \in A \times B | \xi(S \times I) = n]}{P[\sigma \in A | \xi(S \times I) = n]} \\
&\quad = \frac{E\left[P[(\sigma,\tau) \in A \times B | \xi] | \xi(S \times I) = n\right]}{P[\sigma \in A | \xi(S \times I) = n]} \\
&\quad = \frac{E[\xi(A \times B); \xi(S \times I) = n]}{E[\xi(A \times I); \xi(S \times I) = n]} \\
&\quad = \frac{\int_{A \times B} Q_{s,t}[\mu(S \times I) = n] \cdot E\xi(ds \times dt)}{\int_{A \times I} Q_{s,t}[\mu(S \times I) = n] \cdot E\xi(ds \times dt)} \\
&\quad = \frac{\int_B \lambda(dt) \int_A Q'_{s,0}[\mu(S \times I) = n-1]\nu(ds)}{\int_I \lambda(dt) \int_A Q'_{s,0}[\mu(S \times I) = n-1]\nu(ds)} \\
&\quad = \lambda B.
\end{aligned}$$

Therefore, $\tau \perp\!\!\!\perp (\sigma, \xi(S \times I))$. This completes the proof of theorem for the case when $I = [0,1]$. $\square$

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