APPLICATION OF GENERALIZED HAMILTONIAN DYNAMICS

TO MODIFIED COULOMB POTENTIAL

Except where reference is made to the work of others, the work described in this thesis is my own or was done in collaboration with my advisory committee.
This thesis does not include proprietary or classified information.

Julian Antolin Camarena

Certificate of Approval:

Michael S. Pindzola Professor Physics	Eugene A. Oks, Chair Professor Physics
Joseph D. Perez Professor Physics	George T. Flowers Dean Graduate School

APPLICATION OF GENERALIZED HAMILTONIAN DYNAMICS

TO MODIFIED COULOMB POTENTIAL

Julian Antolin Camarena

A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
December 19, 2008

APPLICATION OF GENERALIZED HAMILTONIAN DYNAMICS

TO MODIFIED COULOMB POTENTIAL

Julian Antolin Camarena

____
Signature of Author

____
Date of Graduation

THESIS ABSTRACT

APPLICATION OF GENERALIZED HAMILTONIAN DYNAMICS

TO MODIFIED COULOMB POTENTIAL

Julian Antolin Camarena

Master of Science, December 19, 2008
(B.S. University of Texas at El Paso, 2006)

53 Typed Pages

Directed by Eugene Oks

We apply Dirac's generalized Hamiltonian dynamics (GHD), a purely classical formalism, to spinless particles under the influence of a binomial potential. The integrals of the motion for this potential were chosen as the constraints of GHD, and use Fradkin's unit Runge vector in place of the Laplace-Runge-Lenz vector.

A functional form of the unit Runge vector is derived for the binomial potential. It is shown in accordance with Oks and Uzer (2002) that a new kind of time dilation occurs for stable, nonradiating states. The primary result which is derived is that the energy of these classical stable states agrees exactly with the quantal results for the ground state and all states of odd values of the radial and angular harmonic numbers.

ACKNOWLEDGEMENTS

I would like to thank the Auburn University Department of Physics for having me as a graduate student for the past two years and for the education I have received under them.

I wish to extend my heartfelt thanks to Prof. Eugene A. Oks for having been such a wonderful teacher and mentor to. You have been a very important person in my educational refinement, I thank you for this.

I dedicate this thesis to my parents, Cecilia and Antonio, and Cadmiel for all their love and support throughout the time it has taken me to get this far. I thank you most of all. I never could have done this without you.

Last, but not least, I thank all my friends. Whatever it is you have done, rest assured it means a lot to me.

This thesis has been prepared in accordance with Auburn University Graduate School's Guide to Preparation and Submission of Thesis and Dissertations 2005, using Microsoft Word 2007.

TABLE OF CONTENTS

1. INTRODUCTION

In 1950, Dirac developed a generalized Hamiltonian dynamics (hereafter GHD) [1-3]. The conventional Hamiltonian dynamics is based on the assumption that the momenta are independent functions of velocities. Dirac analyzed a more general situation where momenta are not independent functions of velocities [1-3]. Physically, the GHD is a purely classical formalism for constrained systems; it incorporates the constraints into the Hamiltonian. Dirac designed the GHD with applications to quantum field theory in mind [3].

The present work, where GHD is applied to atomic and molecular systems by choosing integrals of the motion as the constraints of the system, stems from a paper in which this idea was applied to hydrogenic atoms treated non-relativistically on the basis of the Coulomb potential [4]. Using this purely classical formalism, Oks and Uzer demonstrated the existence of non-radiating states and found their energy to be in exact agreement with the corresponding results of quantum mechanics. They employed two fundamental experimental facts, but did not "forcefully" quantize any physical quantity describing the atom. In particular, this amounted to classically deriving Bohr's postulate on the quantization of the angular momentum rather than accepting it on an axiomatic basis.

It important to point out that the physics behind classical non-radiating states is a new kind of time-dilation found by Oks and Uzer.

The content of this thesis differs from the above mentioned paper by Oks and Uzer in that the dynamics analyzed are of a more general nature: a term proportional to $1/r^2$ is added to the Coulomb potential. This more complicated potential we call here the *binomial potential*. Then the generalized unit Laplace-Runge-Lenz vector [5,6], or as named by Fradkin, the unit Runge vector [5], is utilized instead of the classical Laplace-Runge-Lenz vector.

This binomial potential has interesting applications. The primary application considered here is to pionic atoms. We will classically obtain results corresponding to the solution of the quantal (relativistic) Klein-Gordon equation, which is appropriate because pions are spinless particles. Another application concerns the precession of planetary orbits: for this phenomenon Einstein's equations of general relativity are equivalent to non-relativistic equations for the motion in the binomial potential [7]. We shall also briefly mention an application furnished by the description of the energy of nonradiating states of the so-called nanoplasmas [14].

An outline of the remainder of the thesis is in order:

In section 2, we briefly outline Dirac's generalized Hamiltonian dynamics. Section 3 serves to describe with more detail the applications of the binomial potential given in the above paragraph. In sections 4 and 5 we discuss the dynamical symmetries or Fradkin and the generalization of the Laplace-Runge-Lenz vector.

We present our new results in section 6 and appendices A, B, and C. Section 7, contains the conclusions.

2. DIRAC'S GENERALIZED HAMILTONIAN DYNAMICS.

Dirac [1-3] considered a dynamical system of N degrees of freedom characterized by generalized coordinates q_n and velocities $v_n = \frac{dq_n}{dt}$, where $n = 1, 2, ..., N$. If the Lagrangian of the system is

$$L = L(q, v), \tag{2.1}$$

then momenta are defined as

$$p_n = \frac{\partial L}{\partial v_n}. \tag{2.2}$$

Each of the quantities q_n, v_n, p_n can be varied by δq_n, δv_n, δp_n, respectively. The latter small quantities are of the order of ε, the variation being worked to the accuracy of ε. As a result of the variation, eq. (2.2) would not be satisfied any more, since their right-hand side would differ from the corresponding left side by a quantity of the order of ε as can be seen from:

$$\delta L = \left(v_n - \frac{\partial H}{\partial v_n} \right) \delta p_n = (v_n - v_n) \delta p_n = 0$$

for an arbitrary variation in the momenta. In the above, Hamilton's canonical equations of motion were invoked. Further, Dirac distinguished between two types of equations. To one type belong equations such as eqs. (2.2), which does not hold after the variation (he called them "weak" equations). In what follows, for weak equations, adopting Dirac's nomenclature, we use a different equality sign $\approx$ from the usual. Another type constitute equations such as eq. (2.1), which holds exactly even after the variation (he called them "strong" equations).

If quantities $\partial L/\partial v_n$ are *not* independent functions of velocities, one can exclude velocities v_n from Eqs. (2.2) and obtain one or several weak equations

$$\phi(q,p) \approx 0, \tag{2.3}$$

containing only q and p. In his formalism, Dirac [1-3] used the following complete system of independent equations of the type (3):

$$\phi_m(q,p) \approx 0, \qquad (m = 1,2,...M). \tag{2.4}$$

Here the word "independent" means that neither of the ϕ's can be expressed as a linear combination of the other ϕ's with coefficient depending on q and p. The word "complete" means that any function of q and p, which would become zero allowing for eqs. (2.2) and which would change by ε under the variation, should be a linear combination of the functions $\phi_m(q, p)$ from (4) with coefficients depending on q and p.

Finally, proceeding from the Lagrangian to a Hamiltonian, Dirac [1-3] obtained the following central result:

$$H_g = H(q,p) + u_m \phi_m(q,p) \tag{2.5}$$

(here and below, the summation over a twice repeated suffix is understood). Equation (2.5) is a strong equation expressing a relation between the generalized Hamiltonian H_g and the conventional Hamiltonian $H(q, p)$. Quantities u_m are coefficients to be determined. Generally, they are functions of q, v, and p; by using Eqs. (2.2), they could be made functions of q and p. It should be emphasized that $H_g \approx H(q, p)$ would be only a weak equation - in distinction to Eq. (2.5).

Equation (2.5) shows that the Hamiltonian is not uniquely determined, because a linear combination of ϕ's may be added to it. Equations (2.4) are called *constraints*. The above distinction between constraints (i.e., weak equations) and strong equations can be reformulated as follows.

Constraints must be employed in accordance to certain rules. Constraints can be added. Constraints can be multiplied by factors (depending on q and p), but only on the left side, so that these factors must not be used inside Poisson brackets.

If f is some function of q and p, then $\frac{df}{dt}$ (i.e., a general equation of motion) in the Dirac's GHD is

$$\frac{df}{dt} = [f, H] + u_m [f, \phi_m], \tag{2.6}$$

where $[f, g]$ is the Poisson bracket defined for two functions f and g of the canonical variables p and q as:

$$[f, g] = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}. \tag{2.7}$$

where r is an index put to stress the fact that in general there will be several generalized coordinates and momenta. Here and throughout we adopt the summation convention so that a sum is understood over any repeated index unless it is explicitly stated otherwise. Substituting $\phi_{m'}$ in (2.6) instead of f and taking into account eqs. (2.4), one obtains:

$$[\phi_{m'}, H] + u_m [\phi_{m'}, \phi_m] \approx 0. \qquad (m' = 1, 2, \ldots, M). \tag{2.8}$$

These consistency conditions allow determining the coefficients u_m.

Last of all, we note that the GHD was designed by Dirac specifically for applications to quantum field theory [3], that is, for the purpose totally different from our purpose.

3. APPLICATIONS OF THE BINOMIAL POTENTIAL

A. *Pionic atoms described by the Klein-Gordon equation of relativistic quantum mechanics.*

Relativistic treatments of the hydrogenic atoms are typically presented working with the Dirac equation, which is a relativistic wave equation that is particularly suited well for spin-1/2 particles. However, in the literature one may also find a treatment of hydrogen and hydrogenlike atoms ignoring spin; that is, working with the Klein-Gordon equation (hereafter, the KG equation) [8,10-13].

The radial KG equation for the problem of the hydrogenic atom is given by:

$$\frac{d^2R}{d\rho^2}+\frac{2}{\rho}\frac{dR}{d\rho}+\left[\frac{\lambda}{\rho}-\frac{1}{4}-\frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right]R=0\,. \tag{3.1}$$

where Z is the atomic number and $\alpha=\frac{e^2}{c\hbar}\cong\frac{1}{137}$ is the fine structure constant.

Thus, the radial KG equation for the Coulomb potential is equivalent to the radial Schrödinger equation for the binomial potential $-\lambda/\rho - \gamma^2/\rho^2$.

For usual hydrogenic atoms, the fine structure splitting predicted by the KG equation is greater than what is observed experimentally [8]. However, for *pionic* atoms, the KG equation becomes *exact*. Indeed, the pionic atom is an exotic hydrogenic atom, where the atomic electron is substituted by a negative pion. Negative pions are *spinless* particles of the same charge as electrons, but 273 times heavier than electrons. Due to the spinless nature of pions, the KG equation for pionic atoms becomes exact.

B. *Precession of planetary orbits*

In his seminal paper, *Die Grundlange der allgemeinen Relativitästhoerie* [7], Einstein showed that general relativistic effects perturb the Kepler potential by an additive term proportional to $1/r^2$ and used it to calculate the precession of Mercury's orbit around the sun. His calculations for the precession yielded 43''/century, which was later confirmed by observations. There are many good textbooks on general relativity that derive this result [15-17].

C. *Radiation of nonrelativistic particles in a central field*

Karnakov et al. [14] derive the spectrum and expressions for the intensity of dipole radiation for a classical nonrelativistic particle executing nonperiodic motion. The potential in which the particles under consideration move is of the form $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$. The authors of this paper apply their results to the description of the radiation and the absorption of a classical collisionless electron plasma in nanoparticles irradiated by an intense laser field. Also, they find the rate of collisionless absorption of electromagnetic wave energy in equilibrium isotropic nanoplasma.

4. DYNAMICAL SYMMETRIES OF FRADKIN

Fradkin [5] has shown that all classical dynamical problems of both the relativistic and non-relativistic type, dealing with a central potential, necessarily possess *O(4)* and *SU(3).* This led him to a generalization of the Runge-Lenz vector in the Kepler problem. Fradkin also found a generalization of the conserved symmetric tensor for the harmonic oscillator problem, and constructs a systematic way of imbedding the Lorentz and the *SU(3)* group in and infinite-dimensional Lie algebra. Here we will only be concerned with the results relating to the generalization of the Runge-Lenz vector and the construction of the elements of the Lie algebra of *O(4)* and *SU(3)* in terms of canonical variables.

In the non-relativistic Kepler problem the force on the affected particle is an inverse square force given by:

$$\dot{\mathbf{p}} = -\frac{\lambda}{r^2}\hat{r}\,; \quad \mathbf{p} = m\dot{\mathbf{r}}\,,\ \hat{r} = \frac{\mathbf{r}}{r} \tag{4.1}$$

and the overdot denotes total differentiation with respect to time. In the Kepler problem, the Hamiltonian and the angular momentum (vector **L** and magnitude L^2) are the conserved quantities. There also exists another conserved vector quantity, namely the Laplace-Runge-Lenz vector, or simply the Runge-Lenz vector. It is defined to be:

$$\mathbf{A} = (-2mE)^{-\frac{1}{2}}(\mathbf{p}\times\mathbf{L} - \lambda m\hat{r}) \tag{4.2}$$

For negative energies (*E<0*) **A** is a real vector. This vector, which is a constant of the motion, lies in the plane of the orbit and points from the center of motion to perihelion (that is, along the major axis from one focus to the closest point of the orbit); some authors refer to it as the eccentricity vector [10].

Fradkin found, by differentiation via the standard Poisson bracket formalism, that for the Kepler problem, and indeed for all central potential problems, that **A, L**, and *H* satisfy the following closed Lie algebra:

$$\begin{aligned} \left[A_i, H\right] &= \left[L_i, H\right] = 0 \\ \left[L_i, L_j\right] &= \varepsilon_{ijk} L_k \\ \left[L_i, A_j\right] &= \varepsilon_{ijk} A_k \\ \left[A_i, A_j\right] &= \varepsilon_{ijk} L_k \end{aligned} \tag{4.3}$$

It is seen that the Lie algebra given above is isomorphic to that of the generator of the *O*(*4*) symmetry group, which is the group of orthogonal transformations representing rotations in four dimensions. Fradkin also concluded that if the existence of the Runge-Lenz vector is simply to ensure that the plane of the motion is conserved, then it should always be possible to find a vector analogous to the Runge-Lenz vector for all central potentials.

Fradkin proposed a generalization for the Runge-Lenz vector choosing $\hat{r}, \hat{L},$ and $\hat{r} \times \hat{L}$ as a mutually orthogonal triad of unit vectors. This unit Runge vector is

$$\hat{k} = (\hat{k} \cdot \hat{r})\hat{r} + (\hat{k} \cdot \hat{L})\hat{L} + (\hat{k} \cdot \hat{r} \times \hat{L})\hat{r} \times \hat{L}\,, \tag{4.4}$$

but since the unit Runge vector is in the plane of the orbit and the angular momentum vector is perpendicular to the plane of motion, then the second term is identically zero ($\hat{k} \perp \mathbf{L}$). $\hat{k}$ may be chosen to be the direction from which the azimuthal angle θ is measured (with the positive sense given by a right-handed rotation about $\hat{L}$), then we have:

$$\hat{r}\cdot\hat{k}=\cos\theta \quad \text{and} \quad \hat{k}\cdot\hat{r}\times\hat{L}=\sin\theta \tag{4.5}$$

thus

$$\hat{k}=(\cos\theta)\hat{r}+(\sin\theta)\hat{r}\times\hat{L} \tag{4.6}$$

Defining *u=1/r*, we may write the following differential equation for *u* and the azimuthal angle θ in terms of the energy *E*, potential *V* and angular momentum *L*:

$$\left(\frac{du}{d\theta}\right)^2=\left(\frac{2m}{L^2}\right)(E-V)-u^2 \tag{4.7}$$

At this point we note the following relations and definition:

$$\begin{aligned}\cos\theta &= f(u,L^2,E)\\ \sin\theta &= \left(\frac{\partial f}{\partial u}\right)\frac{(\hat{r}\cdot\mathbf{p})}{L}\cdot\end{aligned} \tag{4.8}$$

Further, the putting $V=-\lambda u$ for the potential of the Kepler problem, the orbit equation becomes:

$$f=\cos\theta=\left[2mEL^2+(\lambda m)^2\right]^{-1/2}\left(L^2u-\lambda m\right). \tag{4.9}$$

The unit Runge vector may be expressed as:

$$\hat{k}=\left[f-u\frac{\partial f}{\partial u}\right]\hat{r}+L^{-2}\frac{\partial f}{\partial u}\mathbf{p}\times\mathbf{L} \tag{4.10}$$

and it's Poisson bracket with the total energy *E* (or, more importantly, the Hamiltonian) vanishes.

Lastly, all of its entries have mutually vanishing Poisson brackets and it satisfies the following relation with the angular momentum:

$$\left[\hat{k}_i, H\right] = 0; \quad \left[\hat{k}_i, \hat{k}_j\right] = 0; \quad \left[L_i, \hat{k}_j\right] = \varepsilon_{ijk}\hat{k}_k; \text{ for } i, j, k = 1,2,3. \tag{4.11}$$

5. FURTHER TOPICS ON THE GENERALIZATION OF THE LAPLACE-RUNGE-LENZ VECTOR

We now turn to a brief discussion of further results that were utilized in our work. They are the results of Holas and March [6] on a further treatment of the unit Runge vector of Fradkin discussed in the previous section. These results, however, are centered on the construction and time dependence of the vector itself rather than on the dynamical symmetries of central potentials or the algebras satisfied by the unit Runge vector.

Holas and March using

$$\mathbf{p} \times \mathbf{L} = \frac{\mathbf{r}L^2}{r^2} - \frac{(\mathbf{p} \cdot \mathbf{r})}{r^2 L} \mathbf{L} \times \mathbf{r} \tag{5.1}$$

they rewrite the unit Runge vector, eq. (4.10), as:

$$\hat{k} = f\, \hat{r} - \frac{(\mathbf{p} \cdot \mathbf{r})}{Lr} \frac{\partial f}{\partial u} \hat{L} \times \hat{r} \tag{5.2}$$

where the function f is specified in the next section. This is the form of the unit Runge vector with which we shall work.

6. APPLICATION OF GENERALIZED HAMILTONIAN DYNAMICS TO THE BINOMIAL POTENTIAL

In our work, the angular momentum vector and the unit Runge vector are constants of the motion for a centrally symmetric potential and consequently have vanishing Poisson brackets with the Hamiltonian for the system and are thus suitable constraints for the application of GHD. Following Oks and Uzer [4], the Hamiltonian for this system is:

$$H_g = \frac{p^2}{2\mu} - \frac{Ze^2}{r} + \frac{\Lambda}{2\mu r^2} + \mathbf{u} \cdot \left(\mathbf{L} - \mathbf{L}_0\right) + \mathbf{w} \cdot \left(\hat{k} - \hat{k}_0\right), \qquad (6.1)$$

where Λ is the strength of the binomial potential, Ze is the nuclear charge, e is the charge of an electron, μ is the reduced mass, **u** and **w** are the yet unknown constant vectors (to be determined later) of the GHD formalism, $\mathbf{L}_0$ and $\hat{k}_0$ are the values of the angular momentum and unit Runge vector in a particular state of the motion so that in those states

$$\mathbf{L} \approx \mathbf{L}_0 \qquad (6.2)$$

and

$$\hat{k} \approx \hat{k}_0 . \qquad (6.3)$$

We may define the following quantities:

$$H_0 = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$
$$H_B = H_0 + \frac{\Lambda}{2\mu r^2} \tag{6.4}$$

where the subscript B is for binomial. The consistency conditions for this system are:

$$\lfloor \mathbf{L}, H_g \rfloor \approx 0$$
$$\left[\hat{k}, H_g\right] \approx 0 \quad . \tag{6.5}$$

First we must derive the form of the unit Runge vector in this problem. It is derived in Appendix A. We arrive at the result:

$$\hat{k} = \frac{1 + g\, g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}}\hat{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{Lr}\left(\frac{g_0 f}{1 + g\, g_0} - \frac{g\left(1 + g_0^2\right)}{\left(1 + g\, g_0\right)^2} f^3\right)\left(\frac{g - g^3}{u - u_3}\right)\hat{L}\times\hat{r}\,. \tag{6.6}$$

where

$$f = \frac{1 + g\, g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}}\,. \tag{6.7}$$

and

$$\frac{\partial f}{\partial u} = \left[\frac{g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}} - \frac{\left(1 + g\, g_0\right)\left(g + g\, g_0^2\right)}{\left(1 + g^2 + g_0^2 + g^2 g_0^2\right)^{3/2}}\right]\left(\frac{\partial g}{\partial u}\right), \tag{6.8}$$

The functions g and are g_0 defined in Appendix A. The unit Runge vector as appears in eq. (6.6) is a general result, valid for any value of the parameter Λ. Hereafter, however, we only consider a small perturbation in the binomial potential such that $\Lambda << L^2$. We therefore perform a Taylor series expansion about $\Lambda = 0$ and keep only terms linear in Λ:

$$\hat{k} = \hat{k}^{(\Lambda=0)} + \Lambda \left. \frac{d\hat{k}}{d\Lambda} \right|_{\Lambda=0} \tag{6.9}$$

where $\hat{k}^{(\Lambda=0)}$ denotes the unperturbed unit Runge vector, which, by definition, is equal to the normalized classical Laplace-Runge-Lenz vector. The derivative term is fully worked out in Appendix A.

The next step is to calculate the Poisson brackets given in eq. (6.5) to arrive at a functional form of the consistency conditions and thus solve for the unknown vector coefficients **u** and **w**. We begin with the angular momentum bracket:

$$\left[L_i, H_g\right] = \left[L_i, H_0\right] + \left[L_i, \frac{\Lambda}{2\mu r^2}\right] + u_j\left[L_i, \left(L_j - L_{0_j}\right)\right] + w_j\left[L_i, \left(\hat{k}_j - \hat{k}_{0j}\right)\right] \tag{6.10}$$

clearly the first two terms vanish since the angular momentum is conserved in any centrally symmetric potential in the absence of external forces. We now have:

$$\left[\mathbf{L}, H_g\right] = \mathbf{u} \times \mathbf{L} + \mathbf{w} \times \hat{k} \approx 0 \tag{6.11}$$

We now proceed to the calculation of the time derivative of the unit Runge vector via the Poisson bracket:

$$\left[\hat{k}, H_g\right] = \left[\hat{k}_i, H_0\right] + \left[\hat{k}_i, \frac{\Lambda}{2\mu r^2}\right] + u_j\left[\hat{k}_i, \left(L_j - L_{0_i}\right)\right] + w_j\left[\hat{k}_i, \left(\hat{k}_j - \hat{k}_{0j}\right)\right] \approx 0\,. \tag{6.12}$$

The following result is obtained:

$$\left[\hat{k}, H_g\right] = \mathbf{u} \times \hat{k} \approx 0 \tag{6.13}$$

where **A** is that given in (6.3) with the identifications $\lambda = Ze^2$ and the energy E as the Coulomb Hamiltonian. The magnitude of **A** is found to be:

$$A = \sqrt{1 + \frac{2H_0 L^2}{\mu Z e^2}} \tag{6.14}$$

We seek the unknown vector coefficients in the following form:

$$\begin{aligned} \mathbf{u} &= a_1 \hat{k}_0 + a_2 \mathbf{L}_0 + a_3 \hat{k}_0 \times \mathbf{L}_0 \\ \mathbf{w} &= b_1 \hat{k}_0 + b_2 \mathbf{L}_0 + b_3 \hat{k}_0 \times \mathbf{L}_0 \end{aligned} \tag{6.15}$$

Substituting eq. (6.15) into eq. (6.11) yields:

$$a_1 \hat{k}_0 \times \mathbf{L}_0 + a_3 \left(\hat{k}_0 \times \mathbf{L}_0\right) \times \mathbf{L}_0 - b_2 \hat{k}_0 \times \mathbf{L}_0 + b_3 \left(\hat{k}_0 \times \mathbf{L}_0\right) \times \hat{k}_0 \approx 0\,. \tag{6.16}$$

For this expression to vanish and from eq. (6.31) we conclude that

$$a_1 = b_2 \qquad \text{and} \qquad a_2 = a_3 = b_3 = 0\,. \tag{6.17}$$

We now have:

$$\begin{aligned} \mathbf{u} &= a_1 \hat{k}_0 \\ \mathbf{w} &= b_1 \hat{k}_0 + a_1 \mathbf{L}_0 \end{aligned}\,. \tag{6.18}$$

At this point it is required to find a_1 and b_1 in terms of the coordinates, momenta and integrals of the motion. It is first pointed out that Oks and Uzer [4] achieved this by calculation of the equations of motion of **r** and **p**. However, it is found that the use of the unit Runge vector makes these calculations very tedious and that an alternative is available. Instead, the coefficients sought may be found in a much simpler and straightforward manner by the calculation of the frequency of precession of the Runge-Lenz vector which, by definition, is equivalent to the precession of the unit Runge vector. For the sake of completeness, the calculation of the equations of motion has been included in Appendix C. The calculations pertaining to the derivation of the frequency of precession of the Laplace-Runge-Lenz vector and finding the unknown vector coefficients are given in Appendix B.

We begin by recalling that for a vector that precesses in the plane of motion:

$$\frac{d\mathbf{D}}{dt} = \boldsymbol{\omega}_{precession} \times \mathbf{D} \Rightarrow \omega_{precession} = \frac{1}{D}\frac{dD}{dt} \tag{6.19}$$

since the frequency vector is perpendicular to the plane of the orbit. Also, it should be noted that from eq. (6.19) we need only deal with magnitudes and not vectors.

It was already stated in the context of general relativity in section 3, that the appearance of the binary term in the Hamiltonian leads to a precession of the orbit. In particular, we are interested in the fact that the Runge-Lenz vector precesses, which put in physical terms means that the eccentricity of the orbit, given by

$$\varepsilon = \frac{A}{Ze^2} \tag{6.20}$$

oscillates. Alternatively, we may say that for $\Lambda = 0$,

$$A^2 = 1 + \frac{2H_0 L^2}{\mu Z e^2} = 1 + \frac{2L^2}{\mu Z e^2}\left(H_B - \frac{\Lambda}{2\mu r^2} \right) \tag{6.21}$$

will oscillate. Following Oks and Uzer [4], we will investigate the case of radiationless states, i.e. states in which there are no transitions or other mechanisms that cause radiation of the electron in its orbit. These states should satisfy:

$$\frac{dA^2}{dt} = 0\,. \tag{6.22}$$

This condition will impose further constraints on the coefficients we seek thus allowing to find them in terms of constants of the motion.

The calculation of the left side of eq. (6.22) is, of course, carried out through the Poisson bracket formalism and is given by the following expression:

$$\frac{dA^2}{dt}=\left[A^2,H_g\right]=\frac{2L^2}{\mu Ze^2}\left[\left(H_B-\frac{\Lambda}{2\mu r^2}\right),H_g\right]=-\frac{\Lambda L^2}{\mu^2 Ze^2}\left[\frac{1}{r^2},H_g\right] \tag{6.23}$$

where the first Poisson bracket is of the binomial Hamiltonian with the generalized Hamiltonian and must vanish since the binary potential is conservative, and the bracket containing the square of the angular momentum must necessarily vanish in a central potential. Since we are concerned only with the first order contributions in terms of Λ, then in the right side of Eq. (6.23) it is sufficient to calculate all factors next to Λ in the zeroth order. The details of the calculations of eq. (6.23) are included in Appendix B. The result obtained for the frequency, hereafter the generalized frequency ω_g, is:

$$\begin{aligned}\omega_g &= \frac{\Lambda L^2}{\mu^2 Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\mathbf{r}\cdot\mathbf{p}\left(\left\langle\frac{1}{r^4}\right\rangle+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)\\ &= \frac{\Lambda L^2}{\mu^2 Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right). \\ &\equiv \omega_0\left|1+B(H_0,L_0)\right|\end{aligned} \tag{6.24}$$

The last step is to stress that the generalized frequency varies from the classical frequency by a factor of $\left|1+B(H_0,L_0)\right|$. This may be understood by the time transformation

$$t\to t'=t\left(1+B(H_0,L_0)\right) \tag{6.25}$$

which is a time dilation dependent upon b_1, for if $b_1=0$, the time would remain unaltered as B would vanish. Upon substitution of this scaled time into all calculations, all quantities regain their standard functional form. This is in complete agreement with the results of Oks and Uzer in [4]. With eq. (6.25) in mind, we note that the generalized period of the motion of the electron about the nucleus is

$$T_g = \frac{T}{\left|1 + B(H_0, L_0)\right|} . \tag{6.26}$$

At this point it is necessary to point out that the equations relevant to the results derived for the generalized frequency resulted independent of a_1 and, therefore, without loss of generality we may set $a_1=0$ in the generalized Hamiltonian. Also, as in Appendix B, we opt to substitute eq. (B.19) into the generalized Hamiltonian, this yields:

$$\begin{aligned} H_g &= H_B + \frac{\mu Z e^2 B(H_0, L_0)\left(1 + \frac{2H_0 L_0}{\mu Z e^2}\right)}{8L^2 \frac{2Z^4 e^8 + 3Z^2 e^4 A^2}{8Z^4 e^8 + 24 Z^2 e^4 A^2 + 3A^4} - \frac{L_0^2}{\sqrt{1 + \frac{2H_0 L_0}{\mu Z e^2}}}} \left(\hat{k} \cdot \hat{k}_0 - 1\right) \\ &= H_B + \frac{\mu Z e^2 B(H_0, L_0) A^2}{8L^2 \frac{2Z^4 e^8 + 3Z^2 e^4 A^2}{8Z^4 e^8 + 24 Z^2 e^4 A^2 + 3A^4} - \frac{L_0^2}{A}} \left(\hat{k} \cdot \hat{k}_0 - 1\right) \end{aligned} . \tag{6.27}$$

With the goal of a generalized frequency for non-radiating states of motion of the electron, we note that the generalized frequency of (6.24), following Oks and Uzer [4], may be rewritten as:

$$\begin{aligned} \omega_g &= \omega_0 \left|1 + B(H_0, L_0)\right| = \left(\frac{\left|2H_0\right|^3}{\mu Z e^2}\right)^{\frac{1}{2}} \left|1 + B(H_0, L_0)\right| \\ &\equiv \left(\frac{\left|2H_0\right|^3}{D}\right)^{\frac{1}{2}} \left|1 + B(H_0, L_0)\right| \end{aligned} \tag{6.28}$$

where, again, the quantities differ from their standard value by the factor $1+B$. The last step of eq. (6.28) defines D.

In (6.28) we see on the central results of the formalism: we may have generalized frequency vanish, i.e. $\omega_g = 0$ despite $\omega_0 \neq 0$. This is in stark contrast to the classical formalism in which $\omega_g = 0$ if and only if $H_0 = 0$, as can be seen in eq. (6.28) when $B = 0$. In this state $B(H_0,L_0) = -1$, and this is a stable, nonradiating state of the classical atom since there is no radiation and consequently no energy is lost throughout the motion.

Of particular interest for the determination of $B(H_0,L_0)$ is an experimental fact used by Oks and Uzer [4]. It is as follows: highly excited atoms primarily emit radiation at a non-zero and finite frequency determined by the limit $H_0 \to 0$. Thus, it is expected that there exists a limiting value for the generalized frequency as the Coulomb Hamiltonian approaches zero. We have

$$\lim_{H_0 \to 0} \omega_g = \lim_{H_0 \to 0} \left(\frac{|2H_0|^3}{D} \right)^{\frac{1}{2}} |1 + B(H_0, L_0)| \equiv \Omega \tag{6.29}$$

and this yields:

$$B(H_0, L_0) = \Omega \left(\frac{D}{|2H_0|^3} \right)^{\frac{1}{2}} - 1 \approx \Omega \left(\frac{D}{|2H_0|^3} \right)^{\frac{1}{2}} = \frac{\Omega}{\omega_0} \tag{6.30}$$

the contribution of -1 is negligible since in the limit as H_0 approaches zero, the ratio becomes much larger than 1.

Now we consider a particular state of the motion in which there is no radiation from the electron and we have $\omega_g = 0$. In this state let $H_0 = H_S$ (the subscript S is for "stable"). We must have:

$$B(H_S, L_s) = -1, \tag{6.31}$$

thus

$$\Omega = -|2H_S|^{\frac{3}{2}} D^{-\frac{1}{2}} = -\sqrt{\frac{8}{D}}|H_S|^{\frac{3}{2}} = -\omega_0(H_S, L_S) \tag{6.32}$$

and

$$B(H_0, L_0) = -\left|\frac{H_S}{H_0}\right|^{\frac{3}{2}} = -\frac{\omega_0(H_S, L_S)}{\omega_0(H_0, L_0)}. \tag{6.33}$$

Upon substitution of eqs. (6.32) and (6.33) into eq. (6.28) we arrive at:

$$\omega_g = \omega_0(H_S, L_S) - \omega_0(H_0, L_0). \tag{6.34}$$

We find, then, that the average frequency in the classical process of radiation in a weakly bound state is given by:

$$\langle \omega_g \rangle \approx \frac{\omega_g^{initial} + \omega_g^{final}}{2} \approx \frac{\omega_g^{initial}}{2} = \frac{\omega_0(H_S, L_S) - \omega_0(H_0, L_0)}{2} \approx \frac{\omega_0(H_S, L_S)}{2} \tag{6.35}$$

where the final frequency is taken to vanish since there should no longer be any radiation in the final state of motion and $\omega_0(H_S, L_S) >> \omega_0(H_0, L_0)$.

At this point, in keeping with the development of the problem as in [4], we should introduce Planck's hypothesis, whereby we assume that the smallest possible change in energy is proportional to the frequency of the motion, and the proportionality constant is the reduced Planck's constant $h \equiv \frac{h}{2\pi} \approx 1.05 \times 10^{-34} Js$ in SI units. In our particular problem, however, this is not so simple because, as is established in Holas and March [6], the unit Runge vector is only piecewise continuous reflecting the well-known fact that the motion in the modified Coulomb potential is only conditionally periodic (as opposed to periodic). Given this fact, the relation between changes of the energy and of the angular momentum should be refined as follows:

$$\int_0^{T_r} \Delta E dt = \oint \Delta L d\theta = \int_0^{T_\theta} \omega \Delta L dt \tag{6.36}$$

where T_r is the period of radial motion and T_θ is the period angular motion. Eq. is justified as the change in energy correlates with the change, in this case a decrease, of the size of the orbit. Therefore, the integral containing the energy is over the period of radial motion. On the right-hand side of eq. (6.36), the integral contains the angular momentum which is the variable canonically conjugate to the angular variable θ, therefore the integration is performed over the period of angular motion.

Combining eq. (6.36) with Planck's hypothesis we get:

$$\int_0^{T_r} \Delta E dt = \oint \Delta L d\theta = \int_0^{T_\theta} \omega \Delta L dt \quad \Rightarrow \quad \frac{T_r}{T_\theta} \Delta E = \langle \omega \Delta L \rangle = h \langle \omega \rangle \tag{6.37}$$

In eq. (6.37), the change in energy must, of course, satisfy

$$\Delta E = |H_S| - |H_0| \approx |H_S| = h \langle \omega_g \rangle = \frac{h}{2} \omega_0 (H_S, L_S) \tag{6.38}$$

or

$$|H_S| \approx \frac{h}{2} \omega_0 (H_S, L_S). \tag{6.39}$$

We note that on both sides of the eq. (6.39) only physical quantities pertaining to the stable states are present. Also, in eq. (6.37) we have

$$\frac{T_r}{T_\theta} = \frac{\omega_\theta}{\omega_r} = \frac{1}{\gamma} \quad ; \quad \gamma = \sqrt{1 + \frac{\Lambda}{L^2}} \tag{6.40}$$

(note that as $\Lambda \to 0$, $\gamma = \sqrt{1 + \frac{\Lambda}{L^2}} \to 1$, which implies that $T_r = T_\theta$, as known from the Coulomb potential)

and therefore

$$\begin{aligned} \Delta E &= \frac{1}{\gamma}|H_S| \approx \frac{\hbar}{2}\omega_0(H_S, L_S) = \frac{\hbar}{2}\frac{(n\omega_r + m\omega_\theta)}{2} = \frac{\hbar}{2}\left(n + \frac{m}{\gamma}\right)\frac{\omega_r}{2} \\ &= \hbar\left(n + \frac{m}{\gamma}\right)\sqrt{\frac{1}{2\mu Z^2 e^4}}|H_S|^{\frac{3}{2}} \end{aligned} \tag{6.41}$$

where *n,m* = *1,2,3…* . In the third step of eq. (6.41) we used the relation between the frequencies given in eq. (6.40) and we have substituted $\omega_0(H_S, L_S)$ for the term $\frac{(n\omega_r + m\omega_\theta)}{2}$, which is the average of the two frequencies throughout the motion (hence the *1/2*); and, further, the expression must be valid not only for the first harmonic, but for all occurring harmonics of the radial and angular frequencies, hence the integer factors *n* and *m*. We have also used:

$$\omega_{0\gamma} = \frac{\omega_{0r}}{\gamma} = \sqrt{\frac{8}{\mu Z^2 e^4}} \frac{|H_S|^{\frac{3}{2}}}{\sqrt{1 + \frac{2\mu Z e}{L^2}}} . \tag{6.42}$$

We now take notice that from eq. (6.41) we may obtain an expression for the Hamiltonian in the radiationless state of motion in terms of the integers *n* and *m*, we find:

$$|H_S| = \frac{2\mu Z^2 e^4}{\hbar^2 (n\gamma + m)^2} \; ; \quad n,m = 1,2\ldots . \tag{6.43}$$

We compare this classically-derived result with the known quantal result as may be found in, say, *Quantum Mechanics: Nonrelativistic Theory* of Landau and Lifschitz [18] in problem 3 after section 36:

$$\left|H_{quantal}\right| = \frac{2\mu Z^2 e^4}{h^2\left(\left(2\ell+1\right)\gamma+\left(2n_r+1\right)\right)^2}\;;\quad n_r,\ell = 0,1,2\ldots\;. \tag{6.44}$$

where n_r and ℓ are the radial and angular momentum quantum numbers, respectively. We see that in the quantal result, the ground state ($\ell, n_r = 0$), agrees exactly with our derived expression for *n,m=1*. Furthermore, the correspondence between the quantal result and ours agrees for all odd *n* and *m*, i.e. when these integers are of the form *n=2k+1* and *m=2q+1, q,k=0,1,2…*. We may identify *n* and *m* as the radial and angular harmonic numbers.

7. CONCLUSIONS

We close with a brief recapitulation of the work put forth in the preceding.

In the section of application, motivation was given for the use and interest of the binomial potential. The well-known and interesting applications mentioned were that of the solution to the Klein-Gordon equation governing the dynamics of pionic atoms; radiation of particles in nanoplasmas; and the advance of the perihelion of planets orbiting in a central potential as can be shown by means of general relativity. The main new results obtained for the binomial potential are as follows.

1. We obtained an explicit expression for the additional (to the angular momentum) vector integral of the motion: the unit Runge-Lenz vector.
2. Beginning with Dirac's generalized Hamiltonian dynamics, a purely classical formalism, a (generalized) Hamiltonian was set up that described the dynamics of a spinless particle in a Coulomb potential perturbed by the presence of a binomial potential, i.e. one that varies inversely with the square of the distance from the center of force. With this Hamiltonian and the use of consistency conditions, in this case the necessity that the angular momentum, energy (Hamiltonian), and the unit Runge-Lenz vector be the seven conserved quantities of the central potential it was shown that the use of GHD leads to an effective time dilation.

3. We derived classical energies of radiationless states in the system of bound spinless particles and found that they agree with quantum theory for the ground state and with all states of odd principle and angular momentum quantum numbers.

4. We derived the explicit expression for the generalized Hamiltonian. It leads to a dynamics that is much richer than the usual classical dynamics. This can be seen from the many additional terms in the equations of motion derived in Appendix C.

It is worth emphasizing some interesting physics of classical nonradiating stable states following Oks and Uzer [4]: In those states, $\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{p}}{dt} = 0$, so that $\mathbf{r}(t) = \mathbf{r}_0$ and $\mathbf{p}(t) = \mathbf{p}_0$, where $\mathbf{r}_0$ and $\mathbf{p}_0$ are some constant vectors. Thus, the particle (for example, the pion) is motionless, but its momentum is nonzero. This is not surprising: for example, for a charge in an electromagnetic field characterized by a vector potential **A**, it is also possible to have $\mathbf{v} = \frac{\mathbf{p} - \frac{e}{mc}\mathbf{A}}{m} = 0$, while $\mathbf{p} = \frac{e}{mc}\mathbf{A} \neq 0$.

BIBLIOGRAPHY

1. Dirac, P. A. M 1950 *Canad. J. Math.* **2**, 129
2. Dirac, P. A. M. 1958 *Proc. R. Soc.* A **246** 326
3. Dirac, P. A. M 1964 *Lectures on Quantum Mechanics* (New York: Academic). Reprinted by Dover Publications, 2001.
4. Oks E. and Uzer T. 2002 *J. Phys. B: At. Mol. Opt. Phys* **35** 165
5. Fradkin D. M. 1967 *Prog. Theor. Phys.* **37** 798
6. Holas A. and March N. H. 1990 *J. Phys. A: Math. Gen.* **23** 735
7. Einstein A. 1916 *Annalen der Physik* **49**. It is reprinted in *The Principle of Relativity* (Dover 1952) with other landmark papers by Weyl H., Lorentz H., and Minkowski H.
8. Josephson J. 1980 *Found. of Phys.* **10** 243
9. Landau L. D. and Lifschitz E. M. 1982 *Mechanics* 3rd Edition Butterworth-Heinneman
10. Sokolov A. A., Ternov I. M., and Zhukovskii V. Ch. 1984 1st edition *Quantum Mechanics* Mir Publishers.
11. Greiner W. 1990 *Relativistic Quantum Mechanics: Wave Equations*
12. Schiff L. I. 1968 *Quantum Mechanics (International Pure and Applied Physics Series)* 3rd edition McGraw-Hill Companies.

13. Capri A. 2002 *Relativistic Quantum Mechanics and Introduction to Quantum Field Theory* 1st edition World Scientific Publishing Company.

14. B. M. Karnakov, Ph. A. Korneev, and S. V. Popruzhenko 2008 *J. of Exp. and Theor. Phys.,* **106***, No. 4, 650*

15. Landau L. D. and Lifschitz E. M. 1980 *Classical Theory of Fields* 2nd Edition Butterworth-Heinneman

16. Walecka J. D. 2007 *Introduction to General Relativity* 1st edition World Scientific Publishing Company

17. Schutz S. 1985 *A First Course in General Relativity* Cambridge University Press.

18. Landau L. D. and Lifschitz E. M. 1981 *Quantum Mechanics: Nonrelativistic Theory* 3rd Edition Butterworth-Heinneman

APPENDIX A

DERIVATION OF THE FUNCTIONAL FORM OF THE UNIT RUNGE VECTOR

The function f, given by

$$f = \cos\theta; \quad \theta = \frac{L}{\mu}\int_{u_0}^{u}\left[\frac{2}{\mu}\left(E - V\left(1/u'\right)\right) - \left(\frac{L_{eff}u'}{\mu}\right)^2\right]^{-1/2} du' \qquad \text{(A.1)}$$

where

$$L_{eff}^2 = L^2 - \Lambda \qquad \text{(A.2)}$$

is the effective angular momentum and shows a correction due to the binomial potential. The integral in eq. (A.1), upon the substitution of the Coulomb potential, may be rewritten as:

$$\int_{u_0}^{u}\left[\frac{2}{\mu}\left(E - V\left(1/u'\right)\right) - \left(\frac{L_{eff}u'}{\mu}\right)^2\right]^{-1/2} du' = \int_{u_0}^{u}\frac{du'}{\sqrt{\left(\frac{2}{\mu}\right)\left(E + Ze^2u'\right) - \left(\frac{Lu'}{\mu}\right)^2}}\,. \qquad \text{(A.3)}$$

If we now introduce the substitutions

$$u_1 = \frac{Ze^2\mu}{L^2}\left(1+\sqrt{\frac{2EL^2}{Z^2e^4\mu}}\right)$$
$$u_2 = \frac{Ze^2\mu}{L^2}\left(1-\sqrt{\frac{2EL^2}{Z^2e^4\mu}}\right), \tag{A.4}$$

then the left-hand side of eq. (A.3), in the indefinite form of the integral, becomes:

$$\int \frac{du'}{\sqrt{(u_1-u)(u-u_2)}} = \tan^{-1}\left(\frac{u-\frac{u_1+u_2}{2}}{\sqrt{(u_1-u)(u-u_2)}}\right) \tag{A.5}$$

after some simplifications. It is convenient to define

$$u_3 = \frac{u_1+u_2}{2} = \frac{\mu Ze^2}{L^2} \tag{A.6}$$

and thus eq. (A.5) reduces to:

$$\tan^{-1}\left(\frac{u-\frac{u_1+u_2}{2}}{\sqrt{(u_1-u)(u-u_2)}}\right) = \tan^{-1}\left(\frac{u-u_3}{\sqrt{(u_1-u)(u-u_2)}}\right). \tag{A.7}$$

Putting in the limits of integration yields:

$$f = \cos\left(\tan^{-1}\left(\frac{u-u_3}{\sqrt{(u_1-u)(u-u_2)}}\right) - \tan^{-1}\left(\frac{u_0-u_3}{\sqrt{(u_1-u_0)(u_0-u_2)}}\right)\right). \tag{A.8}$$

It is convenient to define:

$$g = g(u) \equiv \frac{u-u_3}{\sqrt{(u_1-u)(u-u_2)}}; \quad g(u_0) \equiv g_0. \tag{A.9}$$

Using the identity

$$\cos\left(\tan^{-1}(A) - \tan^{-1}(B)\right) = \frac{1+AB}{\sqrt{1+A^2+B^2+A^2B^2}}, \tag{A.10}$$

we may then write

$$f = \frac{1 + g\,g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}}\,. \tag{A.11}$$

Consequently, the partial derivative in the unit Runge vector becomes:

$$\frac{\partial f}{\partial u} = \left[\frac{g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}} - \frac{(1 + g\,g_0)(g + g\,g_0^2)}{(1 + g^2 + g_0^2 + g^2 g_0^2)^{3/2}} \right] \left(\frac{\partial g}{\partial u} \right), \tag{A.12}$$

where

$$\frac{\partial g}{\partial u} = \frac{1}{\sqrt{(u_1 - u)(u - u_2)}} - \frac{1}{2} \frac{\left(u - \frac{u_1 + u_2}{2} \right)(-2u + u_1 + u_2)}{((u_1 - u)(u - u_2))^{3/2}}\,. \tag{A.13}$$

We may use the definitions (A.9) and (A.11) to rewrite eq. (A.13) and put it into eq. (A.12) to get the following compact form:

$$\frac{\partial f}{\partial u} = \left(\frac{g_0 f}{1 + g\,g_0} - \frac{g(1 + g_0^2)}{(1 + g\,g_0)^2} f^3 \right) \left(\frac{g - g^3}{u - u_3} \right). \tag{A.14}$$

where the term in the second set of parenthesis is the simplification of $\frac{\partial g}{\partial u}$. We thus arrive at:

$$\hat{k} = \frac{1 + g\,g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}}\,\hat{r} - \frac{\mathbf{p} \cdot \mathbf{r}}{Lr} \left(\frac{g_0 f}{1 + g\,g_0} - \frac{g(1 + g_0^2)}{(1 + g\,g_0)^2} f^3 \right) \left(\frac{g - g^3}{u - u_3} \right) \hat{L} \times \hat{r}\,. \tag{A.15}$$

This is a general result valid for any value of Λ. However, since we are considering a small perturbation in the binomial potential, such that $\Lambda << L^2$, then we may perform a Taylor series expansion of the unit Runge vector with respect to Λ about $\Lambda = 0$:

$$\hat{k} = \hat{k}^{(\Lambda=0)} + \Lambda \left. \frac{d\hat{k}}{d\Lambda} \right|_{\Lambda=0} \tag{A.16}$$

where $\hat{k}^{(\Lambda=0)}$ denotes the unperturbed unit Runge vector, which, by definition, is equal to the normalized classical Laplace-Runge-Lenz vector. Differentiation with respect to Λ yields:

$$\frac{d\hat{k}}{d\Lambda} = \frac{df}{d\Lambda}\hat{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{L_{eff}\,r}\left(\frac{d}{d\Lambda}\left(\frac{\partial f}{\partial u}\right) + \frac{1}{2L_{eff}^2}\frac{\partial f}{\partial u}\right)\hat{L}\times\hat{r}\,. \tag{A.17}$$

The second term in the parenthesis is due to $\frac{\partial L_{eff}}{\partial \Lambda} = -\frac{1}{2L_{eff}}$. We now proceed to calculate the above quantities. For the first term:

$$\frac{df}{d\Lambda} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial \Lambda} + \frac{\partial f}{\partial g_0}\frac{\partial g_0}{\partial \Lambda} \tag{A.18}$$

where

$$\frac{\partial f}{\partial g} = \frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^2}f^3;\quad \frac{\partial f}{\partial g_0} = \frac{gf}{1+g\,g_0} - \frac{g_0\left(1+g^2\right)}{\left(1+g\,g_0\right)^2}f^3 \tag{A.19}$$

as can be seen in eq. (A.14); and

$$\frac{\partial g}{\partial \Lambda} = -\frac{1}{2L_{eff}}\frac{\partial g}{\partial u_i}\frac{\partial u_i}{\partial L_{eff}};\quad \frac{\partial g_0}{\partial \Lambda} = -\frac{1}{2L_{eff}}\frac{\partial g_0}{\partial u_i}\frac{\partial u_i}{\partial L_{eff}}\qquad i=1,2,3 \tag{A.20}$$

We find:

$$\frac{\partial g}{\partial u_1} = -\frac{1}{2}\frac{(u-u_2)}{(u-u_3)^2}g^3; \quad \frac{\partial u_1}{\partial L_{eff}} = -\frac{2u_1}{L_{eff}} + \frac{\mu}{L_{eff}^2}\sqrt{2\mu E}$$

$$\frac{\partial g}{\partial u_1} = -\frac{1}{2}\frac{(u_1-u)}{(u-u_3)^2}g^3; \quad \frac{\partial u_2}{\partial L_{eff}} = -\frac{2u_2}{L_{eff}} - \frac{\mu}{L_{eff}^2}\sqrt{2\mu E} \tag{A.21}$$

$$\frac{\partial g}{\partial u_3} = -\frac{g}{u-u_3}; \quad \frac{\partial u_3}{\partial L_{eff}} = -\frac{2u_3}{L_{eff}}$$

and

$$\frac{\partial g_0}{\partial u_1} = -\frac{1}{2}\frac{(u-u_2)}{(u-u_3)^2}g_0^3; \quad \frac{\partial u_1}{\partial L_{eff}} = -\frac{2u_1}{L_{eff}} + \frac{\mu}{L_{eff}^2}\sqrt{2\mu E}$$

$$\frac{\partial g_0}{\partial u_1} = -\frac{1}{2}\frac{(u_1-u)}{(u-u_3)^2}g_0^3; \quad \frac{\partial u_2}{\partial L_{eff}} = -\frac{2u_2}{L_{eff}} - \frac{\mu}{L_{eff}^2}\sqrt{2\mu E}\ . \tag{A.22}$$

$$\frac{\partial g_0}{\partial u_3} = -\frac{g_0}{u-u_3}; \quad \frac{\partial u_3}{\partial L_{eff}} = -\frac{2u_3}{L_{eff}}$$

The second derivative term is found to be:

$$\begin{aligned}\frac{d}{d\Lambda}\left(\frac{\partial f}{\partial u}\right) = &\left[\frac{g_0}{1+g\,g_0} - 3\frac{g(1+g_0^2)}{(1+g\,g_0)^2}f^2\right]\frac{df}{d\Lambda}\left(\frac{g-g^3}{u-u_3}\right)\\ &+\left[-\frac{g_0}{(1+g\,g_0)^2} - \frac{(1+g_0^2)}{(1+g\,g_0)^2}f^3 - 2\frac{g_0 g(1+g_0^2)}{(1+g\,g_0)^3}\right]\frac{dg}{d\Lambda}\left(\frac{g-g^3}{u-u_3}\right)\\ &+\left[\frac{g_0 f}{1+g\,g_0} - \frac{g(1+g_0^2)}{(1+g\,g_0)^2}f^3\right]\left(\frac{1-3g^2}{u-u_3}\right)\frac{dg}{d\Lambda}\\ &-\left[\frac{g_0 f}{1+g\,g_0} - \frac{g(1+g_0^2)}{(1+g\,g_0)^2}f^3\right]\left[\frac{g-g^3}{(u-u_3)^2}\right]\left(\frac{du}{d\Lambda} - \frac{du_3}{d\Lambda}\right)\end{aligned}\ . \tag{A.23}$$

We note that substituting $\Lambda = 0$ amounts to the substitution $L_{eff} \to L$ in accordance with eq. (A.2).

APPENDIX B

DERIVATION OF THE FREQUENCY OF PRECESSION OF THE LAPLACE-RUNGE-LENZ VECTOR

We start by explicitly writing the generalized Hamiltonian in eq. (6.23), substituting the vectors **u** and **w** from eq. (6.18):

$$\begin{aligned} H_g &= \frac{p^2}{2\mu} - \frac{Ze^2}{r} + \frac{\Lambda}{2\mu r^2} + a_1 \mathbf{L}_0 \cdot \hat{k} + a_1 \mathbf{L} \cdot \hat{k}_0 + b_1 \hat{k} \cdot \hat{k}_0 - b_1 \\ &= H_B + a_1 \mathbf{L}_0 \cdot \hat{k} + a_1 \mathbf{L} \cdot \hat{k}_0 + b_1 \hat{k} \cdot \hat{k}_0 - b_1 \end{aligned} \tag{B.1}$$

The Poisson bracket of the binomial term, which is simply a term inversely proportional to the square of the distance, with the generalized Hamiltonian (B.1) is, in accordance with eq. (6.23):

$$\frac{dA^2}{dt} = -\frac{\Lambda L^2}{\mu^2 Ze^2}\left[\frac{1}{r^2}, H_g\right] = -\frac{\Lambda L^2}{\mu^2 Ze^2}\left\{-\frac{2\mathbf{r}\cdot\mathbf{p}}{r^4} + b_1 \hat{A}_0 \left[\frac{1}{r^2}, \frac{\mathbf{A}}{A}\right]\right\} \tag{B.2}$$

where we have made the substitution

$$\hat{k}^{(\Lambda=0)} = \frac{\mathbf{A}}{A},$$

which is justified by definition; it has also been taken into account that the Poisson bracket of the binomial potential with the angular momentum vanishes and the bracket with b_1 must vanish as b_1 is constant. It is worthwhile to recall that the quantities indexed by a 0 are constant and consequently may be factored out of any bracket in which they appear. In (B.2) we note that:

$$\left[\frac{1}{r^2},\frac{A_q}{A}\right]=\frac{1}{A}\left[\frac{1}{r^2},A_q\right]+A_q\left[\frac{1}{r^2},\left(A_jA_j\right)^{-\frac{1}{2}}\right]=\frac{1}{A}\left[\frac{1}{r^2},A_q\right]-\frac{A_qA_j}{A^3}\left[\frac{1}{r^2},A_j\right] \quad \text{(B.3)}$$

and

$$\left[\frac{1}{r^2},A_j\right]=-\frac{2x_i}{r^4}\frac{\partial A_j}{\partial p_i} \quad \text{(B.4)}$$

since the second term of the bracket is zero because the coordinates and momenta are assumed to be independent of each other. Thus, the entire calculation of the frequency of oscillation of the Runge-Lenz vector has been reduced to the calculation of the derivative of the of the Runge-Lenz vector with respect to the momenta. This is as follows:

$$\begin{aligned}\frac{\partial A_j}{\partial p_i}&=\frac{\partial}{\partial p_i}\left(\frac{1}{\mu Ze^2}\varepsilon_{jkl}p_kL_l-\frac{x_j}{r}\right)=\frac{\partial}{\partial p_i}\left(\frac{1}{\mu Ze^2}\varepsilon_{jkl}\varepsilon_{lmn}p_kx_mp_n\right)\\&=\frac{1}{\mu Ze^2}\left(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km}\right)\left(\delta_{ik}x_mp_n+\delta_{in}x_mp_k\right)\\&=\frac{1}{\mu Ze^2}\left(2x_jp_i-x_ip_j-\mathbf{r}\cdot\mathbf{p}\delta_{ij}\right)\end{aligned} \quad \text{(B.5)}$$

so that (B.4) becomes:

$$\left[\frac{1}{r^2},A_j\right]=-\frac{2}{\mu Ze^2r^4}\left(x_j\mathbf{r}\cdot\mathbf{p}-p_jr^2\right). \quad \text{(B.6)}$$

With the simple result found in (B.6), we may now calculate the last part of (B.4) in order to get a final form for (B.3). We have:

$$\frac{1}{A}\left[\frac{1}{r^2}, A_q\right] - \frac{A_q A_j}{A^3}\left[\frac{1}{r^2}, A_j\right] = -\frac{2}{\mu Z e^2 r^4 A}\left(x_q \mathbf{r}\cdot\mathbf{p} - p_q r^2\right) + \frac{2}{\mu Z e^2 r^4}\frac{A_q}{A^3}\left(\mathbf{r}\cdot\mathbf{A}\mathbf{r}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{A} r^2\right)$$

$$. \qquad \text{(B.7)}$$

Then, finally, we arrive at:

$$\frac{dA^2}{dt} = \left[A^2, H_g\right] = -\frac{\Lambda L^2}{\mu^2 Z e^2}\left(-\frac{2\mathbf{r}\cdot\mathbf{p}}{\mu r^4} + b_1 \hat{A}_0\left[\frac{1}{r^2}, \frac{\mathbf{A}}{A}\right]\right)$$

$$= -\frac{2\Lambda L^2}{\mu^2 Z e^2 r^4}\mathbf{r}\cdot\mathbf{p}\left(1 + \frac{b_1}{1 + \frac{2H_0 L_0}{\mu Z e^2}}\left(2r - \frac{L_0^2}{\mu Z e^2\sqrt{1 + \frac{2H_0 L_0}{\mu Z e^2}}}\right)\right) \qquad \text{(B.8)}$$

This expression was significantly simplified in form by carrying out the dot product of the coordinates and momenta with the Runge-Lenz vector:

$$\mathbf{r}\cdot\mathbf{A} = \frac{L^2}{\mu Z e^2} - r \ ; \qquad \mathbf{p}\cdot\mathbf{A} = \frac{\mathbf{p}\cdot\mathbf{r}}{r}. \qquad \text{(B.9)}$$

Furthermore, since our calculations are limited to the first order in terms of Λ, then every factor next to Λ in the right side of (B.8) can be calculated in the zeroth order in terms of Λ. Therefore it is legitimate to replace the quantities $1/r^4$ and $1/r^3$ in the right side of (B.8) by their averages over the unperturbed Kepler ellipse.

These averages are determined as follows:

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{p}{1+\varepsilon\cos\theta}\right)^3 d\theta = \frac{2+3\varepsilon^2}{2p^3} \qquad \text{(B.10a)}$$

$$\left\langle \frac{1}{r^4} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{p}{1+\varepsilon\cos\theta}\right)^4 d\theta = \frac{8+24\varepsilon^2+3\varepsilon^4}{8p^4} \qquad \text{(B.10b)}$$

is the average orbital radius, and

$$p = \frac{L^2}{\mu Z e^2}; \quad \text{and} \qquad \varepsilon = \frac{A}{Ze^2} \tag{B.11}$$

We shall keep the definitions of the momentum and the eccentricity as in (B.11) for the sake of brevity, but it is to be understood that these quantities are in terms of constants of the motion. Now substitution of (B.11a,b) into (B.8) yields:

$$\begin{aligned}
&\frac{2\Lambda L^2}{\mu^2 Z e^2}\mathbf{r}\cdot\mathbf{p}\left(\left\langle\frac{1}{r^4}\right\rangle + \frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\left\langle\frac{1}{r^3}\right\rangle - \frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\left\langle\frac{1}{r^4}\right\rangle\right)\right)\\
&= \frac{2\Lambda L^2}{\mu^2 Z e^2}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle} - \frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)\\
&= 0
\end{aligned} \tag{B.12}$$

Next we find the frequency of precession. Since we calculated the time derivative of the square of the magnitude of the Runge-Lenz vector rather than the magnitude to the first power, we make a small correction to (B.12) to arrive at our desired result:

$$\frac{dA^2}{dt} = 2A\frac{dA}{dt} \Rightarrow \omega_{precession} = \frac{1}{A}\frac{dA}{dt} = \frac{1}{2A^2}\frac{dA^2}{dt} \tag{B.13}$$

we therefore arrive at:

$$\omega_{precession} = \frac{2\Lambda L^2}{\mu^2 Ze^2 A^2}\left\langle\frac{1}{r^4}\right\rangle \mathbf{r}\cdot\mathbf{p}\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

$$= \frac{2\Lambda L^2}{\mu^2 Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left\langle\frac{1}{r^4}\right\rangle \mathbf{r}\cdot\mathbf{p}\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

(B.14)

We may now rewrite this as:

$$\omega_{precession} = \frac{\Lambda L^2\mathbf{r}\cdot\mathbf{p}}{\mu^2 Ze^2 A^2}\left\langle\frac{1}{r^4}\right\rangle\left(1+B\left(H_0,L_0\right)\right) \tag{B.15}$$

where

$$B\left(H_0,L_0\right)=\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)$$

$$=\frac{b_1}{\mu Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(8L^2\frac{2+3\varepsilon^2}{8+24\varepsilon^2+3\varepsilon^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right), \tag{B.16}$$

$$=\frac{b_1}{\mu Ze^2 A^2}\left(8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{A}\right)$$

where in the last step we substituted the expression for the eccentricity and the classical Runge-Lenz vector. We see that in for the case $b_1=0$, we recover the well-known classical expression. Thus, the appearance of the function $B(H_0,L_0)$ of eq. (B.18) is a result characteristic of Dirac's formalism for the central potential as it depends directly on b_1, the one remaining coefficient from the formalism's constant vectors introduced in the generalized Hamiltonian.

Furthermore, we may solve for b_1 in terms of the Coulomb Hamiltonian, the angular momentum and *B:*

$$b_1 = \frac{\mu Z e^2 B(H_0, L_0)\left(1 + \frac{2H_0 L_0}{\mu Z e^2}\right)}{8L^2 \frac{2Z^4 e^8 + 3Z^2 e^4 A^2}{8Z^4 e^8 + 24 Z^2 e^4 A^2 + 3A^4} - \frac{L_0^2}{\sqrt{1 + \frac{2H_0 L_0}{\mu Z e^2}}}} \tag{B.17}$$

or

$$b_1 = \frac{\mu Z e^2 B(H_0, L_0) A^2}{8L^2 \frac{2Z^4 e^8 + 3Z^2 e^4 A^2}{8Z^4 e^8 + 24 Z^2 e^4 A^2 + 3A^4} - \frac{L_0^2}{A}} \tag{B.18}$$

in terms of the classical Runge-Lenz vector. We may now substitute this result into the generalized Hamiltonian to obtain:

$$H_g = H_B + a_1 \mathbf{L}_0 \cdot \hat{k} + a_1 \mathbf{L} \cdot \hat{k}_0 + \frac{\mu Z e^2 B(H_0, L_0)\left(1 + \frac{2H_0 L_0}{\mu Z e^2}\right)}{8L^2 \frac{2Z^4 e^8 + 3Z^2 e^4 A^2}{8Z^4 e^8 + 24 Z^2 e^4 A^2 + 3A^4} - \frac{L_0^2}{\sqrt{1 + \frac{2H_0 L_0}{\mu Z e^2}}}} \left(\hat{k} \cdot \hat{k}_0 - 1\right) \tag{B.19}$$

Furthermore, since the equations relevant to all results are independent of a_1, we may, without loss of generality, set $a_1=0$, and the Hamiltonian reduces to:

$$
\begin{aligned}
H_g &= H_B + \frac{\mu Ze^2 B(H_0, L_0)\left(1+\dfrac{2H_0L_0}{\mu Ze^2}\right)}{8L^2 \dfrac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4} - \dfrac{L_0^2}{\sqrt{1+\dfrac{2H_0L_0}{\mu Ze^2}}}}\left(\hat{k}\cdot\hat{k}_0 - 1\right) \\
&= H_B + \frac{\mu Ze^2 B(H_0, L_0)A^2}{8L^2 \dfrac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4} - \dfrac{L_0^2}{A}}\left(\hat{k}\cdot\hat{k}_0 - 1\right)
\end{aligned} \quad . \qquad \text{(B.20)}
$$

The last step was obtained from substitution of (B.19), where we use the magnitude of the classical Runge-Lenz vector in terms of the Coulomb Hamiltonian and the angular momentum. Furthermore, it should be noted that the generalized Hamiltonian is expressed solely as a function of conserved quantities.

APPENDIX C

DERIVATION OF THE EQUATIONS OF MOTION VIA THE POISSON BRACKET FORMALISM

In classical mechanics the equations of motion for any quantity are given by the Poisson bracket of the quantity with the Hamiltonian for the system. In this formalism, this is extended to the generalized Hamiltonian for the system. Thus we have:

$$\begin{aligned}\dot{\mathbf{r}} &= \left[\mathbf{r}, H_g\right] = \left[\mathbf{r}, H_B\right] + u_i\left[\mathbf{r}, L_i - L_{0_i}\right] + w_i\left[\mathbf{r}, \hat{k}_i - \hat{k}_{0_i}\right] \\ \dot{\mathbf{p}} &= \left[\mathbf{p}, H_g\right] = \left[\mathbf{p}, H_B\right] + u_i\left[\mathbf{p}, L_i - L_{0_i}\right] + w_i\left[\mathbf{p}, \hat{k}_i - \hat{k}_{0_i}\right]\end{aligned}. \tag{C.1}$$

The equations of motion are known for the pure Coulomb potential, so we only have to calculate the term due to the binomial potential. The calculations yield:

$$\begin{aligned}\left[x_i, H_B\right] &= \frac{p_{0_i}}{\mu} + \left(b_1\hat{k}_j + a_1 L_j\right)\left[x_i, \hat{k}_j\right] \\ &= \frac{\mathbf{p}_0}{\mu} + b_1\left(\delta_{iq}\frac{\partial f}{\partial p_q} rf + \frac{\mathbf{p}\cdot\mathbf{r}}{rL}\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right) + \frac{L^2 - r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\left[\hat{L}\times\hat{r}\right]_j\right) \\ &\quad + b_1\Lambda\frac{d}{d\Lambda}\left(\delta_{iq}\frac{\partial f}{\partial p_q} rf + \frac{\mathbf{p}\cdot\mathbf{r}}{rL}\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right) + \frac{L^2 - r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\left[\hat{L}\times\hat{r}\right]_j\right)\end{aligned} \tag{C.2}$$

Here we have interchanged the order of differentiation since, by assumption, all functions involved are, at minimum, piecewise continuous and differentiable.

For the momentum we have:

$$[p_i, H_g] = -\frac{Ze^2}{r^2}\frac{x_i}{r} + (b_1\hat{k}_j + a_1 L_j)[p_i, \hat{k}_j]$$

$$= -\frac{Ze^2}{r^2}\frac{x_i}{r} - (b_1\hat{k}_j + a_1 L_j)\begin{pmatrix} \delta_{iq}\frac{\partial f}{\partial x_q}\hat{r} - \delta_{ik}\frac{f}{r^3}x_k - \delta_{iq}\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)[\hat{L}\times\hat{r}]_j \\ -\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\left(\left(\frac{1}{r} + \frac{6p^2}{L^2}\right)\frac{x_i}{r}[\hat{L}\times\hat{r}]_j + \frac{1}{rL}(\mathbf{p}\cdot\mathbf{r}\delta_{ij} - 2p_j x_i + p_i x_j)\right) \end{pmatrix}$$

$$-(b_1\hat{k}_j + a_1 L_j)\Lambda\frac{d}{d\Lambda}\begin{pmatrix} \delta_{iq}\frac{\partial f}{\partial x_q}\hat{r} - \delta_{ik}\frac{f}{r^3}x_k - \delta_{iq}\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)[\hat{L}\times\hat{r}]_j \\ -\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\left(\left(\frac{1}{r} + \frac{6p^2}{L^2}\right)\frac{x_i}{r}[\hat{L}\times\hat{r}]_j + \frac{1}{rL}(\mathbf{p}\cdot\mathbf{r}\delta_{ij} - 2p_j x_i + p_i x_j)\right) \end{pmatrix}$$

(C.3)

In eq. (C.3), we note that the dot product with the angular momentum, the coefficient of a_1, vanishes in all terms except for the term with δ_{ij}. Furthermore, it is important to note that the equation of motion for the momentum and the position vector should be contained in the plane of motion, and therefore, the only acceptable value for a_1 is zero because there occur no other terms proportional to a_1 and the angular momentum that would make the expression perpendicular to the plane of the orbit vanish. In eqs. (C.2) and (C.3) only the derivatives of the vectors have been worked out fully, this is because the derivatives of scalar quantities are best dealt with as follows:

$$\frac{\partial f}{\partial x_q} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_q} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial x_q} = -\frac{\partial f}{\partial u}\frac{x_k}{r^3}\delta_{kq} + \frac{6p^2}{L}\frac{\partial f}{\partial L}x_k\delta_{kq}$$
$$\Rightarrow \delta_{iq}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_q} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial x_q}\right) = -\frac{1}{r^3}\frac{\partial f}{\partial u}x_i + \frac{6p^2}{L}\frac{\partial f}{\partial L}x_i \tag{C.4}$$

and

$$\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)=\left(\frac{p_k}{rL}\frac{\partial f}{\partial u}-\left(\frac{x_k}{r^2}+\frac{6p^2}{L^2}x_k\right)\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}+\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial}{\partial x_q}\left(\frac{\partial f}{\partial u}\right)\right)\delta_{kq}$$

$$\Rightarrow \delta_{iq}\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)=\left(\frac{p_i}{rL}\frac{\partial f}{\partial u}-\left(\frac{x_i}{r^2}+\frac{6p^2}{L^2}x_i\right)\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}-\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\left(\frac{x_i}{r^3}\frac{\partial^2 f}{\partial u^2}+3\frac{\partial f}{\partial u}\frac{x_i}{r^2}+\frac{6p^2}{L}\frac{\partial^2 f}{\partial L\partial u}x_i\right)\right) \tag{C.5}$$

where

$$\frac{\partial^2 f}{\partial u^2}=\left(\begin{array}{l}\frac{g_0}{1+g\,g_0}\frac{\partial f}{\partial u}-\frac{g_0^2 f}{(1+g\,g_0)^2}\frac{\partial g}{\partial u}-\frac{(1+g_0^2)}{(1+g\,g_0)^2}f^3\frac{\partial g}{\partial u}\\ +2\frac{g(1+g_0^2)}{(1+g\,g_0)^3}f^3\frac{\partial g}{\partial u}-3\frac{g(1+g_0^2)}{(1+g\,g_0)^3}f^2\frac{\partial f}{\partial u}\end{array}\right)\left(\frac{g-g^3}{u-u_3}\right). \tag{C.6}$$
$$+\left(\frac{g_0 f}{1+g\,g_0}-\frac{g(1+g_0^2)}{(1+g\,g_0)^2}f^3\right)\left(\frac{1-3g^2}{u-u_3}\frac{\partial g}{\partial u}-\frac{g-g^3}{(u-u_3)^2}\right)$$

For the derivative with respect to the momentum, we find:

$$\frac{\partial f}{\partial p_q}=\frac{\partial f}{\partial L}\frac{\partial L}{\partial p}\frac{\partial p}{\partial p_q}=\frac{6r^2}{L}\frac{\partial f}{\partial L}p_k\delta_{kq}\Rightarrow \delta_{iq}\left(\frac{6r^2}{L}\frac{\partial f}{\partial L}p_k\delta_{kq}\right)=\frac{6r^2}{L}\frac{\partial f}{\partial L}p_i \tag{C.7}$$

and similarly

$$\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)=\frac{x_i}{rL}\frac{\partial f}{\partial u}-\frac{6r\mathbf{p}\cdot\mathbf{r}}{L^3}\frac{\partial f}{\partial u}p_i+\frac{6r\mathbf{p}\cdot\mathbf{r}}{L^2}\frac{\partial^2 f}{\partial L\partial u}p_i \tag{C.8}$$

After setting a_1 to zero and carrying out the dot products, the equations of motion become:

$$\begin{aligned}[\mathbf{r},H_B]&=\frac{p_{0_i}}{\mu}+b_1\hat{k}_j[x_i,\hat{k}_j]\\ &=\frac{\mathbf{p}_0}{\mu}+b_1\left(\left(\frac{6r^3 f}{L}\frac{\partial f}{\partial L}-\frac{6(\mathbf{p}\cdot\mathbf{r})^2}{L^3}\left(\frac{1}{L}\frac{\partial f}{\partial u}-\frac{\partial^2 f}{\partial L\partial u}\right)\right)\mathbf{p}+\frac{\mathbf{p}\cdot\mathbf{r}}{(rL)^2}\frac{\partial f}{\partial u}\mathbf{r}+\frac{(\mathbf{p}\cdot\mathbf{r})^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\hat{L}\times\hat{r}\right)\\ &\quad+b_1\Lambda\frac{d}{d\Lambda}\left(\frac{6r^2}{L}\frac{\partial f}{\partial L}rf\mathbf{p}+\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\left(\frac{1}{rL}\frac{\partial f}{\partial u}\mathbf{r}-\frac{6r\mathbf{p}\cdot\mathbf{r}}{L^3}\frac{\partial f}{\partial u}\mathbf{p}+\frac{6r\mathbf{p}\cdot\mathbf{r}}{L^2}\frac{\partial^2 f}{\partial L\partial u}\mathbf{p}\right)+\frac{L^2-r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\hat{L}\times\hat{r}\right)\end{aligned} \tag{C.9}$$

$$\left[\mathbf{p},H_g\right]=-\frac{Ze^2}{r^2}\frac{x_i}{r}+b_1\hat{k}_j\left[p_i,\hat{k}_j\right]$$

$$=-\frac{Ze^2}{r^2}\hat{r}-b_1\begin{pmatrix}-\left(\frac{\partial f}{\partial u}+\frac{6rp^2}{L}\frac{\partial f}{\partial L}+1\right)\frac{f}{r^2}\hat{r}\\+\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\frac{1}{rL}\frac{\partial f}{\partial u}-\left(\frac{1}{r^2}+\frac{6p^2}{L^2}\right)-\frac{\mathbf{p}\cdot\mathbf{r}}{L}\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial u^2}+\frac{3}{r}\frac{\partial f}{\partial u}+\frac{6rp^2}{L}\frac{\partial^2 f}{\partial L\partial u}\right)\right)\hat{r}\\+\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\left(\frac{1}{r}+\frac{6p^2}{L^2}\right)\hat{r}+\frac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\hat{k}-2\left(\mathbf{p}\cdot\mathbf{r}f-\frac{\mathbf{p}\cdot\mathbf{r}L}{r^2p}\frac{\partial f}{\partial u}\right)\hat{r}+rf\right)\mathbf{p}\right)\end{pmatrix}$$

$$-b_1\Lambda\frac{d}{d\Lambda}\begin{pmatrix}-\left(\frac{\partial f}{\partial u}+\frac{6rp^2}{L}\frac{\partial f}{\partial L}+1\right)\frac{f}{r^2}\hat{r}\\+\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\frac{1}{rL}\frac{\partial f}{\partial u}\mathbf{p}-\left(\frac{1}{r^2}+\frac{6p^2}{L^2}\right)-\frac{\mathbf{p}\cdot\mathbf{r}}{L}\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial u^2}+\frac{3}{r}\frac{\partial f}{\partial u}+\frac{6rp^2}{L}\frac{\partial^2 f}{\partial L\partial u}\right)\right)\hat{r}\\+\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\left(\frac{1}{r}+\frac{6p^2}{L^2}\right)\hat{r}+\frac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\hat{k}-2\mathbf{p}\cdot\mathbf{r}\left(f-\frac{L}{r^2p}\frac{\partial f}{\partial u}\right)\hat{r}+rf\mathbf{p}\right)\right)\end{pmatrix}.$$

(C.10)

As expected, the equations of motion are coplanar with the position vector and the momentum vector.