

PATH CURVATURES ON A CONVEX ROOF

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PATH CURVATURES ON A CONVEX ROOF

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A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
December 17, 2007

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Robert Ford

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VITA

Robert Ford was born December 20th, 1968 to parents Jo and Ralph Ford, both mathematicians at Auburn University. He is the youngest of four siblings and his two brothers still live in Auburn. He has spent time in Pennsylvania, Ohio, and England, but has always returned to his hometown. In 1995, he graduated with a BA in History at Auburn University. He added a Masters in Mathematics four years later. He has never married and has no children.

DISSERTATION ABSTRACT

PATH CURVATURES ON A CONVEX ROOF

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Doctor of Philosophy, December 17, 2007
(Masters of Mathematics, Auburn University)
(Bachelor of Arts, History, Auburn University)

40 Typed Pages

Directed by Krystyna Kuperberg

Given a set of rectangles, R_1 through R_k , where R_i and R_{i+1} share a common edge and these common edges are congruent and parallel to each other. The resulting "roof" is part of the surface of a convex body. We'll consider paths from one corner of this "roof" to the opposite corner. Extending the common edges to lines we'll call ridges and rectangles to planar strips, we can allow such paths to go "off the roof". Path curvature is computed for a polygonal path by simple adding up the curvatures of each intermediate vertex. More general paths use a sequence of polygonal approximations to compute curvature. We'll discern when the path of least curvature goes off the roof or when it is the shortest path.

ACKNOWLEDGMENTS

I would like to acknowledge Krystyna Kuperberg for her tireless support. Her patience in guiding my research made this material possible.

Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliography follows van Leunen’s *A Handbook for Scholars*.

Computer software used The document preparation package T_EX (specifically L^AT_EX) together with the departmental style-file auphd.sty.

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CHAPTER 1

INTRODUCTION

The *total curvature* of a C^2 path C parameterized by arc length s in \mathbf{R}^n , $r(s)$, is defined as $\int_C |r''(s)| ds$. W. Fenchel proved in [4] for \mathbf{R}^3 and K. Borsuk in [3] for any \mathbf{R}^n that the total curvature of a closed curve is bounded from below by 2π , with the equality holding only for convex simple closed curves in \mathbf{R}^2 . The *total curvature* of a polygonal path $P = [z_0, z_1, \dots, z_n]$ is defined as

$$t(P) = \sum_{i=1}^{n-1} (\pi - \angle z_{i-1} z_i z_{i+1}).$$

One way you might compute these angles would be to replace the segments of the path with vectors, all heading in the same direction along the path. The angle between consecutive vectors would give you the curvature at each point. Recall the angle between two vectors, u and v , is the $\arccos(\frac{u \cdot v}{|u||v|})$. For more general paths, approximate polygonal paths can be used to approximate the curvature. As you develop a sequence of polygonal paths which better approximate the curve, you get a sequence of curvatures that approach the curvature of the path. This is essentially the definition for path curvature.

Polygonal paths form the building blocks of all path curvature. Just as polygonal paths may approximate a general path, polyhedral surfaces approximate a general surface. We know the path of least curvature on a convex body can not be greater

than 2π . Currently the total curvature of the shortest path between two points on the surface of a convex body is not known to have an upper bound as discussed in [1], [6], and [7]. It has recently been proved in [2] that such an upper bound, should it exist, must be greater than 2π .

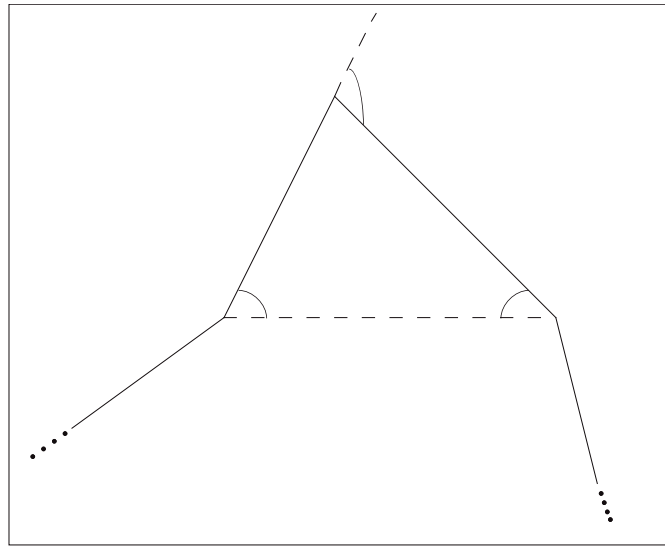


Figure 1.1: Milnor's Proof

J. Milnor proved in [5] if a segment of a polygonal path is replaced by two segments then the curvature of the path will either increase or remain the same. His argument is a very simple one. There are three places where the curvature can change: At the endpoints of the replaced segment and at the point common to the two new segments. The new point of curvature will obviously add to the curvature of the path (assuming an actual change has occurred). The old points of curvature can be altered at most by the measure of the angle between the corresponding new segment and the

replaced segment. Since these angles will add up to the curvature of the new point, the net decrease in the curvature at these points can not be more than the increase in curvature at the new point. For paths of least curvature on these polyhedral roofs, we can focus our attention on polygonal paths comprised of the minimum number of segments.

CHAPTER 2
CONVEX PATHS

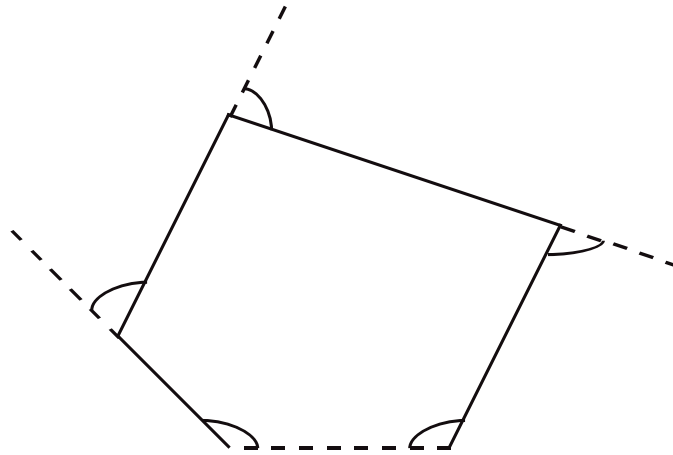


Figure 2.1: Convex Path

We'll call a planar polygonal path convex if by adding a line segment between the initial and final point of the path we get a convex polygon. The interior angle at a point of curvature is supplemental to that curvature. The interior angles of an n sided polygon add up $(n - 2)\pi$. If we add the total curvature to the corresponding interior angles we get $(n - 2)\pi$. So the two interior angles of the polygon formed by the new segment will have a sum equal to the curvature of the original path.

In addition to proving curvature is increased or preserved when replacing one segment with two in a polygonal path, Milnor also observed, but did not prove, that

if the segments involved in the change in the curvature were not co-planar then the curvature would be increased.

To see this, we'll analyze the curvature a two segmented path as one segment is rotated about a line. The rotated segment can be seen as the slant height of a cone. Let α be the angle between the slant height and height (the line of rotation for the segment). The other segment will be attached to the vertex of the cone. Orienting this segment and the line of revolution into the xy -plane so the vertex is the origin and the attached segment has non-positive x -coordinates, we'll let β be the angle between the attached segment and negative x -axis. Finally, let ϕ be the angle of rotation of the slant height from the xy -plane.

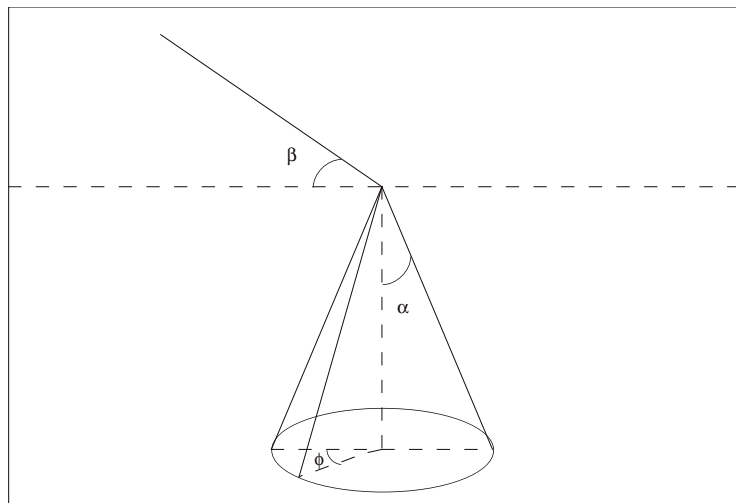


Figure 2.2: Rotating Part of a Path

This same argument works with the segment immediately after the replaced segment in the path.

Milnor's first observation, that adding vertices never decreases the curvature of the path, is easily extendable to replacing a single segment with multiple segments. Likewise the relation is reversible (replacing multiple segments with one segment will not increase the curvature). Milnor's second observation, that the segments involved in the change in the path must be coplanar to preserve curvature, requires a simple argument to see it extends to multiple segments.

Theorem 2.1 *Replacing a single segment with multiple segments only preserves curvature in the path if the new segments along the segments immediately before and after are coplanar.*

Proof. Imagine we have a path and we replace one segment with multiple segments (more than two). We'll call the shorter path O and the longer N. For simplicity, further suppose N has the minimum number of added vertices, b_0 to b_k , which is to say the path does not have curvature zero at any new vertex. Last we'll assume $cur(N) = cur(O)$. Working backwards, starting with N and ending in O, an iteration will consist of replacing two segments with one segment. While it's possible that N has no segment immediately following or preceding the change in the path, it is not possible for both segments to be missing. We will label the segment which for certain exists a_0, a_1 . We'll then label the added vertices that follow or precede depending b_0, b_1, \dots, b_k . If there exist the other segment immediately following or preceding it

will be labeled a_2, a_3 , otherwise a_2 will exist as the initial or final point of the path. So the first iteration will replace a_0, a_1, b_0, b_1, b_2 with a_0, a_1, b_1, b_2 , the second iteration will replace a_0, a_1, b_1, b_2, b_3 with a_0, a_1, b_2, b_3 , and so forth.

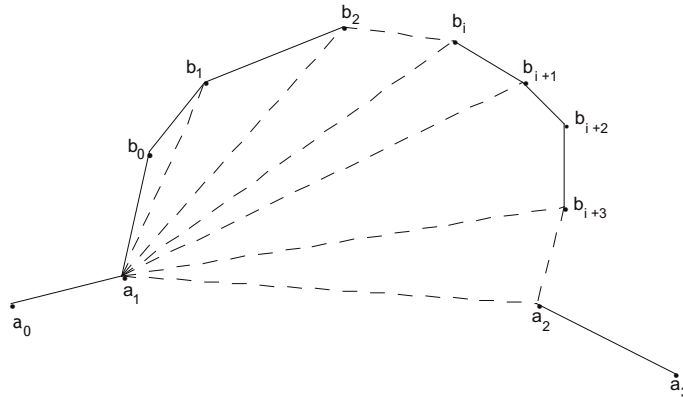


Figure 2.4: The Iterative Process

For curvature to be preserved, it must be preserved at each iteration. This means the vertices of each iteration must be co-planar.

Case 2.1: There are three or more iterations.

Assume there are three or more iterations, but the first iteration is collinear. If a path is collinear then each point of curvature is either zero or π . Since we assumed the curvature at b_0 and b_1 are both positive, they each must be π . If the curvature is to be preserved then the new curvature at a_1 and b_1 must also add up to 2π . So in the second iteration, a_1, b_1 , and b_2 initially add up to more than 2π since the curvature

at b_2 can not be zero. This leads to the second iteration failing to preserve curvature. So the first iteration is not collinear.

For the induction hypothesis, let the $i + 1$ st iteration be non-collinear, but coplanar with all of the vertices from previous iterations, and is not the second to last iteration.

The $i + 2$ nd iteration is not the last iteration. The $i + 1$ st and $i + 2$ nd iterations share four vertices in common a_0, a_1, b_{i+1} , and b_{i+2} . Suppose these vertices are collinear. In the $i + 1$ st iteration, a_1, b_i , and b_{i+1} must then initially have positive curvature. When b_i is removed, curvature at a_1 and b_{i+1} must be increased for curvature to be preserved. Since the remaining vertices are collinear, the new curvatures at a_1 and b_{i+1} must add up to 2π . This will prevent the $i + 2$ nd iteration from preserving curvature. Since the four common vertices to the two consecutive iterations can not be collinear, they must uniquely determine a plane. This means the $i + 2$ nd iteration is not collinear and is coplanar with all of the vertices of the previous iterations.

For the last iteration then we can consider three subcases. If a_3 does not exist then we're done. If a_3 does exist but is not collinear with the four common vertices then the second to last iteration argument can be employed to show the four common vertices cannot be collinear. Finally, if a_3 exists and is collinear with the four common vertices then we're done.

Case 2.2: There are two iterations.

This is an easier case. If the first iteration is collinear then it will be coplanar with the last iteration. If the first iteration is not collinear then we use the same argument as above for the last iteration. \square

CHAPTER 3

PATH CURVATURE ON A CONVEX ROOF.

Given a set of rectangles, R_1 through R_k , where R_i and R_{i+1} share a common edge and these common edges are congruent and parallel to each other. Further suppose the resulting "roof" is part of the surface of a convex body. We'll consider paths from one corner of this roof to the opposite corner. Extending the common edges to lines we'll call ridges and rectangles to planar strips, we can allow such paths to go off the roof. For a polygonal path, curvature is computed by simple adding up the curvatures of each intermediate vertex. More general paths use a sequence of polygonal approximations to compute curvature. We'll attempt to discern when the path of least curvature goes off the roof or when it is the shortest path.

3.1 The Simplest Roofs

We'll begin with $k = 2$. The roof is simple enough, it's comprised of two rectangles with dimension $a \times c$ and $b \times c$ and has a dihedral angle of γ . We'll even consider extreme scenarios like the roof being impossibly thin ($c = 0$) or impossibly steep ($\gamma = 0$). In the case where $\gamma = 0$ the two half planes would coincide, but we'll consider each side of the roof distinct and any path plotted has to go through the ridge line.



Figure 3.1: Simple Roof

Let S be the shortest path and P be the path of least curvature. If we flatten the roof, S would become a single line segment connecting the opposite corners of a rectangle.

Theorem 3.1 *S is the unique path of least curvature in four basic cases: When $\gamma = \pi$, when $a = b$ and $0 < \gamma < \pi$, when $b < a$, $c = 0$ and $\gamma \geq \arccos(\frac{b}{a})$, and when $a = b$, $\gamma = 0$, $c > 0$.*

Proof.

Case 1

$$\gamma = \pi$$

If the roof is actually flat then S has curvature of zero. In truth, this is really the $k = 1$ case.

Case 2

$$a = b \text{ and } 0 < \gamma < \pi$$

We'll call a two segmented path isosceles if the two segments are of equal length. If we compare two isosceles paths with the same initial point and final point, then the one with the greater total length has the greater curvature. All that's left to show is that an isosceles path has less curvature than a non isosceles path of the same length.

Let x and y be the lengths of the two segments of our path. Let z be the distance from the initial point to the final point. By the law of cosines $z^2 = x^2 + y^2 - 2xy \cos \theta$ (where $\pi - \theta$ is the curvature).

$$z^2 + 2xy = (x + y)^2 + 2xy \cos(\pi - \theta)$$

$$\frac{z^2 - (x + y)^2}{2xy} + 1 = \cos(\pi - \theta)$$

$$\arccos\left(\frac{z^2 - (x + y)^2}{2xy} + 1\right) = \pi - \theta$$

$\pi - \theta$ is minimized when xy is maximized. Given $x + y$ is fixed, xy is maximized when $x = y$.

If $a = b$ then the shortest path is also isosceles. A longer path will be non isosceles so its curvature is greater than an isosceles path with the same length which in turn has greater curvature than the shortest path.

Case 3

$$b < a, c = 0 \text{ and } \gamma \geq \arccos\left(\frac{b}{a}\right)$$

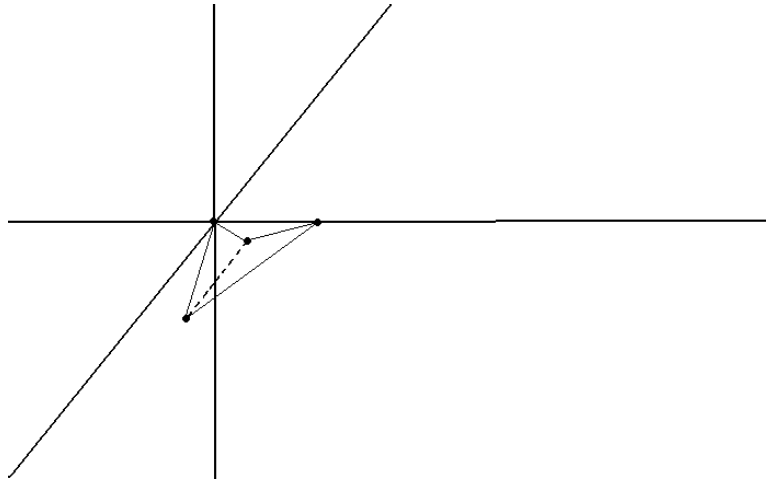


Figure 3.2: A Simple Folding of the Longer Path

We can visualize this case by orienting the roof in the following way. Let the ridge line be the z axis. Let the short side of length b lie on the negative, y -axis. if γ is equal to the $\arccos\left(\frac{b}{a}\right)$ then the initial point on the long side will have the same y -coordinate as the final point on the short side, so the line passing through those two points is parallel to the x -axis. If we plot a longer two segmented path, one whose ridge point has a positive or negative z -coordinate, then we can revolve that path

about the line passing through the initial and final point until it also lies in the xy plane. Its curvature is clearly greater.

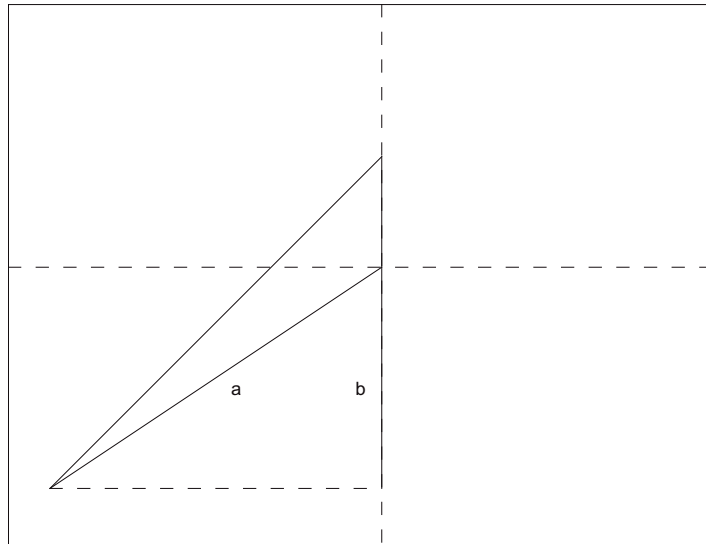


Figure 3.3: The Longer Path has Greater Curvature

If $\gamma > \arccos(\frac{b}{a})$ then the initial point has a greater y -coordinate than $-b$, but if we rotate the roof about the z axis until the initial and final points have matching y -coordinates then the proof follows as before.

Case 4

$$a = b, \gamma = 0, c > 0.$$

If we orient the path in the xy -plane so that the ridge point is the origin and the initial and final points have the same negative y -coordinate then we'll observe the x -coordinates of the initial and final points have the same absolute value, but are on opposite sides of the y -axis. So the center of the circle passing through the

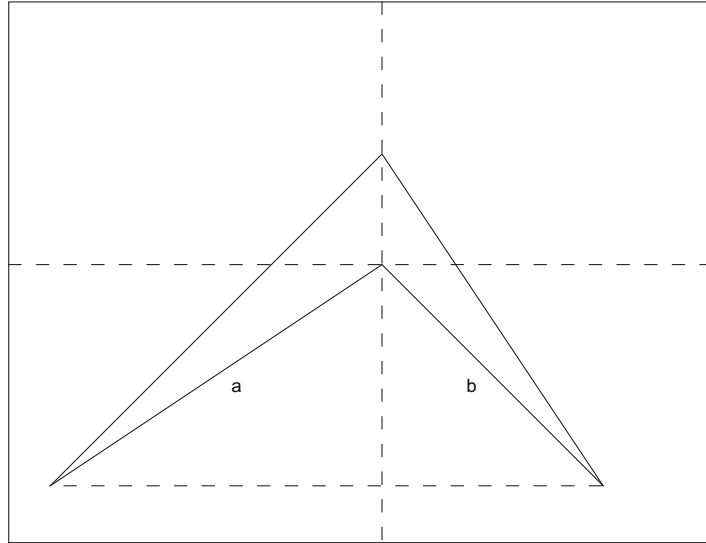


Figure 3.4: In the Wider Case the Longer Path also has Greater Curvature

ridge, initial, and final points has an x -coordinate of zero. This means the ridge line is tangent to the circle and so all other paths have greater curvature. \square

Observe when $c = 0$, $a = b$, and $\gamma = 0$, all two segmented paths have the same curvature in this case. So S is a path of least curvature, but not the only path of least curvature.

Also observe that case 3 is not optimal. There are smaller γ that allow for agreement, but the proof of this is more involved and will held to the end. We can however establish a visual criteria for agreement. If the ridge point is not co-linear with the initial and final points, then these points are co-circular. If this circle does not intersect the positive y -axis then S has the minimum curvature. It is also interesting to notice when $c = 0$ and S is not the path of least curvature that there are two paths

of least curvature. If a path whose ridge point has a positive z -coordinate is a path of least curvature then by symmetry there is a path whose ridge point has a negative, but same absolute valued, z -coordinate with the same curvature.

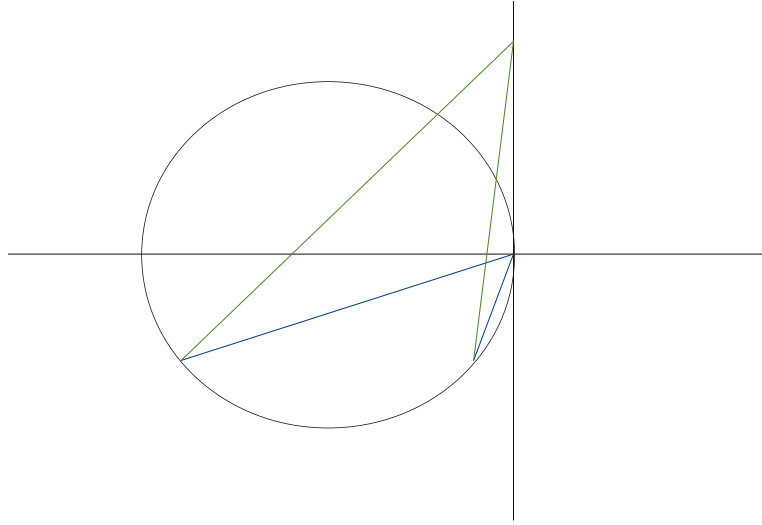


Figure 3.5: The Optimal Case

Once we've finalized case 3, in all other cases S will not be the path of least curvature. If the ridge line is oriented to be parallel to the z -axis, we give the initial point a z -coordinate of 0 or more specifically make it the origin, and give the ridge of the roof a positive orientation then the z -coordinate of S 's ridge point is $\frac{ac}{a+b}$. Computing the path of least curvature's ridge point will be more involved.

If $a > b$, $c > 0$ and $\gamma = 0$ then the z -coordinate of P 's ridge point is less than the z -coordinate of S 's ridge point.

$(a - h)^2$, reduces to a quadratic when we plug in $(0, 0)$ and $(a - b, c)$ and eliminate h as a variable:

$$(a - b)k^2 - 2(ac)k + (ac^2 - ba^2 + ab^2) = 0.$$

$$k = \frac{ac \pm \sqrt{ab((a - b)^2 + c^2)}}{a - b}.$$

Observe both local minimums are equidistant to the collinear ridge point or local maximum. The two circles tangent to the ridge line are not congruent. The left circle has a smaller radius and produces a smaller curvature in the path. So the left local minimum is an absolute minimum and the right local minimum is not.

In the specific case where $a = b$, $c = 0$, and $\gamma = 0$, S is a path of least curvature, but all paths comprised of two segments have the equally bad curvature of π .

For $a > b$, $c > 0$, and $0 < \gamma < \pi$, we must look at the curvature in three dimensions. Let the initial point again be the origin and the ridge line be parallel to the z -axis and lie in yz -plane with a positive orientation. Then the coordinates of the final point will be $(-b(\sin(\gamma)), a - b(\cos(\gamma)), c)$. Given a ridge point of $(0, a, z)$, we can compute the vectors and then the curvature.

Let $u = \langle 0, a, z \rangle$, and $v = \langle -b(\sin(\gamma)), -b(\cos(\gamma)), c - z \rangle$. Let β equal the curvature.

$\cos(\beta) = (u \cdot v) / (|u||v|) \longrightarrow \beta = \arccos((u \cdot v) / (|u||v|))$. If we differentiate β with respect to z we get $\beta' = (-1/\sqrt{1 - (\cos(\beta))^2})(\cos(\beta)')$

$-1/\sqrt{1 - (\cos(\beta))^2}$ is never zero and is undefined when $\beta = 0$ or π , which can only happen when $\gamma = \pi$ or 0 . $\cos(\beta)'$, once simplified, has a denominator based on the magnitudes of u and v and cannot be zero. So the only critical points we get come from the numerator which simplifies to:

$$(2ab \cos(\gamma) - a^2 - b^2)z^3 + (3a^2c - 3abc \cos(\gamma))z^2 \\ + (ab \cos(\gamma)(a^2 + b^2 + c^2) - 3a^2c^2 - 2a^2b^2)z + (a^2c^3 + a^2b^2c - a^3bc \cos(\gamma))$$

If we plug in S 's ridge point, we do not get zero, so disagreement exists in this case. We're also interested in when P tries to leave the roof. This solvable cubic produces one local extremum when $\gamma \geq \pi/2$. We see this by taking the derivative of the cubic, we get a quadratic with no real roots. If we plug in $x = 0$ and $x = c$ into the cubic we'll observe the local extreme is a minimum and occurs on the roof when $\gamma \geq \pi/2$. By analyzing this cubic we can see $c \geq \sqrt{b(a-b)}$ also guarantees P stays on the roof. Finally, shifting back to $c = 0$, we can optimize our original Case 3 for agreement. Notice the coefficient of z is zero in this case when $\gamma = \arccos(\frac{2ab}{a^2+b^2})$. Notice also that $\frac{2ab}{a^2+b^2} = \frac{b}{a} \frac{a^2+a^2}{a^2+b^2} > \frac{b}{a}$ if $b < a$. So $\arccos(\frac{2ab}{a^2+b^2}) < \arccos(\frac{b}{a})$, making the following theorem an improvement.

Theorem 3.2 *S and P will agree if $c = 0$ and $\gamma \geq \arccos(\frac{2ab}{a^2 + b^2})$.*

3.2 More Complex Roofs

If we add to the number of ridges, we can still partially determine when P and S will agree.

Theorem 3.3 *Let $c = 0$ and suppose we have k ridges where $k > 1$. Further suppose that the shortest path has the following properties: 1. It is a convex path. 2. When we orient the path in the xy -plane so that the y -coordinates of the initial and final point are the same and less than the y -coordinates of the interior points of the path, and no two points of this path share the same x -coordinate. Under these conditions S and P will agree.*



Figure 3.7: Before Folding with More Ridges

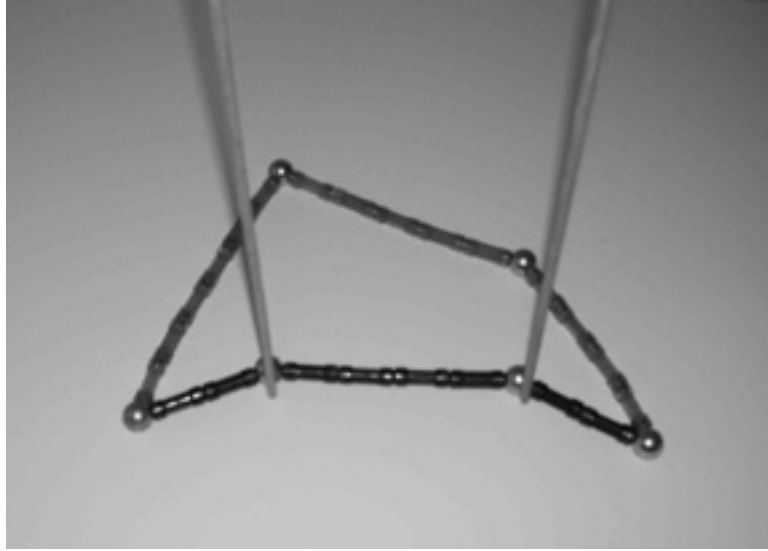


Figure 3.8: After Folding with More Ridges

Proof.

Let S be the shortest path and L be an arbitrary longer path.

Subcase 3.1: Let the initial and final segment of L be a planar.

We'll do two things to the path L . First we'll replace the intermediate segments with one segment, and then fold L about the line passing through the initial and final point, a line which is parallel to the x -axis, into the xy -plane above the line of revolution. We'll call the new path L' . If the original L was planar then $cur(L) \geq cur(L')$ and the folding allows us to easily see L' has greater curvature than S (the two interior angles whose sum is the curvature of the path are larger for the polygon formed by L'). If the original path was not planar then $cur(L) > cur(L')$ and the folding allows us to see L' has curvature no less than S .

Subcase 3.2: Let the initial and final segments of L be non planar.

L can once again be polygonally approximated with a planar path, but this time it will take three steps. First we can replace the intermediate segments with one segment (this may or may not produce a change in L). This step will reduce the curvature or maintain it depending. Second we'll rotate the path about the line passing through the initial and final point until the final ridge point lies in the xy -plane with a y -coordinate greater than the initial or final point (again this may or may not produce a change in the perspective of the path). This step produces no change in curvature. At this point, the initial segment is not in the xy -plane. We'll rotate the initial and intermediate segments about the line passing through the initial point and the last ridge point until the first ridge point of L lies in the xy -plane in the half plane not containing the final segment of the path. This last step is certain to be non-trivial (it will produce a change in the path). We'll call this new path L' . If we consider the intermediate segment of L' to be the slant height of a cone centered on the axis of revolution in the third step then we see that third step decreased the curvature at the final ridge point. So $cur(L') < cur(L)$.

The first ridge point of L' must fall in the interior of the half plane formed by the line passing through the initial point and last ridge point which does not contain the last segment of the path. Since the axis of revolution has a positive slope, the x -coordinates of the effected points in step three will decrease. Since the x -coordinate of the first ridge point had been unaffected during the first two steps, the first ridge point of L' must be to the left of the vertical passing though the first ridge point of

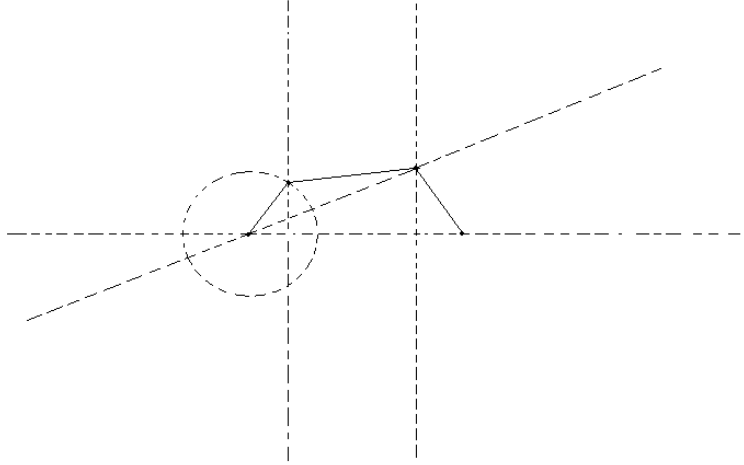


Figure 3.9: A Partial Folding

S . Finally the length of the initial segment of L' is not less in length than the initial segment of S , therefore the first ridge point of L' may not lie within that distance of the initial point of the path. So the angle the initial segment of L' makes with the segment between the initial and final point is greater than the corresponding angle for S . The other angle needed to compute the curvature for L' is no less than its corresponding angle for S . So $cur(S) < cur(L')$. \square

In the one ridge case we were able to expand our folding argument to include thin roofs with some overhang. We used both folding and a circle to see this improvement. With multiple ridges, folding and partial folding along with the circle does not always work to see S is also P , but we can get a partial result using this argument. Imagine we orient the shortest path as above, but that some points share the same x -coordinate.

Agreement between S and P is still provable using a folding argument if the following conditions are added. If we replace the intermediate segments with one segment then last ridge point has curvature less than or equal to $\pi/2$. We'll call this condition 1. If we extend the initial and final segments into lines then they intersect at a point whose y -coordinate is greater than the y -coordinate the initial or final point. We'll call this condition 2. If we label the distance between the initial point and intersection point a and the distance between the final point and intersection point b then the included angle at point of intersection is greater than or equal to $\arccos(\frac{2ab}{a^2+b^2})$. We'll call this condition 3.

Theorem 3.4 *Let $c = 0$ and suppose we have k ridges where $k > 1$. Further suppose that the shortest path has the following properties:*

1. *It is a convex path.*
2. *If the path is oriented in the xy -plane so that the y -coordinates of the initial and final point are the same and less than the y -coordinates of the interior of the path then conditions 1, 2, and 3 are met.*

Under these conditions S and P will agree.

Proof.

Suppose L is an arbitrary longer path and S is the shortest path. In each case we can replace the intermediate segments of both paths with single segments, thus creating L' and S' . The curvature of S is equal to the curvature of S' .

Case 4.1 Suppose L' has the same initial and final segments as S' .

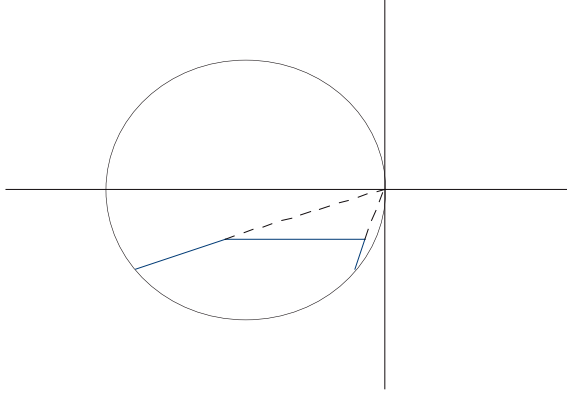


Figure 3.10: Optimal Case

In this case, $cur(L) > cur(L') = cur(S') = cur(S)$, since L would be non-coplanar.

Case 4.2 Suppose the initial and final segments of L' are coplanar but do not lie in the xy -plane.

If we construct a pseudo ridge line where the lines containing the initial and final segments of S' meet then the lines containing the initial and final segments of L' also intersect at the pseudo ridge, but not on the xy -plane. Now a simple folding argument works as it did with the one ridge case (here the pseudo ridge is the one ridge).

Case 4.3 Suppose the initial and final segments of L are not coplanar.

For purposes of closeness, we'll look at the plane containing a ridge point, the initial point, and final point and compute it's dihedral angle with the xy -plane. The smaller the dihedral angle, the "closer" the ridge point will be. The planar path for

a ridge point will be the intersection of the plane containing that ridge point, the initial and final points with the roof, but with intermediate segments replaced with one as usual (this last step does not change the curvature of the planar path). If we compute the planar paths for the initial and final ridge points, the one for the closer ridge point will be a shorter path than the original path. That is it's other ridge point will be closer to the xy -plane. So the corresponding segment, either initial or final, is shorter in the planar approximation than in the original longer path. So let L'' be the "closer" planar approximation. $cur(L'') \geq cur(S)$, with equality occurring when L'' is S . We'll fold L'' into the xy -plane and fold L' by the same amount. The common segment to both will lie in the xy -plane, but the rest of L' will not. If the final segment was the closer segment then a second partial folding, similar to case 3.2, will allow us to see $cur(L') > cur(L'')$. If the initial segment is closer then a more difficult argument is necessary.

We'll again let L'' be the planar approximation of L' , but this time using the initial segment and final point to form the plane. We'll then do a partial folding first. We'll fold the intermediate segment and final segment about the line passing through the first ridge point and final point until they are in the same plane as L'' , but away from the first segment. If we look at the projected image of this partial folding in the xy -plane, then we'll see the shadow of the final ridge point falls underneath the shadow of the line containing the final segment of L'' . This partially folded path, we'll call L''' , has curvature less than L' but more than L'' . Since we know the curvature of L'' is at least as great as S , we're done. \square

Unlike in the one ridge case, congruency of the rectangles does not generally yield agreement between S and P , even if we add congruency of the dihedral angles as an added condition. In the case of a two ridge symmetric roof, we can see the problem when we divide the path into two halves. If $c > 0$ then neither the first nor the second half of the shortest path represents the path of least curvature between the midpoint and initial or final point. Because of the symmetry in fact, the paths of least curvature between the midpoint and the initial or final point add no curvature at the midpoint if joined. So it is easy to see a path with less curvature than S . It is also possible to construct counter examples with more than two ridges.

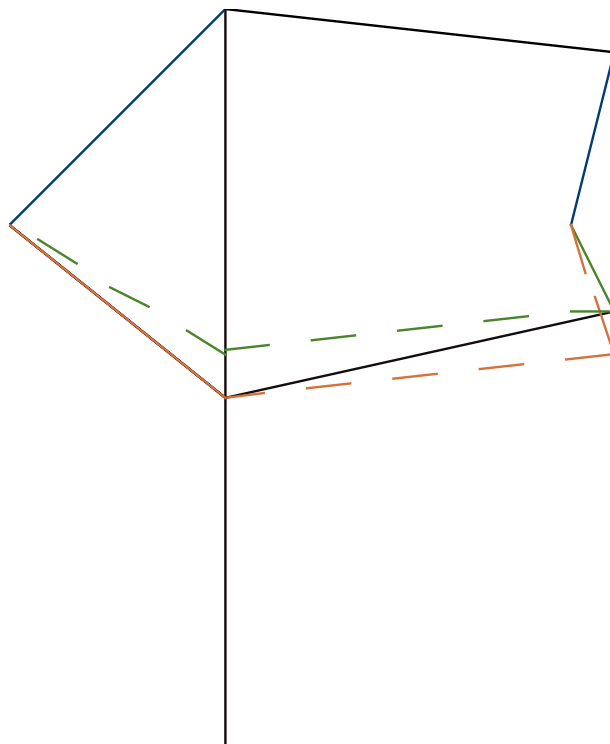


Figure 3.11: The Closer Plane

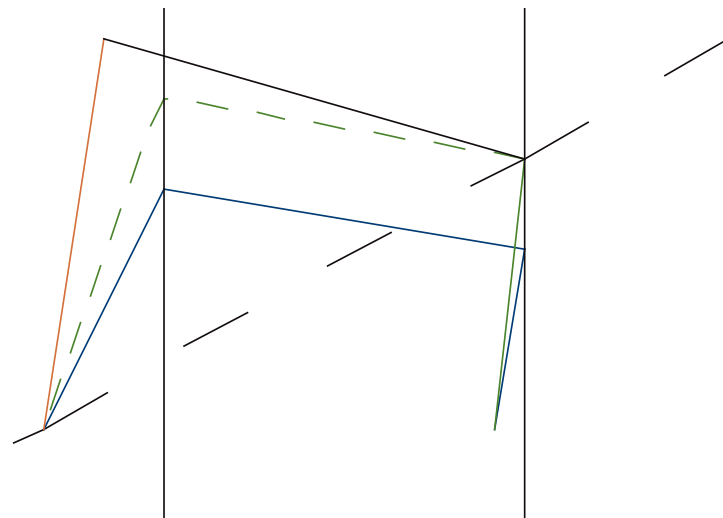


Figure 3.12: After Partial Folding

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