Disjoint Intersection Problem For Steiner Triple Systems

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Vita

Sangeetha Srinivasan was born on January 30^{th} 1984, in Mayiladuthurai, a small town in South India. She grew up for the most part in Chennai and finished high school at Padma Seshadri Bala Bhavan Senior Secondary School, K. K. Nagar. She went on to pursue a B.Tech in Electrical and Electronics Engineering at Pondicherry Engineering College, Pondicherry University from July 2001 to May 2005. Soon after in August 2005, she came to pursue a PhD in Math at Auburn University, Auburn, Alabama, U.S.A. Sangeetha was also the recipient of the Baskervill Fellowship, awarded by the College of Sciences and Mathematics, Auburn University, in Spring 2007.

Thesis Abstract

Disjoint Intersection Problem For Steiner Triple Systems

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Let (S, T_1) and (S, T_2) be two Steiner Triple systems on the set S of symbols with the set of triples T_1 and T_2 respectively. They are said to *intersect in m blocks* if $|T_1 \cap T_2| = m$. Further, if the blocks in $|T_1 \cap T_2|$ are pairwise disjoint then (S, T_1) and (S, T_2) are said to *intersect in m pairwise disjoint blocks* and are said to have *disjoint intersection.*

The *Disjoint Intersection Problem* for Steiner Triple Systems is to completely determine $\mathrm{Int_d}(v) = \Big\{ m \;\Big|\; \exists$ two Steiner triple systems of order v intersecting in m pairwise disjoint blocks$\Big\}$. $\mathrm{Int_d}(v)$ was determined by Chee. Here we describe a different proof of his result using a modification of the Bose and Skolem Constructions.

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Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliography follows van Leunen's *A Handbook for Scholars*.

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CHAPTER 1

INTRODUCTION

1.1 Definitions

A *Steiner Triple System* is an ordered pair (S, T), where S is a finite set of *points* or *symbols*, and T is a set of 3-element subsets of S called *triples*, such that each pair of distinct elements of S occurs together in exactly one triple of T. The *order* of a Steiner Triple System (S, T) is the size of the set S, denoted by $|S|$.

Let (S, T_1) and (S, T_2) be two Steiner Triple Systems on the same of set of symbols S, with the set of triples T_1 and T_2 respectively. The two systems are said to *intersect in m triples* if $|T_1 \cap T_2| = m$. Further, if the triples in $T_1 \cap T_2$ are pairwise disjoint, then they are said to *intersect in m pairwise disjoint triples* and are said to have *disjoint intersection.*

The Disjoint Intersection Problem for Steiner Triple Systems is to completely determine $\mathrm{Int}_{\mathrm{d}}(v) = \Big\{ m \;\Big|\; \exists$ two non-trivial Steiner Triple Systems of order v (so $v > 1$) intersecting in m pairwise disjoint triples$\Big\}$. A related well-studied problem is the Intersection Problem, which is to determine $\mathrm{Int}(v) = \Big\{ m \;\Big|\; \exists$ two Steiner Triple systems of order v intersecting in m triples$\Big\}$.

1.2 History of the Intersection Problem

The intersection problem for Steiner Triple Systems was completely solved by Lindner and Rosa [1]. The problem was then generalized to require more of the intersecting triples, namely that both systems have all the triples containing one specified point in common. Such a set of common triples is called a flower, so this problem was known as the flower intersection problem. This is also equivalent to solving the Intersection problem for Group Divisible Designs of block size 3 and group size 2 (that is, $\{3\}-$GDDs of the type 2^t). This was solved by Lindner and Hoffman [2].

The Intersection problem has also been studied in other settings such as, Group Divisible Designs of block size 3, having 3 groups of size g (that is, $\{3\}-$GDDs of type g^3), and Group Divisible Designs of block size 3 and group size g (that is, $\{3\}-$GDDs of type g^t). The former was solved by Fu [3] and the latter was by Butler and Hoffman [4].

In the spirit of requiring more of the intersection triples, Chee [5] specified that the intersecting triples also have to be pairwise disjoint. So, in a sense this is at the opposite end of the spectrum from the flower intersection problem. This is the Disjoint Intersection Problem for Steiner Triple Systems and it was completely solved by Chee. He proved that (recall $\mathrm{Int_d}(v)$ only considers systems that are non-trivial)

$$\mathrm{Int_d}(v) = \begin{cases} \left\{0,\ 1,\ \ldots,\ \frac{v}{3}\right\} & \text{if } v \equiv 3 \pmod 6, \quad \text{and} \\ \left\{0,\ 1,\ \ldots,\ \frac{v-1}{3}\right\} & \text{if } v \equiv 1 \pmod 6, \end{cases}$$

$$\text{except that, } \mathrm{Int_d}(7) = \{0,\ 1\} \text{ and } \mathrm{Int_d}(9) = \{0,\ 1,\ 3\}.$$

In this paper, we provide a different proof of Chee's result based on the Bose and Skolem constructions. The paper is self contained in that these constructions are clear from the proofs given in this paper; but the interested reader can find a good description of these constructions in [6]. The proof naturally breaks down into several cases, each being covered in one of the following sections. The cases depend on both the order v and the values of $int \in \mathrm{Int_d}(v)$. Using this method, all except a few of the largest values in $\mathrm{Int_d}(v)$ are found very quickly (see the next two chapters), using no more than some basic latin squares in the proof.

The symbols used in the construction often contain $\mathbb{Z}_a \times \mathbb{Z}_3$. It helps to think of the vertices being arranged with a on each of 3 levels; so that the edges $(i,j),(i,k)$ maybe thought of as being vertical edges. These vertical edges play a pivotal role in the upcoming sections. We also observe that, $\mathrm{Int}_{\mathrm{d}}(1) = \{0\}$ and $\mathrm{Int}_{\mathrm{d}}(3) = \{1\}$.

Chapter 2

$v \equiv 3 \pmod{6}$

In this chapter it is shown that when $v \equiv 3 \pmod{6}$, $\text{Int}_{\text{d}}(v) \subseteq \left\{0,\ 1,\ \ldots,\ \frac{v}{3} - 1,\ \frac{v}{3}\right\}$. Let $v = 6x + 3$, and let the set of symbols be $S = \mathbb{Z}_{2x+1} \times \mathbb{Z}_3$.

2.1 The First System

For the first system STS_1, the set of triples $T_1 = \tau_{1a} \cup \tau_{1b}$ are defined below according to the Bose construction and are shown in Figure 2.1. Here, the subscripts a and b denote the vertical triples and the mixed level triples respectively. Also, in defining the mixed level triples, we are using an idempotent commutative quasigroup of order $2x + 1$, $Q_B = (\mathbb{Z}_{2x+1}, \circ)$, whose entry in the i^{th} row and j^{th} column is denoted by $i \circ j$ (the subscript B stands for Bose).

$$\tau_{1a} = \left\{ \left\{(i,\ 0),\ (i,\ 1),\ (i,\ 2)\right\} \;\middle|\; i \in \mathbb{Z}_{2x+1} \right\} \text{ and}$$

$$\tau_{1b} = \left\{ \left\{(i,\ k),\ (j,\ k),\ (i \circ j,\ k+1)\right\} \;\middle|\; i,\ j \in \mathbb{Z}_{2x+1},\ i \neq j,\ k \in \mathbb{Z}_3 \right\},$$

reducing the sum modulo 3.

2.2 The Second System

We now define the set of triples $T_2 = \tau_{2a} \cup \tau_{2b}$ of the second Steiner Triple system, STS_2, such that the two systems STS_1 and STS_2 intersect in the disjoint set of vertical triples as shown in Figure 2.1. Hence, $2x + 1 = \frac{v}{3} \in \text{Int}_{\text{d}}(v)$.

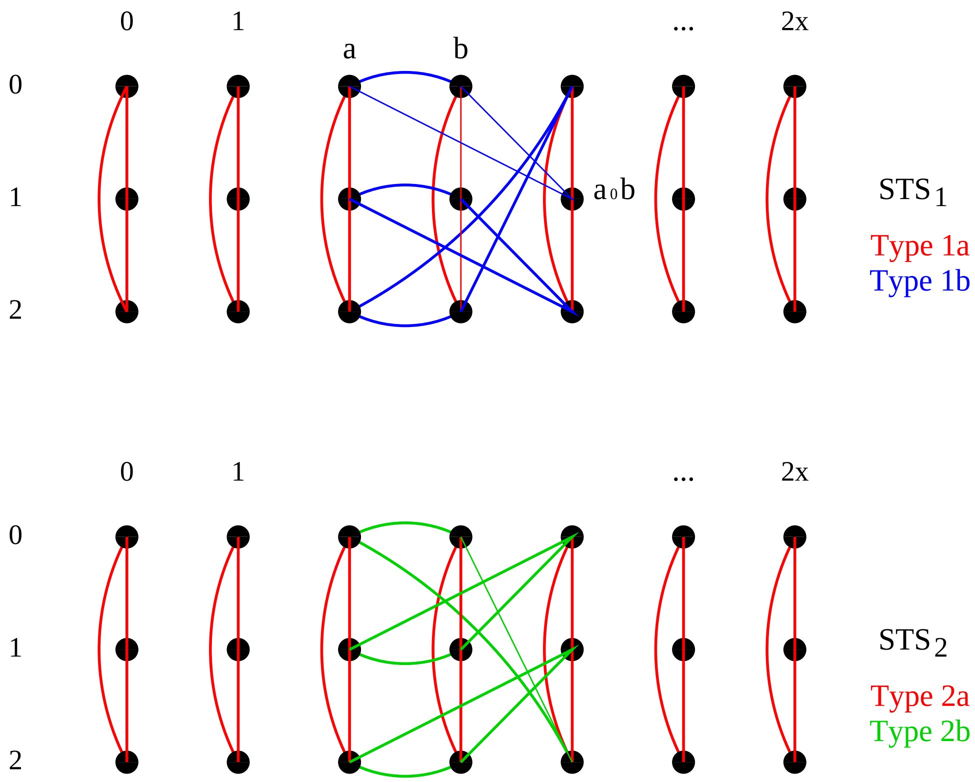

Figure 2.1: STS_1 and STS_2 based on the Bose Construction: $v = 21$

$\tau_{2a} = \tau_{1a}$, and

$$\tau_{2b} = \left\{ \left\{ (i,\, k),\ (j,\, k),\ (i \circ j,\, k+2) \right\} \,\middle|\, i,\, j \in \mathbb{Z}_{2x+1},\ i \neq j,\ k \in \mathbb{Z}_3 \right\},$$

reducing the sum modulo 3.

Thus, $T_1 \cap T_2 = \tau_{2a} = \tau_{1a}$.

2.3 Modifying the First System

We have already shown that $\frac{v}{3} \in \mathrm{Int}_{\mathrm{d}}(v)$. In order to get the other values in $\mathrm{Int}_{\mathrm{d}}(v)$, we make modifications in STS_1 to get STS_1^* with the set of triples $T_1^* = \tau_{1a}^* \cup \tau_{1b}^*$. By breaking down $k > 1$ of the vertical triples in STS_1, we get $\frac{v}{3} - k \in \mathrm{Int}_{\mathrm{d}}(v)$ as shown in Figure 2.2.

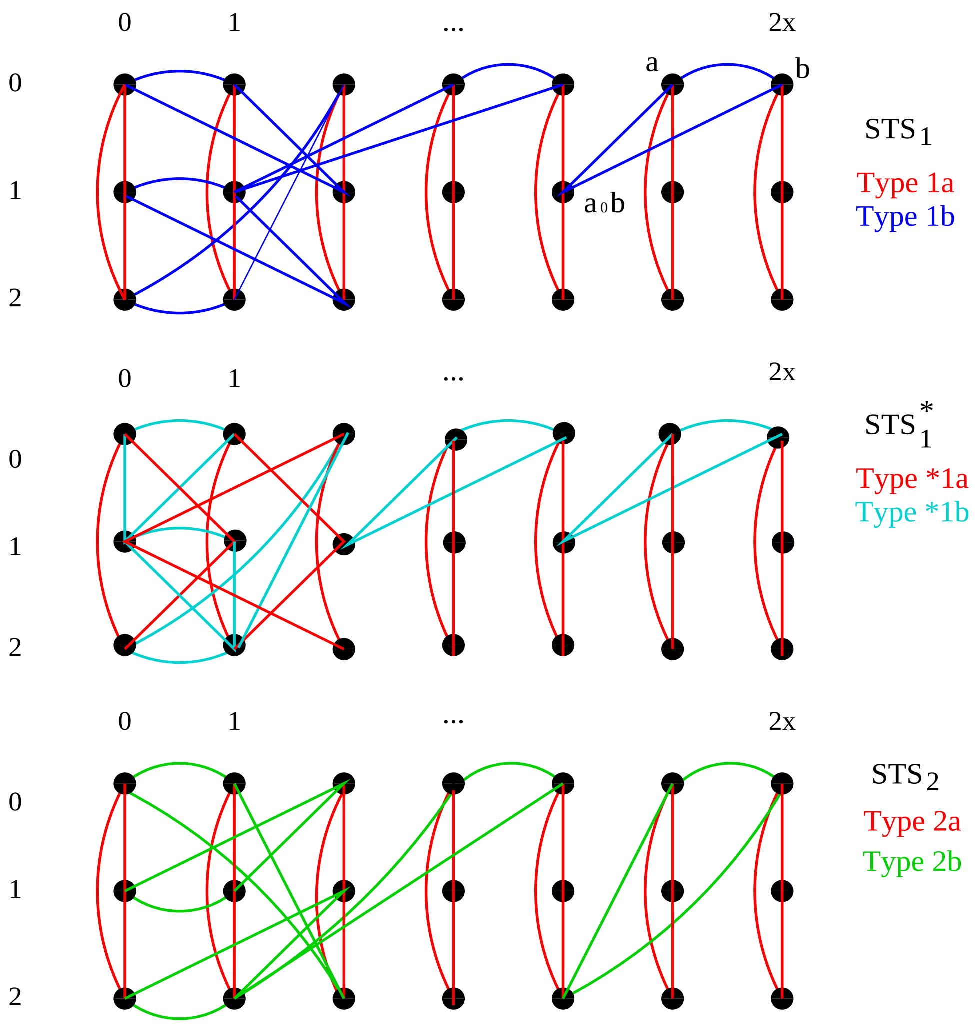

Figure 2.2: STS_1, STS_1^* and STS_2 based on the Bose construction: $v = 21$, $k = 3$ and $int = (\frac{v}{3} - 3) = 4$

For $0 \le i < k$, let $\alpha^+(i) = i + 1 \pmod k$ and $\alpha^-(i) = i - 1 \pmod k$. (So $\alpha^+(i)$ and $\alpha^-(i)$ are in $\{0, 1, \ldots, k-1\}$). Otherwise, let $\alpha^+(i) = \alpha^-(i) = i$. Then, the triples in STS_1^* are given by;

$$
\begin{aligned}
\tau_{1a}^* &= \Big\{ \Big\{(i,\,0),\ (\alpha^+(i),\,1),\ (i,\,2)\Big\} \,\Big|\, i \in \mathbb{Z}_{2x+1} \Big\}, \text{and} \\
\tau_{1b}^* &= \Big\{ \Big\{(i,\,0),\ (j,\,0),\ (\alpha^+(i \circ j),\,1)\Big\} \,\Big|\, i,\,j \in \mathbb{Z}_{2x+1},\ i \neq j \Big\} \bigcup \\
&\quad \Big\{ \Big\{(i,\,1),\ (j,\,1),\ (\alpha^-(i \circ j),\,2)\Big\} \,\Big|\, i,\,j \in \mathbb{Z}_{2x+1},\ i \neq j \Big\} \bigcup \\
&\quad \Big\{ \Big\{(i,\,2),\ (j,\,2),\ (i \circ j,\,0)\Big\} \,\Big|\, i,\,j \in \mathbb{Z}_{2x+1},\ i \neq j \Big\}.
\end{aligned}
$$

The main idea is to break down two or more of the vertical triples in STS_1, taking care that the new triples formed are unlike any of the triples in STS_2. All the edges from level 1 to the first k vertices on level 2, are cyclically shifted to the right (modulo k). This is done by the permutation $\alpha^+(i)$. Thus, the first k vertical triples are broken up and are no longer triples, while all the mixed level triples are intact (though some have their vertex on the lower level shifted). To make the broken triples whole again, we apply a similar permutation $\alpha^-(i)$ to all the edges going from level 2 to the first k vertices of level 3, which shifts them cyclically to the left (modulo k) this time. Thus, a new set of vertical triples τ_{1a}^* is formed. The mixed level triples going from the third level to the first level remain unchanged.

In this way, we can break down $2 \le k \le \frac{v}{3}$ vertical triples of STS_1, and hence get $\mathrm{Int_d}(v) \subseteq \Big\{0,\ 1,\ \ldots,\ \frac{v}{3} - 2,\ \frac{v}{3}\Big\}$, when $v \equiv 3 \pmod 6$. The missing value is dealt with in the next section.

2.4 The case where $v = 6x + 3$, and $int = 2x$

So far, we have proved that when $v \equiv 3 \pmod 6$, $\mathrm{Int_d}(v) \subseteq \left\{0, 1, \ldots, \frac{v}{3} - 2, \frac{v}{3}\right\}$. In order to get the missing value $int = \frac{v}{3} - 1$, we start looking at the cyclic Steiner Triple System of order 15; the missing value here would be 4 (we don't start with $v = 9$ because the missing value 2 doesn't exist for this case and for $v = 3$, the case is trivially not satisfied). We generate STS_2 by simply choosing a different base block for the same set of edge differences as in STS_1. Then, we modify some of the triples in STS_1 to get STS_1^* such that we achieve the missing value 4 in $\mathrm{Int_d}(15)$ (shown in Tables 2.1 through 2.4). We will need the following theorem to conclude our proof.

Table 2.1: Cyclic STS(15): Triples covering edge differences $\{1, 3, 4\}$

STS_1	STS_1^*	STS_2
$\{1, 2, 5\}$	—	$\{1, 4, 5\}$
$\{2, 3, 6\}$	—	$\{2, 5, 6\}$
$\{3, 4, 7\}$	$\{3, 4, 7\}$	$\{3, 6, 7\}$
$\{4, 5, 8\}$	$\{4, 5, 8\}$	$\{4, 7, 8\}$
$\{5, 6, 9\}$	$\{5, 6, 9\}$	$\{5, 8, 9\}$
$\{6, 7, 10\}$	$\{6, 7, 10\}$	$\{6, 9, 10\}$
$\{7, 8, 11\}$	$\{7, 8, 11\}$	$\{7, 10, 11\}$
$\{8, 9, 12\}$	$\{8, 9, 12\}$	$\{8, 11, 12\}$
$\{9, 10, 13\}$	$\{9, 10, 13\}$	$\{9, 12, 13\}$
$\{10, 11, 14\}$	$\{10, 11, 14\}$	$\{10, 13, 14\}$
$\{11, 12, 15\}$	$\{11, 12, 15\}$	$\{11, 14, 15\}$
$\{12, 13, 1\}$	$\{12, 13, 1\}$	$\{12, 15, 1\}$
$\{13, 14, 2\}$	$\{13, 14, 2\}$	$\{13, 1, 2\}$
$\{14, 15, 3\}$	$\{14, 15, 3\}$	$\{14, 2, 3\}$
$\{15, 1, 4\}$	$\{15, 1, 4\}$	$\{15, 3, 4\}$

Table 2.2: Cyclic STS(15): Triples covering edge differences $\{2,\ 6,\ 7\}$

STS_1	STS_1^*	STS_2
$\{1,\ 3,\ 9\}$	$\{1,\ 3,\ 9\}$	$\{1,\ 7,\ 9\}$
$\{2,\ 4,\ 10\}$	$\{2,\ 4,\ 10\}$	$\{2,\ 8,\ 10\}$
$\{3,\ 5,\ 11\}$	—	$\{3,\ 9,\ 11\}$
$\{4,\ 6,\ 12\}$	$\{4,\ 6,\ 12\}$	$\{4,\ 10,\ 12\}$
$\{5,\ 7,\ 13\}$	$\{5,\ 7,\ 13\}$	$\{5,\ 11,\ 13\}$
$\{6,\ 8,\ 14\}$	$\{6,\ 8,\ 14\}$	$\{6,\ 12,\ 14\}$
$\{7,\ 9,\ 15\}$	$\{7,\ 9,\ 15\}$	$\{7,\ 13,\ 15\}$
$\{8,\ 10,\ 1\}$	$\{8,\ 10,\ 1\}$	$\{8,\ 14,\ 1\}$
$\{9,\ 11,\ 2\}$	$\{9,\ 11,\ 2\}$	$\{9,\ 15,\ 2\}$
$\{10,\ 12,\ 3\}$	$\{10,\ 12,\ 3\}$	$\{10,\ 1,\ 3\}$
$\{11,\ 13,\ 4\}$	$\{11,\ 13,\ 4\}$	$\{11,\ 2,\ 4\}$
$\{12,\ 14,\ 5\}$	$\{12,\ 14,\ 5\}$	$\{12,\ 3,\ 5\}$
$\{13,\ 15,\ 6\}$	$\{13,\ 15,\ 6\}$	$\{13,\ 4,\ 6\}$
$\{14,\ 1,\ 7\}$	$\{14,\ 1,\ 7\}$	$\{14,\ 5,\ 7\}$
$\{15,\ 2,\ 8\}$	$\{15,\ 2,\ 8\}$	$\{15,\ 6,\ 8\}$

Table 2.3: Cyclic STS(15): Disjoint set of triples

STS_1	STS_1^*	STS_2
$\{1,\ 6,\ 11\}$	—	$\{1,\ 6,\ 11\}$
$\{2,\ 7,\ 12\}$	$\{2,\ 7,\ 12\}$	$\{2,\ 7,\ 12\}$
$\{3,\ 8,\ 13\}$	$\{3,\ 8,\ 13\}$	$\{3,\ 8,\ 13\}$
$\{4,\ 9,\ 14\}$	$\{4,\ 9,\ 14\}$	$\{4,\ 9,\ 14\}$
$\{5,\ 10,\ 15\}$	$\{5,\ 10,\ 15\}$	$\{5,\ 10,\ 15\}$

Table 2.4: Cyclic STS(15): Newly added triples

STS_1	STS_1^*	STS_2
—	$\{1,\ 6,\ 2\}$	—
—	$\{6,\ 11,\ 3\}$	—
—	$\{11,\ 1,\ 5\}$	—
—	$\{2,\ 3,\ 5\}$	—

Theorem 2.1 ***(Allan B. Cruse)*** *The necessary and sufficient conditions for embedding an incomplete symmetric latin square L of order n in a symmetric latin square S of order t are:*

1. *$N_L(\sigma) \geq 2n - t$ for all $1 \leq \sigma \leq t$ and*

2. *The number of symbols occurring on the diagonal of L with the wrong parity (that is $N_L(\sigma)$ not congruent to $t\,(mod 2)$) is at most $t - n$.*

Consider an idempotent and commutative latin square $Q_B = (\mathbb{Z}_{\frac{v}{3}}, \circ)$ of order $\frac{v}{3} \geq 10$, which produces a $STS(v)$ formed by the Bose construction. By Cruse's Theorem 2.1 we can ensure that Q_B contains an idempotent and commutative sub-square $(\mathbb{Z}_5, \circ)$ of order 5. (To see this, note that in our case, $N_L(\sigma) \in \{0, 5\}$ and hence we require $t \geq 2n$ (by Condition 1). Now, according to whether t is odd or even, the number of symbols with the wrong parity is 0 or 5. Hence (by Condition 2), $t \geq 5$. But, since our $n = 5$, this requirement is already met by the constraint $t \geq 2n$.)

We remove all the triples in $\mathbb{Z}_5 \times \mathbb{Z}_3$, a subsystem of order 15 in $STS_1(v)$ and $STS_2(v)$, and replace them respectively with the triples from $STS_1^*(15)$ and $STS_2(15)$ (obtained from tables 2.1 through 2.4 and renamed accordingly), to get $STS_1^*(v)$ and $STS_2^*(v)$ respectively. Now, $STS_1^*(v)$ and $STS_2^*(v)$ intersect in $\frac{v}{3} - 1$ disjoint triples.

This leaves us with proving that $6 \in \mathrm{Int_d}(21)$ and $8 \in \mathrm{Int_d}(27)$. We can use examples to show that these values exist (The interested reader can obtain these from the appendices A and B given in [5]).

Chapter 3

$v \equiv 1 \pmod 6$: All except six cases

In this chapter it is shown that when $v = 6x+1$, $\mathrm{Int_d}(v) \subseteq \left\{0, 1, \ldots, \frac{v-1}{3}\right\} \setminus M$, where,

$$M = \begin{cases} \{2x-1\} & if\ x \equiv 0 \pmod 3), \\ \{2x-2,\ 2x\} & if\ x \equiv 1 \pmod 3), \quad \text{and} \\ \{2x-3,\ 2x-1,\ 2x\} & if\ x \equiv 2 \pmod 3). \end{cases}$$

Let the set of symbols be $S = \mathbb{Z}_{2x} \times \mathbb{Z}_3 \cup \{\infty\}$.

3.1 The First Half of the System

For the first system STS_1, we define the set of triples $T_1 = \tau_{1a} \cup \tau_{1b} \cup \tau_{1c} \cup \tau_{1d} \cup \tau_{1e}$ according to the Skolem construction as shown in Figure 3.1. Here, the subscripts a and b denote the vertical triples and the mixed level triples respectively, while c, d and e denote diagonal triples. Also, in defining the mixed level triples, we are using a half-idempotent commutative quasigroup of order $2x$, $Q_S = (\mathbb{Z}_{2x}, \circ)$, whose entry in the i^{th} row and j^{th} column is denoted by $i \circ j$ (the subscript S stands for Skolem). A quasigroup is said to be half idempotent if $i \circ i = i$ when $i \leq x$ and $i \circ i = i - x$ when $i > x$.

$$\tau_{1a} = \left\{ \left\{ (i,\,0),\ (i,\,1),\ (i,\,2) \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

$$\tau_{1b} = \left\{ \left\{ (i,\,k),\ (j,\,k),\ (i \circ j,\,k+1) \right\} \,\middle|\, i,\,j \in \mathbb{Z}_{2x},\ i \neq j,\ k \in \mathbb{Z}_3 \right\},$$

reducing the sum modulo 3.

$$\tau_{1c} = \left\{ \left\{ (x+i,\,0),\ (i,\,1),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

$$\tau_{1d} = \left\{ \left\{ (x+i,\,1),\ (i,\,2),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

$$\tau_{1e} = \left\{ \left\{ (x+i,\,2),\ (i,\,0),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

Similar to the $v = 6x + 3$ case, we define the triples of the second system STS_2, such that the two systems intersect in the disjoint set of x vertical triples, namely, $\tau_{2a} = \tau_{1a}$.

$$\tau_{2a} = \tau_{1a},$$

$$\tau_{2b} = \left\{ \left\{ (i,\,k),\ (j,\,k),\ (i \circ j,\,k+2) \right\} \,\middle|\, i,\,j \in \mathbb{Z}_{2x},\ i \neq j,\ k \in \mathbb{Z}_3 \right\},$$

reducing the sum modulo 3.

$$\tau_{2c} = \left\{ \left\{ (x+i,\,0),\ (i,\,2),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

$$\tau_{2d} = \left\{ \left\{ (x+i,\,1),\ (i,\,0),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

$$\tau_{2e} = \left\{ \left\{ (x+i,\,2),\ (i,\,1),\ \infty \right\} \,\middle|\, i \in \mathbb{Z}_x \right\}$$

We then use permutations, just like in the $v = 6x + 3$ case, on the first x vertical triples of STS_1 (or the first half of the system) to get STS_1^*. By breaking down k (where $x \geq k > 1$) of the vertical triples in STS_1, we get $x - k = \frac{v-1}{6} - k \in \mathrm{Int_d}(v)$.

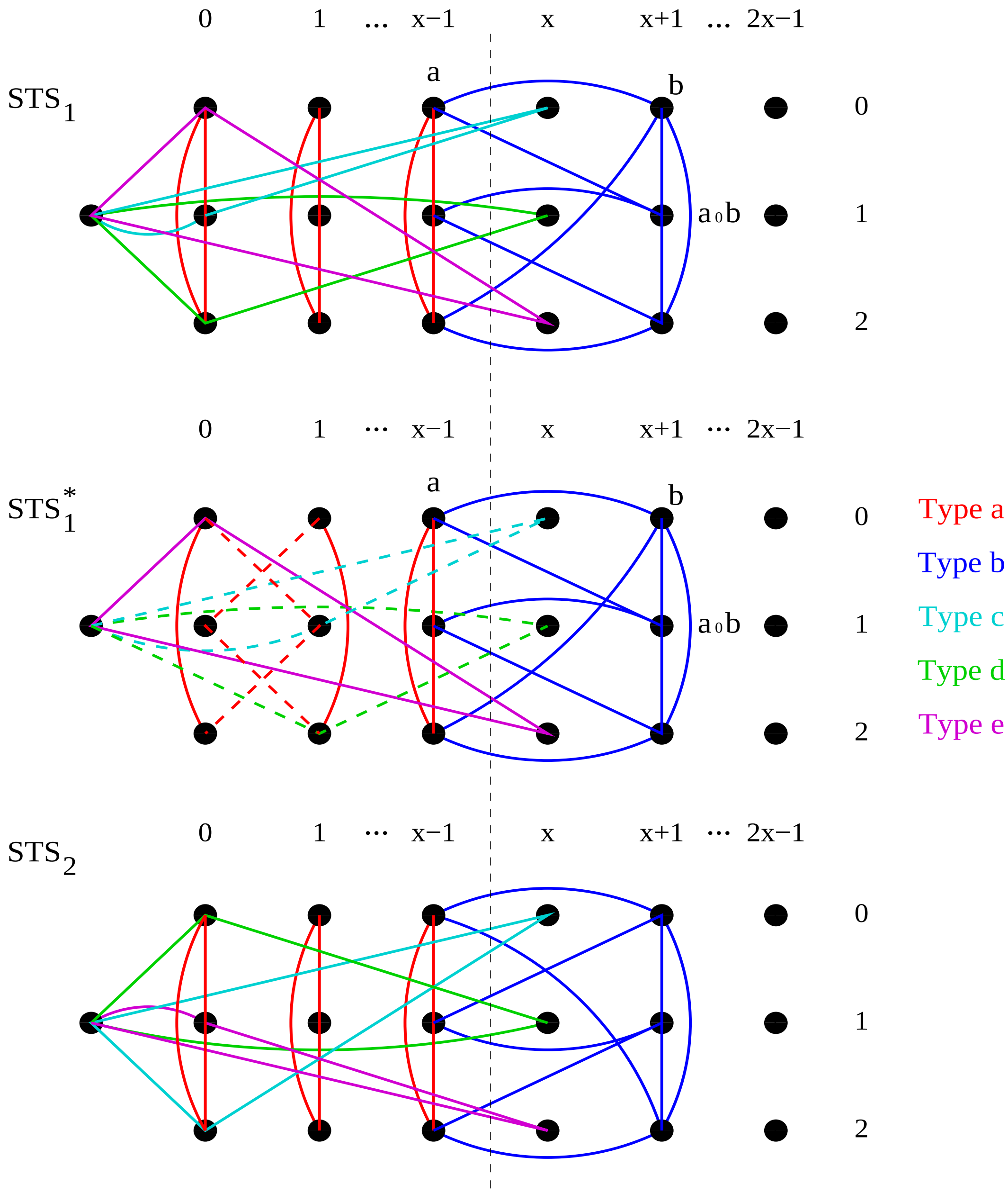

Figure 3.1: STS_1, STS_1^* and STS_2 based on the Skolem construction: $v = 19$, $k = 2$ and $int = \frac{v-1}{6} - 2 = 1$

The strategy behind the following choice of triples for STS_1^* is the same as described in Section 2.3. Here we need to take care that *all* the edges from level 1 to the first k vertices on level 2, are cyclically shifted to the right (modulo k) by the permutation $\alpha^+(i)$; and a similar permutation $\alpha^-(i)$ shifts *all* the edges going from level 2 to the first k vertices of level 3 cyclically to the left (modulo k) this time. We notice that $\tau_{1e}^* = \tau_{1e}$, and the mixed level triples containing edges joining vertices on the third level to vertices on the first level remain unchanged.

If $\alpha^+(i)$ and $\alpha^-(i)$ are as defined in Section 2.3, then the triples in STS_1^* are;

$$
\begin{aligned}
\tau_{1a}^* =& \Big\{ \Big\{(i,\, 0),\ (\alpha^+(i),\, 1),\ (i,\, 2)\Big\} \Big| \ i \in \mathbb{Z}_x \Big\} \\
\tau_{1b}^* =& \Big\{ \Big\{(i,\, 0),\ (j,\, 0),\ (\alpha^+(i \circ j),\, 1)\Big\} \Big| \ i,\, j \in \mathbb{Z}_{2x},\ i \neq j \Big\} \bigcup \\
& \Big\{ \Big\{(i,\, 1),\ (j,\, 1),\ (\alpha^-(i \circ j),\, 2)\Big\} \Big| \ i,\, j \in \mathbb{Z}_{2x},\ i \neq j \Big\} \bigcup \\
& \Big\{ \Big\{(i,\, 2),\ (j,\, 2),\ (i \circ j,\, 0)\Big\} \Big| \ i,\, j \in \mathbb{Z}_{2x},\ i \neq j \Big\}, \\
\tau_{1c}^* =& \Big\{ \Big\{\ (x+i,\, 0),\ (\alpha^+(i),\, 1),\ \infty \Big\} \Big| \ i \in \mathbb{Z}_x\ \Big\} \\
\tau_{1d}^* =& \Big\{ \Big\{\ (x+i,\, 1),\ (\alpha^-(i),\, 2),\ \infty \Big\} \Big| \ i \in \mathbb{Z}_x\ \Big\} \\
\tau_{1e}^* =& \tau_{1e}
\end{aligned}
$$

This procedure shows that $\Big\{\ 0,\ 1,\ \ldots,\ \frac{v-1}{6} - 2,\ \frac{v-1}{6} \Big\} \subseteq \mathrm{Int_d}(v)$. The missing value $\frac{v-1}{6} - 1$ occurs because we can't break apart only one of the vertical triples (hence we require $k > 1$).

3.2 The Other half of the System

To get most of the larger values in $\mathrm{Int_d}(v)$, in this section we introduce a method where we create, then break down, vertex-disjoint near subsystems of order 9 in the right half of STS_1 and STS_2. The modified systems, labeled STS_1^* and STS_2^*, are defined such that we get either 0 or 3 additional disjoint triples per near subsystem in their intersection.

Before we get to work on the triples in a near subsystem, we need to create appropriate near subsystems; that is, we need sets of 9 triples on 9 vertices in the right halves of STS_1 and STS_2. In graph theoretical terminology, each near subsystem is formed from K_9 by removing a particular parallel class. This can be achieved by manipulating the quasigroup of order $2x$ as shown in Figures 3.2 and 3.3.

While manipulating the quasigroup Q_S we note that such an embedding of a 3×3 incomplete latin square in one of order $2x$ is possible as long as $x \geq 3$, by Cruse's Theorem (Theorem 2.1). We can extend the argument to make more than one near subsystem of order 9 on the right half of the system, each arising from its own 3×3 incomplete latin square. (To see that we can incorporate $s \leq \frac{x}{3}$ such 3×3 disjoint squares, note that in our case, $N_L(\sigma) \in \{0,\ 1,\ 2\}$ and hence we require $t \geq 2s$ (by Condition 1 of Theorem 2.1). Also, t is always even and the number of symbols with the wrong parity is $3s$. Hence (by Condition 2 of Theorem 2.1), $t \geq 3s$; this requirement is already met by the constraint $t = 2x$.)

More formally, for $0 \leq s \leq \frac{x}{3}$, let $Q_S = (\mathbb{Z}_{2x}, \circ)$ be the half-idempotent commutative quasigroup (used in the Skolem construction for $v = 6x+1$) which for $1 \leq i \leq s$ contains the following 4 sub-squares defined by the operators $\circ_i$, $\star_{(i,\,0)}$, $\star_{(i,\,1)}$ and $\star_{(i,\,2)}$. Let $B_{(1,\,i)}$ and $B_{(2,\,i)}$ each represent a set of 9 triples on the symbols $\{x+3i-3,\ x+3i-2,\ x+3i-1\} \times \mathbb{Z}_3$

$Q_S =$

	0	1	...	x−1	x	x+1	x+2		2x−1
0	0								
1		1							
x−1				x−1					
x					0	x+2	x+1		
x+1					x+2	1	x		
x+2					x+1	x	2		
2x−1									x−1

Half Idempotent Commutative Quasigroup of Order 2x

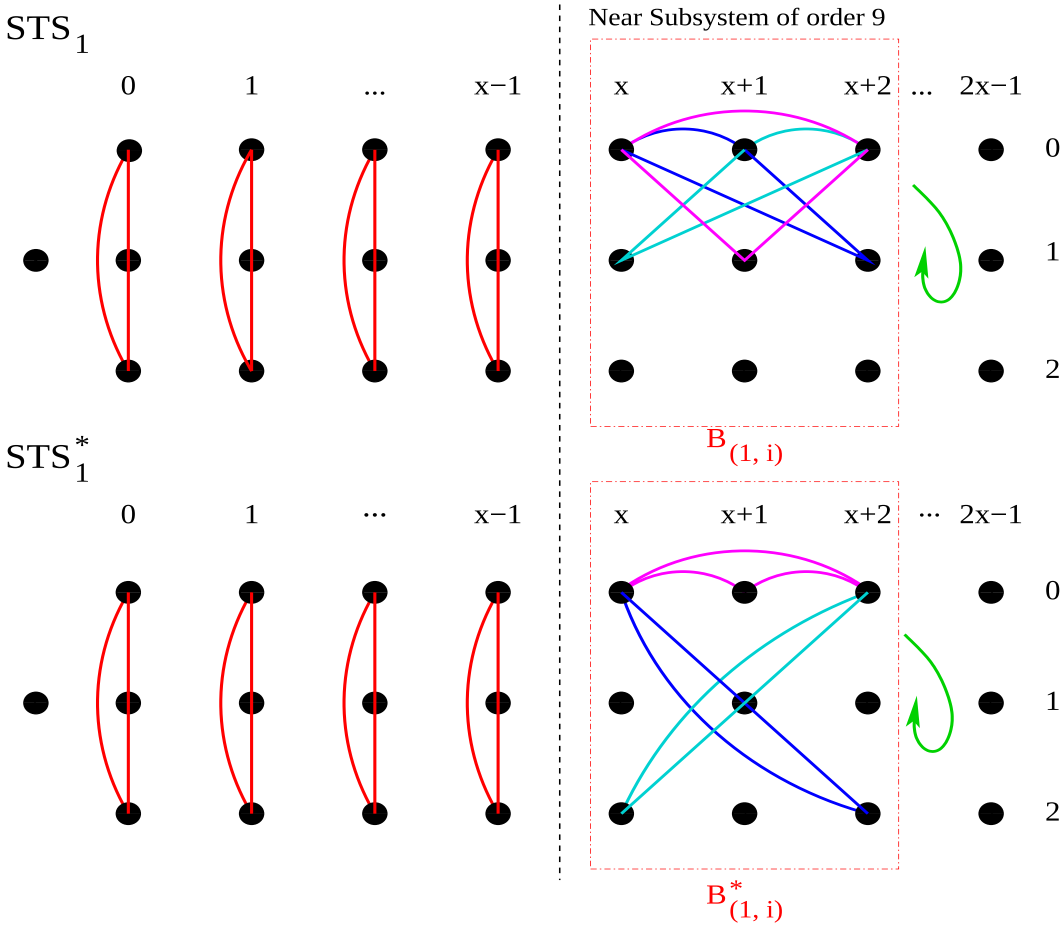

Figure 3.2: A near subsystem of order 9 in STS_1 and STS_1^* : $v = 25$

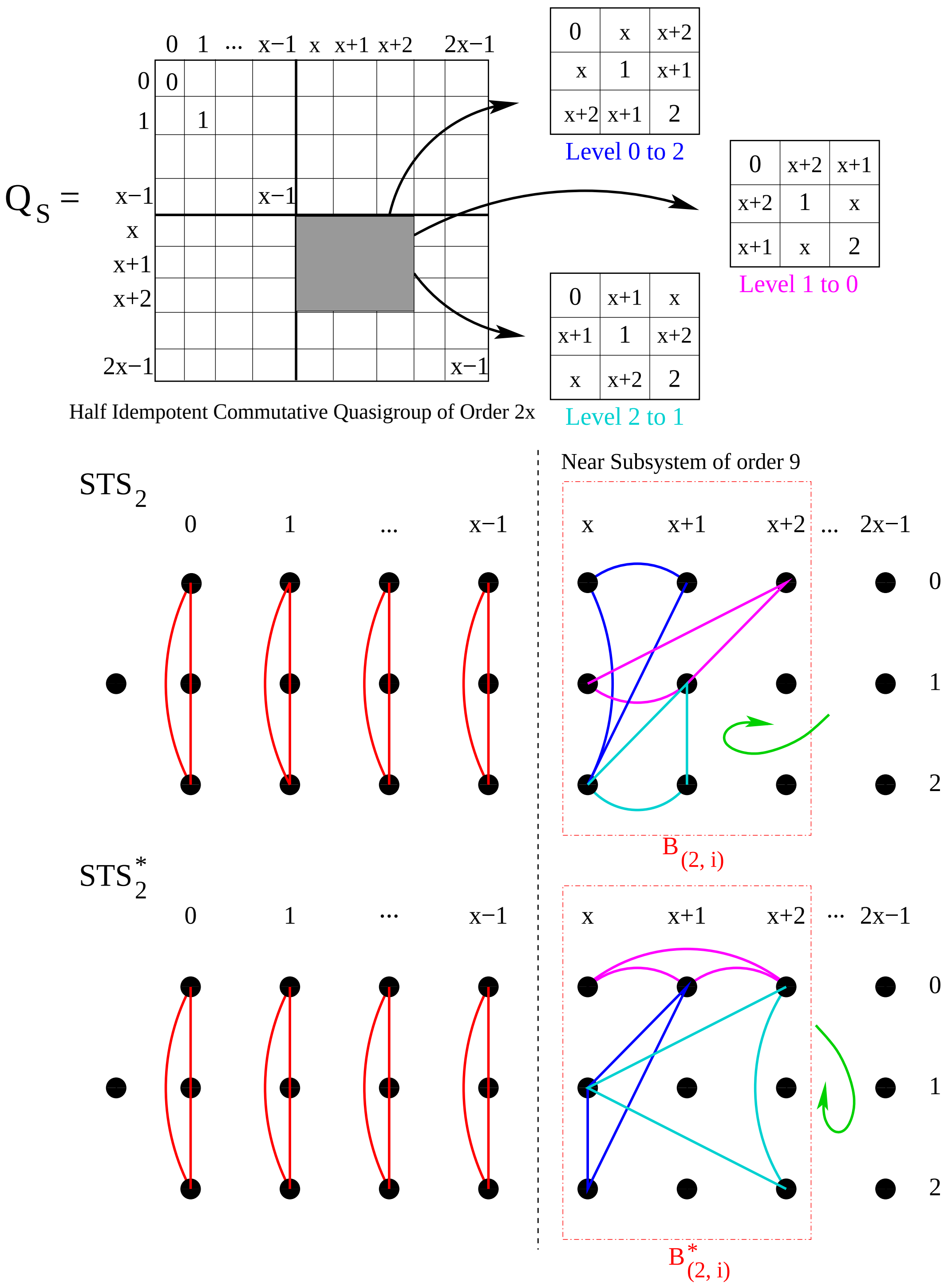

Figure 3.3: A near subsystem of order 9 in STS_2 and STS_2^* : $v = 25$

for the i^{th} near subsystem in STS_1 and STS_2 respectively. $B_{(1,\, i)}$ and $B_{(2,\, i)}$ are given as follows.

$$B_{(1,\, i)} = \Big\{(p,\ l),\ (q,\ l),\ (p \circ_i q,\ l+1)\ \Big|\ l \in \mathbb{Z}_3,\ x+3i-3 \le p < q \le x+3i-1\Big\},\ \text{and}$$
$$B_{(2,\, i)} = \Big\{(p,\ l),\ (q,\ l),\ (p \star_{(i,\, l)} q,\ l-1)\ \Big|\ l \in \mathbb{Z}_3,\ x+3i-3 \le p < q \le x+3i-1\Big\},$$

reducing the sums $(l+1)$ and $(l+2)$ modulo 3.

$\circ_i$	$x+3i-3$	$x+3i-2$	$x+3i-1$
$x+3i-3$	$3i-3$	$x+3i-1$	$x+3i-2$
$x+3i-2$	$x+3i-1$	$3i-2$	$x+3i-3$
$x+3i-1$	$x+3i-2$	$x+3i-3$	$3i-1$

$\star_{i,\,0}$	$x+3i-3$	$x+3i-2$	$x+3i-1$
$x+3i-3$	$3i-3$	$x+3i-3$	$x+3i-1$
$x+3i-2$	$x+3i-3$	$3i-2$	$x+3i-2$
$x+3i-1$	$x+3i-1$	$x+3i-2$	$3i-1$

$\star_{i,\,1}$	$x+3i-3$	$x+3i-2$	$x+3i-1$
$x+3i-3$	$3i-3$	$x+3i-1$	$x+3i-2$
$x+3i-2$	$x+3i-1$	$3i-2$	$x+3i-3$
$x+3i-1$	$x+3i-2$	$x+3i-3$	$3i-1$

We then remove the set of triples $B_{(1,\, i)}$ and $B_{(2,\, i)}$ from STS_1 and STS_2 and replace them with $B^*_{(1,\, i)}$ and $B^*_{(2,\, i)}$ to form STS^*_1 and STS^*_2 respectively. STS^*_1 and STS^*_2 are the required Steiner Triple Systems Intersecting in $3s$ disjoint triples, where s is the number of

$\star_{i,\,2}$	$x+3i-3$	$x+3i-2$	$x+3i-1$
$x+3i-3$	$3i-3$	$x+3i-2$	$x+3i-3$
$x+3i-2$	$x+3i-2$	$3i-2$	$x+3i-1$
$x+3i-1$	$x+3i-3$	$x+3i-1$	$3i-1$

near subsystems formed. $B^*_{(1,\,i)}$ and $B^*_{(2,\,i)}$ are given as follows.

$$\begin{aligned} B^*_{(1,\,i)} = &\Big\{(p,\ l),\ (q,\ l),\ (r,\ l)\ \Big|\ l \in \mathbb{Z}_3,\ x+3i-3 \le p < q < r \le x+3i-1\Big\} \bigcup \\ &\Big\{(p,\ l),\ (p+1,\ l+1),\ (p+2,\ l+2)\ \Big|\ p = x+3i-3,\ l \in \mathbb{Z}_3\Big\} \bigcup \\ &\Big\{(p,\ l),\ (p-1,\ l+1),\ (p-2,\ l+2)\ \Big|\ p = x+3i-1,\ l \in \mathbb{Z}_3\Big\}, \end{aligned}$$

reducing the sums $(l+1)$ and $(l+2)$ modulo 3, and

$$\begin{aligned} B^*_{(2,\,i)} = &\Big\{(p,\ l),\ (q,\ l),\ (r,\ l)\ \Big|\ l \in \mathbb{Z}_3,\ x+3i-3 \le p < q < r \le x+3i-1\Big\} \bigcup \\ &\Big\{(p+1,\ l),\ (p,\ l+1),\ (p,\ l+2)\ \Big|\ p = x+3i-3,\ l \in \mathbb{Z}_3\Big\} \bigcup \\ &\Big\{(p,\ l),\ (p-2,\ l+1),\ (p,\ l+2)\ \Big|\ p = x+3i-1,\ l \in \mathbb{Z}_3\Big\}, \end{aligned}$$

reducing the sums $(l+1)$ and $(l+2)$ modulo 3.

3.3 A Combination of Techniques

We can use a combination of the methods explained above (in Sections 3.1 and 3.2) in order to get almost all of the intermediate values, leaving us with the following exceptions when $v = 6x+1$:

$$int \neq \begin{cases} \{2x-1\} & if\ x \equiv 0 \pmod 3 \\ \{2x-2,\ 2x\} & if\ x \equiv 1 \pmod 3 \\ \{2x-3,\ 2x-1,\ 2x\} & if\ x \equiv 2 \pmod 3 \end{cases}$$

Exceptions occur due to the combined limitations of our techniques on the right and left halves of the triple system. On the left we can't achieve $int = x - 1$. Since we can achieve disjoint intersections only in multiples of 3 from the right half of the system, if $x \equiv 1 \text{ or } 2 \pmod 3$, we can't get disjoint intersections $(x-1)$ or $(x-2)$ respectively from the right half alone.

Table 3.1: $x \equiv 0 \pmod 3$

To realise an intersection int, break down k vertical triples in the left half, and for $1 \le i \le s$, replace $B_{(1,\,i)}$ and $B_{(2,\,i)}$ with $B^*_{(1,\,i)}$ and $B^*_{(2,\,i)}$ respectively in the right half.

int	k	s	Exceptions
$0, 1, \ldots, x-2, x$	$x, x-1, \ldots, 2, 0$	—	—
$x - 1$	4	1	$v = 19$
$2x, 2x-2, \ldots, x+i, \ldots, x+1$	$0, 2, \ldots, x-i, \ldots, x-1$	$\frac{x}{3}$	—
$2x - 1$	—	—	$v \ge 19$

The ensuing discussion is summarized in Table 3.1. When $x \equiv 0 \pmod 3$, (that is, when $x = \{0,\ 3,\ 6,\ 9,\ \ldots\}$ and so $v = \{1,\ 19,\ 37,\ 55,\ \ldots\}$), we can always have at least one near subsystem of order 9 on the right (except when $v = 1$; which is not a concern since there are no triples in an $STS(1)$ to begin with). We already know how to get $int \in \{0,\ 1,\ 2,\ \ldots,\ x-2,\ x\}$ from Section 3.1. To get $int = x - 1$, we use the construction from Section 3.1 where $k = 4$ (giving $x - 4$ disjoint triples on the left), combined with 3 disjoint triples in the intersection from the right half using exactly $s = 1$ of the near subsystems explained in Section 3.2. The exception here will be $v = 19$, where there are not enough triples on the left, since $k = 4 > x = 3$. Now, to get $int \in \{2x-1,\ 2x-3,\ \ldots,\ x+i,\ \ldots,\ x+2,\ x+1\}$, we use $s = \frac{x}{3}$ near subsystems to get x disjoint triples in the intersection on the right half. We also use the construction in Section 3.1 (on the left half of the system) with $k = \{0,\ 2,\ \ldots,\ x-i,\ \ldots,\ x-2,\ x-1\}$ corresponding respectively to each of the values in int. Finally, we note that no combination of the techniques in Sections 3.1 and 3.2 can get the value $int = 2x-1$.

Using arguments similar to those described above and inputs from the following tables one can achieve all the required disjoint intersection values except if $(v,\ int) \in E = \Big\{(7,\ 0),\ (13,\ 1),\ (19,\ 1),\ (13,\ 3)\Big\}$. Note that in the following tables, k stands for the number of triples to be broken down in the construction from Section 3.1 and s represents the number of near subsystems to be altered by the construction in Section 3.2. The exceptional cases in E are considered below.

Table 3.2: $x \equiv 1 \pmod 3$

To realise an intersection int, break down k vertical triples in the left half, and for $1 \le i \le s$, replace $B_{(1,\ i)}$ and $B_{(2,\ i)}$ with $B^*_{(1,\ i)}$ and $B^*_{(2,\ i)}$ respectively in the right half.

int	k	s	Exceptions
$0, 1, \ldots, x-2, x$	$x, x-1, \ldots, 2, 0$	—	—
$x-1$	4	1	$v = 7$
$2x-1, 2x-3, \ldots, x+i, \ldots, x+1$	$0, 2, \ldots, x-i-1, \ldots, x-2$	$\frac{x-1}{3}$	—
$2x-2$	—	—	$v \ge 7$
$2x$	—	—	$v \ge 25$

When $(v,\ int) = (7,\ 0)$, let the set of symbols be $\mathbb{Z}_7$. Consider the cyclic Steiner Triple System with base block $\{0,\ 1,\ 3\}$ as STS_1. For the same set of symbols, we take the base block as $\{0,\ 6,\ 4\}$ and generate the rest of the 6 triples for STS_2, such that STS_1 and STS_2 intersect in none of the blocks.

When $(v,\ int) = (13,\ 1)$, let the cyclic Steiner Triple System of order 13 on the set of symbols $\mathbb{Z}_{13}$ with base blocks $\{0,\ 1,\ 4\}$ and $\{0,\ 2,\ 7\}$ represent STS_1. We choose

Table 3.3: $x \equiv 2 \pmod 3$

To realise an intersection int, break down k vertical triples in the left half, and for $1 \le i \le s$, replace $B_{(1,\,i)}$ and $B_{(2,\,i)}$ with $B^*_{(1,\,i)}$ and $B^*_{(2,\,i)}$ respectively in the right half.

int	k	s	Exceptions
$0, 1, \ldots, x-2, x$	$x, x-1, \ldots, 2, 0$	—	—
$x-1$	4	1	$v = 13$
$2x-2, 2x-4, \ldots, x+i, \ldots, x+1$	$0, 2, \ldots, x-i-2, \ldots, x-3$	$\frac{x-2}{3}$	$v = 13$
$2x-3$	—	—	$v \ge 13$
$2x-1$	—	—	$v \ge 13$
$2x$	—	—	$v \ge 13$

different base blocks $\{0,\ 1,\ 10\}$ and $\{0,\ 2,\ 8\}$ to generate STS_2, such that STS_1 and STS_2 intersect in none of the triples. We apply the permutation $\alpha = (7\ 8)$ on the vertices of STS_2 to achieve STS_2^*. Then, STS_1 and STS_2^* intersect in exactly one disjoint triple (namely, $\{0,\ 2,\ 7\}$). When $(v,\ int) = (13,\ 3)$, we apply the permutation $\alpha' = (7\ 9\ 8\ )(11\ 12)$ on the vertices of STS_2 to achieve STS_2^*. Now, STS_1 and STS_2^* intersect in 3 disjoint triples (namely, $\{0,\ 2,\ 7\}$, $\{1,\ 3,\ 8\}$ and $\{4,\ 6,\ 11\}$).

Similarly, when $(v,\ int) = (19,\ 1)$, let the cyclic Steiner Triple System of order 19 on the set of symbols $\mathbb{Z}_{19}$ with base blocks $\{0,\ 1,\ 5\}$, $\{0,\ 2,\ 8\}$ and $\{0,\ 3,\ 10\}$ represent STS_1. We choose different base blocks $\{0,\ 1,\ 15\}$, $\{0,\ 2,\ 13\}$ and $\{0,\ 3,\ 12\}$ to generate STS_2, such that STS_1 and STS_2 intersect in none of the triples. Then, we apply the permutation $\alpha'' = (10\ 12)$ on the vertices of STS_2 to achieve STS_2^*. Now, STS_1 and STS_2^* intersect in exactly one disjoint triple (namely, $\{0,\ 3,\ 10\}$).

Chapter 4

Conclusion

Thus we have shown a different proof of Chee's result (Conditions 4.1 and 4.2) for the Disjoint Intersection problem, except possibly for a few cases given by Condition 4.3. The main advantage of our method is that $\mathrm{Int_d}(v)$ is determined very quickly and easily. In this thesis we have shown that

$$\mathrm{Int_d}(v) = \begin{cases} \left\{0,\ 1,\ \ldots,\ \frac{v}{3}\right\} & \text{if } v \equiv 3 \pmod{6}, \quad \text{and} \\ \left\{0,\ 1,\ \ldots,\ \frac{v-1}{3}\right\} & \text{if } v \equiv 1 \pmod{6}, \end{cases} \tag{4.1}$$

except that, $\mathrm{Int_d}(7) = \{0,\ 1\}$, $\mathrm{Int_d}(9) = \{0,\ 1,\ 3\}$, and except whether or not (4.2)

$$\mathrm{Int_d}(6x+1) \supseteq \begin{cases} \{2x-1\} & if\ x \equiv 0 \pmod{3}, \\ \{2x-2,\ 2x\} & if\ x \equiv 1 \pmod{3}, \quad \text{and} \\ \{2x-3,\ 2x-1,\ 2x\} & if\ x \equiv 2 \pmod{3}. \end{cases} \tag{4.3}$$

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