SOME TECHNIQUES IN THE CONTROL OF DYNAMIC SYSTEMS WITH PERIODICALLY VARYING COEFFICIENTS

Except where reference is made to the work of others, the work described in this dissertation is my own or was done in collaboration with my advisory committee. This dissertation does not include proprietary or classified information.

Yandong Zhang

Certificate of Approval:

George T. Flowers
Professor
Mechanical Engineering

Subhash C. Sinha, Chair
Professor
Mechanical Engineering

Dan B. Marghitu
Professor
Mechanical Engineering

Amnon J. Meir
Professor
Mathematics and Statistics

Joe F. Pittman
Interim Dean
Graduate School

SOME TECHNIQUES IN THE CONTROL OF DYNAMIC SYSTEMS WITH PERIODICALLY VARYING COEFFICIENTS

Yandong Zhang

A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
December 17, 2007

SOME TECHNIQUES IN THE CONTROL OF DYNAMIC SYSTEMS WITH

PERIODICALLY VARYING COEFFICIENTS

Yandong Zhang

____
Signature of Author

____
Date of Graduation

DISSERTATION ABSTRACT

SOME TECHNIQUES IN THE CONTROL OF DYNAMIC SYSTEMS WITH PERIODICALLY VARYING COEFFICIENTS

Yandong Zhang

Doctor of Philosophy, December 17, 2007
(MS, SJTU (Shanghai), China, 2000)
(BS, HUST (Wuhan), China, 1996)

208 Typed pages.

Directed by Subhash C.Sinha

The goal of this dissertation is to develop control system design methods for dynamic systems with time periodic coefficients. These systems arise naturally in many branches of science and engineering. As a rule, time periodic ordinary differential equations can not be integrated explicitly in general cases. Therefore, even for the linear time periodic systems, the controller design problem is much more challenging than its time invariant counterpart. In this dissertation, several strategies are developed for the controller design problem associated with dynamic systems with time periodic coefficients.

First, a linear feedback control system design techniques is developed for stabilizing the linear systems. The Floquet multipliers of the closed loop system can be assigned to desired locations in the complex plane via this method. This method is also used to design robust controllers for linear time periodic systems with parameter uncertainty. For the nonlinear time periodic system, we design a linear feedback controller such that the local stability of the closed system is guaranteed.

Second, the feedback linearization problem of nonlinear time periodic systems is addressed. The classical feedback linearization theory is generalized to the time periodic case. Two approaches are developed for the feedback linearization of such systems. The first approach is based upon the time dependent Lie derivative and the second method utilizes the time dependent Poincaré normal form theory.

Finally, observer design methods for time periodic systems are developed. It is shown that observer and feedback control design problems are dual to each other. The symbolic control design method is applied to design identity observers for linear time periodic systems. For nonlinear time periodic systems, an observer design method is developed based on the time invariant manifold theory. A fast computation method is also suggested to obtain the Fourier-Taylor series expression for the invariant manifold of the nonlinear time periodic system. Then the identity observers for nonlinear time periodic systems are constructed.

ACKNOWLEDGEMENTS

The author is grateful to his advisor Dr.Subhash C. Sinha, Professor, Mechanical Engineering for his support, direction and encouragement. The author is also thankful to the members of his doctoral committee Dr. George T. Flowers, Progessor, Mechanical Engineering, Dr. Dan B. Marghitu, Professor, Mechanical Engineering and Dr. Amnon J. Meir, Professor, Mathematics, for serving on the committee. Financial support for this work was provided by the National Science Foundation and Department of Mechanical Engineering, Auburn University is gratefully acknowledged.

This work is dedicated to the author's wife, Jiehui Wan, whose love, support and understanding during the past four years have been the motivation to reach this goal.

Style manual or Journal used: ASME Journal of Computational and Nonlinear Dynamics

Computer software used: Microsoft Word

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

Ordinary differential equations with time periodic coefficients are of great importance in science and engineering. First, systems modeled by nonlinear ordinary differential equations with time periodic coefficients appear in many branches of science and engineering. For example, such equations arise naturally in the modeling of structures subjected to periodic loading, helicopter rotor blades, asymmetric rotor bearing systems, and electrical circuits, etc， [1]. Second, it is often convenient to introduce ordinary differential equations with time periodic coefficients for studying properties of the periodic orbits of dynamics systems. Third, it is often possible to improve system's performance by introducing time periodic feedback controls. The system equations of resulting closed loop systems are ordinary differential equations with time periodic coefficients.

Since 19th century, many authors have contributed to the *analysis* of ordinary differential equations with time periodic coefficients (See [1] and references there). For analyzing a given particular time periodic system, there are three main methodologies: rigorous analytical analysis, asymptotic analysis and numerical methods. Among them, rigorous analysis is only feasible for some special systems. Numerical methods are very useful for systems with fixed parameters. However, it is impractical to analyze the evolution of solution as parameters change by using numerical computations.

Asymptotic analysis can be viewed as a mixture of rigorous analysis and numerical techniques. Though the problem of justification of asymptotic analysis is still far from being solved, it is a very fruitful technique and widely used in science and engineering. People often use experiments or experiences instead of rigorous mathematical proof to support results obtained through asymptotic analysis techniques.

Since rigorous results are difficult (if not impossible) to obtain, the system *synthesis* problem associated with periodic systems is quite challenging compared to its time-invariant counterpart. Established results for control system design of time invariant systems generally can not carry over to time periodic systems. By considering linear time periodic systems as general time varying systems and taking advantage of the properties of particular systems, many successful feedback controller design examples [2-5] had been reported since 1980s. On the other hand, several authors (See [6] and references there) developed control design method by using a discontinuous feedback signal. In this approach, the system states' value at time: $0, T, 2T, \ldots$ are viewed as a discrete dynamic system and then the control theory for linear time invariant systems can be applied. Such methods usually lead to a non-periodic closed loop system. Due to the difficulty of designing feedback controller for general linear time periodic systems, several authors [4, 6] have developed control methods under special assumptions. Sinha and Joseph [7] have proposed a control design method by employing the Floquet theory. A time invariant auxiliary system is constructed and stabilized with the pole placement method. Then a feedback controller is designed for the original linear time periodic system. However, this method uses a least square approximation approach and in certain parameter range, asymptotic stability may not be guaranteed. Recently, Sinha *et al.* [8, 9] have suggested a feedback control strategy for linear time periodic systems by using the symbolic computation technique introduced by Sinha and Butcher [10]. Under mild assumptions, this controller design method is applicable to nonlinear time periodic systems also. On the other hand, Deshmukh and Sinha [11] have presented a local control method based on the Lyapunov-Floquet transformation and backstepping technique.

The primary goal of this dissertation is to develop some new strategies as well as practical algorithms for the control system design of time periodic systems. First, a linear feedback controller design method based on the symbolic computation is developed. Robust controller design problem for linear time periodic systems is also addressed. This methodology is also applicable for the local stabilization of nonlinear time periodic system. As an example, we applied this design methodology to the chaos control problem. For driving a chaotic system to a desired time periodic orbit, we first design a feedback forward controller and then the error dynamics between the chaotic system and the desire orbit is formulated as a nonlinear time periodic system. A feedback controller is designed from the error dynamics by requiring that the errors go to zero as time goes to infinity.

Second, we generalize the geometric control theory of time invariant systems to time periodic cases. Two approaches for the feedback linearization of nonlinear time periodic systems are developed. In the first approach, time t is viewed as an auxiliary variable and a time dependent Lie derivative is defined. Then the feedback linearization method for time invariant systems can be applied. However, there are some restrictions in this approach due to the difficulty in characterizing the rank of a time periodic matrix. The second approach is based on the Poincaré normal form technique. Two computational procedures are developed for the approximate feedback linearization of nonlinear time periodic systems. The first computation procedure is similar to the classical Poincaré normal techniques. A sequence of near identity transformations is computed to obtain the linearized closed loop system. The second compoutational procedure can be viewed as a symbolic computation method for the Poincaré normal form. By using this method, the normal form of a differential equation with parameters can be evaluated. It is also possible to develop a suboptimal control design method by using this technique.

Third, observer design methods for time periodic systems are developed. It is shown that the observer design problem of linear time periodic system is a dual to the control design problem. Therefore, we can design the linear observer by using the symbolic computation techniques developed earlier. For nonlinear systems, the relationship between the observer and the original system states can be viewed as the invariant manifold of an auxiliary system. Therefore, by applying the invariant manifold theory, we can construct the observer for the nonlinear time periodic systems. However, the invariant manifold for a nonlinear time periodic systems is often characterized by a set of semi-linear time period partial differential equations (PDEs). It is rather difficult to solve these PDEs by the usual methods. To alleviate this difficulty, a practical algorithm is developed for obtaining an approximate solution using the Fourier-Taylor series.

The contents of this dissertation are arranged as follows: Chapter 2 introduces some background mathematical preliminaries. In Chapter 3, the control design methods for linear time periodic systems are developed. The robust control of linear time periodic systems under parameter uncertainty is also discussed. In Chapter 4, control design methods for nonlinear time periodic systems are developed. Observer design methods are presented in Chapter 5. Chapter 6 concludes this research and gives future perspectives.

2. MATHEMATICAL BACKGROUND

2.1 Existence of Solutions of Linear Time Periodic Ordinary Differential Equations and the Picard Iteration

Consider a system of linear time periodic ordinary differential equations of the form

$$\dot{x}(t) = A(t)\,x(t) \tag{2.1}$$

where $A(t)$ is a $n \times n$ matrix of period T and $x(t)$ is an n-dimensional vector-valued function. Due to the importance of the Picard iteration for the following chapters, we give the proof its convergence and at the same time the existence of solutions of (2.1) is proven [12].

The Picard iteration of equation (2.1) is defined by

$$P_k(t) = \int_0^t A(s)P_{k-1}(s)ds \tag{2.2}$$

$$P_0(t) = I \tag{2.3}$$

where I is the identity matrix of order n and $k = 1, 2, \ldots$;

We define $P(t)$ as the formal summation of $P_k(t)$,

$$P(t) := I + \sum_{k=1}^{+\infty} P_k(t) \tag{2.4}$$

It is easy to verify that $P(t)$ is the solution of (2.1) in a formal sense.

The (Frobenius) norm of a real time periodic matrix $A(t)$ is defined by

$$|A(t)| := \sqrt{Tr(AA^H)} = \sqrt{\sum\sum a_{ij}^2} \tag{2.5}$$

If we set

$$a(t) = \int_0^t |A(s)|\,ds \tag{2.6}$$

then by induction we have

$$P_k(t) \le a^k(t)/k!, \quad k = 1, 2, \ldots \tag{2.7}$$

Therefore, the series in (2.4) is majoranted by

$$I\,(1 + a(t)/1 + a^k(t)/2 + \ldots + a^k(t)/k! + \ldots) = \exp(a(t))\,I \tag{2.8}$$

Thus, the series converges uniformly in arbitrary finite time interval and the solution of equation (2.1) with initial condition $x(0)$ can be expressed as

$$x(t) = P(t)x(0) \tag{2.9}$$

2.2 Lyapunov-Floquet Theory

In this section, we briefly introduce the Lyapunov-Floquet theory. More details can be found in [13].

The solutions of equation (2.1) can be written as

$$x(t) = \psi(t, t_0)x(t_0) \tag{2.10}$$

$\psi(t, t_0)$ is known as the State Transition Matrix (STM). The first column of $\psi(t, t_0)$ is the solution of equation (2.1) with initial condition $x(t_0) = (1, 0, \ldots, 0)^T$, the second column of

$\psi(t,t_0)$ is the solution of equation (2.1) with initial condition $x(t_0)=(0,1,...,0)^T$, and so on. $\psi(T+t_0,t_0)$ is called the Floquet Transition Matrix (FTM).

Let

$$y(t)=x(t+T) \tag{2.11}$$

and use equation (2.1). Then we have

$$\dot{y}(t)=\dot{x}(t+T)=A(t+T)x(t+T)=A(t)y(t) \tag{2.12}$$

Therefore,

$$y(t)=\psi(t,t_0)y(t_0)=\psi(t+T,t_0)x(t_0)=\psi(t+T,t_0)y(t_0-T) \tag{2.13}$$

Let $t_0=0, t=T$, we have

$$\begin{aligned} &y(T)=\psi(T,0)y(0)=\psi(2T,0)y(-T)=\psi(2T,T)\psi(T,0)\psi(0,T)y(0) \\ &=\psi(2T,T)y(0) \end{aligned} \tag{2.14}$$

Thus, $\psi(T,0)=\psi(2T,T)$.

Similarly, it is easy to prove that

$$\psi(t,t_0)=\psi(t+NT,t_0+NT) \tag{2.15}$$

and

$$\begin{aligned} &x(h+NT+t_0,t_0)=\psi(h+NT+t_0,t_0)x(t_0) \\ &=[\psi(T+t_0+h,t_0+h)]^N\psi(h+t_0,t_0)x(t_0) \\ &=\psi(h+t_0,t_0)[\psi(T+t_0,t_0)]^N x(t_0) \end{aligned} \tag{2.16}$$

By defining $\Phi(t)=\psi(t,0)$, we have

$$x(h+NT)=\Phi(h)\Phi(T)^N x(0) \tag{2.17}$$

For brevity, $\Phi(t)$ is called the STM of equation (2.1) also and $\Phi(T)$ is called the FTM. Therefore, if the value of $\Phi(t)$ for $0 < t \leq T$ is known, the solution of equation (2.1) for $-\infty < t < \infty$ can be obtained by using (2.17).

From the Liouville formula

$$\det(\Phi(t)) = \exp[\int_0^t tr(A(s))ds] \tag{2.18}$$

we have $\det(\Phi(t)) > 0, \forall t$. Because a nonsingular matrix always has the logarithm, we can set

$$\exp(TC) = \Phi(T) \tag{2.19}$$

Now we prove that $\Phi(t)$ is the product of a T-time periodic matrix-valued function and the exponential of a matrix.

If

$$Q(t) := \Phi(t)\exp(-tC) \tag{2.20}$$

then

$$Q(t+T) = \Phi(t)\Phi(T)\exp(-TC)\exp(-tC) = Q(t) \tag{2.21}$$

Thus we have

$$\Phi(t) = Q(t)\exp(tC) \tag{2.22}$$

Furthermore, it is not difficult to verify that, the time periodic change of coordinates

$$x = Q(t)y \tag{2.23}$$

transforms equation (2.1) to a constant coefficient linear system

$$\dot{y}(t) = C\,y(t) \tag{2.24}$$

Therefore, $Q(t)$ is named as the L-F (Lyapunov-Floquet) transformation matrix.

We would like to point out that, if and only if each Jordan block of $\Phi(T)$ belonging to a negative eigenvalue occurs an even number of times, the matrix C defined by equation (2.19) can be a real matrix [13]. In other cases, to transform general equation (2.1) to a real linear time constant system, we can view equation (2.1) as a 2T-time periodic system. Then we set

$$\exp(2TR) = \Phi(2T) = \Phi(T)\Phi(T) \tag{2.25}$$

The matrix R can always be selected to be a real matrix because the square of a nonsingular matrix always has a real logarithm. Without losing generality, we assume $\Phi(T)$ has a real logarithm in this dissertation.

2.3 Chebyshev Polynomials and the Operational Matrices

The shifted Chebyshev Polynomials of the first kind may be obtaining using:

$$s_r(t) = \cos[r \cos^{-1}(2t-1)],\ r = 1, 2, ... \tag{2.26}$$

A function $y(\bar{t})$ defined over the interval $[a,b]$, can be approximated by using the shifted Chebyshev polynomials with the aid of an auxiliary map $\bar{t} \to a+(b-a)t$. The function is approximated by

$$y(a+(b-a)t) = \sum_{i=1}^{N} s_i(t) b_i \tag{2.27}$$

where b_i are constant coefficients. b_i can be calculated using:

$$\begin{aligned} b_r &= \frac{1}{\pi}\int_0^{\pi} y[a+0.5(b-a)(\cos t+1)]dt,\ r>0 \\ b_0 &= \frac{2}{\pi}\int_0^{\pi} y[a+0.5(b-a)(\cos t+1)]\cos rtdt \end{aligned} \tag{2.28}$$

For convenience in algebraic manipulation, a $2n \times 2nN$ Chebyshev polynomial matrix is defined as

$$S(t) = I_{2n\times 2n} \otimes [s_0(t), s_1(t), ..., s_N(t)]^T \tag{2.29}$$

where $\otimes$ represents the Kronecker product. Using the above definition, an $n \times n$ matrix of functions $Y(t)$ can be expressed as

$$Y(t) = BS(t)^T \tag{2.30}$$

where B is the coefficients matrix. The coefficients matrix of $\int_0^t Y(s)ds$ is given by GB, where G is the integration operational matrix. The coefficients for product of two functions can be obtained by using matrix manipulation. Let B_1, B_2 are coefficients matrix of $Y_1(t), Y_2(t)$, the coefficients matrix of $Y_1(t) \times Y_2(t)$ is given by $L_{B_1} \times B_2$. L_{B_2} is the operational matrix. All its elements are linear combination of elements of B_2.

The values of entries of the operational matrices are appended at the end of this dissertation. Other details can be found in [14] and [15].

2.4 Computaion of FTM, STM and L-F transformation

From equation (2.1), we have

$$x(t) = \int_0^t A(s)x(s)ds + x(0) \tag{2.31}$$

We assume that coefficients matrix of $x(t)$ is B. Then, after using the operational matrix given in the last section, we have

$$(I - GL_A)B = \overline{I} \tag{2.32}$$

where $\overline{I}$ is the coefficients matrix of the identity matrix of order n.

Therefore,

$$B = (I - GL_A)^{-1}\overline{I} \tag{2.33}$$

where I is the identity matrix of order $N \times n$. The FTM of (1) is given by

$$\Phi(t) = BS^T(t) \tag{2.34}$$

There are two approaches to calculate the coefficients matrix of $\Phi(t)^{-1}$. First, by using the product operational matrix directly and the relationship $\Phi(t)^{-1}\Phi(t) = I$, the coefficients matrix of $\Phi(t)^{-1}$ is given by

$$(L_B)^{-1}\overline{I} \tag{2.35}$$

Second, because $\Phi(t)^{-1}$ is the transpose of FTM of the adjoint system (corresponding to equation (2.1)) [13]

$$\dot{x}(t) = -A^T(t)x(t) \tag{2.36}$$

the coefficients matrix of $\Phi(t)^{-1}$ can be obtained by using equation (2.33) (We notice that (2.36) is itself a linear time periodic system also). It is easy to verify that these two approaches lead to the same result. After we get the FTM of equation (1) and its inverse, the L-F transformation $Q(t)$ and its inverse can be obtained by using equation (2.19), (2.20). Because $Q(t)$ is a time periodic matrix, after obtaining its Chebyshev expansion, the Fourier expansion of $Q(t)$ can be obtained by using the Fast Fourier Transformation algorithm.

2.5 Symbolic Computation of FTM and STM

Consider the linear time periodic system with parameters

$$\dot{x}(t) = A(t, p)x(t) \tag{2.37}$$

where $A(t, p) = A(t+T, p)$ is a time periodic matrix and p is the parameters vector.

Because of the parameters embedded in (2.37), the computation method given in the last section is not applicable. For this case, equation (2.32) is written as

$$(I - G(p)L_{A(p)})B(p) = \overline{I} \tag{2.38}$$

where $G(p), L_{A(p)}$ are function matrices. The coefficients matrix $B(p)$ of $x(t)$ can not be solved from (2.38) explicitly.

Historically, for some well known time periodic systems with parameters (Mathieu equation, etc.), the FTMs had been calculated by using the asymptotic methods. However, such kinds of methods are only effective for systems with small parameters. In 1996, Sinha and Butcher [10] developed a symbolic computation method to compute the FTM of (2.37) by using the Picard iteration and Chebyshev polynomials. The method is applicable for systems with large parameters. The idea is to translate the integration and product computations of (2.2), (2.3) and (2.4) into Chebyshev coefficients which can be manipulated using operational matrices in last section.

If we set

$$r(p) = G(p)L_{A(p)} \tag{2.39}$$

then, from equations (2.2),(2.3),and (2.4), we have

$$B(p) = [1 + [...[1 + [1 + r(p)]r(p)]r(p)...]]\overline{I} \tag{2.40}$$

We notice the correspondence between equation (2.40) and the successive integrations expression resulted from the Picard iteration:

$$\Phi(t,p) = I + \int A(s,p)[I + \int A(s,p)[I + \int A(s,p) + \ldots]ds \ldots ds \qquad (2.41)$$

3. CONTROLLER DESIGN FOR LINEAR TIME PERIODIC SYSTEMS

3.1 Problem Statement and Background

Consider a linear time periodic system with controls and outputs given by

$$\dot{x}(t) = A(t)\,x(t) + B(t)u(t) \tag{0.42}$$

$$y(t) = C(t)x(t) \tag{0.43}$$

where $A(t), B(t), C(t)$ are T-periodic matrices, $x(t)$ is the system states vector and $u(t)$ is the control needed. Given $A(t), B(t), C(t)$, our task is to find the control $u(t)$ such that system (3.1) meets desired control objectives. Before developing control design methods, we introduce some definitions, notations and background material.

3.1.1 Controllability and Floquet Multipliers Assignment

Definition 1 (Controllability Gramian)**:** The controllability Gramian of system (3.1) on time interval $[t_2, t_1]$ is defined by

$$W(t_2, t_1) \triangleq \int_{t_1}^{t_2} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)ds. \tag{0.44}$$

Definition 2 (Controllability index)**:** Given two $n \times n$ and $n \times r$ real constant matrices J and K, let

$$L_k = [K \mid JK \mid ... \mid J^k K]. \tag{0.45}$$

Then the controllability index of pairs $[J, K]$ is the smallest integer k_0 such that L_{k_0-1} has full rank.

Definition 3 (Floquet Multiplier, Floquet Exponent)**:** The Floquet multipliers of (3.1) are defined as the eigenvalues of its STM $\Phi(T)$. The Floquet exponents of (3.1) are eigenvalues of C matrix defined in equation (2.19).

The Floquet multipliers of a linear time periodic system are unique. However, the Floquet exponents are usually not unique. The stability of system (3.1) is characterized by its Floquet Multipliers completely. Only when all its Floquet multipliers lie in the unit circle of the complex plane, system (3.1) is asymptotic stable.

It is easy to see that system (3.1) is controllable over $[t_2, t_1]$ if and only if $W(t_2, t_1)$ is non-singular. Obviously, system (3.1) is controllable if and only if the pair $[\Phi(T), W(T, 0)]$ is controllable. Kabamba [4] gave the following sharper condition for controllability.

Theorem 1: Let k_0 be the controllability index of $[\Phi(T), W(T, 0)]$. If and only if $W(k_0 T, 0)$ is positive definite, system (3.1) is controllable.

We notice that, controllability of system (3.1) does not imply that all type of Floquet multipliers can be assigned by employing a time periodic feedback control. In 1969, Brunovsky [17] gave the following theorem about time periodic states feedback of system (3.1).

Theorem 2: If systems (3.1) controllable, then there is an $r \times n$ T-periodic real matrix $K(t)$ such that the Floquet multipliers of the closed loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) \tag{0.46}$$

are of real type and their product is positive (We say complex numbers $\lambda_1, \lambda_2, ..., \lambda_n$ is of real type if they are eigenvalues of some matrices.).

3.1.2 Time Periodic Lyapunov Equation and Riccati Differential Equation

Time periodic Lyapunov equations and Riccati differential equations play important roles in control system analysis and design. We give some fundament results here. More details can be found in [18] and [19].

The T-time periodic Lyapunov equation is defined as

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) + R(t) = 0 \tag{0.47}$$

where all matrices are of T-time periodic. For the solvability of (3.6), we have the following theorem [19].

Theorem 3: Assume $\dot{x} = A(t)x$ is asymptotically stable and $R(t)$ is a symmetric positive definite T-periodic matrix. Then there exists a unique T-periodic positive definite solution of (3.6).

The time periodic differential Riccati equation of (3.1), (3.2) is defined as

$$-\dot{P}(t) = A(t)^T P(t) + P(t)A(t) + C^T(t)C(t) - P(t)B(t)B(t)^T P(t) \tag{0.48}$$

In 1986, Bittanti *et al* [18] gave the following fundamental result.

Theorem 4: The periodic differential Riccati equation (3.7) admits a unique positive semidefinite periodic solution $P(t)$ and the resulting closed loop system

$$\dot{x}(t) = [A(t) - B(t)B^T(t)P(t)]x(t) \tag{0.49}$$

is stable if system (3.1),(3.2) is controllable and observable.

There are two ways to solve equation (3.7).

First, we notice that equation (3.7) is a semi-linear equation. A Newton-type algorithm can be developed [18]. The iteration scheme is given as in the following:

$$-\dot{P}_{i+1}(t) = A_i(t)^T P_{i+1}(t) + P_{i+1}(t)A_i(t) + C^T(t)C(t) - P_i(t)B(t)B(t)^T P_i(t) \qquad (0.50)$$

where $A_i(t) = A(t) - B(t)B^T(t)P_i(t)$. The algorithm convergences when system (3.1) is stabilizable and $P_0(t)$ is selected such that $\dot{x}(t) = [A(t) - B(t)B^T(t)P_0(t)]x(t)$ is asymptotic stable.

Second, we consider the Hamiltonian matrix $H(t)$ associated with the T-time periodic equation. We define

$$H(t) \triangleq \begin{bmatrix} A(t) & -B(t)B(t)^T \\ -C(t)^T C(t) & -A(t)^T \end{bmatrix} \qquad (0.51)$$

Let the FTM of $\dot{z}(t) = H(t)z(t)$ be

$$\Phi_H(t) = \begin{bmatrix} \Phi_{H11}(t) & \Phi_{H12}(t) \\ \Phi_{H21}(t) & \Phi_{H22}(t) \end{bmatrix} \qquad (0.52)$$

Then, the solution of (3.7) with initial condition $P(0) = P_0$ is given by

$$P(t) = (\Phi_{21} + \Phi_{22}P_0)(\Phi_{11} + \Phi_{12}P_0)^{-1} \qquad (0.53)$$

We would like to point out that T-time periodic Riccati equation (48) can be transformed into a Lyapunov equation. More precisely, if we set

$$F(t) = A(t) - B(t)B(t)^T P(t) \qquad (0.54)$$

$$H(t)^T H(t) = C(t)^T C(t) + P(t)B(t)B(t)^T P(t) \qquad (0.55)$$

then the T-time periodic equation (3.7) is written as

$$\dot{P}(t) + F(t)^T P(t) + P(t)F(t) + H(t)^T H(t) = 0 \qquad (0.56)$$

which is a Lyapunov equation.

3.2 Stabilization of Linear Time Periodic Systems

Unlike the linear time invariant systems, the solution of the T-time periodic Riccati Equation may not guarantee the stability of the closed loop system. Only when the initial points/solutions are selected properly, the subsequent solutions will lead to a stable loop system. Unfortunately, there is no general way to select the initial solution in the Newton-type method or the initial/final points for the Hamilton equation. This fact reflects the complexity of time periodic systems.

The difficulty of control design for linear time periodic systems is mainly caused by the complex relationship between the FTM and system parameters/control gains. Only for special systems, it is possible to obtain a clear picture of the relationship. In the following we develop two techniques to remove the obstacle. First, we use a zero-order state feedback such that the monodromy matrix of the resulting closed loop system is a linear function of the unknown control gains. Second, we calculate the symbolic expression of the monodromy matrix in terms of unknown control gains by the aids of the symbolic computation technique indicated earlier.

3.2.1 Stabilization of Linear Time Periodic Systems by Using the Zero Order State Feedback

Consider the T-time periodic system

$$\dot{x}(t) = A(t)\,x(t) + B(t)u(t) \tag{0.57}$$

We define a sample hold function as follows:

$$S(x(t)) = x(sT),\ \text{for}\ t \in (sT, sT+T], s = 0,1,2,... \tag{0.58}$$

And we set the control as

$$u(t) = E(t)KS(x) \tag{0.59}$$

where $E(t)$ is a known time periodic function matrix and K includes the unknown constant control gains. For example, if $u(t)$ is a scalar, then we can assume

$$\begin{aligned} u(t) &= (k_{10} + k_{11}\sin t + k_{12}\cos t + k_{13}\sin 2t + k_{14}\cos 2t + \ldots)x_1 \\ &(k_{20} + k_{21}\sin t + k_{22}\cos t + k_{23}\sin 2t + k_{24}\cos 2t + \ldots)x_2 + \ldots \end{aligned} \tag{0.60}$$

Furthermore, $E(t)$ includes terms of $\sin t, \cos t, \sin 2t, \ldots$, etc. and $K(t)$ includes all k_{ij} .

Now the closed loop system is written as

$$\dot{x}(t) = A(t)x(t) + B(t)E(t)KS(x(t)) \tag{0.61}$$

Let $\Phi(t)$ be the STM of $\dot{x} = Ax$; Then the solution of (3.20) in the first time period is given by

$$x(t) = [\Phi(t)\int_0^t \Phi^{-1}(\tau)B(\tau)E(\tau)d\tau\, K + \Phi(t)]x(0),\ t \in [0,T] \tag{0.62}$$

If we define a map $\varphi(t)$ as following

$$\varphi(t) \triangleq \Phi(t)\int_0^t \Phi^{-1}(\tau)B(\tau)E(\tau)d\tau\, K + \Phi(t),\ t \in [0,T] \tag{0.63}$$

Then we have

$$\begin{aligned} &x(t) = \varphi(t)x(0), t \in (0,T] \\ &x(t) = \varphi(t-T)x(T) = \varphi(t-T)\varphi(T)x(0),\ t \in (T,2T] \\ &x(t) = \varphi(t-2T)x(2T) = \varphi(t-2T)\varphi(T)^2 x(0), t \in (2T,3T] \\ &\qquad \ldots\ldots\ldots\ldots \\ &x(t) = \varphi(t-mT)\varphi(T)^m x(0), t \in (mT, mT+T] \end{aligned} \tag{0.64}$$

Therefore, if the eigenvalues of the monodromy matrix

$$\varphi(T) = \Phi(T)\int_0^T \Phi^{-1}(\tau)B(\tau)E(\tau)d\tau\, K + \Phi(T) \tag{0.65}$$

all lie in the unit circle of the complex plane, the closed loop system (3.20) is stable. The controller given by (3.18) can be designed by using the pole placement method for linear constant systems if the pair

$$[\Phi(T), \Phi(T)\int_0^T \Phi^{-1}(\tau)B(\tau)E(\tau)d\tau\,] \tag{0.66}$$

is stablizable by a constant state feedback.

For a linear constant system, the dimension of the control gain matrix is $r \times n$. In order to design a controller we have to determine the nr unknown variables. For linear time periodic systems, we have much more freedom. For a given $B(t)$, the number of unknown variables is decided by the dimension of the basis matrix $E(t)$. The extra freedoms may make the time periodic feedback outperforming the constant feedback even for the linear time invariant systems. We will go back to this topic later.

3.2.2 Stabilization of Linear Time Periodic System by Using the Symbolic Computation Method

Consider the linear T-time periodic system with parameters

$$\dot{x}(t) = A(t, p)x(t) \tag{0.67}$$

where $p = \{p_1, p_2, ..., p_m\}$ is the set of parameters. Assuming $A(t, p)$ an analytic function of t and p, Lyapunov [20] proved that the STM of (3.26) is also the analytic function of t and p. Furthermore, the monodromy matrix $\Phi(T)$ is an analytic function of $p = \{p_1, p_2, ..., p_m\}$. As discussed in the last Chapter, the approximation of $\Phi(T)$ in terms of p can be obtained by using the Picard iteration and Chebyshev polynomials.

Consider the closed loop system

$$\dot{x}(t) = A(t)x(t) + B(t)E(t)Kx(t) \quad (0.68)$$

where $A(t), B(t)$ are given matrices and $E(t), K$ are matrices with same definition as in equation (3.19). We assume $E(t)$, K contain a finite number of elements. Now the control object is to select the proper K such that all eigenvalues of $\Phi_K(T)$ lie in the unit cycle of the complex plane.

By using the symbolic method given in last Chapter, all elements of $\Phi_K(T)$ can be written as monomials of $k_{10}, k_{11,...}$. The eigenvalues of $\Phi_K(T)$ are given by

$$\det[\lambda I - \Phi_K(T)] = 0 \quad (0.69)$$

which is an algebraic equation and its coefficients are monomials of $k_{10}, k_{11,...}$. If and only if all eigenvalues of $\Phi_K(T)$ lie in the unit circle of the complex plane, system (3.27) is asymptotic stable. By using the Schur criterion, we obtain a system of inequalities in terms of $k_{10}, k_{11,...}$. These inequalities yield necessary and sufficient conditions for the asymptotic stability of the closed loop system.

On the other hand, for desired Floquet multiplier $\lambda_1, \lambda_2, ...$ satisfying the conditions in **Theorem 2**, we can obtain the control gains $k_{10}, k_{11,...}$ such that $\lambda_1, \lambda_2, ...$ lie in the desire location of the unit circle. This is analogy to the "pole placement" method for the linear time invariant system.

We give some examples to demonstrate applications of the symbolic computation method.

Example 1:

Consider the well known damped Mathieu equation with control

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a-2b\cos(2\pi t) & -d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{0.70}$$

where $a = 1, b = 10, d = 1$. The open loop system is

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a-2b\cos(2\pi t) & -d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{0.71}$$

By using the algorithm given above, the coefficients of the shifted Chebyshev polynomials expansion of the FTM are given by

$$\begin{aligned}
&\{\{0.148331, -0.830604, 0.107095, 0.0412027, -0.0156108, 0.0310719, -0.00305733, -0.0042078, \\
&\quad 0.000670072, -0.000134344, -0.0000149406, 0.0000848349, -9.08309\times 10^{-6}, -8.09355\times 10^{-6}, \\
&\quad 1.24312\times 10^{-6}, -6.60533\times 10^{-8}, -3.51284\times 10^{-8}, 9.19208\times 10^{-8}, -8.93812\times 10^{-9}, -9.66647\times 10^{-9}, \\
&\quad 1.31736\times 10^{-9}, 2.8332\times 10^{-10}, -6.80839\times 10^{-11}, 4.43292\times 10^{-11}, -2.89694\times 10^{-12}, -6.70807\times 10^{-12}, \\
&\quad 7.90731\times 10^{-13}, 4.01714\times 10^{-13}, -6.25014\times 10^{-14}, 6.73626\times 10^{-16}, 1.4751\times 10^{-15}, -2.17235\times 10^{-15}\}\} \\
&\{\{0.510236, 0.573936, 0.0670018, -0.0463908, -0.0697844, -0.0179481, 0.00454266, 0.0033379, \\
&\quad 0.00104872, -0.0000524168, -0.000215639, -0.0000479145, 0.0000104684, 6.58332\times 10^{-6}, \\
&\quad 1.51676\times 10^{-6}, -2.22661\times 10^{-7}, -2.97122\times 10^{-7}, -4.25373\times 10^{-8}, 1.92697\times 10^{-8}, 6.96136\times 10^{-9}, \\
&\quad 4.8467\times 10^{-10}, -4.21403\times 10^{-10}, -2.11097\times 10^{-10}, -7.63888\times 10^{-12}, 1.9274\times 10^{-11}, 3.83558\times 10^{-12}, \\
&\quad -5.8446\times 10^{-13}, -3.53917\times 10^{-13}, -6.36808\times 10^{-14}, 1.28118\times 10^{-14}, 1.00478\times 10^{-14}, 6.18452\times 10^{-16}\}\} \\
&\{\{-1.16294, 0.554089, 0.996529, -0.302673, 0.502096, -0.0529005, -0.119341, 0.0204755, -0.00152293, \\
&\quad -0.000966791, 0.00331345, -0.000369166, -0.000419285, 0.0000668219, 1.58001\times 10^{-6}, \\
&\quad -2.79257\times 10^{-6}, 5.54321\times 10^{-6}, -5.4435\times 10^{-7}, -7.07402\times 10^{-7}, 9.91949\times 10^{-8}, 2.72495\times 10^{-8}, \\
&\quad -6.19405\times 10^{-9}, 3.45067\times 10^{-9}, -2.02672\times 10^{-10}, -6.27613\times 10^{-10}, 7.54335\times 10^{-11}, 4.31938\times 10^{-11}, \\
&\quad -6.80249\times 10^{-12}, -1.91231\times 10^{-13}, 1.97669\times 10^{-13}, -2.69371\times 10^{-13}, 2.06575\times 10^{-14}\}\} \\
&\{\{0.734942, -0.446008, -0.825859, -0.982023, -0.269169, 0.134527, 0.0897924, 0.0255033, \\
&\quad -0.00366867, -0.00805573, -0.00178166, 0.000569816, 0.000326574, 0.0000673319, -0.0000157589, \\
&\quad -0.0000176064, -2.39923\times 10^{-6}, 1.4094\times 10^{-6}, 4.9331\times 10^{-7}, 2.19806\times 10^{-8}, -3.57537\times 10^{-8}, \\
&\quad -1.6793\times 10^{-8}, -3.55879\times 10^{-10}, 1.78352\times 10^{-9}, 3.46898\times 10^{-10}, -6.67898\times 10^{-11}, \\
&\quad -3.66601\times 10^{-11}, -6.006\times 10^{-12}, 1.56286\times 10^{-12}, 1.12624\times 10^{-12}, 7.66881\times 10^{-14}, -7.949\times 10^{-14}\}\}
\end{aligned}$$

The four elements of its FTM are given by:

$$\Phi 11=1.+3.02485\times 10^{-13}\,t-10.5t^2+3.5t^3+50.3987t^4-13.7547t^5-214.454t^6+70.4064t^7+ 701.156t^8-240.786t^9-1969.17t^{10}+174.889t^{11}+9389.35t^{12}-28455.9t^{13}+126121.t^{14}-588807.t^{15}+ 2.13928\times 10^6t^{16}-6.28827\times 10^6t^{17}+1.55467\times 10^7t^{18}-3.23766\times 10^7t^{19}+5.61686\times 10^7t^{20}- 8.04728\times 10^7t^{21}+9.47738\times 10^7t^{22}-9.14528\times 10^7t^{23}+7.19746\times 10^7t^{24}-4.58279\times 10^7t^{25}+ 2.3288\times 10^7t^{26}-9.23852\times 10^6t^{27}+2.76098\times 10^6t^{28}-585260.t^{29}+78491.3t^{30}-5009.09t^{31}$$

$$\Phi 12=8.27459\times 10^{-16}+1.t-0.5t^2-3.33333t^3+1.70834t^4+22.8974t^5-11.5869t^6-72.1517t^7+ 38.6089t^8+206.853t^9+68.5476t^{10}-2032.22t^{11}+9050.81t^{12}-43007.6t^{13}+187521.t^{14}- 667749.t^{15}+1.96395\times 10^6t^{16}-4.85213\times 10^6t^{17}+1.008\times 10^7t^{18}-1.74969\times 10^7t^{19}+ 2.51892\times 10^7t^{20}-2.98428\times 10^7t^{21}+2.87944\times 10^7t^{22}-2.22501\times 10^7t^{23}+1.33632\times 10^7t^{24}- 5.86861\times 10^6t^{25}+1.59189\times 10^6t^{26}-48072.9t^{27}-171233.t^{28}+77980.1t^{29}-16311.7t^{30}+1426.05t^{31}$$

$$\Phi 21=-2.81067\times 10^{-14}-21.t+10.5t^2+201.595t^3-68.7739t^4-1286.71t^5+492.616t^6+5613.5t^7- 2227.14t^8-19029.3t^9-3905.82t^{10}+154286.t^{11}-614061.t^{12}+2.95506\times 10^6t^{13}-1.36833\times 10^7t^{14}+ 5.08999\times 10^7t^{15}-1.554\times 10^8t^{16}+3.99661\times 10^8t^{17}-8.67032\times 10^8t^{18}+1.5741\times 10^9t^{19}-2.37591\times 10^9t^{20}+ 2.9708\times 10^9t^{21}-3.06971\times 10^9t^{22}+2.61234\times 10^9t^{23}-1.81995\times 10^9t^{24}+1.02749\times 10^9t^{25}- 4.62585\times 10^8t^{26}+1.61991\times 10^8t^{27}-4.24413\times 10^7t^{28}+7.80276\times 10^6t^{29}-893594.t^{30}+47633.t^{31}$$

$$\Phi 22=1.-1.t-10.t^2+6.83333t^3+114.488t^4-69.5562t^5-504.18t^6+292.5t^7+2092.8t^8-1863.64t^9+ 77.7828t^{10}-51519.2t^{11}+380324.t^{12}-1.95138\times 10^6t^{13}+8.65121\times 10^6t^{14}-3.27285\times 10^7t^{15}+ 1.04137\times 10^8t^{16}-2.79629\times 10^8t^{17}+6.36774\times 10^8t^{18}-1.2306\times 10^9t^{19}+2.01295\times 10^9t^{20}- 2.77497\times 10^9t^{21}+3.20656\times 10^9t^{22}-3.08458\times 10^9t^{23}+2.4478\times 10^9t^{24}-1.58247\times 10^9t^{25}+ 8.1888\times 10^8t^{26}-3.30657\times 10^8t^{27}+1.00265\times 10^8t^{28}-2.14533\times 10^7t^{29}+2.88522\times 10^6t^{30}-183291.t^{31}$$

The Floquet multipliers of the open loop system are -1.88042,-0.195637. Therefore, system (3.30) is unstable. For stabilizing the system (3.30), we assume a static state feedback control

$$u=-kx_2(t) \tag{0.72}$$

Then the closed loop system can be written as

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ -a-2b\cos(2\pi t) & -d-k\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{0.73}$$

By using the symbolic computation method, the monodromy matrix of (73) can be computed. The four elements are given by:

$$\Phi(T)\_11=-0.497553+0.916129K-0.284976K^2+ 0.0570156K^3-0.0173115K^4+0.000468781K^5-0.000356244K^6...$$

$\Phi(T)\_12=1.02347 - 0.44072\,K + 0.133758\,K^2 - 0.0268409\,K^3 + 0.00474924\,K^4 - 0.00122771\,K^5 + 0.0000248069\,K^6 - 0.0000248008\,K^7 \ldots$

$\Phi(T)\_21=0.418798 - 1.14678\,K - 0.109878\,K^2 + 0.0100608\,K^3 - 0.0325558\,K^4 + 0.0119503\,K^5 - 0.000112537\,K^6 + 0.000356244\,K^7 \ldots$

$\Phi(T)\_22=-1.5162 + 0.333299\,K + 0.0211313\,K^2 - 0.0497896\,K^3 + 0.00471635\,K^4 - 0.00305127\,K^5 + 0.000845869\,K^6 - 6.05508\times 10^{-9}\,K^7 + 0.0000248008\,K^8 \ldots$

Then by using the Schur criterion, we select $k = 2.5$. The Floquet multipliers of the closed loop system are -0.106835+0.137058i and -0.106835-0.137058i. Therefore, the closed loop system is asymptotic stable.

The following graphs show the states of the uncontrolled systems and the controlled system.

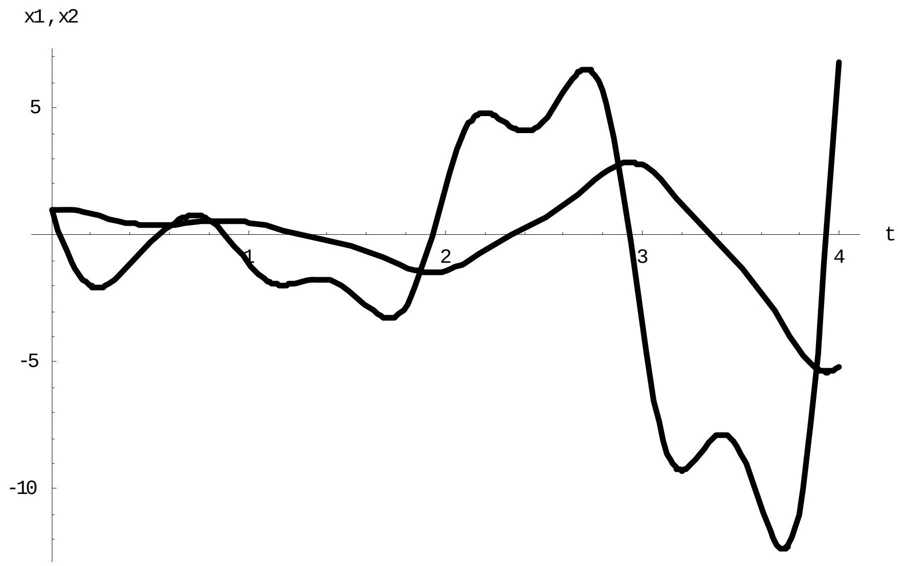


Figure 1: System states of the uncontrolled system

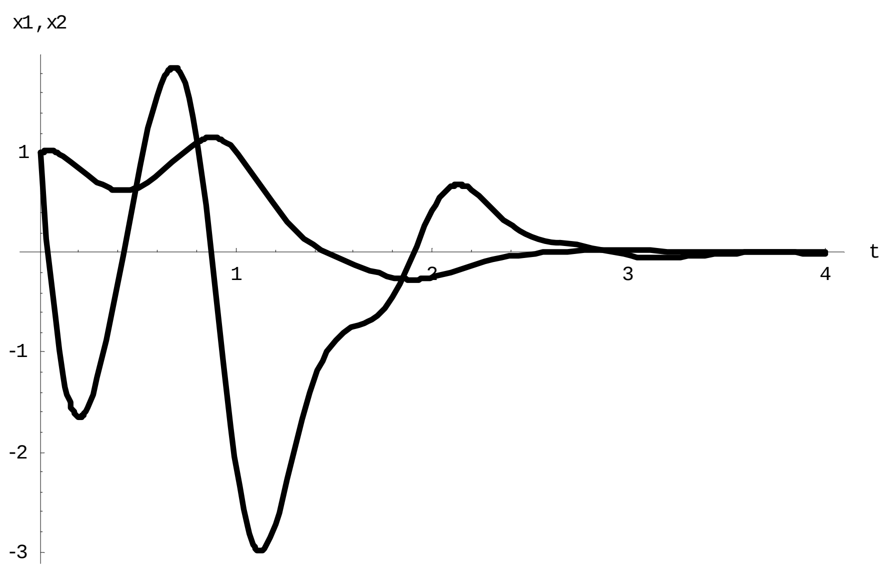


Figure 2: System states of the controlled system

3.3 Robust Control of Linear Time Periodic Systems

In this section, we develop the robust control design method for systems with parameters uncertainty. An example is used to explain how to design a robust controller by using the symbolic design method.

Example 2:

Consider the well known damped Mathieu equation with control

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a-2b\cos(2\pi t) & -d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{0.74}$$

where $b = 10, d = 2.5, a$ is an uncertainty constant and $1 \le a \le 20$.

If we set

$$u = -kx_1 \cos 2\pi t \tag{0.75}$$

where k is an unknown control gain, then the closed loop system is given by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a-2(10+k)\cos(2\pi t) & -2.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{0.76}$$

Now our object is to select a proper k such that (3.33) is stable for $a \in [1, 20]$. The uncertainty parameter is rather so big that the classical perturbation method is impractical to computer the FTM of (3.33). By using the symbolic computation method, we obtain the monodromy matrix of (3.33) as following:

$$\begin{aligned}\Phi\_T11 = {} & 0.550965 - 0.223898\, a + 0.0164873\, a^2 - 0.000510945\, a^3 + 8.70189\times10^{-6}\, a^4 - 9.23421\times10^{-8}\, a^5 + \\ & 4.51861\times10^{-10}\, a^6 - 0.127478\, k + 0.00833645\, a\, k - 0.000216298\, a^2\, k + 3.10725\times10^{-6}\, a^3\, k - \\ & 2.78896\times10^{-8}\, a^4\, k - 0.00784006\, k^2 + 0.000527018\, a\, k^2 - 0.0000138607\, a^2\, k^2 + 2.00724\times10^{-7}\, a^3\, k^2 - \\ & 1.81625\times10^{-9}\, a^4\, k^2 + 0.0000585866\, k^3 - 2.02722\times10^{-6}\, a\, k^3 + 3.29436\times10^{-8}\, a^2\, k^3 - \\ & 8.92251\times10^{-10}\, a^3\, k^3 + 1.66517\times10^{-6}\, k^4 - 5.92411\times10^{-8}\, a\, k^4 + 9.08732\times10^{-10}\, a^2\, k^4 - 2.08084\times10^{-9}\, k^5\end{aligned}$$

$$\begin{aligned}\Phi\_T12 =\ & 0.637331 - 0.0729462\, a + 0.00311852\, a^2 - 0.0000685671\, a^3 + 9.08199\times 10^{-7}\, a^4 - 7.68249\times 10^{-9}\, a^5 + \\ & 0.021314\, k - 0.00137789\, a\, k + 0.0000368875\, a^2\, k - 5.50959\times 10^{-7}\, a^3\, k + 5.09799\times 10^{-9}\, a^4\, k - \\ & 0.000622184\, k^2 + 0.0000375657\, a\, k^2 - 8.78866\times 10^{-7}\, a^2\, k^2 + 1.11117\times 10^{-8}\, a^3\, k^2 - 4.33251\times 10^{-6}\, k^3 + \\ & 2.02978\times 10^{-7}\, a\, k^3 - 3.62699\times 10^{-9}\, a^2\, k^3 + 8.91949\times 10^{-8}\, k^4 - 2.75788\times 10^{-9}\, a\, k^4 + 2.56567\times 10^{-10}\, k^5\end{aligned}$$

$$\begin{aligned}\Phi\_T21 =\ & -1.0299 + 0.212406\, a + 0.0244893\, a^2 - 0.00193262\, a^3 + 0.000051997\, a^4 - \\ & 7.51237\times 10^{-7}\, a^5 + 5.63675\times 10^{-9}\, a^6 + 0.0866677\, k + 0.0689505\, a\, k - 0.00384203\, a^2\, k + \\ & 0.0000891436\, a^3\, k - 1.18345\times 10^{-6}\, a^4\, k + 9.7101\times 10^{-9}\, a^5\, k + 0.0396418\, k^2 + 0.000197099\, a\, k^2 - \\ & 0.0000551419\, a^2\, k^2 + 1.37829\times 10^{-6}\, a^3\, k^2 - 1.66199\times 10^{-8}\, a^4\, k^2 + 0.00219675\, k^3 - 0.0000957961\, a\, k^3 + \\ & 1.86008\times 10^{-6}\, a^2\, k^3 - 2.10948\times 10^{-8}\, a^3\, k^3 + 0.0000112992\, k^4 - 5.0672\times 10^{-7}\, a\, k^4 + \\ & 9.64529\times 10^{-9}\, a^2\, k^4 - 1.0226\times 10^{-10}\, a^3\, k^4 - 1.6709\times 10^{-7}\, k^5 + 4.9763\times 10^{-9}\, a\, k^5 - 7.92797\times 10^{-10}\, k^6\end{aligned}$$

$$\begin{aligned}\Phi\_T22 =\ & -1.04236 - 0.0415331\, a + 0.00869094\, a^2 - 0.000339521\, a^3 + 6.42794\times 10^{-6}\, a^4 - 7.21248\times 10^{-8}\, a^5 + \\ & 3.22633\times 10^{-10}\, a^6 - 0.180763\, k + 0.0117812\, a\, k - 0.000308515\, a^2\, k + 4.48401\times 10^{-6}\, a^3\, k - \\ & 4.05393\times 10^{-8}\, a^4\, k - 0.0062846\, k^2 + 0.000433103\, a\, k^2 - 0.0000116625\, a^2\, k^2 + 1.72062\times 10^{-7}\, a^3\, k^2 - \\ & 1.54569\times 10^{-9}\, a^4\, k^2 + 0.0000694158\, k^3 - 2.52956\times 10^{-6}\, a\, k^3 + 3.81244\times 10^{-8}\, a^2\, k^3 + \\ & 1.13876\times 10^{-10}\, a^3\, k^3 + 1.44286\times 10^{-6}\, k^4 - 5.27978\times 10^{-8}\, a\, k^4 + 1.02907\times 10^{-9}\, a^2\, k^4 - 2.84903\times 10^{-9}\, k^5\end{aligned}$$

Then we draw the stability chart of (3.33).

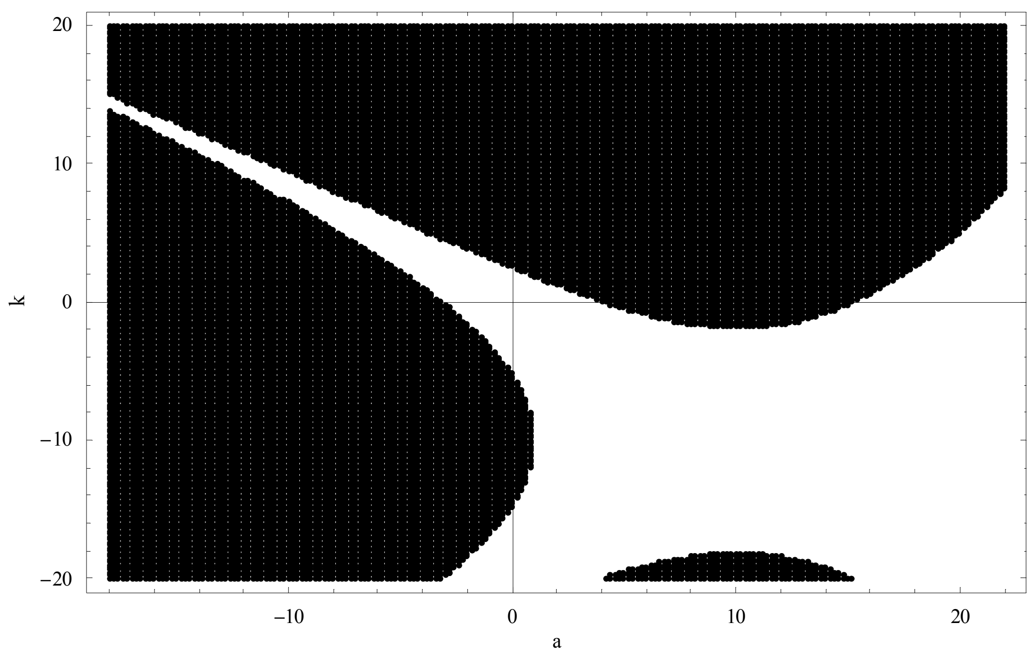


Figure 3: The stability chart of the closed loop system

In Figure 3, the black area denotes the unstable zone. Therefore, it is easy to see that (3.33) is stable for $a \in [1, 20]$ is we select

$$k = -2.5 \tag{0.77}$$

We would like to point out the advantages and restrictions of the symbolic control design method. The symbolic expressions of the monodromy matrix in term of the uncertainty parameters and the unknown control gains can be viewed as an estimation of the exact monodromy matrix. However, it is very demanding in computing time for drawing the stablility chart of a high-dimensional MIMO (multiple-input multiple-output) system. The computational complexity is also affected by the number of unknown control gains. The more control gains we assumed, the more time we need to select a proper controller. A possible way to save computation time is to compute the symbolic expression of the monodromy matrix by using the multi-Chebyshev polynomials collection method.

3.4 Local Control of the Nonlinear Systems

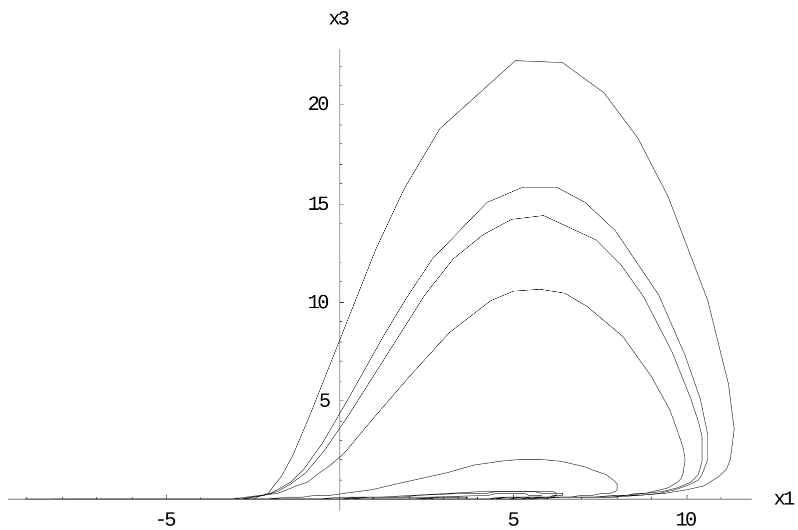
A nonlinear time periodic system is locally stable if all the Floquet multipliers of its linear part lie in the unit circle of the complex plane. Therefore, the symbolic control design method can be applied to general nonlinear systems with controllable linear parts. Two examples are presented in the following to demonstrate this local control design method.

Example 3 (The Rössler's system):

The Rössler's equations form a nonlinear system notable for its recognizable strange attractor and the example it presents of the low degree of nonlinearity necessary to induce chaos. The Rössler's equations are given by

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 - 0.2x_2 \\ \dot{x}_3 = 0.2 + x_3(x_1 - a) \end{cases} \tag{0.78}$$

where a is the critical parameter. There is only a single mild nonlinearity, located in the third state equation. Nonetheless, for $a = 5.7$, equation (3.37) possesses a familiar chaotic attractor shown in the following figure.



Consider an arbitrary desired periodic trajectory

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 + \cos 2\pi t \\ \sin 2\pi t \\ \sin 2\pi t \end{bmatrix} \tag{0.79}$$

We assume that we can apply feedforward controls on each equation of (3.37) but the feedback control can only be applied to the first equation. We aim to design control to drive the system to the desired periodic orbit. Let f_1, f_2, f_3 denote the right hand side of equation (3.37), the feedforward control u_1, u_2, u_3 is chosen as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 - f_1(y_1, y_2, y_3, t) \\ \dot{y}_2 - f_2(y_1, y_2, y_3, t) \\ \dot{y}_3 - f_3(y_1, y_2, y_3, t) \end{bmatrix} \tag{0.80}$$

Inculding this feedforward control and linearizing about the goal trajectory given by equation (3.38), we can express the linearized error dynamics in $e = x - y$ as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = E(t) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} u_f \\ 0 \\ 0 \end{bmatrix} \tag{0.81}$$

Where

$$E(t) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ \sin 2\pi t & 0 & -0.7 + \cos 2\pi t \end{bmatrix} \tag{0.82}$$

Example 4:

The dynamics of this example has been studied by Dorning et al. [21]. We consider a horizontal thermal convection layer subjected to a high temperature at the bottom boundary and a low temperature at the top boundary. Frequently, this simple geometric model provides a relevant representation of the dynamics of the natural cooling of a heat-generating component whose top supplies heat to the lower boundary of the fluid layer. In many cases of technological interest, this lower surface is not retained at a constant temperature, rather varies in some time-dependent way which is generally periodic. Hence, in order to ensure the safety of those components in electrical power generating and distribution systems that are dependent upon convective cooling, it is important to be able to control those systems. We use Navier-Stokes equations for mass and momentum conversation. Then, following Lorenz method and making Fourier expansions of the stream function and temperature field and truncating them at the same low order as Lorenz did, it is straightforward to arrive the nonautonomous parametrically forced Lorenz equations. These are given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= [\rho_0 + \rho_1 \cos \omega_1 t]x - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{0.83}$$

where x, y and z are the time-dependent Fourier coefficients of the first term in the expansion of the stream function and the first two terms in the expansion of the temperature field. For $\rho_0 = 26.5; \rho_1 = 5; \omega_1 = 2\pi; \sigma = 2; b = 0.6,$ we obtain the "Shaken butterfly", a chaotic attractors as shown in Figure 5 in x-z projection and Figure 20 in y-z projection.

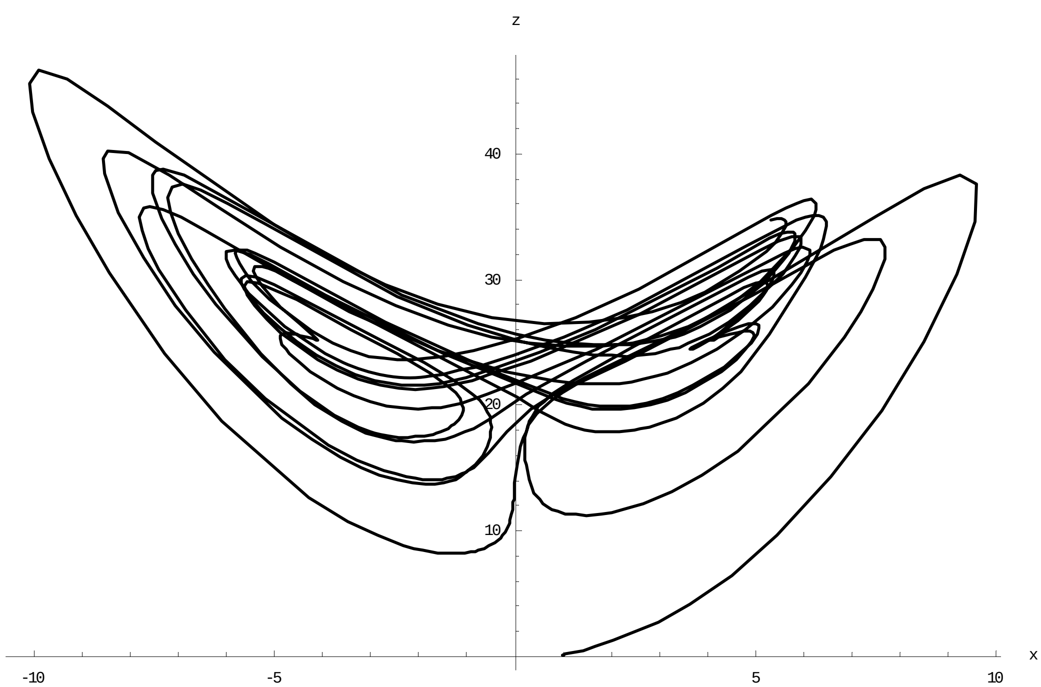


Figure 5: The "Shaken butterfly" chaotic attractor of

the nonautonomous parametrically forced Lorenz equations

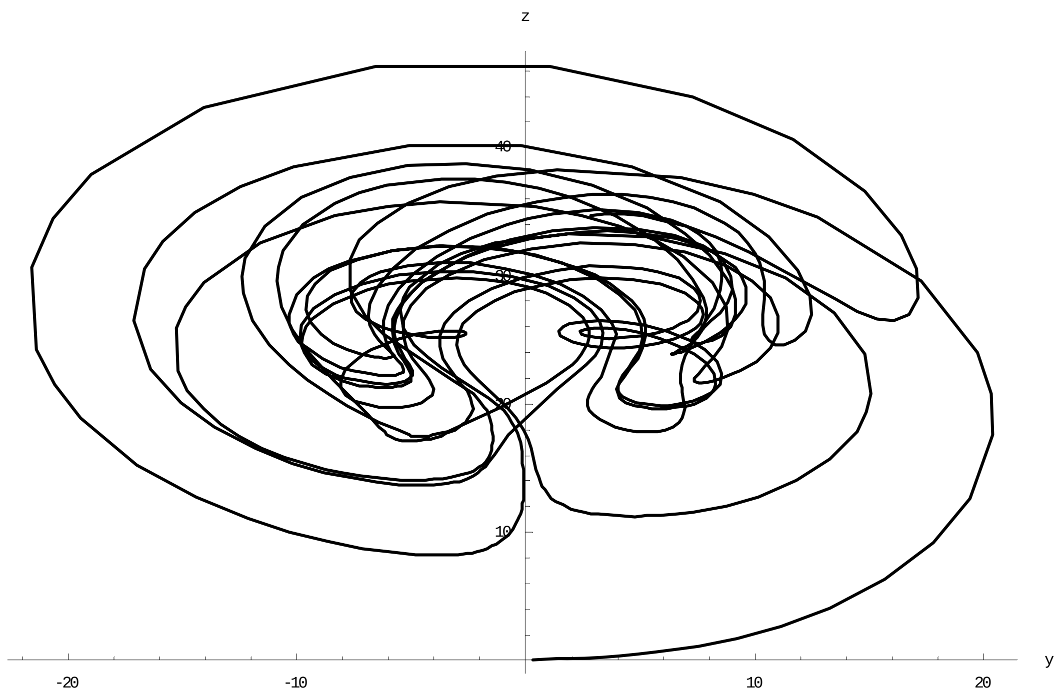


Figure 6: The uncontrolled trajectory of

We assume that we can apply feedforward controls on each equation of (3.54) but the feedback control can only be applied to the second equation. We aim to design control to drive the system to an arbitrary desired periodic orbit. Let the desired periodic orbit be given by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \sin \omega_2 t \\ \sin \omega_2 t \\ 20 + \cos \omega_2 t \end{bmatrix} \tag{0.84}$$

Let $u_1(t), u_2(t), u_3(t), u_f(x, y, z, t)$ denote the feedforward controls and the feedback control respectively, the closed loop system is written as

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y + u_1 \\ \dot{y} &= [\rho_0 + \rho_1 \cos \omega_1 t] x - y - xz + u_2 + u_f \\ \dot{z} &= xy - bz + u_3 \end{aligned} \tag{0.85}$$

Now our object is to find proper controls $u_1(t), u_2(t), u_3(t), u_f(x, y, z, t)$ such that the system (3.56) is stable around the time periodic orbit given by (3.55). Let f_1, f_2, f_3 denote the right hand side of the equation (3.54). By expanding f_1, f_2, f_3 around (3.55), we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(x_1, y_1, z_1) \\ f_2(x_1, y_1, z_1) \\ f_3(x_1, y_1, z_1) \end{bmatrix} + J \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_f \end{bmatrix} \tag{0.86}$$

where J is the Jacobian matrix of $(f_1, f_2, f_3)^T$ evaluated at $(y_1(t), y_2(t), y_3(t))^T$.

The feedforward controllers are chosen as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 - f_1(y_1, y_2, y_3, t) \\ \dot{y}_2 - f_2(y_1, y_2, y_3, t) \\ \dot{y}_3 - f_3(y_1, y_2, y_3, t) \end{bmatrix} \tag{0.87}$$

For the desired periodic orbit given by (3.55), equation (3.57) becomes

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \omega_2 \cos \omega_2 t \\ \omega_2 \cos \omega_2 t + (21 - \rho_0) \sin \omega_2 t + \dfrac{\sin \omega_2 t}{2} + \dfrac{\rho_1}{2} [\sin(\omega_1 - \omega_2) t - \sin(\omega_1 + \omega_2) t] \\ (20b - 0.5) + b \cos \omega_2 t - \omega_2 \sin \omega_2 t + 0.5 \cos 2\omega_2 t \end{bmatrix}$$

(0.88)

By including this feedforward control, we can express the linearized error dynamics in e as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = E(t) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 \\ u_f \\ 0 \end{bmatrix} \tag{0.89}$$

where

$$E(t)=\begin{bmatrix} -\sigma & \sigma & 0 \\ (\rho_0+\rho_1\cos\omega_1 t-20-\cos\omega_2 t) & -1 & -\sin\omega_2 t \\ \sin\omega_2 t & \sin\omega_2 t & -b \end{bmatrix} \tag{0.90}$$

We assume that

$$u_f=-k_1e_1-k_2e_2-k_3e_3 \tag{0.91}$$

Then we can compute the FTM for the closed-loop system if ω_1 and ω_2 are commensurate. Let $\omega_2=2\pi=\omega_1$ and, for simplicity, take $k_2=k_3=0$, then with the symbolic computation we obtain that $k_1\geq 5.18$ to guarantee the asymptotic stability. The controlled trajectories are shown in Figure 7, Figure 8 and Figure 9.

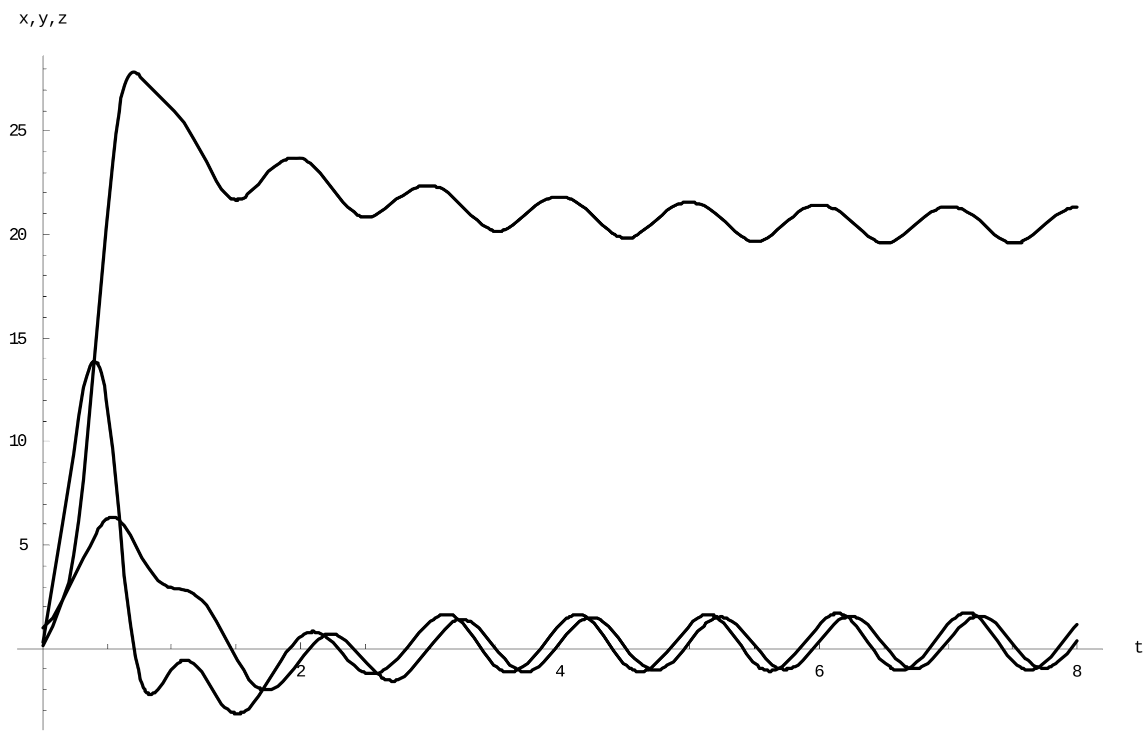


Figure 7: The system states of the controlled system

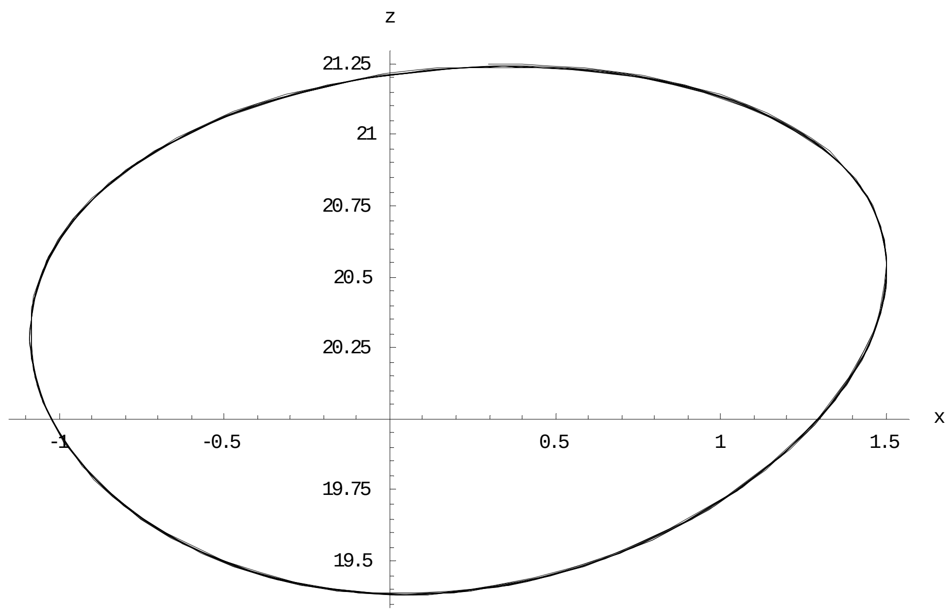


Figure 8: The x-z projection of the phase portrait of the controlled system (t=10~20)

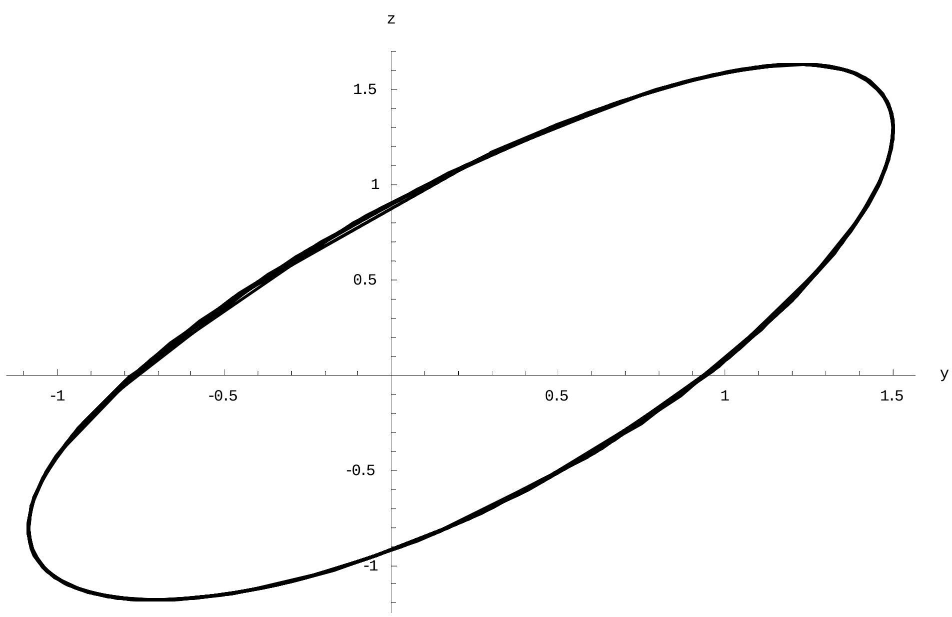


Figure 9: The y-z projection of the phase portrait of the controlled system (t=10~20)

4. CONTROL SYSTEM DESIGN FOR NONLINEAR TIME PERIODIC SYSTEMS

4.1 Control of Nonlinear Time Periodic Systems via Feedback Linearization

In the past twenty years or so, the feedback linearization of nonlinear system has been studied by many authors [22-24]. Therein, the nonlinear control system is first transformed into the Brunovsky form by a change of coordinates and state feedback. Then a linear controller is designed to control the linearized system. Unfortunately, most nonlinear controllable systems do not admit Brunovsky form and even when one does, a transformation may be very difficult to find. Hence, various approaches to approximate feedback linearization have been developed in recent years [25, 26]. However, to the author's best knowledge, almost all researches in this direction are limited to autonomous systems.

By treating the time as a new independent variable, the classical exact feedback linearization method for autonomous system can be naturally generalized to the time variant case. However, as we will see in the next section, this approach has serious restrictions when applied to nonlinear time periodic systems. In the present work, we propose a new methodology for feedback linearization of time periodic systems. The idea is to find a coordinate transformation and a state feedback under which the nonlinear time periodic system can be (approximately) transformed into a linear time periodic system.

By extending Poincaré's time-dependent normal form theory [27] to nonlinear time periodic control systems, we suggest an algorithm by which the system is feedback linearizable up to the r^{th} order. Then the controller is designed to stablize the linear time periodic system by using the symbolic design method given in [9].

4.1.1 Exact Feedback Linearization of Time Dependent Systems

In this section, we generalize the classical exact feedback linearization theory for the autonomous system by treating time as an augment variable. We only consider the feedback linearization of the single input state-space system. Results for the multi-input system can be drawn easily in a similar way.

Consider the nonlinear time-varying single input system

$$\dot{x} = f(x,t) + g(x,t)u(x,t) \tag{0.92}$$

where $x \in R^n$ $f(x,t) = (f_1(x,t), f_2(x,t), ..., f_n(x,t))^T$, $g(x,t) = (g_1(x,t), g_2(x,t), ..., g_n(x,t))^T$ and $u(x,t)$ is a scalar function. The equilibrium point is x_0.

Consider the feedback and the coordinate transformation given by

$$\begin{aligned} u(x,t) &= \alpha(x,t) + \beta(x,t)v \\ y &= \Phi(x,t) \end{aligned} \tag{0.93}$$

where $\alpha(x,t)$ and $\beta(x,t)$ are scalar functions and $\Phi(x,t) = (\varphi_1(x,t), \varphi_2(x,t), ..., \varphi_n(x,t))$.

We attempt to find a proper transformation (4.2) under which system (4.1) can be transformed into the Brunovsky form

$$\dot{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & \dots & \dots & \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix} v \tag{0.94}$$

We define

$\tilde{f}(x,t) = (f_1(x,t), f_2(x,t),..., f_n(x,t),1)^T, \tilde{g}(x,t) = (g_1(x,t), g_2(x,t),..., g_n(x,t),0)^T$. Then by treating t as an augment variable and applying the feedback linearizable conditions for autonomous system [23], we have the following proposition

Proposition 1: The single input control system (4.1) is exact feedback linearizable near x_0 if the following conditions are satisfied for every t

(i) $g(x_0,t), ad_{\tilde{f}(x_0,t)}\tilde{g}(x_0,t),..., ad^{n-1}_{\tilde{f}(x_0,t)}\tilde{g}(x_0,t)$ are linearly independent,

(ii) $span\{\tilde{g}(x,t), ad_{\tilde{f}(x,t)}\tilde{g}(x,t),..., ad^{n-2}_{\tilde{f}(x,t)}\tilde{g}(x,t)\}$ is involutive near x_0 ,where

$ad^k_{\tilde{f}(x_0,t)}\tilde{g}(x_0,t)$ is recursively defined by $ad^0_{\tilde{f}}\tilde{g} = \tilde{g}$,

$$ad^k_{\tilde{f}}\tilde{g} = \sum_{i=1}^{n} \frac{\partial(ad^{k-1}_{\tilde{f}}\tilde{g})}{\partial x_i} f_i + \frac{\partial(ad^{k-1}_{\tilde{f}}\tilde{g})}{\partial t} - \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x_i}(ad^{k-1}_{\tilde{f}}\tilde{g})_i \ .$$

If the conditions (i) and (ii) are satisfied, we can design the linear controller v and then design the nonlinear controller u using the transformation (4.2). However, this approach has severe restrictions in practical applications. Firstly, the conditions given in proposition 1 are very strong and hard to meet for nonlinear time periodic systems. In fact, even most linear time periodic systems cannot be transformed into the Brunovsky form. Secondly, the conditions are difficult to verify because we have to evaluate the rank of the time periodic matrix and decide the involutivness of a time periodic distribution near x_0. We use the following example to demonstrate the limitations of this approach.

Example 5:

Consider the well-known Mathieu-equation with cubic nonlinearity and a scalar control u given by

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & \pi \\ -\pi(a+b\sin(2\pi t)) & -\pi d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{1}{3}x_1^3 \end{Bmatrix} + \begin{Bmatrix} u \\ 0 \end{Bmatrix} \tag{0.95}$$

For $a = 1.1, b = 1, d = 0.3$, the Floquet multipliers are (-1.3) and (-0.3) and thus the system exhibits unbounded behavior. It is easy to verify that the conditions of proposition 1 are satisfied. In fact, if we let

$$u = \frac{-v + 2\pi^2 \cos(2\pi t)x_1 + 0.3\pi^2(1.1+\sin(2\pi t))x_1 + 0.3\pi(x_2 + (1/3)x_1^3)}{\pi(1.1+\sin(2\pi t)) + x_1^2} \tag{0.96}$$

$$y_1 = x_2,\ y_2 = -\pi(1.1+\sin(2\pi t))x_1 - 0.3\pi x_2 \tag{0.97}$$

the original system is transferred into $\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ v \end{Bmatrix}$. From (122), it is obvious that the approach is applicable only when $\pi(a+b\sin(2\pi t)) + x_1^2 \neq 0, \forall x_1, t$.

If we let

$$v = -2y_1 - 4y_2 \tag{0.98}$$

then we have

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \tag{0.99}$$

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}$ are -3.41 and -0.59 and therefore the system (125) is asymptotic stable.

By using (4.5), (4.6) and (4.7), we can obtain the expression of the controller u in terms of x_1 and x_2. The system states of uncontrolled and controlled systems are shown in the following figures.

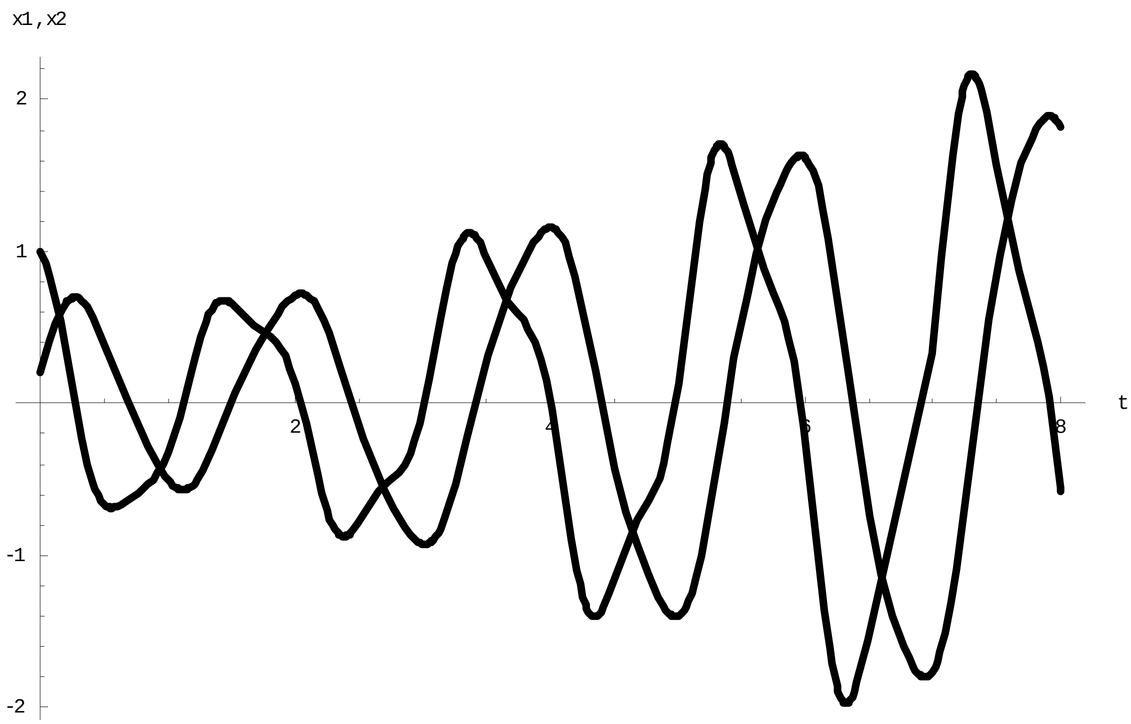


Figure 10: The system states of the uncontrolled system

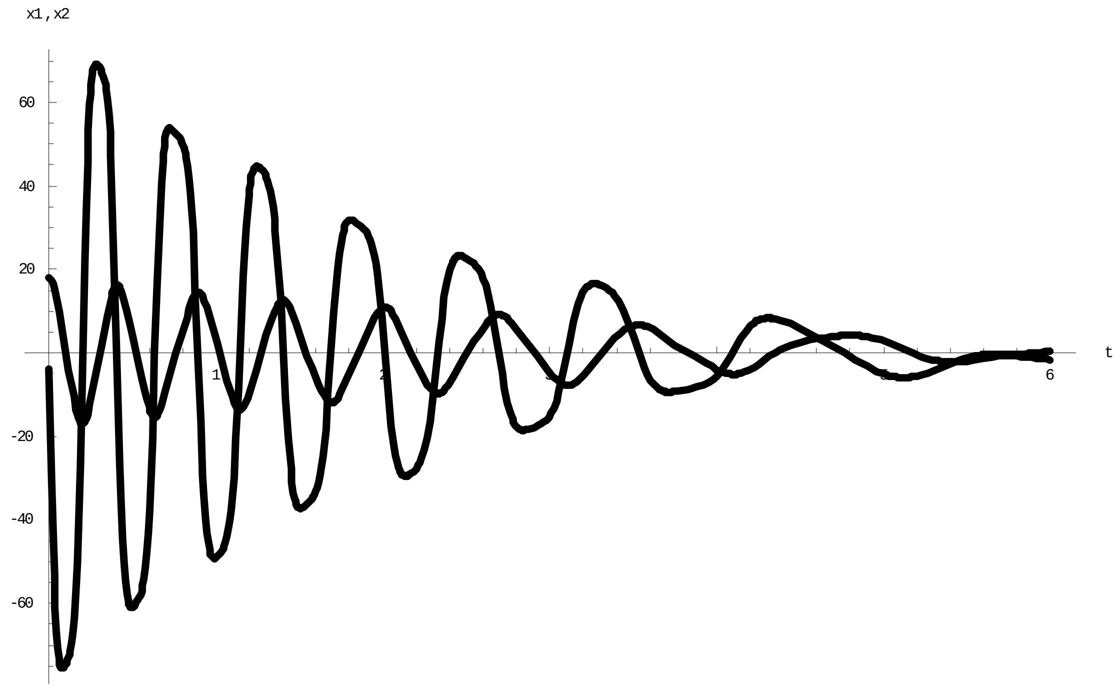


Figure 11: System states of the controlled system

4.2.2 Approximate Feedback Linearization via the Extended Normal Form Theory

In this section we propose to find a coordinate transformation and state feedback that can convert the original nonlinear time-periodic system into a dynamically equivalent linear time-periodic system. Assuming this is possible, the control of the linear time periodic system can be designed by the symbolic control method.

Consider the affine nonlinear time periodic system

$$\dot{z} = f(z,t) + g(z,t)u \tag{0.100}$$

where, $z \in R^n$, $f(z,t)$ is a n dimensional vector field, $g(z,t)$ is a $n \times p$ function matrix, u is a p dimensional vector, $f(0,t) = g(0,t) = 0$, $f(z,t) = f(z,t+T)$, $g(z,t) = g(z,t+T)$.

Assuming that $f(z,t)$, $g(z,t)$ are smooth, we design a controller for the above system desired to be driven to a periodic orbit $\overline{z}(t) = \overline{z}(T+t)$ or to a fixed point.

After expanding the closed-loop system into Taylor series around the fixed point z_0 (or a time periodic orbit $\overline{z}(t)$), we have

$$\dot{x} = A(t)x + f_2(x,t) + f_3(x,t) + ... + f_r(x,t) + ... + (g_0(t) + g_1(x,t) + ... g_{r-1}(x,t) + ...)u$$

(0.101)

where f_2, f_3 and f_r are quadratic, cubic and r^{th} order monomial forms of x with T-periodic coefficients, respectively, and x is the perturbation variable. Equation (4.10) defines the local control problem of stabilization of z_0 or the periodic orbit $\overline{z}(t)$.

Now applying the Lyapunov-Floquet transformation $L(t)$ and modal transformation M via $x = ML(t)\overline{x}$, we have

$$\dot{\overline{x}} = J\overline{x} + \overline{f}_2(\overline{x},t) + \overline{f}_3(\overline{x},t) + ... + \overline{f}_r(\overline{x},t) + ... + (\overline{g}_0(t) + \overline{g}_1(\overline{x},t) + ... \overline{g}_{r-1}(\overline{x},t) + ...)u \quad (0.102)$$

where J is the Jordan canonical form of a time invariant matrix.

At this stage, the idea is to use a sequence of transformations and state feedback to convert equation (4.10) into a linear control system step by step. It is important to note that at each step, a near identity transformation of order r does not affect any lower order nonlinear terms. In the following, we show this by considering the effect of an r^{th} order near identity transformation on equation (4.10).

We denote the eigenvalues of J as λ_j, $j = 1,...,n$. These are the Floquet exponents of the system and we assume that these are not critical, i.e., the system is hyperbolic. At this stage we define a set of r^{th} order near identity transformations by

$$\begin{cases} \overline{x} = y + \varphi_r(y,t) \\ u = v + \alpha_r(x,t) + \beta_{r-1}(x,t)v \end{cases} \quad (0.103)$$

where φ_r and α_r are the r^{th} order monomial forms of y with T -periodic coefficients and β_{r-1} is the $(r-1)^{th}$ order monomial forms of y .

Proposition 2: The system

$$\dot{x} = J\overline{x} + \overline{f}_2(\overline{x},t) + \overline{f}_3(\overline{x},t) + \ldots + \overline{f}_r(\overline{x},t) + \ldots + (\overline{g}_0(t) + \overline{g}_1(\overline{x},t) + \ldots \overline{g}_{r-1}(\overline{x},t) + \ldots)u$$

can be transformed into

$$\dot{y} = Jy + \overline{f}_2(y,t) + \ldots + \overline{f}_{r-1}(y,t) + O(|y|^{r+1},t) + (\overline{g}_0(t) + \overline{g}_1(y,t) + \ldots + \overline{g}_{r-2}(y,t) + O(|y|^r,t))v$$

by an r^{th} order near identity transformation if and only if there exist a set of φ_r , α_r and β_{r-1} that satisfy the following equations

$$\frac{\partial \varphi_r(y,t)}{\partial t} + [Jy, \varphi_r(y,t)] = \overline{f}_r(y,t) + \overline{g}_0(t)\alpha_r(y,t) \qquad (0.104)$$

$$\frac{\partial \varphi_r(y,t)}{\partial y}\overline{g}_0(t) = \overline{g}_0(t)\beta_{r-1}(y,t) + \overline{g}_{r-1}(y) \qquad (0.105)$$

where [ ] denotes the Lie bracket of two vector fields $d_1(x,t)$ and $d_2(x,t)$ defined by

$$[\ ] \equiv \frac{\partial d_2(x,t)}{\partial x} d_1 - \frac{\partial d_1(x,t)}{\partial x} d_2 \ .$$

Proof:

Substituting (4.11) into the system equation (4.10) and collecting the terms up to r^{th} order, we have

$$\dot{y} = Jy + \overline{f}_2(y,t) + ... + \overline{f}_{r-1}(y,t) - \frac{\partial \varphi_r(y,t)}{\partial t} - [Jy, \varphi_r(y,t)] + \overline{g}_0(t)\alpha_r(y,t) + \overline{f}_r(y,t)$$
$$+(\overline{g}_0(t) + \overline{g}_1(y,t) + ... + \overline{g}_{r-2}(y,t))v +$$
$$(-\frac{\partial \varphi_r(y,t)}{\partial y}\overline{g}_0(t) + \overline{g}_0(t)\beta_{r-1}(y,t) + \overline{g}_{r-1}(y))v + O(|y|,|v|)^{r+1}$$

(0.106)

Proposition 2 can be deduced immediately if the r^{th} order terms are to be eliminated.

At this point, consider the linear space with a basis

$\{e^{ik\frac{2\pi}{T}} y^{j_1}{}_1 y^{j_2}{}_2 ... y^{j_n}{}_n \mid j_1 + j_2 + ... j_n = r, k = 0, \pm 1, \pm 2, ...\}$ .Then $\varphi_r(y,t)$, $\alpha_r(y,t)$ can be expressed as [27]

$$\varphi_r(y,t) = \sum\sum \varphi_{k,(j_1 \cdots j_n)} e^{ik\frac{2\pi}{T}t} y_1^{j_1} y_2^{j_2} ... y_n^{j_n}$$

$$\alpha_r(y,t) = \sum\sum \alpha_{k,(j_1 \cdots j_n)} e^{ik\frac{2\pi}{T}t} y_1^{j_1} y_2^{j_2} ... y_n^{j_n}$$

Also consider the linear space with a basis

$\{e^{ik\frac{2\pi}{T}} y^{\overline{j}_1}{}_1 y^{\overline{j}_2}{}_2 ... y^{\overline{j}_n}{}_n \mid \overline{j}_1 + \overline{j}_2 + ... \overline{j}_n = r-1, k = 0, \pm 1, \pm 2, ...\}$

and express $\beta_{r-1}(y,t)$ as

$$\beta_{r-1}(y,t) = \sum\sum \beta_{k,(j_1 \cdots j_n)} e^{ik\frac{2\pi}{T}t} y_1^{\overline{j}_1} y_2^{\overline{j}_2} ... y_n^{\overline{j}_n}$$

$\overline{f}_r(y,t)$ and $\overline{g}_{r-1}(y)$, $\overline{g}_0(t)$ can also be expressed in similar monomials with periodic coefficients. Then equation (4.12), (4.13) and (4.14) yields a set of linear algebraic equations given by

$$(ik\frac{\pi}{T} + \sum_{p=1}^{n} m_p \lambda_p - \lambda_j)\varphi_{k,(j_1 \cdots j_n)} = f_{k,(j_1 \cdots j_n)} - (\sum g_{0,k}\alpha_{l,(j_1 \cdots j_n)}) \qquad (0.107)$$

$$\sum ik\frac{\pi}{T}\varphi_{k,(j_1\cdots j_n)}g_{0,l}=(\sum g_{0,m}\alpha_{l,(j_1\cdots j_n)})+g_{r-1,l} \tag{0.108}$$

Depending upon the number of Fourier coefficients, the degree of the nonlinearity and the system dimension, one may obtain a large set of underdetermined equations. It is observed that such a system may have no solution or has an infinite number of solutions. Assuming that a solution exists, we use standard software like MATHEMATICA to find a solution. It is noticed that the left side of equation of (4.16) and (4.17) is similar to the "homological equation" found in the time-dependent normal form theory. Even though we have assumed the system to be hyperbolic, in certain resonance cases, we may be able to find sufficient number of α_r that could cancel the resonance terms via feedback. This will be investigated in the future.

As shown, an r^{th} order near identity transformation does not affect terms of order less than *r* and therefore, we can use a sequence of transformations to eliminate all the nonlinearities step by step. First, we apply a quadratic transformation, then a cubic transformation (which will not produce new quadratic nonlinearity), and so on. The original system can be linearized up to r^{th} order if and only if all set of equations (4.16), (4.17) obtained for $k=2,3,......r$ are solvable. Finally, we arrived at

$$\dot{y}=A(t)y+(g_0(t))v+O(y,v)^{r+1} \tag{0.109}$$

Then a controller can be designed for this system following the approach in last chapter. We use the following examples to demonstrate the control design method via linearization process.

Example 6:

Consider the well-known Mathieu-equation with cubic nonlinearity with a scalar control u.

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & \pi \\ -\pi(a + bCos(2\pi t)) & -\pi d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{3} x_1^{\ 3} \end{Bmatrix} + \begin{Bmatrix} u \\ 0 \end{Bmatrix} \tag{0.110}$$

For $a = 1, b = 4, d = 0.3$, the Floquet multipliers are (-0.058) and (-6.7) and thus the system exhibits unbounded behavior.

It should be observed that due to the structure of the closed loop system, one cannot select u to cancel out the nonlinearity directly, as suggested in the literature by some authors.

After applying the 2T Lyapunov-Floquet transformation ($L(t)$) and modal (M) transformation to equation (4.19), we obtain

$$\begin{Bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} + \begin{Bmatrix} \bar{f}_3^{\,1}(\bar{x}_1, t) \\ \bar{f}_3^{\,2}(\bar{x}_2, t) \end{Bmatrix} + M^{-1} L(t)^{-1} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u \tag{0.111}$$

where $\lambda_1 = -2.85, \lambda_2 = 1.91$. Depending upon the accuracy desired, only a finite number of terms are kept in the Fourier representation of $L(t)$. For the problem under consideration, 11 terms were sufficient to provide a reasonable accuracy. Consider the basis

$$\{e^{ik\frac{\pi}{T}t} y_1^{j_1} y_2^{j_2} \mid j_1 + j_2 = 3, \quad k = 0, \pm 1, \pm 2, .. \pm 11\} \tag{0.112}$$

and

$$\{e^{ik\frac{\pi}{T}t} y_1^{j_1} y_2^{j_2} \mid j_1 + j_2 = 2, \quad k = 0, \pm 1, \pm 2, .. \pm 11\} \tag{0.113}$$

Now we define a pair of near identity transformations

$$\begin{cases} \bar{x} = y + \varphi_3(y,t) \\ u = v + \alpha_3(x,t) + \beta_2(x,t)v \end{cases} \quad (0.114)$$

Then equation (4.16), (4.17) takes the form

$$\frac{\partial \varphi_3(y,t)}{\partial t} + [Jy, \varphi_3(y,t)] = \bar{f}_3(y,t) + \bar{g}_0(t)\alpha_3(y,t) \quad (0.115)$$

$$\frac{\partial \varphi_3(y,t)}{\partial y}\bar{g}_0(t) + \bar{g}_0(t)\beta_2(y,t) = 0 \quad (0.116)$$

where $\bar{g}_0(t) = M^{-1}L(t)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We express the equation (4.24) in terms of the basis (4.21) and the equation (4.25) in terms of basis (4.22) and obtain a set of linear algebraic equations where the unknown variables are the coefficients of φ_3, α_3 and β_2. φ_3 has $2\times 4\times 23$ unknown coefficients. α_3 has 4×23 coefficients. β_2 has 3×23 unknown coefficients. In total, we have $2\times 4\times 23 + 4\times 23 + 3\times 23$ unknown coefficients, however φ_3 and α_3 are related.

Using the subroutine '*LinearSolve*' in MATHEMATICA, we find a set of solution. In the following, we record expressions for α_3 and β_2 only, for brevity,

$$\alpha_3(\bar{x},t) = -0.3021x_1^3 Cos(\pi t) - 0.099x_1^2 x_2 Cos(\pi t) - 0.02769x_1 x_2^2 Cos(\pi t)$$
$$-0.191836x_2^3 Cos(\pi t) + \ldots$$

$$\beta_2(\bar{x},t) = -0.0402694x_1^2 + 0.0109554x_1 x_2 - 0.0257654x_2^2 +$$
$$0.101718x_1^2 Cos(2\pi t) + \ldots\ldots$$

Thus the system is transformed to

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + M^{-1} L(t)^{-1} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} v + O(y,v)^4 \qquad (0.117)$$

Then by using the symbolic design method, as discussed earlier, a control v can be designed in the y domain. The following graphs show the states of the uncontrolled and the controlled systems.

It should be observed that the methodology used here is more general widely applicable in all values of system parameters which is not the case of the approach presented in last example.

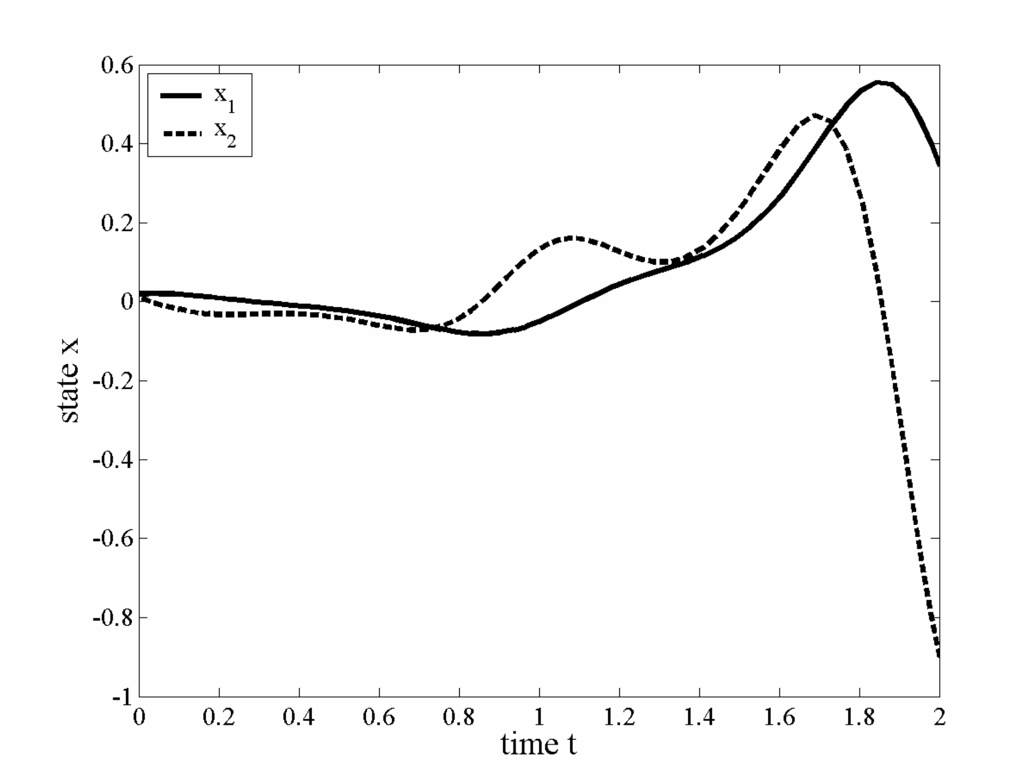


Figure 12: The uncontrolled System

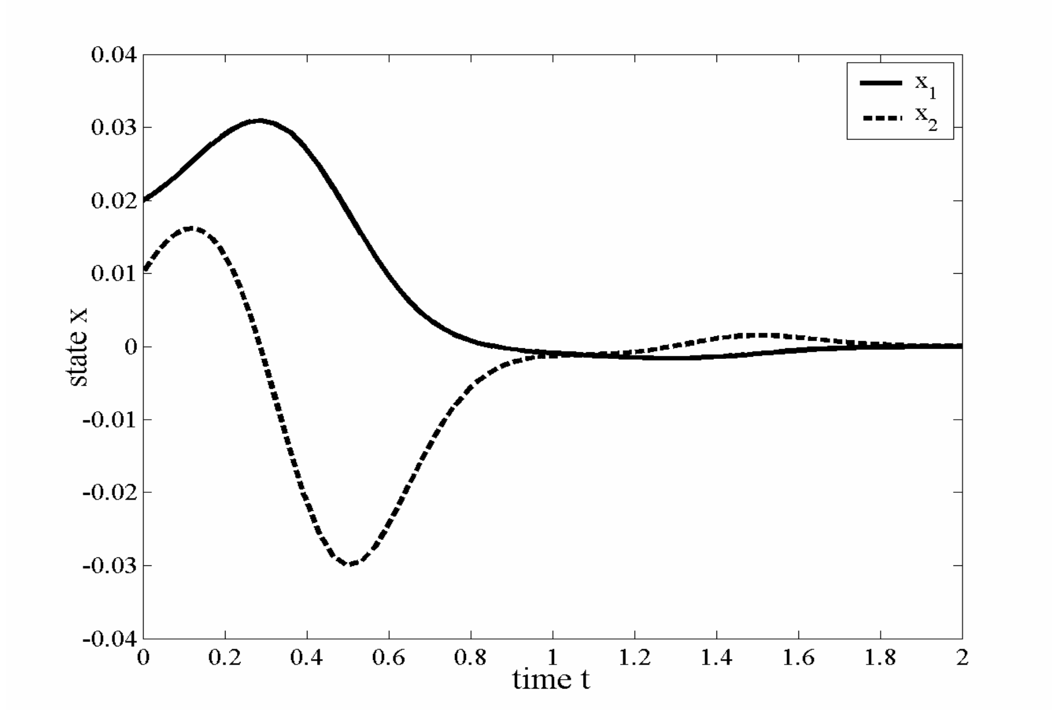


Figure 13: The controlled system

Example 7:

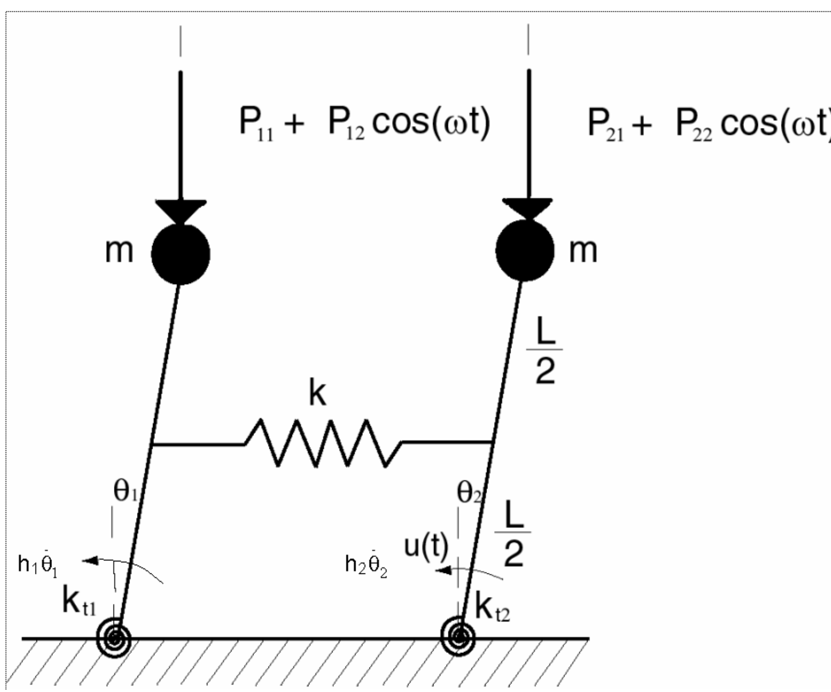


Figure 14: Coupled pendulums

In this example, we design a nonlinear controller for a system consisting of two inverted pendulums moving in the horizontal plane with time-dependent load acting on each of the pendulums. The structural diagram of the system considered is shown in Figure 14. The equations of motion can be shown to be

$$\begin{aligned} ml^2\ddot{\theta}_1 + h_1\dot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}q_1(\theta_1,\theta_2) - P_1(t)l\sin\theta_1 = 0 \\ ml^2\ddot{\theta}_2 + h_2\dot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}q_2(\theta_1,\theta_2) - P_2(t)l\sin\theta_2 = 0 \end{aligned} \quad (0.118)$$

where $q_1(\theta_1,\theta_2)$ and $q_2(\theta_1,\theta_2)$ are nonlinear functions of $(\theta_1 - \theta_2)$, $P_1(t) = P_{11} + P_{12}\cos(\omega t)$ and $P_2(t) = P_{21} + P_{22}\cos(\omega t)$. m, l, k_{ti} and k denote mass, length, torsional stiffness and coupling stiffness, respectively. h_1, h_2 are the torsional damping constants. The local dynamics can be obtained by expanding these equations of motion about the fixed point $(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) = (0,0,0,0)$. The linearized equations are

$$\begin{aligned} ml^2\ddot{\theta}_1 + h_1\dot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}(\theta_1 - \theta_2) - P_1(t)l\theta_1 = 0 \\ ml^2\ddot{\theta}_2 + h_2\dot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}(\theta_1 - \theta_2) - P_2(t)l\theta_2 = 0 \end{aligned} \quad (0.119)$$

On the other hand, if terms up to the cubic order are retained then these equations may be approximated as

$$\begin{aligned} \frac{\ddot{\theta}_1}{ml^2} + \frac{h_1}{ml^2}\dot{\theta}_1 + \frac{k_{t1}}{ml^2}\theta_1 + \frac{k}{4m}[c_1(\theta_1-\theta_2) + c_2(\theta_1-\theta_2)^2 + c_3(\theta_1-\theta_2)^3] - \frac{P_1(t)}{ml}(\theta_1 - \frac{\theta_1^3}{6}) = 0 \\ \frac{\ddot{\theta}_2}{ml^2} + \frac{h_2}{ml^2}\dot{\theta}_2 + \frac{k_{t2}}{ml^2}\theta_2 + \frac{k}{4m}[c_1(\theta_2-\theta_1) + c_2(\theta_2-\theta_1)^2 + c_3(\theta_2-\theta_1)^3] - \frac{P_2(t)}{ml}(\theta_2 - \frac{\theta_2^3}{6}) = 0 \end{aligned}$$

(0.120)

Setting P_{11} and P_{21} equal to zero, the equations can be written as

$$\begin{aligned}
&\ddot{\theta}_1+\bar{h}_1\dot{\theta}_1+(\omega_{n_1}^2-\varepsilon p_1\cos(\omega t))\theta_1+\varepsilon p_1\frac{\theta_1^3}{6}-b\theta_2-c(\theta_1-\theta_2)^2-d(\theta_1-\theta_2)^3=0\\
&\ddot{\theta}_2+\bar{h}_2\dot{\theta}_2+(\omega_{n_2}^2-\varepsilon p_2\cos(\omega t))\theta_2+\varepsilon p_2\frac{\theta_2^3}{6}-b\theta_1-c(\theta_1-\theta_2)^2-d(\theta_1-\theta_2)^3=0
\end{aligned} \tag{0.121}$$

where

$$\omega_{n_1}^2=[\frac{k_{t1}}{ml^2}+\frac{k}{4m}c_1],\omega_{n_2}^2=[\frac{k_{t2}}{ml^2}+\frac{k}{4m}c_1],\varepsilon p_1=\frac{P_{12}l}{ml^2},\varepsilon p_2=\frac{P_{22}l}{ml^2},\omega=2\pi$$

$$b=\frac{k}{4m}c_1,c=\frac{k}{4m}c_2,d=\frac{k}{4m}c_3,\bar{h}_1=\frac{h_1}{ml^2},\bar{h}_2=\frac{h_2}{ml^2}$$

Using the following typical parameter values:

$\omega_{n_1}^2=5,\omega_{n_2}^2=4,\varepsilon p_1=2.5,\varepsilon p_2=5,\omega=2\pi,b=2.5,c=0,d=1.5,\ \bar{h}_1=-3.2,\bar{h}_2=1.2$, the state space form of equation (4.27) is given by

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{Bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -(5-2.5\cos(2\pi t)) & 2.5 & -3.2 & 0\\ 2.5 & -(4-5\cos(2\pi t)) & 0 & 1.2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ -2.5\cos(2\pi t)\dfrac{x_1^3}{6}+1.5(x_1-x_2)^3\\ -2.5\cos(2\pi t)\dfrac{x_2^3}{6}-1.5(x_1-x_2)^3+u\end{Bmatrix} \tag{0.122}$$

where $\{x_1,x_2,x_3,x_4\}^T=\{\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2\}^T$.

After applying 2T L-F transformation ($L(t)$) and modal

(M) transformation $x = L(t)M\,\overline{x}$, we obtain

$$\begin{Bmatrix} \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \\ \dot{\overline{x}}_3 \\ \dot{\overline{x}}_4 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{Bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \\ \overline{x}_4 \end{Bmatrix} + \begin{Bmatrix} \overline{f}_3^{\,1}(\overline{x}_1, t) \\ \overline{f}_3^{\,2}(\overline{x}_1, t) \\ \overline{f}_3^{\,3}(\overline{x}_1, t) \\ \overline{f}_3^{\,4}(\overline{x}_1, t) \end{Bmatrix} + M^{-1}L(t)^{-1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} u \qquad (0.123)$$

where $\lambda_1 = 0.41 + 1.99i, \lambda_2 = 0.41 - 1.99i, \lambda_3 = -1.41 + 1.42i, \lambda_4 = -1.41 - 1.42i$. Depending upon the accuracy desired, only a finite number of terms are kept in the Fourier representation of $L(t)$. For the problem under consideration, 11 terms were sufficient to provide a reasonable accuracy. Here we use the same basis as given in the last example except $k = 0, \pm 1, ..., \pm 5$. We define a pair of near identity transformations that cancels the cubic terms

$$\begin{cases} \overline{x} = y + \varphi_3(y,t) \\ u = v + \alpha_3(x,t) + \beta_2(x,t)v \end{cases} \qquad (0.124)$$

then we have

$$\frac{\partial \varphi_3(y,t)}{\partial t} + [Jy, \varphi_3(y,t)] = \overline{f}_3(y,t) + \overline{g}_0(t)\alpha_3(y,t)$$
$$\frac{\partial \varphi_3(y,t)}{\partial y}\overline{g}_0(t) + \overline{g}_0(t)\beta_2(y,t) = \overline{g}_2(y)$$

We express the first equation in terms of the given basis and obtain a set of linear algebraic equations where the unknown variables are the coefficients of φ_3 , α_3 and β_2 . In total, we have $4 \times 4 \times 11 + 4 \times 11 + 3 \times 11$ unknown coefficients. Once again using the '*LinearSolve*' in MATHEMATICA, we find a set of solution. For brevity we record expressions for α_3 and β_2 only.

$$\alpha_3(\bar{x},t) = 9.32x_1^3Cos(\pi t) - 0.0029x_1^2x_2Cos(\pi t) - 1.0782x_1x_2^2Sin(\pi t) - 0.4632x_2^3Sin(\pi t) + \ldots$$

$$\beta_2(\bar{x},t) = -0.693x_1^2 + 0.615x_1x_2 - 1.83x_1^2Sin(\pi t) + 0.0176x_1x_2Cos(\pi t) + \ldots\ldots$$

Finally, the original system is transformed into

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} + M^{-1}L(t)^{-1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} v + O(y,v)^4 \qquad (0.125)$$

Then a nonlinear controller is designed as discussed before. The following graphs show the states of the uncontrolled and the controlled systems.

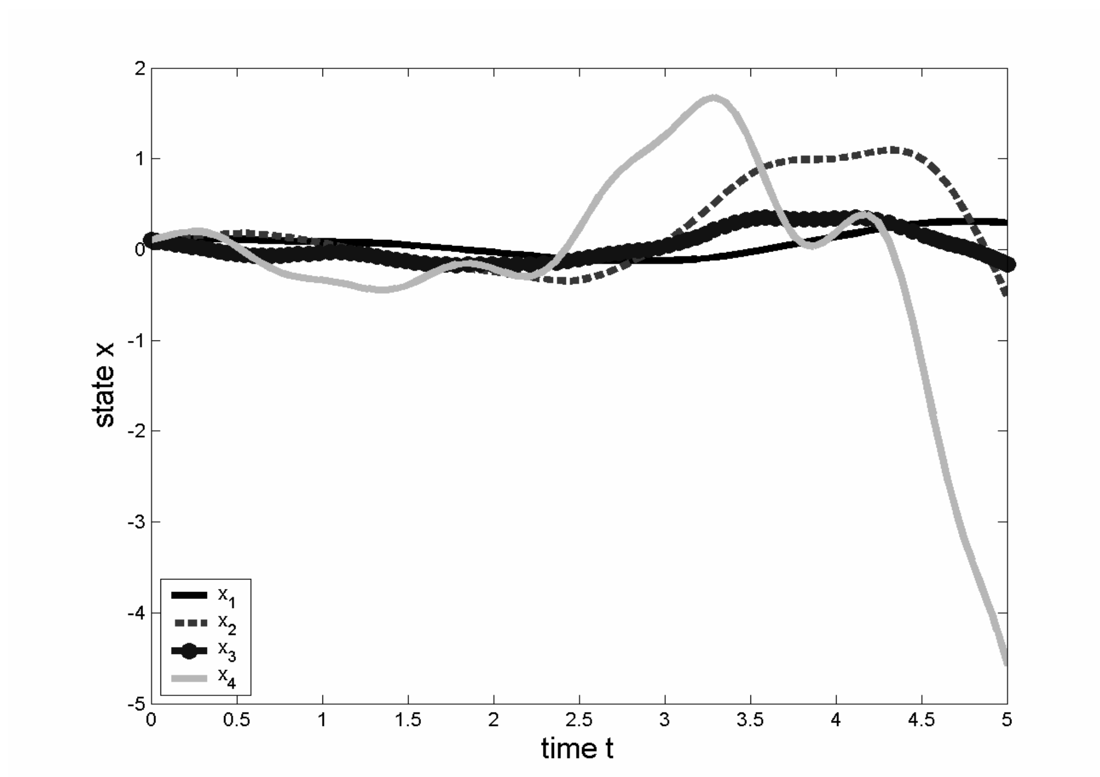


Figure 15: The uncontrolled System

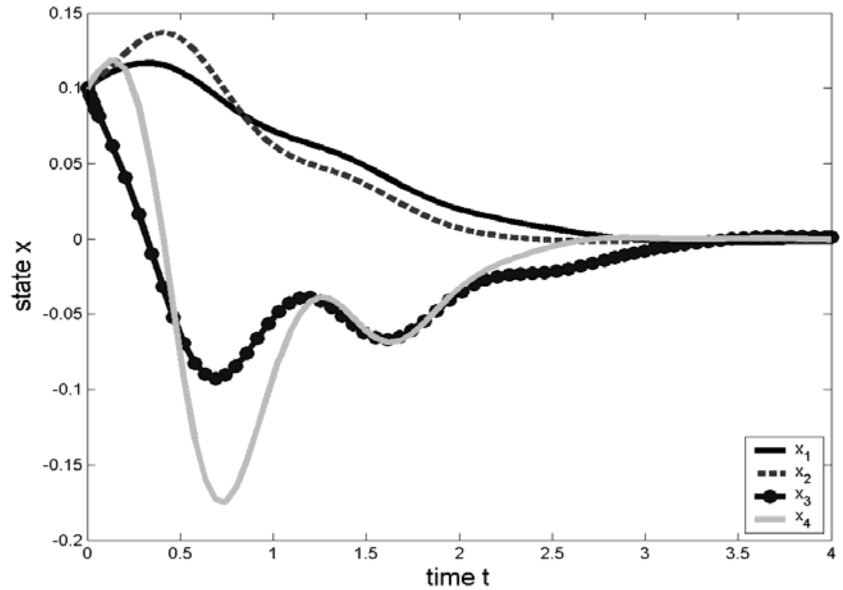


Figure 16: The controlled system

4.3 The Poincaré Normal form for Underdetermined Ordinary Differential Equations

In last section, we generalized the Poincaré normal from techniques to the nonlinear systems with controls. However, there exist some drawbacks in the control design algorithm given in last section. First, for designing a control in the domain of x, we design a local control in the domain of y and then transfer the control back to the x domain. The advantage of the design method is not very clear comparing with the classical local design method. Second, it is also easy to see that that algorithm is very demanding for computer speed and memory storage. For overcoming these difficulties and some other considerations, we develop the Poincaré normal form technique for the underdetermined ordinary differential equation. By employing this technique, we can draw (formally) the simplest form of an ordinary differential equation with parameters and unknown controllers in a systematic way. It is a powerful tool for control design as well as analysis of nonlinear systems. The feedback linearization problem studied in last section can be solved by this technique more effectively. We also outlined some other important applications of this technique.

4.3.1 The Poincaré Normal Form for Nonlinear Ordinary Differential Equations with Parameters and Controls

Consider the nonlinear T-time periodic equations with parameters and controls

$$\dot{x} = A(t)x + f(x, p, q, u, t) \tag{0.126}$$

where f is a T-time periodic vector-valued function , p, q denotes the parameters and u denotes the controls. f only includes nonlinear terms of x. We noticed that we can assume $A(t)$ is a constant matrix because we can compute the L-F transformation of (4.35) effectively by using the algorithm given earlier. After regrouping the terms of $f(x, p, q, u, t)$ we have

$$\dot{x} = Ax + f_1(x, p, t) + B(t)u + f_2(x, q, u, t) \tag{0.127}$$

where $f_1(x, p, t)$ includes monomials of x and $f_2(x, q, u, t)$ includes terms constituted by both x and u.

Now we aim to reduce the (4.59) into the simplest form. We first state the computation scheme and then explain its mechanics and why it is more effectively than the algorithms given in last section. Without losing generality, we assume that A is a constant matrix in diagonal form:

$$A = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \quad (0.128)$$

We also assume that

$$\sum_{i=1}^{n} \lambda_i m_i - \lambda_j + k\frac{2\pi}{T}\sqrt{-1} \neq 0,\ j = 1, 2, ..., n \quad (0.129)$$

are satisfied for any integers k and positive integers satisfying $\sum_{i=1}^{n} m \geq 1$.

The computation scheme for the Poincaré normal form of (4.35):

Step 1: Substitute all terms of $f_1(x, p, t)$ with a form of $a(p)\, x_1^{m_1} x_2^{m_2} ... x_n^{m_n} e^{k\frac{2\pi}{T}\sqrt{-1}}$ by

$$\frac{a(p)\, x_1^{m_1} x_2^{m_2} ... x_n^{m_n} e^{k\frac{2\pi}{T}\sqrt{-1}}}{\sum_{i=1}^{n} \lambda_i m_i - \lambda_j + k\frac{2\pi}{T}\sqrt{-1}} \quad (0.130)$$

, where $a(p)$ denotes the constant coefficient (may include parameters) of terms of $f_1(x, p, t)$. We denote the resulted function as $\varphi_1(x, p, t)$.

Step 2: Expand the function

$$f_3(x, p, t) = \frac{\partial \varphi_1(x, p, t)}{\partial x} f_1(x, p, t) \quad (0.131)$$

We noticed that all terms of $f_3(x,p,t)$ are constituted by x. Then we change all terms of $f_3(x,p,t)$ according to the formula (4.39). The resulted function is denoted by $\varphi_2(x,p,t)$.

Step 3: We construct $f_4, f_5, \ldots$ by using the formula

$$f_k(x,p,t) = \frac{\partial \varphi_{k-2}(x,p,t)}{\partial x} f_1(x,p,t) \tag{0.132}$$

In every steps, $\varphi_r(x,p,t)$ is obtained through applying the substitution operation for $f_{r+1}(x,p,t)$.

Step 4: The system (4.35) is transformed into

$$\dot{y} = Ay + B(t)u + f_2(x,q,u,t) \tag{0.133}$$

by the coordinate transformation

$$y = \phi(x,p,t) = x + \varphi_1(x,p,t) + \varphi_2(x,p,t) + \ldots + \varphi_N(x,p,t) \tag{0.134}$$

Step 5: We compute the inverse function of $\phi(x,p,t)$ and substitute it into (4.43).

Then we have

$$\dot{y} = Ay + B(t)u + R_2(y,p,q,u,t) + \textit{higher order terms} \tag{0.135}$$

where

$$R_2(y,p,q,u,t) = f_2(\phi^{-1}, q, u, t) \tag{0.136}$$

The mechanism of the computation scheme is similarly with the classical Poincaré normal form technique. In fact, if we define an operator L as following:

$$L\varphi := \frac{\partial \varphi}{\partial x} Ax + \frac{\partial \varphi}{\partial t} - A\varphi \tag{0.137}$$

then we have

$$
\begin{aligned}
L\varphi_1 &= f_1 \\
L\varphi_2 &= \frac{\partial \varphi_1}{\partial x} f_1 \\
L\varphi_3 &= \frac{\partial \varphi_2}{\partial x} f_1 \\
&\cdots\cdots \\
L\varphi_N &= \frac{\partial \varphi_{N-1}}{\partial x} f_1
\end{aligned} \tag{0.138}
$$

Therefore

$$L\phi = f_1 + \frac{\partial \phi}{\partial x} f_1 - \frac{\partial \varphi_N}{\partial x} f_1 \tag{0.139}$$

We notice that $\varphi_N(x, p, t)$ only includes terms with higher order terms. In the above algorithm, for obtaining $\phi(x, p, t)$, we only need compute the production off Taylor-Fourier series for $N-1$ times. To obtain (4.44), we need compute the inversion and the composition of formal Taylor-Fourier series for only 1 time.

To further improve the computing speed, we develop a fast algorithm for the inversion of a formal power series. For brevity, we state the algorithm for formal power series with only one variables. The inversion of multi-variable Taylor-Fourier series can be obtained by using the same algorithm without difficulty.

Let $X(x) = x + a_1 x^2 + a_2 x^3 + ... + a_{N-1} x^N$, our object is to find a function defined by $Y(y) = y + b_1 y^2 + b_2 y^3 + ... + b_{N-1} y^N$ such that

$$X \circ Y(y) \equiv y \pmod{y^N} \tag{0.140}$$

It is equivalent to solving the equation

$$X(x) - y = 0 \tag{0.141}$$

By using the Newton method, the solution of (4.73) can be obtained by using the following iteration relationships

$$\begin{aligned} &Y_1 = y \\ &Y_i = y - [(X'(Y_{i-1}))^{-1} \bmod y^N][(X(Y_{i-1}) - y) \bmod y^N] \end{aligned} \qquad (0.142)$$

The iteration ends when $Y_k = Y_{k+1} \bmod y^N$.

Let $D = I - X'(Y_{i-1})$, $[(X'(Y_{i-1}))^{-1} \bmod y^N]$ is computed by using the following formula

$$[X'(Y_{i-1})]^{-1} = (...(I + (I + (I + D)D)D) + ...) \qquad (0.143)$$

The mechanism of the above algorithm is same as the Newton method for find roots for polynomial equations.

We would like to point out that the technique developed above is very powerful. It can be viewed as a uniform framework as well as a systematic computational method for many differential problems. For example, for the feedback linearization of T-time periodic nonlinear systems, we can view the coefficients of $\alpha(x,t), \beta(x,t)$ in (4.35) as parameters q. Then the above computation scheme is applicable. If we can select proper parameters such that

$$R_2(y,q,u,t) \equiv 0 \bmod y^N \qquad (0.144)$$

then the system is feedback linearized to N^{th} order. On the other hand, in the algorithm given in last section, we need to compute the inversion and composition of Taylor-Fourier series for many times. Obviously, it is much slower than the algorithm here.

4.3.2 Suboptimal Control Design of Nonlinear Systems

In this section, we outline a possible methodology for designing the suboptimal control of nonlinear systems by using the techniques given in last two sections.

Usually, the nonlinear optimal control problem's exact solution is characterized a set of partial differential equation (the Hamilton-Jacoby-Bellman equation). However, even with the aids of computer, it is extremely difficult to solve the HJB equation if possible. Consequently, people often apply the linear optimal control method to the linear parts of nonlinear systems to obtain a linear feedback control. It is clearly that the control obtained this way has serious restrictions because it does not employ systems' nonlinear characters at all. Therefore, some authors tried to design an extra nonlinear or time varying control near the linear control to further improve the system performance. Especially, it is found that a small time periodic signal $\varepsilon \sin t$ may improve the system performance significantly. The pioneer work of this direction in the chemical engineer literature appeared in the late 1960's [28]. It used a variational approach to derive necessary conditions for optimality under small time periodic perturbation. After that, many theoretical and application results are reported [29-31]. However, as my best knowledge, most of the efforts are about optimal control over the finite horizon and authors only considered systems' stead state response. Furthermore, because the high order variational calculus is very complicated, people usually use the first approximation to select the controller. By using the normal form computation method given earlier, we can develop a systematic methodology for the suboptimal control design problem.

The contributions are two folds. First, the algorithm we given here is applicable for the finite time horizon optimal control problem as well as the infinite time horizon time periodic optimal control problems. Second, the methodology is also applicable for nonlinear time invariant systems. For the nonlinear time invariant system, many authors have investigated the problem of improving system performance by using sinusoidal forces. Their results are all based on the Pontryagin's minimum principle and variational calculus. Due to the computation difficulty of high order variational calculus formula, most of works only deal with second order variational derivative of the system performance index. There is no practical and systematic method to attack the more general problem. Our method is based on the normal form technique discussed earlier. By the aids of the symbolic computation tools, we gave a systematic way to solve this problem.

Consider the optimal control problem

$$\dot{x}(t) = A(t)x(t) + f(x,t) + B(t)u \tag{0.145}$$

where $A(t), B(t)$ are T-time periodic matrix and $f(x,t)$ is T-time periodic vector-valued function which includes high order terms. The performance index to be minimized is of the form

$$J(u) = \int_0^{+\infty} [x(t)^T Qx + u^T Ru]dt \tag{0.146}$$

where Q and R are positive definite weighting matrices. Our strategy is to search the local optimal control near the controller design from the related linear optimal control problem. Consider the linear optimal problem

$$\min J(u(t)) = \int_0^{+\infty} [x(t)^T Qx + u^T Ru]dt \tag{0.147}$$

subjected to

$$\dot{x}(t) = A(t)x(t) + B(t)u \tag{0.148}$$

The solution of this linear time periodic optimal control problem is characterized by the T-time periodic Riccati equation or a linear $2n$ dimension Hamilton equation. This Hamilton equation can be solved by using the Chebyshev spectral method we developed earlier. Let $u_0 = K(t)x(t)$ be the solution of this linear optimal control problem. Let

$$u = u_0 + \delta u \tag{0.149}$$

Then we have

$$\dot{x}(t) = \bar{A}(t)x(t) + f(x,t) + B(t)\,\delta u \tag{0.150}$$

where $\bar{A}(t) = A(t) + B(t)K(t)$ is known matrix. Because δu only includes high order terms of $x(t)$, we parameterized δu as

$$\delta u = \sum_{j,m} \varepsilon_{j,m}\, x^m \exp(i\frac{2\pi}{T} jt) \tag{0.151}$$

where $\varepsilon_{j,m}$ are unknown constant control gains. Let ε denotes all $\varepsilon_{j,m}$ and consider the companion system

$$\begin{cases} \dot{x}(t) = \bar{A}(t)x(t) + f(x,t) + B(t)[\sum_{j,m} \varepsilon_{j,m}\, x^m \exp(i\frac{2\pi}{T} jt)] \\ \dot{\varepsilon} = 0 \end{cases} \tag{0.152}$$

Therefore, if we view $[x_1, x_2, ..., x_n, \varepsilon]^T$ as independent variables, (4.59) can be transferred into a linear system

$$\begin{cases} \dot{\tilde{x}}(t) = S\,\tilde{x}(t) \\ \dot{E} = 0 \end{cases} \tag{0.153}$$

where S is a constant matrix. The transformation between x and $\tilde{x}$ is given by

$$x = L^{-1}(t)\tilde{x} + p_0(\varepsilon_{j,m}, \tilde{x}) + p_1(\varepsilon_{j,m}, \tilde{x})\exp(i\frac{2\pi}{T}t) + p_{-1}(\varepsilon_{j,m}, \tilde{x})\exp(-i\frac{2\pi}{T}t) + ... \quad (0.154)$$

where $L(t)$ is the Lyapunov-Floquet transformation matrix.

In (4.63), $p_0(\varepsilon_{j,m}, x), p_1(\varepsilon_{j,m}, x), p_2(\varepsilon_{j,m}, x), ...$ are high order monomials which can be obtained by the normal form technique we have discussed earlier.

Because the solution of (4.61) can be expressed in terms of the basis

$$\exp(s_j t), t\exp(s_j t), ..., t^{n-1}\exp(s_j t), \; j = 1, 2, ..., n \quad (0.155)$$

where s_j are eigenvalues of S. Therefore, the solution of (4.61) can be expressed in terms of

$$\{\exp(s_j t), t\exp(s_j t), ..., t^{n-1}\exp(s_j t)...\} \times \{\varepsilon, \varepsilon^2, ...\} \times \{\exp(i\frac{2\pi}{T}jt), \; j = 0, \pm 1, ...\} \quad (0.156)$$

Thus we can write the solution of (4.61) as

$$x(t) = \sum_{j,k} r_{(j,k)}(\varepsilon) t^j \exp[(\alpha_k + \beta_k i)t] \quad (0.157)$$

where $r_{(j,k)}(\varepsilon)$ are monomials of ε.

By substituting (4.65), (4.66) and $u_0 = K(t)x(t)$ into $x(t)^T Qx + u^T Ru$, we can see that $x(t)^T Qx + u^T Ru$ can be express as $\sum_{j,k} \overline{r}_{(j,k)}(\varepsilon) t^j \exp[(\overline{\alpha}_k + \overline{\beta}_k i)t]$. We notice that

$$\int t^j \exp[(\overline{\alpha}_k + \overline{\beta}_k i)t]dt = \frac{t^j \exp[(\overline{\alpha}_k + \overline{\beta}_k i)t]}{\overline{\alpha}_k + \overline{\beta}_k i} - \frac{j}{\overline{\alpha}_k + \overline{\beta}_k i}\int t^{j-1}\exp[(\overline{\alpha}_k + \overline{\beta}_k i)t]\, dt$$

(0.158)

Therefore, every terms of $x(t)^T Qx + u^T Ru$ can be **integrated out explicitly**. Therefore the performance index can be expressed as monomials of ε explicitly

$$J = J_0 + \frac{\varepsilon^2}{2} J_2 + \frac{\varepsilon^4}{4!} J_4 + ... \qquad (0.159)$$

where $J_0 = J(u_0)$. (Because J is in the quadratic form, only terms with even order of ε appear). Now the nonlinear optimal control problem is transferred into a function optimal problem. By using numerical method, we can select the proper control gains ε. The total controller is given by

$$u = K(t)x(t) + \sum_{j,m} \varepsilon_{j,m} x^m \exp(i \frac{2\pi}{T} jt) \qquad (0.160)$$

We note that our method have more advantages than the classical variational calculus approach. For example, the J_2 in (4.68) is corresponding to the "Pi criteria" given in [30]. Obviously our method is more effective because we also consider high order terms.

5. OBSERVER DESIGN FOR NONLINEAR TIME PERIODIC SYSTEMS

5.1 Introduction

Generally speaking, it is not possible to obtain all system states of a complex dynamics system by using a direct measurement approach. Therefore, there is a definite need for a reliable estimation of the immeasurable variables, especially when they are used for model-based controller design or for process monitoring. For this purpose, a state observer is usually employed in order to reconstruct the immeasurable system states. The idea is to construct an auxiliary system such that the error dynamics between the original and auxiliary systems goes to zero as time approaches infinity. In the case of linear time invariant systems, the Luenberger observer theory offers a complete answer to this problem [32]. In the field of nonlinear time-invariant systems, numerous attempts have been made in the development of the nonlinear observer design methods. Some approaches are based on local linearization techniques which are extensions of popular Kalman filter [33] and the Luenberg observer. Many authors have investigated the observer design by exploiting the Lipschitz property of nonlinear systems [34-38]. Another approach is to design the observer by using the coordinate transformation technique. In [39], Krener and Isidori considered the problem of synthesis of observers yielding error dynamics that are linear in transformed coordinates. Kazantis and Kravaris

[40] designed the observers by using the Lyapunov auxiliary theorem. In [41, 42], Krener and Xiao developed an observer design method for systems in the Siegel domain. There are other notable works on this topic as well. For a detailed survey of the observer design, the interested reader is referred to the paper by Krener [43]. On the other hand, the observer design problem associated with time periodic systems is quite challenging and has not been addressed as frequently as its time-invariant counterpart. Sinha and Joseph [2] have proposed a methodology in the design of observers for linear time periodic system. However, this method uses a least square approximation approach and in certain parameter range, the convergence may not be guaranteed. As already shown in [2], the observer design problem for the time periodic system can be translated into the controller design for the dual feedback control system. In this work, we propose an observer design method for the linear time periodic system by using the symbolic design method presented in previous chapters. For nonlinear time periodic systems, we propose a method of observer design by using the time dependent Poincaré normal form. We aim to design an auxiliary nonlinear time periodic system forced by the output of the original system such that the errors between it and the original system goes to zero as time approaches infinity. First, we design an observer for the linear part of the system by using the method mentioned above. Second, we propose an auxiliary linear periodic system whose input is the output of the original system. The original n dimension system together with the auxiliary linear system constitutes a $2n$ dimension state space. In that space, we prove the existence and regularity of an invariant manifold by using the Poincaré normal form theory. The existence of the invariant manifold implies that the auxiliary system is an observer of the original system. Then the Taylor-Fourier series

expression of the invariant manifold is obtained with the aid of the symbolic software **MATHEMATICA** and the identity observer is designed.

5.2 Observer Design for the Linear Time Periodic Systems

Consider the linear time periodic system with n state variables and m outputs

$$\begin{aligned} \dot{x} &= A(t)x \\ \dot{y} &= B(t)y \end{aligned} \tag{0.161}$$

where $A(t), B(t)$ are T-time periodic matrices of dimensions $n \times n$, $m \times n$, respectively. The system model is known but the initial condition $x(0)$ is unknown all $x_i(t)$ can not be measured directly. Therefore, all information of system states can only be obtained through $y(t)$ which is measurable. In order to achieve this task, we construct a new dynamic system with $y(t)$ as its input such that the error between it and (5.1) goes to zero as time approaches infinity. Then observer design problem for the linear time periodic system is defined as in the following.

Definition 4: A linear time periodic system

$$\dot{z} = C(t)z + D(t)y \tag{0.162}$$

is called an observer for (5.1), if there is an invertible periodic map $E(x,t)$ such that:

(i) if $z(0) = E(x(0),t)$ then $z(t) = E(x(t),t)$ $\forall t > 0$.

(ii) $\lim_{t \to \infty} (E(x(t),t) - z(t)) = 0$.

In particular when $E(x,t) = x$ (the identity map), (5.2) is called the identity observer.

Now our objective is to find proper $C(t)$ and $D(t)$ satisfying the above conditions. From (5.1) and (5.2) we have

$$d(x-z)/dt = \dot{x} - \dot{z} = A(t)x - C(t)z - D(t)B(t)x \tag{0.163}$$

Taking $e = x - z$ as the auxiliary variable, we hope to construct $D(t)$ such that $\lim_{t\to\infty} e(t) = 0$.

In (5.3), we let $C(t) = A(t) - D(t)B(t)$ to yield

$$\dot{e} = [A(t) - D(t)B(t)]e = C(t)e \tag{0.164}$$

Thus the observer design problem is reduced to the design of a periodic matrix $D(t)$ such that $C(t) = A(t) - D(t)B(t)$ is a Hurwitz matrix.

Consider the linear T-periodic systems

$$\dot{e} = (A(t) - D(t)B(t))e \tag{0.165}$$

and the adjoint system

$$\dot{\hat{e}} = -(A^T(t) - B^T(t)D^T(t))\hat{e} \tag{0.166}$$

If we let the FTM of (5.4) be $\alpha(T)$ and the FTM of (5.6) be $\beta(T)$, then we have the following relationship,

$$\alpha^{-1}(T) = \beta^T(T) \tag{0.167}$$

When $[-A^T(t), B^T(t)]$ is controllable, we can find a $D(t)$ such that all Floquet multipliers of (5.6) do not lies in the unit circle. Obviously, from (5.7), all the eigenvalues of $\alpha(T)$ lie in the unit circle of the complex plane when all the eigenvalues of $\beta(T)$ do not lie there. Therefore, it is possible to design a $D(t)$ such that $A(t) - D(t)B(t)$ is a Hurwitz matrix.

We summarize the observer design procedure as in the following

(1) Design a $D(t)$ such that $A(t)-D(t)B(t)$ is a Hurwitz matrix by using the symbolic controller design method.

(2) $\dot{z}=[A(t)-D(t)B(t)]z+D(t)y$ is the identity observer of (5.1). The error dynamics is given by $\dot{e}=(A(t)-D(t)B(t))e$.

Example 8:

As an example, we consider the Mathieu-equation with one output, given by

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & \pi \\ -\pi(a+bcos(2\pi t) & -\pi d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{0.168}$$

$$y=[0 \quad 1]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{0.169}$$

We Note that $A(t)=\begin{bmatrix} 0 & \pi \\ -\pi(a+bcos(2\pi t) & -\pi d \end{bmatrix}, B(t)=[0 \quad 1]$.

We select $a=1, b=4, d=0.3$ for numerical computation. The Floquet exponents are –2.9, 1.9 and thus the system is unstable. The first step is to seek a 2×1 matrix $D(t)=[d_1(t) \quad d_2(t)]^T$ such that $A(t)-D(t)B(t)$ is Hurwitz. Using the symbolic control design method, one of the simplest choice is $D(t)=[-0.3sin(2\pi t) \quad 7]^T$. Then the observer is given by

$$\begin{aligned} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} &= \begin{bmatrix} 0 & \pi-d_1(t) \\ -\pi(a+bcos(2\pi t) & -\pi d-d_2(t) \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} y \\ &= \begin{bmatrix} 0 & \pi+0.3sin(2\pi t) \\ -\pi(1+4cos(2\pi t) & -0.3\pi-7 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{bmatrix} -0.3sin(2\pi t) \\ 7 \end{bmatrix} y \end{aligned} \tag{0.170}$$

The Floquet exponents of the error dynamics are –6.87, -1.08.

System states and observer states are shown in Figure 17. The error dynamics is shown in Figure 18.

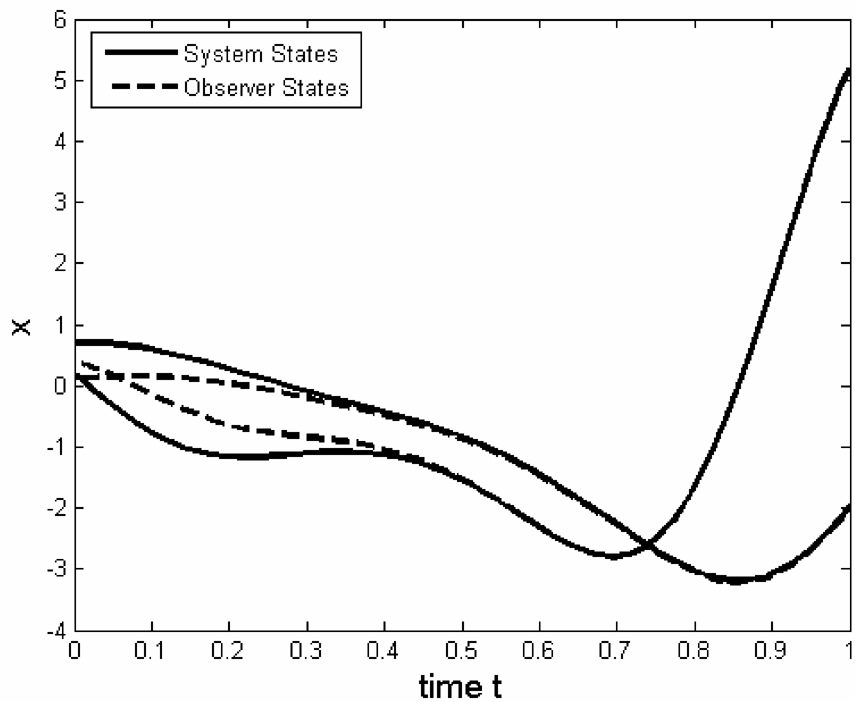


Figure 17: Dynamics of the System and its observer

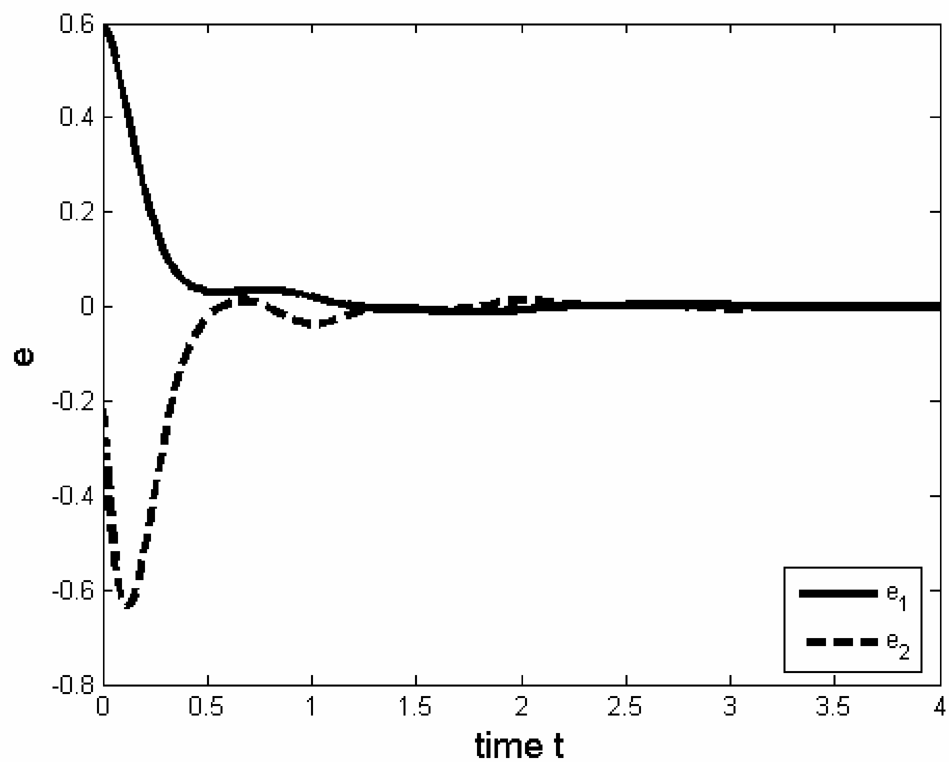


Figure 18: The error dynamics

5.3 Convergence Theorem of the Poincaré Normal Form

The observer design problem for nonlinear time periodic systems is much more challenging than the linear systems. Here, we propose a local observer design method for nonlinear time periodic systems. The idea is to map the nonlinear system into a linear system for which the observer design problem is simpler. Our main tool is the Poincaré normal form technique. In this section, we only review main results of the Poincaré normal form that will be used in the following sections. For more details, readers may refer to [27].

Consider the n dimension nonlinear analytic T-periodic system given by

$$\dot{x} = A(t)x + f(x,t) \tag{0.171}$$

where $A(t) = A(T+t)$, $f(x,t) = f(x,t+T)$ and $f(x,t)$ contains monomials of x with periodic coefficients of the order ≥ 2. The Floquet exponent of T-periodic matrix $A(t)$ is crucial to the local dynamics of (5.11).

For an n dimension index vector $m = (m_1, m_2, ..., m_n)$, we define $|m| = \sum m_i$ and $x^m = x_1^{m_1} x_2^{m_2} ... x_n^{m_n}$. Let the Floquet exponents of $A(t)$ be $\lambda_1, \lambda_2, ..., \lambda_n$, then $A(t)$ is resonant if

$$\sum_{i=1}^{n} m_i \lambda_i - \lambda_s + k\frac{2\pi}{T} j = 0; \;\; j = \sqrt{-1} \tag{0.172}$$

for some nonnegative integers m_i satisfying $\sum m_i \geq 2$ and an integer k. When $A(t)$ is not a resonant matrix, we have the following theorem which is the straightforward conclusion of the Floquet theory and the Poincaré normal form.

Theorem 10: If $A(t)$ is not resonant, (5.11) can be transformed to

$$\dot{\bar{x}} = A(t)\bar{x} \tag{0.173}$$

by a formal T-periodic coordinates transformation given by

$$\begin{aligned} x = \varphi(\bar{x}, t) &= \bar{x} + (\sum \exp(a_{k,l}^{m}\, k\, i\frac{2\pi}{T} t))\bar{x}^T \bar{x} \\ &+ ...high\ order\ terms, l = 1, 2..., n \end{aligned} \tag{0.174}$$

where $a_{k,l}^{(m_1, \cdot m_2 \cdot \cdot, m_n)}$ are constant coefficients.

We notice that the coordinates transformation proposed above is written as a formal Fourier-Taylor series which may be not convergent. In fact, there are examples that the series is divergent [27]. Before stating the convergence theorem, we have the following definition.

Definition 5: Let $(\lambda_1, \lambda_2, ..., \lambda_n)$ be a vector of Floquet exponents of an $n \times n$ periodic matrix $A(t)$. We say $(\lambda_1, \lambda_2, ..., \lambda_n)$ is (C, ν) type if for some positive real number C and ν,

$$|\sum_{i=1}^{n} m_i \lambda_i - \lambda_s + k\frac{2\pi}{T} j| > \frac{C}{(\sum m_i + |k|)^{\nu}}, \quad s = 1, 2, ..., n \qquad (0.175)$$

for any integer k and nonnegative integers m_i satisfying $\sum m_i \geq 2$. We say $A(t)$ is (C, ν) type if the vector of its Floquet exponents is (C, ν) type.

Theorem 11 (Siegel's theorem): If $A(t)$ is (C, ν) type, (5.11) can be transformed into (5.13) by an T-periodic near identity coordinates transformation $x = \varphi(\bar{x}, t)$ satisfying $\varphi(0, t) = 0,\ \left.\frac{\partial \varphi}{\partial x}\right|_{x=0} = I \ , \forall t$.

The proof of the theorem is rather complicated. Interested readers may refer to [17, 21]. Because (5.11) can be transformed into a nonlinear T-periodic system with constant linear part by using the L-F transformation, without any loss of generality, we assume $A(t)$ is a constant matrix in the Jordan canonical form. Then we may assume the transformation in the form of equation (5.14) and substitute it into (5.11). After collecting coefficients, we have

$$(\sum_{i=1}^{n} m_i \lambda_i - \lambda_s + k\frac{2\pi}{T} j)\, a_{k,l}^{(m_1, m_2 .., m_n)} = \gamma \qquad (0.176)$$

where γ is a polynomial in the coefficients of the terms of x of degree less than $\sum m_i$.

Therefore, if all $\sum_{i=1}^{n} m_i \lambda_i - \lambda_s + k\frac{2\pi}{T}j$ are not equal to zero, the coefficients of (5.14) can be evaluated recursively. The denominators of the coefficients of (5.14) are $\sum_{i=1}^{n} m_i \lambda_i - \lambda_s + k\frac{2\pi}{T}j$. The (C,ν) condition gives a proper bound for the denominators to guarantee the convergence of (5.14).

Siegel also proved that for almost all $\lambda_1, \lambda_2, ..., \lambda_n$, the Poincaré normal form transformation (5.14) converges. We would like to point out that the (C,ν) condition is not the sharpest condition for the convergence of (5.14). However it is sufficient for this study.

5.4 Observer Design for Nonlinear Time Periodic Systems

Consider the nonlinear analytic T-periodic system with outputs given by

$$\begin{aligned} \dot{x} &= A(t)x + f(x,t) \\ y &= B(t)x(t) + g(x,t) \end{aligned} \tag{0.177}$$

where x is the system state n vector and y is the output m vector. $A(t), B(t)$ are T-periodic matrices of dimensions $n \times n$, $m \times n$, respectively. $f(x,t)$ and $g(x,t)$ are T-periodic vectors and contain terms that are quadratic in x or higher. $x(0)$ is unknown and $x(t)$ can not be obtained by a direct measurment. The output vector $y(t)$ is measurable. Similar with the linear case, we construct an auxiliary dynamic system with $y(t)$ as the input such that the error between its states and $x(t)$ goes to zero when time approaches infinity. The observer is defined as in the following.

Definition 6: An auxiliary T-periodic dynamic system $\dot{z} = h(z, y, t)$

is called an observer for (21), if there is an invertible periodic map $E(x,t)$ such that:

(i) if $z(0)=E(x(0),t)$ then $z(t)=E(x(t),t)$, $\forall t>0$.

(ii) $\lim_{t\to\infty}(E(x(t),t)-z(t))=0$.

For further consideration, we consider the following auxiliary dynamic system

$$\dot{z}=C(t)z+D(t)\,y(t) \tag{0.178}$$

where $C(t)$, $D(t)$ are unknown T-periodic matrices and $z(0)$ is unknown. It is easy to see that

$$\dot{\hat{z}}=C(t)\hat{z}+D(t)y(t) \tag{0.179}$$

is an observer of (5.17) if $C(t)$ is a Hurwitz matrix. In fact, form (5.18) and (5.19), we have $d(z-\hat{z})/dt=C(t)(z-\hat{z})$ and $\lim_{t\to+\infty}(z(t)-\hat{z}(t))=0$. If we can find an invertible map $z=\varphi(x,t)$ under which (5.17) with initial condition $x(0)$ is transformed to

$$\begin{aligned}&\dot{z}=C(t)z+D(t)y(t)\\&z(0)=\varphi(x(0),t)\end{aligned} \tag{0.180}$$

then we have $\lim_{t\to+\infty}(\varphi(x,t)-\hat{z}(t))=0$. This implies that (5.19) is an observer of (5.17) also. Thus the observer design problem is reduced to finding a proper map $z=\varphi(x,t)$. After putting (5.17) and (5.20) together, we have

$$\begin{cases}\dot{x}=A(t)x+f(x,t)\\ \dot{z}=C(t)z+D(t)B(t)x(t)+D(t)g(x,t)\end{cases} \tag{0.181}$$

From the geometrical viewpoint, it is easy to see that the graph of $z=\varphi(x,t)$ is an invariant manifold of (5.21). By using the Floquet theory and the Poincaré normal form technique, we can prove that an invariant manifold does exist (as shown below) under some mild assumptions.

Proposition 3: Consider an n dimension nonlinear T-periodic system

$$\dot{x} = A(t)x + f(x,t), x(0) = x_0 \tag{0.182}$$

where $A(t)$, $f(x,t)$ are T-periodic and $f(x,t)$ contains only quadratic or higher order terms of x. We assume that the T-periodic matrices $A(t), C(t), D(t)$ satisfy

(i) $A(t) = C(t) + D(t)B(t)$.

(ii) $C(t)$ is a Hurwitz matrix.

(iii) Denoting $\lambda_1, \lambda_2, ..., \lambda_n$ as the Floquet exponents of $A(t)$ and $\sigma_1, \sigma_2, ..., \sigma_n$ as the Floquet exponents of $C(t)$;the $2n$ dimension vector $(\lambda_1, \lambda_2, ..., \lambda_n, \sigma_1, \sigma_2, ..., \sigma_n)$ is (C, ν) type.

If these conditions are satisfied, then there is an invertible T-periodic coordinates transformation $z = \varphi(x,t)$ which transforms (5.17) to

$$\dot{z} = C(t)z + D(t)(B(t)x + g(x,t)) = C(t)z + D(t)y,\ z(0) = \varphi(x_0, 0) \tag{0.183}$$

and the origin is preserved under this transformation (i.e., $\varphi(0,t) = 0$).

Proof: Since $(\lambda_1, \lambda_2, ..., \lambda_n, \sigma_1, \sigma_2, ..., \sigma_n)$ satisfies the (C, ν) condition, (5.22) can be transformed into

$$\begin{cases} \dot{\overline{x}} = A(t)\overline{x} \\ \dot{\overline{z}} = C(t)\overline{z} + D(t)B(t)\overline{x} \end{cases} \tag{0.184}$$

by a near identity transformation which preserves the origin. Let the near identity transformation be

$$\overline{x} = x + \varphi_1(x,z,t) \tag{0.185}$$

$$\overline{z} = z + \varphi_2(x,z,t) \tag{0.186}$$

where φ_1, φ_2 are T-periodic monomials in x and z. From (5.24), in $(\bar{x}, \bar{z})$ domain, obviously $\bar{z} - \bar{x} = 0$ is an invariant manifold passing the origin if $A(t) = C(t) + D(t)B(t)$. Therefore, in the (x, z) domain there exists a T-periodic invariant manifold passing through the origin.

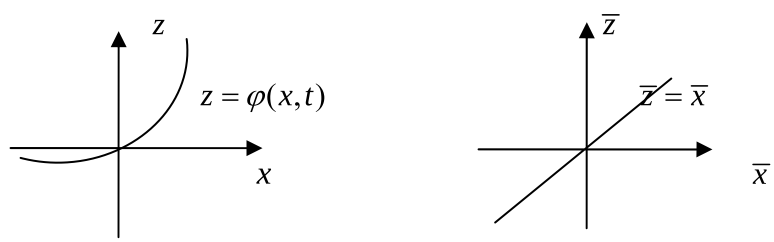


Figure 19: Invariant manifold in two coordinate systems

Substituting (5.25), (5.26) into $\bar{z} - \bar{x} = 0$, yields

$$H(x,z,t) \equiv x + \varphi_1(x,z,t) - z - \varphi_2(x,z,t) = 0 \tag{0.187}$$

It can be obtained that $\left.\frac{\partial H}{\partial x}\right|_{x=0,z=0} \neq 0$ and $\left.\frac{\partial H}{\partial z}\right|_{x=0,z=0} \neq 0$. Therefore, by the implicit function theorem, there exists an invertible T-periodic function

$$z = \varphi(x,t) = x + \ldots + \text{higher order terms} \tag{0.188}$$

which is an invariant manifold of (5.21). This proves the proposition.

Thus if all three conditions of proposition are satisfied, then (5.19) is an observer of (5.17). Obviously, the first two conditions can be satisfied if we follow the design procedure given in Section 1. From **Theorem 11**, the measure of point sets for which

$(\lambda_1, \lambda_2, ..., \lambda_n, \sigma_1, \sigma_2, ..., \sigma_n)$ is not of (C, ν) type is zero [19]. Therefore, in almost all cases, the third condition is satisfied if $(\lambda_1, \lambda_2, ..., \lambda_n, \sigma_1, \sigma_2, ..., \sigma_n)$ is not resonant and $A(t)$ is (C, ν) type. For the exceptional case, parallel with the proof of **Proposition 3**, we can prove the existence of an C^∞ invariant manifold by using the Sternberg-Chen theorem [8], and therefore the Taylor-Fourier series is still an effective approximation of $z = \varphi(x,t)$. We notice that, (5.19) is not the identity observer of (5.20). To construct an identity observer, we have the following proposition.

Proposition 4: Assuming that $z = \varphi(x,t)$ is the T-periodic coordinate transformation which transforms (5.17) into (5.18), the dynamic system

$$\dot{\hat{x}} = A(t)\hat{x} + f(\hat{x},t) + (\frac{\partial \varphi(\hat{x},t)}{\partial \hat{x}})^{-1} D(t)(y - B(t)\hat{x} - g(\hat{x},t)) \tag{0.189}$$

is the identity observer of (5.17).

Proof: $z = \varphi(x,t)$ being the invariant manifold implies that

$$\frac{\partial \varphi(x,t)}{\partial t} + \frac{\partial \varphi(x,t)}{\partial x}(A(t)x + f) = C(t)\varphi(x,t) + D(t)B(t)x + D(t)g(x,t) \tag{0.190}$$

Then we have

$$\begin{aligned}
\frac{d\varphi(\hat{x},t)}{dt} &= \left.\frac{\partial \varphi}{\partial t}\right|_{\hat{x}} + \frac{\partial \varphi}{\partial \hat{x}}\dot{\hat{x}} = \left.\frac{\partial \varphi}{\partial t}\right|_{\hat{x}} + \frac{\partial \varphi}{\partial \hat{x}}(A\hat{x} + f) + \\
&D(t)(y - B(t)\hat{x} - g(\hat{x},t)) \\
&= C(t)\varphi(\hat{x},t) + D(t)B(t)\hat{x} + D(t)g(\hat{x},t) + D(t)(y - B(t)\hat{x} - g(\hat{x},t)) \\
&= C(t)\varphi(\hat{x},t) + D(t)y
\end{aligned} \tag{0.191}$$

On the other hand,

$$\frac{d\varphi(x,t)}{\partial t} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}(A(t)x + f(x,t)) \tag{0.192}$$

Therefore,

$$\frac{d(\varphi(x,t)-\varphi(\hat{x},t))}{dt}=C(t)(\varphi(x,t)-\varphi(\hat{x},t)) \qquad (0.193)$$

Since $C(t)$ is a Hurwitz matrix, we have $\lim_{t\to\infty}(\varphi(x(t),t)-\varphi(\hat{x}(t),t))=0$ and

$\lim_{t\to\infty}(x(t)-\hat{x}(t))=0$. This proves the proposition.

We'd like to point out that, from the viewpoint of geometry, **Proposition 4** implies that the invariant manifold proposed in **Proposition 3** is attractive.

Now we summarize the identity observer design procedure as in the following.

(1) Design $D(t)$ such that $C(t)=A(t)-D(t)B(t)$ is a Hurwitz matrix and $Diag[A(t);C(t)]$ is not resonant.

(2) Calculate the truncated Taylor-Fourier series expression of $\varphi(x,t)$ defined by (5.30).

(3) The identity observer of (5.17) is given by (5.29).

For proving the existence of $z=\varphi(x,t)$, we transformed the original nonlinear observer design problem to a linear problem by using the Poincaré normal form technique. Then the invariant manifold of the linear system is transformed back. However, once the existence of the invariant manifold is guaranteed, the truncated Taylor-Fourier series for $\varphi(x,t)$ should be calculated in an efficient way. We discuss the computational procedure below.

We have to solve (5.30) with the boundary condition $\left.\frac{\partial\varphi}{\partial x}\right|_{x=0}=I$, $\varphi(0,t)=0$.

However, it is very difficult to solve it directly, if not impossible, because $A(t)$ and $C(t)$ are T-periodic matrices. Instead of solving (5.30) directly, we use the L-F transformation to simplify the situation and then we obtain a truncated Taylor-Fourier series solution. In the following proposition, we use a transformation such that equation (5.30) is transformed to a PDE whose linear parts have constant coefficients.

Proposition 5: Let $\dot{x} = A(t)x$ be transformed to $\dot{\overline{x}} = R\overline{x}$ under the coordinate transformation $\overline{x} = Q(t)x$, while $\dot{x} = C(t)x$ be transformed to $\dot{\overline{x}} = M\overline{x}$ using $\overline{x} = P(t)x$ (where R, M are constant matrices in the Jordan canonical form). Let $\psi(x,t)$ be the solution of

$$\frac{\partial \psi(x,t)}{\partial t} + \frac{\partial \psi(x,t)}{\partial x}(Rx + f(Q^{-1}(t)x,t)) = M\psi(x,t) + P(t)D(t)g(Q^{-1}(t)x,t) \quad (0.194)$$

with the boundary condition $\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0$, $\psi(0,t) = 0$. Then the solution of (229) is given by

$$\varphi(x,t) = x + P^{-1}(t)\psi(Q(t)x,t) \quad (0.195)$$

<u>Proof</u> :

From the Floquet theory, we have

$$\dot{Q}(t)Q^{-1}(t) + Q(t)A(t)Q^{-1}(t) = R \quad (0.196)$$

$$\dot{P}(t)P^{-1}(t) + P(t)C(t)P^{-1}(t) = M \quad (0.197)$$

Using the relationships, (5.34), (5.35) is obtained by straightforward computation. This proves the proposition.

Therefore, $\varphi(x,t)$ can be obtained by the solution of (5.34) and the L-F transformation matrices $Q(t)$ and $P(t)$. In order to write (5.34) in a compact form, we define

$$L \circ \psi = \frac{\partial \psi(x,t)}{\partial t} + \frac{\partial \psi(x,t)}{\partial x}Rx - M\psi(x,t) \quad (0.198)$$

where the operator L is linear with respect to ψ. Now (5.34) can be written as

$$L \circ \psi(x,t) = P(t)D(t)g(Q^{-1}(t)x,t) - \frac{\partial \psi(x,t)}{\partial x} f(Q^{-1}(t)x,t) \tag{0.199}$$

At this point, let $\exp(a_{k,l}^{m}\; k\; j\frac{2\pi}{T}t)$ be the basis of the T periodic analytic functions. In terms of this basis, the eigenvalues of L are $\sum_{i=1}^{n} m_i \lambda_i - \sigma_s + k\,\frac{2\pi}{T}\,j$. Therefore, for any T-periodic function $h(x,t)$ defined by

$$h(x,t) \equiv [\sum_{k.m} \exp(a_{k,1}^{m}\; k\; j\frac{2\pi}{T}t)\, x^m, ..., \sum_{k,m} \exp(a_{k,n}^{m}\; k\; j\frac{2\pi}{T}t)\, x^m]^T \tag{0.200}$$

the formal Taylor-Fourier solution of

$$L \circ \psi = h(x,t) \tag{0.201}$$

is given by

$$\psi(x,t) = L^{-1} \circ h(x,t) = \begin{bmatrix} \sum_{k,m} \exp(\frac{a_{k,1}^{m}}{\sum_{i=1}^{n} m_i \lambda_i - \sigma_1 + k\,\frac{2\pi}{T}\,j} k\; j\frac{2\pi}{T}t)\; x^m \\ \sum_{k,m} \exp(\frac{a_{k,2}^{m}}{\sum_{i=1}^{n} m_i \lambda_i - \sigma_2 + k\,\frac{2\pi}{T}\,j}\; k\; j\frac{2\pi}{T}t)\; x^m \\ .... \\ .... \\ \sum_{k,m} \exp(\frac{a_{k,n}^{m}}{\sum_{i=1}^{n} m_i \lambda_i - \sigma_n + k\,\frac{2\pi}{T}\,j}\; k\; j\frac{2\pi}{T}t)\; x^m \end{bmatrix} \tag{0.202}$$

where $a_{k,s}^m$ are the constant coefficients (where m indicates the order of the monomials of x, k indicates the frequency and s indicates the row number of $h(x,t)$).

It is easily seen that the lowest order term of $\psi(x,t)$ has the same order as the lowest order term of $P(t)D(t)g(Q^{-1}(t)x,t)$. Without any loss of generality, we can expand $\psi(x,t)$ as

$$\psi(x,t) = \psi_2(x,t) + \psi_3(x,t) + ... + \textit{higher order terms} \qquad (0.203)$$

where $\psi_2(x,t)$ are the quadratic terms, $\psi_3(x,t)$ are the cubic terms, etc.

Similar to the computation of the Poincaré normal form transformation discussed in last section, $\psi_2(x,t)$, $\psi_3(x,t)$,…can be obtained recursively. Before stating the recursive relations of $\psi_2(x,t)$, $\psi_3(x,t)$,…, the following definitions are helpful.

Definition 7: Let $h(x,t)$ be an arbitrary analytic T-periodic vector valued function. We define

$$\mathrm{P}^m \circ h(x,t) \equiv \left.\frac{\partial h(x,t)}{\partial^m x}\right|_{x=0} x^m \qquad (0.204)$$

and

$$\mathrm{P}_k^m \circ h(x,t) \equiv \exp(-i\frac{2\pi}{T}k\,t)\int_0^T \exp(-i\frac{2\pi}{T}k\,t)(\mathrm{P}^m \circ h(x,t))\,dt \qquad (0.205)$$

It is easy to see that, for a given $h(x,t)$ (not necessary written in the Taylor-Fourier series form), $\mathrm{P}_k^m \circ h(x,t)$ has the form of $x^m \exp(i\frac{2\pi}{T}kt)$.

Now we have the following proposition.

Proposition 6: From (5.41), $\psi(x,t)=\psi_2(x,t)+\psi_3(x,t)+...+\textit{Higher order terms}$ can be expressed as

$$\psi_2(x,t)=\sum_{|m|=2} L^{-1}\circ(\mathrm{P}^m\circ P(t)D(t)g(Q^{-1}(t)x,t)) \tag{0.206}$$

$$\begin{aligned}\psi_{k+1}(x,t)=&\sum_{|m|=k+1} L^{-1}\circ[\mathrm{P}^m\circ P(t)D(t)g(Q^{-1}(t)x,t)]\\ &-\sum_{|j|+|l|=k+2,|l|>1} L^{-1}\circ[\frac{\partial\psi_j(x,t)}{\partial x}(\mathrm{P}^l\circ f(Q^{-1}(t)x,t)]\end{aligned} \tag{0.207}$$

where L^{-1} is given in (5.42).

The **proposition 6** can be proved by a straightforward computation and comparison of coefficients of terms having the same order. We omit it. We notice that only $\psi_j(x,t), |j|<k+1$ appears in the right hand side of (5.47). Therefore, $\psi_2,\psi_3,....$ can be evaluated recursively.

As a summary, the computation procedure of $\varphi(x,t)$ defined in (5.30) is given as follows:

(1) For given $A(t)$ and $C(t)$, compute the corresponding L-F transformation matrices $P(t)$ and $Q(t)$, respectively.

(2) By using Proposition 3, compute $\psi(x,t)$ to the desired order of accuracy.

(3) Then, $\varphi(x,t)=x+P^{-1}(t)\psi(Q(t)x,t)$.

5.5 Illustrative Examples

We present two examples to demonstrate our nonlinear observer design method.

Example 9:

Consider the observer design problem for the Mathieu-equation with cubic nonlinearity

$$\begin{gathered}\begin{Bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\end{Bmatrix}=\begin{bmatrix}0 & \pi\\ -\pi(a+b\,cos(2\pi t) & -\pi d\end{bmatrix}\begin{Bmatrix}\bar{x}_1\\ \bar{x}_2\end{Bmatrix}+\begin{Bmatrix}0\\ \bar{x}_1^{\,3}/3\end{Bmatrix},\\ y=[0\quad 1]\begin{Bmatrix}\bar{x}_1\\ \bar{x}_2\end{Bmatrix}\end{gathered} \tag{0.208}$$

where $a=4, b=1, d=0.3$. The Floquet exponents are $-0.47\pm 0.012i$ (stable). Using the L-F transformation and the modal transformation, (5.48) is transformed to

$$\begin{gathered}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix}=\begin{bmatrix}-0.47+0.012i & 0\\ 0 & -0.47-0.012i\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}+\begin{Bmatrix}f_1(x,t)\\ f_2(x,t)\end{Bmatrix},\\ y=[0\quad 1]Q_1(t)M_1\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}\\ R=\begin{bmatrix}-0.47+0.012i & 0\\ 0 & -0.47-0.012i\end{bmatrix},\\ B(t)=[0\quad 1]Q_1(t)M_1,\ f(x,t)=\begin{Bmatrix}f_1(x,t)\\ f_2(x,t)\end{Bmatrix}\end{gathered} \tag{0.209}$$

where $f_1(x,t), f_2(x,t)$ are cubic nonlinear functions and $Q_1(t)$ is the L-F transformation matrix and M_1 is the modal matrix. Following the procedure given earlier, $D(t)$ is designed such that $C(t)=A-D(t)B(t)$ is Hurwitz. The long expressions for $Q_1(t), D(t), C(t), f(x,t)$ are omitted here for brevity. The Floquet exponents of $C(t)$ are $-1.53\pm 0.205i$. The dynamic system

$$\dot{z}=C(t)z+D(t)y \tag{0.210}$$

is an observer of (5.48) but it is not an identity observer. Equation (5.34) for this case is given by

$$\frac{\partial \psi(x,t)}{\partial t} + \frac{\partial \psi(x,t)}{\partial x}(Rx + f(M_1^{-1}Q_1^{-1}(t)x,t)) = M\psi(x,t) + P(t)M_2 f(M_1^{-1}Q_1^{-1}(t)x,t)$$

(0.211)

where $M = \begin{bmatrix} -1.53-0.205i & 0 \\ 0 & -1.53+0.205i \end{bmatrix}$, M_2 is the modal transformation matrix of

$\dot{P}(t)P^{-1}(t) + P(t)C(t)P^{-1}(t)$ (a constant matrix) where $P(t)$ is the L-F transformation

matrix of $C(t)$. Then we have

$$\varphi(x,t) = x + M_2^{-1}P^{-1}(t)\psi(Q(t)M_1x,t) \quad (0.212)$$

A series solution of (5.51) is obtained using **Mathematica^{TM}** and then the identity observer is constructed. The observer is given by

$$\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} = \begin{bmatrix} 0 & \pi \\ -\pi(4+cos(2\pi t) & -0.3\,\pi \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix} +$$
$$\begin{Bmatrix} 0 \\ \hat{x}_1^3/3 \end{Bmatrix} + (\frac{\partial \varphi(\hat{x},t)}{\partial \hat{x}})^{-1} \begin{bmatrix} -0.2cos(2\pi t) \\ 4 \end{bmatrix} (y - [0 \quad 1]\,\hat{x})$$

where

$$\varphi(\hat{x},t) = \begin{bmatrix} \hat{x}_1 - 0.0789\hat{x}_1^3 - 0.000250285\hat{x}_1^3\ cos(2\pi t) + ... \\ \hat{x}_2 - 0.00159047\hat{x}_1^3 - 0.0499661\hat{x}_1^3\ cos(2\pi t) + ... \end{bmatrix}$$

and

$$\left[\frac{\partial \varphi(\hat{x},t)}{\partial \hat{x}}\right]^{-1} = \begin{bmatrix} 1 + 0.23695\hat{x}_1^2 - 0.000750855\hat{x}_1^2\ cos(2\pi t)... & 0 \\ -0.004771\hat{x}_1^2 - 0.149898\hat{x}_1^2\ cos(2\pi t)... & 1 \end{bmatrix}$$

The error dynamics between system and observer states is shown in Figure 20. In Figure 21 we show the system dynamics and the observer dynamics.

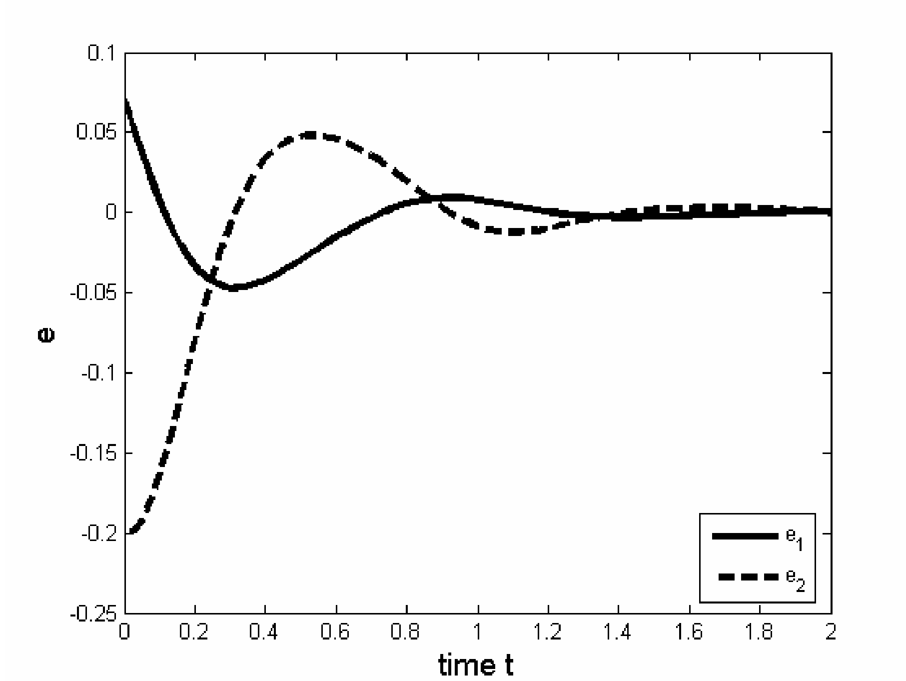


Figure 20: Error dynamics

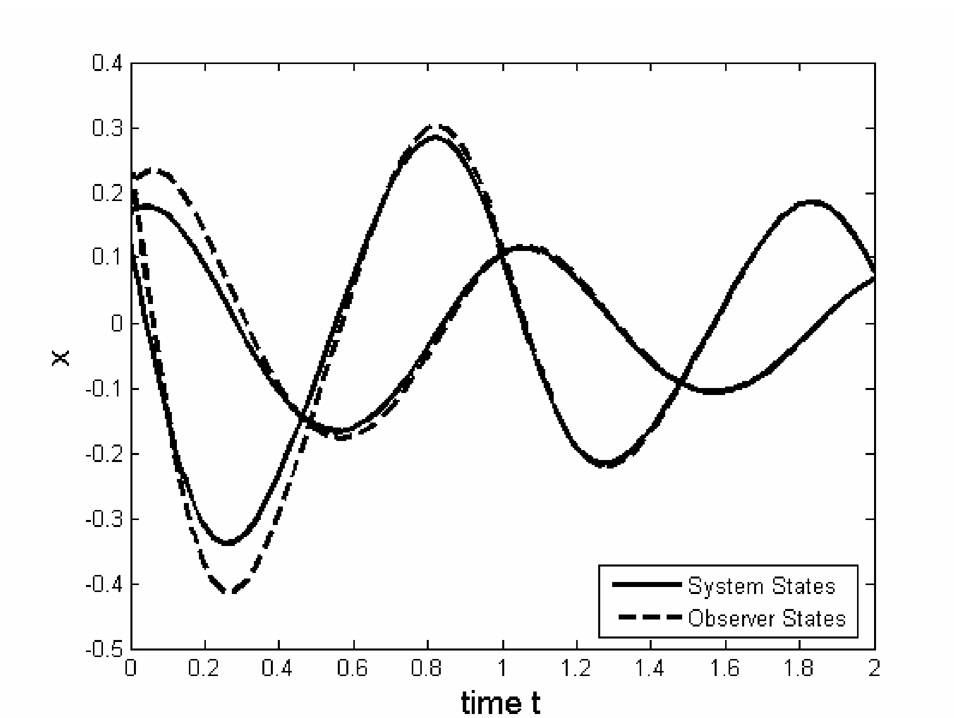


Figure 21: Dynamics of the System and its observer

Example 10:

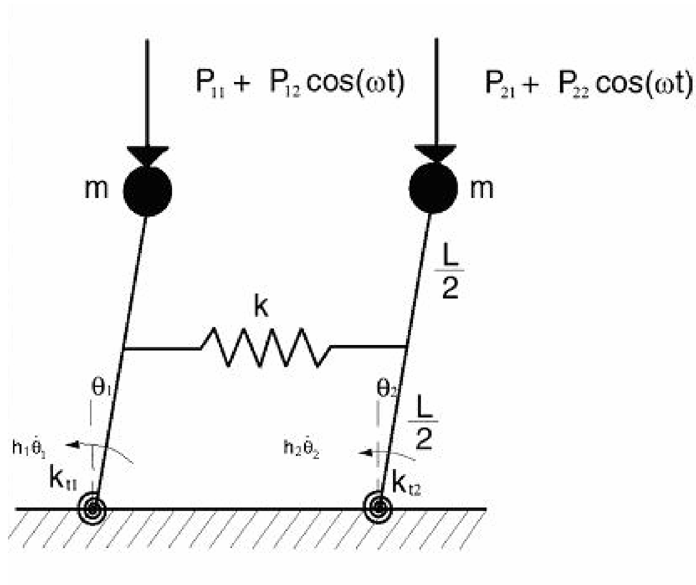


Figure 22: Coupled pendulums

In this example, we design an observer for a system consisting of two inverted pendulums moving in the horizontal plane with time-dependent load acting on each of the pendulums. The structural diagram of the system considered is shown in Figure 22. The equations of motion can be shown to be

$$
\begin{aligned}
ml^2\ddot{\theta}_1 + h_1\dot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}q_1(\theta_1,\theta_2) - P_1(t)l\sin\theta_1 = 0 \\
ml^2\ddot{\theta}_2 + h_2\dot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}q_2(\theta_1,\theta_2) - P_2(t)l\sin\theta_2 = 0
\end{aligned}
\tag{0.213}
$$

where $q_1(\theta_1,\theta_2)$ and $q_2(\theta_1,\theta_2)$ are nonlinear functions of $(\theta_1-\theta_2)$, $P_1(t) = P_{11} + P_{12}\cos(\omega t)$ and $P_2(t) = P_{21} + P_{22}\cos(\omega t)$. m, l, k_{ti} and k denote mass, length, torsional stiffness and coupling stiffness, respectively. h_1, h_2 are the torsional damping constants. The local dynamics can be obtained by expanding these equations of motion about the fixed point $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (0,0,0,0)$. The linearized equations are

$$\begin{aligned} ml^2\ddot{\theta}_1 + h_1\dot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}(\theta_1 - \theta_2) - P_1(t)l\theta_1 = 0 \\ ml^2\ddot{\theta}_2 + h_2\dot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}(\theta_1 - \theta_2) - P_2(t)l\theta_2 = 0 \end{aligned} \quad (0.214)$$

On the other hand, if terms up to the cubic order are retained, then these equations may be approximated as

$$\begin{aligned} &\frac{\ddot{\theta}_1}{ml^2} + \frac{h_1}{ml^2}\dot{\theta}_1 + \frac{k_{t1}}{ml^2}\theta_1 + \frac{k}{4m}[c_1(\theta_1 - \theta_2) + c_2(\theta_1 - \theta_2)^2 + c_3(\theta_1 - \theta_2)^3] \\ &-\frac{P_1(t)}{ml}(\theta_1 - \frac{\theta_1^3}{6}) = 0 \\ &\frac{\ddot{\theta}_2}{ml^2} + \frac{h_2}{ml^2}\dot{\theta}_2 + \frac{k_{t2}}{ml^2}\theta_2 + \frac{k}{4m}[c_1(\theta_2 - \theta_1) + c_2(\theta_2 - \theta_1)^2 + c_3(\theta_2 - \theta_1)^3] \\ &-\frac{P_2(t)}{ml}(\theta_2 - \frac{\theta_2^3}{6}) = 0 \end{aligned} \quad (0.215)$$

Setting P_{11} and P_{21} equal to zero, the equations can be written as

$$\begin{aligned} &\ddot{\theta}_1 + \overline{h}_1\dot{\theta}_1 + (\omega_{n_1}^2 - \varepsilon p_1\cos(\omega t))\theta_1 + \varepsilon p_1\frac{\theta_1^3}{6} \\ &-b\theta_2 - c(\theta_1 - \theta_2)^2 - d(\theta_1 - \theta_2)^3 = 0 \\ &\ddot{\theta}_2 + \overline{h}_2\dot{\theta}_2 + (\omega_{n_2}^2 - \varepsilon p_2\cos(\omega t))\theta_2 + \varepsilon p_2\frac{\theta_2^3}{6} \\ &-b\theta_1 - c(\theta_1 - \theta_2)^2 - d(\theta_1 - \theta_2)^3 = 0 \end{aligned} \quad (0.216)$$

where

$$\begin{aligned} &\omega_{n_1}^2 = [\frac{k_{t1}}{ml^2} + \frac{k}{4m}c_1], \omega_{n_2}^2 = [\frac{k_{t2}}{ml^2} + \frac{k}{4m}c_1], \\ &\varepsilon p_1 = \frac{P_{12}l}{ml^2}, \varepsilon p_2 = \frac{P_{22}l}{ml^2}, \omega = 2\pi \end{aligned}$$

$$b = \frac{k}{4m}c_1, c = \frac{k}{4m}c_2, d = \frac{k}{4m}c_3, \overline{h}_1 = \frac{h_1}{ml^2}, \overline{h}_2 = \frac{h_2}{ml^2}$$

Using the following typical parameter values:

$\omega_{n_1}^2 = 5, \omega_{n_2}^2 = 4, \varepsilon p_1 = 2.5, \varepsilon p_2 = 5, \omega = 2\pi, b = 2.5, c = 0, d = 1.5, \overline{h}_1 = -3.2, \overline{h}_2 = 1.2$, the state space form of equation (5.56) is given by

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(5-2.5\cos(2\pi t)) & 2.5 & -3.2 & 0 \\ 2.5 & -(4-5\cos(2\pi t)) & 0 & 1.2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -2.5\cos(2\pi t)\frac{x_1^3}{6} + 1.5(x_1 - x_2)^3 \\ -2.5\cos(2\pi t)\frac{x_2^3}{6} - 1.5(x_1 - x_2)^3 \end{Bmatrix} \qquad (0.217)$$

where $\{x_1, x_2, x_3, x_4\}^T = \{\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\}^T$. At this point, let us introduce the output equation $y = [0 \quad 1 \quad 0 \quad 1]\{x_1, x_2, x_3, x_4\}^T$.

The Floquet exponents of (5.57) are

$$\begin{aligned} \lambda_1 &= 0.41 + 1.99i, \quad \lambda_2 = 0.41 - 1.99i, \\ \lambda_3 &= -1.41 + 1.42i, \quad \lambda_4 = -1.41 - 1.42i \end{aligned}.$$

This indicates an unstable system. It is to be noted that x_1 and x_3 are the immeasurable states. Once again equation (5.58) has the form

$$\frac{\partial \psi(x,t)}{\partial t} + \frac{\partial \psi(x,t)}{\partial x}(Rx + f(M_1^{-1}Q_1^{-1}(t)x,t)) = M\psi(x,t) + P(t)M_2 f(M_1^{-1}Q_1^{-1}(t)x,t)$$

and we have

$$\varphi(x,t) = x + M_2^{-1}P^{-1}(t)\psi(Q(t)M_1x,t)$$

The series solution obtained using **Mathematica™** and then the identity observer has the following form

$$\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(5-2.5\cos(2\pi t)) & 2.5 & -3.2 & 0 \\ 2.5 & -(4-5\cos(2\pi t)) & 0 & 1.2 \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{Bmatrix}$$

$$+\begin{Bmatrix} 0 \\ 0 \\ -2.5\cos(2\pi t)\dfrac{\hat{x}_1^3}{6} + 1.5(\hat{x}_1 - \hat{x}_2)^3 \\ -2.5\cos(2\pi t)\dfrac{\hat{x}_2^3}{6} - 1.5(\hat{x}_1 - \hat{x}_2)^3 \end{Bmatrix} + \tag{0.218}$$

$$\left[\frac{\partial\varphi(\hat{x},t)}{\partial\hat{x}}\right]^{-1} \begin{Bmatrix} 1.2 \\ 0.4 \\ -2.5\cos(2\pi t) \\ 2.2\sin(2\pi t) \end{Bmatrix} (y - [0 \quad 1 \quad 0 \quad 1]\hat{x})$$

Where

$$\varphi(\hat{x},t) = \begin{bmatrix} \hat{x}_1 - 0.892\hat{x}_1^3 - 0.000750245\hat{x}_1^3\sin(2\pi t) + \\ 1.2305\,\hat{x}_1\hat{x}_2^2 + 0.000654\hat{x}_2\hat{x}_1^2 + \ldots \\ \\ \hat{x}_2 - 0.459347\hat{x}_1^3 - 0.149721\hat{x}_1^3\cos(2\pi t) + \\ 2.1375\,\hat{x}_1\hat{x}_2^2 + 0.7842\hat{x}_2\hat{x}_1^2 + \ldots \\ \\ \hat{x}_3 - 1.00907\hat{x}_1^3 - 0.946161\hat{x}_2^3\sin(2\pi t) + \\ 0.00215437\,\hat{x}_1\hat{x}_2^2 + 0.97242\hat{x}_2\hat{x}_1^2 + \ldots \\ \\ \hat{x}_4 - 0.00956562\hat{x}_1^3 - 0.745361\hat{x}_2^3\cos(2\pi t) + \\ 3.2256\,\hat{x}_1\hat{x}_2^2 + 0.042568\,\hat{x}_2\hat{x}_1^2 \end{bmatrix} \tag{0.219}$$

In Figure 23 we show the dynamics of the original system. The error dynamics between system and observer states is shown in Figure 24.

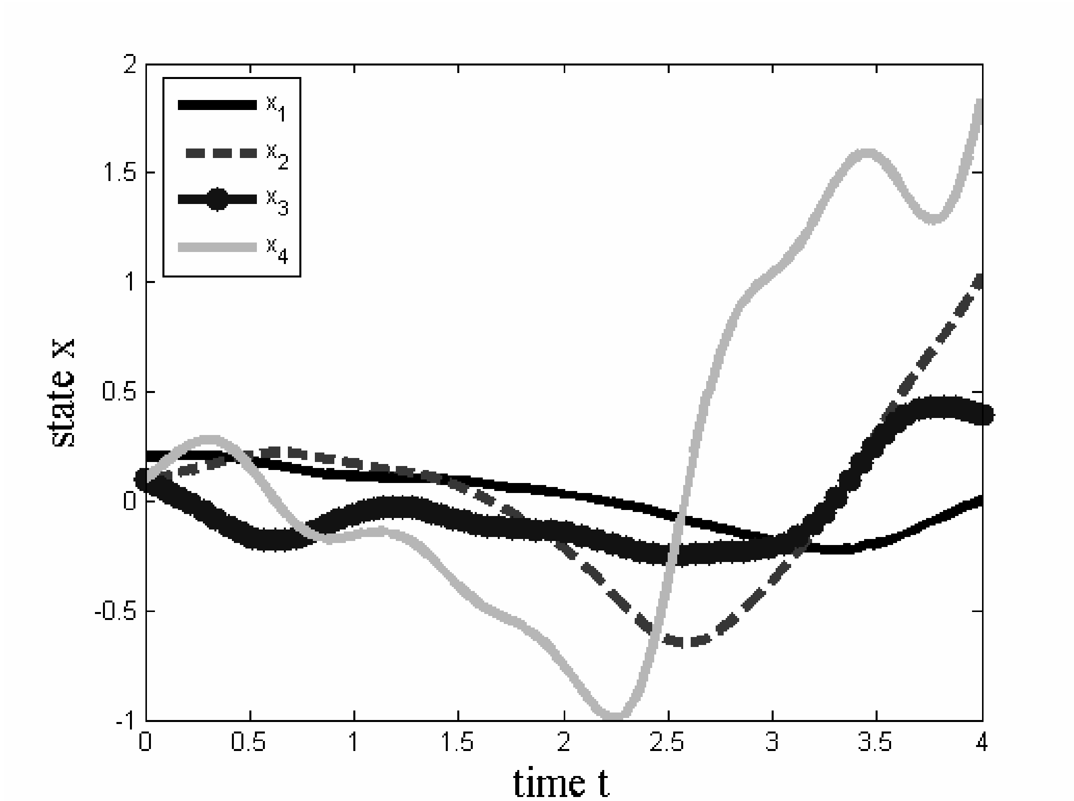


Figure 23: States of the original system

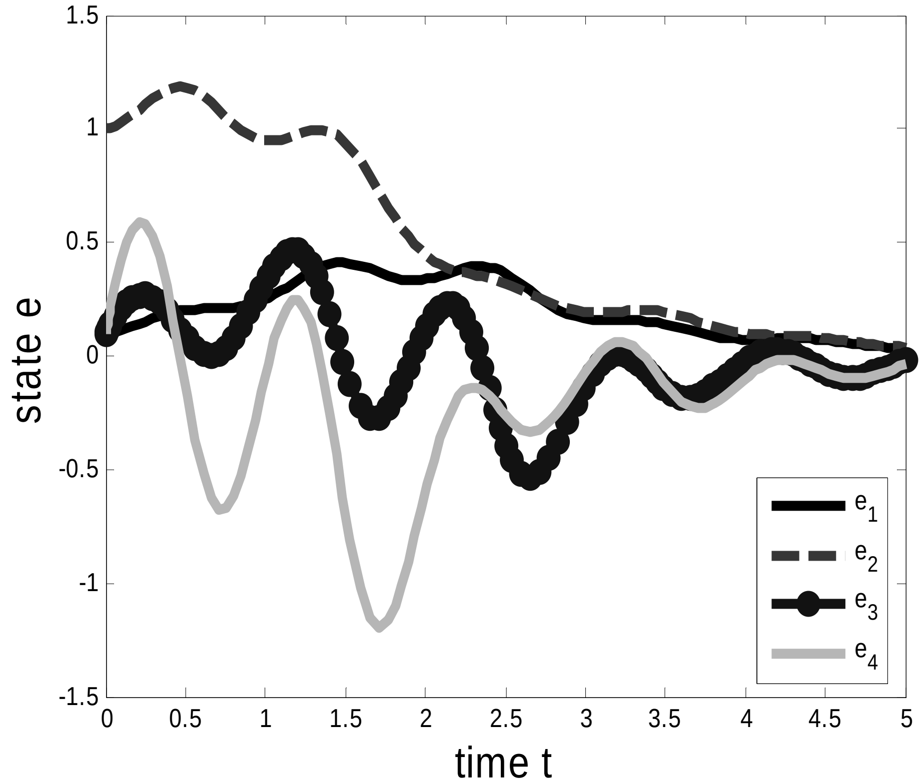


Figure 24: States of the error dynamics

5.6 Conclusions

In this section, the observer design problems associated with linear as well as nonlinear free system with time periodic coefficients have been addressed. For the linear systems a symbolic computational technique (used earlier to design controllers) has been used to guarantee the error between the system and observer states to be zero (as time approaches infinity) by placing the Floquet multipliers in the desired location within the unit circle of the complex plane. For the nonlinear problem, the resulting system is embedded in a *2n* dimensional space. We prove the existence and analytic properties of the invariant manifold in this space. This is crucial for the observer design and since the invariant manifold is attractive, it guarantees the convergence. However, since the PDE characterizing the manifold is nonlinear and contains periodic coefficients, it is rather difficult to obtain a truncated Fourier-Taylor representation of the invariant manifold by equating the coefficients of the like monomials in a simple fashion. In order to alleviate this problem, the original PDE is transformed to a new PDE using L-F transformation such that the linear parts of the resulting PDE have constant coefficients. Then a truncated Fourier-Taylor series expression is obtained through a recursive computation.

The results are very satisfactory for all examples considered in this paper. However, the computational algorithm given here is by no means optimal or efficient. It can be improved in two ways. First, new formal power series manipulation algorithm can be applied here to save CPU time and memory usage greatly. Second, the symbolic manipulation can be implemented numerically to accelerate the computation. It should be observed that we have only dealt with observer design problem for free systems. For time periodic systems with controls, the problem is more challenging. However, similar method can be developed if we use the symbolic computation method for normal forms given in the last chapter. We will discuss these topics in the future.

6. DISCUSSIONS AND CONCULSIONS

In this dissertation, some new techniques for control of dynamics systems with time periodic coefficients are presented. It has been shown that feedback controllers can be designed and implemented for linear and nonlinear systems successfully employing symbolic, geometric and computational methods. The suggested control design methods are mainly based on the ideas that the state transition matrix of a linear time periodic system with unknown control gains can be evaluated symbolically and the Poincaré normal form of a nonlinear time periodic system with unknown gains can be obtained by an iterative computational procedure.

The first control design method is based on the symbolic computation of the state transition matrix of the linear time periodic systems. For a linear time periodic system with unknown control gains, the series expression of the STM in terms of these unknown control gains is obtained by the aid of the symbolic computation software (MATHEMATICA). Then the control theory for linear time invariant system is applied to assign the Foloquet multipliers for the closed loop system. We also develop a robust control approach for linear time periodic systems under parameter certainty. This control system design method can be also applied to nonlinear time periodic systems under mild assumptions to obtain the local stability.

Then the domain of attraction of the resulting closed loop system can be estimated by viewing the nonlinear part of the resulting closed system as a perturbation. For demonstrating the effectiveness of the control design method, we apply this technique to chaos control problems. We first design a feedback forward controller and then the error dynamics between the chaotic system and the desire orbit is formulated as a nonlinear time periodic system. Then a feedback controller is designed from the error dynamics to guarantee that the errors go to zero as time goes to infinity.

The second approach of the control system design is based on the feedback linearization idea for the time invariant system (or the geometric control theory). Two approaches are developed to generalize the feedback linearization technique to time periodic systems. The first approach is based on the properties of time dependent Lie derivative. Treating time as an auxiliary variable, a time dependent Lie derivative is defined and the feedback linearization theory for the time invariant system can be applied. Because it is difficult to determine the rank of a time periodic matrix, this exact feedback linearization approach has serious restrictions. The second approach is based on the Poincaré normal form techniques. Two computational procedures are developed in order to linearize nonlinear time periodic systems in an approximate sense. The Poincaré normal form of a nonlinear time periodic system with unknown control gains can be obtained symbolically through this procedure. A suboptimal control design method is also discussed using the suggested symbolic computational method.

In Chapter 5, observer design methods are discussed since for most complex dynamic systems it is not possible to obtain all states by direct measurements. For linear time periodic system, it is shown that the identity observer can be designed by using the symbolic computation method. For nonlinear time periodic system, the relationship between the observer states and the original system states is viewed as the invariant manifold of a $2n$ dimension auxiliary system. Then by using the invariant manifold theory, the existence and attraction of the invariant manifold is proven. However, this invariant manifold is characterized by a set of semi-linear time periodic partial differential equations which is rather difficult to solve by using the usual methods. By using the Lyapunov-Floque theory, these equations are transformed such that the coefficients of linear parts of the PDEs are constant. Then the Fourier-Taylor series solutions of the PDEs are obtained through simple algebraic manipulations. Finally the identity observers are constructed for nonlinear time periodic systems.

It is anticipated that the methodologies presented here would be of significant value in the controller and observer design of linear and nonlinear time periodic systems.

In summary, the main contributions and developments of this study can be listed as follows:

1. Improvements on the algorithm for the symbolic computation of the state transition matrices (and thus FTMs) of linear time periodic systems.

2. Feedback controller design method for stabilizing linear time periodic systems by using the symbolic computation method.

3. Robust controller design method for the linear time periodic system under parameter certainty.

4. Controller design method for the local stabilization of the nonlinear time periodic systems (including chaos systems) and the attraction domain estimation for the resulting closed system.

5. Exact and approximate feedback linearization for the nonlinear time periodic systems. Symbolic computation procedure for the Poincaré normal for of systems including parameters and dependent variables (controls).

6. Observer design method for linear time periodic systems by using the symbolic computation method. Observer design method for nonlinear time periodic systems based on the invariant manifold theory. Fast algorithm for the Taylor-Fourier series solution of the invariant manifold of the nonlinear time periodic system.

6.1 Scope for Future Studies

(1) Improve the performance and robustness of the controllers.

(2) Actual implementation through hardware on a real engineering system. This will serve as a test for the methods developed in this study.

(3) Study the convergence of the matching procedure given in Chapter 4.

(4) Study the advantages of the time periodic feedback controller for time invariant systems.

(5) Develop observer design methods for linear and nonlinear time periodic systems involving feedback controls.

REFERENCES

1. Richards, J.A., 1983, *Analysis of periodically time-varying systems*, Springer-Verlag, Berlin.

2. Calico, RA. and Wiesel, WE., 1984, "Control of time-periodic systems", J. Guidance, **7(6)**, pp.671-676.

3. Calise, AJ., Wasikowski, ME. And Schrage, DP., 1992, "Optimal output feedback for linear time-periodic systems", J. Guidance, **15(4)**, pp. 416-423.

4. Kabamba, PT., 1996, "Monodromy eigenvalues assignment in linear periodic systems", IEEE Transactions of Automatic Control, **31**, pp 950-952..

5. Torambe, A. and Valigi, P., 1996, "Asymptotic stabilization of a class of continuous-time linear periodic systems", System control letter, **28**, pp. 189-196.

6. Montagnier, P., Spiteri, Raymond J., and Angeles, J., 2004, "The Control of Linear Time periodic Systems Using Flopquet-Lyapunov Theory", Int. J. Control, **77**, pp 472-490.

7. Sinha, S.C. and Joseph, P.， 1994, "Control of general dynamic systems with periodically varying parameters via Liapunov-Floquet transformation", ASME J. Dyn. Syst. Measur. Control ,**116**, pp.650-658.

8. Sinha, S.C., Henrichs, J.T. and Ravindra, B.， 2000, "A general approach in the design of active controllers for nonlinear systems exhibiting chaos", International Journal of Bifurcation and Chaos, **10**(1), pp.165-178

9. Sinha.S.C., Gourdon.E and Zhang. Y.，2005, “Control of time-periodic systems via symbolic computation with application to chaos control”, Communications in Nonlinear Science and Numerical Simulation, **10**(8), pp.835-854.

10. Sinha, S. C. and Eric A. Butcher，1997, “Symbolic Computation of Fundamental Solution Matrices for Time-Periodic Dynamical Systems,” Journal of Sound and Vibration, **206**(1), pp.61-85.

11. Deshmukh, V.S. and Sinha, S.C.，2004, “Control of Dynamic Systems with Time-Periodic Coefficients via the Lyapunov-Floquet Transformation and Backstepping Technique”, Journal of Vibration and Control ,**10**, pp. 1517-1533.

12. Hartman, P., 2002, *Ordinary Differential Equations*, Classics in Applied Mathematics. **38**. Philadelphia, SIAM.

13. Yakubovich, V.A. and Starzhinskii, V.M., 1975, *Linear Differential Equations with Periodic Coefficients, Part I and Part II*, John Wiley & Sons, New York.

14. Sinha,S.C. and Wu, D.-H., 1991, “An Efficient Computational Scheme for the Analysis of Periodic Systems”, Journal of Sound and Vibration, **151**, pp. 91-117.

15. Pandiyan, R., Bibb, J.S., and Sinha, S.C., 1993, “Liapunov-Floquet Transformation: Computation and Applications to Periodic Systems”, *Proceedings of the 14th Bienial Conference on Mechanical Vibration and Noise*, Albuquerque, NM, Sept. 19-22, DE-Vol.56, pp. 337-348. S.C. Sinha and R.M. Evan-Iwanowski, eds.

16. Lee YJ. and Bala MJ., 1998, “Control design of periodic time-varying systems via time invariant methods”, J. Guidance, **74(2)**, pp. 486-488.

17. Brunovsky, P., 1969, “Controllability and Linear Closd-Loop Controls in Linear Periodic Systems”, Journal of Differential Equations, **6**, pp 296-313.

18. Bittani, S., Laub, Alan J. and Willems, Jan C., 1991, *The Riccati equation*, Springer.

19. Gajic, Z. and Qureshi, M., 1995, *Lyapunov Matrix Equation in System Stability and Control*, Academic Press, San Diego, CA.

20. Lyapunov, A.M., 1992, *The General Problem of the Stability of Motion*, Taylor & Francis.

21. Dorning JJ., Decker WJ., and Holloway JP., 1992, "Controlling the dynamics of chaotic convection flows", In: Kim JH., Stringer J.,(Eds)., Applied chaos., pp. 189.

22. Cheng,D., Isidori, A., Respondek,W. and Tarn,T.J., 1988, "Exact linearization of nonlinear system with outputs", Math.Systems Theory, **21**, pp.63-83.

23. Nijmeijer,N. and Van der schaft,A.J., 1995, *Nonlinear Dynamical Control Systems*, Springer-Verlag.24. Marino, R., and Tomei, P., 1998, *Nonlinear Control Design: Geometric, Adaptive, Robust*, Prentice-Hall, London.

25. Krener, A.J., 1988, "Nonlinear controller design via approximate normal forms", Proc. IMA Conf.on Signal Processing, Minneapolis.

26. Guardabassi,G.O. and Savaresi,S.M.,2001, "Approximate linearization via feedback—an overview", Automatica ,37, pp.1-15.

27. Arnold, V.I.,1988, *Geometrical methods in the theory of ordinary differential equations*, New York: Springer-verlag.

28. Horn, F.J.M. and Lin, R.C., 1967, "Periodic processes: a variational approach", *Ind. Engng Chem Process Des. Dev.* **6**, pp. 21-30.

29. Bailey, J.E., 1973, "Periodic operation of chemical reactors: a review", Chem. Eng. Commun. **1**, pp.111-124.

30. Bittani, S., Fronza, G., and Guadarabassi, G., 1973, “Periodic control: a frequency domain approach”, IEEE Trans. Autom. Control, **18**, pp 33-38.

31. Berstein, D.S., and Gilbert, E.G., 1980, “Optimal periodic control: the pi test revisited”, IEEE Trans. Autom. Control, **25**, pp 673-684.

32. Luenberger, D.G., 1964, “Observing the state of a linear system”, IEEE trans. Military Electronics, **8,** pp. 74-80.

33. Simon, D., 2006, Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches, John Wiley & Sons.

34. Thau, F.E., 1973, “Observing the state of nonlinear dynamic system”, Int. Journal of Control, 17**,** pp. 471-479.

35. Gauthier, J.P. and Kupka, I.A.K., 1992, “A simple observer for nonlinear systems, application to bioreactors”, IEEE Trans. Automat. Control, **37,** pp. 875-880.

36. Raghavan, S. and Hedrick, J.K., 1994, “Observer design for a class of nonlinear system”, Int. Journal of Control, **59**, pp. 515-528.

37. Rajamani, R. and Cho, Y. M., 1998, “Existence and design of observers for nonlinear systems: relation to distance to unobservability”, Int. Journal of Control, **69**, pp. 717-731.

38. Arcak, M. and Kokotovic, P., 2001, “Nonlinear observers: a circle criterion design and robustness analysis”, Automatica, **37**, pp.1923-1930.

39. Krener, A.J. and Isidori, A., 1983, “Linearization by output injection and nonlinear observers”, Systems Control Lett., **3**, pp. 47-52.

40. Kazantzis, N. and Kravaris, C. 1998, “Nonlinear observer design using Lyapunov’s auxiliary theorem”, Systems Control Lett. **34**, pp. 241-247.

41. Krener, A.J. and Xiao, M., 2002, “Nonlinear observer design in the Siegel domain”, SIAM J. Control Optim. **41**, pp. 932-953.

42. Krener, A.J. and Xiao, M., 2004, Erratum: nonlinear observer design in the Siegel domain, SIAM J. Control Optim. **43**, pp. 377-378.

43. Krener, A.J., 2004, “Nonlinear observers”, in *Control systems, Robotics and Automation*, EOLSS Publishers, Oxford, UK, 2004.

APPENDIX A: THE OPERATIONAL MATRICES

The $n \times n$ integration operational matrix is given by

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & \dots & \dots & 0 \\ -\frac{1}{8} & 0 & \frac{1}{8} & \dots & \dots & \dots & 0 \\ -\frac{1}{6} & -\frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{16} & 0 & -\frac{1}{8} & \dots & \dots & \dots & \dots \\ -\frac{1}{30} & 0 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \frac{1}{4(n-1)} \\ \frac{(-1)^n}{2n(n-2)} & 0 & 0 & \dots & 0 & \frac{-1}{4(n-2)} & 0 \end{bmatrix}$$

Let $a_1, a_2, \dots$ be the coefficients of the Chebyshev expansion of a function $f(t)$. The product operational matrix is given by

$$\begin{bmatrix} a_0 & \frac{a_1}{2} & \frac{a_2}{2} & \dots & \frac{a_n}{2} \\ a_1 & a_0 + \frac{a_2}{2} & \frac{a_1 + a_3}{2} & \dots & \frac{a_{n-2} + a_n}{2} \\ a_2 & \frac{a_1 + a_3}{2} & a_0 + \frac{a_4}{2} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & \frac{a_{n-2} + a_n}{2} & \dots & \dots & a_0 + \frac{a_{2(n-1)}}{2} \end{bmatrix}$$

APPENDIX B: MATHEMATICA PROGRAMS FOR CHAPTER 2 AND CHAPTER 3

```
(****Solve ODE numerically by using the Chebyshev Polynomial***)

Off[General::spell1]

SetDirectory[ToFileName[

    Extract["FileName"/.NotebookInformation[EvaluationNotebook[]],

     {1},FrontEnd`FileName

     ]]];

<<Utilities`MemoryConserve`

<<LinearAlgebra`MatrixManipulation`

fQp[p_]:=Module[{i,j},

  MQp:=Array[Qp,{Length[p],Length[p]}];

  Do[Qp[i,1]=p[[i]],{i,1,Length[p]}];

  Do[Qp[1,j]=p[[j]]/2,{j,2,Length[p]}];

  Do[If[(i   j),If[((2 j-1)   Length[p]),Qp[i,j]=p[[2 i-
1]]/2+p[[1]],Qp[i,j]=p[[1]]],If[(i   1&&j   1),If[(i+j-1)   Length[p],Qp[i,j]=p[[i+j-
1]]/2+p[[Abs[i-j]+1]]/2,Qp[i,j]=p[[Abs[i-j]+1]]/2]]],{i,1,Length[p]},{j,1,Length[p]}];

  Qp[1,1]=p[[1]];

  MQp

  ]
```

```
fGT[p_]:=Module[{i,j}
```

```
   MGp:=Array[G,{Length[p],Length[p]}];
   Do[If[j    1,If[i    2,G[i,j]=-1/8,G[i,j]=Cos[i Pi]/(2i(i-2))]
      ,If[Abs[i-j]    1,If[(i-j)    1,G[i,j]=-1/(4 (i-2)),If[j    2,G[i,j]=1/(4 i),G[i,j]=1/2]]
        ,G[i,j]=0]
      ] ,{i,1,Length[p]},{j,1,Length[p]}];
   Transpose[MGp];
   ]
fQD[m_,n_,tensorP_]:=Module[{i,j,k,l,A,B},
   MQD:=Array[QD,{n m,n m}];
   Do[QD[(i-1)*m+k,(j-1)*m+l]=fQp[tensorP[[i,j ]]][[k,l]],
     {i,1,n},{j,1,n},{k,1,m},{l,1,m}
     ];
   ]
fGG[GG_,m_,n_]:=Module[{i,j,k,l},
   GGT:=Array[GGTD,{n m,n m}];
   Do[GGTD[i,j]=0,{i,1,m n},{j,1,m n}];


   Do[GGTD[(i-1)*m+k,(i-1)*m+l]=GG[[k,l]],
     {i,1,n},{k,1,m},{l,1,m}
     ];


   (*GGT=Outer[Times,IdentityMatrix[n],GG]*)
```

```
   ]
fXMN[m_,n_]:=Module[{i},
   XXMN:=ZeroMatrix[n m,n];
   Do[XXMN=ReplacePart[XXMN,1,{1+m(i-1),i}],{i,1,n}];
   ]
fPhi[m_,i_,j_,x_]:=Module[{k,aaaa},aaaa:=Transpose[Take[B2,{1+m (i-1),m i},{j}]];
    Expand[aaaa.Table[ChebyshevT[k-1,2x-1],{k,m}]]
    ];
MyQC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K5=Chop[J[[1]]];
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    K2
    ];
MyRC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    Re[K2]
    ];
```

```
<<Calculus`FourierTransform`

r=32;

SetDirectory[ToFileName[

    Extract["FileName"/.NotebookInformation[EvaluationNotebook[]],

      {1},FrontEnd`FileName

      ]]];

Off[NIntegrate::ncvb]

Off[NIntegrate::ploss]

Off[General::stop]

(*Chebyshev coefficients of cos*)

pp3=Chop[Table[NFourierCosCoefficient[-Cos[Pi*Cos[t]],t,n,FourierParameters    {-
1,1/( 2*Pi)}],{n,0,r-1}]]

(***********************************************)

(*(*Chebyshev coefficients of sin*)

  Chop[Table[NFourierCosCoefficient[-Sin[Pi*Cos[t]],t,n,FourierParameters    {-1,1/(
2*Pi)}],{n,0,15}]]*)


(*pp3={0.30424217764409384`,0,0.9708678652630196`,0,-
0.30284915526269945`,0,0.02909193396501055`,0,-
0.0013922439911775739`,0,0.000040189944511023766`,0,-7.782767012882281`*^-
7,0,1.0826530292730342`*^-8,0}; *)(*pp3:Coefficients of Cos*)

(*pp4= {0,-0.5692306863595052`,0,0.6669166724059815`,0,-
0.10428236873423727`,0,0.006840633536991538 5`,0,-
```

```
0.000250006884950304760`,0,5.850248308364251`*^-6,0,-9.534772755231072`*^-
8,0,1.1456385093017214`*^-9 };
  (*pp4:Coefficients of Sin*)*)
(*************************************************************)
a=1;b=10;d=1;
r=Length[pp3];
n=2;
p3=-2 b pp3;
p3[[1]]=-a-2 b pp3[[1]];
p1=Chop[p3-p3];
p2=p1;
p2[[1]]=1;
p4=p1;
p4[[1]]=- d;
fGT[p1];

fGG[Transpose[MGp],r,n]


tensorP:=Array[tP,{n,n,r}];
Do[tP[1,1,k]= p1[[k]],{k,1,r}];
Do[tP[1,2,k]= p2[[k]],{k,1,r}];
Do[tP[2,1,k]= p3[[k]],{k,1,r}];
Do[tP[2,2,k]=p4[[k]],{k,1,r}];
```

```
fQD[r,n,tensorP];

fXMN[r,n];

B2:=Inverse[(IdentityMatrix[n r]-GGT.MQD)].XXMN

Phi[t_]:=Table[(fPhi[r,i,j,x][[1]])/.x    t,{i,1,n},{j,1,n}];

S1=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-d

x2[t],x1[0]    1,x2[0]    0},{x1,x2},{t,0,1.2}];

S2=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-d

x2[t],x1[0]    0,x2[0]    1},{x1,x2},{t,0,1.2}];

SS=IdentityMatrix[2];

SS[[1,1]]=(x1[1]/.S1)[[1]];

SS[[2,1]]=(x2[1]/.S1)[[1]];

SS[[1,2]]=(x1[1]/.S2)[[1]];

SS[[2,2]]=(x2[1]/.S2)[[1]];

Phi[1]

Print["errors=",SS-Phi[1]]

Eigenvalues[Phi[1]]//Abs

f11[x_]:=Evaluate[fPhi[16,1,1,x]];

f12[x_]:=Evaluate[fPhi[16,1,2,x]];

f21[x_]:=Evaluate[fPhi[16,2,1,x]];

f22[x_]:=Evaluate[fPhi[16,2,2,x]];

AAA:={{f11[1][[1]],f12[1][[1]]},{f21[1][[1]],f22[1][[1]]}};

CCC=MyQC[AAA,1];

 Phi[t_]:={{f11[t][[1]],f12[t][[1]]},{f21[t][[1]],f22[t][[1]]}};
```

```
Q1[t_]:=Chop[Phi[t].MatrixExp[- CCC t]]/;t    0

Q1[t_]:=Chop[Phi[(t+1)].MatrixExp[- CCC (t+1)]]/;t<0

<<Calculus`FourierTransform`

Chop[NFourierTrigSeries[Q1[t][[1,1]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working
Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[1,2]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working
Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[2,1]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working
Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[2,2]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working
Precision    4],0.00000001]

Psi[t_]:={{f11[t][[1]],f21[t][[1]]},{f12[t][[1]],f22[t][[1]]}};

Q1i[t_]:=Chop[MatrixExp[CCC t].Psi[t]]/;t    0

Q1i[t_]:=Chop[MatrixExp[CCC(t+1)].Psi[t+1]]/;t<0

Print["    11=",Phi[t][[1,1]]]//Chop

Print["    12=",Phi[t][[1,2]]]//Chop

Print["    21=",Phi[t][[2,1]]]//Chop

Print["    22=",Phi[t][[2,2]]]//Chop


(******Example 2******)

<<LinearAlgebra`MatrixManipulation`

fQp[p_]:=Module[{i,j},

   MQp:=Array[Qp,{Length[p],Length[p]}];
```

```
   Do[Qp[i,1]=p[[i]],{i,1,Length[p]}];
   Do[Qp[1,j]=p[[j]]/2,{j,2,Length[p]}];
   Do[If[(i    j),If[((2 j-1)    Length[p]),Qp[i,j]=p[[2 i-
1]]/2+p[[1]],Qp[i,j]=p[[1]]],If[(i    1&&j    1),If[(i+j-1)    Length[p],Qp[i,j]=p[[i+j-
1]]/2+p[[Abs[i-j]+1]]/2,Qp[i,j]=p[[Abs[i-j]+1]]/2]]],{i,1,Length[p]},{j,1,Length[p]}];
   Qp[1,1]=p[[1]];
   MQp
   ]
fGT[p_]:=Module[{i,j},
   MGp:=Array[G,{Length[p],Length[p]}];
   Do[If[j    1,If[i    2,G[i,j]=-1/8,G[i,j]=Cos[i Pi]/(2i(i-2))]
     ,If[Abs[i-j]    1,If[(i-j)    1,G[i,j]=-1/(4 (i-2)),If[j    2,G[i,j]=1/(4 i),G[i,j]=1/2]]
      ,G[i,j]=0]
     ] ,{i,1,Length[p]},{j,1,Length[p]}];
   Transpose[MGp];
   ]
fQD[m_,n_,tensorP_]:=Module[{i,j,k,l,A,B},
   MQD:=Array[QD,{n m,n m}];
   Do[QD[(i-1)*m+k,(j-1)*m+l]=fQp[tensorP[[i,j ]]][[k,l]],
    {i,1,n},{j,1,n},{k,1,m},{l,1,m}
    ];
   ]
fGG[GG_,m_,n_]:=Module[{i,j,k,l},
```

```
   GGT:=Array[GGTD,{n m,n m}];
   Do[GGTD[i,j]=0,{i,1,m n},{j,1,m n}];


   Do[GGTD[(i-1)*m+k,(i-1)*m+l]=GG[[k,l]],
     {i,1,n},{k,1,m},{l,1,m}
     ];


   (*GGT=Outer[Times,IdentityMatrix[n],GG]*)
   ]
fXMN[m_,n_]:=Module[{i},
   XXMN:=ZeroMatrix[n m,n];
   Do[XXMN=ReplacePart[XXMN,1,{1+m(i-1),i}],{i,1,n}];
   ]
fPhi[m_,i_,j_,x_]:=Module[{k,aaaa},aaaa:=Transpose[Take[B2,{1+m (i-1),m i},{j}]];
     Expand[aaaa.Table[ChebyshevT[k-1,2x-1],{k,m}]]
     ];
MyQC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
     J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
     J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
     K1:=J2[[2]]/T;
     K5=Chop[J[[1]]];
     K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
     K2
```

```
    ];
MyRC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    Re[K2]
    ];
<<Calculus`FourierTransform`
r=8;
(**********Pr: times of the Picard iteration****)
Pr=8;


Off[NIntegrate::ncvb]
Off[NIntegrate::ploss]
Off[General::stop]
(*Chebyshev coefficients of cos*)
pp3=Chop[Table[NFourierCosCoefficient[-Cos[Pi*Cos[t]],t,n,FourierParameters  {-
1,1/( 2*Pi)}],{n,0,r-1}]]
(**************************************************************)
(*(*Chebyshev coefficients of sin*)
  Chop[Table[NFourierCosCoefficient[-Sin[Pi*Cos[t]],t,n,FourierParameters  {-1,1/(
2*Pi)}],{n,0,15}]]*)
```

```
(*****************************************************************)
a=1;
b=10;
d=1;

r=Length[pp3];
n=2;
p3=-2  b pp3;
p3[[1]]=- a-2  b pp3[[1]];
p1=Chop[p3-p3];
p2=p1;
p2[[1]]=1;
p4=p1;
p4[[1]]=- d-K;
fGT[p1];

fGG[Transpose[MGp],r,n]

tensorP:=Array[tP,{n,n,r}];
Do[tP[1,1,k]= p1[[k]],{k,1,r}];
Do[tP[1,2,k]= p2[[k]],{k,1,r}];
Do[tP[2,1,k]= p3[[k]],{k,1,r}];
Do[tP[2,2,k]=p4[[k]],{k,1,r}];
```

```
fQD[r,n,tensorP];

fXMN[r,n];

(***************The Picard Iteration*****************)

R1=GGT.MQD;

f[p_]:=f[p]=IdentityMatrix[Length[R1]]+R1. f[p-1];

f[0]=IdentityMatrix[Length[R1]];

(********************************)

(*B2:=Inverse[(IdentityMatrix[n r]-GGT.MQD)].XXMN*)

B2=f[Pr].XXMN;

Phi[t_]:=Table[(fPhi[r,i,j,x][[1]])/.x    t,{i,1,n},{j,1,n}];

Phi[1]//Chop

S1=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-d

x2[t],x1[0]    1,x2[0]    0},{x1,x2},{t,0,1.2}];

S2=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-d

x2[t],x1[0]    0,x2[0]    1},{x1,x2},{t,0,1.2}];

SS=IdentityMatrix[2];

SS[[1,1]]=(x1[1]/.S1)[[1]];

SS[[2,1]]=(x2[1]/.S1)[[1]];

SS[[1,2]]=(x1[1]/.S2)[[1]];

SS[[2,2]]=(x2[1]/.S2)[[1]];

Phi[1]

Print["errors=",SS-Phi[1]];

f11[x_]:=Evaluate[fPhi[16,1,1,x]];
```

```
f12[x_]:=Evaluate[fPhi[16,1,2,x]];

f21[x_]:=Evaluate[fPhi[16,2,1,x]];

f22[x_]:=Evaluate[fPhi[16,2,2,x]];

AAA:={{f11[1][[1]],f12[1][[1]]},{f21[1][[1]],f22[1][[1]]}};

CCC=MyQC[AAA,1];

 Phi[t_]:={{f11[t][[1]],f12[t][[1]]},{f21[t][[1]],f22[t][[1]]}};

Q1[t_]:=Chop[Phi[t].MatrixExp[- CCC t]]/;t    0

Q1[t_]:=Chop[Phi[(t+1)].MatrixExp[- CCC (t+1)]]/;t<0

<<Calculus`FourierTransform`

Chop[NFourierTrigSeries[Q1[t][[1,1]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working

Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[1,2]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working

Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[2,1]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working

Precision    4],0.00000001]

Chop[NFourierTrigSeries[Q1[t][[2,2]],t,16,AccuracyGoal    4,PrecisionGoal    4,Working

Precision    4],0.00000001]

Psi[t_]:={{f11[t][[1]],f21[t][[1]]},{f12[t][[1]],f22[t][[1]]}};

Q1i[t_]:=Chop[MatrixExp[CCC t].Psi[t]]/;t    0

Q1i[t_]:=Chop[MatrixExp[CCC(t+1)].Psi[t+1]]/;t<0

<<Calculus`FourierTransform`

r=32;

SetDirectory[ToFileName[
```

```
   Extract["FileName"/.NotebookInformation[EvaluationNotebook[]],
     {1},FrontEnd`FileName
     ]]];
Off[NIntegrate::ncvb]
Off[NIntegrate::ploss]
Off[General::stop]
(*Chebyshev coefficients of cos*)
pp3=Chop[Table[NFourierCosCoefficient[-Cos[Pi*Cos[t]],t,n,FourierParameters  {-
1,1/( 2*Pi)}],{n,0,r-1}]]
(**************************************************************)
(*(*Chebyshev coefficients of sin*)
 Chop[Table[NFourierCosCoefficient[-Sin[Pi*Cos[t]],t,n,FourierParameters  {-1,1/(
2*Pi)}],{n,0,15}]]*)
(**************************************************************)
(*a=1;b=10;*)
(*d=1;*)
d=3.5;
r=Length[pp3];
n=2;
p3=-2 (k2+10)  pp3;
p3[[1]]=-(k1 +1)-2 ( k2 +10) pp3[[1]];
p1=Chop[p3-p3];
p2=p1;
```

```
p2[[1]]=1;

p4=p1;

p4[[1]]=- d;

fGT[p1];


fGG[Transpose[MGp],r,n]


tensorP:=Array[tP,{n,n,r}];

Do[tP[1,1,k]= p1[[k]],{k,1,r}];

Do[tP[1,2,k]= p2[[k]],{k,1,r}];

Do[tP[2,1,k]= p3[[k]],{k,1,r}];

Do[tP[2,2,k]=p4[[k]],{k,1,r}];

fQD[r,n,tensorP];

fXMN[r,n];

(***************The Picard Iteration******************)

R1=GGT.MQD//N//Expand//Chop;

SetDirectory[ToFileName[

     Extract["FileName"/.NotebookInformation[EvaluationNotebook[]],

       {1},FrontEnd`FileName

       ]]];

<<B2.nb;

Points1=Flatten[Table[{-10+(i-1)/10,-10+ (j-1)/10},{i,1,201},{j,1,201}]//N,1];

Sp[i_]:=SS/.{k1    Points1[[i,1]],k2    Points1[[i,2]]};
```

```
Condition1[i_]:=Abs[((Sp[i][[1,1]]) (Sp[i][[2,2]])+(Sp[i][[1,2]]) (Sp[i][[2,1]])-1)];

Condition2[i_]:=Abs[Sp[i][[1,1]]+Sp[i][[2,2]]];

Condition3[i_]:=Max[Abs[Eigenvalues[Sp[i]]]]>1;

Condition4[i_]:=(Max[Abs[Eigenvalues[Sp[i]]]]-1)<0.0001;

Trajectory1={};

(*Do[If[(Condition1[i]<0.0001)&&(Condition2[i]<2),Trajectory1=Append[Trajectory1,P
oints1[[i]]]] ,{i,1,201 201}]*)

Do[If[Condition3[i],Trajectory1=Append[Trajectory1,Points1[[i]]]] ,{i,1,201 201}]

ListPlot[Trajectory1,AxesLabel    {k1,k2},PlotRange    All,Frame    True]

SetDirectory[ToFileName[
    Extract["FileName"/.NotebookInformation[EvaluationNotebook[]],
     {1},FrontEnd`FileName
     ]]];

a=1;b=10;d=1;k=2.5;

S1=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-d
x2[t],x1[0]    1,x2[0]    1},{x1,x2},{t,0,6}];

S2=NDSolve[{x1'[t]    x2[t],x2'[t]    -(a+2 b Cos[2     t])x1[t]-(d+k)
x2[t],x1[0]    1,x2[0]    1},{x1,x2},{t,0,6}];

Plot[{x1[t]/.S1,x2[t]/.S1},{t,0,4},AxesLabel    {"t","x1,x2"},PlotRange    All,PlotStyle
{Thickness[0.008]}]

B2 = {{0.6865719342481182 - 0.08069108166227373*k1 + 0.003822611590545506 7*
     k1^2 - 0.00009013360708742051*k1^3 + 1.2675797625331755*^-6*k1^4 -
    1.1793044661390972*^-8*k1^5 - 0.04524538847569097*k2 +
```

0.0010288540955975904*k1*k2 - 3.97816068261472*^-6*k1^2*k2 -

1.2537699093278255*^-7*k1^3*k2 + 2.245434889859783*^-9*k1^4*k2 -

0.0023863906112368415*k2^2 + 0.00010508498332291639*k1*k2^2 -

2.0758565095518293*^-6*k1^2*k2^2 + 2.466621658376707*^-8*k1^3*k2^2 -

2.699955963178532*^-10*k1^4*k2^2 - 3.309644330378697*^-10*k1^5*k2^2 -

0.000013461210500320205*k2^3 + 4.277606064069221*^-7*k1*k2^3 -

6.1687296022377365*^-9*k1^2*k2^3 + 1.5285519094432519*^-10*k1^3*k2^3 +

1.498741013648239*^-7*k2^4 - 4.9729714906783115*^-9*k1*k2^4 -

5.612536720388906*^-10*k1^2*k2^4 + 1.5426350676919247*^-10*k2^5 +

2.2696445646465925*^-10*k2^6, 0.2474791142399776 -

0.01784685428076378*k1 + 0.0005730178937937426*k1^2 -

0.000010285718747589271*k1^3 + 1.1709653723893585*^-7*k1^4 -

9.155126844391113*^-10*k1^5 + 0.0065697635392013756*k2 -

0.00040106947117079904*k1*k2 + 9.363727897416942*^-6*k1^2*k2 -

1.221307386491147*^-7*k1^3*k2 + 1.039149898124552*^-9*k1^4*k2 -

0.00007173318439933516*k2^2 + 3.3281730516126428*^-6*k1*k2^2 -

6.898934036253412*^-8*k1^2*k2^2 + 8.349164907118895*^-10*k1^3*k2^2 -

3.342113696910365*^-10*k1^4*k2^2 - 1.4641172700881379*^-6*k2^3 +

4.746332029845172*^-8*k1*k2^3 - 7.682352776771196*^-10*k1^2*k2^3 +

2.325105953647929*^-9*k2^4 + 1.7087189550762184*^-10*k1*k2^4 -

2.1317022679807038*^-10*k2^5}, {-0.14608913209930285 -

0.09829516345045115*k1 + 0.005787472397713169*k1^2 -

0.00014914715018517829*k1^3 + 2.2007873225273463*^-6*k1^4 -

2.1099974777561947*^-8*k1^5 - 0.03834898982550318*k2 +

0.0013750245852832714*k1*k2 - 0.000011707614318506453*k1^2*k2 -

8.80784998269825*^-8*k1^3*k2 + 2.635081400398317*^-9*k1^4*k2 -

0.0035763084873020835*k2^2 + 0.00017000938132054729*k1*k2^2 -

3.5759107584560046*^-6*k1^2*k2^2 + 4.425987548900092*^-8*k1^3*k2^2 -

4.362875786163933*^-10*k1^4*k2^2 - 5.079295284822199*^-10*k1^5*k2^2 -

0.000016028251068860976*k2^3 + 5.416942330274015*^-7*k1*k2^3 -

8.344590103782614*^-9*k1^2*k2^3 + 2.2629959255816933*^-10*k1^3*k2^3 +

2.877918309361939*^-7*k2^4 - 9.74174056530765*^-9*k1*k2^4 -

8.415070727081383*^-10*k1^2*k2^4 + 1.4757433124923814*^-10*k2^5 +

3.654148737328856*^-10*k2^6, 0.22484615457256246 -

0.025046477217384185*k1 + 0.0009138116817440475*k1^2 -

0.0000174798132451161*k1^3 + 2.065569269385665*^-7*k1^4 -

1.6569386108430485*^-9*k1^5 + 0.01094275197425743*k2 -

0.0006271393303526463*k1*k2 + 0.000015010287446995741*k1^2*k2 -

2.021160740012899*^-7*k1^3*k2 + 1.7676605076860872*^-9*k1^4*k2 -

0.00012018419214645767*k2^2 + 6.204265938065328*^-6*k1*k2^2 -

1.3388175336779916*^-7*k1^2*k2^2 + 1.640142106305592*^-9*k1^3*k2^2 -

5.491777917023574*^-10*k1^4*k2^2 - 2.206852479460763*^-6*k2^3 +

7.745564141319949*^-8*k1*k2^3 - 1.291782494315402*^-9*k1^2*k2^3 +

7.217819305231364*^-9*k2^4 - 3.01120037562835*^-10*k2^5},

{0.18902553553601925 - 0.010651155635327782*k1 + 0.0022582212531815932*

k1^2 - 0.00008183718311484685*k1^3 + 1.4221600008227732*^-6*k1^4 -

1.500034009026173*^-8*k1^5 + 0.010364890805578598*k2 +

0.0006652545649601205*k1*k2 - 0.000023695428690447*k1^2*k2 +

2.8288640760394445*^-7*k1^3*k2 - 1.6129592723041613*^-9*k1^4*k2 -

0.0009205362828245027*k2^2 + 0.00008124848976629302*k1*k2^2 -

2.238706831629147*^-6*k1^2*k2^2 + 3.186639178286482*^-8*k1^3*k2^2 -

2.6969386473118883*^-10*k1^4*k2^2 - 2.0267232278492077*^-10*k1^5*k2^2 +

7.369211804018406*^-6*k2^3 - 1.5086384268683893*^-7*k1*k2^3 +

7.21597508626859*^-10*k1^2*k2^3 + 2.5755162887231244*^-7*k2^4 -

9.024753839402215*^-9*k1*k2^4 - 2.8864350740174383*^-10*k1^2*k2^4 -

2.3770166001118024*^-10*k2^5 + 1.7950529338349352*^-10*k2^6,

-0.012665127402091941 - 0.006975153408791228*k1 +

0.0004386483974352188*k1^2 - 0.000010518763518865865*k1^3 +

1.4055447503731173*^-7*k1^4 - 1.2183467040281543*^-9*k1^5 +

0.003419486963211281*k2 - 0.00022054346636002112*k1*k2 +

6.518684854963199*^-6*k1^2*k2 - 1.0360628599705529*^-7*k1^3*k2 +

1.0156224886716408*^-9*k1^4*k2 - 0.0000727266775059048*k2^2 +

5.186832002717018*^-6*k1*k2^2 - 1.2199732840347404*^-7*k1^2*k2^2 +

1.5358244061520102*^-9*k1^3*k2^2 - 2.898152736207847*^-10*k1^4*k2^2 -

6.581854486236083*^-7*k2^3 + 3.7697513292597015*^-8*k1*k2^3 -

7.336224211862382*^-10*k1^2*k2^3 + 1.2502589738748881*^-8*k2^4 -

2.7437880631903894*^-10*k1*k2^4},

{-0.06560509045434293 + 0.011701317755914207*k1 +

0.00008106687365241238*k1^2 - 0.000025330800350698 77*k1^3 +

6.516772486791577*^-7*k1^4 - 8.297855718933467*^-9*k1^5 +

0.00010981110934023004*k2 + 0.0008568563888403279*k1*k2 -

0.00003594527123974286*k1^2*k2 + 5.877068644196066*^-7*k1^3*k2 -

5.346114363083575*^-9*k1^4*k2 + 0.0009362753673958152*k2^2 +

8.279230406935716*^-6*k1*k2^2 - 9.431101340104063*^-7*k1^2*k2^2 +

1.8203040770137222*^-8*k1^3*k2^2 - 1.8089949914677303*^-10*k1^4*k2^2 +

0.000024891137326984 91*k2^3 - 7.064999130390062*^-7*k1*k2^3 +

8.652374355567557*^-9*k1^2*k2^3 + 2.1414229304725694*^-7*k2^4 -

7.569943637958198*^-9*k1*k2^4 - 5.809072761543526*^-10*k2^5,

-0.01919986129083429 + 0.0014370638894713578*k1 +

0.00008759501912611781*k1^2 - 4.120792230562658*^-6*k1^3 +

7.156917312536278*^-8*k1^4 - 7.15046768861106*^-10*k1^5 -

0.0038308844651978 14*k2 + 0.00012430292777491613*k1*k2 -

1.0545030136346094*^-6*k1^2*k2 - 8.714462801215442*^-9*k1^3*k2 +

2.4200635723382443*^-10*k1^4*k2 - 0.000048358628434828956*k2^2 +

4.159295178585824*^-6*k1*k2^2 - 1.0244490018227333*^-7*k1^2*k2^2 +

1.3162203580203456*^-9*k1^3*k2^2 + 5.322614087012994*^-7*k2^3 +

1.577268800545411*^-9*k1*k2^3 - 2.090179939620689*^-10*k1^2*k2^3 +

1.5469561069073275*^-8*k2^4 - 4.0273360778694214*^-10*k1*k2^4},

{-0.07238525406917634 + 0.007784566489363763*k1 -

0.0003485888702237614*k1^2 - 2.1265164502288696*^-7*k1^3 +

1.8063616775016348*^-7*k1^4 - 3.41655321868897*^-9*k1^5 -

0.0009101970220332495*k2 + 0.0007903392448182499*k1*k2 -

0.000033182863888137304*k1^2*k2 + 6.115689729721845*^-7*k1^3*k2 -
6.39910003396242*^-9*k1^4*k2 + 0.0008617492131782916*k2^2 -
0.000015025200595766529*k1*k2^2 - 2.0486160094909012*^-7*k1^2*k2^2 +
7.984933553350342*^-9*k1^3*k2^2 - 1.1830036937844717*^-10*k1^4*k2^2 +
0.00002531959782662861*k2^3 - 7.919288111941197*^-7*k1*k2^3 +
1.1062177798341639*^-8*k1^2*k2^3 + 1.5245364620697545*^-7*k2^4 -
5.298831992561518*^-9*k1*k2^4 + 1.1330992471989609*^-10*k1^2*k2^4 -
6.552935563937015*^-10*k2^5, -0.03406024759083293 +
0.0017956218295441645*k1 - 0.000029753716505485522*k1^2 -
7.082360550188495*^-7*k1^3 + 2.5110761608639193*^-8*k1^4 -
3.268138650447771*^-10*k1^5 - 0.0037198269500360037*k2 +
0.00019864917904854296*k1*k2 - 3.891335791708753*^-6*k1^2*k2 +
3.885740015650152*^-8*k1^3*k2 - 2.1438172036294043*^-10*k1^4*k2 -
0.00005203038301965852*k2^2 + 3.127890679021161*^-6*k1*k2^2 -
7.477237909419038*^-8*k1^2*k2^2 + 9.746874456251645*^-10*k1^3*k2^2 +
7.542365022286381*^-7*k2^3 - 1.436576951457329*^-8*k1*k2^3 +
1.3059231427937338*^-8*k2^4 - 3.5946584340206547*^-10*k1*k2^4},
{0.015568568683628332 + 0.0025944352301556653*k1 -
0.00016452818176828342*k1^2 + 3.5567899174160653*^-6*k1^3 +
5.7844534100928265*^-9*k1^4 - 9.215877251265704*^-10*k1^5 +
0.0015990237748969454*k2 + 0.0002133010971982127*k1*k2 -
0.000016341602686966673*k1^2*k2 + 3.888227759551692*^-7*k1^3*k2 -
4.84711092888425*^-9*k1^4*k2 + 0.00012124093763963097*k2^2 -

7.956037658026953*^-6*k1*k2^2 + 2.9138056221612708*^-8*k1^2*k2^2 +
2.328417628752355*^-9*k1^3*k2^2 + 0.000012346354577533964*k2^3 -
4.873737942469483*^-7*k1*k2^3 + 8.261638496702983*^-9*k1^2*k2^3 +
7.282146695912064*^-8*k2^4 - 2.5971346316923448*^-9*k1*k2^4 -
4.919273464862767*^-10*k2^5, -0.002000327439812378 +
0.0006560733079334591*k1 - 0.000025910657244238277*k1^2 +
2.2673092595518207*^-7*k1^3 + 4.408385472371323*^-9*k1^4 -
1.0821984298362866*^-10*k1^5 - 0.0006539376898791263*k2 +
0.0000906933814368835 2*k1*k2 - 2.746444354248204*^-6*k1^2*k2 +
3.8648866164184414*^-8*k1^3*k2 - 3.1323651679580423*^-10*k1^4*k2 -
0.000038446348413744*k2^2 + 1.760068871529251*^-6*k1*k2^2 -
4.1810429922516166*^-8*k1^2*k2^2 + 5.786283947647797*^-10*k1^3*k2^2 +
3.414458510642609*^-7*k2^3 - 1.2811211372557961*^-8*k1*k2^3 +
1.6704191797827615*^-10*k1^2*k2^3 + 6.754058231724862*^-9*k2^4 -
2.2619053185201984*^-10*k1*k2^4},
{0.00021902716501885243 - 0.0009653313165655964*k1 -
0.000016771437900173875*k1^2 + 1.4103723513663441*^-6*k1^3 -
1.8471539827366394*^-8*k1^4 - 0.0010084369288008636*k2 -
0.00009290130819124341*k1*k2 - 1.7247180084081628*^-6*k1^2*k2 +
1.3264949630400734*^-7*k1^3*k2 - 2.389692136004509*^-9*k1^4*k2 -
0.0001076195512621311*k2^2 - 1.003161909766653*^-6*k1*k2^2 +
5.8612178337422105*^-8*k1^2*k2^2 - 4.3831769748533763*^-7*k2^3 -
1.2586895225982893*^-7*k1*k2^3 + 3.767721977870702*^-9*k1^2*k2^3 -

2.5558971059203353*^-9*k2^4 - 2.845587622995181*^-10*k1*k2^4 -

2.8683643621755217*^-10*k2^5, 0.0036433033957657414 -

0.00002774515792056787*k1 - 6.051943543335589*^-6*k1^2 +

1.7652934412248727*^-7*k1^3 - 7.895525348894021*^-10*k1^4 +

0.0003380693709316469*k2 - 3.1641879928739845*^-7*k1*k2 -

7.612304965229284*^-7*k1^2*k2 + 1.8619120288232406*^-8*k1^3*k2 -

2.1032029808449542*^-10*k1^4*k2 - 3.4394868856128186*^-6*k2^2 +

3.3256733496437664*^-7*k1*k2^2 - 1.2473307015638355*^-8*k1^2*k2^2 +

2.3019491260643945*^-10*k1^3*k2^2 - 1.7475266838356367*^-8*k2^3 -

6.305419986903187*^-9*k1*k2^3 + 1.3995234900429787*^-10*k1^2*k2^3 +

7.729795486680639*^-10*k2^4}, {0.0011819191687029718 -

0.0005926544551812647*k1 + 0.000020895863136228 47*k1^2 +

1.462597437926607*^-8*k1^3 - 6.803149484717117*^-9*k1^4 +

0.0000979931230037442*k2 - 0.000006899485403057641*k1*k2 +

2.603627260383165*^-6*k1^2*k2 - 3.6045563102499083*^-9*k1^3*k2 -

6.082251303708046*^-10*k1^4*k2 - 0.000003906151551787698*k2^2 -

5.362201662198358*^-7*k1*k2^2 + 5.3805041095683185*^-8*k1^2*k2^2 -

7.799752061244822*^-10*k1^3*k2^2 - 4.107522542402518*^-6*k2^3 +

5.067197112905324*^-8*k1*k2^3 + 6.870944739369824*^-10*k1^2*k2^3 -

4.1920947431349836*^-8*k2^4 + 9.161817724163123*^-10*k1*k2^4 -

1.7039210824025783*^-10*k2^5, 0.0010495036661260252 -

0.0001351363349221007*k1 + 1.5640115346312334*^-6*k1^2 +

3.1201861501200435*^-8*k1^3 - 7.005700647968115*^-10*k1^4 +

0.00024068029281987 55*k2 - 0.000017162686474622182*k1*k2 +

1.8012820420043855*^-7*k1^2*k2 + 3.1043647157831823*^-9*k1^3*k2 +

0.000013370871306373435*k2^2 - 4.1692292141079103*^-7*k1*k2^2 +

3.957319737744039*^-9*k1^2*k2^2 - 4.34626998734875*^-8*k2^3 -

2.8784145438608177*^-9*k1*k2^3 - 1.882361556262116*^-9*k2^4},

{0.0013165930338620887 - 0.000061028929391251 23*k1 +

0.000010668597427025 68*k1^2 - 2.0816096922821026*^-7*k1^3 +

3.1584540249209054*^-10*k1^4 + 0.00020545750394208 71*k2 -

0.000016560009626005255*k1*k2 + 1.4329486675801186*^-6*k1^2*k2 -

2.6832782363612387*^-8*k1^3*k2 - 6.441324042070605*^-6*k2^2 -

6.082589860661423*^-7*k1*k2^2 + 3.5083567815687146*^-8*k1^2*k2^2 -

6.673693823851489*^-10*k1^3*k2^2 - 1.7305275785637302*^-6*k2^3 +

5.2928765302707835*^-8*k1*k2^3 - 3.214938615852799*^-10*k1^2*k2^3 -

3.610578692772065*^-8*k2^4 + 9.477919212963374*^-10*k1*k2^4 -

1.181824931618037*^-10*k2^5, 0.0002774890135655474 -

0.0000343014322485322*k1 + 1.5846732027423107*^-6*k1^2 -

1.4622514597337195*^-8*k1^3 - 1.0220986308118861*^-10*k1^4 +

0.00008860841312090514*k2 - 6.877776082006838*^-6*k1*k2 +

2.218316533860155*^-7*k1^2*k2 - 1.9059470745130726*^-9*k1^3*k2 +

6.692179008396649*^-6*k2^2 - 3.652024489148935*^-7*k1*k2^2 +

6.662404823872721*^-9*k1^2*k2^2 + 4.77782492331468*^-8*k2^3 -

2.085589086633591*^-9*k1*k2^3 - 1.479419471176439*^-9*k2^4},

{-0.0004995639879485391 + 0.000027338919703328816*k1 +

5.754383126231044*^-7*k1^2 - 8.309024831703801*^-8*k1^3 +

1.140782233620856*^-9*k1^4 - 0.00006567494606291447*k2 +

2.6293209299393184*^-6*k1*k2 + 1.6422557941143536*^-7*k1^2*k2 -

1.1308557620791008*^-8*k1^3*k2 + 1.4808867447376958*^-10*k1^4*k2 +

1.8451040976801006*^-6*k2^2 + 6.153799006830931*^-8*k1*k2^2 +

8.657749188710278*^-9*k1^2*k2^2 - 2.917563507306511*^-10*k1^3*k2^2 +

2.482290611981015*^-7*k2^3 + 1.1159649581576714*^-8*k1*k2^3 -

2.5780335579052853*^-10*k1^2*k2^3 - 9.959658377192536*^-9*k2^4 +

4.139642858230944*^-10*k1*k2^4, -0.00011635012013934226 +

3.4835173909156256*^-6*k1 + 3.5208171849339113*^-7*k1^2 -

1.0325256333275847*^-8*k1^3 - 0.000021832339100382015*k2 -

2.603613624060919*^-7*k1*k2 + 6.219576556423533*^-8*k1^2*k2 -

1.4199245021559259*^-9*k1^3*k2 - 5.311077616210216*^-7*k2^2 -

7.411780623424885*^-8*k1*k2^2 + 2.9871945888520324*^-9*k1^2*k2^2 +

4.701253848764328*^-8*k2^3 - 1.1700916999382466*^-9*k1*k2^3 -

2.2902640082298321*^-10*k2^4}, {-0.0001497855016896467 +

0.000026694049172090454*k1 - 1.012359407928965*^-6*k1^2 -

1.902501099522302*^-9*k1^3 + 4.009399461849645*^-10*k1^4 -

6.233622751858356*^-6*k2 + 5.3512432189748235*^-6*k1*k2 -

1.4808463095716798*^-7*k1^2*k2 - 4.1053616748976504*^-10*k1^3*k2 +

3.958775972712404*^-6*k2^2 + 2.2508454890591554*^-7*k1*k2^2 -

5.220083758257882*^-9*k1^2*k2^2 + 3.684790554993311*^-7*k2^3 -

4.477975630360526*^-9*k1*k2^3 + 6.078268446870244*^-9*k2^4,

-0.00010256885444719134 + 5.110195729473194*^-6*k1 -

8.543277156134285*^-8*k1^2 - 1.908866522818*^-9*k1^3 -

0.000022071132072411017*k2 + 9.97945436352563*^-7*k1*k2 -

9.01004951672994*^-9*k1^2*k2 - 2.9801026948009925*^-10*k1^3*k2 -

1.2478180580358085*^-6*k2^2 + 4.924256750663165*^-8*k1*k2^2 -

3.772737607537116*^-9*k2^3 + 2.9839584398769877*^-10*k2^4},

{0.000039820410418994479 + 5.756464762388507*^-6*k1 -

4.10784115678185*^-7*k1^2 + 9.421064170027428*^-9*k1^3 +

4.568937493938613*^-6*k2 + 1.2702099634070224*^-6*k1*k2 -

8.522584796878088*^-8*k1^2*k2 + 1.4950340544027648*^-9*k1^3*k2 +

1.9814377897804972*^-7*k2^2 + 6.016369293236779*^-8*k1*k2^2 -

4.614772920797102*^-9*k1^2*k2^2 + 6.801968837908764*^-8*k2^3 -

2.0479370743254216*^-9*k1*k2^3 + 5.792538668374704*^-9*k2^4 -

1.1984096647657936*^-10*k1*k2^4, -5.152907955787769*^-8 +

1.7259471759358894*^-6*k1 - 6.615707192395944*^-8*k1^2 +

6.208126482914444*^-10*k1^3 - 1.8527794533196355*^-6*k2 +

4.424521692545122*^-7*k1*k2 - 1.242527017452164*^-8*k1^2*k2 -

3.7342041375840755*^-7*k2^2 + 3.247894059143559*^-8*k1*k2^2 -

6.491913836727287*^-10*k1^2*k2^2 - 1.770952646174356*^-8*k2^3 +

5.398306197657714*^-10*k1*k2^3 + 1.4073572949790806*^-10*k2^4},

{-1.0424054699740614*^-6 - 2.649543148678606*^-6*k1 -

2.8427141193917528*^-8*k1^2 + 3.15715559560141*^-9*k1^3 -

3.13254649214306*^-6*k2 - 4.448993824724501*^-7*k1*k2 -

1.0303674266785298*^-8*k1^2*k2 + 6.200468599910334*^-10*k1^3*k2 -

5.5977622782946*^-7*k2^2 - 1.1911026751431348*^-8*k1*k2^2 -

8.783401224281662*^-10*k1^2*k2^2 - 1.9536857532232218*^-8*k2^3 +

1.558331476602278*^-10*k1*k2^3 + 8.644040071146774*^-10*k2^4,

8.519872630852165*^-6 - 1.6238702903902745*^-7*k1 -

1.2588189078225454*^-8*k1^2 + 4.184119606575382*^-10*k1^3 +

1.3758747662137388*^-6*k2 - 6.821334393985533*^-9*k1*k2 -

3.384936701576806*^-9*k1^2*k2 + 1.2141775513890609*^-8*k2^2 +

3.641163254243375*^-9*k1*k2^2 - 2.65006132364564*^-10*k1^2*k2^2 -

4.642282936138123*^-9*k2^3 + 2.753897257407621*^-10*k1*k2^3},

{2.868677120854997*^-6 - 1.0679796424743014*^-6*k1 +

4.4383011363119055*^-8*k1^2 + 8.903854262341056*^-7*k2 -

2.2629893070085874*^-7*k1*k2 + 8.805329690925846*^-9*k1^2*k2 +

1.0139782334109736*^-9*k2^2 - 1.2321502207873293*^-8*k1*k2^2 +

4.394172624374197*^-10*k1^2*k2^2 - 1.5024372734325912*^-8*k2^3 -

9.006887741051478*^-10*k2^4, 1.773785680610865*^-6 -

2.6697973254060735*^-7*k1 + 3.469183962951295*^-9*k1^2 +

6.801448192423934*^-7*k2 - 5.801149838266143*^-8*k1*k2 +

5.219811837278272*^-10*k1^2*k2 + 7.867003568971174*^-8*k2^2 -

3.402629381685462*^-9*k1*k2^2 + 2.3799266501724864*^-9*k2^3},

{1.7698360027368502*^-6 - 1.2621290922712189*^-8*k1 +

1.6232422204825155*^-8*k1^2 - 3.585675931270632*^-10*k1^3 +

5.064452456461355*^-7*k2 - 2.934672558237598*^-8*k1*k2 +

3.926168243847217*^-9*k1^2*k2 + 2.13195521615323*^-8*k2^2 -

4.692444578839109*^-9*k1*k2^2 + 2.8729111724378645*^-10*k1^2*k2^2 -

4.577469074637808*^-9*k2^3 - 4.1486309844702445*^-10*k2^4,

2.3888735951253825*^-7 - 4.2577870754719866*^-8*k1 +

2.6250187598313635*^-9*k1^2 + 1.6711297000872748*^-7*k2 -

1.5130772696324853*^-8*k1*k2 + 6.124887635917871*^-10*k1^2*k2 +

2.754234814242243*^-8*k2^2 - 1.7702642727617086*^-9*k1*k2^2 +

1.4173175106945746*^-9*k2^3}, {-8.547203533935696*^-7 +

6.483033157010924*^-8*k1 - 6.70757576961258*^-10*k1^2 -

1.0752920047793402*^-10*k1^3 - 1.9463853689907832*^-7*k2 +

1.157307854323379*^-8*k1*k2 + 1.2041706764273074*^-10*k1^2*k2 -

4.728214497341464*^-9*k2^2 + 4.0460900819389693*^-10*k1*k2^2 +

1.1698906709813783*^-9*k2^3, -2.4222963006213527*^-7 +

1.2203392318035233*^-8*k1 + 3.5022388784390427*^-10*k1^2 -

7.105591633053183*^-8*k2 + 1.1794640831086018*^-9*k1*k2 +

1.2032371814289955*^-10*k1^2*k2 - 5.458958386339526*^-9*k2^2 -

1.6774152995470454*^-10*k1*k2^2},

{-1.7844223417625078*^-7 + 3.221059801165259*^-8*k1 -

1.6375922907630177*^-9*k1^2 - 2.346569059467358*^-8*k2 +

1.0533106834996619*^-8*k1*k2 - 3.7078355162633715*^-10*k1^2*k2 +

5.157956024090509*^-9*k2^2 + 1.034632946585541*^-9*k1*k2^2 +

1.0889556568228067*^-9*k2^3, -1.210032228233588*^-7 +

7.0758098833958205*^-9*k1 - 1.7753675082857249*^-10*k1^2 -

4.0728629231402676*^-8*k2 + 2.1304162504064735*^-9*k1*k2 -

4.509409800413198*^-9*k2^2 + 2.04416017689288*^-10*k1*k2^2 -

1.5705776891529699*^-10*k2^3}, {6.789000122736629*^-8 +

2.9536529942115594*^-9*k1 - 3.836925265780288*^-10*k1^2 +

1.1794991738704581*^-8*k2 + 1.3684758262380906*^-9*k1*k2 -

1.3179157261022437*^-10*k1^2*k2 + 1.7908820922925178*^-10*k1*k2^2,

1.1792404397149276*^-8 + 1.3268810727129906*^-9*k1 +

1.0062900000846961*^-9*k2 + 5.973150241257226*^-10*k1*k2 -

4.5254407549096197*^-10*k2^2}, {3.3077167395589628*^-9 -

3.777777383622085*^-9*k1 - 2.753829245830814*^-9*k2 -

1.0282238457334049*^-9*k1*k2 - 1.0125850808029243*^-9*k2^2,

1.1720064782559736*^-8 - 4.3912035970377414*^-10*k1 +

3.222597937410066*^-9*k2 + 2.3831008637615887*^-10*k2^2},

{6.893354703451351*^-10 - 9.225896511046787*^-10*k1 +

9.585879494557266*^-10*k2 - 3.019954489659425*^-10*k1*k2 +

1.3369799079761165*^-10*k2^2, 9.45570393546005*^-10 -

2.684566191386621*^-10*k1 + 6.852876656122355*^-10*k2 +

1.4370154262004875*^-10*k2^2}, {1.0588635479410036*^-9 +

1.497125518287265*^-10*k1 + 5.458862536296939*^-10*k2,

-2.7590808815229425*^-10}, {-6.806624696125504*^-10 -

2.4495455128623444*^-10*k2, -2.2173673258445025*^-10},

{-1.3039232894809*^-10, 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},

{0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},

{-0.5216411015722338 - 0.1081422415786229*k1 + 0.010711049580049361*
k1^2 - 0.0004157982608985171*k1^3 + 8.294487617220413*^-6*k1^4 -
1.0050861861866727*^-7*k1^5 + 1.0989896478911721*^-9*k1^6 -
0.05974086500316877*k2 + 0.009128042367456788*k1*k2 -
0.0003647423784155024*k1^2*k2 + 7.017091456804912*^-6*k1^3*k2 -
8.120737711975361*^-8*k1^4*k2 + 5.853194556354424*^-10*k1^5*k2 -
0.0008319555786624432*k2^2 + 0.0003047546973735876*k1*k2^2 -
0.000011699250838806513*k1^2*k2^2 + 2.0624755446554948*^-7*k1^3*k2^2 -
1.87194437615144*^-9*k1^4*k2^2 + 1.565957875935477*^-9*k1^5*k2^2 +
0.00018885744219607916*k2^3 - 7.1634180622663965*^-6*k1*k2^3 +
1.2260104267889153*^-7*k1^2*k2^3 - 1.4525540718925086*^-9*k1^3*k2^3 +
2.1813936582432376*^-10*k1^4*k2^3 + 1.927552959194928*^-6*k2^4 -
7.30690854266138*^-8*k1*k2^4 + 3.385122626484897*^-9*k1^2*k2^4 -
1.0561429518243235*^-10*k1^3*k2^4 - 9.085186283223133*^-9*k2^5 +
1.4447288381068421*^-10*k1*k2^5 - 6.062675243205101*^-10*k2^6,
0.32712676244247385 - 0.03670761221332851*k1 + 0.0021209633842483935*
k1^2 - 0.000057151244129368036*k1^3 + 8.782026050054193*^-7*k1^4 -
8.686356230598511*^-9*k1^5 - 0.004687790427364982*k2 +
0.0001617867683597301*k1*k2 - 3.864435426018995*^-7*k1^2*k2 -
4.98176116174484*^-8*k1^3*k2 + 9.511034401885035*^-10*k1^4*k2 -
0.0007436954974680387*k2^2 + 0.000048417537678891179*k1*k2^2 -
1.2051162030862181*^-6*k1^2*k2^2 + 1.649256085061826*^-8*k1^3*k2^2 +
8.360507603832819*^-10*k1^4*k2^2 + 2.1043384092238325*^-6*k2^3 -

1.4206428682111307*^-8*k1*k2^3 - 6.091463845095541*^-10*k1^2*k2^3 +

1.4650459262616284*^-7*k2^4 - 5.695277896123595*^-9*k1*k2^4 +

7.991736352372634*^-10*k2^5}, {0.39547439519283645 +

0.015164620981482059*k1 + 0.01238617720558177*k1^2 -

0.000630856069487837*k1^3 + 0.000013848120369157428*k1^4 -

1.7598093657724229*^-7*k1^5 + 2.0398987367853294*^-9*k1^6 +

0.0503551220288178*k2 + 0.015399565824040697*k1*k2 -

0.0007222497093314125*k1^2*k2 + 0.000014382445077149 62*k1^3*k2 -

1.6803258683813276*^-7*k1^4*k2 + 1.22144158317882*^-9*k1^5*k2 +

0.0037676446115619583*k2^2 + 0.0003742738711582 1056*k1*k2^2 -

0.000018894578505907677*k1^2*k2^2 + 3.6016142013109855*^-7*k1^3*k2^2 -

3.543249312341395*^-9*k1^4*k2^2 + 2.0719557891575182*^-9*k1^5*k2^2 -

1.0275899445481102*^-10*k1^6*k2^2 + 0.00041177847794964347*k2^3 -

0.000015380631358268 45*k1*k2^3 + 2.6000763227101576*^-7*k1^2*k2^3 -

2.892181155337544*^-9*k1^3*k2^3 + 3.283890219593594*^-10*k1^4*k2^3 +

3.5478432063018143*^-6*k2^4 - 1.3610611586817768*^-7*k1*k2^4 +

5.290796366231226*^-9*k1^2*k2^4 - 2.2924646192558455*^-10*k1^3*k2^4 -

1.9643978600523926*^-8*k2^5 + 3.5157350129523015*^-10*k1*k2^5 -

9.155947254245074*^-10*k2^6, -0.5536534004551461 -

0.02864015797845602*k1 + 0.0029347045028362403*k1^2 -

0.00009177088939233034*k1^3 + 1.5055812168911018*^-6*k1^4 -

1.544456269817233*^-8*k1^5 + 1.531696605433135*^-10*k1^6 -

0.022022021552904985*k2 + 0.0012252279768402259*k1*k2 -

0.0000217736725816295*k1^2*k2 + 1.7016474059733155*^-7*k1^3*k2 -

3.5281477057224106*^-10*k1^4*k2 - 0.0013307549348591282*k2^2 +

0.00008989413031972798*k1*k2^2 - 2.2700830855342216*^-6*k1^2*k2^2 +

3.13005054006344*^-8*k1^3*k2^2 + 1.118593694412773*^-9*k1^4*k2^2 +

7.60737701616187*^-6*k2^3 - 1.3419317576543108*^-7*k1*k2^3 +

4.6494185014820494*^-10*k1^2*k2^3 + 2.9003287477706284*^-7*k2^4 -

1.0503383267475853*^-8*k1*k2^4 + 1.529940906111082*^-10*k1^2*k2^4 +

1.135743178592326*^-9*k2^5}, {-0.45892567474725976 +

0.1768961706445578*k1 - 0.0017277904307538207*k1^2 -

0.000235007921056328*k1^3 + 7.785825944331786*^-6*k1^4 -

1.1661733801091594*^-7*k1^5 + 1.3683644989837853*^-9*k1^6 +

0.03391422929567504*k2 + 0.012755986393780411*k1*k2 -

0.0006826542995569794*k1^2*k2 + 0.000014386496912917683*k1^3*k2 -

1.7295507984109215*^-7*k1^4*k2 + 1.3079410023318786*^-9*k1^5*k2 +

0.012641322791883437*k2^2 - 0.000070528130535501823*k1*k2^2 -

9.09485864230618*^-6*k1^2*k2^2 + 2.35455607141001*^-7*k1^3*k2^2 -

2.8172798409401848*^-9*k1^4*k2^2 + 3.702239775240105*^-10*k1^5*k2^2 +

0.00044182788866760093*k2^3 - 0.000016493613056642543*k1*k2^3 +

2.785804457729142*^-7*k1^2*k2^3 - 2.822405684498775*^-9*k1^3*k2^3 +

1.295502942391051*^-10*k1^4*k2^3 + 2.7039385946450652*^-6*k2^4 -

1.0717120859199367*^-7*k1*k2^4 + 2.29550731877031*^-9*k1^2*k2^4 -

1.4605473761463192*^-10*k1^3*k2^4 - 2.177480539017558*^-8*k2^5 +

4.721932200790794*^-10*k1*k2^5 - 1.0591298005557997*^-10*k1^2*k2^5 -

2.4150740560289875*^-10*k2^6 + 1.0494545134500323*^-10*k1*k2^6,

-0.2451310934053034 + 0.02677068444288042*k1 + 0.0005866800415205202*

k1^2 - 0.00004438323527826953*k1^3 + 9.30177502256578*^-7*k1^4 -

1.0744958104886073*^-8*k1^5 + 1.1773647707573092*^-10*k1^6 -

0.05314658875175977*k2 + 0.0028321308581301303*k1*k2 -

0.0000608140368731925*k1^2*k2 + 7.088290727703619*^-7*k1^3*k2 -

5.1684351500948264*^-9*k1^4*k2 - 0.0010066542263502485*k2^2 +

0.00007201801160552519*k1*k2^2 - 1.874705392701344*^-6*k1^2*k2^2 +

2.642455327601251*^-8*k1^3*k2^2 - 1.1934018590936397*^-10*k1^5*k2^2 +

0.000013036086736290 81*k2^3 - 3.3823542301702357*^-7*k1*k2^3 +

3.9491181386635236*^-9*k1^2*k2^3 + 2.641379080314111*^-7*k2^4 -

8.547406655241897*^-9*k1*k2^4 + 2.435208188811712*^-10*k1^2*k2^4 +

2.0689819121523454*^-10*k2^5},

{-1.1167298890953201 + 0.1003738660641037*k1 - 0.005679592819870886*

k1^2 + 0.00023841395430930483*k1^3 + 2.470840362575637*^-6*k1^4 -

5.597821574084389*^-8*k1^5 + 5.406263310586539*^-10*k1^6 -

0.03256400441581103*k2 + 0.010077529304359664*k1*k2 -

0.0005326862798078377*k1^2*k2 + 0.000012119353816317833*k1^3*k2 -

1.5512891265968616*^-7*k1^4*k2 + 1.252513328396919*^-9*k1^5*k2 +

0.011131934874157982*k2^2 - 0.0002757140469721379*k1*k2^2 -

9.849238528744355*^-7*k1^2*k2^2 + 1.0523028603264826*^-7*k1^3*k2^2 -

1.6499384871494439*^-9*k1^4*k2^2 - 3.7925971182280507*^-10*k1^5*k2^2 +

0.0003528247835174949*k2^3 - 0.000014173720616773821*k1*k2^3 +

2.5423485220199586*^-7*k1^2*k2^3 - 2.591868092413522*^-9*k1^3*k2^3 +

1.4874301753233131*^-6*k2^4 - 6.390808515295366*^-8*k1*k2^4 +

3.2377088857948816*^-10*k1^2*k2^4 - 2.0657903548956065*^-8*k2^5 +

5.343500662118088*^-10*k1*k2^5 - 1.1507185191630817*^-10*k1^2*k2^5 +

2.1451699267006644*^-10*k2^6, -0.4523323812384123 +

0.027161069291874514*k1 - 0.0005744826766455903*k1^2 -

7.620781241400967*^-6*k1^3 + 3.8114541659263093*^-7*k1^4 -

5.697789140161297*^-9*k1^5 - 0.049377917258595354*k2 +

0.0029895757077204764*k1*k2 - 0.00007392315142134155*k1^2*k2 +

9.990150285739184*^-7*k1^3*k2 - 8.477794679668801*^-9*k1^4*k2 -

0.0007489415148118905*k2^2 + 0.000048399474297994366*k1*k2^2 -

1.2941044583065322*^-6*k1^2*k2^2 + 1.9013910151417084*^-8*k1^3*k2^2 -

6.022840003392592*^-10*k1^4*k2^2 + 0.000012872860605150816*k2^3 -

4.3577328210622893*^-7*k1*k2^3 + 6.334193084669391*^-9*k1^2*k2^3 +

1.9001215686708167*^-7*k2^4 - 6.240909866655718*^-9*k1*k2^4 +

1.4090413007107 22*^-10*k1^2*k2^4 - 4.3877000564042747*^-10*k2^5},

{0.3283354107048551 + 0.03648035757358733*k1 - 0.0027005929145827684*

k1^2 + 0.00006896168315205526*k1^3 - 3.430103981787076*^-8*k1^4 -

1.7043069349937416*^-8*k1^5 + 1.567919354039772*^-10*k1^6 +

0.032596495983592315*k2 + 0.002473709727696465*k1*k2 -

0.0002513110446800669*k1^2*k2 + 7.334014539882223*^-6*k1^3*k2 -

1.0880170748407615*^-7*k1^4*k2 + 9.785550270083383*^-10*k1^5*k2 +

0.0014060183831336621*k2^2 - 0.00016987889541824843*k1*k2^2 +

2.2224629658187693*^-6*k1^2*k2^2 + 1.7019117949625627*^-8*k1^3*k2^2 -

5.74646810908354*^-10*k1^4*k2^2 - 2.3053293578065036*^-10*k1^5*k2^2 +

0.00014313424074378166*k2^3 - 8.015614100174407*^-6*k1*k2^3 +

1.7475195350609016*^-7*k1^2*k2^3 - 2.0477432855950153*^-9*k1^3*k2^3 +

1.3423107807799675*^-7*k2^4 - 1.633188493648809*^-8*k1*k2^4 -

1.5669814223315367*^-8*k2^5 + 4.866138664935836*^-10*k1*k2^5 +

2.9341827215449644*^-10*k2^6, -0.014732757915292655 +

0.00952591776922442*k1 - 0.00046446018799292936*k1^2 +

5.066271488483729*^-6*k1^3 + 7.134742475224483*^-8*k1^4 -

2.1643968910334188*^-9*k1^5 - 0.007175975169386018*k2 +

0.001340495724831155*k1*k2 - 0.000048160000709579896*k1^2*k2 +

8.134026263850621*^-7*k1^3*k2 - 8.072511436827408*^-9*k1^4*k2 -

0.0004263506851323001*k2^2 + 0.000022106469462494905*k1*k2^2 -

6.453665905140862*^-7*k1^2*k2^2 + 1.062990897976914*^-8*k1^3*k2^2 -

5.293261616957651*^-10*k1^4*k2^2 + 6.648949831875168*^-6*k2^3 -

3.5716264862361835*^-7*k1*k2^3 + 6.457413694756736*^-9*k1^2*k2^3 +

7.850317520253067*^-8*k2^4 - 3.5036595781272743*^-9*k1*k2^4 +

1.4476750966296994*^-10*k1^2*k2^4 - 7.00137058385763*^-10*k2^5},

{0.041434176011501844 - 0.02417919776571639*k1 - 0.00010217089629072361*

k1^2 + 0.000027243821751297796*k1^3 - 4.193383214268998*^-7*k1^4 -

1.3133642709489572*^-9*k1^5 - 0.018000852063278994*k2 -

0.0025678986127323222*k1*k2 - 1.7604575976426292*^-6*k1^2*k2 +

2.3342502487629063*^-6*k1^3*k2 - 5.274331211627833*^-8*k1^4*k2 +

5.976085546088536*^-10*k1^5*k2 - 0.0026560525366946842*k2^2 -

0.000035310837439873576*k1*k2^2 + 2.2928617623110043*^-6*k1^2*k2^2 -

2.2528650865776046*^-8*k1^3*k2^2 - 0.00005228878170856269*k2^3 -

1.502859637667783*^-6*k1*k2^3 + 7.724000742851227*^-8*k1^2*k2^3 -

1.252231786954272*^-9*k1^3*k2^3 - 9.518281639882589*^-7*k2^4 +

2.0873226728034162*^-8*k1*k2^4 - 1.3105918282853452*^-10*k1^2*k2^4 -

9.654295430274627*^-9*k2^5 + 3.6394537238766803*^-10*k1*k2^5 +

1.2926079185152337*^-10*k2^6, 0.09263158021491505 -

0.0015688799808321829*k1 - 0.0000984232125578255*k1^2 +

3.7109956389011636*^-6*k1^3 - 2.0626769145598312*^-8*k1^4 -

4.687672751250414*^-10*k1^5 + 0.010139313941980784*k2 -

0.00018881115705623883*k1*k2 - 0.000011661778754001025*k1^2*k2 +

3.772966260699535*^-7*k1^3*k2 - 5.047687153960755*^-9*k1^4*k2 +

0.00008354461350264755*k2^2 - 1.6467765663464987*^-6*k1*k2^2 -

9.774639279943071*^-8*k1^2*k2^2 + 3.418911021416528*^-9*k1^3*k2^2 -

1.610835764693248*^-10*k1^4*k2^2 + 8.050765694924364*^-7*k2^3 -

2.0592096987311444*^-7*k1*k2^3 + 4.978575928571343*^-9*k1^2*k2^3 -

1.893554597992535*^-8*k2^4 - 8.329754018199822*^-10*k1*k2^4 -

7.039868959076321*^-10*k2^5}, {0.016964037032288812 -

0.015408347029525987*k1 + 0.0005899707207828936*k1^2 -

2.1741151962648305*^-6*k1^3 - 1.4999010801971792*^-7*k1^4 +

1.3886851137867142*^-9*k1^5 + 0.000616020485653449*k2 -

0.0017923122162677952*k1*k2 + 0.00007552100905926573*k1^2*k2 -

4.4244097922105515*^-7*k1^3*k2 - 1.185948890638646*^-8*k1^4*k2 +
2.5040942167423876*^-10*k1^5*k2 - 0.001018800369658958*k2^2 -
0.000010758142257709697*k1*k2^2 + 1.6397018413864007*^-6*k1^2*k2^2 -
2.954923468942857*^-8*k1^3*k2^2 + 2.2887857343003958*^-10*k1^4*k2^2 -
0.00010379285080689762*k2^3 + 1.7318617847646354*^-6*k1*k2^3 +
9.519183572015962*^-9*k1^2*k2^3 - 5.336889378110747*^-10*k1^3*k2^3 -
1.3221982611043733*^-6*k2^4 + 3.5610807697357416*^-8*k1*k2^4 -
3.055555561572787*^-10*k1^2*k2^4 - 5.24827293252897*^-9*k2^5 +
2.2612423872065776*^-10*k1*k2^5, 0.025273790880955403 -
0.0035955483894448796*k1 + 0.000053752956891840094*k1^2 +
5.316529693803266*^-7*k1^3 - 1.6820284695194224*^-8*k1^4 +
0.005902778628196576*k2 - 0.00047337190390654135*k1*k2 +
6.768886375385323*^-6*k1^2*k2 + 4.0425303101397295*^-8*k1^3*k2 -
1.807781101049367*^-9*k1^4*k2 + 0.00034257628314258114*k2^2 -
0.000013094907968091885*k1*k2^2 + 1.908420079363104*^-7*k1^2*k2^2 -
9.426589155248595*^-10*k1^3*k2^2 - 1.7996718941022074*^-7*k2^3 -
1.0093842117249768*^-7*k1*k2^3 + 3.116503245375275*^-9*k1^2*k2^3 -
5.65779894319754*^-8*k2^4 + 7.162364133137506*^-10*k1*k2^4 -
5.341576995657975*^-10*k2^5}, {0.03617752405104938 -
0.0010112461681421356*k1 + 0.0003003436133134532*k1^2 -
6.605114681494121*^-6*k1^3 + 2.3978634429897294*^-8*k1^4 +
5.062800725084398*^-10*k1^5 + 0.006201634227941753*k2 -
0.00033826721614249497*k1*k2 + 0.000039632774604153155*k1^2*k2 -

8.493376625332279*^-7*k1^3*k2 + 4.60929914783222*^-9*k1^4*k2 -
0.00007318330640353763*k2^2 - 0.000011234951605474498*k1*k2^2 +
8.861694822127569*^-7*k1^2*k2^2 - 2.0565287378527988*^-8*k1^3*k2^2 +
2.11008368924017*^-10*k1^4*k2^2 - 0.00004176915696891466*k2^3 +
1.517995216568133*^-6*k1*k2^3 - 1.3185320040392246*^-8*k1^2*k2^3 -
1.1290385054497164*^-10*k1^3*k2^3 - 8.9048663344614*^-7*k2^4 +
2.7702637023219933*^-8*k1*k2^4 - 3.158801815523948*^-10*k1^2*k2^4 -
2.6013886479012536*^-9*k2^5 + 1.0644753720581988*^-10*k1*k2^5,
0.005192298716537443 - 0.0009029961907385915*k1 +
0.000046823432482230586*k1^2 - 5.257086200384492*^-7*k1^3 -
1.6775083082586819*^-9*k1^4 + 0.0020256490396212815*k2 -
0.00018121710587334916*k1*k2 + 6.6077531625498445*^-6*k1^2*k2 -
6.956226084762055*^-8*k1^3*k2 + 0.0001660922987573551*k2^2 -
9.628392605492069*^-6*k1*k2^2 + 2.016129755759307*^-7*k1^2*k2^2 -
2.10576688113756*^-9*k1^3*k2^2 + 1.2244829736128885*^-6*k2^3 -
5.4590890187449947*^-8*k1*k2^3 + 1.619690166318434*^-9*k1^2*k2^3 -
3.74870551479628*^-8*k2^4 + 9.397408746703774*^-10*k1*k2^4 -
3.193705607848407*^-10*k2^5}, {-0.016129699691394446 +
0.0011859777155494011*k1 + 4.8865529684984226*^-6*k1^2 -
2.5836424788844183*^-6*k1^3 + 4.049807755236922*^-8*k1^4 -
1.2932228192645543*^-10*k1^5 - 0.002127786958451386*k2 +
0.0001395436965883396*k1*k2 + 2.6194457685371517*^-6*k1^2*k2 -
3.415134025340714*^-7*k1^3*k2 + 5.17081474399734*^-9*k1^4*k2 +

0.00007492206484159783*k2^2 + 4.256022396445583*^-6*k1*k2^2 +

1.3316069070723315*^-7*k1^2*k2^2 - 7.709928922603805*^-9*k1^3*k2^2 +

0.000011217780380372921*k2^3 + 3.130465931511482*^-7*k1*k2^3 -

9.719461698221119*^-9*k1^2*k2^3 - 1.48411733026568*^-7*k2^4 +

9.957718069699315*^-9*k1*k2^4 - 1.9250574134861768*^-10*k1^2*k2^4 -

5.186712296116444*^-10*k2^5, -0.004112311770573317 +

0.00018826898837394433*k1 + 9.96063392216595*^-6*k1^2 -

3.419991526533066*^-7*k1^3 + 2.795677119117555*^-9*k1^4 -

0.0008362695707599395*k2 + 7.183317382880833*^-6*k1*k2 +

1.7252966577731487*^-6*k1^2*k2 - 4.649690894053718*^-8*k1^3*k2 +

3.8588380999429083*^-10*k1^4*k2 - 0.000031808113435875504*k2^2 -

1.4210661685896295*^-6*k1*k2^2 + 8.003705527948324*^-8*k1^2*k2^2 -

1.3332279126703735*^-9*k1^3*k2^2 + 1.0369884070474048*^-6*k2^3 -

2.0342813944391466*^-8*k1*k2^3 + 4.910194040625389*^-10*k1^2*k2^3 -

3.871865856636498*^-9*k2^4 + 4.5232901735887735*^-10*k1*k2^4},

{-0.005953453032537451 + 0.0009416795723779074*k1 -

0.000041051504351368371*k1^2 + 5.6036333808485745*^-8*k1^3 +

1.3871581550154578*^-8*k1^4 - 1.822841055349258*^-10*k1^5 -

0.0003730058982050363*k2 + 0.00019165309188967527*k1*k2 -

6.221582758410564*^-6*k1^2*k2 + 9.311373102354085*^-9*k1^3*k2 +

1.699821568739752*^-9*k1^4*k2 + 0.00013293906294272164*k2^2 +

8.229335948642259*^-6*k1*k2^2 - 2.3650468788922034*^-7*k1^2*k2^2 +

7.905328641751314*^-10*k1^3*k2^2 + 0.000013607725545124718*k2^3 -

1.7572527311851815*^-7*k1*k2^3 - 2.897516469662224*^-9*k1^2*k2^3 +

2.6489854824091684*^-7*k2^4 - 2.6267044582631623*^-9*k1*k2^4 +

1.1368701026899467*^-9*k2^5, -0.003687349717560112 +

0.00019464964121444542*k1 - 3.886110005523447*^-6*k1^2 -

5.778815292369053*^-8*k1^3 + 1.5932073103411832*^-9*k1^4 -

0.0008098201802476688*k2 + 0.000038871728750871 42*k1*k2 -

4.90859745802706*^-7*k1^2*k2 - 8.571954463206933*^-9*k1^3*k2 +

2.025921435532623*^-10*k1^4*k2 - 0.0000480574295113378*k2^2 +

2.0580857597846947*^-6*k1*k2^2 - 1.158397878800281*^-8*k1^2*k2^2 -

2.790423517117948*^-10*k1^3*k2^2 - 3.0442100184779544*^-7*k2^3 +

1.2147960584830942*^-8*k1*k2^3 - 1.5655724288302699*^-10*k1^2*k2^3 +

9.85436792968408*^-9*k2^4}, {0.001854603874752966 +

0.00020177660622957273*k1 - 0.000015829226285933575*k1^2 +

4.0760646052894545*^-7*k1^3 - 5.700828579817952*^-10*k1^4 +

0.00023651109981353142*k2 + 0.000044888143110526286*k1*k2 -

3.2926750902744873*^-6*k1^2*k2 + 6.559467181440455*^-8*k1^3*k2 -

1.6037753705849582*^-10*k1^4*k2 + 8.498317325113959*^-6*k2^2 +

2.0406547539865455*^-6*k1*k2^2 - 1.7851828008632295*^-7*k1^2*k2^2 +

2.7932997078529196*^-9*k1^3*k2^2 + 2.2815341772412675*^-6*k2^3 -

8.8700791785618*^-8*k1*k2^3 - 4.385408897609878*^-10*k1^2*k2^3 +

2.1013596855235954*^-7*k2^4 - 4.944996219932117*^-9*k1*k2^4 +

1.603890795352811*^-9*k2^5, 0.0000762925544429894 +

0.00006286236230098423*k1 - 2.714307943596308*^-6*k1^2 +

2.971007534461894*^-8*k1^3 + 2.385331071625818*^-10*k1^4 -

0.00005030536314618965*k2 + 0.000016556326429500847*k1*k2 -

5.137509025393734*^-7*k1^2*k2 + 4.620373137075576*^-9*k1^3*k2 -

0.000012688234017518796*k2^2 + 1.2471748558434018*^-6*k1*k2^2 -

2.7501949919194256*^-8*k1^2*k2^2 + 1.9405293313099255*^-10*k1^3*k2^2 -

6.554629785077454*^-7*k2^3 + 2.178048725338814*^-8*k1*k2^3 -

3.122631220224847*^-10*k1^2*k2^3 + 4.373084572991441*^-9*k2^4 +

1.0119375745734686*^-10*k2^5}, {0.00003796703504841054 -

0.00012608239450570847*k1 - 5.571280342099545*^-7*k1^2 +

1.3213637778940144*^-7*k1^3 - 2.1660162972450563*^-9*k1^4 -

0.00012366098813070328*k2 - 0.000022396636869317353*k1*k2 -

2.9819752012385757*^-7*k1^2*k2 + 2.5732819801948098*^-8*k1^3*k2 -

3.364315791713067*^-10*k1^4*k2 - 0.000025411975965774544*k2^2 -

7.740460075943967*^-7*k1*k2^2 - 2.7701337558902262*^-8*k1^2*k2^2 +

1.3790047153598094*^-9*k1^3*k2^2 - 1.1314366748485186*^-6*k2^3 +

3.393752095899185*^-9*k1*k2^3 + 2.1767810366106638*^-8*k2^4 -

1.937990904609451*^-9*k1*k2^4 + 8.395745796075558*^-10*k2^5,

0.0004154044603275401 - 9.75818796448185*^-6*k1 -

4.6879914306982477*^-7*k1^2 + 1.8566507989036024*^-8*k1^3 -

1.2642886078439993*^-10*k1^4 + 0.00007302510264875191*k2 -

1.0460887032310247*^-6*k1*k2 - 1.30457765133524*^-7*k1^2*k2 +

3.3484563159982565*^-9*k1^3*k2 + 1.8552928100944677*^-6*k2^2 +

8.838305951943132*^-8*k1*k2^2 - 1.0440592200623121*^-8*k1^2*k2^2 +

1.527544599289248*^-10*k1^3*k2^2 - 1.5351149754631328*^-7*k2^3 +

9.712698119001327*^-9*k1*k2^3 - 1.9539132662331408*^-10*k1^2*k2^3 -

2.081465829823892*^-9*k2^4}, {0.00010250581631719439 -

0.00005150784331552294*k1 + 2.245274803906521*^-6*k1^2 -

6.920362952254081*^-9*k1^3 - 6.722003820813342*^-10*k1^4 +

0.00003547785008023294*k2 - 0.00001100109527938313*k1*k2 +

4.5726222035185596*^-7*k1^2*k2 - 1.8682657931594638*^-10*k1^3*k2 -

1.2219375482212703*^-10*k1^4*k2 - 2.2000894992014605*^-7*k2^2 -

6.065477350376724*^-7*k1*k2^2 + 2.453172842874765*^-8*k1^2*k2^2 -

7.113321114385849*^-7*k2^3 + 1.4084394847000004*^-9*k1*k2^3 +

1.875499905993966*^-10*k1^2*k2^3 - 4.4735732856126976*^-8*k2^4 +

3.280063050373792*^-10*k1*k2^4 - 1.1226043792133263*^-10*k2^5,

0.0000785598339435377 - 0.000013079313440195105*k1 +

1.9660322105790374*^-7*k1^2 + 2.394318819798805*^-9*k1^3 +

0.000031216932799874 57*k2 - 2.911569017697781*^-6*k1*k2 +

3.296098513958109*^-8*k1^2*k2 + 5.924448217983902*^-10*k1^3*k2 +

3.7422641878511287*^-6*k2^2 - 1.8189853017977617*^-7*k1*k2^2 +

1.0624709624062557*^-9*k1^2*k2^2 + 1.237561858089685*^-7*k2^3 -

1.9720600163038295*^-9*k1*k2^3 - 1.8192875249166722*^-9*k2^4},

{0.0000880024976071 6725 + 1.0956766308638776*^-6*k1 +

8.073747430981073*^-7*k1^2 - 1.940709079946849*^-8*k1^3 +

0.00002670124349216411*k2 - 1.0414665106400557*^-6*k1*k2 +

1.9637884468183596*^-7*k1^2*k2 - 4.0294294776218995*^-9*k1^3*k2 +

1.4572829700395584*^-6*k2^2 - 2.0231672352569236*^-7*k1*k2^2 +
1.4458988317648668*^-8*k1^2*k2^2 - 2.3628274307771514*^-10*k1^3*k2^2 -
1.9366751330137368*^-7*k2^3 - 4.086238991790843*^-9*k1*k2^3 +
2.502024812509003*^-10*k1^2*k2^3 - 1.9723581975397734*^-8*k2^4 +
5.21773154950121*^-10*k1*k2^4 - 3.7333165709596444*^-10*k2^5,
6.4505740466372625*^-6 - 1.9636105706087547*^-6*k1 +
1.3543393268501414*^-7*k1^2 - 1.517266122526008*^-9*k1^3 +
6.983113870492739*^-6*k2 - 7.186646523197819*^-7*k1*k2 +
3.2019196542166725*^-8*k1^2*k2 - 2.7910950312923437*^-10*k1^3*k2 +
1.27248758542774*^-6*k2^2 - 8.639277668425662*^-8*k1*k2^2 +
2.2797021528762672*^-9*k1^2*k2^2 + 6.931808338831704*^-8*k2^3 -
3.5060087165554744*^-9*k1*k2^3 + 3.8189428133678493*^-10*k2^4},
{-0.000046665393967264954 + 4.027098093140822*^-6*k1 -
6.264178697565329*^-8*k1^2 - 5.617745640409991*^-9*k1^3 -
0.000010822192083940539*k2 + 7.664491170615712*^-7*k1*k2 -
6.149235762845644*^-10*k1^2*k2 - 1.405470386217083*^-9*k1^3*k2 -
2.727358180575313*^-7*k2^2 + 3.4170379771746613*^-8*k1*k2^2 +
1.682030782001952*^-9*k1^2*k2^2 + 6.993527074636104*^-8*k2^3 +
3.5910900949341076*^-10*k1*k2^3 + 2.1000833973406874*^-9*k2^4 +
1.0287406615317582*^-10*k1*k2^4 - 1.4795338893147403*^-10*k2^5,
-0.000013677021448227l1 + 8.036326519164156*^-7*k1 +
1.620565498444224*^-8*k1^2 - 7.98238651503579*^-10*k1^3 -
4.150597800729819*^-6*k2 + 1.0502889820060172*^-7*k1*k2 +

5.817963585735184*^-9*k1^2*k2 - 1.6460881927056756*^-10*k1^3*k2 -
3.485776680138697*^-7*k2^2 - 4.961802332132612*^-9*k1*k2^2 +
6.950547481456194*^-10*k1^2*k2^2 - 9.100946011918122*^-10*k1*k2^3 +
6.712520669146236*^-10*k2^4}, {-0.000011108318546096625 +
1.802468922535887*^-6*k1 - 1.016409003721001*^-7*k1^2 +
6.726944156468978*^-10*k1^3 - 1.659690264019568*^-6*k2 +
6.01950121973029*^-7*k1*k2 - 2.348657697360714*^-8*k1^2*k2 +
2.633880489937412*^-7*k2^2 + 6.046017288929956*^-8*k1*k2^2 -
1.6293142480031667*^-9*k1^2*k2^2 + 6.267075487834357*^-8*k2^3 +
1.2758628252064941*^-9*k1*k2^3 + 3.5087515376356173*^-9*k2^4,
-6.927118086064966*^-6 + 4.207501916555816*^-7*k1 -
1.1567117865543361*^-8*k1^2 - 2.3752124499960183*^-6*k2 +
1.2865861867441764*^-7*k1*k2 - 2.2801742189736705*^-9*k1^2*k2 -
2.698839105479158*^-7*k2^2 + 1.274202259039997*^-8*k1*k2^2 -
1.0051697210578934*^-8*k2^3 + 3.6812944729439084*^-10*k1*k2^3},
{4.617827236349101*^-6 + 1.3727819893427136*^-7*k1 -
2.2396332357980338*^-8*k1^2 + 8.340063882662365*^-10*k1^3 +
8.561201300041417*^-7*k2 + 7.20644044675407*^-8*k1*k2 -
7.839947634848847*^-9*k1^2*k2 + 2.0983649883598613*^-10*k1^3*k2 +
1.0957051782958005*^-8*k2^2 + 9.893839280111748*^-9*k1*k2^2 -
8.964885497146293*^-10*k1^2*k2^2 - 2.581695125213491*^-10*k2^3 +
2.8934110503665363*^-10*k1*k2^3 + 7.569184818269182*^-10*k2^4,
8.567563555009472*^-7 + 7.142911283431902*^-8*k1 -

4.807778286193104*^-9*k1^2 + 1.1275717910207268*^-7*k2 +

3.426105321409015*^-8*k1*k2 - 1.401459502839035*^-9*k1^2*k2 -

2.104016483350024*^-8*k2^2 + 5.102689465150051*^-9*k1*k2^2 -

1.421397730050892*^-10*k1^2*k2^2 - 4.02990386594349*^-9*k2^3 +

2.7299671288961564*^-10*k1*k2^3 - 1.5486634270732633*^-10*k2^4},

{3.1198444118344482*^-7 - 2.5900935020989483*^-7*k1 +

3.1650062367322116*^-9*k1^2 + 1.5872451530351888*^-10*k1^3 -

1.5788606596044755*^-7*k2 - 7.21687154667594*^-8*k1*k2 +

2.435703304781425*^-10*k1^2*k2 - 6.672113654804965*^-8*k2^2 -

5.756335692175592*^-9*k1*k2^2 - 7.022407158315981*^-9*k2^3 -

1.6758723368970985*^-10*k2^4, 8.170881746299867*^-7 -

3.210164088175108*^-8*k1 - 2.0476581251481555*^-10*k1^2 +

2.3141982081374828*^-7*k2 - 7.68802135159687*^-9*k1*k2 -

1.7888007602972577*^-10*k1^2*k2 + 1.8718316678527952*^-8*k2^2 -

3.406025417145145*^-10*k1*k2^2},

{1.307152888211608*^-9 - 6.3570204672117*^-8*k1 +

3.6947594493257928*^-9*k1^2 + 5.406069177223385*^-8*k2 -

2.0991951716649653*^-8*k1*k2 + 1.1218793026464192*^-9*k1^2*k2 +

8.503757412506006*^-9*k2^2 - 2.2841589474773045*^-9*k1*k2^2 +

1.1422818239735319*^-10*k1^2*k2^2 - 7.029447026371338*^-10*k2^3 -

1.7099263041304545*^-10*k2^4, 5.487285649480081*^-8 -

1.8798800110166863*^-8*k1 + 4.316194910642449*^-10*k1^2 +

4.432945909631428*^-8*k2 - 6.356368426459723*^-9*k1*k2 +

1.1046154389753668*^-10*k1^2*k2 + 9.7328322998851 6*^-9*k2^2 -

7.26969642453059*^-10*k1*k2^2 + 8.014888624001164*^-10*k2^3},

{7.382883593520126*^-8 + 1.2990621410896902*^-8*k1 +

7.185945266792934*^-10*k1^2 + 4.038963973936978*^-8*k2 +

1.8634014260463347*^-9*k1*k2 + 2.8583777255400147*^-10*k1^2*k2 +

6.184989269760621*^-9*k2^2 - 1.8021735337382023*^-10*k1*k2^2 +

1.1022179471229168*^-10*k2^3, -2.6756489714311214*^-8 -

4.849749830796946*^-10*k1 + 1.5556853064974508*^-10*k1^2 -

6.072306797754431*^-10*k2 - 6.224988482133711*^-10*k1*k2 +

1.5599904594446836*^-9*k2^2 - 1.6112942676971476*^-10*k1*k2^2 +

2.339459653476407*^-10*k2^3}, {-5.1082342858020834*^-8 +

6.546608811838977*^-9*k1 - 2.349932733091769*^-10*k1^2 -

1.8791992386401963*^-8*k2 + 1.9597024047620677*^-9*k1*k2 -

1.6572898881125162*^-9*k2^2 + 1.882057384984757*^-10*k1*k2^2,

-1.6990493414695258*^-8 + 1.603902944371725*^-9*k1 -

7.752403490215377*^-9*k2 + 4.3696645821613025*^-10*k1*k2 -

1.1884849392384823*^-9*k2^2}, {-1.0880247900078996*^-8 +

1.01361726459865*^-9*k1 - 1.1449799500320853*^-10*k1^2 -

3.28126055100561*^-9*k2 + 6.182412718394156*^-10*k1*k2 +

1.0441263148938522*^-10*k1*k2^2, -4.683842662127961*^-9 +

3.8137084826020974*^-10*k1 - 2.554467147794183*^-9*k2 +

1.5870638791997076*^-10*k1*k2 - 4.914193155615168*^-10*k2^2},

{6.093304589433636*^-9 - 2.1927953716287238*^-10*k1 +

1.7841899216418068*^-9*k2 + 1.0528234757742122*^-10*k2^2,

1.6353921223984435*^-9 + 5.58357535815584*^-10*k2},

{5.942770473529257*^-10 - 2.418062578401636*^-10*k1,

7.899206098807802*^-10 + 3.330931918423561*^-10*k2},

{-2.59035945755205*^-10, 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},

{0., 0.}, {0., 0.}, {0., 0.}}

SS = {{0.6091972072132368 - 0.1691199412402733*k1 + 0.011450229876802713*

k1^2 - 0.0003419606457356784*k1^3 + 5.705207833992923*^-6*k1^4 -

6.052935619196365*^-8*k1^5 - 0.07320511908595312*k2 +

0.004759746178365423*k1*k2 - 0.000122606122820763*k1^2*k2 +

1.7501376749473977*^-6*k1^3*k2 - 1.6174596899877852*^-8*k1^4*k2 -

0.005111628121965394*k2^2 + 0.0003398126850893788*k1*k2^2 -

8.863135730359162*^-6*k1^2*k2^2 + 1.2756977486863244*^-7*k1^3*k2^2 -

1.2751769081906551*^-9*k1^4*k2^2 - 1.0415662843050103*^-9*k1^5*k2^2 +

0.000034808320099596267*k2^3 - 1.1846901675361067*^-6*k1*k2^3 +

1.8059987687650563*^-8*k1^2*k2^3 + 3.7915478350249455*^-10*k1^3*k2^3 +

1.0555123367943074*^-6*k2^4 - 3.7331837906840584*^-8*k1*k2^4 -

1.5780943274288767*^-9*k1^2*k2^4 - 2.2394030386466943*^-9*k2^5 +

7.718846235810383*^-10*k2^6, 0.4091612239339726 -

0.046167041190911126*k1 + 0.0019546995301192305*k1^2 -

0.000042704679078418 35*k1^3 + 5.6370392695842*^-7*k1^4 -

4.940878476199826*^-9*k1^5 + 0.013351071459988963*k2 -

0.0008583596811459546*k1*k2 + 0.00002287977670368471*k1^2*k2 -

3.409616919701228*^-7*k1^3*k2 + 3.3265007164728646*^-9*k1^4*k2 -

0.0003888998814654743 5*k2^2 + 0.000023323076332121264*k1*k2^2 -

5.436767167139941*^-7*k1^2*k2^2 + 7.110614114186221*^-9*k1^3*k2^2 -

1.1732044350141785*^-9*k1^4*k2^2 - 2.689842975782832*^-6*k2^3 +

1.25392467945833*^-7*k1*k2^3 - 2.695663920158255*^-9*k1^2*k2^3 +

5.4949669420255783*^-8*k2^4 - 1.0918968938524445*^-9*k1*k2^4 -

5.142902643609054*^-10*k2^5}, {-1.298964020221379 +

0.18233258641011138*k1 + 0.01372771595192251*k1^2 -

0.0011724127240669876*k1^3 + 0.00003187058481801589*k1^4 -

4.65858183769722*^-7*k1^5 + 5.204671150122918*^-9*k1^6 +

0.011040307401004944*k2 + 0.045479353049442034*k1*k2 -

0.0024468200444220383*k1^2*k2 + 0.000056035586982321 21*k1^3*k2 -

7.398675323758882*^-7*k1^4*k2 + 6.1937883728344915*^-9*k1^5*k2 +

0.024559061637366507*k2^2 + 0.00011864326323457062*k1*k2^2 -

0.000033903718229338624*k1^2*k2^2 + 8.484874384078966*^-7*k1^3*k2^2 -

1.0017171885136759*^-8*k1^4*k2^2 + 3.39834499501355*^-9*k1^5*k2^2 -

1.0275899445481102*^-10*k1^6*k2^2 + 0.0013657673801194216*k2^3 -

0.00005942873903668506*k1*k2^3 + 1.1511320308052495*^-6*k1^2*k2^3 -

1.3705576865047683*^-8*k1^3*k2^3 + 6.760786820227882*^-10*k1^4*k2^3 +

6.926441407858336*^-6*k2^4 - 3.1099902851458023*^-7*k1*k2^4 +

1.0350196538179095*^-8*k1^2*k2^4 - 4.809154947226488*^-10*k1^3*k2^4 -

1.0190752629280905*^-7*k2^5 + 2.685720686204532*^-9*k1*k2^5 -

2.2098483197188816*^-10*k1^2*k2^5 - 1.12617359867183*^-9*k2^6 +

1.0494545134500323*^-10*k1*k2^6, -0.8228670762523063 -

0.007535296957092534*k1 + 0.004608781907947641*k1^2 -

0.00019249437047283967*k1^3 + 3.731830592024761*^-6*k1^4 -

4.320683033997667*^-8*k1^5 + 2.7090613761904443*^-10*k1^6 -

0.11993386884148952*k2 + 0.007764005796589902*k1*k2 -

0.00020268516800710015*k1^2*k2 + 2.9428022175816837*^-6*k1^3*k2 -

2.7387444898437344*^-8*k1^4*k2 - 0.003750478566108594*k2^2 +

0.0002581814866325101*k1*k2^2 - 6.960261516967505*^-6*k1^2*k2^2 +

1.0296646101188327*^-7*k1^3*k2^2 + 6.61950716291706*^-10*k1^4*k2^2 -

1.1934018590936397*^-10*k1^5*k2^2 + 0.00004422282598449166*k2^3 -

1.6234699445885134*^-6*k1*k2^3 + 2.6138097436527065*^-8*k1^2*k2^3 +

8.635432302412281*^-7*k2^4 - 3.321530636010131*^-8*k1*k2^4 +

6.821865492263216*^-10*k1^2*k2^4 - 4.534134577822906*^-10*k2^5}}

APPENDIX C: MATHEMATICA PROGRAMS FOR CHAPTER 4

```
Off[General::spell1 ];

<<LinearAlgebra`MatrixManipulation`

fQp[p_]:=Module[{i,j},

   MQp:=Array[Qp,{Length[p],Length[p]}];

   Do[Qp[i,1]=p[[i]],{i,1,Length[p]}];

   Do[Qp[1,j]=p[[j]]/2,{j,2,Length[p]}];

   Do[If[(i==j),If[((2 j-1)≤Length[p]),Qp[i,j]=p[[2 i-
1]]/2+p[[1]],Qp[i,j]=p[[1]]],If[(i≠1&&j≠1),If[(i+j-1)≤Length[p],Qp[i,j]=p[[i+j-
1]]/2+p[[Abs[i-j]+1]]/2,Qp[i,j]=p[[Abs[i-j]+1]]/2]]],{i,1,Length[p]},{j,1,Length[p]}];

   Qp[1,1]=p[[1]];

   MQp

   ]

fGT[p_]:=Module[{i,j},

   MGp:=Array[G,{Length[p],Length[p]}];

   Do[If[j==1,If[i==2,G[i,j]=-1/8,G[i,j]=Cos[i Pi]/(2i(i-2))]

      ,If[Abs[i-j]==1,If[(i-j)==1,G[i,j]=-1/(4 (i-2)),If[j≠2,G[i,j]=1/(4 i),G[i,j]=1/2]]

       ,G[i,j]=0]

      ] ,{i,1,Length[p]},{j,1,Length[p]}];

   Transpose[MGp];
```

```
   ]
fQD[m_,n_]:=Module[{i,j,k,l,A,B},
   MQD:=Array[QD,{n m,n m}];
   Do[QD[(i-1)*m+k,(j-1)*m+l]=fQp[tensorP[[i,j ]]][[k,l]],
     {i,1,n},{j,1,n},{k,1,m},{l,1,m}
     ];
   ]
fGG[GG_,m_,n_]:=Module[{i,j,k,l},
   GGT:=Array[GGTD,{n m,n m}];
   Do[GGTD[i,j]=0,{i,1,m n},{j,1,m n}];

   Do[GGTD[(i-1)*m+k,(i-1)*m+l]=GG[[k,l]],
     {i,1,n},{k,1,m},{l,1,m}
     ];

   (*GGT=Outer[Times,IdentityMatrix[n],GG]*)
   ]
fXMN[m_,n_]:=Module[{i},
   XXMN:=ZeroMatrix[n m,n];
   Do[XXMN=ReplacePart[XXMN,1,{1+m(i-1),i}],{i,1,n}];
   ]
fPhi[m_,i_,j_,x_]:=Module[{k,aaaa},aaaa:=Transpose[Take[B2,{1+m (i-1),m i},{j}]];
     Expand[aaaa.Table[ChebyshevT[k-1,2x-1],{k,m}]]
```

```
    ];
MyQC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K5=Chop[J[[1]]];
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    K2
    ];
MyRC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    Re[K2]
    ];
(*f0:Nonlinear Part*)
f0={0,0,-2.5 Cos[2 Pi t] x1^3 /6+1.5 (x1-x2)^3,0};
 (* Transformation: x=F11.M1.z *)
 {x1,x2,x3,x4}=F11[t].M1.{z1,z2,z3,z4}//TrigToExp//Expand;


 f0=f0 //TrigToExp//Expand//Chop;
 (*f1:Nonlinear part in terms of z*)
```

```
f1=Inverse[M1].iF11[t].f0//Chop//TrigToExp//Expand//Chop;
g1=Inverse[M1].iF11[t].{0,0,0,1}//Chop//TrigToExp//Expand//Chop;
Save["f1.m",f1];
Save["g1.m",g1];
(* Calculate the max frequency in f1 and g1.The function "Exponent" has a bug.We
have to use it indirectly.*)
(* frq:The max frequency in f1 and g1 *)
(*temp1:always used as temporary variable and released after the cell *)
frqlist=Join[Flatten[Exponent[f1,E^(I Pi t),List]],Flatten[Exponent[g1,E^(I Pi t),List]]];
frq=frqlist//Abs//Max;
(*Z:independent variables; v1: number of independent variables; poly: nonlinear order
  *)
Z={z1,z2,z3,z4,z5,z6,z7,z8};
XX={x1,x2,x3,x4,x5,x6,x7,x8};
v1=4;
poly=3;
<<DiscreteMath`Combinatorica`
(* Produce the monomials *)
(*For n independent variable numbers and order q frequency m,
  the number of monomials=(2m+1)n!/[(n-1)!q!]*)
p1=Flatten[Compositions[poly,v1]];
```

```
(* Gbase1:fourier-tylor basis for frequency=frq,independent variable
number=v1,nonlinear order=poly *)

Gbase=With[{n=v1,q=poly,fr=frq,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]
Exp[I Pi t j]//Expand,{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten];
Print["frequency1=",frq];
Share[];
Alpha=Array[aoa,{Length[Gbase]}];
Length[Gbase];
(*Transformation y=M1 z in Alpha*)
RL0=Table[Z[[i]]->(M1.{z1,z2,z3,z4})[[i]],{i,1,4}];
temp1=Chop[Alpha.Gbase/.RL0]//Expand;
f2=Chop[Expand[Chop[f1+g1 (temp1),10^-6]],10^-6];
Clear[temp1];
Save["f2.m",f2];
(*Truncate the f2 upto desired frq*)
frqlist1=Join[Flatten[Exponent[f2,E^(I Pi t),List]],Flatten[Exponent[g1,E^(I Pi
t),List]]];
frq1=frqlist1//Abs//Max
Print["frequency1=",frq1];
f3=f2;
(*f3=Table[Total[Table[If[Abs[Exponent[f2[[j,i]],E^(Pi I
t)]]≤10,f2[[j,i]],0],{i,1,Length[f2[[j]]]}]],{j,1,v1}];
```

```
  Save["f3.m",f3];*)
 (*Computer the position of corresponding Gbase; n1: numbe of the
monomial;n2:number of the frequency*)
 Gmonomial=Compositions[poly,v1]
 n1=With[{n=v1,q=poly,fr=frq,p=p1},(n+q-1)!/((q!)(n-1)!)];
 n2=2*frq+1;
 Gbase1=With[{n=v1,q=poly,fr=0,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]
Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten];

 (* f4:the transformation Phi expressed in terms of alpha *)
 f4=Table[i,{i,1,v1}];


 (*The mathematica can not decide if aoa[1]≠0 is true; We have to define DaoaD *)
 test=Array[DaoaD,{Length[Gbase]}];
 Do[DaoaD[i]=1,{i,1,Length[Gbase]}];
 Share[]
 FloquetMultiplier=Table[J[[i,i]],{i,1,4}]
 (* a:the frequency position; b: the monomial position; *)
 With[{n=1},
  ClearAll[a,b];
  a=Table[Exponent[f3[[n,i]],E]/t,{i,1,Length[f3[[n]]]}];
  b=Table[i,{i,1,Length[f3[[n]]]}];
```

```
Do[Do[If[Coefficient[f3[[n,j]]/.{t→0,aoa→DaoaD},Gbase1[[i]]]≠0,b[[j]]=i],{i,1,Length[
Gbase1]}],{j,1,Length[f3[[n]]]}];

   (*Resonance check*)
   For[i=1,i≤Length[f3[[n]]],If[((a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]])==0),Print["@RESONANCE@"]];i++];

   f4[[n]]=Total[Table[f3[[n,i]]/(a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]]),{i,1,Length[f3[[n]]]}]]//Chop;
   Share[];
   ]

  With[{n=2},
   ClearAll[a,b];
   a=Table[Exponent[f3[[n,i]],E]/t,{i,1,Length[f3[[n]]]}];
   b=Table[i,{i,1,Length[f3[[n]]]}];

Do[Do[If[Coefficient[f3[[n,j]]/.{t→0,aoa→DaoaD},Gbase1[[i]]]≠0,b[[j]]=i],{i,1,Length[
Gbase1]}],{j,1,Length[f3[[n]]]}];

   (*Resonance check*)
```

```
   For[i=1,i≤Length[f3[[n]]],If[((a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]])==0),Print["@RESONANCE@"]];i++];

   f4[[n]]=Total[Table[f3[[n,i]]/(a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]]),{i,1,Length[f3[[n]]]}]]//Chop;
   Share[];
   ]

  With[{n=3},
   ClearAll[a,b];
   a=Table[Exponent[f3[[n,i]],E]/t,{i,1,Length[f3[[n]]]}];
   b=Table[i,{i,1,Length[f3[[n]]]}];

Do[Do[If[Coefficient[f3[[n,j]]/.{t→0,aoa→DaoaD},Gbase1[[i]]]≠0,b[[j]]=i],{i,1,Length[
Gbase1]}],{j,1,Length[f3[[n]]]}];

   (*Resonance check*)
   For[i=1,i≤Length[f3[[n]]],If[((a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]])==0),Print["@RESONANCE@"]];i++];

   f4[[n]]=Total[Table[f3[[n,i]]/(a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]]),{i,1,Length[f3[[n]]]}]]//Chop;
   Share[];
```

```
]

With[{n=4},
  ClearAll[a,b];
  a=Table[Exponent[f3[[n,i]],E]/t,{i,1,Length[f3[[n]]]}];
  b=Table[i,{i,1,Length[f3[[n]]]}];

Do[Do[If[Coefficient[f3[[n,j]]/.{t→0,aoa→DaoaD},Gbase1[[i]]]≠0,b[[j]]=i],{i,1,Length[
Gbase1]}],{j,1,Length[f3[[n]]]}];
  (*Resonance check*)
  For[i=1,i≤Length[f3[[n]]],If[((a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]])==0),Print["@RESONANCE@"]];i++];

  f4[[n]]=Total[Table[f3[[n,i]]/(a[[i]]+FloquetMultiplier.(Gmonomial[[b[[i]]]])-
FloquetMultiplier[[n]]),{i,1,Length[f3[[n]]]}]]//Chop;
  Share[];
  ]

Save["f4.m",f4];

(*<<"f4.m";*)
Z={z1,z2,z3,z4}
T1=Inverse[M1].Z;
```

```
  Rl=Table[Z[[i]]→T1[[i]],{i,1,4}]
  ff4=Expand[M1.(f4/.Rl)]//Expand//Chop;
  Save["ff4.m",ff4];
  ff5=ExpToTrig[ff4]//Expand//Chop;
  Save["ff5.m",ff5];
  Clear[f4];
  f4=ff4//Chop;
  Clear[g1];
  g1=iF11[t].{0,0,0,1}//Chop//TrigToExp//Expand//Chop;
  Share[];

  (*Truncate the imagine part of ff4 and high frequency terms*)
  dAcc=5;
  ff6=Table[Total[Table[If[Abs[Exponent[f4[[j,i]],E^(Pi I
t)]]≤dAcc,f4[[j,i]],0],{i,1,Length[f4[[j]]]}]],{j,1,v1}];
  Save["ff6.m",ff6];
  Clear[f4];
  f4=ff6;
  f5=Table[0,{i,1,4}]
f7=Table[Total[Table[If[Abs[Exponent[f5[[j,i]],E^(Pi I
t)]]≤dAcc,f5[[j,i]],0],{i,1,Length[f5[[j]]]}]],{j,1,4}];
Save["f7.m",f7];
(* Produce the base and combination number for Beta *)
```

```
(*Gbase2 : base with frq, v1, poly-1 *)

ClearAll[p1];

(* DesiredAccuracy:desiredAcc=5*)

p1=Flatten[Compositions[poly-1,v1]]

Gbase2=With[{n=v1,q=poly-1,fr=frq,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]

Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten];

Gbase3=With[{n=v1,q=poly-1,fr=0,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]

Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten]

Length[Gbase2]

(* Define the Beeta *)


Beeta=Array[bob,{Length[Gbase2]}];

f8=Table[i,{i,1,v1}];

f8=g1*(Beeta.Gbase2)//Expand;

Save["f8.m",f8];


f9=Table[Total[Table[If[Abs[Exponent[f8[[j,i]],E^(Pi I

t)]]≤dAcc,f8[[j,i]],0],{i,1,Length[f8[[j]]]}]],{j,1,4}];

Save["f9.m",f9];

f10=Chop[f7-f9]//Expand;

Save["f10.m",f10];

(*For the truncated result f9, we define the new base Gbase4 and Gbase5 *)
```

```
(*Map f10 into Polynomial*)
dAcc=5;
f99=Table[Total[Table[If[Abs[Exponent[f10[[j,i]],E^(Pi I
t)]]≤dAcc,f10[[j,i]],0],{i,1,Length[f10[[j]]]}]],{j,1,4}];

(*Should be very careful because we can not map the 1→something *)
Rl4=Join[Table[E^(I Pi t k)→X^(k+dAcc),{k,-dAcc,-1}],Table[E^(I Pi t
k)→X^(k+dAcc),{k,1,dAcc}]];

f12=f99/.Rl4;
Save["f12.m",f12];
Table[PolynomialQ[f12[[i]],{z1,z2,z3,z4,X}],{i,1,4}]
Equation1=Table[Flatten[CoefficientList[f12[[i]],{z1,z2,z3,z4,X}]],{i,1,4}];
Equation2=Flatten[Equation1];
k=0;
Do[If[Equation2[[i]]==0,,k++,k++],{i,1,Length[Equation2]}];
Equation3=Table[0,{i,1,k}];
k=0;
Do[If[Equation2[[i]]==0,,k++;Equation3[[k]]=Equation2[[i]],k++;Equation3[[k]]=Equatio
n2[[i]]],{i,1,Length[Equation2]}];
Save["Equation3.m",Equation3];
Unknowns=Join[Alpha,Beeta];
Save["Unknowns.m",Unknowns];
```

```
Print[Length[Equation3]," equations with ",Length[Unknowns]," Unknowns"];
temp1=Array[DaoaD,{Length[Alpha]}];
temp2=Array[DbobD,{Length[Beeta]}];
Do[DaoaD[i]=0,{i,1,Length[Alpha]}];
Do[DbobD[i]=0,{i,1,Length[Beeta]}];
b=Equation3/.{aoa→DaoaD,bob→DbobD};
Equation4=(Equation3-b)//Chop;
m=Table[0,{i,1,Length[Equation3]},{j,1,Length[Unknowns]}];
Do[m[[i,j]]=Re[Coefficient[Equation3[[i]],Unknowns[[j]]]],{i,1,Length[Equation3]},{j,1,
Length[Unknowns]}];

(*Use the Linear Solve*)
Print[Length[Equation3]," equations with ",Length[Unknowns]," Unknowns"];
Clear[temp1];
Clear[temp2];
temp1=Array[DaoaD,{Length[Alpha]}];
temp2=Array[DbobD,{Length[Beeta]}];
Do[DaoaD[i]=0,{i,1,Length[Alpha]}];
Do[DbobD[i]=0,{i,1,Length[Beeta]}];
b=Equation3/.{aoa→DaoaD,bob→DbobD};
Equation4=(Equation3-b)//Chop;
m=Table[0,{i,1,Length[Equation3]},{j,1,Length[Unknowns]}];
```

```
Do[m[[i,j]]=Re[Coefficient[Equation3[[i]],Unknowns[[j]]]],{i,1,Length[Equation3]},{j,1,
Length[Unknowns]}];
Save["m1.m",m];
Save["b.m",b];
ls1=LinearSolve[m,b];
Save["ls1.m",ls1];
Alpha1=ExpToTrig[Table[ls1[[i]],{i,1,Length[Alpha]}].Gbase]//Chop;
Beeta1=ExpToTrig[Table[ls1[[Length[Alpha]+i]],{i,1,Length[Beeta]}].Gbase2]//Chop;
Save["Alpha1.m",Alpha1];
Save["Beeta1.m",Beeta1];
Table[PolynomialQ[f12[[i]],{z1,z2,z3,z4,X}],{i,1,4}]
Equation1=Table[Flatten[CoefficientList[f12[[i]],{z1,z2,z3,z4,X}]],{i,1,4}];
Equation2=Flatten[Equation1];
SetDirectory["C:\Data\Dataoct22"];
ClearAll[Equation3];
k=0;
Do[If[Equation2[[i]]==0,,k++,k++],{i,1,Length[Equation2]}];
Equation3=Table[0,{i,1,k}];
k=0;
Do[If[Equation2[[i]]==0,,k++;Equation3[[k]]=Equation2[[i]],k++;Equation3[[k]]=Equatio
n2[[i]]],{i,1,Length[Equation2]}];
Save["Equation3.m",Equation3];
(*Inital values*)
```

```
v1=4;frq=20;poly=3;

<<DiscreteMath`Combinatorica`

(* Produce the monomials *)

(*For n independent variable numbers and order q frequency m,

  the number of monomials=(2m+1)n!/[(n-1)!q!]*)

p1=Flatten[Compositions[poly,v1]]

Z={z1,z2,z3,z4,z5,z6,z7,z8};

Gbase=With[{n=v1,q=poly,fr=frq,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]

Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten];

Print["frequency1=",frq];

Share[];

Alpha=Array[aoa,{Length[Gbase]}];

Dimensions[Alpha]

ClearAll[p1];

(* DesiredAccuracy:desiredAcc=10*)

dAcc=10;

p1=Flatten[Compositions[poly-1,v1]]

Gbase2=With[{n=v1,q=poly-1,fr=frq,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]

Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten];

Gbase3=With[{n=v1,q=poly-1,fr=0,p=p1},Table[Product[Z[[i]]^p[[n k-n+i]],{i,1,n}]

Exp[I Pi t j],{j,-fr,fr},{k,1,(n+q-1)!/((q!)(n-1)!)}]//Flatten]

Length[Gbase2]

Beeta=Array[bob,{Length[Gbase2]}];
```

```
(*Solutions=Solve[Table[Equation3[[i]]==0,{i,1,Length[Equation3]}],Unknowns];
Save["Solutions.m",Solutions];
Solutions1=NSolve[Table[Equation3[[i]]==0,{i,1,Length[Equation3]}],Unknowns];
Save["Solutions1.m",Solutions1];*)
(*Use the Linear Solve*)
Print[Length[Equation3]," equations with ",Length[Unknowns]," Unknowns"];
temp1=Array[DaoaD,{Length[Alpha]}];
temp2=Array[DbobD,{Length[Beeta]}];
Do[DaoaD[i]=0,{i,1,Length[Alpha]}];
Do[DbobD[i]=0,{i,1,Length[Beeta]}];
b=Equation3/.{aoa→DaoaD,bob→DbobD};
Equation4=(Equation3-b)//Chop;
m=Table[0,{i,1,Length[Equation3]},{j,1,Length[Unknowns]}];
Do[m[[i,j]]=Re[Coefficient[Equation3[[i]],Unknowns[[j]]]],{i,1,Length[Equation3]},{j,1,
Length[Unknowns]}];
Do[aoa[i]=ls1[[i]],{i,1,Length[Alpha]}];
Do[bob[i]=ls1[[820+i]],{i,1,Length[Beeta]}];
Alpha1=ExpToTrig[Alpha.Gbase]//Chop;
Beeta1=ExpToTrig[Beeta.Gbase2]//Chop;
Save["Alpha1.m",Alpha1];
Save["Beeta1.m",Beeta1];
p2={0.30424217764409384`,2.5026601968371524`*^-
16,0.9708678652630196`,1.3252311172764654`*^-16,-
```

```
0.30284915526269945`,1.511229668379056`*^-16,0.02909193396501055`,0,-
0.0013922439911775739`,0,0.000040189944511023766`,0,-7.782767012882281`*^-
7,0,1.0826530292730342`*^-8,0,-1.1351103472439703`*^-10};
(*p2 is the chebysheve terms of COS(x)*)
fGT[p2];
fGG[Transpose[MGp],16,4]

D1=4;
D2=16;
(*D1 is the number of the states*)

tensorP:=Array[tP,{D1,D1,D2}];
(*the first row*)
Do[tP[1,1,k]=0,{k,1,D2}];
Do[tP[1,2,k]=0,{k,1,D2}];
Do[tP[1,3,k]=0,{k,2,D2}];
tP[1,3,1]=1;
Do[tP[1,4,k]=0,{k,1,D2}];

(*the second row*)
Do[tP[2,1,k]=0,{k,1,D2}];
Do[tP[2,2,k]=0,{k,1,D2}];
Do[tP[2,3,k]=0,{k,1,D2}];
```

```
Do[tP[2,4,k]=0,{k,2,D2}];

tP[2,4,1]=1;


(*the third row*)

Do[tP[3,1,k]=2.5*p2[[k]],{k,2,D2}];

tP[3,1,1]=-5+2.5*p2[[1]];

Do[tP[3,2,k]=0,{k,2,D2}];

tP[3,2,1]=0.5;

Do[tP[3,3,k]=0,{k,2,D2}];

tP[3,3,1]=3.2;

Do[tP[3,4,k]=0,{k,1,D2}];


(*the fouth row*)

Do[tP[4,2,k]=5*p2[[k]],{k,2,D2}];

tP[4,2,1]=-4+5*p2[[1]];

Do[tP[4,1,k]=0,{k,2,D2}];

tP[4,1,1]=3.5;

Do[tP[4,3,k]=0,{k,1,D2}];

Do[tP[4,4,k]=0,{k,2,D2}];

tP[4,4,1]=1.2;


fQD[D2,D1];

fXMN[D2,D1];
```

```
B2:=Inverse[(IdentityMatrix[D2 D1]-GGT.MQD)].XXMN;
PhiMatrix=Table[f[i,j,x_],{i,1,D1},{j,1,D1}];
Do[f[i,j,x_]=fPhi[D2,i,j,x][[1]],{i,1,D1},{j,1,D1}]
Phi[t_]=Table[f[i,j,t],{i,1,D1},{j,1,D1}];
Share[];
J=JordanDecomposition[Phi[1]];
Jtest=J[[2]];
Do[Jtest=ReplacePart[Jtest,Log[J[[2,i,i]]],{i,i}],{i,1,D1}];
CMatrix=Chop[J[[1]].Jtest.Inverse[J[[1]]],10^-5];
RMatrix=Chop[(CMatrix+Conjugate[CMatrix])/2];
FloquetMultiplier=Eigenvalues[RMatrix]
Print["FloquetMultiplier= ", FloquetMultiplier[[1]]," ",FloquetMultiplier[[2]],"
",FloquetMultiplier[[3]]," ",FloquetMultiplier[[4]]]
Q1[t_]=Phi[t].(MatrixExp[-RMatrix t])//Expand//Chop;
(*Q2[t_]=(Phi[t-1].Q1[1].(MatrixExp[-RMatrix
t].MatrixExp[RMatrix]))//*)(*Expand//Chop;*)
FFF=1/2(1+Sign[t])Phi[t].(MatrixExp[-RMatrix t])+1/2(1-
Sign[t])Phi[t+1].Q1[1].(MatrixExp[-RMatrix t].MatrixExp[-RMatrix]);
(*F11 is the Fourier form of the 2T transformation*)
<<Calculus`FourierTransform`
F11[t_]=Chop[NFourierTrigSeries[FFF,t,16,FourierParameters→{-1,1/2}]];
(*The comparsion of Q1[t],Q2[t] and F11*)
(*Itor=Flatten[Table[F11[t][[i,j]],{i,1,4},{j,1,4}]];
```

```
  Itorq1=Flatten[Table[Q1[t][[i,j]],{i,1,4},{j,1,4}]];
  Itorq2=Flatten[Table[Q2[t][[i,j]],{i,1,4},{j,1,4}]];
  Plot[Evaluate[Itor],{t,0,2}];
  Plot[Evaluate[Itorq1],{t,0,1}];
  Plot[Evaluate[Itorq2],{t,1,2}]*)
(*The modal transformation M in the paper=M1*)
M1=JordanDecomposition[RMatrix][[1]];
(*iPhi[t]=Inversion of the Phi[t]*)
(*iF11[t]=Inversion of the F11[t]*)
(*fGT[p2];
  fGG[Transpose[MGp],16,4];*)
tensorP=-Transpose[tensorP];
fQD[D2,D1];
fXMN[D2,D1];
B2=Inverse[(IdentityMatrix[D2 D1]-GGT.MQD)].XXMN;
Share[];
ClearAll[f];
iPhiMatrix=Table[f[i,j,x_],{i,1,D1},{j,1,D1}];
Do[f[i,j,x_]=fPhi[D2,j,i,x][[1]],{i,1,D1},{j,1,D1}]
iPhi[t_]=Table[f[i,j,t],{i,1,D1},{j,1,D1}];
iQ1[t_]=MatrixExp[RMatrix t].iPhi[t];
iFFF=1/2(1+Sign[t])(MatrixExp[RMatrix t].iPhi[t])+1/2(1-Sign[t])MatrixExp[RMatrix
t].MatrixExp[2 RMatrix].iPhi[1].iPhi[t+1];
```

```
<<Calculus`FourierTransform`

iF11[t_]=Chop[NFourierTrigSeries[iFFF,t,16,FourierParameters→{-1,1/2}]];

Share[];

ClearAll[J];

{M1,J}=JordanDecomposition[RMatrix];

(*J is the matrix of the linear part*)

SetDirectory["C:\Data\Dataoct19"];

(*Save useful Data*)

Save["F11.m",F11];

Save["iF11.m",iF11];

Save["M1.m",M1];

Save["J.m",J];

(*f0:Nonlinear Part*)

f0={0,0,-2.5 Cos[2 Pi t] x1^3 /6+1.5 (x1-x2)^3,0};

(* Transformation: x=F11.M1.z *)

{x1,x2,x3,x4}=F11[t].M1.{z1,z2,z3,z4}//TrigToExp//Expand;

f0=f0 //TrigToExp//Expand//Chop;

(*f1:Nonlinear part in terms of z*)

f1=Inverse[M1].iF11[t].f0//Chop//TrigToExp//Expand//Chop;

g1=Inverse[M1].iF11[t].{0,0,0,1}//Chop//TrigToExp//Expand//Chop;

Save["f1.m",f1];

Save["g1.m",g1];
```

APPENDIX D: MATHEMATICA PROGRAMS FOR CHAPTER 5

```
<<LinearAlgebra`MatrixManipulation`
fQp[p_]:=Module[{i,j},
   MQp:=Array[Qp,{Length[p],Length[p]}];
   Do[Qp[i,1]=p[[i]],{i,1,Length[p]}];
   Do[Qp[1,j]=p[[j]]/2,{j,2,Length[p]}];
   Do[If[(i==j),If[((2 j-1)≤Length[p]),Qp[i,j]=p[[2 i-
1]]/2+p[[1]],Qp[i,j]=p[[1]]],If[(i≠1&&j≠1),If[(i+j-1)≤Length[p],Qp[i,j]=p[[i+j-
1]]/2+p[[Abs[i-j]+1]]/2,Qp[i,j]=p[[Abs[i-j]+1]]/2]]],{i,1,Length[p]},{j,1,Length[p]}];
   Qp[1,1]=p[[1]];
   MQp
   ]
fGT[p_]:=Module[{i,j},
   MGp:=Array[G,{Length[p],Length[p]}];
   Do[If[j==1,If[i==2,G[i,j]=-1/8,G[i,j]=Cos[i Pi]/(2i(i-2))]
      ,If[Abs[i-j]==1,If[(i-j)==1,G[i,j]=-1/(4 (i-2)),If[j≠2,G[i,j]=1/(4 i),G[i,j]=1/2]]
        ,G[i,j]=0]
      ] ,{i,1,Length[p]},{j,1,Length[p]}];
   Transpose[MGp];
   ]
```

```
fQD[m_,n_]:=Module[{i,j,k,l,A,B},
   MQD:=Array[QD,{n m,n m}];
   Do[QD[(i-1)*m+k,(j-1)*m+l]=fQp[tensorP[[i,j ]]][[k,l]],
     {i,1,n},{j,1,n},{k,1,m},{l,1,m}
     ];
   ]
fGG[GG_,m_,n_]:=Module[{i,j,k,l},
   GGT:=Array[GGTD,{n m,n m}];
   Do[GGTD[i,j]=0,{i,1,m n},{j,1,m n}];


   Do[GGTD[(i-1)*m+k,(i-1)*m+l]=GG[[k,l]],
     {i,1,n},{k,1,m},{l,1,m}
     ];


   (*GGT=Outer[Times,IdentityMatrix[n],GG]*)
   ]
fXMN[m_,n_]:=Module[{i},
   XXMN:=ZeroMatrix[n m,n];
   Do[XXMN=ReplacePart[XXMN,1,{1+m(i-1),i}],{i,1,n}];
   ]
fPhi[m_,i_,j_,x_]:=Module[{k,aaaa},aaaa:=Transpose[Take[B2,{1+m (i-1),m i},{j}]];
     Expand[aaaa.Table[ChebyshevT[k-1,2x-1],{k,m}]]
```

```
];
MyQC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K5=Chop[J[[1]]];
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    K2
    ];
MyRC[QT_,T_]:=Module[{J,J1,J2,K1,K2}, J:=JordanDecomposition[QT];
    J1:=ReplacePart[J,Log[J[[2,1,1]]],{2,1,1}];
    J2:=ReplacePart[J1,Log[J1[[2,2,2]]],{2,2,2}];
    K1:=J2[[2]]/T;
    K2:=Chop[J[[1]].K1.Inverse[J[[1]]],10**-8];
    Re[K2]
    ];
F11[t_]:=Chop[1.0238946668034958` Cos[π t]-0.010578809229942313` Cos[3 π t]-
0.012452725747756913` Cos[5 π t]-0.0008219143493066704` Cos[7 π
t]+0.10151486201482154` Sin[π t]+0.027435843446217828` Sin[3 π
t]+0.0021043567003638936` Sin[5 π t]+0.00007593812644268017` Sin[7 π t],0.00001];
F12[t_]:=Chop[7.3833570887853195`*^-6 Cos[π t]+3.15965597590817`*^-7 Cos[3 π t]-
1.1016476429579142`*^-6 Cos[5 π t]-6.649766582176165`*^-7 Cos[7 π
```

t]+0.6769734366684271` Sin[π t]+0.18299739117822617` Sin[3 π

t]+0.014058498114366429` Sin[5 π t]+0.0005196341624796523` Sin[7 π t],0.00001];

F21[t_]:=Chop[-2.024145365209229`*^-10-0.09814468407923488` Cos[π t]-

3.143426239790581`*^-10 Cos[2 π t]+0.0843803457557818` Cos[3 π t]-

1.5855281165200825`*^-10 Cos[4 π t]+0.01296270759045827` Cos[5 π

t]+0.0007033726545787643` Cos[7 π t]-2.334863006713653` Sin[π

t]+2.136118643081275`*^-10 Sin[2 π t]-0.3226392153323683` Sin[3 π

t]+2.5899644928717436`*^-10 Sin[4 π t]+0.03504039439136719` Sin[5 π

t]+2.219293498972874`*^-10 Sin[6 π t]+0.004746822036022211` Sin[7 π

t]+1.9302779210098042`*^-10 Sin[8 π t],0.00001];

F22[t_]:=Chop[0.36976912648278565` Cos[π t]-1.1700785479407116`*^-10 Cos[2 π

t]+0.5521708409925531` Cos[3 π t]+0.07403354475255694` Cos[5 π

t]+0.003888867518600427` Cos[7 π t]-0.10154149005952358` Sin[π t]-

0.027445700127792436` Sin[3 π t]+1.019012837300123`*^-10 Sin[4 π t]-

0.002101827218856671` Sin[5 π t]-0.00007268956264423754` Sin[7 π t],0.00001];

iF11[t_]:=Chop[-2.3278313304730958`*^-10+0.3697781217198411` Cos[π t]-

4.0269817428578714`*^-10 Cos[2 π t]+0.5521829922327093` Cos[3 π t]-

2.7234755943517097`*^-10 Cos[4 π t]+0.07403543734202646` Cos[5 π t]-

1.7739855764919987`*^-10 Cos[6 π t]+0.003889109975626077З` Cos[7 π t]-

1.1524169292453923`*^-10 Cos[8 π t]-0.10154348100447448` Sin[π

t]+1.4998091484030862`*^-10 Sin[2 π t]-0.027450916835203974` Sin[3 π

t]+2.126076823275036`*^-10 Sin[4 π t]-0.002109252737006817` Sin[5 π

```
t]+2.0333594877297578`*^-10 Sin[6 π t]-0.00007731168175179222` Sin[7 π
t]+1.8203423360688142`*^-10 Sin[8 π t],0.00001];
iF12[t_]:=Chop[4.971283664322692`*^-7 Cos[π t]-1.7264813771001492`*^-6 Cos[3 π
t]-5.411557529935718`*^-7 Cos[5 π t]-1.250387881845949`*^-7 Cos[7 π t]-
0.6769880086023852` Sin[π t]-0.18300196273497005` Sin[3 π t]-
0.014059231635335779` Sin[5 π t]-0.0005199131319093384` Sin[7 π t],0.00001];
iF21[t_]:=Chop[5.124611564093218`*^-10+0.09812600773449506` Cos[π
t]+8.70034247771323`*^-10 Cos[2 π t]-0.08438001413399737` Cos[3 π
t]+5.788089703709365`*^-10 Cos[4 π t]-0.01296169461316749` Cos[5 π
t]+3.6243019696513556`*^-10 Cos[6 π t]-0.0007020950655365257` Cos[7 π
t]+2.3582644170794254`*^-10 Cos[8 π t]+2.3349079887606443` Sin[π t]-
4.712354118030504`*^-10 Sin[2 π t]+0.32264562169263306` Sin[3 π t]-
6.647762618161579`*^-10 Sin[4 π t]-0.03503896054836174` Sin[5 π t]-
6.419578757466127`*^-10 Sin[6 π t]-0.004745270764427489` Sin[7 π t]-
5.861210648772097`*^-10 Sin[8 π t],0.00001];
iF22[t_]:=Chop[1.0239108203691816` Cos[π t]-0.010579283162141013` Cos[3 π t]-
0.012452575989254365` Cos[5 π t]-0.000821681758807806` Cos[7 π
t]+0.10151342821267317` Sin[π t]+0.027429397491313387` Sin[3 π
t]+0.002100460529368267` Sin[5 π t]+0.00007281359501330037` Sin[7 π t],0.00001];
a=1;b=4;d=0.3;
R={{-0.6126001202082464`,-0.9425998793697027`},{-5.991875874791725`,-
0.3298370868713718`}};
Q[t_]:={{F11[t],F12[t]},{F21[t],F22[t]}};
```

```
iQ[t_]:={{iF11[t],iF12[t]},{iF21[t],iF22[t]}};
J=JordanDecomposition[R][[2]];
M1=JordanDecomposition[R][[1]];
J1=J[[1,1]];
J2=J[[2,2]];
Print["Eigenvalues of the Linear Part=",J1," ",J2];
<<DiscreteMath`Combinatorica`
f0={0,1/3 x1^3};
Rule1=Q[t].M1.{x1,x2};


f1=Chop[Expand[TrigToExp[Chop[Expand[Inverse[M1].iQ[t].(f0/.{x1→Rule1[[1]],x2→
Rule1[[2]]})]]]],0.0001];
g1=Chop[TrigToExp[Inverse[M1].iQ[t].{1,0}],0.0001];


Save["c:/a5/f1.m",f1];
Save["c:/a5/g1.m",g1];
<<DiscreteMath`Combinatorica`
X={x1,x2,x3,x4,x5,x6,x7,x8};
v1=2;
poly=3;
frequency=Max[Abs[Exponent[f1[[1]],E^(I Pi t),List]]];
Print["frequency1=",frequency];
TT[j_]:=Module[{i},
```

```
   Terms4:=Array[tpp,v1];
   tpp[1]=X[[1]]^(Compositions[poly,v1][[j]][[1]]);Do[tpp[i]=tpp [i-1]
X[[i]]^(Compositions[poly,v1][[j]][[i]]),{i,2,v1}];
   tpp[v1]
   ]
G21={};
G1={TT[1]};
Do[AppendTo[G1,TT[i]],{i,2,Length[Compositions[poly,v1]]}];
Do[Do[AppendTo[G21,G1[[j]] Exp[I Pi t i]],{j,1,Length[Compositions[poly,v1]]}],{i,-
frequency,frequency}];
MyCoefficient[pp_,GM_]:=Coefficient[Coefficient[pp,E^(Exponent[GM,E])],GM/E^(Ex
ponent[GM,E])];
Length[G21];
Print["Length of Alpha=",Length[G21]];
Alpha:=Array[aoa,{Length[G21]}];
f2=Chop[Expand[TrigToExp[Chop[Expand[Inverse[M1].iQ[t].(f0/.{x1→Rule1[[1]],x2→
Rule1[[2]]})+Inverse[M1].iQ[t].{G21.Alpha,0}]]]],0.0001];

frequency=Max[Abs[Exponent[f2[[1]],E^(I Pi t),List]]];
Print["frequency2=",frequency];
G2={};
G1={TT[1]};
Do[AppendTo[G1,TT[i]],{i,2,Length[Compositions[poly,v1]]}];
```

```
Do[Do[AppendTo[G2,G1[[j]] Exp[I Pi t i]],{j,1,Length[Compositions[poly,v1]]}],{i,-
frequency,frequency}];
Print["Length[G2]=",Length[G2]]
PowerVector[GM_]:=Module[{G3,i},
    G3={};
    Do[AppendTo[G3,Exponent[GM,X[[i]]]],{i,1,v1}];
    G3
    ];
Reasonance[GM_,jj_]:=(Eigenvalues[R][[jj]]-PowerVector[GM ].Eigenvalues[R]-
Coefficient[Exponent[GM,E],t]);
Do[Do[If[(Reasonance[G2[[i]],j]==0),Print["Reasonance"]],{i,1,Length[G2]}],{j,1,v1}];
NormalForm:=Array[NM,{v1}];
Do[NM[i]=0,{i,1,v1}]
Do[Do[If[(Reasonance[G2[[i]],j]≠0),If[(Exponent[G2[[i]],E]/t≠0),NM[j]=NM[j]+
G2[[i]](MyCoefficient[f2[[j]],G2[[i]]])],Print["Reasonance"]],{i,1,Length[G2]}],{j,1,v1}
];
tempf:=Array[TMF,{v1}];
Do[TMF[i]=NM[i],{i,1,v1}];
Do[NM[i]=0,{i,1,v1}];
Do[Do[If[(Reasonance[G2[[i]],j]≠0),If[(Exponent[G2[[i]],E]/t≠0),NM[j]=NM[j]+
G2[[i]](MyCoefficient[f2[[j]],G2[[i]]])/Reasonance[G2[[i]],j]],Print["Reasonance"]],{i,1,
Length[G2]}],{j,1,v1}];
```

```
Do[Do[If[(Reasonance[G2[[i]],j]≠0),If[(Exponent[G2[[i]],E]/t==0),NM[j]=NM[j]+
G2[[i]](Coefficient[Chop[f2[[j]]-
tempf[[j]]],G2[[i]]])/Reasonance[G2[[i]],j]],Print["Reasonance"]],{i,1,Length[G2]}],{j,1,
v1}];

KL=JordanDecomposition[R][[2]].NormalForm-
{{D[NormalForm[[1]],x1],D[NormalForm[[1]],x2]},{D[NormalForm[[2]],x1],D[Normal
Form[[2]],x2]}}.JordanDecomposition[R][[2]].{x1,x2}-
{D[NormalForm[[1]],t],D[NormalForm[[2]],t]}-f2;
Print["************************************test*************************
**************"]
Print["testnormalform=",Chop[Simplify[KL],0.000001]]
Print["************************************test*************************
**************"]
Share[]
Print["************************************test*************************
**************"]
pphi11=D[NormalForm[[1]],x1];
pphi12=D[NormalForm[[1]],x2];
pphi21=D[NormalForm[[2]],x1];
pphi22=D[NormalForm[[2]],x2];
pphi={{D[NormalForm[[1]],x1],D[NormalForm[[1]],x2]},{D[NormalForm[[2]],x1],D[N
ormalForm[[2]],x2]}};
```

```
frequency=Max[Abs[Exponent[pphi,E^(I Pi t),List]]];
Print["frequency=",frequency];
v1=2;
poly=2;
TT[j_]:=Module[{i},
   Terms4:=Array[tpp,v1];
   tpp[1]=X[[1]]^(Compositions[poly,v1][[j]][[1]]);Do[tpp[i]=tpp [i-1]
X[[i]]^(Compositions[poly,v1][[j]][[i]]),{i,2,v1}];
   tpp[v1]
   ]
G31={};
G1={TT[1]};
Do[AppendTo[G1,TT[i]],{i,2,Length[Compositions[poly,v1]]}];
Do[Do[AppendTo[G31,G1[[j]] Exp[I Pi t i]],{j,1,Length[Compositions[poly,v1]]}],{i,-
frequency,frequency}];
Print["************************************test************************
**************"]
Share[]
Print["************************************test************************
**************"]
Beta2=Array[bob,{Length[G31]}];
Equation2=-pphi.g1+(Beta2.G31)g1;
frequency=Max[Abs[Exponent[Equation2,E^(I Pi t),List]]]
```

```
v1=2;
poly=2;
TT[j_]:=Module[{i},
   Terms4:=Array[tpp,v1];
   tpp[1]=X[[1]]^(Compositions[poly,v1][[j]][[1]]);Do[tpp[i]=tpp [i-1]
X[[i]]^(Compositions[poly,v1][[j]][[i]]),{i,2,v1}];
   tpp[v1]
   ]
G3={};
G1={TT[1]};
Do[AppendTo[G1,TT[i]],{i,2,Length[Compositions[poly,v1]]}];
Do[Do[AppendTo[G3,G1[[j]] Exp[I Pi t i]],{j,1,Length[Compositions[poly,v1]]}],{i,-
frequency,frequency}];
NormalForm:=Array[NM,{v1}];
Do[NM[i]=0,{i,1,v1}]
Do[Do[If[(Exponent[G3[[i]],E]/t≠0),NM[j]=NM[j]+
G3[[i]](MyCoefficient[Equation2[[j]],G3[[i]]])],{i,1,Length[G3]}],{j,1,v1}];
tempf:=Array[TMF,{v1}];
Do[TMF[i]=NM[i],{i,1,v1}];
tempf1=Equation2-tempf;
L:=Array[lol,{Length[G3],v1}];
Print["**********************************test************************
*************"]
```

```
Share[]
Print["***********************************test*************************
**************"]
Do[Do[If[(Exponent[G3[[i]],E]/t≠0),lol[i,j]=Chop[(MyCoefficient[tempf[[j]],G3[[i]]]),0.
0001],lol[i,j]=Chop[(Coefficient[tempf1[[j]],G3[[i]]])],0.0001],{i,1,Length[G3]}],{j,1,v1
}];
Save["c:\L.m",L];
equation4:=Array[ro,{Length[G3] 2,Length[Alpha]+Length[Beta2]}];
Do[ro[i,k]=Coefficient[L[[i-IntegerPart[(i-1)/Length[G3]] Length[G3],IntegerPart[(i-
1)/Length[G3]]+1]],Alpha[[k]]],{i,1,Length[G3] 2},{k,1,Length[Alpha]}];
Do[ro[i,k+Length[Alpha]]=Coefficient[L[[i-IntegerPart[(i-1)/Length[G3]]
Length[G3],IntegerPart[(i-1)/Length[G3]]+1]],Beta2[[k]]],{i,1,Length[G3]
2},{k,1,Length[Beta2]}];
Print["***********************************test*************************
**************"]
Share[]
Print["***********************************test*************************
**************"]
tempf2:=Array[tmf2,{Length[G3] 2}];
Do[tmf2[i]=L[[i,1]],{i,1,Length[G3]}];
Do[tmf2[i]=L[[i-Length[G3],2]],{i,Length[G3]+1,Length[G3] 2}];
Print["*************************************"];
tempf3=Chop[Simplify[tempf2-Simplify[equation4.Join[Alpha,Beta2]]]];
```

```
equation4=Chop[Simplify[equation4]];

Save["result1",equation4];

Save["tempf3",tempf3];

Print["***************************************"]

<<LinearAlgebra`MatrixManipulation`


tempf4=Array[tmf4,{Length[G3] 2,1}];

Do[tmf4[i,1]=tempf3[[i]],{i,1,Length[G3] 2}]

Dimensions[equation4]

Dimensions[AppendRows[equation4,tempf4]]

MatrixRank[AppendRows[equation4,tempf4]]

MatrixRank[equation4]

Print["***************************************"]

Print["**************************************************************

***************"]

i=31;

aaa=Join[TakeRows[equation4,{i,Length[G3]-(i-

1)}],TakeRows[equation4,{i+Length[G3],Length[G3]-(i-1)+Length[G3]}]];

bbb=Join[TakeRows[tempf4,{i,Length[G3]-(i-

1)}],TakeRows[tempf4,{i+Length[G3],Length[G3]-(i-1)+Length[G3]}]];

Solution1=LinearSolve[aaa,bbb];

Print["**************************************************************

***************"]
```

```
checkdim[i_]:=Module[{aaa,bbb},
    aaa=Join[TakeRows[equation4,{i,Length[G3]-(i-
1)}],TakeRows[equation4,{i+Length[G3],Length[G3]-(i-1)+Length[G3]}]];
    bbb=Join[TakeRows[tempf4,{i,Length[G3]-(i-
1)}],TakeRows[tempf4,{i+Length[G3],Length[G3]-(i-1)+Length[G3]}]];
    MatrixRank[aaa]-MatrixRank[AppendRows[aaa,bbb]]
    ];
Print["*************************************************************
***************"]


Do[Print[checkdim[i],"-------",Length[G3]/3-IntegerPart[(i-1)/3],"------",i],{i,1,30}]
Rm={{-0.6126001202082464`,-0.9425998793697027`},{-5.991875874791725`,-
0.3298370868713718`}};
Phi[1]={{-3.2050915011896213`,1.324798857991513`},{8.421420891183407`,-
3.602507363259747`}};
iQ1[1]=Inverse[Phi[1].MatrixExp[-Rm]]
FFF=1/2(1+Sign[t])(MatrixExp[Rm t]).iPhi[t]+1/2(1-Sign[t])(MatrixExp[Rm
t].MatrixExp[Rm]).iQ1[1].iPhi[t+1];
<<Calculus`FourierTransform`
F11=Chop[NFourierTrigSeries[FFF,t,8,FourierParameters→{-1,1/2}]]
p2={0.30424217764409384`,2.5026601968371524`*^-
16,0.9708678652630196`,1.3252311172764654`*^-16,-
```

```
0.30284915526269945`,1.511229668379056`*^-16,0.02909193396501055`,0,-
0.0013922439911775739`,0,0.000040189944511023766`,0,-7.782767012882281`*^-
7,0,1.0826530292730342`*^-8,0,-1.1351103472439703`*^-10};
fGT[p2];
fGG[Transpose[MGp],16,2]
tensorP:=Array[tP,{2,2,16}];
Do[tP[1,1,k]=0,{k,1,16}];
Do[tP[2,1,k]=0,{k,2,16}];
tP[2,1,1]=-Pi;
aaa=1;
bbb=4;
ddd=0.3;
Do[tP[1,2,k]=Pi*bbb*p2[[k]],{k,2,16}];
tP[1,2,1]=Pi*aaa+bbb*Pi*p2[[1]];
Do[tP[2,2,k]=0,{k,2,16}];
tP[2,2,1]=Pi*ddd;
fQD[16,2];
fXMN[16,2];
B2:=Inverse[(IdentityMatrix[32]-GGT.MQD)].XXMN
if11[x_]:=fPhi[16,1,1,x];
if12[x_]:=fPhi[16,1,2,x];
if21[x_]:=fPhi[16,2,1,x];
if22[x_]:=fPhi[16,2,2,x];
```

```
iPhi[t_]:=Transpose[{{if11[t][[1]],if12[t][[1]]},{if21[t][[1]],if22[t][[1]]}}];
ResultTemp = {{9.459460812676904*^-7, {0.7039247177226323,
0.614060626788138,
-0.22491461398805415, -0.11087344546143868, 0.13588998804447416,
0.009150832073654542, 0.005123622118513337, 0.00003485366463928426,
0.4635757468771665, 0.6772463993074569, 0.5053318551613496,
0.22861340792108945, 0.012704745692849895, 0.009116662288150952,
-0.015225729749111195, -0.000054065549780501396, 1.332718865604059,
4.817480276057992}}, {1.574851981518728*^-8, {0.20812107954537376,
-0.5631505311163653, -0.007586107092019185, 0.013069890168860593,
0.019644787545837736, -0.0004629921363702432, -0.0017344785712967948,
1.3704072289755973*^-6, 0.8576292072514765, -0.19547462940655205,
0.11097492062600657, -0.06955579644281587, -0.011386533855981644,
-0.005620382268333169, -0.004328520772144943, 2.593717314670443*^-6,
0.45497417490133335, 6.754574573114714}}, {2.1525600665684757*^-6,
{1.2392023534578984, 0.24639543370155087, -0.01919278056072649,
0.40933241383523417, 0.1239253826217059, -0.06484894899823197,
0.051416943051486565, -0.00006083167537215881, 0.14557366887500772,
0.5499070182042283, 0.466484299216122, 0.430156275750982,
-0.10687475172206762, 0.12494753963201909, -0.08749627072919249,
-0.00008383482905514384, 0.9874907152248631, 5.786145059257675}},
{4.793973257927716*^-8, {-0.45265821406283807, 0.13998564552366255,
-0.307160361241143, -0.05267016836730426, -0.009539169835937117,
```

-0.0020457131735826363, -0.0006488969445352661,

-0.000014153114862805802, 1.0465593984331372, 0.06690997506944274,

-0.21620666220942766, -0.0028933725120094196, 0.0035329955458558784,

0.0038904412591224385, 0.0010895800625278194,

-0.000014406257974242402, -0.42666660807015555, 4.23758830172676}},

{3.8502636908761605*^-8, {0.6959516721997071, 0.41572441699693224,

-0.1444523192133281, -0.0063268696621810415, -0.0049207313108212344,

-0.004923137967518572, -0.005694289291790561, -9.657078994932829*^-6,

0.5024586539049648, -0.17947772707187631, -0.1510129405231804,

0.053960135882274135, -0.0032561985997142367, 0.015867267818902367,

-0.0026255239178456895, 7.639556053037768*^-6, -0.1818350541933153,

3.9983393520278576}}, {5.175754781626553*^-7,

{-0.030340569844071075, -0.05134071631282341, 0.1121054037330687,

0.038961512797572394, -0.07581427177917273, -0.033150848472496316,

-0.15028031585118615, -1.5791066482923998*^-7, 0.017394467696515672,

-0.04332143650916207, -0.030290316469748578, -0.08086258546154333,

0.02274481846562126, -0.10144433566341227, -0.06545671191405246,

-1.0562435669300922*^-8, 0.021354422395528507, 4.890380870496636}},

{1.7302999701294413*^-8, {0.00045606619960884744, 0.1484880088122732,

0.015473360053038055, 0.04670236167332846, -0.0136378433684295,

0.00038456593175294174, 0.002392302341412711, 2.2220066618078857*^-7,

0.1983154664958103, -0.15996893483698263, -0.05086195616421787,

-0.011664788251500843, 0.014164128281560572, -0.005670200365213842,

0.0020164036418320206, -1.3329242514336991*^-7, -0.08420246886482052,
6.644097134477675}}, {1.1549132912855608*^-7,
{-0.4179839576215574, -0.1934456157031891, -0.01378496825960234,
-0.014770905384556793, -0.0031272135778036797, -0.003458090025508305,
0.0050569200141654495, 4.694017499524044*^-6, 0.3895900908625163,
0.48133212617711163, 0.10501653755626168, 0.13858282539501215,
-0.043794650138405104, -0.025035974615778233, 0.013097066120347516,
5.684205118275426*^-6, -0.4059964429805249, 4.728791607190955}},
{1.3139889147385096*^-7, {-0.07952728790590675, -0.4864008596082756,
-0.1891307058889629, 0.05239792036775826, 0.019896188490716823,
0.010888801942220 22, 0.0015802016736723254, 1.2024662572402656*^-6,
1.0347459025204389, -0.09999891131694055, -0.1544545016747794,
0.1206240394303742, 0.012583059761966425, 0.007280741260132286,
-0.002174129468815898, 0.00002082138820859517, -0.13874410963603556,
4.639584165092001}}, {1.6584370888367436*^-7,
{-0.3507234803448114, -0.31665576259528677, 0.22420725953072074,
-0.048281920449412284, -0.0032658312192840617, -0.016761514267419047,
-0.008324214263780737, 5.525049427858722*^-6, 0.15333604645298096,
-0.3852042677687783, 0.06380789679817045, 0.056214595866820725,
0.051836144049683584, 0.015483207027789068, 0.013243043642785406,
-6.239498236983669*^-7, 0.2534110881591648, 5.495832273036208}},
{5.1212094824345275*^-8, {0.19357881691502457, -0.020955611898127,
-0.0615861286977735, 0.020409785325495543, -0.03374699119052932,

-0.020129737457456373, -0.003325062380460136, -2.8878781530654604*^-7,
0.21386169625884918, -0.2475697352414916, 0.005294266765286255,
-0.05603093740580228, 0.0006085195860797818, -0.0030287355062018594,
0.0020385645036643733, -1.953665397783194*^-6, -0.28880623079713885,
4.235414318843983}}, {1.8523190633455435*^-6,
{0.9090851361756779, -0.05822682423901338, 0.27692609745774555,
0.4278865516791624, -0.11767650098447262, 0.01184899336039271,
-0.0007491567642237485, 0.00009272472644599105, 0.5425918665103963,
-0.11606206288504434, -0.5501968958639153, -0.16839023265694764,
-0.06140545568316439, 0.08580658178146554, 0.0010969662750838066,
0.000060153870933756075, 0.9755159902873319, 5.0015892840022635}},
{4.458360758514489*^-7, {0.33690642563073264, 1.0671571032080087,
0.332701944330696, -0.010795700411398802, -0.0050134739386725365,
0.0031746669283866814, -0.00009344566743266161,
-0.0004971816601532319, 0.6208828881640612, 1.0764715641833331,
0.5689335714376332, -0.06306672752391433, 0.014677185907940392,
-0.004321189637759419, 0.0012104447501964284, -0.0004616389092499289,
0.8088430887604872, 2.3587996468714483}}, {2.541207043957278*^-7,
{-0.30524872025318905, -0.16373474393275744, 0.10246947101973615,
0.030063428097931055, -0.10530293958132977, -0.09895354759322589,
-0.02157158583881092, 5.561682823053342*^-7, -0.30011181105319723,
0.13831707919521255, 0.045881303729141495, -0.042873875853778554,
0.0917699176583438, -0.02059131994773417, -0.02527836329612874,

-1.6108360250809342*^-6, 0.22163762133668868, 6.581312164728857}},
{3.7466657836401366*^-9, {0.0025040193945495832, -0.017706915697495162,
-0.03108692237103356, 0.008767119182132157, -0.0027584063217898957,
0.003373784561521294, -0.0016469169273887099, 4.828514534583127*^-8,
0.047287036414322296, -0.010587955195146714, -0.03190716970017414,
-0.0005960757448502426, -0.0023410318366799436,
-0.0037384398502163753, -0.0018470014138866747, 3.277302678763529*^-8,
0.025977466585093353, 2.5708812420750657}}, {9.265742165864106*^-9,
{-0.10087349665832027, -0.5021444563576625, -0.08633015455268879,
-0.038745976651845884, 0.00780095425859966, 0.002786659819151955,
-0.001389080379894007, -9.03423537687966*^-6, 0.11285269537119604,
-0.042712333846191455, 0.33443238443570683, -0.019404005340765883,
-0.009508833911841652, 0.003499422642888397, 0.0005173035358272749,
0.000010170594860687793, 0.17236825987528845, 4.724855645073426}},
{8.682743508494359*^-8, {0.8581935183822874, 0.2692422574693941,
-0.024491894460958424, 0.05790207682769462, 0.013673780196873073,
0.00030933990721359 09, -0.0024915488220897783,
-0.000054738438062787 22, 1.0161246159391917, 0.4983140090086436,
-0.01961750581990442, 0.0011829371095987168, 0.008115526760709825,
-0.004649736059161377, 0.0020945235272263733, -3.376905676185963*^-6,
0.18956292945895414, 2.3084497103153}}, {5.219746100450563*^-8,
{-0.2274255526332585, 0.01339084442687666, 0.13352599133351617,
-0.010842127134593428, -0.008046968820834041, -0.00025523671465043865,

-0.011350888482631771, 6.012122398475178*^-7, -0.011585845865470654,
0.10127433920638142, 0.10142464948607179, 0.09074108260110096,
-0.012375365535186422, 0.019513441730903604, -0.010718544069103964,
4.1915402404722614*^-7, -0.12186438537494303, 5.198325076120963}},
{1.4363965069178464*^-8, {-0.9075715536788296, -0.5337957866214456,
-0.08821020309247758, -0.012222785449428937, -0.008160140773978303,
0.0015905085591491762, -0.00011376438324947243, 6.308140143828487*^-6,
1.0201815382742885, -0.21520334906275548, -0.15819036889073199,
-0.02421247884476762, 0.009108430115235492, -0.00017718738148374077,
0.0002529350904584503, 2.8576545762596492*^-6, -0.5443745927851241,
3.918369873087868}}, {1.264552918925276*^-8, {0.8893196446846683,
0.4424671517859442, 0.01359309124673451, 0.10955890769237303,
-0.01725135897929542, 0.003765341428193575, -0.0009887621981848472,
-0.000013274681450952958, 0.8375811692732037, 0.7597750790411179,
0.09561311996250689, -0.048874235431465705, 0.0019529593806242027,
0.006597707519402017, -0.0009123443881124302, -7.460375922790581*^-6,
-0.2831686129273203, 5.638316969622036}}}