Non-metric Continua that support Whitney maps

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# Non-metric Continua that support Whitney maps

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## Vita

Jennifer Williamson Stone, daughter of Cecil and Peggy(Duke) Williamson, was born on October 22, 1973 in Selma, Alabama. She graduated from Morgan Academy in 1992. In March of 1997, she graduated (with distinction) from Auburn University with a B.S. in mathematics. In June of 1998, she graduated (Sum Cum Laude) from Auburn University with a M.S. in Mathematics. She and her husband, Russ Stone, have a son Nicholas(13) and a daughter Emily(7).

## Dissertation Abstract

## Non-metric Continua that support Whitney maps

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An example of a non-metric continuum is constructed, where every non-degenerate subcontinuum is non-metric, that supports a Whitney map. Additional non-metric examples are given and examinations of conditions under which non-metric continua support Whitney maps are made.

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Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliography follows van Leunen's *A Handbook for Scholars.*

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

## Table of Contents

# Chapter 1

## Introduction and Background

The purpose of this paper is to explore non-metric continua that support Whitney maps. An indecomposable non-metric continuum, where each proper non-degenerate subcontinuum is non-metric, that supports a Whitney map is constructed. Several other examples of non-metric continua that support Whitney maps are given. In addition, topological properties that will prevent non-metric continua from supporting a Whitney map are examined.

A non-metric analog of the Cantor set and the Solenoid are used in the construction of an example of a non-metric continuum that supports a Whitney map. This example will then be used to construct an indecomposable non-metric continuum, where each proper non-degenerate subcontinuum is non-metric, that supports a Whitney map. Thus a short discussion of the Cantor set and the Solenoid will be useful.

A metric Cantor set is any uncountable topological space that is compact, each point of the set is a limit point of the set and the only connected subsets are singleton points. The most common example of a Cantor set is the "middle third" Cantor set. This Cantor set is in $[0, 1]$ and has the following structure:

$M_1 = [0, 1/3] \cup [2/3, 1]$

$M_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

...etc.

At each stage, $M_j$ is formed by removing the middle third open interval from each segment of $M_{j-1}$. The middle third Cantor set is $\bigcap_{i=1}^{\infty} M_i$. Every Cantor set is homeomorphic to the middle third Cantor set. In Chapter 2 an example $S$ is constructed. $S$ is an non-metric continuum (A *continuum* is a compact, connected space) that supports a Whitney map. $S$ will be formed using a subset $Z$ of the lexicographic arc $Lx$. The lexicographic arc, $Lx$, is a non-metric linearly ordered connected compact Hausdorff space. ( A space is *Hausdorff* if given points $x \neq y$ there exist open disjoint sets $U$ and $V$ such that $x \in U$ and $y \in V$.) The standard lexicographic arc is the topological space $Lx$ defined as follows. $Lx = [0,1] \times [0,1]$. If $x_1 = (p_1, q_1)$ and $x_2 = (p_2, q_2)$, then $x_1 <_{Lx} x_2$ if and only if $p_1 < p_2$ or $p_1 = p_2$ and $q_1 < q_2$ where "$<$" denotes the standard order on $[0,1]$. The space $Lx$ together with the order topology induced by the order "$<_{Lx}$" is called the lexicographic arc. $Z$ is defined as $\{(p,q) \in Lx | q = 0, q = 1\} \backslash \{(0,0),(1,1)\}$. $Z$ is a non-metric analog of the Cantor set. $Z$ is an uncountable compact space where every point of the set is a limit point of the set and the only connected subsets are singleton points; however this set is non-metric and so is not homeomorphic to the Cantor set.

As an example of an indecomposable continuum we construct a "solenoid" in Euclidean three space. A solenoid can be embedded in $R^3$ and has roughly the following structure:

Let $M_1$ =solid torus

$M_2$= a smaller solid torus lying inside of $M_1$ wrapped twice around the "hole" inside of $M_1$ before joining up with itself.

$M_3$= a still smaller solid torus wrapped twice inside of $M_2$,

...etc.

The solenoid $= \bigcap_{i=1}^{\infty} M_i$. (The solenoid is more commonly defined in terms of inverse limit see pages 9-10.)

One of the central cross-sections of the solenoid will yield the Cantor set. Each point from this Cantor set is connected to another point from the Cantor set with a unique arc in the solenoid. This fact can be used to give another construction of the solenoid. In that construction, begin with the Cantor set cross a metric arc, then assign a "gluing pattern" between $C \times \{0\}$ and $C \times \{1\}$ to reproduce the solenoid according to how theses arcs are connected. In Chapter 2 in order to construct $S$ we will replace the Cantor set with $Z$ and assign a "gluing" that will create $S$. Thus $S$ is a non-metric analog of the solenoid. Both $S$ and the solenoid are indecomposable continua. (An *indecomposable continuum* is one that is not the union of two proper subcontinua.) They both have uncountably many composants. (If $X$ is a continuum and $p \in X$ then the *composants of $X$ at $p$* is the set to which $x$ belongs if and only if there is a proper subcontinuum of $X$ containing $x$ and $p$). Each composant can be linearly matched with the real line. Also both the solenoid and $S$ have the property that each proper subcontinuum is a metric arc. A good understanding of $S$ will be needed since our main example in Chapter 3 will use $S$ with a dense set of points "blown up" into copies of S-like continua.

A *Whitney map* $g$ is a measure on the space of compact subsets of a continuum that has the property that if $A \subsetneq B$ then $g(A) < g(B)$, and $g(\{x\}) = 0$ for each

$x \in X$ [9]. A Whitney map on the space of subcontinua of a space will be denoted by $\mu$.

In the 1930's H. Whitney first constructed Whitney functions (now called Whitney maps) to study families of curves [12],[13]. In 1942, J.L. Kelly was the first to use Whitney maps to study hyperspaces [5]. It is known that if $X$ is a compact metric space then there exist a Whitney map for $2^X$ and $C(X)$, where $2^X$ is the space of compact subsets of $X$ and $C(X) = \{K \in 2^X | K \text{ is a continuum }\}$. Whitney defined one as follows: let $A \in 2^X$. Let $F_n(X) = \{H \in 2^X | H$ has at most n points$\}$. Let $n \geq 2$ be a fixed natural number. Define $\lambda_n : F_n(A) \rightarrow [0, \infty)$ by given $K = \{a_1, a_2, ...a_n\} \in F_n(A)$ [where the enumeration of $K$ may not be one-to-one], then

$$\lambda_n(K) = \min\{d(a_i, a_j) | i \neq j\}.$$

Note that $\lambda_n(K) \leq \text{diam}[A]$ for each $K \in F_n(A)$; hence $\omega_n(A)$ given by

$$\omega_n(A) = l.u.b.\{\lambda_n(K) | K \in F_n(A)\}$$

is a real number. This defines $\omega_n(A)$ for each natural number $n \geq 2$. Since $\omega_n(A) \leq \text{diam}[A]$ for each $n = 2, 3, ...$, the series $\sum_{n=2}^{\infty} 2^{1-n}\omega_n(A)$ converges; define $\omega(A)$ by

$$\omega(A) = \sum_{n=2}^{\infty} 2^{1-n}\omega_n(A).$$

Since $A$ was an arbitrary member of $2^X$, we have defined a function $\omega : 2^X \to [0, \infty)$. Whitney proved that this function has all the properties that are now defined as a Whitney map [9](p.24-26). It has recently been shown that local connectivity plus the existence of a Whitney map implies metrizablity. [7]

V. E. Šneĭder showed that if $X$ is a Hausdorff compact space and the diagonal of $X \times X$ is a $G - delta$ set then $X$ is metric [11]. A corollary is that if there is a continuous function $f : X \times X \to \mathbb{R}$ so that $f(x, y) = 0$ if and only if $x = y$, and $X$ is compact then $X$ is metric. Using those facts it is easily shown that no Whitney map for $2^X$ exists when $X$ is compact and non-metric. In 2000 J.J. Charatonik and W.J. Charatonik [1] showed that the non-metric indecomposable continuum example given by Gutek and Hagopian [3] will support a Whitney map on $C(X)$. They also provided examples of non-metric continua that will not support a Whitney map. They posed the question of characterizing non-metric continua for which there exist a Whitney map for $C(X)$ [1]. From this point forward since it is known that no Whitney map on $2^X$ exists, where $X$ is a non-metric compact space, a Whitney map will refer to a Whitney map on $C(X)$. In this paper the author will first construct an example of a non-metric indecomposable continuum that supports a Whitney map. It will be shown that its hyperspace is an example of a non-metric continuum, which is arcwise connected by metric arcs, that does not support a Whitney map. Thus it shows that arcwise connectedness by metric arcs is not a sufficient condition for admitting a Whitney map. The author will also give an example of a hereditarily non-metric indecomposable continuum that supports a Whitney map. In this paper a space is hereditarily non-metric if every non-degenerate proper subcontinuum is non-metric. In addition, a method that proves that there exist continuum-many decomposable non-metric continua, each of which supports a Whitney map, is given. An example of a hereditarily indecomposable non-metric continuum that supports a Whitney map is also constructed.

While the question of characterizing which non-metric continua will support a Whitney map is still unsolved, several theorems are given showing when non-metric continuum cannot support a Whitney map. For example, if $X$ is a non-metric continuum which has a specific contraction then $X$ cannot support a Whitney map. It is also shown that given two non-metric continua that support Whitney maps, their union does not necessarily support a Whitney map. Lastly the author will examine Whitney properties for non-metric spaces. It is shown that $\mu^{-1}(t)$ is also a continuum even if $X$ is non-metric. It is also shown that hereditary indecomposablity is a Whitney property in the non-metric case.

The work on Whitney maps of non-metric continua opens up an area of interesting research for two reasons. First, it will be a challenge to prove what properties are Whitney properties for non-metric continuum. Second, properties that are trivial to prove in the metric case can be considered, for example separability. It is not known if a non-metric continuum that support Whitney maps must be separable. Lastly the existence of Whitney maps on non-metric continua will be useful in that it will provide an additional tool for the study of the hyperspaces of a large class of non-metric continua.

**Statements of needed background Definitions and Theorems**

Definition: A compact connected Hausdorff space is said to be a *continuum.* Note: It need not be metric.

**Theorem 1.1.** *If $H$ and $K$ are closed subsets of the compact set $M$ but no subcontinuum of $M$ intersects both $H$ and $K$, then $M$ is the union of two closed sets one containing $H$ and the other containing $K$.*

**Theorem 1.2.** *Suppose that $M$ is a continuum and $U$ is an open set intersecting but not containing $M$. If $L$ is a component of $M - U$ then $L$ contains a point of the boundary of $U$.*

Definition: If $X$ is a continuum and $p \in X$ then the *composant of $X$ at $p$* is the set to which $x$ belongs if and only if there is a proper subcontinuum of $X$ containing $x$ and $p$.

In the solenoid and the example $S$ outlined above, all points connected to a given point by an arc is a composant.

**Theorem 1.3.** *If $K$ is a composant of the continuum $X$, then $K$ is dense in $X$.*

Definition: The continuum $X$ is said to be *irreducible from the point $p$ to the point $q$* if and only if no proper subcontinuum of $X$ contains $p$ and $q$.

We define the $sin 1/x$ curve as the closure of $\{(x, y)|x \in (0, 1], y = sin 1/x\}$. The $sin 1/x$ curve is irreducible between the points $(0, 0)$ and $(1, sin 1)$.

Definition: If $X$ is a set then the subset $K$ of $X$ is said to be *nowhere dense in* $X$ if and only if every open set intersecting $X$ contains an open set that contains no point of $K$.

Definition : A space $X$ is *separable* if $X$ contains a countable dense set.

Definition: A space $X$ is *completely separable* if $X$ has a countable basis.

The set $Z$ as defined above is not completely separable; this implies that it is non-metric.

Definition: The continuum $X$ is said to be *indecomposable* if and only if it is not the union of two proper subcontinua. As previously mentioned the solenoid is an example of an indecomposable continuum.

**Theorem 1.4.** *The continuum $X$ is indecomposable if and only if every proper subcontinuum of $X$ is nowhere dense in $X$.*

**Theorem 1.5.** *If $X$ is an indecomposable continuum and each of $C$ and $D$ is a composant of $X$ then either $C = D$ or $C \cap D = \emptyset$.*

**Theorem 1.6.** *If the continuum $M$ intersects each of two closed sets $H$ and $K$ then there is a subcontinuum of $M$ irreducible from $H$ to $K$.*

Definition: A space $X$ is hereditarily indecomposable if every subcontinuum is indecomposable.

**Theorem 1.7.** *If $X$ and $Y$ are non-degenerate hereditarily indecomposable metric chainable continua then they are homeomorphic.*

Definition: A non-degenerate hereditarily indecomposable chainable continuum is called a *pseudo-arc*.

Definition: A *chain* is a finite collection of open sets $G_1, ..., G_n$ such that $G_i \cap G_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A member of this chain is called a *link*.

Definition: A continuum $X$ is *chainable* provided that for each positive $\epsilon$, there is a chain in $X$ such that each link has diameter less than $\epsilon$.

**Theorem 1.8.** *The pseudo-arc can be mapped onto any chainable metric continuum.*

Definition: Let $I$ be an ordered index set. For each $\alpha \in I$ let $X_\alpha$ be a topological space and for each $\alpha < \beta$ let $f_\alpha^\beta : X_\beta \to X_\alpha$ be a continuous map so that for $\alpha < \beta < \gamma$ we have $f_\alpha^\beta \circ f_\beta^\gamma = f_\alpha^\gamma$. Then $\{X_\alpha, f_\alpha^\beta\}_I$ is called the *inverse system* of the spaces $\{X_\alpha\}_{\alpha \in I}$ and mappings $\{f_\alpha^\beta\}_{\alpha<\beta}$.

Definition. Suppose that $\{X_\alpha, f_\alpha^\beta\}$ is an inverse system with index set $I$. Then the *inverse limit space* $X = \varprojlim\{X_\alpha, f_\alpha^\beta\}_I$ is defined as follows.

The element $P = \{P_\alpha\}_{\alpha \in I}$ is a point of $X$ provided for each $\alpha \in I$, $P_\alpha \in X_\alpha$ and for each $\alpha < \beta$, $f_\alpha^\beta(P_\beta) = P_\alpha$. The set $R$ is a basic open set provided there exists an $i \in I$ and an open set $O_i$ in the space $X_i$ so that $R = \{P \in X | P_i \in O_i\}$.

**Theorem 1.9.** *The inverse limit space* $X_I = \varprojlim\{X_\alpha, f_\alpha^\beta\}_I$ *is a subspace of the product space* $\Pi_{\alpha \in I} X_\alpha$.

**Theorem 1.10.** *The inverse limit of arcs is chainable.*

As mentioned previously the solenoid is usually defined as an inverse limit on a circle. The solenoid equals $\varprojlim\{X_i, f_i\}$ where i runs over natural numbers, each $X_i$ is a circle, and $f_i$ wraps the circle $X_{i+1}$ a certain number of times around the circle $X_i$.

Definition: Suppose that $X$ is a topological space. Then the *hyperspace of* $X$ denoted by $2^X$ is the space of compact subsets of $X$. Suppose that $U_1, U_2, U_3, ...U_n$ is a finite collection of open sets. Then $\{K \in 2^X | K \in \cup_{i=1}^{n} U_i$ and for all $1 \leq i \leq n, K \cap U_i \neq \emptyset\}$ is a basic open set for the topology.

Definition: Let $C(X) = \{K \in 2^X|$ K is a continuum$\}$; $C(X)$ is called the *hyperspace of subcontinua of* $X$.

**Theorem 1.11.** *$C(X)$ is a closed subset of $2^X$.*

Definition: Let $X$ be a continuum. A *Whitney map* is defined as a continuous function $\mu : C(X) \to [0, \infty)$ such that $\mu(M) = 0$ if and only if the set $M$ in $X$ is a singleton, and for any $A, B \in C(X)$ such that $A \subsetneq B$ then $\mu(A) < \mu(B)$.

Definition: A *Whitney level* is a subset of $C(X)$ that is of the form $\mu^{-1}(t)$, where $\mu$ is some Whitney map for $C(X)$ and $t \in [0, \mu(X)]$.

Definition: A property $P$ is a *Whitney property* provided that if a continuum $X$ has the property $P$ so does $\mu^{-1}(t)$ for each Whitney map $\mu$ on $C(X)$ and for each $t \in [0, \mu(X)]$.

Considerable work has been done on Whitney properties of metric continua. [9]

Definition: Suppose that $X$ is a Hausdorff space. Then the collection $G$ of subsets of $X$ is upper semi-continuous means that for each $g \in G$ and open set $D$ containing

$g$ there is an open set $D'$ containing $g$ so that each member of $G$ intersecting $D'$ lies in $D$.

Definition: Suppose that $X$ is a Hausdorff space and $G$ is an upper semi-continuous collection so that $\cup G = X$. Then $X/G$ is the space whose points are the elements of $G$; the set $R$ is a basis element for the topology of $X/G$ if $R$ is a subset of $G$ so that $\cup R$ is an open set in $X$.

**Theorem 1.12.** *If $X$ is a Hausdorff compact space and the diagonal of $X \times X$ is a $G -$ delta set, then $X$ is metric [11].*

**Theorem 1.13.** *If there is a continuous function $f : X \times X \to \mathbb{R}$ so that $f(x, y) = 0$ if and only if $x = y$, and $X$ is compact, then $X$ is metric.*

## Chapter 2

## A Non-metric continuum that supports a Whitney map

**Theorem 2.1. There exists a non-metric indecomposable continuum that supports a Whitney map.**

**Part 1**. Definitions. The Lexicographic Arc.

Let $I$ denote the unit interval $I = [0, 1]$ with the usual topology.

Definition. The standard lexicographic arc is the topological space $Lx$ defined as follows. $Lx = [0, 1] \times [0, 1]$. If $x_1 = (p_1, q_1)$ and $x_2 = (p_2, q_2)$, then $x_1 <_{Lx} x_2$ if and only if $p_1 < p_2$ or $p_1 = p_2$ and $q_1 < q_2$ where "$<$" denotes the standard order on $[0, 1]$. The space $Lx$ together with the order topology induced by the order "$<_{Lx}$" is called the lexicographic arc.

Background Theorem 2.1.1. The lexicographic arc is a nonmetric Hausdorff arc. It is first countable at each point but is not separable and not completely separable.

Let $Z$ denote the subspace of $Lx$ defined by $Z = \{(p, q) \in Lx |\ q = 0 \text{ or } q = 1\} \setminus \{(0, 0), (1, 1)\}$. (Note these two points are removed because they turn out to be isolated points.)

Let us denote the set $Z_0 = \{(p, q) \in Z |\ q = 0\}$ as the "bottom" part of $Z$ and $Z_1 = \{(p, q) \in Z |\ q = 1\}$ as the "top" part of $Z$. For each $t \in [0, 1]$ let $t_0$ be the element $(t, 0)$ in $Z$ and similarly let $t_1 = (t, 1)$.

Background Theorem 2.1.2. Properties of the space $Z$:

1. $Z$ is a compact separable subspace of $Lx$.

2. $Z$ is not completely separable (and hence is not metric).

3. If $M$ is a subset of the bottom (or top) of $X$ which is dense with respect to the usual order topology on $[0, 1]$ then it is dense in $Z$ with respect to the lexicographic topology.

4. Every point of $Z$ is a limit point of the space.

5. $Z$ is totally disconnected.

6. The collection $\{[a_1, b_0]|a < b \in \mathbb{R}\}$ is a basis for the topology of $Z$.

□

Let $J \subset [0, 1]$ be an interval.

Background Theorem 2.1.3. Let $f : J \to [0, 1]$ be an order preserving homeomorphism. Then the function $f'$ defined by $f'(t_i) = (f(t), i) = f(t)_i$ is an order preserving homeomorphism on $Z$. Similarly, if $f : [0, 1] \to [0, 1]$ is an order reversing homeomorphism. Then the function $f'$ defined by $f'(t_i) = f(t)_{1-i}$ is an order reversing homeomorphism on $Z$.

Proof. Let $J \subset [0, 1]$ be an interval and let $f$ be an order preserving homeomorphism from $J$ into $f(J) \subset [0, 1]$ with respect to the usual topology of [0,1]. Let $J_{\{0,1\}} = \{t_i \in Z|t \in J\}$ and let $f' : J_{\{0,1\}} \to Z$ be defined by $f'(t_i) = (f(t))_i$.

$(i)$ $f'$ is one-to-one: Suppose $a_k \neq b_i$, then:

Case 1. $a \neq b$; then $f(a) \neq f(b)$ so $f'(a_k) = (f(a))_k \neq (f(b))_i = f'(b_i)$.

Case 2. $a = b$; then $k \neq i$ so $(f(a))_k \neq (f(a))_i$ and hence $f'(a_k) \neq f'(b_i)$.

$(ii)$ $f'$ is onto $f(J)_{\{0,1\}}$: Given $a_k \in J_{\{0,1\}}$ then $a \in J$ so there is a point $b \in f(J)$ such that $f(b) = a$, so $f'(b_k) = (f(b))_k = a_k$.

$(iii)$ $f'$ is order preserving: Assume $a_k < b_i$, $a, b \in R$, and $k, i \in \{0, 1\}$.

Case 1. $a \neq b$. Then $a < b$ implies that $f(a) < f(b)$ and then, $f'(a_k) = (f(a), k) <_Z (f(b), i) = f'(b_i)$.

Case 2. $a = b$. Then $k = 0$ and $i = 1$; which implies that $f'(a_k) = (f(a), 0) <_Z (f(a), 1) = (f(b), 1) = f'(b_i)$.

$(iv)$ $f'$ is continuous: Let $U$ be a basic open set with $f'(t_i) \in U$ and $U = [a_1, b_0]$. Let $V = [(f^{-1}(a))_1, (f^{-1}(b))_0]$. Then $V$ is open in $Z$, $t_i \in V$, and $f'(V) \subset U$ since $f$ is order preserving.

□

This establishes that $f'$ is an order preserving homeomorphism. Now let us consider $f' : J_{\{0,1\}} \to Z$ defined by $f'(t_i) = (f(t), 1 - i) = (f(t))_{1-i}$.

The proof of one-one and onto is similar to the order preserving case.

$(i)$ $f'$ is order reversing: Let $a_k < b_i$.

Case 1. $a \neq b$. Then $a < b$ implies that $f(a) > f(b)$ so, $f'(a_k) = (f(a), 1 - k) >_Z (f(b), 1 - i) = f'(b_i)$.

Case 2. $a = b$. Then $k = 0$ and $i = 1$. So we have $a_k = a_0$ and $b_i = a_1$. Thus, $f'(a_0) = (f(a), 1 - 0) > (f(a), 0) = f'(a_1)$.

$(ii)$ $f'$ is continuous: Let $U$ be a basic open set with $f'(t_i) \in U$ and $U = [a_1, b_0]$. Then $f(a) \leq f(t) \leq f(b)$ implies that $t \in [f^{-1}(b), f^{-1}(a)]$. Let $V =$

$[(f^{-1}(b))_1, (f^{-1}(a))_0]$. So $V$ is open in $Z$, $t_i \in V$, and $f'(V) \subset U$ since $f$ is order reversing.

**Part 2**. Definition of f's. Construction.

Define $H_1^1 = [0_1, (\frac{1}{2})_0]$ and $H_2^1 = [(\frac{1}{2})_1, 1_0]$.

Define $H_1^2 = [0_1, (\frac{1}{4})_0]$, $H_2^2 = [(\frac{1}{4})_1, (\frac{1}{2})_0]$, $H_3^2 = [(\frac{1}{2})_1, (\frac{3}{4})_0]$, and $H_4^2 = [(\frac{3}{4})_1, 1_0]$.

For each positive integer $n$ and $1 \leq i \leq n$ define $H_i^n = [(\frac{i-1}{2^n})_1, (\frac{i}{2^n})_0]$. Note that $H_i^n = H_{2i-1}^{n+1} \cup H_{2i}^{n+1}$.

Claim 2.2.1. For each integer $n$, $\{H_i^n\}_{i=1}^{2n}$ is a partition of $Z$ into disjoint clopen sets and if $U$ is an open set in $Z$ then there exists integers $n$ and $i$ so that $H_i^n \subset U$.

Construction: For each $n$ we wish to find a homeomorphism $f^n$ from $Z$ onto $Z$ by using order preserving maps to map elements of $\{H_i^n\}_{i=1}^{2n}$ onto each other.

Define $f_{i,j}^n : H_i^n \to H_j^n$, for $t_k \in Z$ with $t \in [0,1]$ and $k \in \{0,1\}$, by $f_{i,j}^n(t_k) = (\frac{j-1}{2^n} + t - \frac{i-1}{2^n})_k$. Note by Background Theorem 2.1.3 that $f_{i,j}^n$ is an order preserving homeomorphism.

We define $f^n : Z \to Z$ inductively.

Let $f^1(t) = f_{1,2}^1(t)$ for $t \in H_1^1$; let $f^1(t) = f_{2,1}^1(t)$ for $t \in H_2^1$.

For $n > 0$ let $f^{n+1}(t) = f^n(t)$ for $t \in H_i^n$ for $i > 1$ (i.e. $t \in H_i^{n+1}$ for $i > 2$). Let $f^{n+1}(t) = f_{1,2^{n+1}}^{n+1}(t)$ for $t \in H_1^{n+1}$; let $f^{n+1}(t) = f_{2,2^{n+1}-1}^{n+1}(t)$ for $t \in H_2^{n+1}$. What this does is interchange $H_1^{n+1}$ and $H_2^{n+1}$ before moving them to $H_{2^{n+1}}^{n+1}$ and $H_{2^{n+1}-1}^{n+1}$ but keeps the rest of $Z$ in the same order and preserves the previous assignments made at the $n^{\text{th}}$ level. We have $f^{n+1}(H_1^{n+1}) = H_{2^{n+1}}^{n+1}$, $f^{n+1}(H_2^{n+1}) = H_{2^{n+1}-1}^{n+1}$ and $f^{n+1}(r_k) = f^n(r_k)$ if $r_k \in Z - (H_1^{n+1} \cup H_2^{n+1})$.

Define $F : Z \to Z$ by $F(t) = lim_{n\to\infty} f^n(t)$.

Note:

1. For each $n$ and each point $x \in Z$, the orbit of $x$ under $f^n$ intersects every set in $\{H_i^n\}_{i=1}^{2^n}$.

2. For every point $x_k \in Z$ except $0_1$, $x_k \in H_2^i$ for some $i$. This is true since $x_k \in [(\frac{1}{2^n})_1, (\frac{2}{2^n})_0] = H_2^n$ for some $n$ whenever $x_k \neq 0_1$. Thus $Z = \cup_{n=1}^{\infty} H_2^n \cup \{0_1\}$.

3. Given $i$ is the least integer such that $x_k \in H_2^i$ then $F(x_k) = f^i(x_k)$. This is true since $f^i(x_k) = f^j(x_k)$ for all $j > i$ so $F(x_k) = lim_{n\to\infty} f^n(x_k) = f^i(x_k)$.

4. $f^j(H_1^j) \subset f^i(H_1^i)$ for $i < j$.

5. $f^i(H_2^i) \subset f^{i-1}(H_1^{i-1})$.

**Part 3**. $F$ is a homeomorphism.

Claim 1. $F(0_1) = 1_0$:

We have $0_1 \in H_1^n$ for every $n$.

$f^n(H_1^n) = H_{2^n}^n$, thus $f^n(0_1) = (\frac{2^n-1}{2^n})_1$. By the topology on $Z$,

$F(0_1) = lim_{n\to\infty}(\frac{2^n-1}{2^n})_1 = 1_0$. So $F(0_1) = 1_0$.

Claim 2. $F$ is well-defined:

Given a point $x_k \in Z$, $x_k \neq 0_1$, we have $x_k \in H_2^i$ for some $i$; so $F(x_k) = f^i(x_k)$ which uniquely defines $F(x_k)$. From above we have $0_1$ mapped only to $1_0$.

Claim 3. $F$ is one-to-one:

Case 1. Suppose $a_k$ and $b_h$ such that $a_k \neq b_h$ and $a_k, b_h \neq 0_1$.

Case 1.1: $a \neq b$. Then there exist $i, j, m$ with $j$ and $m$ not equal to 1 such that $a_k \in H^i_j$ and $b_k \in H^i_m$ with $j \neq m$. Then $F(a_k) = f^i(a_k) \neq f^i(b_k) = F(b_k)$ because $f^i$ is one-to-one.

Case 1.2: $a = b$. Then there exist $i, j$ such that $j \neq 1$ so that $a_k, b_h \in H^i_j$. Then $F(a_k) = f^i(a_k) \neq f^i(b_h) = F(b_h)$ because $f^i$ is one-to-one.

Case 2. Suppose $a_k$ and $b_h = 0_1$ such that $a_k \neq 0_1$.

Then there exists an $i$ such that $0_1 \in H^i_1$ and $a_k \in H^i_j$ with $j \neq 1$. Thus $f^i(a_k) \notin f^i(H^i_1)$; therefore $F(a_k) = f^i(a_k) \neq 1_0 = F(0_1)$.

Claim 4. $F$ is onto:

We know that $F(0_1) = 1_0$; so we need only show onto for $a_k \neq 1_0$. Since $\bigcap_{n=1}^{\infty} H^n_{2^n} = \{1_0\}$, then given $a_k \neq 1_0$ there exists $i$ such that $a_k \in H^i_j$ and $j \neq 2^i$ (i.e. not the last partition element.) Since $f^i$ is onto there exists $H^i_m$ and $b_k \in H^i_m$ with $m \neq 1$ such that $f^i(H^i_m) = H^i_j$ such that $a_k = f^i(b_k) = F(b_k)$ since $m \neq 1$.

Claim 5. $F$ is continuous:

Suppose that $F(x_k) \neq 1_0$ and $U$ is an open set such that $F(x_k) \in U$. There exists $H^i_j$ such that $F(x_k) \in H^i_j \subset U$ and $j \neq 2^i$. Let $V = (f^i)^{-1}(H^i_j)$; this set is open since $f^i$ is a homeomorphism and $V \neq H^i_1$ since $j \neq 2^i$. So,

$F((f^i)^{-1}(H^i_j)) = f^i((f^i)^{-1}(H^i_j))$, since $j \neq 2^i$,

$f^i((f^i)^{-1}(H^i_j)) = H^i_j \subset U$.

If $F(x_k) = 1_0$ and $U$ is the open set $[a_1, 1_0]$ then there exists an $i$ such that $H^i_{2^i} \subset [a_1, 1_0]$. Since $H^i_{2^i} \subset [a_1, 1_0]$ we have, $f^i(H^i_1) \subset [a_1, 1_0]$.

Claim: $F(H^i_1) \subset [a_1, 1_0]$.

Suppose for every point $w_m \neq 0_1$ such that $w_m \in H_1^i$ there exists $n$ such that $w_m \in H_2^{i+n}$ for some $n$. This implies by construction (see notes 3, 4, and 5) that $F(w_m) = f^{i+n}(w_m)$.

So $F(H_2^{i+n}) = f^{i+n}(H_2^{i+n}) \subset f^{i+n-1}(H_1^i) \subset f^i(H_1^i) \subset U$.

Hence $F(w_m) \in U$ and we know that $F(0_1) \in U$.

Thus $F(H_1^i) \subset U$, and $F$ is continuous.

$\square$

**Part 4**. Definition of a non-metric continuum $S$ such that $S$ supports a Whitney map.

Let $X = Z \times [0,1]$.

Let $G = \{\{(t,0),(F(t),1)\} \mid t \in Z\} \cup \{\{(t,r)\} \mid r \neq 0,1\}$.

Let $S = X/G$.

Observe that $G$ is an upper semi-continuous collection filling up X.

Lemma 2.4.1. Let $\{x_0, x_1, ..., x_n\}$ be points in $Z$ so that $F(x_i) = x_{i+1}$ for $i = 0, 1, ..n-1$; then there exists open sets $V_0, V_1, ..., V_n$ in $Z$ that are pairwise disjoint and so that $F(V_i) = V_{i+1}$ and $x_i \in V_i$ with $i = 0, 1, ..., n$.

Proof. Step 1. There exists pairwise disjoint sets $U_0, U_1, ..., U_n$, such that $x_i \in U_i$ and each $U_i = [a_1^{x_i}, b_0^{x_i}]$, with $i = 0, 1, ..., n$.

Step 2. Let $W_1 = [F(a_1^{x_0}), F(b_0^{x_0})] \cap [a_1^{x_1}, b_0^{x_1}]$; note $[F(a_1^{x_0}), F(b_0^{x_0})] = F(U_0)$.

So we define:

$W_1 = F(U_0) \cap U_1$;

$$W_2 = F(W_1) \cap U_2;$$

$$\vdots$$

$$W_i = F(W_{i-1}) \cap U_i;$$

$$\vdots$$

$$W_n = F(W_{n-1}) \cap U_n.$$

Step 3. Define:

$$V_n = W_n;$$

$$V_{n-1} = F^{-1}(V_n) = F^{-1}(W_n);$$

$$V_{n-2} = F^{-1}(V_{n-1}) = F^{-1}(F^{-1}(W_n));$$

$$\vdots$$

$$V_0 = F^{-1}(V_1) = \underbrace{F^{-1}(F^{-1}\cdots}_{n\text{-times.}}(W_n)).$$

We have:

- The $V_i$ 's are pairwise disjoint since the $U_i$ 's are pairwise disjoint, $W_i \subset U_i$, and $V_i \subset W_i$.
- $F(V_i) = V_{i+1}$ by construction.
- $x_i \in V_i$.

Proof: $x_0 \in U_0$ so $F(x_0) \in F(U_0)$;

$x_1 \in U_1$ so $x_1 = F(x_0) \in U_1$ and so $x_1 \in W_1 = F(U_0) \bigcap U_1$.

Likewise $x_2 \in W_2$ and so on: $x_i \in W_i$ for each $i$.

Then we have $x_n \in W_n = V_n$. So $x_{n-1} = F^{-1}(x_n) \in F^{-1}(V_n) = V_{n-1}$; $x_{n-1} \in V_{n-1}$; and likewise for all $i$ we have $x_i \in V_i$. Note that $V_i \subset U_i$; so the elements $\{V_i\}_{i=1}^{n}$ are disjoint. This establishes the lemma.

$\square$

**Part 5**. Indecomposablity of S.

Notation:

We have $X = Z \times [0,1]$ and $S = X/G$. Note that the point $(z,0)$ is identified with the point $(F(z),1)$,and that the points $z \in Z$ are written in the form $z = t_i$ for some $t \in [0,1]$ and $i \in \{0,1\}$.

Let $z \in Z$; for each positive integer $n$ we define an arc $A_n^z$. Let $A_0^z$ be the arc $\{z\} \times [0,1] \subset S$. $A_0^z$ is an arc beginning at $(z,1)$ and ending at $(z,0)$. Let $A_1^z$ be the arc $\{z\} \times [0,1] \cup \{F(z)\} \times [0,1] \subset S$. Let $A_n^z = \{z\} \times [0,1] \cup \cup_{i=1}^{n} \{F^n(z)\} \times [0,1]$. Thus for example $A_2^z$ is the arc in $S$ beginning at $(z,1)$ and ending at $(F^2(z),0)$.

Define $\mathcal{A}_z = \bigcup_{i=0}^{\infty} A_i^z$.

Theorem 2.5.1. $\mathcal{A}_z$ is dense and is the union of metric arcs for each $z \in Z$.

Proof. First, since each $A_i^z$ is just finitely many metric arcs glued together, each $A_i^z$ is metric.

Assume that $\mathcal{A}_x$ is not dense for some $x \in Z$. Let $y \in S$ be such that $y \notin \mathcal{A}_x$ and $y$ is not a limit point of $\mathcal{A}_x$. Thus there exists a basic open set $U$ in the form $U = [z_1, z_2] \times (r, s)$ such that $y \in U$ and no point of $\mathcal{A}_x$ is in $U$. Look at the projection $\pi_1(U)$ of $U$ onto $Z \times \{1\}$. Now since $y$ is not a limit point of $\mathcal{A}_x$ then $\pi_1(U) \bigcap \mathcal{A}_x = \emptyset$. We also know that there exist $i$ and $j$ such that $H_j^i \times \{1\} \subset U$ which implies that:

$(H_j^i \times \{1\}) \bigcap \mathcal{A}_x = \emptyset$.

This implies that $(F^{-1}(H_j^i) \times \{1\}) \bigcap \mathcal{A}_x = \emptyset$.

Thus $(F^{-2}(H_j^i)) \times \{1\} \bigcap \mathcal{A}_x = \emptyset$.

$\vdots$

$(F^{-2^i}(H_j^i)) \times \{1\}) \bigcap \mathcal{A}_x = \emptyset$.

But then $\mathcal{A}_x \bigcap (\bigcup_{j=1}^{2^i} H_j^i \times 1) = \emptyset$ which is a contradiction since $\mathcal{A}_x$ is nonempty and $\bigcup_{j=1}^{2^i} H_j^i = Z$. Therefore $\mathcal{A}_x$ is dense.

□

Similarly define $A_{-n}^z = \{z\} \times [0,1] \cup \cup_{i=1}^n \{F^{-n}(z)\} \times [0,1]$ and $\mathcal{A}_z^- = \bigcup_{i=0}^\infty A_{-i}^z$. Then by the same argument we have:

Theorem 2.5.2. $\mathcal{A}_z^-$ is dense and is the union of metric arcs for each $z \in Z$.

Theorem 2.5.3. Every proper subcontinuum of $S$ is a metric arc or a singleton point.

Proof. Let $M$ be a proper subcontinuum of $S$.

Let $(z,t) \in M$. [To use our rough terminology: we wish to find points $(a,r)$ and $(b,s)$ "above" and "below" this point that are not in $M$. This will show that there is a metric arc, which will be denoted as $L$, that is contained in $M$ and then we will show that $L = M$.]

Consider the arc $\{z\} \times [t,1]$. If this arc is not a subset of $M$ then there is a number $r > t$ so that $(z,r) \notin M$. Let $(a,r) = (z,r)$. If $\{z\} \times [t,1] \subset M$ then there is a first integer $n$ so that $A_{-n}^z \not\subseteq M$. Otherwise the dense subset $\mathcal{A}_z^-$ would be a subset of $M$ and this would contradict the fact that $M$ is a proper subcontinuum of $S$. Then there is an integer $n$ and points $(F^{-n}(z), u)$ and $(F^{-n}(z), r)$ so that $(\{z\} \times [t, 1]) \cup A^z_{-(n-1)} \cup (\{F^{-n}(z)\} \times [0, u]) \subset M$ and $(F^{-n}(z), r) = (a, r) \notin M$; and furthermore since $M \bigcap (\{F^{-n}(z)\} \times [0, 1])$ is closed $u$ has the property that for every $w$ between $u$ and $r$ there is a $w'$ so that $u < w' < w$ and $(a, w') \notin M$. We can think of $(a, u)$ as one endpoint of the arc $L$ that is contained in $M$.

Similarly, consider the arc $\{z\} \times [0, t]$. If this arc is not a subset of $M$ then there is a number $s < t$ so that $(z, s) \notin M$. Let $(b, s) = (t, s)$. If $\{z\} \times [0, t] \subset M$ then there is a first integer $j$ so that $A^z_j \not\subseteq M$. Thus, as in the above argument there is a first integer $m$ and points $(F^m(z), v)$ and $(F^m(z), s)$ so that $\{z\} \times [0, t] \cup A^z_{(m-1)} \cup \{F^m(z)\} \times [v, 1] \subset M$ and $(F^m(z), s) = (b, s) \notin M$; and furthermore for every $w$ between $s$ and $v$ there is a $w'$ so that $w < w' < v$ and $(b, w') \notin M$. Thus we can think of $(b, v)$ as being the other endpoint of the metric arc $L \subset M$.

Let $L$ be the arc lying in $\mathcal{A}_z \cup \mathcal{A}^-_z$ with end points $(a, u)$ and $(b, v)$ as defined above. Note that (roughly) $L = A^z_{(m-1)} \cup \{F^m(z)\} \times [v, 1] \cup \{z\} \times [0, 1] \cup A^z_{-(n-1)} \cup \{F^{-n}(z)\} \times [0, u]$. (Roughly in the sense that a slight modification is necessary in the case that $n = 0$ or $m = 0$.)

We now will show that $L = M$. Assume not so that there exists $(q, w)$ such that $(q, w) \in M$ but $(q, w) \notin L$.

Note that the projection of $L$ onto $Z$ is the set $\{F^i(z)\}^m_{i=-n}$.

Case 1. $q \notin \{x | (x, y) \in L \text{ for some } y \in [0, 1]\}$, i.e. $q \notin \{F^i(z)\}^m_{i=-n}$.

Case 2. $q \in \{x | (x, y) \in L \text{ for some } y \in [0, 1]\}$, i.e $q \in \{F^i(z)\}^m_{i=-n}$.

We will now make a tube-like open set in $S$ such that it will contain $L$ but not $(q, w)$ and whose boundary misses $M$ (thus getting a contradiction). We say an open set $U$ is tube-like in $S$ if the projection of $U$ onto $Z$ is a collection of disjoint open sets $\{U_1, U_2, ...U_j\}$ such that $F(U_i) = U_{i+1}$, for $i \in [1, j]$.

Case 1: $q \notin \{x | (x, y) \in L \text{ for some } y \in [0, 1]\}$.

We know that $(a, r)$ and (b,s) are not in $M$; thus there exists $V_a$ and $V_b$, open sets in $S$ containing $(a, r)$ and $(b, s)$ respectively, that do not intersect $M$.

By Lemma 2.4.1 (about the V's), there exists a clopen set $V \subset Z$ containing $z$ so that $V_L = \cup\{F^i(V)\}_{i=-n}^{m}$ does not contain $q$. Note that $V_L$ contains all the points from $\{F^i(z)\}_{i=-n}^{m}$. Furthermore, $V$ can be chosen so that $F^{-n}(V) \times \{r\} \subset V_a$ and $F^m(V) \times \{s\} \subset V_b$.

Thus $F^{-n}(V) \times [0, r) \cup (\cup_{i=-(n-1)}^{m-1}(F^i(V) \times [0, 1])) \cup F^m(V) \times (s, 1]$ is an open set $O$ in $S$ that contains $L$. Moreover $(q, w)$ is not in $O$ and, since $V$ is clopen, $Bd(O) \subset V_a \cup V_b$ and hence $Bd(O)$ does not intersect $M$. But $L \subset M$ and $L$ lies in $O$ and $(q, w)$ is a point in $M$ not in $O$ which contradicts the connectedness of $M$.

Case 2: $q \in \{x | (x, y) \in L \text{ for some } y \in [0, 1]\}$.

There are two possibilities: $q = F^m(z)$ or $q = F^{-n}(z)$. Suppose $q = F^{-n}(z)$; thus $(q, w) = (a, w)$. Since $(a, u)$ is an endpoint of $L$, there exists a number $r'$ so that $u < r' < w$ so that $(a, r') \notin M$.

Then repeat the construction as above but with $(a, r')$ replacing $(a, r)$ and the open set $V_a$ containing $(a, r')$ and no point of $M$. Then the open set $O$ constructed as above contains $L$ and does not contain $(q, w)$. And again $Bd(O)$ does not intersect

$M$ which contradicts the connectedness of $M$. Therefore $(a,w) \notin M$ for $w > u$. If $q = F^m(z)$ then the same argument works by selecting $s'$ so that $w < s' < v$ and $(b,s') \notin M$. Thus we have shown that every proper subcontinuum of $S$ is a metric arc or is a singleton point. Notice that since we showed $M = L$ then the projection of $M$ onto $Z$ is the set $\{F^i(z)\}_{i=-n}^m$. All proper subcontinua will have this feature.

Theorem 2.5.4. $S$ is an indecomposable continuum such that each composant is the union of a countable collection of metric arcs.

Proof. Claim 1. $S$ is a continuum:

We know that, since $G$ is an upper semicontinous collection filling up the compact space $X$, $S$ is compact.

For connectedness we know $\mathcal{A}_x$ is connected for any $x \in S$; thus $\overline{\mathcal{A}_x} = S$ is also connected.

Claim 2. $S$ is indecomposable:

If $S$ were decomposable then it would be the union of two arcs ( since each proper subcontinuum is an arc) but $S$ is not the union of two arcs.

Claim 3. Each composant is the union of a countable collection of metric arcs:

Claim: $\mathcal{A}_x = \bigcup_{i=-\infty}^{\infty} A_i^x$ is the composant of $(x,0)$, $x \in Z$. Recall $A_n^z = \{z\} \times [0,1] \cup (\cup_{i=1}^n \{F^n(z)\} \times [0,1])$ and that $\mathcal{A}_x$ is a countable collection of metric arcs.

Assume $\mathcal{A}_x = \bigcup_{i=-\infty}^{\infty} A_i^x$ is not the composant of $(x,0)$; so there exist a point $(r,w)$ and a proper subcontinuum $B$ of $S$ such that $(x,0),(r,w) \in B$ and $B \backslash \mathcal{A}_x \neq \emptyset$. But any proper subcontinuum is an arc and thus we can say $B$ starts at $(a,b)$ and ends at $(F^p(a),d)$, for some $p \in (-\infty,\infty)$. If $(x,0)$ is on this arc then $x = F^n(a)$

for some n which implies that $a = F^{-n}(x)$. Also $F^p(a) = F^m(x)$ for some $m$. Thus $B \subset \bigcup_{i=-n}^{m} A_i^x \subset \mathcal{A}_x$. This is a contradiction. Thus given any point that is in the composant of $x$, that point is in $\mathcal{A}_x$, and we know that given any point $p \in \mathcal{A}_x$ there exist a subcontinuum that contains $p$ and $(x, 0)$, namely the arc with $p$ and $(x, 0)$ as the starting and ending points respectively. Therefore $\mathcal{A}_x$ is the composant of $(x, 0)$ and actually is the composant of any point in $\mathcal{A}_x$.

**Part 6**. Definition of *len* and continuity.

From Part 5 we have: If $I \subset S$ is a proper subcontinuum of $S$ then there exists a finite number of points $z_1, z_2, ..., z_n$ so that:

$$I \subset \cup_{i=1}^{n} \{z_i\} \times [0, 1]/G.$$

Let $\rho$ denote the usual length of intervals in $[0, 1]$ and let $\pi_2$ denote the projection of $S$ onto the second coordinate (loosely defined by ignoring the decomposition part). Define the "length" of $I$, $len(I)$, as follows:

$$len(I) = \sum_{i=1}^{n} \rho(\pi_2(I \cap \{z_i\} \times [0, 1])).$$

Lemma 2.6.1. *len* is continuous.

Let $M$ be a proper subcontinuum of $S$. Let $U$ be an open set in $\mathbb{R}$ such that $len(M) \in U$. Now there exists an $\epsilon > 0$ such that $(len(M) - \epsilon, len(M) + \epsilon) \subset U$.

$M \subset \cup_{i=0}^{n} A_{x_i}$. [We assume that the sets $A_{x_i}$ are the "vertical" intervals comprising $Z \times [0, 1]$ so that for $x_i$ we have $A_{x_i} = \{x_i\} \times [0, 1]$.]

Thus there exists $x_0, x_1, ..., x_n$ so that $M$ has endpoints $(x_0, y_0)$ and $(x_n, y_n)$.

Using the Lemma 2.4.1, we can find $V_0, ..., V_n$ such that $x_i \in V_i$, and $V_i$ and $V_j$ are pairwise disjoint for $i \neq j$.

Let:

$W_0 = V_0 \times (y_0 - \frac{\epsilon}{4}, y_0 + \frac{\epsilon}{4})$, and

$W_n = V_n \times (y_n - \frac{\epsilon}{4}, y_n + \frac{\epsilon}{4})$.

Let $M'$ be the arc that begins at $(x_0, y_n - \frac{\epsilon}{4})$ and ends at $(x_n, y_n + \frac{\epsilon}{4})$. Cover $M'$ with "balls" of radius $\frac{\epsilon}{4}$, where "ball" around the point $(x_i, s)$ would be the open set $V_i \times (s - \frac{\epsilon}{4}, s + \frac{\epsilon}{4})$. Since $M'$ is compact, there exists finitely many of these open sets, say $G_1, ..., G_m$, that cover $M'$. (Note: each $G_i = V_j \times (y_j - \frac{\epsilon}{4}, y_j + \frac{\epsilon}{4})$ for some $j$ and some $y_j$.) Then $\mathcal{G} = \{G_1, G_2, ..., G_n, W_0, W_n\}$ will cover $M$. $R(\mathcal{G}) = \{K \in C(S) | K$ intersects each element of $\mathcal{G}$ and is a subset of $\cup \mathcal{G}\}$ is open in $C(S)$.

Thus, by definition, if $N \in R(\mathcal{G})$ and the fact that we know that $N$ is an arc [previous result], $N$ must start in $W_0$ and end in $W_n$. Thus $len(N) \in (len(M) - \frac{\epsilon}{2}, len(M) + \frac{\epsilon}{2})$. (Note: this is true since it is at most $\frac{\epsilon}{2}$ longer than $M$ or no shorter than $\frac{\epsilon}{2}$ of $M$.)

So $len(N) \in U$ and hence $len(R(\mathcal{G})) \subset U$. So $len$ is continuous.

**Part 7**. Definition of $\mu$, where $\mu$ is a Whitney map.

Define $\mu : C(S) \to \mathbb{R}$ by

$\mu(I) = \frac{\arctan(len(I))}{\frac{\pi}{2}}$

for $I \neq S$ and $\mu(S) = 1$.

Claim 2.7.1. $\mu$ is a Whitney map.

Part 1. $\mu$ is continuous:

Since *len* and *arctan* are continuous it suffices to show that $\mu$ is continuous at $S$. Let $\mu(S) = 1 \in U$ where $U$ is an open set in $\mathbb{R}$. There exists an $\epsilon > 0$ such that $(\mu(S) - \epsilon, \mu(S)) \subset U$. Furthermore there is a number $N$ such that if $len(K) > N + \epsilon$ then $\mu(K) > \mu(S) - \epsilon$, where $K$ is a proper subcontinuum of $S$. Now let $K$ be a proper subcontinuum so that $len(K) > N + \epsilon$. By a previous argument we know that $K$ is a metric arc beginning at $(x_0, y_0)$ and ending at $(x_n, y_n)$.

Using the same method as in proof that *len* is continuous there exists an open set $R(\mathcal{G})$ in $C(S)$ such that if $J \in R(\mathcal{G})$ then $len(J) > len(K) - \epsilon > N$; so then $len(J) > N + \epsilon$ and thus $\mu(J) > \mu(S) - \epsilon$.

We will now make a new open set $\mathcal{V}$ in $C(S)$. Let $\mathcal{V} = R(\mathcal{G} \cup \{S\})$. Note $\mathcal{V}$ is open since it is made from a collection of finitely many open sets from $S$, and $S \in \mathcal{V}$. Then, given any proper subcontinuum $M \in \mathcal{V}$, $M$ must intersect each open set from $\mathcal{G}$ which means that $len(M) > len(K) - \epsilon > N$. Thus $\mu(M) > \mu(S) - \epsilon$ which implies that $\mu(M) \in U$. Since $\mu(S) \in U$ then $\mu(\mathcal{V}) \subset U$, and thus $\mu$ is continuous.

Part 2: Given $A \subsetneq B$ then $\mu(A) < \mu(B)$.

If $B \neq S$ and $A \subsetneq B$ then $len(A) < len(B)$ which implies that $\mu(A) < \mu(B)$.

If $B = S$ and $A \subsetneq B$ then $\mu(A) < 1 = \mu(B)$.

**Corollary 2.1.** *The same construction can be done using any irreducible continuum that supports a Whitney map.*

# Chapter 3

## A hereditarily non-metric continuum that supports a Whitney map

Note that in this paper, by a hereditarily non-metric continuum we mean a continuum such that every nondegenerate subcontinuum is non-metric.

**Theorem 3.1.** *If for each positive integer $i$ the space $X_i$ supports a Whitney map $\mu_i$ and $f_i : X_{i+1} \to X_i$, then $X = \varprojlim(X_i, f_i)$ supports a Whitney map.*

*Proof.* We will assume that $\mu_i(X_i) = 1$ for all $i$. Let $\pi_i$ be the projection map from $X$ onto $X_i$. Define a map $\Pi_i$ from the hyperspace of $X$ onto the hyperspace of $X_i$ by $\Pi_i(H) = \pi_i(H)$, where $H$ is any subcontinuum of $X$. We first need to show that $\Pi_i$ is continuous.

Let $U$ be an open set in $C(X_i)$ such that $\Pi_i(H) \in U$, where $H$ is a subcontinuum of $X$. Now $U = R(\{U_j\}_{j=1}^n)$ where $U_j$ is an open set in $X_i$. Define an open set $V_j$ in $X$ as $V_j = \overleftarrow{U_j}$. Thus $V_j = \{x \in X |\ x_i \in U_j\}$. Define $\widetilde{V} \subset C(X)$ as $\widetilde{V} = R(\{V_j\}_{j=1}^n)$. Let $K$ be a point in $C(X)$ such that $K \in \widetilde{V}$; then $K \bigcap V_j \neq \emptyset$ for each $j = 1$ to $n$. Thus $\pi_i(K) \bigcap U_j \neq \emptyset$ for each $j = 1$ to $n$ and by the definition of the $V_j's$, $\pi_i(K) \subset \bigcup_{j=1}^n U_j$. Therefore $\pi_i(K) \in U$ in $C(X_i)$; thus $\Pi_i(K) \in U$, which implies that $\Pi_i(\widetilde{V}) \subset U$. Thus $\Pi_i$ is continuous.

Define $\mu : X \to X_i$ by $\mu(H) = \sum_{i=1}^{\infty} \frac{\mu_i(H_i)}{2^i}$ where $H$ is a subcontinuum of $X$ and $H_i = \pi_i(H)$. First it is clear that if $K \subsetneq H$ then $\mu(K) < \mu(H)$ since in order for

$K \subsetneq H$ there exist an $i$ such that $K_i \subsetneq H_i$; so $\mu_i(K_i) < \mu_i(H_i)$ and $\mu_j(K_j) \leq \mu_j(H_j)$ for $i \neq j$.

Second we need to show that $\mu$ is continuous. Let $U$ be an open set in $\mathbb{R}$ such that $\mu(H) \in U$. There exist an $n$ and an $\epsilon$ such that

$[\sum_{i=1}^{n} \frac{\mu_i(H_i)}{2^i} \pm \epsilon] \subset U$ and

$\sum_{i=n}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}$. Note we define $[A \pm \epsilon] = [A - \epsilon, A + \epsilon]$

Since each $\mu_i$ is continuous there exists an open set $V_i \subset C(X_i)$ such that $\mu_i(V_i) \subset [\mu_i(H_i) \pm \frac{\epsilon}{2}]$. Also since $\Pi_i : C(X) \to C(X_i)$ is continuous there exists an open set $V_i^{'}$, containing $H_i$, such that $\Pi(V_i^{'}) \subset V_i$. Define an open set $\widetilde{V} \subset C(X)$ by $\widetilde{V} = \bigcap_{i=1}^{n} V_i^{'}$. If $K \in \widetilde{V}$ then $\Pi_i(K) \in V_i$ for each $i = 1$ to $n$. Thus $\mu_i(K) \in [\mu_i(H_i) \pm \frac{\epsilon}{2}]$ for each $i = 1$ to $n$. Therefore

$\sum_{i=1}^{n} \frac{\mu_i(K_i)}{2^i} < \mu(K) = \sum_{i=1}^{\infty} \frac{\mu_i(K_i)}{2^i} = \sum_{i=1}^{n} \frac{\mu_i(K_i)}{2^i} + \sum_{i=n+1}^{\infty} \frac{\mu_i(K_i)}{2^i} <$
$\sum_{i=1}^{n} \frac{\mu_i(K_i)}{2^i} + \frac{\epsilon}{2} \in \sum_{i=1}^{n} \frac{\mu_i(H_i) \pm \frac{\epsilon}{2}}{2^i} + \frac{\epsilon}{2} \subset [\sum_{i=1}^{n} \frac{\mu_i(H_i)}{2^i} \pm \epsilon] \subset U$.

Thus $\mu(\widetilde{V}) \subset U$ so $\mu$ is continuous and thus a Whitney map. $\square$

- Definition of breaking and gluing a copy of $S$ at a point on the arc $[e, f]$.

Let $I_1 = [a, b]$ and $I_2 = [c, d]$ be two metric arcs. Let $S$ be the space of our example from Chapter 2, and let $p, q$ be two points from $S$ that are in different composants. Define the decomposition space

$$D = (I_1 \cup I_2 \cup S) \diagup \{\{p, b\}, \{q, c\}\}.$$

Now we define what is meant by breaking an arc at a point and gluing in $S$. (For shorthand we will refer to it as breaking and gluing $S$ at a point). Let $[e, f]$ be an arc and $t \in [e, f]$, $t \neq e, f$. There exist natural homomorphisms $h_1 : [a, b) \to [e, t)$ and $h_2 : (c, d] \to (t, f]$. Define a new space

$$S^1 = D \cup ([e, f] \setminus t) \diagup \{\{\{x, h_1(x)\}|x \in [a, b)\} \bigcup \{\{y, h_2(y)\}|y \in (c, d]\}\}$$

where $x \in [a, b)$ and $y \in (c, d]$. Note we use this notation loosely since this is not a decomposition space.

Define the topology $T$ by open sets on $\{x, h_1(x)\}$. If $x \in D$ , $x \in I_1$ or $I_2$, and $x \notin \{p, q\}$ (thus $x$ is on the arc but not the endpoints where $S$ was glued), then the basic open sets are the open sets from the normal topology of an arc.

If $x \in D, x \in S, x \notin \{p, q\}$ (thus $x$ is inside the glued copy of $S$), then use the relative topology from our example $S$.

If $x = p$, or $x = q$ then an open set containing $x$ would be the union of an open set in $S$ containing $p$(or $q$) and a half-open interval on either $I_1$ (or $I_2$). Observe an important fact that since $p$ and $q$ are in different composants of $S$, $D$ is irreducible from $e$ to $f$.

$\square$

Now in the above definition we defined what was meant by breaking and gluing in a copy of $S$ at a point on an arc. Since our space $S$ is $Z \times [0, 1]$ with identifications we can think of $S$ as having uncountably many disjoint arcs (except for the endpoints), namely $z \times [0,1]$ for each $z \in Z$. Given a fixed $z$ we can use the above procedure to break and glue at a point in an arc from $S$. (Note: there are two different $S's$. First, we start with $S$ and then we take another $S$ to glue into the first one). Thus we can glue a copy of $S$ into $S$ at a point on the arc $z \times [0,1]$ for some fixed $z \in Z$.

For example, let $(1/2_0, 1/3)$ be a point in $S$. $(1/2_0, 1/3)$ is on the arc $1/2_0 \times [0,1]$, so we can use the lemma to break and glue a copy of $S$ at the point $(1/2_0, 1/3)$. Our resulting space would be our example $S$ with one copy of $S$ glued at the point $(1/2_0, 1/3)$.

In the next theorem we will not just break and glue in one copy of $S$. We will want to "break and glue across $S$", meaning that for some fixed $t \in (0,1)$, we will break and glue at all points from the collection $\{(z,t)|z \in Z\}$. Using the previous example of $(1/2_0, 1/3)$, the term "break and glue across $S$ at $(1/2_0, 1/3)$" would mean that you would use the lemma and break and glue at each point from the collection $\{(z, 1/3)|z \in Z\}$. The topology for this space would be locally the product topology on $Z \times [0,1]$ for points not in a new glued copy of $S$. For points inside a glued copy we would use the product topology on $Z \times Z \times [0,1]$. Note this idea can be extended if, as we will do in a later step, we glued inside a glued copy; then locally for the points inside the new glued copy the topology would be the product topology of $Z \times Z \times Z \times [0,1]$.

Another fact we will use considers the relationship between two spaces made by using the lemma. Let $S_{1/3}$ be the space made by breaking and gluing across $S$ at $(1/2_0, 1/3)$, and let $S_{1/4}$ be the space made by breaking and gluing across $S$ at the point $(1/2_0, 1/4)$. Notice that each of these spaces look like our original example $S$ but with a copy of $S$ glued into each arc $z \times [0,1]$ for all $z \in Z$. Thus it is easy to see that $S_{1/3} \cong S_{1/4}$. This is true because of our construction of $S$ and the fact that it is again really just uncountably many arcs with identifications. Thus for example the homeomorphism would map the arc $z \times [0,1] \subset S_{1/3}$ onto the arc $z \times [0,1] \subset S_{1/4}$ by mapping $z \times [0, 1/3)$ onto $z \times [0, 1/4)$, the copy of $S$ onto the copy of $S$, and lastly $z \times (1/3, 1]$ onto $z \times (1/4, 1]$. We will use this fact in the future so that if we have constructed a Whitney map on $S_{1/3}$ then we would similarly be able to produce one on $S_{1/4}$.

**Theorem 3.2.** *There exist a hereditarily non-metric continuum that supports a Whitney map.*

*Proof.* It has already been shown that the inverse limit of spaces $\{X_\alpha\}_{\alpha=1}^{\infty}$ will support a Whitney map if each $X_\alpha$ supports a Whitney map.

Using our lemmas and our space $S$ we will construct a system of spaces $\{S_\alpha\}_{\alpha=1}^{\infty}$ and maps $\{f_\alpha^\beta\}_{\alpha<\beta}$ so that the inverse limit space $X$ supports a Whitney map and is hereditarily non-metric.

To construct each $S_\alpha$ we will take specific points from $S_{\alpha-1}$ and using our lemma we will break and glue a copy of $S$ at each of those points. We will rely heavily on the understanding of $S$. Recall that $S = Z \times [0,1] \big/ \{(z,0),(F(z),1) | z \in Z\}$. Locally $S$ is the topological product of an open subset of $Z$ and a metric arc. Any point of $S$ can be represented as $(z,t)$ where $z \in Z$ and $t \in [0,1]$. ( Note: $z \in Z$ is actually the point $x_0$ or $x_1$ from $Z$). If the point $p = (z, 1)$ or $(z, 0)$ then the representation of $p$ is not unique since $(z, 0) = (F(z), 1)$ and $(z, 1) = (F^{-1}(z), 0)$. For this example we will insert multiple copies of $S$ in $S_n$ to obtain $S_{n+1}$; thus representing a point as $(z, t)$ will not be sufficient to identify that point. Denote the point $(z, t) \in S$ as $P \begin{pmatrix} z \\ t \end{pmatrix}$. The $P$ indicates a point in $S$ and then $\begin{pmatrix} z \\ t \end{pmatrix}$ indicates where in $S$ the point is, namely $(z, t)$.

Let $S_0$ be a copy of $S$. Points in $S_0$ will be denoted as $P_0 \begin{pmatrix} z \\ t \end{pmatrix}$. Note the subscript indicates which space the point is in.

Let $R = 1/2_0 \times (0, 1) \subset S_0$.

Let $\left\{ P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}, P_0 \begin{pmatrix} 1/2_0 \\ c_2^0 \end{pmatrix}, P_0 \begin{pmatrix} 1/2_0 \\ c_3^0 \end{pmatrix}, .... \right\}$ be a countable dense subset of $R \subset S_0$.

We will break and glue a copy of $S$ at the point $P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}$. Thus the arc $1/2_0 \times [0, 1]$ in $S$ would now have a copy of $S$ glued in at the point $c_1^0 \in [0, 1]$. We want to not only break and glue on the arc $1/2_0 \times [0, 1]$ but for every arc $z \times [0, 1]$ from $S_0$. Thus at every point $P_0 \begin{pmatrix} z \\ c_1^0 \end{pmatrix}, z \in Z$ break and glue a new copy of $S$. $S_1$ will be this new space made by breaking and gluing at all the points mentioned.

We need notation for the points in $S_1$. In $S_1$ there are two types of points; points that are in new glued copies of $S$ and points of the form $P_0 \begin{pmatrix} z \\ t \end{pmatrix}$ for $t \neq c_1^0$. ( Note:

the point $P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix}$ for any $z \in Z$ does not exist in $S_1$ since these are the points that were "replaced" by a copy of $S$).

Case 1: Points corresponding to $P_0 \begin{pmatrix} z \\ \\ t \end{pmatrix}$ for $t \neq c_1^0$.

These points can still be thought of as a point $(z, t)$ for some $z \in Z, t \in [0, 1], t \neq c_1^0$; thus let $P_1 \begin{pmatrix} z \\ \\ t \end{pmatrix}$ denote the corresponding point in $S_1$ that is the point that is on the arc $z \times [0, 1]$ at the $t^{th}$ coordinate.

Case 2: Points inside a glued copy of $S$.

There are uncountably many new glued copies of $S$ in $S_1$. Thus to name a point inside a glued copy you must indicate which glued copy it is inside and then where on the glued copy the point is located. To indicate this we will use the same type of notation but add another pair of coordinates. A point inside the glued copy will be denoted by

$$P_1 \begin{pmatrix} z_1 & & z_2 \\ \\ c_1^0 & , & t \end{pmatrix}$$

for some $z_1, z_2 \in Z$. In this notation the first column will tell you which glued copy the point is in ( namely the one glued at $P_0 \begin{pmatrix} z_1 \\ \\ c_1^0 \end{pmatrix}$ ) and the second column tells the position of the point on the new glued copy.

For example:

$$P_1 \begin{pmatrix} 1/2_0 & & 1/3_1 \\ & & \\ c_1^0 & , & 1/4 \end{pmatrix}$$

would indicate the point $P \begin{pmatrix} 1/3_1 \\ \\ 1/4 \end{pmatrix}$ from the glued copy of $S$ at the point $P_0 \begin{pmatrix} 1/2_0 \\ \\ c_1^0 \end{pmatrix}$.

The topology of $S_1$ is defined as follows. If $p \in S_1$ and $p$ is of the form $P_1 \begin{pmatrix} z \\ \\ t \end{pmatrix}$ for $t \neq c_1^0$, then a local basic element at that point is a set $\left\{ P_1 \begin{pmatrix} z \\ \\ t \end{pmatrix} \mid P_0 \begin{pmatrix} z \\ \\ t \end{pmatrix} \in O_0 \right\}$ for some open set $O_0 \in S_0$ not containing $P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix}$.

For a point of the form $P_1 \begin{pmatrix} z_1 & & z_2 \\ & & \\ c_1^0 & , & t \end{pmatrix}$ we will use the open set $J$ from the definition that described an open set in an arc that did not contain 0 or 1 with a copy of $S$ glued in at the point $c_1^0$ . Thus an open set in $S_1$ for a point of the form $P_1 \begin{pmatrix} z_1 & & z_2 \\ & & \\ c_1^0 & , & t \end{pmatrix}$ would be $U \times J$ where $U$ is an open set in $Z$. This is topologically an open set containing $P_1 \begin{pmatrix} z_1 & & z_2 \\ & & \\ c_1^0 & , & t \end{pmatrix}$ with the product topology.

Note that, like $S$, $S_1$ is indecomposable. Also recall: given any proper subcontinuum $M \subset S$, $M$ was contained in finitely many arcs joined together ( see explanation of composants of $S$). $S$ has a fiber-like structure with each fiber being an arc $[0, 1]$.

In $S_1$ if we think of a fiber now as an arc with a glued copy of $S$; the composants and the description of proper subcontinua will be the same as it was for $S$. Thus any proper subcontinuum $M$ will be contained in finitely many fibers and hence can only intersect finitely many of the new glued copies of $S$.

Now that we have described $S_1$ and the points from $S_1$ we need to describe the bonding map $f_0^1$ from $S_1$ to $S_0$. This bonding map will take all the glued copies from $S_1$ and collapse them down to the point at which they were glued. On all other points the bonding map will be the identity. In notation this would be represented as

$$f_0^1\left(P_1\begin{pmatrix} z \\ \\ t \end{pmatrix}\right) = P_0\begin{pmatrix} z \\ \\ t \end{pmatrix}, t \neq c_1^0$$

$$f_0^1\left(P_1\begin{pmatrix} z_1 & & z_2 \\ \\ c_1^0 & , & t \end{pmatrix}\right) = P_0\begin{pmatrix} z_1 \\ \\ c_0^1 \end{pmatrix}.$$

In order to define this bonding map we needed notation to describe each point in $S_1$; in order to describe the Whitney map we will need notation to distinguish between the uncountably many copies of $S$ that we glued into $S_0$ when we made $S_1$. The reason this is important goes back again to our example $S$ and what we have already mentioned about proper subcontinua of $S_1$. A composant in $S_1$ is similar to the composant from $S_0 = S$ except with countably many copies of $S$ glued in at specific points. Observe that if $J$ is a composant of $S_0$ then $(f_0^1)^{-1}(J)$ is a composant of $S_1$. Thus any proper subcontinuum of $S_1$ would be contained in what could be thought of as finitely many fibers joined together where each of these is an arc that has a copy of $S$ glued into it. Because of this fact about the composants, a proper subcontinuum $M$ will only intersect finitely many of these new glued copies of $S$ in $S_1$. ( Note: This fact will hold for any $S_\alpha$. If $M$ is a proper subcontinuum then $M$ will only intersect finitely many of the glued copies from $S_1$. This will be used heavily in future levels.)

The easiest way to distinguish between copies of $S$ is to denote the copy by using the point at which the copy was glued. For instance we glued in a copy of $S$ at the point $P_0\begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}$, denote this copy as

$$S_1^{'}\left(P_0\begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}\right).$$

Notice that the $'$ will indicate that we are denoting a copy of $S$, the subscript indicates what space the copy is in ( in our case $S_1$), and $P_0\begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}$ tells at which point from the previous space the copy of $S$ was glued.

Thus $\left\{S_1^{'}\left(P_0\begin{pmatrix} z \\ c_1^0 \end{pmatrix}\right)\right\}$ for all $z \in Z$ would be the collection of all glued copies of $S$ in $S_1$.

Next we will define the Whitney map. Let $\mu$ be the whitney map defined for our example $S$. Since $S_0$ is a copy of $S$ we can define a Whitney map $\mu_0$ for $S_0$ the same way we defined $\mu$.

Let $M$ be a proper subcontinuum of $S_1$.

For each $z \in Z$, let $A_z = M \bigcap S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$. Now since $M$ is a proper subcontinuum, $\{A_z | A_z \neq \emptyset\}$ is finite or empty. (This is true for the previous reasons stated about the composants and proper subcontinua of $S_1$). Let $\{A_i\}_{i=1}^{n_1} = \{A_z | A_z \neq \emptyset\}$. Now $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) \cong S_0$ Therefore each $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ will have a Whitney map defined the same way as $\mu_0$. Call this Whitney map $\mu_0^1$. Note: the superscript indicates that the Whitney map is on $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ and the subscript indicates that $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) \cong S_0$. Now since each $A_i$ is either a proper subcontinuum of or equal to $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ for some $z \in Z$, then each $A_i$ would have a Whitney value for $\mu_0^1$. Note that at most two of $A_i's$ are such that $A_i = A_z \neq S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$; thus $\mu_0^1(A_i) \neq 1$ for at most two $A_i's$.

Define $\mu_1 : C(S_1) \longrightarrow \mathbb{R}$ by

$$\mu_1(M) = \frac{Arctan\left(len(f_0^1(M)) + \sum_{i=1}^{n} \mu_0(A_i)\right)}{\frac{\pi}{2}}, and$$

$$\mu_1(S_1) = 1.$$

Since $\mu_1$ is the composition of continuous functions, it is continuous. To see that the Whitney property is satisfied, notice that given $M, N \in C(S_1)$ if $M \subsetneq N$ then $len(f_0^1(M)) + \sum_{i=1}^{n_M} \mu(A_i^{1_M}) \lneq len(f_0^1(N)) + \sum_{i=1}^{n_N} \mu(A_i^{1_N})$. Thus $\mu_1(M) \lneq \mu_1(N)$.

Now to make $S_2$ we will again want to break and glue at specific points from $S_1$. The first step in this process is again to find a countable dense set. This time (and in infinitely many future steps) we will choose a countable dense set from one of the new glued copies from the previous space. In $S_0$ our countable dense set came from the arc $\{1/2_0\} \times (0,1)$, we will again look at the arc $\{1/2_0\} \times (0,1)$ but this time it be will inside the copy of $S$ glued at the point $P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}$. Recall we denoted this copy as $S_1' \left( P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix} \right)$, and points on this copy would have the form

$$P_1 \begin{pmatrix} 1/2_0 & z \\ c_1^0 & , \ t \end{pmatrix}.$$

Let $\left\{ P_1 \begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 & , \ c_1^1 \end{pmatrix}, P_1 \begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 & , \ c_2^1 \end{pmatrix}, P_1 \begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 & , \ c_3^1 \end{pmatrix}, .... \right\}$ be a countable dense set from the arc $\{1/2_0\} \times (0,1)$ on $S_1' \left( P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix} \right)$. Make the collection so that none of the $c_i^1$'s are 0 or 1.

We want to glue inside $S_1'\left(P_0\begin{pmatrix}1/2_0\\ \\ c_1^0\end{pmatrix}\right)$ ( recall this is the copy of $S$ glued at $\left(P_0\begin{pmatrix}z\\ \\ c_1^0\end{pmatrix}\right)$ ) and we will repeat the procedure used in making $S_1$. We will break and glue at the point $P_1\begin{pmatrix}1/2_0 & 1/2_0\\ & \\ c_1^0\ , & c_1^1\end{pmatrix}$. In a similar way that we made $S_1$, we also want to break and glue all the way across $S_1'\left(P_0\begin{pmatrix}1/2_0\\ \\ c_1^0\end{pmatrix}\right)$. Thus for every $z \in Z$ we have the point $P_1\begin{pmatrix}1/2_0 & z\\ & \\ c_1^0\ , & c_1^1\end{pmatrix}$. Break and glue at those points. At this stage in our construction the arc $\{1/2_0\} \times [0,1]$ from $S_0$ would now look like an arc with a copy of $S$ glued in and then within that copy there are uncountably many copies of $S$ glued on each arc $z \times [0,1]$ . (Hopefully it is becoming clear why you need notation to indicate all the points and all the copies of $S$). We have glued in uncountably many copies of $S$ into the copy of $S$ glued at $P_0\begin{pmatrix}1/2_0\\ \\ c_1^0\end{pmatrix}$. We want to do the same gluing on all other copies of $S$ from $S_1$, using the same $c_1^1 \in (0,1)$. Recall a glued copy was denoted as $S_1'\left(P_0\begin{pmatrix}z\\ \\ c_1^0\end{pmatrix}\right)$. And a point inside was named by $P_1\begin{pmatrix}z_1 & z_2\\ & \\ c_1^0\ , & t\end{pmatrix}$. If we fix $z_1$ then $\left\{P_1\begin{pmatrix}z_1 & z_2\\ & \\ c_1^0\ , & t\end{pmatrix} |z_2 \in Z, t \in [0,1]\right\}$ would be all the points from $S_1'\left(P_0\begin{pmatrix} z_1 \\ c_1^0 \end{pmatrix}\right)$. We want to break and glue at all points where $t = c_1^1$. Thus for a fixed $z_1$ we will break and glue at each point from the collection $\left\{P_1\begin{pmatrix} z_1 & z_2 \\ c_1^0 & , c_1^1 \end{pmatrix} | z_2 \in Z\right\}$. Repeat this procedure for $z_1 = z$, all $z \in Z$ .

Putting this all together and using the notation, what we have done is break and glue a copy of $S$ at each point from the collection

$$\left\{P_1\begin{pmatrix} z_1 & z_2 \\ c_1^0 & , c_1^1 \end{pmatrix} | z_1, z_2 \in Z\right\}.$$

Denote this new space as $S_2$. Using the same notation as before, the new glued copies can be represented by the points at which they were glued. Thus $\left\{S_2'\left(P_1\begin{pmatrix} z_1 & z_2 \\ c_1^0 & , c_1^1 \end{pmatrix} | z_1, z_2 \in Z\right)\right\}$ is the set of all new glued copies of $S$. Building on our previous notation we can denote the three different types of points from $S_2$.

Type 1 : $\left\{P_2\begin{pmatrix} z \\ t \end{pmatrix} | z \in Z, t \in [0,1]/c_1^0\right\}$. These are the points not in any glued copy of $S$.

Type 2 : $\left\{P_2\begin{pmatrix} z_1 & z_2 \\ c_1^0 & , t \end{pmatrix} | z_1, z_2 \in Z, t \in [0,1]/c_1^1\right\}$. These points are in the copies of $S$ that were glued to make $S_1$.

Type 3 : $\left\{P_2\begin{pmatrix} z_1 & z_2 & z_3 \\ c_1^0 & , c_1^1 & , t \end{pmatrix} | z_1, z_2, z_3 \in Z, t \in [0,1]\right\}$. These are the points that lay inside one of the new copies of $S$ from the collection

$$\left\{ S_2' \left( P_1 \begin{pmatrix} z_1 & z_2 \\ c_1^0 \quad , & c_1^1 \end{pmatrix} | z_1, z_2 \in Z \right) \right\}.$$

For example $P_2 \begin{pmatrix} 1/2_0 & 1/3_0 & 1/4_0 \\ c_1^0 \quad , & c_1^1 \quad , & 1/5 \end{pmatrix}$ would be the point $P \begin{pmatrix} 1/4_0 \\ 1/5 \end{pmatrix}$ on the copy of $S$ glued at $P \begin{pmatrix} 1/3_0 \\ c_1^1 \end{pmatrix}$ where $P \begin{pmatrix} 1/3_0 \\ c_1^1 \end{pmatrix}$ is a point inside the copy of $S$ glued at $P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}$. In notation $P_2 \begin{pmatrix} 1/2_0 & 1/3_0 & 1/4_0 \\ c_1^0 \quad , & c_1^1 \quad , & 1/5 \end{pmatrix}$ would be the point $P \begin{pmatrix} 1/4_0 \\ 1/5 \end{pmatrix}$ on $S_2' \left( P_1 \begin{pmatrix} 1/2_0 & 1/3_0 \\ c_1^0 \quad , & c_1^1 \end{pmatrix} \right)$.

The bonding map $f_1^2 : S_2 \to S_1$ will be defined in the same matter as before. New glued copies of $S$ will collapse down to the point at which they were glued and $f_1^2$ will be the identify on all other points. Thus $f_1^2 \left( P_2 \begin{pmatrix} z \\ t \end{pmatrix} \right) = P_1 \begin{pmatrix} z \\ t \end{pmatrix}$ (these are the points not in any glued copy of $S$, they map to themselves), and

$f_1^2 \left( P_2 \begin{pmatrix} z_1 & z_2 \\ c_1^0 \quad , & t \end{pmatrix} \right) = P_1 \begin{pmatrix} z_1 & z_2 \\ c_1^0 \quad , & t \end{pmatrix}$ (these are the points in the glued copy at the first level, they map to themselves), and

$f_1^2 \left( P_2 \begin{pmatrix} z_1 & z_2 & z_3 \\ c_1^0 \quad , & c_1^1 \quad , & t \end{pmatrix} \right) = P_1 \begin{pmatrix} z_1 & z_2 \\ c_1^0 \quad , & c_1^1 \end{pmatrix}$ (these are the points inside the new copy of $S$, they collapse down to point at which $S$ was glued).

As before if $J$ is a composant of $S_2$ then $f_1^2(J)$ is a composant of $S_1$, and conversely if $J$ is a composant of $S_1$ then $(f_1^2)^{-1}(J)$ is a composant of $S_2$. We need to define a Whitney map on $S_2$. In the first level, when we defined our Whitney map, given $M$ a proper subcontinuum we intersected $M$ with the new glued copies of $S$ and then used the sum of these values to define $\mu_1$. But that was possible because $M$ intersected only finitely many of the new glued copies. Notice that given $M$ in $S_2$, $M$ can intersect uncountably many copies of $S$ from the collection $\left\{ S_2^{'}\left( P_1 \begin{pmatrix} z_1 & & z_2 \\ & & \\ c_1^0 & , & c_1^1 \end{pmatrix} |z_1, z_2 \in Z \right) \right\}$ ( this is the collection of all new copies of $S$ glued into $S_1$ in order to make a new space $S_2$). Therefore we can not use these Whitney values since we can not sum this uncountable amount. By the definition of $f_1^2$ if $M$ is a proper subcontinuum of $S_2$ then $f_1^2(M)$ is a proper subcontinuum of $S_1$. Thus a proper subcontinuum of $S_2$ can only intersect finitely many copies of $S$ from the collection $\left\{ S_1^{'}\left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) |z \in Z \right\}$. Thus to define $\mu_2$ we will use the intersection of $M$ with the collection $\left\{ S_1^{'}\left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) \right\}$ for all $z \in Z$. To define the Whitney map on $S_2$ let $M$ be a proper subcontinuum. Since at this level the copies of $S$ were glued inside of $S_1^{'}\left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$, let $A_z = M \bigcap S_1^{'}\left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$. Note that $\{z|A_z \neq \emptyset\}$ is finite, so let $\{A_i\}_{i=1}^{n_2} = \{A_z|A_z \neq \emptyset\}$. Also note that each $A_z$ is a subcontinuum.

Now $S_1^{'}\left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) \cong S_1$ (because both are the original $S$ with a copy of $S$ glued onto each arc; thus as previously explained they are homeomorphic.) Therefore each $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ will have a Whitney map defined the same way as $\mu_1$. Call this Whitney map $\mu_1^1$. Note: the superscript indicates that the Whitney map is on $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ and the subscript indicates that $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right) \cong S_1$. Now since each $A_i$ is either a proper subcontinuum of or equal to $S_1' \left( P_0 \begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix} \right)$ for some $z \in Z$, then each $A_i$ would have a Whitney value for $\mu_1^1$. Note that $\mu_1^1(A_z) = 1$ for all but at most two $i = 1, 2, ...., n_2$. (Notice we used the homeomorphism only to show quickly that $A_z$ does support a Whitney map. If $A_z$ was not homomorphic to a previously constructed space then we could prove directly that $A_z$ supports a Whitney map by using the same techniques used to prove $S_1$ supports a Whitney map. In future stages we may not have spaces homeomorphic to previous stages.)

Let $\mu_2 : C(S_2) \longrightarrow \mathbb{R}$ be defined by

$$\mu_2(M) = \frac{Arctan(\mu_1(f_1^2(M)) + \sum_{i=1}^{n_2} \mu_1^1(A_i))}{\frac{\pi}{2}}, and$$

$$\mu_2(S_2) = 1.$$

Roughly speaking $\mu_2$ divides $M$ into different pieces where each piece has a Whitney map associated with it. Note it is not the same Whitney map for every piece. For the subcontinuum $f_1^2(M)$ we use the Whitney map $\mu_1$, for each $A_i$ we use

$\mu_1^1$. Then we take the sum of the respective Whitney values of each piece and apply the arctan function and divide by $\frac{\pi}{2}$ to bound $\mu_2$ by 1. Thus we can define $\mu_2(S_2) = 1$ and satisfy the Whitney property. This idea will be used in all future levels to define the Whitney map; what will change will be the collection that a subcontinuum $M$ intersects.

To begin the process of defining $S_3$ we again will find a countable dense set from inside one of the glued copies of $S$ used in making $S_2$. Namely $S_2'\left(P_1\begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \end{pmatrix}\right)$. Recall points on $S_2'\left(P_1\begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \end{pmatrix}\right)$ have the form $P_2\begin{pmatrix} 1/2_0 & 1/2_0 & z \\ c_1^0 \;, & c_1^1 \;, & t \end{pmatrix}$ for some $z \in Z, t \in [0,1]$. Let $z = 1/2_0$ and thus let $\left\{P_2\begin{pmatrix} 1/2_0 & 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \;, & c_1^2 \end{pmatrix}, P_2\begin{pmatrix} 1/2_0 & 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \;, & c_2^2 \end{pmatrix},\right.$ $\left. P_2\begin{pmatrix} 1/2_0 & 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \;, & c_3^2 \end{pmatrix}, \ldots\right\}$ be a countable dense set from the arc $1/2_0 \times (0,1)$ on $S_2'\left(P_1\begin{pmatrix} 1/2_0 & 1/2_0 \\ c_1^0 \;, & c_1^1 \end{pmatrix}\right)$.

We will again break and glue but not using the first element from our countable dense set as we have done in previous levels. We need our inverse limit to be hereditarily non-metric; thus all the points from all the countable dense sets must eventually end up with a copy of $S$ glued at that point. If we continue gluing at each level using the first element from the countable dense set obtained at that level it

will not result in hereditarily nonmetric. We will set up a diagonal system using the dense sets that are found at each level and a function $h$ from the positive integers into a diagonal array that will determine which point will be used at which level. For ease in making our diagonal array, think of the dense sets as $\{c_1^0, c_2^0, c_3^0 ...\}$ instead of
$$\left\{ P_0 \begin{pmatrix} 1/2_0 \\ c_1^0 \end{pmatrix}, P_0 \begin{pmatrix} 1/2_0 \\ c_2^0 \end{pmatrix}, P_0 \begin{pmatrix} 1/2_0 \\ c_3^0 \end{pmatrix}, .... \right\}, \text{ and}$$

$\{c_1^1, c_2^1, c_3^1, ...\}$ instead of
$$\left\{ P_1 \begin{pmatrix} 1/2_0 & & 1/2_0 \\ c_1^0 & , & c_1^1 \end{pmatrix}, P_1 \begin{pmatrix} 1/2_0 & & 1/2_0 \\ c_1^0 & , & c_2^1 \end{pmatrix}, P_1 \begin{pmatrix} 1/2_0 & & 1/2_0 \\ c_1^0 & , & c_3^1 \end{pmatrix}, .... \right\}, \text{etc.}$$
Arrange these sets in a diagonal array

$$\begin{pmatrix} c_1^0 & , & c_2^0 & , & c_3^0 & , & c_4^0 & , & c_5^0 & , & .... \\ c_1^1 & , & c_2^1 & , & c_3^1 & , & c_4^1 & , & . & , & .... \\ c_1^2 & , & c_2^2 & , & c_3^2 & , & . & , & . & , & .... \\ c_1^3 & , & c_2^3 & , & . & , & . & , & . & , & .... \\ c_1^4 & , & . & , & . & , & . & , & . & , & .... \\ c_1^5 & , & . & , & . & , & . & , & . & , & .... \\ . & , & . & , & . & , & . & , & . & , & .... \end{pmatrix}$$

We can construct a function $h$ from $\mathbb{N}$ into the diagonal array to indicate which point to choose at which level. We have already completed the first two levels and thus we know that $h(1) = c_1^0$ and $h(2) = c_1^1$. Let $h(3) = c_2^0$, $h(4) = c_1^2$, $h(5) = c_2^1$, $h(6) = c_3^0$, etc.

We know that in order to make $S_3$ we will use the point in $S_2$ that corresponds to $c_2^0$. This point is $P_2\begin{pmatrix} 1/2_0 \\ \\ c_2^0 \end{pmatrix}$. Just as before we will break and glue at this point and all points from the collection $\left\{ P_2\begin{pmatrix} z \\ \\ c_2^0 \end{pmatrix} |z \in Z \right\}$. These points are not in any previous glued copy. This new space is $S_3$.

Notice that what we have done is glue a new copy of $S$ not inside a previously glued copy. Recall in $S_1$ a fiber $z \times [0,1]$ had the form of an arc with a copy of $S$ glued in at the point $(z, c_1^0)$. In $S_2$ a fiber had the form of the arc from $S_1$ but now had uncountably many copies of $S$ glued into the previous glued copy. In $S_3$ a fiber would look like the arc from $S_2$ except with a new copy of $S$ glued at the point $(z, c_2^0)$. In making $S_3$ we notice that the points from the first countable set are special in the sense that when $h(n) = c_j^0$ for some $n, j$ then we are not gluing inside any previous glued copy. This is the only time that will happen; if $h(n) \neq c_j^0$ then we will always be gluing inside a previous copy . But even in this case the points from the first countable set are unique in that we will always be gluing inside an $S$ that was glued at the points from this first dense set. In other words if $h(n) \neq c_j^0$ then the new copies of $S$ will be glued inside $S_l'\left( P_k\begin{pmatrix} z \\ \\ c_m^0 \end{pmatrix} \right)$ for some $l, k, z, m,$. This is important because it enables us to define a Whitney map on any $S_n$. Before we define $\mu_3$ we will define the bounding map $f_1^3 : S_3 \longrightarrow S_2$. This will be similar to previous levels. Define

$$f_1^3\left(P_3\begin{pmatrix} z \\ \\ t \end{pmatrix}\right) = P_2\begin{pmatrix} z \\ \\ t \end{pmatrix}, t \neq c_1^0, t \neq c_2^0,$$

$$f_1^3\left(P_3\begin{pmatrix} z_1 & z_2 \\ \\ c_1^0 \quad , & t \end{pmatrix}\right) = P_2\begin{pmatrix} z_1 & z_2 \\ \\ c_1^0 \quad , & t \end{pmatrix}, t \neq c_1^1,$$

$$f_1^3\left(P_3\begin{pmatrix} z_1 & z_2 & z_3 \\ \\ c_1^0 \quad , & c_1^1 \quad , & t \end{pmatrix}\right) = P_2\begin{pmatrix} z_1 & z_2 & z_3 \\ \\ c_1^0 \quad , & c_1^1 \quad , & t \end{pmatrix}, \text{ and}$$

$$f_1^3\left(P_3\begin{pmatrix} z_1 & z_2 \\ \\ c_2^0 \quad , & t \end{pmatrix}\right) = P_2\begin{pmatrix} z_1 \\ \\ c_2^0 \end{pmatrix}.$$

On the first three types of points $f$ is the identity and then on the last type $f$ collapses the glued copy of $S$ onto the point at which it was glued.

Let $M$ be a proper subcontinuum. As before we will think of $M$ in pieces. The first will be $f_2^3(M)$. But notice at this level we did not glue inside $S_1^{'}\left(P_0\begin{pmatrix} z \\ \\ c_1^0 \end{pmatrix}\right)$ as we did in the previous level; thus the way we define $A_z$ must change. For each $z \in Z$, let $A_z = M \bigcap S_3^{'}\left(P_2\begin{pmatrix} z \\ \\ c_2^0 \end{pmatrix}\right)$. Again each $A_z$ is a subcontinuum and $\{z | A_z \neq \emptyset\}$ is finite. Let $\{A_i\}_{i=1}^{n_3} = \{A_z | A_z \neq \emptyset\}$.

$S_3^{'}\left(P_2\begin{pmatrix} z \\ \\ c_2^0 \end{pmatrix} | z \in Z\right) \cong S_0$, thus there exist a Whitney map on the space, call it $\mu_0^3$. As stated when defining $\mu_2$, if $A_z$ is not homeomorphic to a previous space then we can show directly that $A_z$ will support a Whitney map by using the same techniques.

Define $\mu_3 : C(S_3) \longrightarrow \mathbb{R}$ by

$$\mu_3(M) = \frac{Arctan(\mu_2(f_2^3(M)) + \sum_{i=1}^{n_3} \mu_0^3(A_i))}{\frac{\pi}{2}}, and$$

$$\mu_3(S_3) = 1.$$

To begin on the fourth level we again choose a countable dense set from the arc $1/2_0 \times (0,1)$ in $S_3'\left(P_2\left(\begin{array}{c} 1/2_0 \\ c_2^0 \end{array}\right)\right)$. Recall this is a copy of $S$ glued in the previous level. Let

$\left\{P_3\left(\begin{array}{cc} 1/2_0 & 1/2_0 \\ c_2^0 \ , & c_1^3 \end{array}\right), P_3\left(\begin{array}{cc} 1/2_0 & 1/2_0 \\ c_2^0 \ , & c_2^3 \end{array}\right), P_3\left(\begin{array}{cc} 1/2_0 & 1/2_0 \\ c_2^0 \ , & c_3^3 \end{array}\right), \ldots\right\}$

be a countable dense set from the arc $1/2_0 \times (0,1)$ in $S_3'\left(P_2\left(\begin{array}{c} 1/2_0 \\ c_2^0 \end{array}\right)\right)$.

Since $h(4) = c_1^2$ we will break and glue at the corresponding point which is $P_1\left(\begin{array}{ccc} 1/2_0 & 1/2_0 & 1/2_0 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \end{array}\right)$. Remember this point is inside a copy of $S$ glued inside another copy of $S$. We want to break and glue "across" $S_3$, so break and glue at each point from the collection $\left\{P_3\left(\begin{array}{ccc} z_1 & z_2 & z_3 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \end{array}\right) | z_1, z_2, z_3 \in Z\right\}$. This new space will be $S_4$.

$\left\{S_4'\left(P_3\left(\begin{array}{ccc} z_1 & z_2 & z_3 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \end{array}\right) | z_1, z_2, z_3 \in Z\right)\right\}$ is the collection of all copies of $S$ glued into the space $S_3$ in order to make a new space $S_4$.

Define $f_3^4 : S_4 \longrightarrow S_3$ as the identity on all points except those of the form

$\left(P_4\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \ , & t \end{pmatrix}\right)$.

For these points define $f_3^4$ by

$f_3^4\left(P_4\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \ , & t \end{pmatrix}\right) = \left(P_3\begin{pmatrix} z_1 & z_2 & z_3 \\ c_1^0 \ , & c_1^1 \ , & c_1^2 \end{pmatrix}\right)$.

We want to define the Whitney map at this level. Let $M$ be a proper subcontinuum. In $S_4$ the new copies were again glued inside of $S_1^{'}\left(P_0\begin{pmatrix} z \\ c_1^0 \end{pmatrix}\right)$.

Therefore for each $z \in Z$, let $A_z = M \bigcap S_1^{'}\left(P_0\begin{pmatrix} z \\ c_1^0 \end{pmatrix}\right)$, and $\{A_i\}_{i=1}^{n_4} = \{A_z | A_z \neq \emptyset\}$. At this level $S_1^{'}\left(P_0\begin{pmatrix} z \\ c_1^0 \end{pmatrix}\right) \cong S_2$. There exist a Whitney map $\mu_2^1$ on $S_1^{'}\left(P_0\begin{pmatrix} z \\ c_1^0 \end{pmatrix}\right)$ that is defined in the same manner as $\mu_2$. Again if it was not homoeomorphic to a copy we could have proved directly that $A_z$ will support a Whitney map.

Then define $\mu_4$ by

$$\mu_4(M) = \frac{Arctan(\mu_3(f_3^4(M)) + \sum_{i=1}^{n_4} \mu_2^1(A_i))}{\frac{\pi}{2}}$$

and

$$\mu_4(S_4) = 1.$$

In each of the previous four cases we used the fact that $A_z$ was homeomorphic to a previous $S_i$ in order to show that there was a Whitney map on $A_z$. In some future levels it is the case that $A_z$ will not be homeomorphic to any previously made space. As previously mentioned this will not be a problem though because we do not need $A_z$ to be homeomorphic to a previous copy we just need to know that $A_z$ has a Whitney map. If $A_z$ is not homeomorphic to a previous copy then it is necessary to determine directly that it has Whitney map. This can be done by using the same techniques that were used to show $S_1, ..., S_4$ have Whitney maps.

For example when $n = 8$, $A_z$ is not homeomorphic to any previous copy. $A_z$ though is homeomorphic to a space call it $S_3 \oplus 1$, that is $S_3$ with copies of $S$ glued in at each point from the collection $P_1 \begin{pmatrix} 1/2_0 & & z \\ & & \\ c_1^0 & , & c_2^1 \end{pmatrix}, z \in Z$. We will want to show that a Whitney map can be constructed on this space. First we need to show that a simpler space has a Whitney map. Take $S_1$ and break and glue at each point from the collection $P_0 \begin{pmatrix} z \\ \\ c_2^0 \end{pmatrix}, z \in Z$. Thus each fiber would now have two copies of $S$ glued into it. Call this new space $S_1 \oplus 1$. (Note we have clearly glued uncountably many copies of $S$ not just one, but the $\oplus 1$ indicates we have we have done one more step and glued at just one more collection of points.) Now there exists a bonding map $g : S_1 \oplus 1 \to S_1$ defined as the identity except on the new copies of $S$ and those will collapse down to the point at which they were glued. If $M$ is a proper subcontinuum of $S_1 \oplus 1$ then $A_z$ will be homeomorphic to $S$. Thus $A_z$ will have a Whitney map, namely $\mu_0$. There are only finitely many $A_z$ say $A_1$ to $A_n$. Thus define the Whitney map using out technique on $S_1 \oplus 1$ as $\mu(M)_{S_1 \oplus 1} = \mu_{S_1}(g(M)) + \sum_{i=1}^{n}(\mu_0(A_i))$. This can be shown to be a Whitney map by the previous methods. Now the next step is to prove that $S_3 \oplus 1$ has a Whitney map. There exists a bonding map $g : S_3 \oplus 1 \to S_3$. $g$ will be the identity on all points except the new glued copies which will collapse down to the point at which they were glued. If $M$ is a proper subcontinuum of $S_3 \oplus 1$ then $A_z$ will be $M$ intersected with $\left\{ S_1^{'} \left( P_0 \left( \begin{array}{c} z \\ \\ c_1^0 \end{array} \right) \right) \right\}$ for every $z \in Z$. Only finitely many of these will be non-empty. Denote those as $A_1$ to $A_m$. Now $A_z$ is homomorphic to $S_1 \oplus 1$ which was just shown to have a Whitney map . Thus $A_z$ will have a Whitney map. Let $\mu'_{S_1 \oplus 1}$ denote the Whitney map on $A_z$. Define a Whitney map $\mu_{S_3 \oplus 1}$ as $\mu_{S_3 \oplus 1}(M) = \mu_3(g(M)) + \sum_{i=1}^{m}(\mu'_{S_1 \oplus 1}(A_i))$.

Now the same procedure can be used to show that no matter what the structure of $A_z$, it will have a Whitney map. This fact is what is needed to construct a Whitney map on $S_n$, any $n$. For example, when $n = 9$, then a typical $A_z$ will be homeomorphic to $(S_3 \oplus 1) \oplus 1$. Since we know that $S_3 \oplus 1$ has a Whitney map then it can be proven that $(S_3 \oplus 1) \oplus 1$ has a Whitney map, regardless of where the copies of $S$ are glued. When $n = 10$, $A_z$ is homeomorphic to $S$, thus we know it has a Whitney map.

Define $S_n$ inductively assuming that we have defined $S_{n-1}$. At the $nth$ level first define a countable dense subset from the $1/2_0 \times (0,1)$ arc in the copy of $S$ glued in the previous level. As before we want to make a new space by breaking and gluing in copies of $S$. There are 2 cases that can occur at the $nth$ level.

Case 1. $h(n) = c_m^0$ for some $m$. Thus we will beak and glue at

$\left\{ P_{n-1}\begin{pmatrix} z \\ \\ c_m^0 \end{pmatrix} | z \in Z \right\}$. Note these points are not in any previously glued copy of $S$.

The map $f_{n-1}^n : S_n \longrightarrow S_{n-1}$ will be defined as in previous levels. It will be the identify on all points except those points from the new glued copies and those will collapse down to the point at which the copy was glued.

In this case to define the Whitney map let $M$ be a proper subcontinuum of $S_n$. Since we have not glued inside any previous copy, for each $z \in Z$ let $A_z = M \bigcap S_n^{'}\left( P_{n-1}\begin{pmatrix} z \\ \\ c_m^0 \end{pmatrix}\right)$; $A_z$ is a subcontinuum and $\{z|A_z \neq \emptyset\}$ is finite, so let $\{A_i^1\}_{i=1}^j = \{A_z|A_z \neq \emptyset\}$.

Note that $S_n^{'}\left( P_{n-1}\begin{pmatrix} z \\ \\ c_m^0 \end{pmatrix}\right) \cong S$ so the corresponding Whitney map on $S_n^{'}\left( P_{n-1}\begin{pmatrix} z \\ \\ c_m^0 \end{pmatrix}\right)$ will be labeled $\mu_0^n$.

Let $\mu_n : C(S_n) \longrightarrow \mathbb{R}$ be defined by

$$\mu_n(M) = \frac{Arctan(\mu_{n-1}(f_{n-1}^n(M)) + \sum_{i=1}^j \mu_0^n(A_i))}{\frac{\pi}{2}}$$

and

$$\mu_n(S_n) = 1.$$

Case 2: $h(n) = c_m^j$ some $j \neq 0$. Break and glue at the points

$$\left\{ (P_{n-1} \begin{pmatrix} z_1 & & & & z_{r-1} & & z_r \\ c_k^0 & , & . & . & . & c_a^b & , & c_m^j \end{pmatrix}) | (z_1, z_2, ... z_r \in Z) \right\}.$$

Note that these points are in a previously glued copy of $S$ that was made by breaking and gluing at a point from the first countable dense set, namely $P_{h^{-1}(c_k^0)} \begin{pmatrix} z \\ c_k^0 \end{pmatrix}$ for some $k$.

The bonding map will be as in Case 1.

To define the Whitney map we previously noted that at this level the new copies of $S$ are glued inside a previous copy. Call this copy $S'_{h^{-1}(c_k^0)} \left( P_{h^{-1}(c_k^0)} \begin{pmatrix} z \\ c_k^0 \end{pmatrix} \right)$ for some $k$. Let $A_z = M \bigcap S'_{h^{-1}(c_k^0)} \left( P_{h^{-1}(c_k^0)} \begin{pmatrix} z \\ c_k^0 \end{pmatrix} \right)$, and $\{A_i\}_{i=1}^j = \{A_z | A_z \neq \emptyset\}$. Now we have two cases for $A_z$, either $A_z$ is homomorphic to some previous space (example $n = 4$) or it is not ( example $n = 8$). If $S'_{h^{-1}(c_k^0)} \left( P_{h^{-1}(c_k^0)} \begin{pmatrix} z \\ c_k^0 \end{pmatrix} \right) \cong S_l$ for some $l$, then the Whitney map on $S'_{h^{-1}(c_k^0)} \left( P_{h^{-1}(c_k^0)} \begin{pmatrix} z \\ c_k^0 \end{pmatrix} \right)$ will be defined in the same manner as $\mu_l$ and so will be named $\mu_l^{h^{-1}(c_k^0)}$. Let $\mu_n : C(S_n) \longrightarrow \mathbb{R}$ be defined by

$$\mu_n(M) = \frac{Arctan(\mu_{n-1}(f_{n-1}^n(M)) + \sum_{i=1}^j \mu_l^{h^{-1}(c_k^0)}(A_i))}{\frac{\pi}{2}}$$

and

$$\mu_n(S_n) = 1.$$

If $A_z$ is not homeomorphic to any previous $S_i$, $A_z$ can be shown to have a Whitney map using the same techniques that proved $S_i$ has a Whitney map. Denote that Whitney map by $\mu^{h^{-1}(c_k^0)}$. Let $\mu_n : C(S_n) \longrightarrow \mathbb{R}$ be defined by

$$\mu_n(M) = \frac{Arctan(\mu_{n-1}(f_{n-1}^n(M)) + \sum_{i=1}^{j} \mu^{h^{-1}(c_k^0)}(A_i))}{\frac{\pi}{2}}$$

and

$$\mu_n(S_n) = 1.$$

Let $X = \varprojlim\{S_n, f\}$ To show the inverse limit space $X$ is hereditarily non-metric, let $M$ be a nondegenerate proper subcontinuum of $X$. There exist $a, b \in M$ such that $a \neq b$. By the nature of our inverse limit space and the fact that $M$ is a continuum, we can find an $n$ and an $r$ and points $a_r, b_r \in \pi_{S_r}(M)$, such that $a_r \neq b_r$ and $a_r, b_r \in S_n^{'}\left(P_{n-1}\begin{pmatrix} z_1 & & & & & z_r \\ c_k^0 & , & . & . & . & , & c_k^j \end{pmatrix}\right)$. Furthermore $a_r, b_r$ are such that

$$a_r = P_r\begin{pmatrix} z_1 & & & & z_r & & z_{a_r} \\ c_m^0 & , & . & . & . & c_k^j & , & t_{a_r} \end{pmatrix} \text{ and}$$

$$b_r = P_r\begin{pmatrix} z_1 & & & & z_r & & z_{b_r} \\ c_m^0 & , & . & . & . & c_k^j & , & t_{b_r} \end{pmatrix}, \text{ where } z_{a_r} = z_{b_r}, \text{ and } t_{a_r} \lneq t_{b_r}.$$

By the way we chose our countable dense sets at each level there exists a $c_w^v$ such that $t_{a_r} < c_w^v < t_{b_r}$. Since $M$ is connected there exists an $x \in M$ such that

$$x = P_r \begin{pmatrix} z_1 & & & & z_r & z_{a_r} \\ c_m^0 & , & . & . & . & c_k^j & , & c_w^v \end{pmatrix}.$$

Now at some level, call it $p$, $h(p) = c_w^v$; a copy of $S$ was glued at $x$, namely $S_p' \left( P_r \begin{pmatrix} z_1 & & & & z_r & z_{a_r} \\ c_m^0 & , & . & . & . & c_k^j & , & c_w^v \end{pmatrix} \right)$. Thus in $S_p$, $\pi_p(M)$ must contain a point from this copy of $S$. But it also contains $a_p$, and $b_p$ which, because of the irreducibility of the continua at each level, lies on either side of this copy of $S$; thus the whole copy of $S$ must be in $\pi_p(M)$ . Therefore $\pi_p(M)$ is non-metric and so $M$ must also be non-metric. $\square$

# Chapter 4

## a hereditarily indecomposable non-metric continuum that supports a Whitney map

**Theorem 4.1.** *There exist a hereditarily indecomposable non-metric continuum that supports a Whitney map.*

*Proof.* It has already been shown that the inverse limit of spaces $\{X_\alpha\}_{i=1}^{\infty}$ will support a Whitney map if each $X_\alpha$ supports a Whitney map.

Using the space $S$ from the example we will construct a system of spaces $\{S_\alpha\}_{i=1}^{\infty}$ and maps $\{f_\alpha^\beta\}_{\alpha<\beta}$ so that the inverse limit space supports a Whitney map and is hereditarily indecomposable.

Let $S$ be as in the example and $\mu$ the Whitney map on $S$. Let $S_0 = S$ and $\mu_0 = \mu$.

Recall that in our example $A_0^z = z \times [0,1]$, $A_1^z = z \times [0,1] \bigcup F(z) \times [0,1]$, $A_2^z = z \times [0,1] \bigcup F(z) \times [0,1] \bigcup F^2(z) \times [0,1]$, etc.

Let $Ps$ stand for the pseudo-arc.

First recall that given any chainable continuum, then the pseudo-arc can be mapped onto that continuum. In order to make $S_1$ for each $z \in Z$, replace each $A_0^z$ in $S_0$ with a pseudo-arc in such a way that two "endpoints" $a$ and $b$ of the pseudo-arc map to $(z,0)$ and $(z,1)$ respectively . Another way of thinking of this is that $S_1$ is $Z$ cross $Ps$ with identifications.

To define the bonding map $f_0^1$, first define a function $g : Ps \longrightarrow [0, 1]$ such that $g$ is continuous and $g(a) = 0, g(b) = 1$. A point in $S_1$ can be thought of as a point on the pseudo-arc that is associated with a specific $z \in Z$. Thus the point $(z, x)$ would represent the point $x$ on the pseudo-arc associated with $z$. For ease of notation denote the pseudo-arc associated with $z$ as $Ps(z)$.

Define $f_0^1(z, x) = (z, g(x))$. To show this is continuous let $U \times V$ be an open set in $S_0$ such that $f_0^1(z, x) \in U \times V$. $U$ is an open set in $Z$. Since $g$ is continuous and $V$ is an open set in $Ps$ there exist an open set $B$ such that $x \in B$ and $g(B) \subset V$. Now $U \times B$ is an open set in $S_1$ and by the definition of $U \times B$, $f_0^1(U \times B) \subset U \times V$. Thus $f_0^1$ is continuous.

To define the Whitney map, let $M$ be a subcontinuum in $S_1$. First note that $M$ is contained in the union of finitely many pseudo-arcs joined end to end, say $\{Ps(z_i)\}_{i=1}^n$. Define $M \bigcap Ps(z_i) = M_i$. Let $\mu_p$ be the Whitney map defined on the pseudo-arc.

Define $\mu_1 : C(S_1) \longrightarrow \mathbb{R}$ by

$$\mu_1(M) = \frac{Arctan\left(\sum_{i=1}^n \mu_p(M_i)\right)}{\frac{\pi}{2}}$$

and

$\mu_1(S_1) = 1$.

This is continuous and satisfies the Whitney property since if $N \subsetneq M$ then there exist a $z_i$ such that $N_i = N \bigcap Ps(z) \subsetneq M \bigcap Ps(z) = M_i$ which would mean that $\mu_p(N_i) < \mu_p(M_i)$. So then $\mu_1(N) < \mu_1(M)$.

For $S_2$, for every $z \in H_2^1$ (recall: $H_2^1 = [(\frac{1}{2})_1, 1_0]$ is the second half of $Z$), replace $A_1^z \subset S_0$ with $Ps$. This can be represented as $H_2^1 \times Ps$ with identifications. We think of these pseudo-arcs as twice as long as the pseudo-arcs from $S_1$.

Before we define the bonding map we will define a function from a single pseudo-arc onto two pseudo-arcs glued end to end. Let $Ps$ be a pseudo-arc with endpoints $e$ and $f$. Let $P' = Ps^1 \bigcup Ps^2$, where $Ps^1$ has endpoints $a$ and $b$ and $Ps^2$ has endpoints $b$ and $c$. $P'$ is two pseudo-arcs joined end to end. There exists a continuous function $g_2 : Ps \longrightarrow P'$ such that $g_2(e) = a$, $g_2(f) = c$, the points between $e$ and $g_2^{-1}(b)$, which we will call the bottom half of $Ps$, will all map to $Ps^1$, and the points between $g_2^{-1}(b)$ and $f$, which we will call the top half of $Ps$, will all map to $Ps^2$. Thus this $g_2$ can be thought of as having two parts. The first part would tell which of the 2 pseudo arcs a point will map to and then the second part would tell exactly where on that pseudo arc the point maps. Define $h : P_S \rightarrow \{1, 2\}$ by $h(x) = 1$ if $g_2$ maps $x$ somewhere onto the $Ps^1$, $h(x) = 2$ if $g_2$ maps $x$ somewhere onto $Ps^2$. Note: if $g_2(x) = b$ then $h(x)$ could be thought of as 1 or 2 since the point $b$ is in both $Ps^1$ and $Ps^2$.

Define $f_1^2(z, x) = (z, g_2(x))$ if $h(x) = 1$ and $f_1^2(z, x) = (F(z), g_2(x))$ if $h(x) = 2$, where $F$ is the function used for identifications in $S$. Roughly speaking, instead of thinking of $S_1$ as $Z \times Ps$, think of $S_1$ as $\{z \times Ps\}_{z \in H_2^1} \bigcup \{F(z) \times Ps\}_{z \in H_2^1}$ with identifications. In $S_1$, $Ps(z)$ could be thought of as $Ps^1$ and $Ps(F(z))$ could be thought of as $Ps^2$. Recall because of the identifications they are joined at a point. Then $f_1^2$ would take $Ps(z)$ (recall this is the pseudo-arc in $S_2$ associated with the point $z$) onto $Ps(z) \bigcup Ps(F(z))$ for each $z \in H_2^1$.

To show continuity, let $f_1^2(z,x) \in U \times V$, where $U \times V$ is an open set in $S_1$.

Case 1: $h(x) = 1$ and $g(x) \neq b$.

That means that $f_1^2(z,x) = (z, g_2(x))$; thus the point $(z,x)$ is on the bottom half of the $Ps(z)$ in $S_2$. Now there exists a $U' \times V' \subset S_1$ such that $U' \times V' \subset U \times V$, $S_1^2(z,x) \in U' \times V'$, $U' \subset H_2^1$, and $V'$ is contained in $Ps(z)$. Since $g_2$ is continuous and $V'$ is an open set in $Ps^1$ there exists an open set $B$ such that $g(B) \subset V'$. $U' \times B$ is an open set in $S_2$, and $(z,x) \in U' \times B$.

Claim : $f_1^2(U' \times B) \subset U \times V$.

Recall that, since $g_2$ has two parts, $g(B) \subset V'$ means that all points in $B$ must map to the same pseudo arc as $x$ does. So either they all map to $Ps^1$ or they all map to $Ps^2$; in Case 1 since $h(x) = 1$, they will all map to $Ps^1$ which is actually $Ps(z)$. So given any $(z', x') \in U' \times B$, since $x' \in B$, then $h(x') = 1$ which will imply that $f_1^2(z', x') = (z', g_2(x'))$. We know that $z' \in U' \subset U$ and we know that $g_2(x') \in g_2(B) \subset V' \subset V$; thus $(z', g_2(x') \in U \times V$. Therefore $f_1^2$ is continuous.

Case 2. $h(x) = 2$ and $g_2(x) \neq b$.

A similar argument using a $V'$ such that $V' \subset Ps(F(z))$ can be used to show that $f_1^2$ is continuous in this case.

Case 3. $g_2(x) = b$.

If $g_2(x) = b$ then that point could be thought of as belonging to Case 1 or Case 2; either way it will be continuous at that point since the two pseudo-arcs are glued together at that point and $g_2$ is continuous at each preimage of that point.

Let $M$ be a subcontinuum of $S_2$. We will define $\mu_2$ in a similar manner as $\mu_1$. First $M$ is contained in finitely many pseudo-arcs joined end to end, say $\{Ps(z_i)\}_{i=1}^{n}$. Define $M \bigcap Ps(z_i) = M_i$.

Define $\mu_2 : C(S_2) \longrightarrow \mathbb{R}$ by

$$\mu_2(M) = \frac{Arctan\left(\sum_{i=1}^{n} \mu_p(M_i)\right)}{\frac{\pi}{2}}$$

and

$\mu_2(S_2) = 1.$

This is continuous and by the previous argument used to show $\mu_1$ satisfied the Whitney property, $\mu_2$ can be shown to satisfy the Whitney property.

For $S_3$ replace $A_3^z$ with pseudo-arc for each $z \in H_4^2$ (the last quarter of $Z$). Thus similarly $S_3 = H_4^2 \times Ps$ with identifications where these pseudo-arcs can be thought of as twice as long as those from $S_2$ and four times as long as those from $S_1$.

Now the bonding map will be the same as the bonding map that went from $S_2$ to $S_1$. We once again can think of it as mapping one pseudo-arc onto two pseudo-arcs, if we think of $S_2$ as $\{z \times Ps\}_{z \in H_4^2} \bigcup \{F^2(z) \times Ps\}_{z \in H_4^2}$ with identifications, instead of $\{z \times Ps\}_{z \in H_2^1}$ with identifications. So in this case we will map $Ps(z)$ in $S_3$, where

$z \in H_4^2$, onto the $Ps(z)$ and $Ps(F^2(z))$ in $S_2$, where $z \in H_4^2$. Thus by the same argument $f_2^3$ will be continuous.

The Whitney map on $S_3$ will be defined as it was for $S_1$ and $S_2$.

For $S_n$ replace $A^z_{2^n-1}$ in $S_0$ with a pseudo-arc for each $z \in H^{n-1}_{2^n}$. As before then $S_n = H^{n-1}_{2^n} \times Ps$ with identifications.The bonding map and the Whitney map will be defined similarly as above. Let $X = \varprojlim\{S_n, f\}$

We now have a non-metric inverse limit $X$ that supports a Whitney map. We need to show that $X$ is hereditarily indecomposable. To show the inverse limit space is hereditarily indecomposable, suppose that $M$ is a nondegenerate decomposable subcontinuum, $M = M_1 \cup M_2$. There exist $a, b \in M$, such that $a \in M_1 - M_2$,and $b \in M_2 - M_1$. By the nature of our inverse limit space and the fact that $M$ is a continuum, we can find an $r$ such that $a_r, b_r \in \pi_{S_r}(M)$, $a_r \neq b_r$ and both $a_r, b_r \in Ps(z)$ for some z; thus there exists an irreducible continuum in the form of a pseudo-arc between $a_r$ and $b_r$ that is contained in this $Ps(z)$. Since $M$ is a subcontinuum, this pseudo-arc associated with $a_r$ and $b_r$ must be contained in $\pi_{S_r}(M)$ which would imply that $M$ is indecomposable. $\square$

## Chapter 5

## Property + Whitney map = metric

**Theorem 5.1.** *If the cone over the compact space $X$ supports a Whitney map then $X$ is metric.*

*Proof.* Define Cone$(X) = (X \times [0,1])\diagup(X \times \{1\})$. Let $\mu$ be a Whitney map on $Cone(X)$. Define $F(x,y) : X \times X \to \mathbb{R}$ by $F(x,y) = \mu(\overline{x,t} \cup \overline{t,y}) - min\{\mu(\overline{x,t}), \mu(\overline{y,t})\}$ where $\overline{x,t}$ represents the arc $\{(x,a)|a \in [0,1]\}$ from $x \times \{0\}$ to the top point $t = X \times \{1\}$.

We will show that each part of $F$ is continuous; thus $F$ is continuous. Then we will use the fact that if $F$ is a continuous function from $X \times X$ into $\mathbb{R}$ such that $F(x,y) = 0$ if and only if $x = y$, then $X$ is metric.

Let $f(x,y) = \mu(\overline{x,t} \cup \overline{t,y})$. Let $(x,y) \in X \times X$ and let $U$ be an open set in $\mathbb{R}$ such that $f(x,y) \in U$. Since $\overline{x,t} \cup \overline{t,y}$ is a continuum and $\mu$ is continuous there exists a basic open set $V$ in $C(Cone(X))$ such that $\overline{x,t} \cup \overline{t,y} \in V$ and $\mu(V) \subset U$. Now there exist open sets in Cone$(X)$, $V_1, V_2, ..., V_n$, so that $V = R(\{V_i\}_{i=1}^n)$ which implies that

$\overline{x,t} \cup \overline{t,y} \subset \bigcup_{i=1}^n V_i \subset Cone(X)$.

For each point $(x,a)$ in $\overline{x,t}$, $(x,a) \in V_i$ for some $i$. Therefore we can find a basic open set in the cross product space of $X \times [0,1]$, namely $W_a^x \times (c_a, d_a) \subset V_i$, such that $(x,a) \in W_a^x \times (c_a, d_a)$.

Now $\overline{x,t} \subset \bigcup_{a\in[0,1]}(W_a^x \times (c_a, d_a))$. Since $\overline{x,t}$ is compact there exist a finite subcover so that

$\overline{x,t} \subset \bigcup_{i=1}^m (W_{a_i}^x \times (c_{a_i}, d_{a_i})) \subset \bigcup_{i=1}^n V_i \subset Cone(X)$.

Then consider $\bigcap_{i=1}^m (W_{a_i}^x)$. It is open and nonempty and for any $z \in W_{a_i}^x$ the arc $\overline{z,t} \subset \bigcup_{i=1}^n V_i$.

Doing the same procedure for the arc $\overline{y,t}$ will yield $\overline{y,t} \subset \bigcup_{i=1}^j (W_{b_i}^y \times (c_{a_i}, d_{a_i})) \subset \bigcup_{i=1}^n V_i \subset Cone(X)$ so that for any $r \in \bigcap_{i=1}^j (W_{b_i}^y)$ the arc $\overline{r,t} \subset \bigcup_{i=1}^n V_i$.

Let $\widetilde{V} = (\bigcap_{i=1}^m (W_{a_i}^x)) \times (\bigcap_{i=1}^j (W_{b_i}^y))$; this is open in $X \times X$.

Claim: If $(r,s) \in \widetilde{V}$ then $f(r,s) \in U$.

Proof. If $(r,s) \in \widetilde{V}$ then $\overline{r,t} \subset \bigcup_{i=1}^n V_i$ and $\overline{s,t} \subset \bigcup_{i=1}^n V_i$, which implies that $\overline{r,t} \bigcup \overline{s,t} \subset \bigcup_{i=1}^n V_i$ and it must intersect each $V_i$ from the construction of the $W_i$. Thus the subcontinuum $(\overline{r,t} \bigcup \overline{s,t})$ is a point in $V \subset C(Cone(X))$; therefore $f(r,s) = \mu(\overline{r,t} \bigcup \overline{s,t}) \in U$.

Now to show that the minimum is continuous define $f(x,y) = min\{\mu(\overline{x,t}), \mu(\overline{y,t})\}$.

Case 1. Assume that $\mu(\overline{x,t}) < \mu(\overline{y,t})$.

Given $f(x,y) \in U$, open, there exist $\epsilon > 0$ such that $[f(x,y) \pm \epsilon] \subset U$, where $[f(x,y) \pm \epsilon] = [f(x,y) - \epsilon, f(x,y) + \epsilon]$. Let $\delta = min\{\frac{\epsilon}{4}, \frac{\mu(\overline{y,t}) - \mu(\overline{x,t})}{2}\}$. By the continuity of $\mu$ there exist a

$V_x$, open in $C(ConeX)$, such that $\overline{x,t} \in V_x \subset C(Cone(X))$ and $\mu(V_x) \subset [\mu(\overline{x,t}) \pm \delta]$

and

$V_y$ such that $\overline{y,t} \in V_y \subset C(X)$ and $\mu(V_y) \subset [\mu(\overline{y,t}) \pm \delta]$.

Since $\overline{x,t} \in V_x$ and $V_x = R(\{V_i\}_{i=1}^n)$ we can do the same construction as before and get open sets such that $\overline{x,t} \subset \bigcup_{i=1}^m (W_{a_i}^x \times (c_{a_i}, d_{a_i})) \subset \bigcup_{i=1}^n V_i \subset Cone(X)$.

Let $\widetilde{V} \subset X \times X$ be defined as

$\widetilde{V} = (\bigcap_{i=1}^m (W_{a_i}^x)) \times (\bigcap_{i=1}^j (W_{b_i}^y))$. Now if $(r,s) \in \widetilde{V}$ then $\mu(\overline{r,t}) \in [\mu(\overline{x,t}) \pm \delta]$ and $\mu(\overline{s,t}) \in [\mu(\overline{y,t}) \pm \delta]$.

Therefore $min\{\mu((\overline{r,t}), \mu(\overline{s,t})\} = \mu(\overline{r,t}) \in [\mu(\overline{x,t}) \pm \delta] \subset [f(x,y) \pm \epsilon] \subset U$. Thus $f(\widetilde{V}) \subset U$.

Case 2.$\mu(\overline{x,t}) = \mu(\overline{y,t})$.

Same proof except let $\delta = \frac{\epsilon}{4}$ so that if $\overline{r,t} \in V_x$ then $\mu(\overline{r,t}) \in [f(x,y) \pm \frac{\epsilon}{4}]$, and if $\overline{s,t} \in V_x$ then $\mu(\overline{s,t}) \in [f(x,y) \pm \frac{\epsilon}{4}]$. Therefore $min\{\mu(\overline{r,t}), \mu(\overline{s,t})\} \in [f(x,y) \pm \frac{\epsilon}{4}] \subset U$, regardless of which arc is the minimum. So that $f(\widetilde{V}) \subset U$ and thus $f$ is continuous.

So we have shown that $F(x,y)$ is a difference of two continuous functions and thus is continuous. We will now show that $F(x,y) = 0 \Leftrightarrow x = y$.

Assume that $F(x,y) = 0$. Then $\mu(\overline{x,t} \cup \overline{y,t}) = min\{\mu(\overline{x,t}), \mu(\overline{y,t}\}$, but $\overline{x,t} \subset (\overline{x,t} \cup \overline{y,t})$ so by definition of a Whitney map $\mu(\overline{x,t}) < \mu(\overline{x,t} \cup \overline{y,t})$. Thus $\overline{x,t} = (\overline{x,t} \cup \overline{y,t})$ which implies that $x = y$.

Assume that $x = y$. Then $\overline{x,t} \cup \overline{y,t} = \overline{x,t} = \overline{y,t}$ so $\mu(\overline{x,t}) = \mu(\overline{x,t} \cup \overline{y,t})$ which implies that $F(x,y) = 0$.

Therefore X is metric. $\square$

The next proof is a generalize of Theorem 5.1

**Theorem 5.2.** *Given $X$ is contractible let $\phi : X \times [0,1] \longrightarrow \mathbb{R}$ be the contraction map, where $\phi(x,1) = p$ for any $x$. Let $\overline{xp} = \{\phi(x,t) | t \in [0,1]\}$. If $\overline{xp} = \overline{yp}$ if and only if $x = y$ and $X$ supports a Whitney map, then $X$ is metric.*

*Proof.* Let $\phi : X \times [0,1] \longrightarrow X$ be the contraction map, where $\phi(x,1) = p$ for any $x$. Let $\overline{xp} = \{\phi(x,t) | t \in [0,1]\}$. This will be a subcontinuum of $X$.

Define $G : X \times X \longrightarrow \mathbb{R}$ by $G(x,y) = \mu(\overline{xp} \bigcup \overline{yp}) - min\{\mu(\overline{xp}), \mu(\overline{yp})\}$.

We will show that each part of $G$ is continuous and thus $G$ is continuous. We will then use the fact that if you have a continuous function $G$ from $X \times X$ into $\mathbb{R}$ such that $G(x,y) = 0$ if and only if $x = y$ then, $X$ is metric.

Define $F : X \times X \longrightarrow \mathbb{R}$ by $F(x,y) = \mu(\overline{xp} \bigcup \overline{yp})$.

Let $(x,y) \in X \times X$ and let $U$ be an open set in $\mathbb{R}$ with $F(x,y) \in U$. By the continuity of $\mu$ there exists a basic open set $V'$ in $C(X)$ such that $(\overline{xp} \bigcup \overline{yp}) \in V'$ and $\mu(V') \subset U$. Now $V' = R(\{V_i\}_{i=1}^n)$ where each $V_i$ is an open set in $X$. First make this collection such that for each $i$ there exists a $z \in (\overline{xp} \bigcup \overline{yp})$ such that $z \in V_i$ and $z \notin V_j$ for every $j \neq i$. There exist subcollections $\{V_j^x\}_{j=1}^m$ and $\{V_k^y\}_{k=1}^l$ of $\{V_i\}_{i=1}^n$ such that $\left(\{V_j^x\}_{j=1}^m \bigcup \{V_k^y\}_{k=1}^l\right) = \{V_i\}_{i=1}^n$, $\overline{xp} \subset \{V_j^x\}_{j=1}^m$, and $\overline{yp} \subset \{V_k^y\}_{k=1}^l$.

For each $z = \phi(x,t) \in \overline{xp}$ there exists a $j$ such that $z \in V_j^x$ and by the continuity of $\phi$ there exists a basic open set $W_t = U_t \times (c_t, d_t) \subset X \times [0,1]$ such that $(x,t) \in W_t$ and $\phi(W_t) \subset V_j^x$.

$\bigcup_{t \in [0,1]} W_t$ covers $x \times [0,1]$ thus there exists a finite subcover $\{W_{t_r}\}_{r=1}^s$ that covers $x \times [0,1]$. Furthermore the subcover can be chosen to satisfy the condition that for each $V_j^x$ there exists $W_{t_r}$ such that $\phi(W_{t_r}) \subset V_j^x$. ( If this condition is not satisfied then just add to the collection a finite number of the $W_t$'s so that the needed property is satisfied.)

Let $A = \bigcap_{r=1}^{s} \pi_x(W_{t_r})$, where $\pi_x(W_{t_r})$ is the projection of $W_{t_r}$ onto the space $X$. $A$ is open in $X$ and if $z \in A$ then

1. Given any $t$ there exists $r$ such that $(z,t) \in W_{t_r}$ which implies that $\phi(z,t) \in V_j^x$ for some j. Thus $\overline{zp} \subset \bigcup_{j=1}^{m} V_j^x$.

2. Since we added in the extra condition on the $W_{t_r}$'s, given any $V_j^x$ there exists $W_{t_r}$ such that $\phi(W_{t_r}) \subset V_j^x$, which implies that $\overline{zp}$ intersects each $V_j^x$ for $j = 1$ to $m$.

Therefore, using the above two facts, $\overline{zp} \in R(\{V_j^x\}_{j=1}^m)$.

Repeating the same procedure using $\overline{yp}$ we obtain an open set $B = \bigcap_{q=1}^{d} \pi_x(O_q)$ such that if $z \in B$ then $\overline{zp}$ is contained in $\bigcup_{k=1}^{l} V_k^y$ and intersects each $V_k^y, k = 1$ to $l$.

Now $A \times B$ is an open set in $X \times X$. If $(a,b) \in A \times B$ then $\overline{ap} \bigcup \overline{bp}$ intersects each $V_i$ since $\overline{ap}$ intersects each member of the collection $\{V_j^x\}_{j=1}^m$ and $\overline{bp}$ intersects each $\{V_k^y\}_{k=1}^l$ and $\left(\{V_j^x\}_{j=1}^m \bigcup \{V_k^y\}_{k=1}^l\right) = \{V_i\}_{i=1}^n$.

Also $\overline{ap} \bigcup \overline{bp} \subset \bigcup_{i=1}^n V_i$. Therefore $\overline{ap} \bigcup \overline{bp} \in V'$ thus $\mu(\overline{ap} \bigcup \overline{bp}) \in U$ which implies that $F$ is continuous.

Now it can be shown that if $F(x,y) = min\{\mu(\overline{xp}), \mu(\overline{yp})\}$, then $F$ is continuous. Thus $G$ is continuous.

We just need to show that $G(x,y) = 0$ if and only if $x = y$. If $x = y$ then $\overline{xp} = \overline{xp} \bigcup \overline{yp}$ which implies that $min\{\mu(\overline{xp}, \mu(\overline{yp})\} = \mu(\overline{xp}) = \mu(\overline{xp} \bigcup \overline{yp})$, so $G(x,y) = 0$.

If $G(x,y) = 0$ then $\mu(\overline{xp} \bigcup \overline{yp}) = min\{\mu(\overline{xp}), \mu(\overline{yp})\}$. We know that $\overline{xp} = \overline{yp}$ if and only if $x = y$ thus $\mu(\overline{xp} \bigcup \overline{yp}) = \mu(\overline{xp})$ only if $x = y$.

Therefore $X$ is metric. $\square$

**Theorem 5.3.** *Given $X$ is contractible, let $\phi : X \times [0,1] \longrightarrow X$ be the contraction map, where $\phi(x,1) = p$ for any $x$. Let $\overline{xp} = \{\phi(x,t) | t \in [0,1]\}$. If $X$ supports a Whitney map and if $Q$ is a compact subset of $X$ such that $\overline{xp} = \overline{yp}$ if and only if $x = y$ for every $x$ and $y$ in $Q$, then $Q$ is metric.*

**Theorem 5.4.** *Let $S$ be the example from Chapter 2. $C(S)$ is contractible, and furthermore that contraction satisfies that conditions of Theorem 5.2.*

*Proof.* Before we begin, a lemma will be useful.

**Lemma 5.1.** *Let $V$ be an open set in $C(S)$. Let $M$ be a proper subcontinuum of $S$ that starts at the endpoint $(a,t)$ and ends at the point $(b,s)$, such that $M \in V$. There exists an open set $\widetilde{V}$ lying in $V$ and associated $\epsilon > 0$ so that*

*1.*

$$\widetilde{V} = R(\{V'_a, V'_b, V'_c\}).$$

*The union of $V'_a, V'_b, V'_c$ (the open sets that make up $\widetilde{V}$) is a tube-like set in $S$ (recall definition of tube-like in $S$ from Chapter 2)*

*2. $M \in \widetilde{V}$*

*3. Suppose $(a_N, t_N)$ and $(b_N, s_N)$ are endpoints of a subcontinuum $N$. Let $(a_N, t_N)$ be the starting point and $(b_N, s_N)$ be the ending point. If $N \in \widetilde{V}$ then $|t - t_N| < \epsilon$ and $|s - s_N| < \epsilon$, for some $\epsilon > 0$.*

*Proof.* Let $M$ be a proper subcontinuum. $M$ is a metric arc with endpoints $(a,t)$ and $(b,s)$. Let $(x,y)$ be the midpoint of the proper subcontinuum $M$. Let $F^{-n}(x) = a$, $F^j(x) = b$, and $F^0(x) = x$. Then

$M = (F^{-n}(x) \times [0,t]) \bigcup (F^{-n+1}(x) \times [0,1]) \bigcup \ldots\ldots \bigcup (F^{-1}(x) \times [0,1]) \bigcup (\{x\} \times [0,1]) \bigcup (F(x) \times [0,1]) \bigcup \ldots\ldots \bigcup (F^j(x) \times [s,1])$.

$V = R(\{V_k\}_{k=1}^m)$. Let $\{\pi_Z(V_k)\}_{k=1}^m$ be the projection onto $Z$ of $\{V_k\}_{i=1}^m$. For each $i \in \{-n, ... -1, 0, 1, ...j\}$, let $\{V_k\}_{k\in J_i}$, $J_i \in \{1, 2, ...m\}$, be the collection of $V_k's$ such that $F^i(x) \in \pi(V_k)$. Let $W_i = \bigcap_{k\in J_i} V_k$ for each $i \in \{-n, ... -1, 0, 1, ...j\}$. Note $W_i$ is an open set in $Z$, and $F^i(x) \in W_i$. Using the methods from the proof of lemma 4.1 in Chapter 2, the $W_i's$ can be refined into $U_i's$ so that

1. $U_i$ is open,
2. $F^i(x) \in U_i$,
3. $F(U_i) = U_{i+1}$ for each $i \in \{-n, ... -1, 0, 1, ...j\}$, and
4. $\{U_i\}_{i=-n}^j$ are pairwise disjoint.

We will use these open sets in $Z$ to create open sets in $S$. We will also need open sets in $[0,1]$. The endpoint $(a,t)$ is in at most $m$ open sets from $\{V_k\}_{k=1}^m$. Thus there exists an $\epsilon_a$ such that if $(a,t) \in V_k$ then $(a, t+\epsilon_a) \in V_k$ and $(a, t-\epsilon_a) \in V_k$. Likewise there exists an $\epsilon_b$ such that if $(b,s) \in V_k$ then $(b, s+\epsilon_b) \in V_k$ and $(b, s-\epsilon_b) \in V_k$. Let $\epsilon = min\{\epsilon_a, \epsilon_b\}$. Define $V'_a = U_{-n} \times (t-\epsilon, t+\epsilon)$, and $V'_b = U_j \times (s-\epsilon, s+\epsilon)$. Note $V'_a$ and $V'_b$ are open sets in $S$ and are contained in any $V_k$ that contained $(a,t)$ and $(b,s)$ respectively. Let $V'_c = (U_{-n} \times [0, t-\frac{\epsilon}{2})) \bigcup (U_{-n+1} \times [0,1]) \bigcup \ldots\ldots \bigcup (U_{j-1} \times [0,1]) \bigcup (U_j \times [s, s+\frac{\epsilon}{2}))$. Note that $V'_c$ is tube-like in $S$. Thus by construction

$\bigcup\{V_a', V_b', V_c'\}$ is tube-like in $S$. Define

$$\widetilde{V} = R(\{V_a', V_b', V_c'\}).$$

Given $N$ is a proper subcontinuum of $S$ if $N \in \widetilde{V}$ then $N$ must begin in $V_a'$ and end in $V_b'$. Let $(a_N, t_N)$ and $(b_N, s_N)$ be the endpoints of $N$; then $|t - t_N| < \epsilon$ and $|s - s_N| < \epsilon$. Also, by the construction of $\widetilde{V}$, $N \in V$. Therefore $\widetilde{V}$ lies in $V$. $\square$

Before we begin,we need some notation.

1. If $N$ is a proper subcontinuum of $S$ then $N + \alpha$ is the arc in $S$ made by extending $N$ by $\frac{\alpha}{2}$ in each direction. Thus $len(N + \alpha) = len(N) + \alpha$.

2. If $a_N$ is an endpoint of $N$ then $a_N + \epsilon$, $\epsilon < 1$, is the endpoint on the new arc $N + 2\epsilon$. Given $a_N = (x, t)$ then $a_N + \epsilon$ would be the point $(x, t + \epsilon)$ or $(x, t - \epsilon)$ depending on which direction we want to go. Note, if $t$ is within $\epsilon$ of 0 or 1 then a slight modification must be made. If $t$ is within $\epsilon$ of 1 then $a_N + \epsilon$ , the new point, would be $(F^{-1}(x), t+\epsilon-1)$. If $t$ is within $\epsilon$ of 0 then $a_N - \epsilon$ would be $(F(x), t-\epsilon+1)$.

3. Let $(a, b)$ and $(c, d)$ be two points in $S$. "$(a, b)$ is within $\epsilon$ of $(c, d)$" will mean $|b - d| < \epsilon$.

Now we need to define the contraction map $f : [0,1] \times C(S) \to C(S)$. Let

$$f(\delta, M) = \begin{cases} M & \text{if } \delta \leq \mu(M), \\ M' & \text{if } \delta > \mu(M), \quad \text{where } M' \text{ has the same midpoint as } M \\ & \qquad\qquad\qquad \text{and } \mu(M') = \delta, \\ S & \text{if } \delta = 1, \\ S & \text{if } M = S. \end{cases}$$

Notice that $M'$ is uniquely defined by the nature of $S$ since if $M' = M + \epsilon$ where $\mu(M') = \delta$ then $\delta = \frac{arctan(lenM')}{\frac{\pi}{2}}$.

We wish to prove that $f$ is continuous. Let $V \in C(S)$ be an open set such that $f(\delta, M) \in V$. Let $M$ be a proper subcontinuum and $\delta \in [0,1)$; then we have three cases:

1. $\mu(M) > \delta$,
2. $\mu(M) = \delta$, and
3. $\mu(M) < \delta$.

Case 1. $\mu(M) > \delta$.

If $\mu(M) > \delta$ then $f(\delta, M) = M$. Let $V \in C(S)$ be an open set such that $f(\delta, M) = M \in V$. Using the lemma we can find a tube-like refinement $\widetilde{V}$ with associated $\epsilon$ containing $M$ so that $\widetilde{V}$ is tube-like in $S$, $\widetilde{V} = R(V'_a, V'_b, V'_c)$, and the associated $\epsilon$ is small enough so that if $N \in \widetilde{V}$ then

$$\mu(N) > \delta + \frac{|\delta - \mu(M)|}{2} > \delta.$$

Let $W = (\delta - \frac{|\delta - \mu(M)|}{4}, \delta + \frac{|\delta - \mu(M)|}{4}) \times \widetilde{V}$. If $(j, N) \in W$ then $j$ is within $\frac{|\delta - \mu(M)|}{4}$ of $\delta$. If $N \in \widetilde{V}$ then $\mu(N) > \delta + \frac{|\delta - \mu(M)|}{2} > \delta + \frac{|\delta - \mu(M)|}{4} > j$. So, for any $(j, N) \in W$, $f(j, N) = N$. We already know $N \in \widetilde{V}$ which is contained in $V$. Therefore $f(j, N) \in V$. Thus $f$ is continuous if $\mu(M) > \delta$.

Case 2. $\mu(M) = \delta$

Let $f(\delta, M) \in V$ where $V$ is an open set in $C(S)$. We can again refine $V$ into a tube-like $\widetilde{V}$ with associated $\epsilon$, such that $M \in \widetilde{V}$. Also we can choose $\epsilon$ small enough so that $M + 2\epsilon \in V$ and if $N \in \widetilde{V}$ then the endpoints of $N$ are within $\frac{\epsilon}{2}$ of the endpoints of $M$. (Recall: "within $\epsilon$" and $M + 2\epsilon$ are defined in the notation section at the beginning of this proof).

Let $N \in \widetilde{V}$. If $N'$ is the subcontinuum made by increasing $N$ equidistance at each end then by the definition of $\epsilon$ and $\widetilde{V}$ if $len(N') < len(M + 2\epsilon)$ then $N' \in V$.

There exist an $\alpha \in \mathbb{R}$ such that if $N \in \widetilde{V}$ then

$\delta - \alpha < \mu(N) < \delta + \alpha < \mu(M + 2\epsilon)$.

Let $W = (\delta - \alpha, \delta + \alpha) \times \widetilde{V}$.

Claim: $f(W) \subset V$. Let $(j, N) \in W$. We have two cases

Case 2.1. $\mu(N) \geq j$. If $\mu(N) \geq j$ then $f(j, N) = N \in \widetilde{V} \subset V$.

Case 2.2. $\mu(N) < j$. If $\mu(N) < j$ then $f(j, N) = N'$ where $N'$ is made by increasing $N$ on each end until $\mu(N') = j$. We know if $N \in \widetilde{V}$ and $len(N') < len(M + 2\epsilon)$ then $N' \in V$. Thus we just need to prove that $len(N') < len(M + 2\epsilon)$. Now $j \in (\delta - \alpha, \delta + \alpha)$ so we know that $j = \mu(N') < \mu(M + 2\epsilon)$. Therefore $len(N') < len(M + 2\epsilon)$. Thus $f$ is continuous if $\mu(M) = \delta$.

Case 3. $\mu(M) < \delta$. Now since $\mu(M) < \delta$, $f(\delta, M) = M'$ where $\mu(M') = \delta$. Let $V$ be an open set in $C(S)$ and $f(\delta, M) = M' \in V$. Now assume that $M \notin V$. (If it were we could refine $V$ using the lemma and $\epsilon \leq \frac{len(M')-len(M)}{4}$). Assume that $V$ is tube-like in $S$. Now there exists a $\gamma$ such that for any continuum $N \in V$ then the endpoints of $N$ are within $\gamma$ of the endpoints of $M'$. Note since $M \notin V$ that the endpoints of $M$ are more than $\gamma$ away from the endpoints of $M'$. Let $O = (\mu(M' - \frac{\gamma}{4}), \mu(M' + \frac{\gamma}{4}))$. Recall that $M' + \frac{\gamma}{4}$ was made by adding $\frac{\gamma}{8}$ to each end of $M'$ and likewise $M' - \frac{\gamma}{4}$ was made by subtracting $\frac{\gamma}{8}$ from each end. Now $len$ is continuous so there exist an open set $U$ containing $M$ such that $len(U) \subset (len(M) - \frac{\gamma}{4}, len(M) + \frac{\gamma}{4})$.

Assume that $U$ is a tube. We know that $V$ is also a tube. Let $\widetilde{U} = U \bigcap V$. This makes the tube corresponding to $\widetilde{U}$ have radius less than the tube corresponding to $V$. (We do this because we will extend continua in $\widetilde{U}$ and we want to make sure that when we extend a continuum we stay inside the tube corresponding to $V$.) Notice that if $N \in \widetilde{U}$ then $len(N)$ is within $\frac{\gamma}{4}$ of $len(M)$. Thus the endpoints of $N$ are within $\frac{\gamma}{4}$ of the endpoints of $M$. Also note that the endpoints of $M$ are at least $\gamma$ away from the endpoints of $M'$. Thus if $N \in \widetilde{U}$ then $N \notin V$ which implies that $\mu(N) \in O$.

Let $W = O \times \widetilde{U}$. If $(j, N) \in W$, then $N \in \widetilde{U}$ and $\mu(N) \notin O$. Therefore for all $(j, N) \in W$, $f(j, N) = N'$ where $len(N') > len(N)$.

Let $\alpha_\delta \in [0, \infty)$ be the unique number such that $\mu(M + \alpha_\delta) = \mu(M') = \delta$. For any $j \in O$ there exist a number $\alpha_j$ such that $\mu(M + \alpha_j) = j$.

For $j \in O$, $j = \mu(M + \alpha_j) < \mu(M' + \frac{\gamma}{4}) = \mu(M + \alpha_\delta + \frac{\gamma}{4})$. Thus $\alpha_j < \alpha_\delta + \frac{\gamma}{4}$. Similarly $j = \mu(M + \alpha_j) > \mu(M' - \frac{\gamma}{4}) = \mu(M + \alpha_\delta - \frac{\gamma}{4})$. Thus $\alpha_j > \alpha_\delta - \frac{\gamma}{4})$. Therefore for any $j \in O$, $\alpha_j$ is within $\frac{\gamma}{4}$ of $\alpha_\delta$.

Given $(j, N) \in W$, $f(j, N) = N'$. $N'$ is made by adding a certain length onto each end of $N$. We have three cases.

Case 3.1. If the $len(N) = len(M)$, then we add $\frac{\alpha_j}{2}$ to each end of $N$.

Case 3.2. If $len(N) < len(M)$, then add $\frac{\alpha_j + |len(M) - len(N)|}{2}$ to each end of $N$.

Case 3.3. If $len(N) > len(M)$, then add $\frac{\alpha_j - |len(M) - len(N)|}{2}$ to each end of $N$.

Let $a_N$ and $b_N$ be the endpoints of $N$, $a_{N'}$ and $b_{N'}$ be the endpoints of $N'$, $a_M$ and $b_M$ be the endpoints of $M$, and $a_{M'}$ and $b_{M'}$ be the endpoints of $M'$. We will show in all three cases that $a_{N'}$ is within $\gamma$ of $a_{M'}$, and $b_{N'}$ is within $\gamma$ of $b_{M'}$. Thus $N' \in V$.

Case 3.1. $len(N) = len(M)$.

Now $a_N$ is within $\frac{\gamma}{4}$ of $a_M$ so $a_N + \frac{\alpha_j}{2}$ is within $\frac{\gamma}{4}$ of $a_M + \frac{\alpha_j}{2}$. $a_M + \frac{\alpha_j}{2}$ is within $\frac{\gamma}{4}$ of $a_{M'}$. Thus $a_N + \frac{\alpha_j}{2} = a_{N'}$ is within $\frac{\gamma}{4}$ of $a_{M'}$. Likewise $b_N + \frac{\alpha_j}{2} = b_{N'}$ is within $\frac{\gamma}{4}$ of $b_{M'}$. Therefore $N' \in V$.

Case 3.2. $len(N) < len(M)$.

In this case in order to make $N'$ we will add $\frac{\alpha_j + |len(M) - len(N)|}{2}$ to each end of $N$. Thus $a_{N'} = a_N + \frac{\alpha_j + |len(M) - len(N)|}{2}$. The endpoints of $N$ are within $\frac{\gamma}{4}$ of the endpoints of $M$ so if we add $\frac{\alpha_j + |len(M) - len(N)|}{2}$ to each endpoint then $a_N + \frac{\alpha_j + |len(M) - len(N)|}{2}$ is within $\frac{\gamma}{4}$ of $a_M + \frac{\alpha_j}{2} + \frac{|len(M) - len(N)|}{2}$. If we show that $a_M + \frac{\alpha_j}{2} + \frac{|len(M) - len(N)|}{2}$ is within $\frac{3\gamma}{4}$ of $a_{M'}$ then we will know that $a_{N'}$ is within $\gamma$ of $a_{M'}$. Thus $N' \in V$.

Recall that

1. $|len(M) - len(N)| < \frac{\gamma}{4}$.

2. $\alpha_\delta - \frac{\gamma}{4} < \alpha_j < \alpha_\delta + \frac{\gamma}{4}$.

3. $a_M + \frac{\alpha_\delta}{2} = a_{M'}$.

Using these facts we have

$a_M + \frac{\alpha_j}{2} + \frac{|len(M)-len(N)|}{2} < a_M + \frac{\alpha_j}{2} + \frac{\gamma}{8} < a_M + \frac{\alpha_\delta}{2} + \frac{\gamma}{8} + \frac{\gamma}{8} = a_{M'} + \frac{\gamma}{4}$.

Similarly we have

$a_M + \frac{\alpha_j}{2} + \frac{|len(M)-len(N)|}{2} > a_M + \frac{\alpha_j}{2} + 0 > a_M + \frac{\alpha_\delta}{2} - \frac{\gamma}{8} > a_{M'} - \frac{\gamma}{8}$.

Therefore $a_M + \frac{\alpha_j}{2} + \frac{|len(M)-len(N)|}{2}$ is within $\frac{\gamma}{4}$ of $a_{M'}$. Thus $a_N + \frac{\alpha_j}{2} + \frac{|len(M)-len(N)|}{2} = a_{N'}$ is within $\frac{\gamma}{2}$ of $a_{M'}$. Thus $N \in V$.

Case 3.3. $len(N) > len(M)$.

In this case in order to make $N'$ we will add $\frac{\alpha_j - |len(M)-len(N)|}{2}$ to each end of $N$. Thus $a_{N'} = a_N + \frac{\alpha_j - |len(M)-len(N)|}{2}$. The endpoint of $N$ are within $\frac{\gamma}{4}$ of the endpoint of $M$ so if we add $\frac{\alpha_j - |len(M)-len(N)|}{2}$ to each endpoint then $a_N + \frac{\alpha_j - |len(M)-len(N)|}{2}$ is within $\frac{\gamma}{4}$ of $a_M + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2}$. If we show that $a_M + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2}$ is within $\frac{3\gamma}{4}$ of $a_{M'}$ then we will know that $a_{N'}$ is within $\gamma$ of $a_{M'}$. Thus $N' \in V$.

$a_M + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2} > a_M + \frac{\alpha_j}{2} - \frac{\gamma}{8} > a_M + \frac{\alpha_\delta}{2} - \frac{\gamma}{8} - \frac{\gamma}{8} = a_{M'} - \frac{\gamma}{4}$.

Similarly we have

$a_M + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2} < a_M + \frac{\alpha_j}{2} + 0 < a_M + \frac{\alpha_\delta}{2} + \frac{\gamma}{8} = a_{M'} - \frac{\gamma}{8}$.

Therefore $a_M + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2}$ is within $\frac{\gamma}{4}$ of $a_{M'}$. Thus $a_N + \frac{\alpha_j}{2} - \frac{|len(M)-len(N)|}{2} = a_{N'}$ is within $\frac{\gamma}{2}$ of $a_{M'}$. Thus $N' \in V$. Thus if $\mu(M) < \delta$, then $f$ is continuous.

We still need to show continuity if $M$ is a proper subcontinuum and $\delta = 1$. If $\delta = 1$ then $f(\delta, M) = S$. Let $S \in V$ where $V$ is open in $C(S)$. By definition $V = R(\{V_i\}_{i=1}^n)$ for some open sets $V_i$ in $S$. There exist an $i$ such that $V_i = S$. Also there exist a number $r \in [0,1)$ such that if $\mu(N) \geq r$ then $N \in V$. Since $\mu$ is continuous there exists an open set $U$ in $C(S)$ so that

$M \in U$, and $\mu(U) \subset (\mu(M) - \frac{|r-\mu(M)|}{2}, \mu(M) + \frac{|r-\mu(M)|}{2})$. Let $W = (r,1) \times U$. If $(j, N) \in W$ then

Case 1. $f(j,N) = N$. Thus $\mu(N) = j > r$ which implies that $N \in V$.

Case 2. $f(j,N) = N'$. Thus $\mu(N') = j > r$ which implies that $N' \in V$.

Therefore if $M$ is a proper subcontinuum then $f$ is continuous for all values in $[0,1]$.

Lastly we need to prove continuity if $M = S$. If $M = S$ then $f(\delta, S) = S$ for any $\delta$.

Case 1. $\delta < 1$.

Let $f(\delta, S) = S \in V$, V is open in $C(S)$. Since we know that one of the open sets that make up $V$ must be $S$, if $N \in V$ then $N + \epsilon \in V$ for any $\epsilon$.

Let $W = (\frac{\delta}{2}, 1) \times V$. Given $(j,N) \in W$, then $N \in V$. If $f(j,N) = N$ then $N \in V$. If $f(j,N) = N'$ then since $N$ is in $V$ so is $N'$.

Case 2. $\delta = 1$.

Let $f(\delta, S) = S \in V$, V is open in $C(S)$. Let $W = (r,1] \times V$. If $(j,N) \in W$; then $N \in V$. If $f(j,N) = N$ then $N \in V$. If $f(j,N) = N'$ then since $N \in V$ then

$N' \in V$. Given $f(j, N) = S$, then $S \in V$. Therefore we have proved continuity if $M = S$. Since $f$ is continuous in all cases then $C(S)$ is contractible.

$\square$

Let $Q = \{H \in C(S) | H \text{ is a singleton point in } S\}$; then the previous contraction for $C(S)$ has the property that $\overline{xp} = \overline{yp}$ if and only if $x = y$ for every $x$ and $y$ in $Q$. If we assume $C(S)$ supports a Whitney map then $Q$ is metric by Corollary 5.1, which is a contradiction; thus $C(S)$ can not support a Whitney map. This is an example of a space supporting a Whitney map but whose hyperspace does not. This is also an example of a space that is arcwise connected by metric arcs that does not support a Whitney map thus proving that being arcwise connected by metric arcs is not a sufficient condition for supporting a Whitney map in the non-metric case.

**Theorem 5.5.** *Suppose $X$ and $Y$ are a continua and $Y$ is nondegenerate. If $X \times Y$ supports a Whitney map then $X$ is metric.*

*Proof.* Let $p$ be an arbitrary point in $Y$. Define $F : X \times X \to \mathbb{R}$ as $F(x, y) = \mu((x \times Y) \cup (y \times Y) \cup (X \times p)) - \min \{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\}$. We will show that each part of $F$ is continuous and thus $F$ is continuous. We will then use the fact that if you have a continuous function $F$ from $X \times X$ into $\mathbb{R}$ such that $F(x, y) = 0$ if and only if $x = y$, then $X$ is metric.

Let $f(x, y) = \mu((x \times Y) \cup (y \times Y) \cup (X \times p))$. We want to show $f$ is continuous. Let $(x, y) \in X \times X$ and let $U \in \mathbb{R}$ be an open set such that $f(x, y) \in U$. We need an open set $\widetilde{V} \in X \times X$ such that $f(\widetilde{V}) \subset U$. Since $X \times Y$ supports a Whitney map we know there exist a basic open set $V \in C(X \times Y)$ such that $(x \times Y) \cup (y \times Y) \cup (X \times p) \in V$ and $\mu(V) \subset U$. Now there exist open sets $V_1, V_2, ...V_n \subset X \times Y$ such that $V = R(\{V_j\}_{j=1}^n)$. For each $(x, a) \in x \times Y$ there exist an open set $W_a^x \times O_a$ such that $(x, a) \in W_a^x \times O_a$ and $W_a^x \times O_a \subset V_j$ for some $j$. Since $x \times Y$ is compact and $x \times Y \subset \bigcup_{a \in X} W_a^x \times O_a$ then there exist a finite subcover of $x \times Y$, namely $x \times Y \subset \bigcup_{i=1}^m W_{a_i}^x \times O_{a_i} \subset \bigcup_{j=1}^n V_j$. We also want to ensure that for every $V_j$ intersecting $x \times Y$ there exist a $W_{a_i}^x \times O_{a_i}$ lying in $V_j$. If this does not happen we can just add in finitely many open sets from the collection $\{W_a^x \times O_a\}_{a \in X}$ so that above condition will be met; thus we can just assume that the collection $\{W_{a_i}^x \times O_{a_i}\}_{i=1}^m$ satisfies the fact that for every $V_j$ intersecting $x \times Y$ there exists $W_{a_i}^x \times O_{a_i}$ that lies in it.

Let $\widetilde{W_x} = \bigcap_{i=1}^m W_{a_i}^x$. If $z \in \widetilde{W_x}$, then $z \times Y \subset \bigcup_{i=1}^m W_{a_i}^x \times O_{a_i} \subset \bigcup_{j=1}^n V_j$ and $z \times Y$ intersects each $V_j$, $j = 1$ to $n$.

By a similar construction using $y \times Y$ instead of $x \times Y$ you can find a $\widetilde{W_y}$ so that if $w \in \widetilde{W_y}$ then $w \times Y \subset \bigcup_{i=1}^k W_{b_i}^y \times O_{b_i} \subset \bigcup_{j=1}^n V_j$ and $w \times Y$ intersects each $V_j$ intersecting $y \times Y$, $j = 1$ to $n$.

Let $\widetilde{V} = \widetilde{W_x} \times \widetilde{W_y}$. If $(z, w) \in \widetilde{V}$ then $f(z, w) = ((z \times Y) \cup (w \times Y) \cup (X \times p)) \subset \bigcup_{j=1}^n V_j$ and $f(z, w)$ intersects each $V_j$. Thus if $f(z, w) \in \widetilde{V}$ then $(z \times Y) \cup (w \times Y) \cup (X \times p)$ lies in $V$. Therefore $\mu((z \times Y) \cup (w \times Y) \cup (X \times p)) \in U$. Thus $f$ is continuous.

Let $g(x, y) = \min \{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\}$. Using a similar procedure we can show that $g$ is continuous. Thus $F$ is continuous.

If $F(x, y) = 0$ then $\mu((x \times Y) \cup (y \times Y) \cup (X \times p)) = \min \{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\}$. With out loss of generality assume that $\mu((x \times Y) \cup (X \times p) =$

min $\{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\}$. Now we know that since $Y$ is nondegenerate that if $x \neq y$ then $(x \times Y) \cup (X \times p) \subsetneq (x \times Y) \cup (y \times Y) \cup (X \times p)$; thus by the Whitney property $\mu((x \times Y) \cup (X \times p)) < \mu((x \times Y) \cup (y \times Y) \cup (X \times p))$, which is a contradiction with $\mu((x \times Y) \cup (y \times Y) \cup (X \times p)) = \min \{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\}$; thus $x \times Y = y \times Y$, which implies that $x = y$.

If $x = y$ then $(x \times Y) \cup (y \times Y) \cup (X \times p) = (x \times Y) \cup (X \times p)$ so then $\mu((x \times Y) \cup (X \times p)) = \mu((x \times Y) \cup (y \times Y) \cup (X \times p)) = \min \{\mu((x \times Y) \cup (X \times p)), (\mu((y \times Y) \cup (X \times p))\} = \mu((x \times Y) \cup (X \times p))$. Thus $F(x, y) = 0$.

Therefore $X$ is metric. □

**Theorem 5.6.** *Let $f$ be a function $f : X \longrightarrow C(C(X))$ such that $f(x)= \overline{xX}$, which is the point in $C(C(X))$ where $\overline{xX}$ represents an order arc in $C(X)$ from $\{x\}$ to $X$. If $f$ is continuous and $C(X)$ supports a Whitney map then, $X$ is metric.*

*Proof.* Define $G : X \times X \longrightarrow \mathbb{R}$ by $G(x, y) = \mu((f(x) \bigcup f(y)) - min\{\mu(f(x)), \mu(f(y))\}$.

We will show that each part of $G$ is continuous thus $G$ is continuous. We will then use the fact that if you have a continuous function $G$ from $X \times X$ into $\mathbb{R}$ such that $G(x, y) = 0$ if and only if $x = y$, then $X$ is metric.

Let $F : X \times X \longrightarrow \mathbb{R}$ be defined as $F(x, y) = \mu(f(x) \bigcup f(y))$. We need to show $F$ is continuous. Let $U$ be an open set in $\mathbb{R}$, $(x, y) \in X \times X$ and $F(x, y) \in U$. By the continuity of $\mu$ there exists an open set $V$ in $C(C(X))$ such that $(f(x) \bigcup f(y)) \in V$ and $\mu(V) \subset U$. Now $V = R(\{V_i\}_{i=1}^{n})$ where each $V_i$ is an open set in $C(X)$. There exists subcollections $\{V_j^x\}_{j=1}^m$ and $\{V_k^y\}_{k=1}^l$ of $\{V_i\}_{i=1}^n$ such that $\left(\{V_j^x\}_{j=1}^m \bigcup \{V_k^y\}_{k=1}^l\right) = \{V_i\}_{i=1}^n$,

$\overline{xX} \subset \bigcup_{j=1}^m V_j^x$ and intersects each $V_j$ and

$\overline{yX} \subset \bigcup_{k=1}^l V_k^y$ and intersects each $V_k$.

Now let $R(\{V_j^x\}_{j=1}^m) = V^x$ this is an open set in $C(C(X))$ such that $\overline{xX} \in V^x$, and likewise $V^y$ is an open set that contains $\overline{yX}$. By the continuity of $f$ there exist open sets $A, B$ in $X$ such that

$x \in A$ and $f(A) \subset V^x$ and

$y \in B$ and $f(B) \subset V^y$.

$A \times B$ is an open set in $X \times X$.

Claim: $F(A \times B) \subset U$.

We need to show that if $(a, b) \in A \times B$ then $f(a) \bigcup f(b) \in V$. Thus we need to show that $f(a) \bigcup f(b) \subset \bigcup_{i=1}^n V_i$ and that $f(a) \bigcup f(b)$ intersects each $V_i$. We know that $f(a) \in V^x$ and $f(b) \in V^y$, thus $f(a) \bigcup f(b) \subset \bigcup_{i=1}^n V_i$. Since $f(a) \in V^x$ we have that $f(a)$ intersects each member of the collection $\{V_j^x\}_{j=1}^m$ and likewise $f(b)$ intersects each member of $\{V_k^y\}_{k=1}^l$. Thus $f(a) \bigcup f(b)$ must intersect each member of $\{V_i\}_{i=1}^n$ since $\left(\{V_j^x\}_{j=1}^m \bigcup \{V_k^y\}_{k=1}^l\right) = \{V_i\}_{i=1}^n$. Therefore $f(a) \bigcup f(b) \in V$ which implies that $\mu(f(a) \bigcup f(b)) \in U$. Thus $F$ is continuous.

By a similar argument if $F(x, y) = min\{\mu(f(x)), \mu(f(y))\}$ then $F$ is continuous. Therefore $G$ is continuous.

If $G(x,y) = 0$ then $\mu((f(x) \bigcup f(y)) = min\{\mu(f(x)), \mu(f(y))\}$. But because of the Whitney property and the fact that $\overline{xX} = \overline{yX}$ if and only if $x = y$ the only time that will happen is when $f(x) = f(y)$ which implies that $x = y$.

If $x = y$ then $\mu((f(x) \bigcup f(y)) = min\{\mu(f(x)), \mu(f(y))\}$, which implies that $G(x,y) = 0$.

Therefore $X$ is metric. $\square$

**Theorem 5.7.** *If each $X$ and $Y$ support Whitney maps and $X \bigcap Y$ is a singleton point, then $X \bigcup Y$ supports a Whitney map.*

*Proof.* Suppose that $K$ is a subcontinuum of $X \bigcup Y$ and $X \cap Y = \{z\}$. Let $K^X = K \bigcap X$ and $K^Y = K \bigcap Y$. Let $\mu_X$ be the Whitney map on $X$ and $\mu_Y$ be the Whitney map on $Y$. We will define the Whitney map on $X \bigcup Y$ as

$\mu(K) = \mu_X(K^X) + \mu_Y(K^Y)$.

First it is obvious that if $H \subsetneq K$ then $\mu(H) < \mu(K)$. Since, in order for $H$ to be a proper subset of $K$, then either $H^X \subsetneq K^X$, which implies that $\mu_X(H^X) < \mu_X(K^X)$, or $H^Y \subsetneq K^Y$, which implies that $\mu_Y(H^Y) < \mu_Y(K^Y)$.

In order to show $\mu$ is continuous, let $U$ be an open set in $\mathbb{R}$ and let $\mu(K) \in U$; then there exists an $\epsilon > 0$ such that $[\mu(K) \pm \epsilon] \in U$. Now since both $\mu_X$ and $\mu_Y$ are Whitney maps, for $K^X$ there exist an open set $V^X$ in $C(X)$ such that $\mu_X(V^X) \subset [\mu_X(K^X) \pm \frac{\epsilon}{2}]$ and likewise for $K^Y$.

Next we will refine each $V^X$ and $V^Y$. Given $V^X$ there exist an open refinement $\widetilde{V}^X = R(\{W_l^X\}_{l=1}^m)$ such that

1. $K^X \in \widetilde{V}^X$.

2. If $z$ is the intersection point of $X \bigcap Y$ and $z \in K^X$ then $z \in W_l^X$ for only one $l$, assume that $l = 1$. Thus $z \in W_1^X$.

3. In the open set $W_1^X$ there exist another a point $x \in K^X$ such that $x$ is in the interior of $W_1^X$ and $x \notin W_l^X$ for $l \neq 1$.

4. $z$ is not a limit point of any $W_l^X$ for $l > 1$.

Notice that $W_l^X$ for $l > 1$ is an open set not only in $X$ but in $X \bigcup Y$, but $W_1^X$ is not an open set in $X \bigcup Y$. This is important because that means that we can not use $W_1^X$ to make an open set in $C(X \bigcup Y)$. To fix this problem we will make two open sets in $X \bigcup Y$ using $W_1^X$.

Make a new open set $C^X$ contained in $X$ such that

1. $C^X \subset W_1^X$,

2. $C^X \bigcap K^X \neq \emptyset$,

3. $C^X \bigcap W_l^X = \emptyset$, for any $l > 1$,

4. $z \notin C^X$.

Notice that since $C^X$ does not contain the intersection point that means that $C^X$ is an open set in $X \bigcup Y$. Also since $C^X \subset W_1^X$ that if we add $C^X$ into the collection of open sets that made up $\widetilde{V}^X$ we will just refine $\widetilde{V}^X$ thus refining $V^X$ even more.

Doing the same procedure will yield similar open sets in $Y$.

Now we will make our second open set using $W_1^X$ and $W_1^Y$.

For $z \in (X \bigcap Y) \bigcap K$, let $D_z = W_1^X \bigcup W_1^Y$. Note that since $z \in W_1^X$ and $z \in W_1^Y$ then $z \in D_z$. Also most important $D_z$ is an open set in $X \bigcup Y$.

Let

$$O = R\left(\{W_l^X\}_{l=2}^m, \{W_j^Y\}_{j=2}^r, C^X, C^Y, D_z\right).$$

$O$ is open in $C(X \bigcup Y)$. Given $H \in O$, let $H^X = H \bigcap X$. $H$ must intersect each $W_l^X$ and $W_j^Y$ for every $l > 2$ and $j > 2$ and $C^X, C^Y, D_z$, and be contained in their union. Thus $H^X \in V^X$. Note this was the reason $C^X$ was necessary so that $H^X$ would be forced to intersect $W_1^X$. Similarly $H^Y \in V^Y$.

Thus $\mu_X(H^X) \in [\mu_X(K^X) \pm \frac{\epsilon}{2}]$ and $\mu_Y(H^Y) \in [\mu_Y(K^Y) \pm \frac{\epsilon}{2}]$, so $\mu(H) \in [\mu(K) \pm \epsilon]$. Therefore $\mu$ is continuous and thus is a Whitney map on $X \bigcup Y$. □

**Corollary 5.1.** *There are continuum many nonhomeomorphic decomposable non-metric continua that support Whitney maps.*

**Theorem 5.8.** *There exist $X$ and $Y$, that both support Whitney maps, such that $X \bigcup Y$ does not support a Whitney map.*

*Proof.* Our example $S$ was made by taking the cross product of $Z$ with $[0, 1]$ and then making the proper identifications. If instead of $[0, 1]$ we use $[-1, 0]$ then a similar non metric space can be made that also supports a Whitney map. Denote this space as $S_{[-1,0]}$.

The intersection of $S$ and $S_{[-1,0]}$ has uncountably many points. Observe that $S \bigcap S_{[-1,0]} = Z \times \{0\}$. $S \bigcup S_{[-1,0]}$ is a continuum.

Assume $S \bigcup S_{[-1,0]}$ supports a Whitney map.

Let $Z_{1/2} = \{(z,t) \in S | t = 1/2\}$. Note that $Z_{1/2} \cong Z$ and thus is non-metric. Given a point $x \in S$, $x = (z^x, t^x)$ for some $z^x \in Z, t^x \in [0,1]$. Let $\overline{xS} = \{(z^x, t) | t \leq t^x\}$. Let $F : Z_{1/2} \times Z_{1/2} \to \mathbb{R}$ be defined as

$$F(x,y) = \mu(\overline{xS} \cup \overline{yS} \cup S_{[-1,0]}) - min\{\mu(\overline{xS} \cup S_{[-1,0]}), \mu(\overline{yS} \cup S_{[-1,0]})\}.$$

Using a similar construction to the one in the proof of "If the cone over $X$ supports a Whitney map then $X$ is metric" (where $S_{[-1,0]}$ behaves in a similar manner as the top point of the cone), $F$ can be shown to be continuous and it can be shown that $F(x,y) = 0$ if and only if $x = y$. Thus $Z_{1/2}$ is metric which is a contradiction. Therefore the continuum $S \bigcup S_{[-1,0]}$ can not support a Whitney map. $\square$

# Chapter 6

## Whitney levels

Let $X$ be a continuum and $\mu$ a Whitney map on $\mathbb{C}(X)$. For $t \in [0, \mu(X)]$ and for $x \in X$ let $W_x^t = \{H \in \mu^{-1}(t) | \, x \in H\}$.

**Theorem 6.1.** *$W_x^t$ is a subcontinuum of $C(X)$.[10]*

**Theorem 6.2.** *If $t > 0$ then $\mu^{-1}(t)$ is a subcontinuum of $C(X)$.*

*Proof.* CLOSED: Let $L$ be a continuum such that $\mu(L) \neq t$. Then $\mu(L) = r$ where $r < t$ or $t < r$. Assume $r < t$

Let $U = (r - \frac{t-r}{2}, r + \frac{t-r}{2})$. $\mu(L) \in U$. Since $\mu$ is continuous there exists an open set $V$ containing $L$ such that $\mu(V) \subset U$. That implies that all continua that are in V must have Whitney value in $U$. Thus $U \bigcap \mu^{-1}(t) = \emptyset$. Therefore $V$ is an open set that contains $L$ and misses $\mu^{-1}(t)$. Thus $L$ is not a limit point of $\mu^{-1}(t)$.

CONNECTED: Assume $\mu^{-1}(t)$ in not connected and thus is the union of two disjoint compact sets $A$ and $B$. Then $\mu^{-1}(t) = A \bigcup B$. Now $\mu^{-1}(t) = \bigcup_{x \in X} W_x^t$.

Let $A' = \{x | W_x^t \subset A\}$ and $B' = \{x | W_x^t \subset B\}$. Since $W_x^t$ is connected, $A' \cup B' = \mu^{-1}(t)$.

First: $A' \bigcap B' = \varnothing$.

Assume not. Then $A' \bigcap B' \neq \varnothing$ so there exist a $z$ such that $W_z^t \subset A$ and $W_z^t \subset B$. But $W_z^t$ is connected and thus can not be contained in and intersect two disjoint separated sets, so $A' \bigcap B' = \varnothing$.

Second: $A'$ and $B'$ are separated. Let $y \in B'$ . We want to show that $y$ can not be limit point of $A'$. Notice for each continuum $K_\alpha$ in $A$, $y \notin K$ ( because if so then $W_y$ would intercept A and B). Therefore for each $K_\alpha$ in $X$ such that $K_\alpha \in A$ there exist open sets $U_\alpha$ and $V_\alpha$ in $X$ such that $K_\alpha \subset U_\alpha, y \in V_\alpha$ and $U_\alpha \bigcap V_\alpha = \varnothing$.

In $C(X)$ define open sets $U'_\alpha = R(U_\alpha)$. Then $A \subset \bigcup_{\alpha \in C} U'_\alpha$. A is compact in $C(X)$ so there exist finitely many of these open sets that cover $A$, namely $A \subset \bigcup_{i=1}^n U'_{\alpha_i}$ in $C(X)$.

$A' \subset \bigcup_{i=1}^n U_{\alpha_i}$. Look at the corresponding $V_{\alpha_i}$'s. Now $y \in \bigcap_{i=1}^n V_{\alpha_i}$; this is open and contains no points of $A'$. Thus $y$ is not a limit point of $A'$.

Therefore $A'$ and $B'$ are separated sets but $X = A' \bigcup B'$ which is a contradiction since $X$ is connected. Thus $\mu^{-1}(t)$ is a subcontinuum of $C(X)$. □

**Theorem 6.3.** *If $X$ is hereditarily indecomposable then $\mu^{-1}(t)$ is hereditarily indecomposable.*

*Proof.* First we want to show that $\mu^{-1}(t)$ is indecomposable. Note that if H and K are two subcontinua of X such that $H, K \in \mu^{-1}(t)$, then since $X$ is hereditarily indecomposable, $H \bigcap K = \varnothing$. Now assume that $\mu^{-1}(t)$ is not indecomposable; $\mu^{-1}(t) = A \bigcup B$ where $A$ and $B$ are two proper subcontinua. Let $A' = \{x | x \in H, H \in A\}$ and $B' = \{x | x \in K, K \in B\}$. Now $A' \bigcup B' = X$. Also $A'$ and $B'$ are proper subcontinua since there exists $H \in A \setminus B$ and $K \in B \setminus A$, and we know that $H \bigcap K = \varnothing$. But we have now shown that $X$ is the union of two proper subcontinua, which is a contradiction since $X$ is indecomposable.

Next let $M$ be a proper subcontinuum of $\mu^{-1}(t)$, then define

$M' = \{x | x \in H, H \in M\}$. $M'$ is a subcontinuum of $X$ and thus is indecomposable so the previous argument will show that $M$ must be indecomposable. $\square$

**Theorem 6.4.** *If $X$ supports a Whitney map $\mu$ then the order arcs are metric.*

*Proof.* We will use the same fact as in Theorem 5.1. Let $O$ denote the order arc. We want to find a continuous function f from $O \times O$ into $\mathbb{R}$ such that $f(H, K) = 0$ if and only if $H = K$. Define $f(H, K) = |\mu(H) - \mu(K)|$. First, since $O$ is an order arc it is obvious that $f(H, K) = 0$ if and only if $H = K$, so we need to show $f$ is continuous. Let $U$ be an open set such that $f(H, K) \in U$; there exist an $\epsilon > 0$ such that $f(H, K) \in [f(H, K) \pm \epsilon] \subset U$. Assume that $\mu(H) < \mu(K)$. Let $\delta = min\{\epsilon, \mu(K) - \mu(H)\}$. Since $\mu$ is continuous there exists an open set $V_H$ such that

$\mu(V_H) \subset [\mu(H) \pm \frac{\delta}{4}]$ and likewise there exists a $V_K$ such that

$\mu(V_K) \subset [\mu(K) \pm \frac{\delta}{4}]$.

Define $\widetilde{V} = V_H \times V_K$. If $(R, S) \in \widetilde{V}$ then $\mu(R) \in [\mu(H) \pm \frac{\delta}{4}]$ and $\mu(S) \in [\mu(K) \pm \frac{\delta}{4}]$. Thus $f(R, S) = |\mu(R) - \mu(S)| \in [|\mu(K) - \mu(H)| \pm \frac{\delta}{2}] \subset [|\mu(K) - \mu(H)| \pm \epsilon] = [f(H, K) \pm \epsilon] \subset U$. Therefore $f$ is continuous and thus $O$ is metric. $\square$

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