Factorwise rigidity involving hereditarily indecomposable spaces

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Factorwise rigidity involving hereditarily indecomposable spaces

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A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
December 19, 2008

Factorwise rigidity involving hereditarily

indecomposable spaces

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Kevin Brian Gammon, son of William and Brenda Gammon, was born on August 16, 1982. He attended Gordon Lee High School in Chickamauga, Georgia where he graduated seventh in his class in May, 2000. He entered Berry College under an academic scholarship and graduated Magna cum Laude in May, 2004 with a Bachelor of Science degree in Mathematics. He then entered Auburn University in August, 2004 and was awarded a Master of Science degree in 2006. He then enrolled in the Doctorate of Philosophy program at Auburn University.

Dissertation Abstract

Factorwise rigidity involving hereditarily indecomposable spaces

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Doctor of Philosophy, December 19, 2008
(M.S., Auburn University, 2006)
(B.S., Berry College, 2004)

60 Typed Pages

Directed by Krystyna Kuperberg

The Cartesian product of two spaces is called factorwise rigid if any self homeomorphism is a product homeomorphism. In 1983, D. Bellamy and J. Łysko proved that the Cartesian product of two pseudo-arcs is factorwise rigid. This argument relies on the chainability of the pseudo-arc and therefore does not easily generalize to the products involving pseudo-circles. In this paper the author proves that the Cartesian product of the pseudo-arc and pseudo-circle is factorwise rigid.

Acknowledgments

The author would like to thank his advisor, Krystyna Kuperberg, for her patience and guidance. He would also like to thank the members of his advisory committee for their useful suggestions and corrections during the course of this research. The author would also like to thank the faculty and staff at Berry College for encouraging him to pursue mathematics. The author is also would like to recognize his family for their continued encouragement throughout the years.

Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliography follows van Leunen’s *A Handbook for Scholars.*

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

The following dissertation focuses primarily on two topological spaces. The first, the pseudo-arc, was originally discovered by B. Knaster in [19] in 1922. In 1948, E.E. Moise constructed a pseudo-arc as an indecomposable continuum homeomorphic to each of its non-degenerate subcontinua [34]. He was the first person to use the term pseudo-arc because the arc also has this property. Moise believed, but did not prove, that the hereditarily indecomposable continuum given by B. Knaster in 1922 is a pseudo-arc. In 1948, R.H. Bing [3] proved that Moises example is homogeneous. In 1951, Bing [4] proved that every hereditarily indecomposable chainable continuum is a pseudo-arc and that all pseudo-arcs are homeomorphic. In an attempt to classify homogeneous planar continua, Bing [5] gave another characterization of the pseudo-arc in 1959 as a non-degenerate homogeneous chainable continuum. The pseudo-arc has been the subject of many interesting research questions. The history of many other aspects of the pseudo-arc can be found in survey papers by W. Lewis [30] and [31].

The second space which will be discussed is the pseudo-circle. In 1951, Bing [4] described the pseudo-circle as a planar hereditarily indecomposable circularly chainable continuum which separates the plane. From this definition, it is apparent that every proper subcontinuum of the pseudo-circle is a pseudo-arc. Through a series of papers, L. Fearnley also proved that the pseudo-circle is unique ([9], [10], [12], [13]).

It has been shown by L. Fearnley in [11] and J. T. Rogers, Jr. in [40] that the pseudo-circle is not homogeneous. This answered the question of whether a continuum in which every subcontinuum was homogeneous must itself be homogeneous.

The purpose of the dissertation is to explore the factorwise rigidity on the Cartesian product of the pseudo-arc and pseudo-circle. Factorwise rigidity has also been studied in spaces with a more well behaved local structure. In [25], K. Kuperberg, W. Kuperberg, and W. Transue proved that the Cartesian product of two Menger universal curves is factorwise rigid. This result was later extended to products whose factors consisted of a combination of Menger universal curves and Sierpiński universal curves by J. Phelps [37]. The question of whether pseudo-arcs have this property is due to W. Lewis [27]. This was answered by D. Bellamy and J. Łysko in [8] and extended to arbitrary products of pseudo-arcs in [7] by D. Bellamy and J. Kennedy. As a result, it has been asked by W. Lewis in [29] if the Cartesian product of any hereditarily indecomposable spaces has this property.

The second chapter of this dissertation contains the definitions and background information required to understand the main result. This includes the definitions of a pseudo-circle and a pseudo-arc. The author assume that the reader has a basic background in topology. For a more in depth introduction to topology and covering spaces than that which is presented, the author recommends the introductory topology book by J. Munkres [35].

The third chapter discusses covering spaces of the pseudo-circle. Since the pseudo-circle is neither path connected or locally path connected the usual theorems regarding covering spaces do not apply. This chapter explores how a sequence of circularly crooked chains lift to a connected covering space. This chapter offers an alternative proof of a result due to J. Heath [16] which states that the k-fold connected covering space of a pseudo-circle is a pseudo-circle. In [16], J. Heath focused on properties of confluent maps and not crooked chains to prove this results. Using the methods developed in the alternative proof one can also easily prove a result of D. Bellamy and W. Lewis [6] which states that a Hausdorff two point compactification of the infinite connected covering space of a pseudo-circle is a pseudo-arc.

The fourth chapter illustrates a creative use of the covering spaces developed in Chapter 3. These covering spaces allow for a very short and accessible proof of a well known result: the pseudo-circle is not homogeneous. The original proofs of this result are due to L. Fearnley in [11] and J. T. Rogers, Jr. in [40]. The result also follows from more general theorems by other authors in [15], [18], [28], and [38]. Chapter 4 is joint work with K. Kuperberg discovered while discussing the research involved in this dissertation. It is originally published in the Proceedings of the American Mathematical Society [23].

Chapter 5 contains the main result of this dissertation: the Cartesian product of the pseudo-arc and pseudo-circle is factorwise rigid. It is known that the Cartesian product of two pseudo-arcs is factorwise rigid. As previously mentioned, this result is due to D. Bellamy and J. Łysko in [8]. Since the pseudo-arc and pseudo-circle share many properties it was suspected that the result could be generalized to include pseudo-circles. However, the proof developed by D. Bellamy and J. Łysko relied on the fact that the pseudo-arc is chainable while the pseudo-circle does not have this property. D. Bellamy and J. Kennedy later extended this result to the arbitrary product of pseudo-arcs. This proof requires the fact that the pseudo-arc is homogeneous. It is not known at this time if the main theorem in this chapter can be extended to arbitrary products of pseudo-arcs and pseudo-circles.

In Chapter 6, the author includes other observations made during the research of the main result. These observations include some results on factorwise rigidity where one factor is hereditarily indecomposable. This chapter also includes a generalization of a result due to K. Kuperberg in [21]. In this paper, K. Kuperberg creates homogenous spaces by making certain identifications on the Menger Universal curve. These spaces are topologically distinct from the Cartesian product of Menger manifolds. The author explores this result using higher dimensional Menger manifolds. These manifold are another example of how factorwise rigidity relates the study of homogeneous continua. The question whether every homogeneous space is bihomogeneous was originally raised by B. Knaster approximately around 1921.

The question was restated to continua in 1930 by D. van Dantzig. The previous mentioned example by K. Kuperberg in [21] is locally connected. G. Kuperberg [20] constructed another in order to make an example of a homogeneous, non-bihomogeneous Peano continuum which is both simpler and of lower dimension than that described by K. Kuperberg in [21]. The example constructed by G. Kuperberg uses the notion developed in [25] that certain Cartesian products with the Menger manifolds as one of the factors has a certain rigidity which must be preserved by homeomorphisms. Several of these results depend on the characterization of the Menger Curve developed by R.D. Anderson and k-dimensional Menger compacta developed by M. Bestvina in [1] and [2], respectively. Another example was given by Minc in [33] of a homogeneous, non-bihomogeneous continuum. However, this example is not locally connected.

Chapter 2

Definitions and preliminary information

All topological spaces in this dissertation will be metric spaces. It will also be assumed that any sets are subsets of a metric space. A topological space is *compact* provided that every open cover has a finite subcover. A space X is *connected* if X is not the union of two disjoint sets which are both open and closed.

A *continuum* is a compact connected metric space. Unless specifically stated otherwise, it will also be assumed that a continuum is non-degenerate. If $A \subset X$ and A is a continuum, A is called a subcontinuum of X. A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable. The following is a useful, well known Lemma regarding hereditarily indecomposable continua:

Lemma 2.1. *If X is hereditarily indecomposable and W, M are two subcontinua of X such that $W \cap M \neq \emptyset$, then $W \subset M$ or $M \subset W$.*

If X is a continuum and $x \in X$, the *composant of x in X* is the union of all proper subcontinua of X which contain the point x. The composant of x will be denoted by $K(x)$. Note that an indecomposable space has uncountably many pairwise disjoint composants (see [26] Theorem 7, page 212.) In a indecomposable continuum, any two points in the same composant are contained in a proper subcontinuum.

A homeomorphism $h : X \times Y \to X \times Y$ is called a *product homeomorphism* if $h(x,y)$ can be written as

1. $h(x,y) = (f(x), g(y))$ where $f : X \to X$ and $g : Y \to Y$ are homeomorphisms or
2. $h(x,y) = (f(y), g(x))$ where $f : Y \to X$ and $g : X \to Y$ are homeomorphisms.

If h is a product homeomorphism, such as Case 2, h will often be written as $h = (f,g)$.

The Cartesian product $X \times Y$ of two continua is called *factorwise rigid* provided that if $h : X \times Y \to X \times Y$ is a homeomorphism, then h is a product homeomorphism.

A space is *k-homogeneous* for some integer $k > 0$ provided that given any two collections consisting of k distinct points there is a self homeomorphism which maps one collection onto the other. A *homogeneous* space is a 1-homogeneous space. The study of k-homogeneity and factorwise rigidity are closely related. For example, a Cartesian product which is factorwise rigid can not be k-homogeneous for any $k > 1$.

A *chain* is a finite collection of open sets $U = \{u_1, u_2, \cdots u_m\}$ such that $u_j \cap u_k \neq \emptyset$ if and only if $|i-j| \leq 1$. If U is a chain, then the subchain of U consisting of the links $\{u_i, \cdots, u_k\}$ will be denoted by $U(i,k)$. An ϵ-chain is a chain in which each link has diameter less than ϵ. A continuum X is *chainable* if given any $\epsilon > 0$, there exists a ϵ-chain covering X. The following is a well known theorem (see, for example, section 2.5 and 12.5 of [36]).

Theorem 2.2. *The following conditions are equivalent for a continuum X:*

1. *X is chainable.*

2. *X can be written as the inverse limit of arcs.*

3. *For every $\epsilon > 0$, there exists an ϵ-map from X into an arc.*

A space X is *chainable between the points p and q* provided that p and q are elements of X and X is chainable in such a way that p is always in the first link and q is always in the last link. The following is a well known Lemma:

Lemma 2.3. *If X is chainable between p and q, then no proper subcontinuum of X contains p and q.*

X is said to be *irreducible between the points p and q.*

A chain $E = \{e_1, e_2, \cdots, e_m\}$ is *crooked inside of the chain* $D = \{d_1, d_2, \cdots, d_m\}$ if the following are true:

1. Every link of E is contained inside of a link of D and

2. If e_j and e_k are contained inside of d_J and d_K, respectively, where $|J - K| > 3$ then the subchain $E(j, k)$ can be written as the union of three proper subchains $E(j, r), E(r, s)$, and $E(s, k)$ where $(s - r)(k - j) > 0$ and e_r is in the link of $D(J, K)$ adjacent to d_K and e_s is in the link of $D(J, K)$ adjacent to d_J.

This definition is due to R.H. Bing [3]. Figure 2.1 gives an example of two chains. The first chain, D_1, is the larger chain consisting of the large circular links. The second chain, D_2, is a finite covering of the arc drawn inside of D_1 using connected open sets so that D_2 is contained inside of D_1.

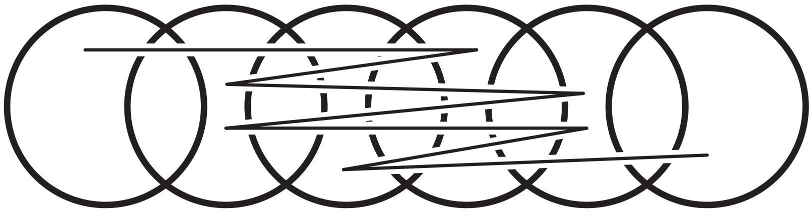

Figure 2.1: An example of a crooked chain.

Let $\mathbb{Z}$ denote the integers. A *circular chain* $U = \{u_i\}_{i\in\mathbb{Z}}$ is a collection open sets so that for some positive $n \in \mathbb{Z}$, where $u_i = u_j$ if and only if $i \bmod n = j \bmod n$ and $u_i \cap u_j \neq \emptyset$ if and only if there exists a $k \in \mathbb{Z}$ so that $u_i = u_k$ and $|k - j| \leq 1$. A circular ϵ-chain is a circular chain in which each link has diameter less than ϵ. A continuum X is *circularly chainable* if given any $\epsilon > 0$, there exists an circular ϵ-chain covering X. The following theorem is well known. Again, the details can be found in the reference book [36].

Theorem 2.4. *The following conditions are equivalent for a continuum X:*

1. *X is circularly chainable.*

2. *X can be written as the inverse limit of simple closed curves.*

3. *For every $\epsilon > 0$, there exists an ϵ-map from X into a simple closed curve.*

The number of distinct links in a chain or circular chain U will be called the *length* of U. If U is a chain or circular chain of length n, a proper subchain F of U is a chain whose links are links of U and whose length is less than n.

Let F be a circular chain contained inside of the circular chain U where U has length n. Suppose that F_1 is a proper subchain of F so that for some fixed

j, F_1 has a link that intersects u_j and if F_1 has a link which intersects u_m, then $j \bmod n \leq m \bmod n$. Next, suppose that F_1 intersects a link u_k such that if F_1 has a link which interests u_l for some l this implies $j \bmod n \leq l \bmod n \leq k \bmod n$. If k is the least such integer greater than j which satisfies these conditions, then F_1 is said to have *span* $|k - j|$ inside of U.

The circular chain E is *crooked* inside the circular chain D if given any proper subchain F of D, each chain of E contained inside of F is crooked inside of F. This definition is also due to R.H. Bing [4]. The following illustration (Figure 2.2) gives an example of two circular chains. The first chain, D, is represented by the large links and the second chain, E, is a finite covering of the arc drawn inside of the picture using connected open sets which are contained inside of the first circular chain. The smaller chain is crooked inside of the larger chain.

In order to check that the E is crooked inside of the chain D, remove a link from D (see Figure 2.3) to create a chain F. Then check each chain inside of E which is contained inside of F to see if it is crooked inside of F. In the following picture, any chain of E which passes through enough links of F to not be trivially crooked must pass through the subchain emphasized by the red links. The chain emphasized by the red links is crooked inside of F, therefore any subchain of E contained inside of F is crooked inside of F.

A *pseudo-arc* is any non-degenerate hereditarily indecomposable chainable continuum. The reader should see Chapter 1 for more details on the history of the

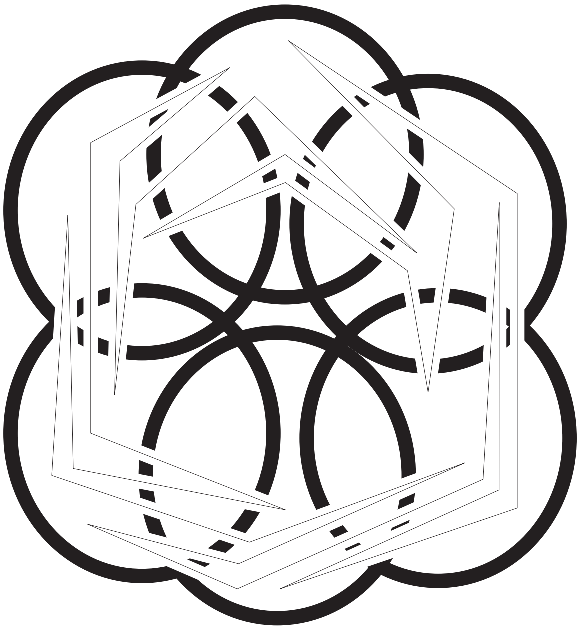

Figure 2.2: A circular chain which is crooked inside another circular chain.

pseudo-arc. In [3], Bing described the pseudo-arc as the intersection of chains D_i between two points p and q satisfying the conditions that

1. D_{i+1} is crooked inside of D_i
2. D_i is an ϵ_i-chain
3. ϵ_i approaches zero as i increases without bound.

Through the remainder of this paper, P will be used to denote the pseudo-arc.

A *pseudo-circle* is a hereditarily indecomposable circularly chainable non-chainable continuum which is emendable inside of the plane. The pseudo-circle was described by R.H. Bing in [4] as a hereditarily indecomposable continuum which separates the

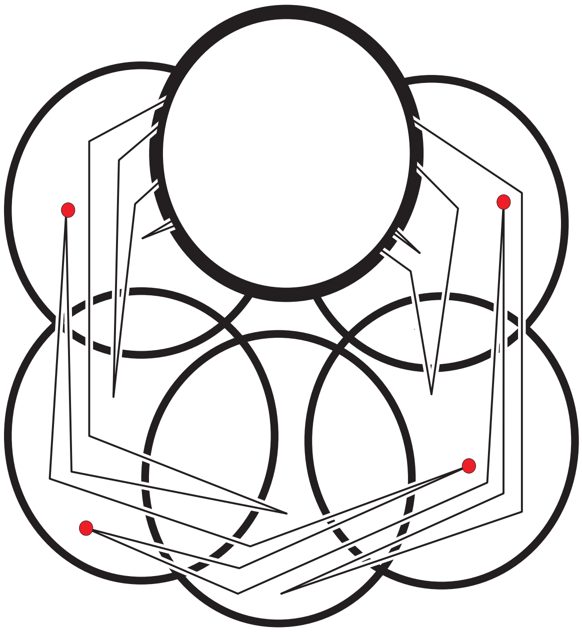

Figure 2.3: Checking the conditions for a circular crooked chain.

plane. In terms of circular chain, Bing described this space as the intersection of circular chains D_i where

1. D_{i+1} is crooked inside of D_i

2. D_{i+1} has winding number ± 1

3. D_i is an ϵ_i-chain

4. ϵ_i approaches zero as i increases without bound.

Throughout the paper, C will denote the pseudo-circle.

The finally chapter briefly explores Menger manifolds. Given n, let K be a PL-manifold of dimension $2n+1$. Let $X_1 = K$. For $i > 1$ define X_i to be a regular neighborhood of the n-skeleton of a triangulation of X_{i-1}. Then $\mu^n_K = \cap_i X_i$ is called a *n-dimensional Menger manifold*.

Chapter 3

Covering spaces of the pseudo-circle

It has been shown by J. Heath [16] that the connected k-fold covering space of a pseudo-circle is itself a pseudo-circle. This proof involved using properties of confluent mappings and did not focus on the lifting of circularly crooked chains. In this chapter it will be shown that given a sequence of circular chains defining a pseudo-circle there is a specific subsequence of circular chains such that the inverse image under a 2^k-fold covering map produces a pseudo-circle. This alternative technique used to prove the result of J.Heath provides extra insight to covering spaces of pseudo-circles that can be used in other applications.

3.1 The connected k-fold covering space of a pseudo-circle

Throughout this chapter, let $\{D_i\}_{i\geq 0}$ will be a collection of circular chains $D_i = \{d_j^i\}_{j\in\mathbb{Z}}$ contained inside of a planar annulus which consists of connected open sets satisfying the following conditions:

1. D_0 contains at least 6 links
2. D_{i+1} is crooked inside of D_i
3. D_{i+1} has winding number 1 inside of D_i
4. $d_0^{(i+1)}$ is contained inside of d_0^i

The first assumption is used to avoid trivialities. The second and third assumptions are typical when describing a pseudo-circle. The fourth assumption is used to ease notation in the following proofs. The length of D_i will be denoted by $n(i)$.

Let p denote the 2-fold covering map from the annulus A onto itself. Denote $p^{-1}(D_i)$ by $F_i = \{f^i_j\}_{j\in\mathbb{Z}}$ and assume that F_i is enumerated so that $p(f^i_j) = d^i_j$. Then F_i is a circular chain of length $2n(i)$ where $p(f^i_j) = p(f^i_k)$ if and only if $j \bmod n(i) = k \bmod n(i)$. It will be shown that in the sequence $\{D_i\}_{i\geq 0}$, as n grows without bound, the span of proper subchains of D_{i+n} becomes so large inside of D_i that for some N, the inverse image of D_{i+N} must be crooked inside of the inverse image of D_i.

When considering the inverse image of D_{i+1} inside of the inverse image of D_i, there is a minimum number of links in $p^{-1}(D_i)$ that one subchain $U \subset p^{-1}(D_{i+1})$ of length $n(i+1)$ must intersect. The following two lemmas find this number by constructing a specific proper subchain of D_{i+1} which has a large span inside of D_i.

Lemma 3.1. *There is a subchain $V = \{v_1, v_2, \cdots, v_m\}$ of F_{i+1} such that*

1. *V contains the link f^{i+1}_0*

2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+1}.*

3. *$p(v_i) = p(v_j)$ if and only if $i = j$.*

4. *V has span at least $2n(i) - 3$ inside of F_i.*

Proof. Since D_{i+1} has winding number 1 inside of D_i, there exists a proper subchain $F_{i+1}(j, m)$ so that $0 < j < m < n(i+1) - 1$, f^{i+1}_m intersects $f^i_{n(i)-1}$, f^{i+1}_j intersects f^i_1, and $F_{i+1}(j,m)$ is contained inside of $F_i(1,n(i)-1)$. Since the chain $p(F_i(1,n(i)-1)$ is a proper subchain of D_i and $F_{i+1}(j,m)$ is contained inside of $F_i(1,n(i)-1)$, the chain $F_{i+1}(j,m)$ must be crooked inside of $F_i(1,n(i)-1)$. This implies that $F_{i+1}(j,m)$ can be written as the union of three subchains

1. $F_{i+1}(j,k)$ where $f_j^{i+1} \cap f_1^i \neq \emptyset$ and $f_k^{i+1} \subset f_{n(i)-2}^i$

2. $F_{i+1}(k,l)$ where f_k^{i+1} is as above and $f_l^{i+1} \subset f_2^i$

3. $F_{i+1}(l,m)$ where f_l^{i+1} is as above and $f_m^{i+1} \cap f_{n(i)-1}^i \neq \emptyset$

where $0 < j < k < l < m$. Let r be an integer such that $-n(i+1) < r < 0$ and $r \bmod n(i+1) = l \bmod n(i+1) = l$.

The chain V will consist of the links $F_{i+1}(r,k)$. The chain $p(V)$ is proper because it does not contain each link of $p(F_{i+1}(k,l))$.

A chain of F_i which contains $F_{i+1}(r,-1)$ must contain at least $n(i)-2$ links. Likewise, a chain of F_i which contains $F_{i+1}(0,k)$ must contain at least $n(i)-1$ links. Therefore, V intersects every link of a subchain of F_i which contain at least $2n(i)-3$ links.

$\square$

The chain V mentioned in the above proof has an additional property that will be used in subsequent proofs. As mentioned previously, the lift of $p(V)$ consist of two distinct, disjoint chains. Each of which intersects all but at most three links of F_i. Since F_i contains at least 12 distinct links, there must be at least 6 links which both of these chains intersect. In particular, the following corollary is true:

Lemma 3.2. *Let V be the chain described in Lemma 3.1. Then there exists a subchain G of D_i consisting of three adjacent links so that for each link g of $p^{-1}(G)$, both chains of $p^{-1}(p(V))$ have a link contained inside of g.*

Lemma 3.3. *For any $l \in \mathbb{Z}$, there is a proper subchain $V = \{v_1, v_2, \cdots, v_m\}$ of F_{i+1} such that*

1. *V contains the link f_l^{i+1}*
2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+1}.*
3. *$p(v_j) = p(v_k)$ if and only if $j = k$.*
4. *V has span at least $2n(i) - 3$ inside of F_i.*

Proof. The chains D_i, D_{i+1}, F_i and F_{i+1} may be renumbered so that Lemma 3.1 may be applied. □

The following two lemmas show that one proper subchain of length $n(i+2)$ in the inverse image of D_{i+2} must intersect every link in the inverse image of D_i. This is done by applying the previous lemma to the circular chains D_{i+1} and D_{i+2}.

Lemma 3.4. *There is a subchain V of F_{i+2} containing the link f_0^{i+2} such that*

1. *V intersects each element of F_i*
2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+2}*
3. *$p(v_j) = p(v_k)$ if and only if $j = k$.*

Proof. Let V_1 be a subchain of F_{i+1} as described in Lemma 3.3 chosen in such a way that d_0^i is the middle link of a chain G as described in Corollary 3.2. Next, apply Lemma 3.1 to the link f_0^{i+2} and the circular chain F_{i+1} to obtain a chain V which intersects all but at most three elements of F_{i+1}.

Notice that since $d_0^{(i+2)} \subset d_0^i$ and d_0^i is the middle link of the chain G, the three links which V may not intersect in F_{i+1} must be contained inside of $p^{-1}(G)$. However, since V must intersect the other links of both chains of $p^{-1}(p(V_1))$, it follows that V must still intersect every element of F_i. □

Lemma 3.5. *For $l \in \mathbb{Z}$, there is a subchain V of F_{i+2} containing the link f_l^{i+2} such that*

1. *V intersects each element of F_i*
2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+2}*
3. *$p(v_j) = p(v_k)$ if and only if $j = k$.*

Proof. The circular chains D_{i+1}, D_{i+2}, F_{i+1}, and F_{i+2} may be renumbered so that Lemma 3.4 may be applied. □

The following Theorem uses the large span of proper subchains in D_{i+2} to show that the inverse image of D_{i+3} must be crooked inside of the inverse image of D_i.

Theorem 3.6. *F_{i+3} is crooked inside of the circular chain F_i.*

Proof. Let E be a proper subchain of F_i and let G be a subchain of F_{i+3} which is contained inside of E. Let H be a subchain of F_{i+2} which contains G. From Lemma

3.5, H is contained inside of a chain in the lift of a proper subchain of D_{i+2} which intersects each element of F_i. Hence G must be crooked inside of H and therefore also crooked inside of E. $\square$

Theorem 3.7. *The sequence of circular chains $\{F_{3(i)}\}_{i\geq 0}$ defines a pseudo-circle. In particular, the connected 2-fold cover of the pseudo-circle is a pseudo-circle.*

Proof. This is a consequence of Theorem 3.6. $\square$

The remaining theorems in this section are used to extend the previous result to n-fold covering spaces for $n > 2$.

Theorem 3.8. *If $p : A \to A$ denotes the 2^k-fold covering of the annulus onto itself, then the sequence of circular chains $\{F_{3^k(i)}\}$ defines a pseudo-circle. In particular, the connected 2^k-fold covering of the pseudo-circle is a pseudo-circle.*

Proof. This follows from the fact that the 2^k-fold covering space is a 2-fold covering space of the $2^{(k-1)}$-fold covering space. $\square$

This leads to the following alternative proof of J. Heath's result originally presented in [16]:

Corollary 3.9. *Let p be j-fold covering map of the annulus to itself, where $2^k < j \leq 2^{k+1}$ for some k. Then for each i, there exists a n such that $3^k(i) < n \leq 3^{k+1}(i)$ and F_n is crooked inside of F_i. In particular, the j-fold connected covering space of the pseudo-circle is a pseudo-circle.*

3.2 Additional Remarks

As in the previous section, let $p : A \to A$ be the 2-fold covering map of the annulus onto itself. A simple example shows that given a sequence of circular chains $\{D_i\}$ defining a pseudo-circle, $i+1$ and $i+2$ will not necessarily produce circular chains whose inverse image is crooked inside of $p^{-1}(D_i)$. In Figure 1, D_0 is represented by large circular links and has length 6. D_1 consists of the smaller links. The first link of D_1 is drawn as a solid black link to easily distinguish where the circular chain begins to repeat. D_2 is not entirely graphed. It consist of a chain which uses the minimal number of connected links in order to be crooked inside of D_1 with one additional property:

Assume that D_2 is enumerate so that increasing the index corresponds to a positive orientation inside of D_1. In the figure, the small dots labeled by p, q, r, and s are links of D_2. Let s be the first link by increasing index which intersects d_3^0 and let t be the first link which intersects d_4^0. Let p be the last link between 1 and t which intersects d_1^0. By the minimality of D_2, this implies that 0 < s < p. The context in which the letter p is used will easily distinguish between the link p and the map p. Consider removing the gray link of D_1 in Figure 1. In order to be crooked, the a subchain of D_2 whose first link is labeled by p and last link is labeled by s must be able to be written as the union of three subchains of D_2: One that will go from the link p to the link q, where q is a subset of d_2^0, one from the link q to link r, where r is a subset of d_2^0, and then from link r to link s. Denote this subchain by $D_2(p, s)$. The additional property that D_2 requires is that q does not intersect d_3^0.

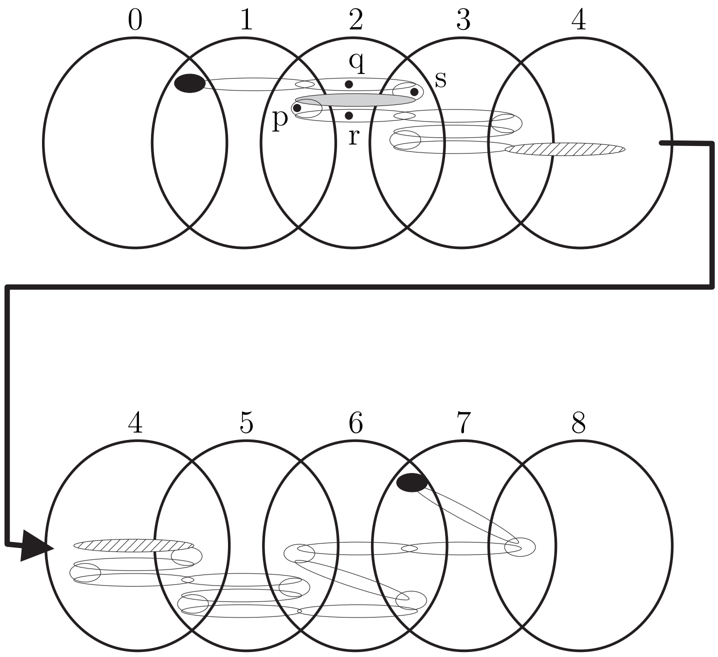

Figure 3.1: Circular chains in the construction of a pseudo-circle

Figure 2 shows the lift of the circular chains D_0 and D_1 to the 2-fold covering space of the annulus.

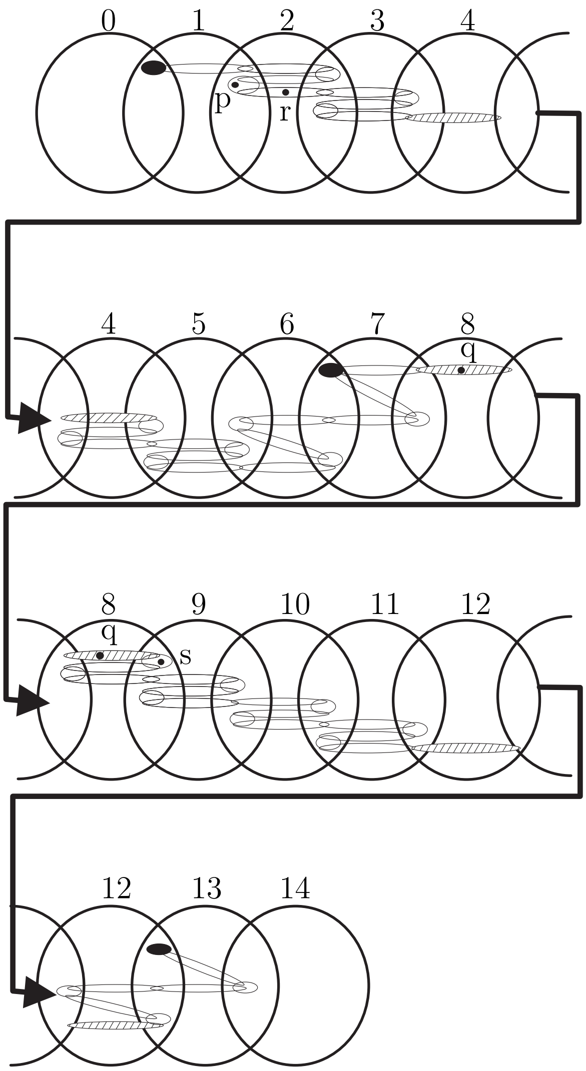

Figure 3.2: The lift of circular chains

Notice that removing link number 11 in Figure 2 provides a proper subchain, F, of $p^{-1}(D_0)$. Let G be a subchain of $p^{-1}(D_2)$ containing the indicated lift of $D_2(1,s)$ which also contains a link which intersects the link 10 of $p^{-1}(D_0)$. In order to be crooked inside of F, G would first have to travel to the 9th link of $p^{-1}(D_0)$, then back to the link 2, and then to the link 10. However, since D_2 was chosen to used the least amount of links possible in order to be crooked inside of D_1 and q does not intersect d_3^0, it is only possible to reach the 8th link and still be able to return to link 2. This can be done by considering the lift of $D_2(p,s)$.

Therefore, G can no be crooked inside of F and $p^{-1}(D_2)$ is not crooked inside of $p^{-1}(D_0)$. This also implies that $p^{-1}(D_1)$ is not crooked inside of $p^{-1}(D_0)$.

3.3 The infinite, connected covering space of a pseudo-circle

The methods of this proof can also be used to provide more insight into a result due to D. Bellamy and W. Lewis in [6] which states that the Hausdorff two point compactification of the infinite, connected covering space of the pseudo-circle is a pseudo-arc. The proof provided by D. Bellamy and W. Lewis uses a specific construction of the pseudo-circle which controls the span of the proper subchains of D_{i+1} inside of D_i. While the underlying idea of the following proof is the similar to the original proof in [6], the author utilizes the methods developed in section 2 to avoid a specific construction of the pseudo-circle and provide more detail to the proof developed by Bellamy and Lewis.

In the following, let $\widetilde{A}$ denote the universal covering space of the annulus with covering map p and $\hat{A}$ the two points compactification of $\widetilde{A}$ obtained by adding points a and b. Then $\widetilde{A}$ contains an infinite, connected covering space of the pseudo-circle. Let $\{D_i\}_{i\geq 0}$ be a sequence of circular chains defining a pseudo-circle satisfying the four conditions listed in Section 2.

Theorem 3.10. *The two points compactification of the infinite, connected covering space of the pseudo-circle is a pseudo-arc.*

Proof. For each i, $p^{-1}(D_i)$ is an infinite chain consisting of infinitely many copies of D_i. Assume, without loss of generality, that proceeding through the links of $p^{-1}(D_i)$ in the direction of a corresponds to traveling through $D(i)$ with a negative orientation.

Arbitrarily select a point $x \in p^{-1}(C)$ such that d_1^0 contains $p(x)$ and select a copy of D_0 in $p^{-1}(D_0)$ which contains x in the first link. Denote this copy by E_0^0. Then E_{-1}^0 will consist of the copy of D_0 that intersect E_0^0 and travels towards a and E_1^0 will consist of the copy of D_0 that intersect E_0^0 and travels towards b. In general, number the copies of D_0 inductively by subtracting one while moving towards the point a and adding one while moving towards the point b.

Let F_0 be the chain from a to b whose links consist of the links of E_0^0 except the first link and E_1^0 except the last two links (See Figure 3). The neighborhood of a will consist of the union of the elements of those chains E_i where $i < 0$ and the first link of E_0^0. The neighborhood of b will consist of the union of those copies of E_i where $i > 1$ and the last two links of E_1. Then this chain has length $2n(0) - 1$, which is one less than the length of the 2-fold cover of D_0.

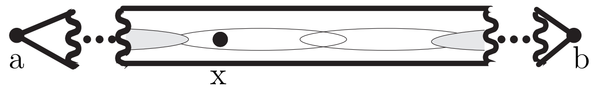

Figure 3.3: First approximation of the infinite covering space

In a similar fashion, let E_0^1 be a copy of D_3 contained inside of $p^{-1}(D_3)$ whose first link is contained inside of the first link of E_0^0. The copies of D_3 will be enumerated inductively similar to the copies of D_0. Let F_1 be a chain from a to b whose links consist of the links of E_{-1}^1 except the first link, the links of E_0^1, the links of E_1^1, and the links of E_2 except the last two links. The links of containing a and b are defined in a similar fashion to those in F_0. Applying the proof of Theorem 2, F_1 is crooked inside of the chain F_0. Notice that F_1 has length $4n(3) - 1$ which is one less than the length of the 4-fold covering of D_3.

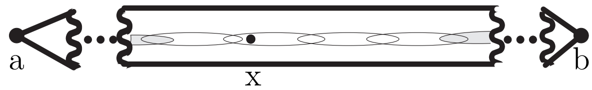

Figure 3.4: Second approximation of the infinite covering space

In general, if F_i has already been constructed using $p^{-1}(D_j)$ for some j, then F_{i+1} will consist of 2^{i+1} copies of $D_{(3^i+j)}$ selected in a similar fashion as those in F_1. Neighborhoods of a and b are also constructed in a similar fashion. Again, by the proof of Theorem 2, F_{i+1} is crooked inside of the chain F_i. Notice that F_{i+1} will have length $2^{(i+1)}n(3^i + j) - 1$ which is one less than the length of the $2^{(i+1)}$-fold cover of $D_{(3^i+j)}$.

Since the mesh of the links of F_i goes to zero as i increases without bound, if follows that $\cap F_i$ is a pseudo-arc. $\square$

Chapter 4

An application of the covering spaces of the pseudo-circle

The following work is joint work with K. Kuperberg and is originally published in the Proceedings of the American Mathematical Society [23]. It provides an interesting application of the infinite covering space of the pseudo-circle described in the previous chapter. The author would like to thank D. Bellamy, W. Lewis, and J. T. Rogers for their useful comments on the results presented in this chapter.

Let A be an annulus and $\widetilde{A}$ be the universal covering space of A with projection p. Let $\widehat{A}$ be the two-point compactification of $\widetilde{A}$ and denote the two added points of the compactification by a and b. Throughout this chapter, consider the pseudo-circle C to be essentially embedded inside of the annulus A. As in the previous Chapter, the infinite connected covering space of C contained in $\widetilde{A}$ will be denoted by $\widetilde{C}$.

4.1 Lifting homeomorphisms to the covering space

Since the pseudo-circle is neither path connected nor locally path connected, the usually Theorems regarding liftings of continuous maps to covering spaces do not apply. In this section we will show how using covering spaces of nice spaces such as the annulus can be used to derive similar lifting lemmas for complicated spaces. This idea will also be used in the following Chapter.

Lemma 4.1. *Let $f : C \to C$ be a homeomorphism. For any $\widetilde{x} \in \widetilde{C}$ and $\widetilde{y} \in p^{-1}(f(p(\widetilde{x})))$ there is a map $\widetilde{f}$ such that the diagram*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{C} \arrow[d, "p"] \\ C \arrow[r, "f"'] & C \end{tikzcd}$$

commutes and $\widetilde{f}(x) = y$.

Proof. Since the annulus is an Absolute Neighborhood Retract, f can be extended to a continuous map $F : U \to A$, where U is a closed, connected annular neighborhood of the pseudo-circle. Let r be a retraction of the annulus onto U. Then $F \circ r$ is a map from the annulus into itself. Since $F \circ r$ agrees with f on the pseudo-circle C, the map $F \circ r$ induces an isomorphism of the fundamental group of A. Therefore, a lift of $F \circ r$ exists which maps $\widetilde{A}$ into $\widetilde{A}$ (see Theorem 16.3 in [17].) Denote the restriction of $F \circ r$ to $p^{-1}(C)$ by $\widetilde{f}$. The commutativity of the diagram holds because $F \circ r$ agrees with f on the pseudo-circle. $\square$

Let $P = \widetilde{C} \cup \{a, b\}$, a two-point compactification $p^{-1}(C)$. As mentioned in Theorem 3.10, this compactification is a pseudo-arc. Then $\tilde{f}$ extends uniquely to a map H from P to P.

Lemma 4.2. *H is a homeomorphism from P to P.*

Proof. Since $\widetilde{A}$ is the universal covering of A and $F \circ r$ induces an isomorphism of the fundamental group, the lift of $F \circ r$ maps fibers bijectively onto fibers (see for example Theorem 54.4 of [35]). Therefore $\tilde{f}$ maps fibers bijectively onto fibers. Since f is a homeomorphism and the diagram in Lemma 4.1 commutes, it follows that $\tilde{f}$ is a bijection. Therefore, the unique extension is also a bijection. Since H is a continuous bijection between continua, H is a homeomorphism. $\square$

It is important to note that the homeomorphism in Lemma 4.2 has the property that the set $\{a, b\}$ is invariant.

4.2 Proof of non-homogeneity of the pseudo-circle

Theorem 4.3. *The pseudo-circle is not homogeneous.*

Proof. Let $K(a)$ and $K(b)$ be the composants of a and b, respectively, in the pseudo-arc P. Let $\widetilde{x}$ and $\widetilde{y}$ be two points in P such that $\widetilde{x} \in (K(a) \cup K(b)) - \{a, b\}$ and $\widetilde{y} \in P - (K(a) \cup K(b))$. If C were homogeneous, then there would be a homeomorphism h of the pseudo-circle such that $h(x) = y$. Therefore, the induced map H as described in Lemma 4.2 maps the set $p^{-1}(x)$ onto $p^{-1}(y)$ and leaves the set $\{a, b\}$ invariant. Since $\widetilde{C}$ is contained inside of the universal covering of the annulus, given any two points in $p^{-1}(y)$ there exists a deck transformation which maps one onto the other. This deck transformation extends uniquely to a homeomorphism of P onto P and leaves the set $\{a, b\}$ invariant. In particular, there is a homeomorphism which maps $\widetilde{x}$ onto $\widetilde{y}$ and leaves the set $\{a, b\}$ invariant.

However, if the set $\{a, b\}$ is invariant under the homeomorphism, then $K(a) \cup K(b)$ would also be invariant. Therefore, this is a contradiction. $\square$

The use of a deck transformation induced by the universal covering space of the annulus can be used to show another interesting result related to the structure of the fibers of the covering space of the pseudo-circle.

Theorem 4.4. *If for some $x \in C$, the composant $K(a)$ intersects the fiber $p^{-1}(x)$, then it contains $p^{-1}(x)$.*

Proof. If $y \in p^{-1}(x) \cap K(a)$, then by the definition of a composant, there is a proper subcontinuum W of P that contains both a and y. Let f be a deck transformation such that $p^{-1}(x) = \{f^n(y)\}_{n\in\mathbb{Z}}$, $\mathbb{Z}$ being the set of integers. Denote by F the extension of f to P. The set $W_n = F^n(W)$ is a continuum containing a and $f^n(y)$. Thus $p^{-1}(x) \subset K(a)$. $\square$

Chapter 5

The cartesian product of the pseudo-arc and pseudo-circle is factorwise rigid

In the following, the projection from a Cartesian product $A \times B$ to the first factor space will be denoted by π_1. Likewise, π_2 will denote the projection to the second factor space. $\check{H}_1(Y)$ will denote the first Čech homology group of the space Y.

Let G be a relation on $P \times P$ which collapses the fiber $P \times \{\alpha\}$ to a single point and $P \times \{\beta\}$ to a single point and consider the quotient space $(P \times P)/G$ with quotient map q. It is useful to notice that if $W \subset (P \times P)/G$ such that $q^{-1}(W)$ intersects $P \times \{\alpha\}$ (or $P \times \{\beta\}$), then $q^{-1}(W)$ contains $P \times \{\alpha\}$ (or $P \times \{\beta\}$.)

Lemma 5.1. *If $B \subset (P \times P)/G$ is a continuum, then $q^{-1}(B)$ is a continuum.*

Proof. If B does not intersect $\{q(P \times \{\alpha\}), q(P \times \{\beta\})\}$, then $q^{-1}(B)$ is homeomorphic to B and hence a continuum.

Suppose that B contains $q(P \times \{\alpha\})$ and assume that $q^{-1}(B)$ is not a continuum. In particular, since q is continuous, this means that $q^{-1}(B)$ is not connected. Then $q^{-1}(B)$ can be written as two disjoint sets which are both closed and open in $q^{-1}(B)$. Let $q^{-1}(B) = U \cup V$ where $U \cap V = \emptyset$. Assume that $P \times \{\alpha\}$ is contained inside of U. Then, since V does not intersect $P \times \{\alpha\}$, the sets $q(U)$ and $q(V)$ are disjoint so that B is not connected.

A similar argument hold if $q^{-1}(B)$ contains $P \times \{\beta\}$ or both of the fibers. □

Let $X = P \times C$. Then X can be essentially embedded inside of the cartesian product, Y, of an annulus A and the disk D^2. Let $\widetilde{Y}$ denote the universal covering space of Y, which contains an infinite, connected covering space $\widetilde{X}$ of X. $\widehat{Y}$ will denote the two points compactification of $\widetilde{Y}$ by adding points $\bar{a}$ and $\bar{b}$. Likewise, $\widehat{X}$ will denote the two points compactification of $\widetilde{X}$ contained inside of $\widehat{Y}$.

Lemma 5.2. *$(P \times P)/G$ is homeomorphic to $\widehat{X}$.*

Proof. In [6], D. Bellamy and W. Lewis have shown that two point compactification of the infinite covering space, $\tilde{C}$, of the pseudo-circle obtained by unwrapping the pseudo-circle is a pseudo-arc. This implies that there is a homeomorphism f_1 from the covering space $\tilde{C}$ to $P - \{\alpha, \beta\}$. Then the map $h_1(x, y) = (f_1(x), id_P(y))$, where id_P is the identity map on P, is a homeomorphism from $\widetilde{X}$ to $(P \times P) - (P \times \{\alpha\} \cup P \times \{\beta\})$. Then this map extends uniquely to a homeomorphism $H : \widehat{X} \to (P \times P)/G$. $\square$

Let $g : X \to X$ be a homeomorphism. Then there exists a lift $\tilde{g}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \widetilde{g} & \\ \widetilde{X} & \rightarrow & \widetilde{X} \\ \downarrow p & & \downarrow p \\ X & \rightarrow & X \\ & g & \end{array}$$

The argument that such a lift exists is similar the lifting argument used by K. Kuperberg and the author in the [23]. First note that since A is an absolute neighborhood retract, g extends to a continuous map f from a closed, connected neighborhood of X homeomorphic to $A \times D^2$ into $A \times D^2$. Then $A \times D^2$ can be retracted to this neighborhood of X. The composition of these maps has a lift, the appropriate restriction of this lift provides the lift of g.

Then $\widetilde{g}$ extends uniquely to a map $H : \widehat{X} \to \widehat{X}$. This map is a continuous bijection and hence a homeomorphism. Any such homeomorphism has the property that the set $\{a, b\}$ is invariant. In particular, since $\widehat{X}$ is homeomorphic to $(P \times P)/G$, the homeomorphism $g : X \to X$ uniquely induces a self homeomorphism of $(P \times P)/G$.

In this section, it will be shown that if $h : (P \times P)/G \to (P \times P)/G$ is such an induced homeomorphism then h has the additional properties that for any points $a \in P$

1. $[q^{-1} \circ h \circ q](P \times \{a\}) = P \times \{b\}$ for some $b \in P$ and
2. $[q^{-1} \circ h \circ q](\{a\} \times P) = (P \times \{\alpha\}) \cup (P \times \{\beta\}) \cup (\{b\} \times P)$ for some $b \in P$.

Throughout this section Φ will denote the set $(P \times \{\alpha\}) \cup (P \times \{\beta\})$. Notice that $q(\Phi)$ is an invariant set under the induced homeomorphism h.

The following Lemma in [8] will be needed:

Lemma 5.3. [Bellamy and Łysko, [8], Lemma 6] *Suppose X and Y are indecomposable continua, and $a \in X$ and $h : X \times Y \to X \times Y$ is a homeomorphism. Then either $\pi_1(h(\{a\} \times Y)) = X$ or $\pi_2(h(\{a\} \times Y) = Y$.*

The following Theorem of J. T. Rogers, Jr. will also be used:

Theorem 5.4. [Rogers, [39], Theorem 14] *The pseudo-circle is not the continuous image of the pseudo-arc.*

Lemma 5.5. *Let $a \in C$. Then $\pi_1(g(P \times \{a\})) = P$.*

Proof. Notice that $\pi_2 \circ g(P \times \{a\}))$ is a continuous mapping of a pseudo-arc into a pseudo-circle. From Theorem 5.4, the pseudo-circle cannot be the continuous image of a pseudo-arc. Therefore that $\pi_2 \circ g(P \times \{a\}))$ cannot be onto. Thus, from Lemma 5.3, it follows that $\pi_1(g(P \times \{a\})) = P$. □

Lemma 5.6. *Let $a \in P$. Then $\pi_1(g(\{a\} \times C)) = C$.*

Proof. Since $\check{H}_1(P)$ is trivial, the restriction $g|_{\{a\}\times C} : \{a\} \times C \to P \times C$ induces an isomorphism between the groups $\check{H}_1(\{a\} \times C)$ and $\check{H}_1(P \times C)$. Likewise, since $\check{H}_1(P)$ is trivial, $\pi_2 : P \times C \to C$ induces an isomorphism between $\check{H}_1(P \times C)$ and $\check{H}_1(C)$. Therefore, the composition of these two maps induces an isomorphism between $\check{H}_1(\{a\} \times C)$ and $\check{H}_1(C)$. In particular, this implies that $\pi_2 \circ g(\{a\} \times C$ must be onto. □

Since the homeomorphism $h : (P \times P)/G \to (P \times P)/G$ is uniquely determined by the homeomorphism $g : P \times C \to P \times C$, the following two corollaries are immediate from the previous two lemmas:

Corollary 5.7. $[\pi_1 \circ q^{-1} \circ h \circ q](P \times \{a\}) = P$ *for every $a \in P$.*

Corollary 5.8. *For every point $a \in P$, $[\pi_i \circ q^{-1} \circ h \circ q](\{a\} \times P) = P$ for $i \in \{1, 2\}$.*

For the following proofs it will be necessary to adapt a Lemma of Bellamy and Łysko in [8]:

Lemma 5.9. [Bellamy and Łysko, [8], Corollary 3] *Let X and Y be chainable continua and suppose W and M are subcontinua of $X \times Y$ such that $\pi_1(W) \subset \pi_1(M)$ while $\pi_2(M) \subset \pi_2(W)$. Then $W \cap M \neq \emptyset$.*

Lemma 5.10. *Suppose that W and M are subcontinua of $(P \times P)/G$ such that $\pi_1 \circ q^{-1}(W) \subset \pi_1 \circ q^{-1}(M)$ and $\pi_2 \circ q^{-1}(M) \subset \pi_2 \circ q^{-1}(W)$, then $M \cap N \neq \emptyset$.*

Proof. Since the inverses image under q of a continuum is a continuum, the inverse image satisfies the conditions of Lemma 5.9. □

With the previous Lemmas in mind, it will now be proven that the induced homeomorphism $h : (P \times P)/G \to (P \times P)/G$ has the additional properties that for any points $p \in P$

1. $[q^{-1} \circ h \circ q](P \times \{p\}) = P \times \{a\}$ for some $a \in P$ and
2. $[q^{-1} \circ h \circ q](\{p\} \times P) = \Phi \cup (\{b\} \times P)$ for some $b \in P$.

Theorem 5.11. *For every $p \in P$, $[q^{-1} \circ h \circ q](P \times \{p\}) = P \times \{b\}$ for some $b \in P$.*

Proof. If $p \in \{\alpha, \beta\}$, the result follows because the set $q(\Phi)$ is invariant under the homeomorphism h.

If $p \notin \{\alpha, \beta\}$, then the observations of the previous Lemmas allow the use of the proof of the main Theorem in [8] developed by Bellamy and Łysko. Suppose that

$\pi_2(q^{-1} \circ h \circ q(P \times \{p\}))$ is non-degenerate. Let $\mathbb{Z}$ denote the set of non-negative integers and let $< W_n >_{n \in \mathbb{Z}}$ be a sequence of non-degenerate, decreasing subcontinua of P such that $\cap W_n = \{p\}$. Since this is a decreasing sequence, assume without loss of generality that $W_n \cap \{\alpha, \beta\} = \emptyset$ for each n. Let $a \in P$ and notice that

$$\bigcap(\{a\} \times W_n) = \{(a,p)\} \subset P \times \{p\}$$

therefore

$$\bigcap[\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_n) =$$

$$[\pi_2 \circ q^{-1} \circ h \circ q](a,p) \in [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\})$$

In particular, $[\pi_2 \circ q^{-1} \circ h \circ q](a,p)$ is an element of

$$[\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\}) \cap [\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_n)$$

for each n. Since P is hereditarily indecomposable, this implies that for each n either

1. $[\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_n) \subset [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\})$ or

2. $[\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\}) \subset [\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_n)$.

Since $\cap([\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_n))$ is degenerate, condition (1) can not be true for each n. Therefore, there exists some N such that $[\pi_2 \circ q^{-1} \circ h \circ q](\{a\} \times W_N) \subset [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\})$.

Let $x_1 \in W_N$ such that $x_1 \neq p$. From the above remarks, $[\pi_2 \circ q^{-1} \circ h \circ q](P \times \{x_1\}) \cap [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\}) \neq \emptyset$.

This implies that either

1. $[\pi_2 \circ q^{-1} \circ h \circ q](P \times \{x_1\}) \subset [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\})$ or

2. $[\pi_2 \circ q^{-1} \circ h \circ q](P \times \{p\}) \subset [\pi_2 \circ q^{-1} \circ h \circ q](P \times \{x_1\})$

We will prove the first case, the proof of the second case is similar. Notice from Lemma 5.7, $[\pi_1 \circ q^{-1} \circ h \circ q](P \times \{x_1\}) = P = [\pi_1 \circ q^{-1} \circ h \circ q](P \times \{p\})$, therefore the conditions of Lemma 5.10 are satisfied. Hence $[h \circ q](P \times \{x_1\}) \cap [h \circ q](P \times \{p\}) \neq \emptyset$. However, this is a contradiction since $[q^{-1} \circ h \circ q]$ restricted to $(P \times P) - \Phi$ is a homeomorphism. $\square$

Theorem 5.12. *$[q^{-1} \circ h \circ q](\{a\} \times P) = \Phi \cup (\{b\} \times P)$ for some $b \in P$.*

Proof. Let $x \in P$ such that $K(x)$ does not contain the set $\{\alpha, \beta\}$. Such a point exists because an indecomposable continuum has uncountably many pairwise disjoint composants (see, for example, K. Kuratowski, [26], Theorems 5 and 7, p. 212). It will first be shown that $[q^{-1} \circ h \circ q](\{a\} \times K(x)) \subset \{b\} \times P$ for some $b \in P$.

Let P_1 be a non-degenerate subcontinuum of $K(x)$. Note that P_1 is a pseudo-arc and consider the subcontinuum of $P \times P_1$ of $P \times P$. From Lemma 5.11, for every point $x_1 \in P_1$, the map $[q^{-1} \circ h \circ q](P \times \{x_1\})$ is mapped homeomorphically onto $P \times \{x_2\}$ for some $x_2 \in P$. Note that x_2 can not equal α or β. In particular, $[q^{-1} \circ h \circ q](P \times P_1)$ is mapped bijectively onto $P \times P_2$ where P_2 is a proper, non-degenerate subcontinuum of P and therefore a pseudo-arc. Similar to the proof of Theorem 5.11, the proof of the main result by Bellamy and Łysko in [8] can be applied to show that $[q^{-1} \circ h \circ q]$ restricted to $P \times P_1$ also preserves horizontal fibers. In particular, $[q^{-1} \circ h \circ q](\{a\} \times P_1) \subset \{b\} \times P$ for some $b \in P$.

Let $p \in P$ and consider $\{p\} \times P_1$. Let $< W_n >_{n \in \mathbb{Z}}$ be sequence of decreasing, non-degenerate subcontinua of P such that $\cap W_n = \{p\}$. Next, Let $a \in P_1$ and notice that

$$\cap(W_n \times \{a\}) = \{(p, a)\} \in \{p\} \times P_1$$

In particular, $\cap[\pi_1 \circ q^{-1} \circ h \circ q](W_n \times \{a\}) = [\pi_1 \circ q^{-1} \circ h \circ q](p, a)$ is an element of $[\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1)$.

Therefore,

$$[\pi_1 \circ q^{-1} \circ h \circ q](W_n \times \{a\}) \cap [\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1) \neq \emptyset$$

for each n. This implies that for each n, either

1. $[\pi_1 \circ q^{-1} \circ h \circ q](W_n \times \{a\}) \subset [\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1)$ or
2. $[\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1) \subset [\pi_1 \circ q^{-1} \circ h \circ q](W_n \times \{a\})$.

However, since $\cap([\pi_1 \circ q^{-1} \circ h \circ q](W_n \times \{a\}))$ is degenerate, condition 2 cannot hold for every n. Thus, there exists some N so that $[\pi_1 \circ q^{-1} \circ h \circ q](W_N \times \{a\}) \subset [\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1)$.

Let $x_1 \in W_N$ such that $x_1 \neq p$. From the above remarks, $[\pi_1 \circ q^{-1} \circ h \circ q](\{x_1\} \times P_1) \cap [\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1) \neq \emptyset$. Since the pseudo-arc is hereditarily indecomposable this implies that either

1. $[\pi_1 \circ q^{-1} \circ h \circ q](\{x_1\} \times P_1) \subset [\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1)$ or

2. $[\pi_1 \circ q^{-1} \circ h \circ q](\{p\} \times P_1) \subset [\pi_1 \circ q^{-1} \circ h \circ q](\{x_1\} \times P_1)$.

The first case will be proven, the second case is similar. Since $[\pi_2 \circ q^{-1} \circ h \circ q](\{x_1\} \times P_1) = P_2 = [\pi_2 \circ q^{-1} \circ h \circ q](\{p\} \times P_1)$, the conditions of Lemma 5.10 are satisfied. Therefore, $[h \circ q](\{x_1\} \times P_1) \cap [h \circ q](\{p\} \times P_1) \neq \emptyset$. This contradicts the fact that $[q^{-1} \circ h \circ q]$ restricted to $(P \times P) - \Phi$ is a homeomorphism. Therefore $[q^{-1} \circ h \circ q](\{a\} \times P_1) \subset \{b\} \times P$ for some $b \in P$.

Next, notice that since P is hereditarily indecomposable any two points in $K(x)$ can be joined by a proper subcontinuum. Therefore, $[q^{-1} \circ h \circ q](\{a\} \times K(x)) \subset \{b\} \times P$.

However, note that $h \circ q(\{a\} \times P) = h \circ q(cl(\{a\} \times K(x))$, since composants in an indecomposable space are dense. From the previous paragraphs, this implies that $h \circ q(\{a\} \times P) = q(\{b\} \times P)$. Therefore it follows that $[q^{-1} \circ h \circ q](\{a\} \times P) = (\{b\} \times P) \cup \Phi$.

□

5.1 Factorwise rigidity of $P \times C$

As in the previous section, let $X = P \times C$ and let $\widehat{X}$ will denote the two points compactification of the infinite covering space $\widetilde{X}$ of X.

Theorem 5.13. *The Cartesian product $P \times C$ is factorwise rigid.*

Proof. Let $h : X \to X$ be a homeomorphism. Then there exists a lift $\tilde{h}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \widetilde{h} & \\ \widetilde{X} & \rightarrow & \widetilde{X} \\ \downarrow p & & \downarrow p \\ X & \rightarrow & X \\ & h & \end{array}$$

Then $\widetilde{h}$ extends uniquely to a map $H : \widehat{X} \to \widehat{X}$. This map is a continuous bijection and hence a homeomorphism. Any such homeomorphism has the property that the set $\{a, b\}$ is invariant. Note that $\widehat{X}$ is homeomorphic to $(P \times P)/G$ and therefore the results of the previous section apply. In particular, for $x \in C$, $h(P \times \{x\}) = p \circ H \circ p^{-1}(P \times \{x\})$. However, from Theorem 5.11, $H \circ p^{-1}(P \times \{x\}) = p^{-1}(P \times \{y\})$ for some $y \in C$. Hence $h(P \times \{x\}) = P \times \{y\}$.

Likewise, from Theorem 5.12, it follows that $h(\{r\} \times C) = p \circ H \circ p^{-1}(\{r\} \times C) = \{s\} \times C$ for some $s \in P$.

Therefore, the cartesian product $P \times C$ is factorwise rigid. $\square$

Chapter 6

Other notes on factorwise rigidity

The following chapter consist of observations made during the research of the main result in this dissertation. As in the previous chapter, if $X \times Y$ is the cartesian product of two spaces, $\pi_1 : X \times Y \to X$ will denote the projection onto the first factor space. Likewise, π_2 will denote the projection onto the second factor space.

6.1 Factorwise rigidity of a cartesian product with one factor space hereditarily indecomposable

This sections deals with the Cartesian product where one factor is hereditarily indecomposable and the other factor space contains arc components. For example, this section shows that if S is the solenoid and C is the pseudo-circle, the $S \times C$ is factorwise rigid with respect to S.

Theorem 6.1. *If X is arcwise connected and Y is hereditarily indecomposable, then any homeomorphism $h : X \times Y \to X \times Y$ preserves X-fibers (i.e. is factorwise rigid with respect to X).*

Proof. Let $b \in Y$ and let $(m_1, b) \in X \times \{b\}$. Let b_1 be the second coordinate of $h(m_1, b)$. Suppose that there exists a point $(m_2, b) \in X \times \{b\}$ such that the second coordinate of $h(m_2, b)$ is b_2 and that $b_1 \neq b_2$. Since X is arcwise connected, there exists an arc A_{m_1,m_2} from (m_1, b) to (m_2, b), so that $h(A_{m_1,m_2})$ is an arc from $h(m_1, b)$

to $h(m_2, b)$. By assumption $\pi_2(h(A_{m_1,m_2}))$ is a non-trivial continuous image of an arc, and hence contains an arc. However, since Y is hereditarily indecomposable, this is a contradiction. $\square$

Remark 6.2. The same proof shows that if X contains any arc A and $b \in Y$, then $h(A \times \{b\}) \subset X \times \{c\}$ for some $c \in Y$.

Theorem 6.3. *If a space X has a dense arc component and Y is hereditarily indecomposable, then $X \times Y$ is factorwise rigid with respect to X.*

Proof. Let $h : X \times Y \to X \times Y$ be a homeomorphism and let $a \in Y$. Let $R \subset X$ be a dense arc component of X. Then R is arcwise connected, hence by Theorem 6.1, it follows that $h(R \times \{a\}) = R \times \{b\} \subset X \times \{b\}$ for some $b \in X_2$. Since $X \times \{b\}$ is closed and h a homeomorphism, it follows that $h(X \times \{a\}) = h(\mathrm{cl}(R \times \{a\}) \subset X \times \{b\}$.

Applying the result to $h^{-1}(X \times \{b\})$, it follows that $h^{-1}(X \times \{b\}) \subset X \times \{a\}$. Therefore, h maps $X \times \{a\}$ one-to-one and onto $X \times \{b\}$. $\square$

6.2 Homogeneous fiber bundles of Menger Manifolds

The following section is another application of how factorwise rigidity relates to the study of homogeneous continua. Let $n \geq 1$ be an integer. A point of $\mu^n_{(S^{2n} \times I)}$ will be denoted $(x_1, x_2, \ldots, x_{2n+1}, y)$ where $(x_1, x_2, \ldots, x_{2n+1}) \in S^{2n}$ and $y \in I$.

Let $\alpha \geq 1$ be an integer and let W be a compact, connected manifold of dimension $2\alpha + 1$. A point of $\mu^\alpha_W \times \mu^n_{(S^{2n} \times I)}$ will be denoted $(a, (x_1, x_2, \ldots, x_{2n+1}, y))$, where

$a \in \mu^{\alpha}_W$ and $(x_1, x_2, \ldots, x_{2n+1}, y) \in \mu^n_{(S^{2n}\times I)}$. For $c \in [0,1]$, let A_c be the subset of $\mu^n_{(S^{2n}\times I)}$ consisting of the collection of points of the form $(x_1, x_2, \ldots, x_{2n+1}, c)$.

Let G denote the quotient space obtained from $\mu^n_{(S^{2n}\times I)}$ by identifying the points $(x_1, x_2, \ldots, x_{2n+1}, 0) \in A_0$ with $(x_1, x_2, \ldots, x_{2n+1}, 1) \in A_1$. The point in G corresponding to $(x_1, x_2, \ldots, x_{2n+1}, y) \in \mu^n_{(S^{2n}\times I)} - A_1$ will be denoted $(x_1, x_2, \ldots, x_{2n+1}, y)_G$. The space G is homeomorphic to $\mu^n_{S^{2n+1}}$. The subsets of G corresponding to A_c will be denoted $\widetilde{A_c}$.

Let $h : \mu^{\alpha}_W \to \mu^{\alpha}_W$ be a fixed point free action of period $k > 1$. The quotient space obtained from $\mu^{\alpha}_W \times \mu^n_{(S^{2n}\times I)}$ by identifying a point $(a, (x_1, x_2, \ldots, x_{2n+1}, 0))$ with $(h(a), (x_1, x_2, \ldots, x_{2n+1}, 1))$ is a fiber bundle determined by the monodromy h whose base space is homeomorphic to $\mu^n_{S^{2n+1}}$ and whose fiber is μ^{α}_W. Denote this space by $\mu^{\alpha}_W \times_h \mu^n_{S^{2n+1}}$. A point of $\mu^{\alpha}_W \times_h \mu^n_{S^{2n+1}}$ will be denoted $(\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)})$.

6.2.1 Homogeneity of the fiber bundles

For each $c \in [0,1)$, define an embedding $\psi_c : \mu^{\alpha}_W \times (G - \widetilde{A_c}) \to \mu^{\alpha}_W \times_h \mu^n_{S^{2n+1}}$ by

$$\psi_c(a, (x_1, x_2, \ldots, x_{2n+1}, y)_G) =$$

$$\begin{cases} (\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) & \text{if } y < c \\ (\overline{h^{-1}(a)}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) & \text{if } y > c \end{cases}$$

Denote the image of ψ_c by $\mathrm{Im}(\psi_c)$.

Theorem 6.4. *The space $\mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$ is homogeneous.*

Proof. Suppose that

$$p = (\ _p\overline{a},\ _p\overline{(x_1, x_2, \ldots, x_{2n+1}, y_p)}) \text{ and}$$

$$q = (\ _q\overline{a},\ _q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)})$$

are two distinct points of $\mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$. Then there exists a point $c \in [0, 1)$ such that y_p and y_q are both less than c. In particular, both p and q are in $\mathrm{Im}(\psi_c)$.

Then $G - \widetilde{A_c}$ is connected. Since G is strongly locally homogeneous, there exists a homeomorphism $g : G \to G$ such that

$$g(\ _p(x_1, x_2, \ldots, x_{2n+1}, y_p)_G) = \ _q(x_1, x_2, \ldots, x_{2n+1}, y_q)_G \text{ and}$$

$$g((x_1, x_2, \ldots, x_{2n+1}, y)_G) = (x_1, x_2, \ldots, x_{2n+1}, y)_G$$

for $(x_1, x_2, \ldots, x_{2n+1}, y)_G \in \widetilde{A_c}$.

Define $\varphi : \mu_W^\alpha \times G \to \mu_W^\alpha \times G$ by

$$\varphi(a, (x_1, x_2, \ldots, x_{2n+1}, y)_G) = (a, g((x_1, x_2, \ldots, x_{2n}, y)_G)).$$

Let $h_1 : \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n \to \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$ be define by

$$h_1(x) = \begin{cases} x & \text{if } y < c \\ \psi_c \circ \varphi \circ \psi_c^{-1}(x) & \text{if } y > c \end{cases}$$

Then $h_1(p)$ is equal to

1. $({}_p\overline{a},\ {}_q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)})$ or

2. $(\overline{h({}_p a)},\ {}_q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)})$ or

3. $(\overline{h^{-1}({}_p\overline{a})},\ {}_q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)})$.

Let $\mathcal{U}$ be a finite open over of connected sets such that if $U \in \mathcal{U}$, then the collection $\{h^i(U)\}_{i=1,2,..k}$ is pairwise disjoint. For each $U \in \mathcal{U}$, the set

$$\{(\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) : a \in \bigcup_{i=1}^{k} h^i(U)\}$$

is homeomorphic to $U \times \mu^n_{S^{2n+1}}$.

It is sufficient to show that for any $U \in \mathcal{U}$ and any two points s and t in U, that there is a homeomorphism $h_2 : \mu^\alpha_W \times_h \mu^n_{S^{2n+1}} \to \mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$ such that $h_2(\overline{s},\ {}_q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)}) = (\overline{t},\ {}_q\overline{(x_1, x_2, \ldots, x_{2n+1}, y_q)})$.

Let $\varphi_2 : \mu^\alpha_W \to \mu^\alpha_W$ be a homeomorphism such that $\varphi_2(s) = t$ and $\varphi_2(x) = x$ for all $x \notin U$. Next, define

$$h_2(\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) =$$

$$\begin{cases} (\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) & \text{if } a \notin \cup_{i=1}^{k} h^i(U) \\ (\overline{h^i \circ \varphi_2 \circ h^{-i}(a)}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) & \text{if } a \in h^i(U) \end{cases}$$

Then h_2 has the desired properties. □

It will now be shown that the fiber bundles created using the above method are not homeomorphic to the trivial fiber bundle. The following lemma of K. Kuperberg appears in [21] as Lemma 1 and will be useful for the proof.

Lemma 6.5. *Let $X = X_1 \times X_2$, where X_i is homeomorphic to μ^n for some n and $i = 1, 2$. Let $U_i \subset X_i$ be a connected open set for $i = 1, 2$. If $\phi : U_1 \times U_2 \to X$ is an open embedding, then*

1. *$\phi(x, y) = (\phi_1(x), \phi_2(y))$, where $\phi_1 : U_1 \to X_1$ and $\phi_2 : U_2 \to X_2$, or*

2. *$\phi(x, y) = (\phi_1(y), \phi_2(x))$, where $\phi_1 : U_2 \to X_1$ and $\phi_2 : U_1 \to X_2$.*

Let $p = ({}_p\overline{a}, {}_p\overline{(x_1, x_2, \ldots, x_{2n+1}, y)})$ be a point of $\mu^{\alpha}_{W} \times_h \mu^{n}_{S^{2n+1}}$. Define the sets

$$M_p = \{(\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) \in \mu^{\alpha}_{W} \times_h \mu^{n}_{S^{2n+1}} : a = h^i({}_pa) \text{ for } i = 1, .., k\}$$

$$N_p =$$

$$\{(\overline{a}, \overline{(x_1, x_2, \ldots, x_{2n+1}, y)}) \in \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n : (x_1, x_2, \ldots, x_{2n+1}, y) =$$

$$_p(x_1, x_2, \ldots, x_{2n+1}, y)\}$$

$$O_p = M_p \cap N_p$$

Note that O_p contains k elements.

The following Lemma appears in [21] as Lemma 1 where the outline for a proof is mentioned. For completeness, a detailed proof is provided and follows the proof given of Lemma 5 in [22], in which K. Kuperberg proved the result for the specific case $\alpha = n = 1$.

Lemma 6.6. *If $\phi : \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n \to \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$ is a homeomorphism, then either*

1. *$\phi(M_p) = M_{\phi(p)}$ and $\phi(N_p) = N_{\phi(p)}$ for all $p \in \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$, or*

2. *$\phi(M_p) = N_{\phi(p)}$ and $\phi(N_p) = M_{\phi(p)}$ for all $p \in \mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$.*

Proof. Since every point of $\mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$ has a neighborhood homeomorphic to an open subset of $\mu_W^\alpha \times \mu_{S^{2n+1}}^n$, every point of $\mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$ has a closed neighborhood homeomorphic to a set of the form $X_1 \times X_2$ where X_1 is homeomorphic to μ^α and

X_2 is homeomorphic to μ^n. Moreover, for every $x_1 \in X_1$ and $x_2 \in X_2$, if follows that $\{x_1\} \times X_2$ and $X_1 \times \{x_2\}$ are in some set M_p or N_p.

Since $\mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$ is compact, there exists a finite collection $\{V_1, \ldots, V_k\}$ of neighborhoods of the above form such that $\mu^\alpha_W \times_h \mu^n_{S^{2n+1}} \subset \cup_{i=1}^k \mathrm{int}(V_i)$.

Likewise, there exists a finite collection of connected open subsets $\{W_1, \ldots, W_l\}$ such that for each $j \in \{1, \ldots, l\}$, there exists an i where $h(W_j) \subset V_i$.

By 6.5, if $p \in \mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$, for every $j \in \{1, \ldots, l\}$ there exists an i such that $\phi(N_p \cap W_j) \subset N_{\phi(p)} \cap V_i \subset N_{\phi(p)}$ or $\phi(N_p \cap W_j) \subset M_{\phi(p)} \cap V_i \subset M_{\phi(p)}$.

Therefore, $\phi(N_p) \subset N_{\phi(p)}$ or $\phi(N_p) \subset M_{\phi(p)}$. In either case, equality is obtained by applying the result to ϕ^{-1}.

Since $\mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$ is connected, if the above holds for one point it must hold for each point in the space. A similar argument holds for M_p. Therefore, since the map is one-to-one, the result follows. □

The following is a corollary of the above lemma:

Corollary 6.7. *If $\phi : \mu^\alpha_W \times_h \mu^n_{S^{2n+1}} \to \mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$ is a homeomorphism, then $\phi(O_p) = O_p$ or $\phi(O_p) \cap O_p = \emptyset$.*

Theorem 6.8. *If h is a fixed action free homeomorphism of period $k > 1$, then $\mu^\alpha_W \times_h \mu^n_{S^{2n+1}}$ is not homeomorphic to $\mu^\alpha_W \times \mu^n_{S^{2n+1}}$.*

Proof. Let $\{x_j : j = 1, \ldots, k\}$ be a finite collection of distinct points in $\mu^\alpha_W \times \mu^n_{S^{2n+1}}$. Let $p_i, i = 1, 2$ denote the projection from $\mu^\alpha_W \times \mu^n_{S^{2n+1}}$ to the first and second factor space, respectively. By assumption, at least one of the sets $\{p_1(x_j) : j = 1, \ldots, k\}$ and

$\{p_2(x_j) : j = 1, \ldots, k\}$ contains more than one element. Without loss of generality, assume that it is $\{p_1(x_j) : j = 1, \ldots, k\}$. Let $p_1(x_n)$ and $p_1(x_m)$ be two distinct element, and let U be an open set about $p_1(x_n)$ which does not contain $p_1(x_m)$ or any other distinct elements of $\{p_1(x_j) : j = 1, \ldots, k\}$. Let y be a point of U other than $p_1(x_n)$ and let $h : \mu_W^\alpha \to \mu_W^\alpha$ be a homeomorphism such that $h(p_1(x_m)) = y$ and $h(x) = x$ for all $x \notin U$. Define $H : \mu_W^\alpha \times \mu_{S^{2n+1}}^n \to \mu_W^\alpha \times \mu_{S^{2n+1}}^n$ by $H(x, y) = (h(x), y)$. Then $\{H(x_j) : j = 1, \ldots, k\} \neq \{x_j : j = 1, \ldots, k\}$ and $\{H(x_j) : j = 1, \ldots, k\} \cap \{x_j : j = 1, \ldots, k\} \neq \emptyset$. Therefore, by 6.7, $\mu_W^\alpha \times \mu_{S^{2n+1}}^n$ can not be homeomorphic to $\mu_W^\alpha \times_h \mu_{S^{2n+1}}^n$. $\square$

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