Inverse Limit Spaces

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Vita

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Thesis Abstract

Inverse Limit Spaces

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This paper is a vast survey of inverse limit spaces. After defining an inverse limit on continuous bonding functions, we prove important theorems about inverse limits, provide examples, and explore various generalizations of traditional inverse limits. In particular, we present original proofs of theorems given by Ingram and Mahavier in "Inverse Limits of Upper Semi-Continuous Set Valued Functions." We then use this new sort of inverse limit to enliven the notion of a "two-sided" inverse limit; finally, we use inverse limits on u.s.c. functions to produce an indecomposable continuum.

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Chapter 1

Introduction

An inverse limit space is a powerful topological tool. Inverse limits not only help us generate complicated continua with interesting properties, but also allow us to represent such continua in a simple and elegant way. Now, a new generalization of inverse limit spaces has opened up more possibilities for topologists to explore. As we will see, this new generalization even breathes new life into a different kind of inverse limit.

This paper provides a vast survey of inverse limits. In Chapter 2, we list basic definitions and theorems that will serve as background material. In Chapter 3, we begin by defining an inverse limit with continuous bonding maps and proving some important preliminary theorems. We demonstrate the power of inverse limits by using them to prove the formidable Tychonoff Theorem; then, we use inverse limits to represent some complicated continua in a simple, straightforward way. Next, in Chapter 4, we consider a generalization of inverse limits from Ingram and Mahavier's "Inverse Limits of Upper Semi-Continuous Set Valued Functions" [1]. After giving original proofs of the theorems from that paper, in Chapter 5 we show how this new notion of inverse limit can revitalize the formerly redundant notion of the "two-sided" inverse limit. Finally, in Chapter 6, we use inverse limits with upper semi-continuous bonding functions to produce an indecomposable continuum, and raise a few questions open for further research.

Chapter 2

Background Definitions and Theorems

Let X be a set and let T be a collection of subsets of X with the following properties:

1. $X \in T$;
2. $\emptyset \in T$;
3. If $\{O_i\}_{i \in \mu}$ is a collection of members of T, then $\bigcup_{i \in \mu} O_i \in T$;
4. If $\{O_i\}_{i=1}^{n}$ is a finite collection of members of T, then $\bigcap_{i=1}^{n} O_i \in T$.

Then the pair (X, T) is called a *topological space* with *topology* T. Such a topological space will often be referred to simply as X when the associated topology T is understood. The members of T are called *open sets*.

A subset K of a topological space X is *closed* if $X - K$ is open.

Suppose M is a subset of a topological space X. A point $p \in X$ is a *limit point* of M if every open set containing p contains a point in M different from p.

Suppose M is a subset of a topological space X. The *closure of* M (denoted $\overline{M}$) is the union of M with the set of all limit points of M.

Suppose a collection B of open sets of a space X satisfies the following property:

If $x \in X$ and O is an open set containing x, then there exists a member b of B such that $x \in b$ and $b \subseteq O$.

Then B is a *basis* for the topology on X and a member b of B is called a *basic open set of* X.

Suppose B is a collection of subsets of a set X such that

1. If $x \in X$, there exists some $b \in B$ with $x \in b$.

2. If b_1 and b_2 are members of B with $x \in b_1 \cap b_2$, then there exists some set b_3 in B with $x \in b_3 \subseteq (b_1 \cap b_2)$.

Then the collection $T = \{\bigcup R | R \subset B\}$ is a topology for X, and B is a basis for this topology. It is said that the topology T is *generated* by the basis B.

A topological space X is called *Hausdorff* if for every pair of distinct points $p, q \in X$, there exist disjoint open sets O_p and O_q containing p and q respectively.

A space X is called *regular* if for every closed set $H \subset X$ and point $p \in X$ not in H, there exist disjoint open sets O_H and O_p containing H and p, respectively.

A space X is called *normal* if for every pair of disjoint closed sets H and K in X, there exist disjoint open sets O_H and O_K containing H and K, respectively.

If $f : X \to Y$ is a function from the set X to Y, and U is a subset of X, we define $f(U) = \{f(u) | \ u \in U\}$.

Let X and Y be topological spaces and let $f : X \to Y$ be a function from X to Y. Then f is said to be *continuous at the point* x if, whenever V is an open set in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq V$. If f is continuous at each point $x \in X$, we say f is *continuous.*

A function $f : X \to Y$ is said to be *onto* if for each $y \in Y$, there exists some $x \in X$ with $f(x) = y$.

A function $f : X \to Y$ is said to be *1-1* if for any pair of distinct points p, q in X, $f(p) \neq f(q)$.

If $f : X \to Y$ is a function and $y \in Y$, then the *preimage of* y (written as $f^{-1}(y)$) is $\{x \in X |\ f(x) = y\}$.

Suppose $f : X \to Y$ is a 1-1 onto function. Then the function $f^{-1} : Y \to X$ given by $f^{-1}(y) = x$ (where x is the unique point in X with the property that $f(x) = y$) is called the *inverse of* f.

If X and Y are topological spaces and $f : X \to Y$ is 1-1, onto, continous, and has a continous inverse, then f is called a *homeomorphism* and the spaces X and Y are said to be *homeomorphic.*

Let X be a topological space. A collection B of open sets of X is a *local basis* at the point $x \in X$ if

1. For each member $b \in B$, $x \in b$;

2. If O is an open set in X containing x, then there exists a member b of B with $x \in b \subseteq O$.

A space X is called *first countable* if for each $x \in X$, there exists a countable local basis at x.

A space X is called *second countable* if X has a basis that is countable.

Let X be a topological space and let $M \subseteq X$. A collection of sets $\{O_i\}_{i \in \mu}$ in X is said to be an *open cover of* M if each O_i is open in X and $M \subseteq \bigcup_{i \in \mu} O_i$.

If $\{O_i\}_{i \in \mu}$ is a cover of X, $\gamma \subseteq \mu$, and $\{O_i\}_{i \in \gamma}$ is also a cover of X, then $\{O_i\}_{i \in \gamma}$ is called a *subcover* of the original cover $\{O_i\}_{i \in \mu}$. A subcover consisting of only finitely many members is called a *finite subcover.*

A space X is *compact* if for every open cover $\{O_i\}_{i \in \mu}$ of X, there exists a finite subcover of X. (I.e., $\{O_{i_j}\}_{j=1}^{n}$ for some natural number n.)

A collection of subsets $\{G_i\}_{i \in \mu}$ of a space X is called a *monotonic collection* if for each pair of members G_j, G_k in the collection, either $G_j \subseteq G_k$ or $G_k \subseteq G_j$.

A space X is *perfectly compact* if whenever $\{G_i\}_{i\in\mu}$ is a monotonic collection of subsets of X, there exists a point p in X that is either a point or a limit point of each G_i.

For each i, $1 \leq i \leq n$, let X_i be a topological space. Define $X = \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n$ to be the set $\{(x_1, x_2, \ldots, x_n)|\ x_i \in X_i \text{ for } 1 \leq i \leq n\}$. Define a topology on X as follows: a basic open set containing $(x_1, x_2, \ldots, x_n)$ is given by $\prod_{i=1}^{n} O_i$, where (for each i) O_i is open in X_i and $x_i \in O_i$.

Then X together with the topology generated by this basis is called a (finite) *product space.*

For each positive integer i, let X_i be a topological space. Define $X = \prod_{i=1}^{\infty} X_i$ to be the set $\{(x_1, x_2, \ldots)|\ x_i \in X_i \text{ for each positive integer } i\}$. Define a topology on X as follows: a basic open set containing $(x_1, x_2, \ldots)$ is given by $\prod_{i=1}^{\infty} O_i$, where (for each i) O_i is open in X_i, $x_i \in O_i$, and for some positive integer N, $O_n = X_n$ if $n \geq N$.

Then X together with the topology generated by this basis is called a (countably infinite) *product space.*

For each i in some arbitrary index set μ, let X_i be a topological space. Define $X = \prod_{i\in\mu} X_i$ to be the set $\{(x_i)_{i\in\mu}|\ x_i \in X_i \text{ for each } i\}$. Define a topology on X as follows: a basic open set containing $(x_i)_{i\in\mu}$ is given by $\prod_{i\in\mu} O_i$, where (for each i) O_i is open in X_i, $x_i \in O_i$, and for all but finitely many i, $O_i = X_i$.

Then X together with the topology generated by this basis may be called a *product space on the index set* μ.

Let $X = \prod_{i\in\mu} X_i$ be a product space (with index set μ either finite or infinite). Then the function $\pi_j : X \to X_j$ defined by $\pi_j((x_i)_{i\in\mu}) = x_j$ is called the *projection map on the* j*th coordinate*.

Suppose X is a topological space with topology T and $S \subset X$. Then the set S together with the topology $\hat{T} = \{S \cap O | \ O \in T\}$ is called a *subspace* of X, where $\hat{T}$ is the *subspace topology*.

Let X be a set. Then the relation $<$ on X is a *linear ordering* on X (and X is said to be *ordered* with respect to $<$) if for any $a, b, c \in X$,

1. If $a \neq b$, either $a < b$ or $b < a$,
2. If $a < b$ then $b \not< a$,
3. If $a < b$ and $b < c$, then $a < c$.

Let X be a set with a linear ordering $<$. Let B be the collection of all subsets of X of the following form:

1. $\{x| \ x < p\}$ for some $p \in X$,
2. $\{x| \ p < x\}$ for some $p \in X$,
3. $\{x| \ p < x < q\}$ for some p,$q \in X$, $p < q$.

Then the topology generated by B is called the *order topology on* X.

Let X and Y be two spaces with linear orderings $<_X$ and $<_Y$, respectively, so that X and Y both have their own respective order topologies. Suppose there exists a function $\phi : X \to Y$ so that if $a, b \in X$, then $a <_X b$ iff $\phi(a) <_Y \phi(b)$. Then ϕ is an *order isomorphism*. If ϕ is onto, then X and Y are said to be *order isomorphic*.

Let X be a set with a linear ordering $<$. Then a subset S of X is called an *initial segment* of X if there exists some element $p \in X$ so that $S = \{x \in X | \ x < p\}$.

Let X be a set with a linear ordering $<$. If S is a subset of X, then S is said to have a *least element* p if $p \in S$ and for each $x \in S$, if $p \neq x$, $p < x$.

Let X be a set with a linear ordering $<$. Then the set X is said to be *well-ordered* if every subset of X has a least element.

Let μ be a well-ordered index set with $h < k \in \mu$. Then the function $f : \prod_{i \in \mu,\ i \leq k} X_i \to \prod_{i \in \mu,\ i \leq h} X_i$ given by $f((x_i)_{i \leq k}) = (x_i)_{i \leq h}$ is called a *generalized projection*.

Suppose X is a topological space and $d : X \times X \to \mathbb{R}$ is a function satisfying the following properties (for all $x, y, z \in X$):

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.

3. $d(x,z) \leq d(x,y) + d(y,z)$.

Then the function d is said to be a *metric* on X. For a given $p \in X$ and $\epsilon > 0$, let $B(p,\epsilon) = \{x \in X \mid d(x,p) < \epsilon\}$. If the collection $\{B(p,\epsilon) \mid p \in X,\ \epsilon > 0\}$ is a basis for the space X, then X is said to be a *metric space.*

Let X be a topological space. Two subsets H and K of X are called *mutually separated* if neither set contains a point or a limit point of the other.

If X is a topological space and $M \subseteq X$, then M is *connected* if M is not the union of two mutually separated non-empty subsets of X.

A topological space X is a *continuum* if X is non-empty, compact, and connected.

A continuum that is Hausdorff (but not necessarily metric) is called a *Hausdorff continuum.*

A continuum that is metric is called a *metric continuum.*

If X is a continuum and A, a subset of X, is also a continuum, then A is called a *subcontinuum* of X. If A is a proper subset of X, then A is a *proper subcontinuum.*

A point p of a space X is called an *isolated point* if there exists an open set $O \subseteq X$ such that $O = \{p\}$.

Let X be a connected set. If $X - \{p\}$ is not connected, then p is a *cut point* of X.

A continuum with exactly 2 non-cut points is called an *arc*.

A *triod* is a union of three arcs whose intersection is exactly one point.

A *fan* is a union of infinitely many arcs, all of which have exactly one point in common.

Let X be a topological space. Suppose that A, a subset of X, is an arc with the property that whenever $O \subseteq X$ is an open set with $O \cap A \neq \emptyset$, there exists some point $p \in O$ with $p \notin A$. Then A is called a *limit arc*.

Background Theorems

Most of the following basic theorems may be found in [5]. The proofs of these theorems are omitted, but may be found in one or more of [2], [3], and [4].

2.1. Let X be a topological space with $M \subseteq X$. If M is compact, M is perfectly compact.

2.2. Let X be a topological space with $M \subseteq X$. If M is closed and perfectly compact, M is compact.

2.3. A closed subset of a compact space is compact.

2.4. Projection maps are continuous.

2.5. The continuous image of a compact set is compact.

2.6. Any finite product of compact sets is compact.

2.7. Let B be a basis for a topological space X. Then every open set of X is a union of members of B.

2.8. The following are equivalent:

i. $f : X \to Y$ is a continuous function from topological space X to topological space Y.

ii. If O is a (basic) open set in Y, then $f^{-1}(O)$ is open in X.

2.9. Suppose X and Y are both well-ordered with respect to the order relations $<_X$ and $<_Y$, respectively. Then exactly one of the following is true:

i. X and Y are order isomorphic.

ii. X is order isomorphic to an initial segment of Y.

iii. Y is order isomorphic to an initial segment of X.

2.10. Suppose μ is a well-ordered index set with $h < k \in \mu$. Then the generalized projection $f : \prod_{i\in\mu,\ i\leq k} X_i \to \prod_{i\in\mu,\ i\leq h} X_i$ given by $f((x_i)_{i\leq k}) = (x_i)_{i\leq h}$ is continuous.

2.11. If X is a compact Hausdorff space, then X is regular.

2.12. If X is a compact Hausdorff space, then X is normal.

2.13. If X is regular, then X is Hausdorff.

2.14. If X is normal, then X is regular.

2.15. The unit interval $[0, 1]$ is a compact subset of the real line.

2.16. Suppose M is a subset of a topological space X. If M is closed and not connected, then M is the union of two disjoint closed sets H and K.

2.17. The continuous image of a connected set is connected.

2.18. The continuous image of a continuum is a continuum.

2.19. Suppose X_i is a connected for each positive integer i. Then $\prod_{i=1}^{n} X_i$ is connected for each positive integer n. Moreover, $\prod_{i=1}^{\infty} X_i$ is connected.

2.20. If $X = A \cup B$, a union of non-empty closed sets, and there exists a connected subset of X that intersects both A and B, then A and B are not mutually separated.

2.21. The common part of a monotonic collection of continua is a continuum.

Chapter 3

Inverse Limits with Continuous Bonding Maps

Suppose that, for each natural number i, X_i is a topological space and f_i is a continuous function from X_{i+1} to X_i. Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be the set of all sequences $x = (x_i)_{i=1}^{\infty}$, where $x_i \in X_i$ and $x_i = f_i(x_{i+1})$ for all i. If O_i is a subset of X_i, define $\overleftarrow{O_i} = \{x \in X \mid x_i \in O_i\}$. Then we say X is an *inverse limit space* and a basis for the topology on X is $\{\overleftarrow{O} \mid O \text{ is open in some } X_i\}$. The X_i's are called the *factor spaces* of X, and the f_i's are continuous *bonding maps*.

In this section, after we prove a few preliminary results, we will use inverse limits to give a straightforward proof of the formidable Tychonoff Theorem. Then we will see how some complicated topological spaces may be represented easily as an inverse limit with a single bonding map f.

We begin with some basic results about inverse limit spaces. First, it is of interest to determine whether a given topological property is taken on by X if each factor space X_i has that property.

Theorem 3.1. *If $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space and each X_i is Hausdorff, then X is Hausdorff.*

Proof: Suppose X_i is Hausdorff for all i, where $p = (p_i)_{i=1}^{\infty}$ and $q = (q_i)_{i=1}^{\infty}$ are distinct elements of X. Then for some i, $p_i \neq q_i$. Since X_i is Hausdorff, there exist disjoint open subsets O_p and O_q of X_i containing p_i and q_i, respectively. Thus, $\overleftarrow{O_p}$ and $\overleftarrow{O_q}$ are disjoint open sets in X containing p and q, respectively. So X is Hausdorff. •

It is also easily shown that

a) if each X_i is regular, X is regular;

b) if each X_i is first countable, X is first countable;

c) if each X_i is second countable, X is second countable.

However, X does not always inherit the topological properties possessed by each X_i. For example, if each X_i is non-empty, it need not follow that X is non-empty. Consider $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ where X_i is the open interval $(0, \frac{1}{i})$ in the real line, and $f_i : X_{i+1} \to X_i$ is the identity map. Each X_i is non-empty, but $X = \emptyset$.

Again suppose $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space. If i, j are positive integers with $i < j$, define $f_i^j : X_j \to X_i$ by $f_i^j = f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$.

Theorem 3.2. *Suppose $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space, $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of positive numbers, $g_i = f_{n_i}^{n_{i+1}}$ for each i, and $Y = \varprojlim\{X_{n_i}, g_i\}_{i=1}^{\infty}$. Then X is homeomorphic to Y.*

Proof: Define $h : X \to Y$ by $h((x_i)_{i=1}^{\infty}) = (x_{n_i})_{i=1}^{\infty}$. We need to show that h is a homeomorphism.

h is easily seen to be onto. To show that h is 1-1, let $p = (p_i)_{i=1}^{\infty}$ and $q = (q_i)_{i=1}^{\infty}$ be distinct points in X. (So $p_j \neq q_j$ for some positive integer j.) Suppose by way of contradiction that $h(p) = h(q)$, i.e., $(p_{n_i})_{i=1}^{\infty} = (q_{n_i})_{i=1}^{\infty}$. If n_k is the first n_i with $n_i > j$, then $p_j = f_j^{n_k}(p_{n_k}) = f_j^{n_k}(q_{n_k}) = q_j$. This is a contradiction, so h is 1-1.

To show h is continuous, let $\overleftarrow{O}$ be basic open in Y. So O is open in some X_{n_j}, where $n_j \in \{n_i\}_{i=1}^{\infty}$. Thus, $h^{-1}(\overleftarrow{O}) = \overleftarrow{O}$, which is open in X, so h is continuous.

To show h^{-1} is continuous, suppose $\overleftarrow{O}$ is basic open in X containing $(x_i)_{i=1}^{\infty}$. If O is open in some X_{n_i}, then $(h^{-1})^{-1}(\overleftarrow{O}) = h(\overleftarrow{O}) = \overleftarrow{O}$ is open in Y. On the other hand, suppose O is open in some X_j, where $j \neq n_i$ for all i. Then if n_k is the first n_i with $n_i > j$, $A = (f_j^{n_k})^{-1}(O)$ is an open set in X_{n_k} with $\overleftarrow{A}$ containing $(x_{n_i})_{i=1}^{\infty}$; moreover, $h^{-1}(\overleftarrow{A}) \subseteq \overleftarrow{O}$. So, in either case, h^{-1} is continuous.

Thus, h is a homeomorphism and the proof is complete. •

Theorem 3.3. *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit space. If there is a natural number N so that f_n is an onto homeomorphism for each $n \geq N$, then X is homeomorphic to X_N.*

Proof: Define a function $h : X \to X_N$ by $h((x_i)_{i=1}^{\infty}) = x_N$. We must show that h is a homeomorphism.

i) h is onto:

Because f_n is an onto homeomorphism for $n \geq N$, for each $x_N \in X_N$ and each $n > N$, $(f_N^n)^{-1}(x_N) = x_n$ for some $x_n \in X_n$. Also, for each $n < N$, $f_n^N(x_N) = x_n$ for some $x_n \in X_n$. It follows that for each $x_N \in X_N$, there exists a sequence $x = (x_1, x_2, \ldots, x_N, x_{N+1}, \ldots) \in X$ with $h(x) = x_N$. So h is onto.

ii) h is 1-1:

Suppose $h(x) = h(y)$; we must show that $x = y$. Since $h(x) = h(y)$, we know $x_N = y_N$, so that $x_n = f_n^N(x_N) = f_n^N(y_N) = y_n$ for all $n < N$. Moreover, since f_n is a homeomorphism for $n \geq N$, $x_n = (f_N^n)^{-1}(x_N) = (f_N^n)^{-1}(y_N) = y_n$ for all $n > N$.

So $x_n = y_n$ for each positive integer n. That is, $x = y$, and h is 1-1.

iii) h is continuous:

Let O be open in X_N. Then $h^{-1}(O) = \overleftarrow{O}$, which is open in X; thus, h is continuous.

iv) h^{-1} is continuous:

Let $\overleftarrow{O}$ be a basic open set in X containing $x = (x_i)_{i=1}^{\infty}$. We must show there exists an open set G in X_N containing $h(x) = x_N$ with $h^{-1}(G) \subset \overleftarrow{O}$. If $O \subset X_N$, then clearly $h^{-1}(O) \subset \overleftarrow{O}$, and $x_N \in O$, so $G = O$. Similarly, if $O \subset X_n$ with $n > N$, $f_N^n(O) = G$ is open in X_N containing x_N, and $h^{-1}(G) \subset \overleftarrow{O}$. Finally, if $O \subset X_n$ with $n < N$, then (since f_n^N is continuous, and O contains x_n) there is an open $G \subset X_N$ containing x_N with $f_n^N(G) \subset O$, so that $\overleftarrow{G} \subseteq \overleftarrow{O}$. It follows that $h^{-1}(G) \subset \overleftarrow{O}$.

All cases are accounted for, so h^{-1} is continuous.

Therefore, h is a homeomorphism and the proof is complete. •

It may be shown that, if each X_i is compact, the inverse limit space $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is a closed subspace of the product space $\prod_{i=1}^{\infty} X_i$. Thus, by the Tychonoff Theorem, if each X_i is compact, then X (a closed subspace of a compact space) is compact. However, we will prove directly that an inverse limit on compact spaces is compact (Theorem 3.4); we will then use this result to prove the Tychonoff Theorem for countable products $\prod_{i=1}^{\infty} X_i$. Finally, we will generalize the notion of inverse limit in order to prove the Tychonoff Theorem in its full generality.

Theorem 3.4. *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit space with X_i non-empty and compact for each i. Then X is non-empty and compact.*

Proof: Since each X_i is compact, each X_i is perfectly compact. We intend to show that X is perfectly compact.

Let $\{G_i\}_{i\in\mu}$ be a monotonic collection of non-empty subsets of X. (Need to show: there is a point $p = (p_i)_{i=1}^{\infty}$ in X such that p is either a point or limit point of every G_i.)

Define $G_{ij} = \pi_j(G_i)$.

Since $\{G_{i1}\}_{i\in\mu}$ is a monotonic collection of non-empty subsets in X_1, a perfectly compact space, there exists $p_1 \in \bigcap_{i\in\mu} \overline{G_{i1}}$, i.e., a point p_1 in X_1 that is a point or limit point of every G_{i1}. For convenience, let us say $f_1 = f$.

(We need to show that there exists some element in $f^{-1}(p_1)$ that is also in $\bigcap_{i\in\mu} \overline{G_{i2}}$.)

Because $\pi_1(G_i) = f \circ \pi_2(G_i)$, clearly $G_{i1} = f(G_{i2})$ for each i. Thus,

$p_1 \in \bigcap_{i\in\mu} \overline{G_{i1}} = \bigcap_{i\in\mu} \overline{f(G_{i2})} \subseteq \bigcap_{i\in\mu} \overline{f(\overline{G_{i2}})}$.

But the continuous image of $\overline{G_{i2}}$, a compact set, is compact (and hence, closed); therefore:

$p_1 \in \bigcap_{i\in\mu} \overline{G_{i1}} = \bigcap_{i\in\mu} \overline{f(G_{i2})} \subseteq \bigcap_{i\in\mu} \overline{f(\overline{G_{i2}})} = \bigcap_{i\in\mu} f(\overline{G_{i2}})$.

(We need to show that $\bigcap_{i\in\mu} f(\overline{G_{i2}}) \subseteq f(\bigcap_{i\in\mu} \overline{G_{i2}})$.)

Let $a \in \bigcap_{i\in\mu} f(\overline{G_{i2}})$. Since $f(\bigcap_{i\in\mu} \overline{G_{i2}})$ is closed, it will suffice to show that a is a point or limit point of $f(\bigcap_{i\in\mu} \overline{G_{i2}})$.

Proof by Contradiction: Suppose $O \subseteq X_1$ is an open set containing a but missing $f(\bigcap_{i\in\mu} \overline{G_{i2}})$. Then $f^{-1}(O)$ is open, contains $f^{-1}(a)$, and misses $\bigcap_{i\in\mu} \overline{G_{i2}}$. In particular, $f^{-1}(a)$ misses $\bigcap_{i\in\mu} \overline{G_{i2}}$, i.e., $f^{-1}(a) \cap (\bigcap_{i\in\mu} \overline{G_{i2}}) = \emptyset$.

Claim: There exists some $\overline{G_{j2}}$ for which $f^{-1}(a) \cap \overline{G_{j2}} = \emptyset$.

Justification: Suppose not. Then $f^{-1}(a) \cap \overline{G_{i2}} \neq \emptyset$ for all i.

Thus, since $\{f^{-1}(a) \cap \overline{G_{i2}}\}_{i\in\mu}$ is a monotonic collection of non-empty closed sets, by perfect compactness, we have that $\bigcap_{i\in\mu}(f^{-1}(a) \cap \overline{G_{i2}}) \neq \emptyset$. However, by set theory, we have $\bigcap_{i\in\mu}(f^{-1}(a) \cap \overline{G_{i2}}) = f^{-1}(a) \cap (\bigcap_{i\in\mu} \overline{G_{i2}})$, so that $f^{-1}(a) \cap (\bigcap_{i\in\mu} \overline{G_{i2}}) \neq \emptyset$. This is a contradiction.

So, there exists some $\overline{G_{j2}}$ for which $f^{-1}(a) \cap \overline{G_{j2}} = \emptyset$. But $a \in \bigcap_{i\in\mu} f(\overline{G_{i2}})$, so $a \in f(\overline{G_{i2}})$ for all i. That means for each i, there exists $y_i \in f^{-1}(a)$ such that $y_i \in \overline{G_{i2}}$.

In particular, for j, there exists some $y_j \in f^{-1}(a)$ such that $y_j \in \overline{G_{j2}}$. But $f(y_j) = a$, so $f^{-1}(a) \cap \overline{G_{j2}} \neq \emptyset$. (Contradiction.)

Thus, if $a \in \bigcap_{i\in\mu} f(\overline{G_{i2}})$, then $a \in f(\bigcap_{i\in\mu} \overline{G_{i2}})$. That is, $\bigcap_{i\in\mu} f(\overline{G_{i2}}) \subseteq f(\bigcap_{i\in\mu} \overline{G_{i2}})$, and that means $\bigcap_{i\in\mu} \overline{G_{i1}} \subseteq f(\bigcap_{i\in\mu} \overline{G_{i2}})$.

So, since there exists some point $p_1 \in \bigcap_{i\in\mu} \overline{G_{i1}}$, there exists some $p_2 \in \bigcap_{i\in\mu} \overline{G_{i2}}$ with $f(p_2) = f_1(p_2) = p_1$. The same argument shows that there is a point $p_3 \in \bigcap_{i\in\mu} \overline{G_{i3}}$ with $f_2(p_3) = p_2$, etc.

It is therefore easy to see that $(p_1, p_2, p_3, \ldots)$ is a point in X that is a point or limit point of each G_i. Thus, X is non-empty and perfectly compact. Since X is closed in itself and perfectly compact, X is compact. This completes the proof. •

Theorem 3.5. *Suppose $X = \prod_{i=1}^{\infty} X_i$ is a product space, $Y_n = \prod_{i=1}^{n} X_i$, and $f_n : Y_{n+1} \to Y_n$ is the continuous function defined by $f_n(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_n)$. Then $Y = \varprojlim\{Y_i, f_i\}_{i=1}^{\infty}$ is homeomorphic to X.*

Proof: Let $F : Y \to X$ be defined by

$$F((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots, (x_1, x_2, x_3, \ldots, x_n), \ldots) = (x_1, x_2, x_3, \ldots, x_n, \ldots).$$

Clearly F is 1-1 and onto; now we must show F is continuous. Let $O = \prod_{i=1}^{\infty} O_i$ be basic open in X, so that for some positive integer k, $O_i = X_i$ for all $i > k$. Then $F^{-1}(O) = \{((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots) \in Y \mid x_i \in O_i \text{ for } 1 \leq i \leq k\}$, i.e., the set of all points in Y whose kth coordinate $(x_1, x_2, \ldots, x_k)$ lies in $\prod_{i=1}^{k} O_i$. Since $\prod_{i=1}^{k} O_i$ is open in Y_k, $F^{-1}(O) = \overleftarrow{\prod_{i=1}^{k} O_i}$ is open in Y, and F is continuous.

To show F^{-1} is continuous at a given point $x = (x_1, x_2, x_3, \ldots)$ in X, let $\overleftarrow{O}$ be basic open in Y (where O is open in some Y_k) so that $F^{-1}(x) \in \overleftarrow{O}$. Since O is open in Y_k, there exists a basic open set $\prod_{i=1}^{k} O_i$ in Y_k that is a subset of O and contains the point $(x_1, x_2, \ldots, x_k)$. It follows that $(\prod_{i=1}^{k} O_i) \times (\prod_{i=k+1}^{\infty} X_i)$, which is open in X, contains x. However, $F^{-1}[(\prod_{i=1}^{k} O_i) \times (\prod_{i=k+1}^{\infty} X_i)] \subset \overleftarrow{O}$; F^{-1} is therefore continuous.

So F is a homeomorphism. •

Tychonoff Theorem for Countable Products. *Let $X = \prod_{i=1}^{\infty} X_i$ be a topological product space with X_i compact for each i. Then X is compact.*

Proof: By Theorem 3.5, if $Y_n = \prod_{i=1}^{n} X_i$, $Y = \varprojlim\{Y_i, f_i\}_{i=1}^{\infty}$ is homeomorphic to X. Any finite product of compact spaces is compact, so each Y_i is compact; thus, by Theorem 3.4, Y is compact. It follows that X is compact also. •

Thus, we have used inverse limits to prove the Tychonoff Theorem for countable products, i.e., that a product of countably many compact spaces is compact. We will now introduce a more general form of inverse limit that, among other things, will allow us to prove the general Tychonoff Theorem, i.e., that *any* product of compact spaces is compact.

A *directed set* I is a set with a partial order $<$ such that for each pair $\alpha, \beta \in I$, there exists some $\gamma \in I$ such that $\alpha < \gamma$ and $\beta < \gamma$. Suppose that for each $i \in I$, X_i is a topological space; also suppose there exists a collection of functions $\{f_i^j\}_{i<j}$ such that if $i < j < k$ then $f_i^j \circ f_j^k = f_i^k$. Define $X = \varprojlim\{X_i, f_i^j\}_{i<j\in I}$ to be the collection of all points $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ that satisfy $f_j^k(x_k) = x_j$ for all j, k (with $j < k$) in I. As before, if O_i is a subset of X_i, define $\overleftarrow{O_i} = \{x \in X | \; x_i \in O_i\}$. Then $X = \varprojlim\{X_i, f_i^j\}_{i<j\in I}$ is an *inverse limit space on the directed set* I, and a basis for the topology on X is $\{\overleftarrow{O} | \; O \text{ is open in some } X_i\}$.

Theorem 3.6. *Let $X = \varprojlim\{X_i, f_j^k\}_{i\in I,\ j<k\in I}$ be an inverse limit space on the directed set I, with f_j^k continuous for all $j, k \in I$, $j < k$. If each X_i is compact, then X is compact.*

Proof: Following the strategy of Theorem 3.4, we will show that X is perfectly compact: Let $\{G_\alpha\}_{\alpha\in\mu}$ be a monotonic collection of non-empty subsets of X. We must show that $\bigcap_{\alpha\in\mu} \overline{G_\alpha} \neq \emptyset$.

If the directed set I is order isomorphic with the natural numbers, then the proof of Theorem 3.4 shows how to obtain a point $(p_i)_{i\in I} \in \bigcap_{\alpha\in\mu} \overline{G_\alpha}$. However, if I is order isomorphic to some subset Φ of the ordinals containing (at least) one limit ordinal, we must use transfinite induction. Assume that all entries of the point $(p_i)_{i\in I}$ have been defined up to but not including the kth entry, so that, if $i, j < k$, $p_i \in \bigcap_{\alpha\in\mu} \overline{\pi_i(G_\alpha)}$ and $f_i^j(p_j) = p_i$. Now we must show how to find a kth entry p_k with the needed properties, namely, $p_k \in \bigcap_{\alpha\in\mu} \overline{\pi_k(G_\alpha)}$ and $f_i^k(p_k) = p_i$ for all $i < k$.

If the kth entry of I corresponds to a non-limit ordinal in Φ, then k has an immediate predecessor j in I, and p_j has already been defined. Clearly the same argument given in the proof of Theorem 3.4 (i.e., the argument that finds p_2 given p_1) suffices here to find p_k given p_j.

However, suppose the kth entry of I corresponds to a limit ordinal β in Φ. By the argument given in Theorem 3.4, for each $i < k \in I$, the set $(f_i^k)^{-1}(p_i) \cap \bigcap_{\alpha\in\mu} \overline{\pi_k(G_\alpha)} \neq \emptyset$.

We note that, if $i < j$, $(f_j^k)^{-1}(p_j) \subseteq (f_i^k)^{-1}(p_i)$. Thus, $\{(f_i^k)^{-1}(p_i)\}_{i<k}$ is a monotonic collection of non-empty closed subsets of X_k. Moreover, for each $i < k$, $f_i^k(\bigcap_{\alpha\in\mu} \overline{\pi_k(G_\alpha)})$ contains p_i, so that $(f_i^k)^{-1}(p_i) \cap \bigcap_{\alpha\in\mu} \overline{\pi_k(G_\alpha)}$ is closed and non-empty. Thus, $\{(f_i^k)^{-1}(p_i) \cap \bigcap_{\alpha\in\mu} \overline{\pi_k(G_\alpha)}\}_{i<k}$ is a monotonic collection of non-empty closed subsets of X_k. Since X_k is perfectly compact, there is a point $p_k \in X_k$ that lies in each set in this collection. Thus, the kth entry of $(p_i)_{i\in I}$ has been defined, and (by transfinite induction) the needed point $(p_i)_{i\in I}$ exists. That means X is perfectly compact and hence, compact. •

It follows from Theorem 3.6 that such an inverse limit (with continuous bonding maps) on a directed set I of *any* size is compact, provided that each factor space X_i is compact. Now, after a few more lemmas, we will be ready to prove the general Tychonoff Theorem.

Lemma 3.7. *Let I be a well-ordered directed set. For each $i \in I$, define $Y_i = \prod_{j\leq i} X_j$. If $k > h$, let $f_h^k : Y_k \to Y_h$ be the continuous function defined by $f_h^k((x_i)_{i\leq k}) = (x_i)_{i\leq h}$. Then the inverse limit $Y = \varprojlim\{Y_i, f_j^k\}_{i\in I,\ j<k\in I}$ is homeomorphic to $X = \prod_{i\in I} X_i$.*

Proof: Let $F : Y \to X$ be defined by $F(\prod_{\alpha\in I}(x_i)_{i\leq\alpha}) = (x_i)_{i\in I}$. We need to show that F is a homeomorphism.

Clearly F is onto and 1-1; we must show F is continuous.

Let O be basic open in X, so that $O = \prod_{i \in I} O_i$ where O_i is open in X_i for each i, and $O_i = X_i$ for all but finitely many i. Rename those finitely many open sets as $O_{i_1}, O_{i_2}, \ldots, O_{i_n}$; thus, for $j = 1, 2, \ldots, n$, $O_{i_j} \subsetneq X_{i_j}$.

Then $F^{-1}(O) = \bigcap_{j=1}^{n} \overleftarrow{\{x \in Y_{i_j} | x_{i_j} \in O_{i_j} \subsetneq X_{i_j}\}}$ is an open set in Y. So F is continuous.

To show F^{-1} is continuous, let $\overleftarrow{O}$ be basic open in Y (so that O is open in Y_α for some $\alpha \in I$). Since O is open in Y_α, there exists a basic open set $\prod_{i \leq \alpha} O_i$ in Y_α that is a subset of O. It follows that $F^{-1}[(\prod_{i \leq \alpha} O_i) \times (\prod_{i > \alpha} X_i)] \subset \overleftarrow{O}$, and F^{-1} is continuous. Thus, F^{-1} is a homeomorphism. •

Lemma 3.8. *Let I be an arbitrary index set, and let $\tilde{I}$ be the set I with a well-ordering placed upon it. Then $X = \prod_{i \in I} X_i$ is homeomorphic to $\tilde{X} = \prod_{i \in \tilde{I}} X_i$.*

Proof: Let $F : X \to \tilde{X}$ be defined by $F((x_i)_{i \in I}) = (x_i)_{i \in \tilde{I}}$. Clearly F is onto and 1-1; it remains to show that F and F^{-1} are continuous.

Let O be basic open in $\tilde{X}$, so that $O = \prod_{i \in \tilde{I}} O_i$ where $O_i = X_i$ for all but finitely many i. Then $F^{-1}(O) = \prod_{i \in I} O_i$, which is open in X. So F is continuous, and the argument is easily altered to show that F^{-1} is also continuous. So F is a homeomorphism. •

With these two lemmas in hand, we are finally prepared to prove the general Tychonoff Theorem.

Tychonoff Theorem: *Suppose for each i in some index set I, X_i is a compact topological space. Then $\prod_{i \in I} X_i$ is compact.*

Proof: The result is already known if the index set I is finite, so assume I is infinite. By Lemma 3.8, without loss of generality we may assume that the index set I is well-ordered. Since I is easily re-ordered in a way that keeps I well-ordered without having a last element, we may also assume without loss of generality that I is a directed set. We will use transfinite induction: Suppose that, for any given $\alpha \in I$, $\prod_{i \leq \alpha} X_i$ is compact. We will show that $X = \prod_{i \in I} X_i$ is compact.

For each $i \in I$, let $Y_i = \prod_{j \leq i} X_j$. Define f_h^k for $h < k \in I$ as in Lemma 3.7. Then $Y = \varprojlim \{Y_i, f_h^k\}_{i \in I,\ h<k \in I}$ is homeomorphic to X, by Lemma 3.7. Since, by the induction hypothesis, Y_i is compact for each $i \in I$, by Theorem 3.6, Y is compact. Thus, X is also compact and the proof is complete. •.

We conclude Chapter 3 with two examples of complex topological spaces that may be easily characterized using an inverse limit.

1) For each i, let $X_i = [0,1]$. Suppose $f : [0,1] \to [0,1]$ is given by

$$f(t) = \left\{ \begin{array}{c} \frac{3}{2}t, 0 \leq t \leq \frac{2}{3} \\ \frac{5}{3} - t, \frac{2}{3} \leq t \leq 1 \end{array} \right\}$$

Let $f_i = f$ for each i. Then $\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \varprojlim\{[0,1], f\}_{i=1}^{\infty}$ is homeomorphic to the topologist's sine curve, i.e., $\{(x, \sin(\frac{1}{x})) \mid x \in [-1,0)\} \cup \{(0,x) \mid x \in [-1,1]\}$.

2) For each i, let $X_i = [0,1]$. Suppose $f : [0,1] \to [0,1]$ is given by

$$f(t) = \left\{ \begin{array}{c} 2t,\ 0 \leq t \leq \frac{1}{2} \\ \\ 2-2t, \frac{1}{2} \leq t \leq 1 \end{array} \right\}$$

Let $f_i = f$ for each i. Then $\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \varprojlim\{[0,1], f\}_{i=1}^{\infty}$ is homeomorphic to the Knaster continuum, a.k.a., "the bucket handle."

Chapter 4

Inverse Limits of Upper Semi-Continuous Set Valued Functions

Suppose X and Y are compact Hausdorff spaces, and define 2^Y to be the set of all non-empty compact subsets of Y. A function $f : X \to 2^Y$ is called *upper semi-continuous* (u.s.c.) if for any $x \in X$ and open V in Y containing $f(x)$, there exists an open U in X containing x so that $f(u) \subset V$ for all $u \in U$. Upper semi-continuity is a generalization of continuity; hence, using upper semi-continuous bonding functions instead of continuous bonding maps provides us with a more generalized notion of an inverse limit.

Suppose that, for each positive integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function. We define $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ to be the set of all points in $\prod_{i=1}^{\infty} X_i$ with $x_i \in f_i(x_{i+1})$ for all i. (For convenience, we shall abbreviate $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ by $\varprojlim \mathbf{f}$.) Then we say $\varprojlim \mathbf{f}$ is an *inverse limit space with u.s.c. bonding functions*, and a basis for the topology on $\varprojlim \mathbf{f}$ is $\{O \cap \varprojlim \mathbf{f} \mid O \text{ is basic open in } \prod_{i=1}^{\infty} X_i\}$. As in Chapter 3, if O_i is a subset of X_i, we define $\overleftarrow{O_i} = \{x \in \varprojlim \mathbf{f} \mid x_i \in O_i\}$; if O_i is open in X_i, then $\overleftarrow{O_i}$ is open in $\varprojlim \mathbf{f}$.

Remark: However, unlike the inverse limit spaces seen in Chapter 3, in general, the collection $\{\overleftarrow{O} \mid O \text{ is open in some } X_i\}$ is **not** a basis for $\varprojlim \mathbf{f}$. An example is given in the appendix to explain why this is so.*

In [1], Ingram and Mahavier not only prove generalizations of the sorts of theorems already seen in Chapter 3, but also provide examples to show when such results do *not* generalize. In this chapter, I give my own proofs of their theorems and explain their counterexamples in detail. (Note that the theorems are numbered here in a way that is consistent with the original numbering in [1]; for example, Theorem 2.1 from [1] has been relabeled 4.2.1, etc.)

First, Ingram and Mahavier introduce the useful notion of the graph of an upper semi-continuous function. If X and Y are compact Hausdorff spaces and $f : X \to 2^Y$ is u.s.c., the *graph of* f (abbreviated $G(f)$) is the set $\{(x, y) \in X \times Y \mid y \in f(x)\}$.

Theorem 4.2.1. *Suppose each of X and Y is a compact Hausdorff space and M is a subset of $X \times Y$ such that if x is in X then there is a point y in Y such that (x, y) is in M. Then M is closed if and only if there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$.*

Proof: Assume the hypothesis.

($\Leftarrow$) Suppose there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$. We need to show that M is closed.

Proof by contradiction: Let (x, y) be a limit point of M with $(x, y) \notin M$. We know the set $f(x)$ is compact, and hence closed; moreover, $\{x\} \times f(x)$ is a subset of M. Because $(x, y) \notin M$, we have $y \notin f(x)$.

Y is a compact Hausdorff space, so Y is regular. Thus, there exist disjoint open O_1, O_2 in Y with $f(x) \subset O_1$ and $y \in O_2$. O_1 contains $f(x)$, so by u.s.c. there exists an open U in X containing x so that $f(U) \subset O_1$.

$U \times O_2$ is open in $X \times Y$ and contains (x, y), a limit point of M, so $U \times O_2$ must contain some other point $(x_0, y_0) \in M$. We note that not every point in $U \times O_2$ can have x as its first coordinate, for otherwise, $U \times O_2$ would have no points in M. (For, each point would be of form (x, z) where $z \notin f(x)$, so that $(x, z) \notin G(f) = M$.) Thus, there is some $(x_0, y_0) \in U \times O_2$ with $x_0 \neq x$.

However, $x_0 \in U$, so $f(x_0) \subseteq O_1$. But $(x_0, y_0) \in M$, so $y_0 \in f(x_0)$. It follows that a point in $f(x_0)$ (namely, y_0) lies in O_2, which was disjoint from O_1. This is a contradiction, so M is closed.

($\Rightarrow$) Suppose that M is closed. We must show that there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$.

For each $x \in X$, consider $\{x\} \times Y$. This set is closed in $X \times Y$, so that $K_x = (\{x\} \times Y) \cap M$ is also closed and non-empty. A closed subset of $X \times Y$ is compact, so K_x is compact. Thus, $\pi_2(K_x)$ is compact in Y.

Define $f : X \to 2^Y$ by $f(x) = \pi_2(K_x)$; we must show f is an upper semi-continuous function.

Let V be open in Y with $\pi_2(K_x) = f(x) \subseteq V$. We need to show there exists an open set U in X with $x \in U$ such that $f(U) \subset V$.

Proof by contradiction: Suppose no such U exists. Let $\{u_\alpha\}_{\alpha\in\mu}$ be the set of all points u_α in X with $f(u_\alpha) \not\subset V$. Then every open set in X containing x must contain infinitely many u_α's. (For, suppose not. Then some open O containing x contains only finitely many u_α's, say, $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_k}$. Thus, because X is regular, there exists an open set R containing x that misses $(X - O) \cup \{u_{\alpha_i}\}_{i=1}^k$, and hence, misses all u_α's.)

Claim: the collection of points $W = \{(u_\alpha, y_\alpha) | \alpha \in \mu, y_\alpha \in f(u_\alpha), y_\alpha \notin V\}$ has a limit point (x, z) with $z \notin V$.

For, suppose not. Then $\{x\} \times (Y - V)$ and $\overline{W}$ are disjoint closed sets in $X \times Y$. So, since $X \times Y$ is normal, there exist disjoint open O_1 and O_2 containing $\{x\} \times (Y - V)$ and $\overline{W}$ respectively. Hence, for each $(x, z) \in \{x\} \times (Y - V)$, we may find a basic open set $(A \times B)_{(x,z)}$ about (x, z) lying in O_1. By the compactness of $X \times Y$, a finite number (say, n) of these open sets covers $\{x\} \times (Y - V)$, so that $\{(\bigcap_{i=1}^n A_i) \times B_j\}_{j=1}^n$ also covers $\{x\} \times (Y - V)$. We note that $\overline{W}$ misses the union of the members of this finite open cover.

$\bigcap_{i=1}^{n} A_i$ is open in X, contains x, and contains no u_α such that $f(u_\alpha) \not\subset V$. This means there does exist an open set (namely, $\bigcap_{i=1}^{n} A_i$) in X with $f(\bigcap_{i=1}^{n} A_i) \subset V$. This is a contradiction, so there does exist a point $(x, z) \in \{x\} \times Y$, with $z \notin V$, that is a limit point of W. But $W \subseteq M$, which is closed, so any limit point of W is an element of M. That is, $(x, z) \in M$, and $z \in f(x)$. But $f(x) \subset V$, and $z \notin V$. A contradiction has been reached, so the proof is complete. •

For the next two theorems, let the following be a standing hypothesis: suppose that for each positive integer n, X_n is a non-empty compact Hausdorff space and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continous bonding function. If $\prod = \prod_{i=1}^{\infty} X_i$, let $G_n = \{x \in \prod \mid x_i \in f_i(x_{i+1})$ for $i \leq n\}$.

Theorem 4.3.1. *For each positive integer n, G_n is a non-empty compact set.*

Proof: We first show that G_n is non-empty for each positive integer n. Pick any point $x_{n+1} \in X_{n+1}$. $f_n(x_{n+1})$ is compact (and non-empty) in X_n; we may pick a point $x_n \in f_n(x_{n+1})$, so that $f_{n-1}(x_n)$ is compact (and non-empty) in X_{n-1}; next, pick a point $x_{n-1} \in f_{n-1}(x_n)$, and so forth. By finishing this process at x_1 and then (for $i > n+1$) choosing x_i from X_i arbitrarily, we find that $\{x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots\} \in G_n$. So $G_n \neq \emptyset$.

Now we must show that G_n is compact. Since G_n is a subspace of $\prod$, which is compact, it will suffice to show that G_n is closed.

Proof by contradiction: Let $p = \{p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+2}, \ldots\}$ be a limit point of G_n in $\prod$, with $p \notin G_n$. Since $p \notin G_n$, it follows that $p_i \notin f_i(p_{i+1})$ for some i, $1 \leq i \leq n$.

X_i is regular, so (in X_i) there exist disjoint open sets O_{p_i} and $O_{f_i(p_{i+1})}$ containing p_i and $f_i(p_{i+1})$ respectively. By the upper semi-continuity of f_i, there exists an open U in X_{i+1} containing p_{i+1} so that $f(U) \subset O_{f_i(p_{i+1})}$. That is, $p_i \notin f(U)$. Hence, for all $u \in U$, $f(U) \subset O_{f_i(p_{i+1})}$ and $f(u) \cap O_{p_i} = \emptyset$.

Thus, $X_1 \times X_2 \times \cdots \times X_{i-1} \times O_{p_i} \times U \times X_{i+2} \times \cdots$ is open in $\prod$, contains p, but misses G_n. However, p was a limit point of G_n, so this is a contradiction.

So G_n is closed in $\prod$, and therefore, G_n is compact. •

Theorem 4.3.2. $K = \varprojlim \mathbf{f}$ is non-empty and compact.

Proof: $K = \varprojlim \mathbf{f} = \bigcap_{n=1}^{\infty} G_n$. But $\{G_n\}_{n=1}^{\infty}$ is a monotonic collection of non-empty closed (compact) subsets of $\prod$; so, since $\prod$ is perfectly compact, $\bigcap_{n=1}^{\infty} G_n$ is non-empty. Moreover, any intersection of closed sets is closed, so $\bigcap_{n=1}^{\infty} G_n$ is also closed, and therefore compact. •

Having dealt with the issue of compactness, we now turn to theorems about connectedness.

Theorem 4.4.1. *Suppose X, Y are compact Hausdorff spaces, X is connected, $f : X \to 2^Y$ is u.s.c., and for each x in X, $f(x)$ is connected. Then the graph $G(f)$ is connected.*

Proof: Suppose by way of contradiction that $G(f)$ is not connected. Then, since $G(f)$ is closed, $G(f) = H \cup K$, a union of disjoint closed sets. For a given x, define $(x, f(x)) = \{(x, y) |\ y \in f(x)\}$. We note that $K \cap (x, f(x))$ and $H \cap (x, f(x))$ cannot both be non-empty. (For, if they were, then $(x, f(x)) = [K \cap (x, f(x))] \cup [H \cap (x, f(x))]$, a union of two disjoint closed point sets. But $f(x)$ was connected.) Thus, for all $x \in X$, either $K \cap (x, f(x)) = \emptyset$ or $H \cap (x, f(x)) = \emptyset$. That is, for each x, $(x, f(x))$ must lie either in H or K but not both.

Because $X \times Y$ is compact and Hausdorff, $X \times Y$ is normal. So there exist disjoint open sets O_H and O_K containing H and K respectively. Without loss of generality, consider a given set $(x, f(x))$ that is a subset of K. We may find a union of basic open sets of form $B_i = A_{i_j} \times R_{i_j}$ in $X \times Y$ that contains $(x, f(x))$ and lies in O_K. By the compactness of $(x, f(x))$, only finitely many (say, n) B_i's cover $(x, f(x))$. Thus, $\{(\bigcap_{j=1}^n A_{i_j}) \times R_{i_t}\}_{t=1}^n$ is an open cover of $(x, f(x))$.

Since $R = \bigcup_{t=1}^n R_{i_t}$ is open in Y and contains $f(x)$, by u.s.c. there exists an open U in X containing x so that $f(U) \subset R$. Then $V = U \cap \bigcap_{j=1}^n A_{i_j}$ is also open in X, contains x, and clearly $f(V) \subset R$. (Indeed, $\{(x, f(x)) \mid x \in V\} \subset \bigcup_{t=1}^n((\bigcap_{j=1}^n A_{i_j}) \times R_{i_t})$.).

Hence, no points z in V can be such that $(z, f(z)) \subset H$. (For that would contradict the fact that $(z, f(z)) \subset O_K$, where $O_K \cap H = \emptyset$.) So, we have found an open set $V = V_x$ in X containing x so that $V_x \cap \pi_1(H) = \emptyset$. Such an open V_x can be found for each x with $(x, f(x)) \subset K$, so that the union of all such V_x's is open in X and contains the set $\{x \mid (x, f(x)) \subset K\}$.

But such an open set (disjoint from the union of the V_x's) can also be found containing the set $\{x \mid (x, f(x)) \subset H\}$. So, we have disjoint (non-empty) open sets in X whose union equals X itself, and this contradicts the fact that X was connected. •

Theorem 4.4.2. *Suppose that X and Y are compact Hausdorff spaces, Y is connected, and f is an upper semi-continuous function from X into 2^Y such that for each y in Y, $\{x \in X \mid y \in f(x)\}$ is a non-empty, connected set. Then $G(f)$ is connected.*

Proof: Suppose by way of contradiction that $G(f) = H \cup K$, a union of disjoint closed sets. For each $y \in Y$, let $A_y = \{x \in X \mid y \in f(x)\}$, and let $(A_y, y) = \{A_y\} \times \{y\}$. Each A_y is connected, so for each y, either $(A_y, y) \subset H$ or $(A_y, y) \subset K$ but not both. We know that $H = \bigcup_{(A_y,y)\subset H}\{(A_y, y)\}$ and $K = \bigcup_{(A_y,y)\subset K}\{(A_y, y)\}$; thus, the sets $\pi_2(H) = \{y \mid (A_y, y) \subset H\}$ and $\pi_2(K) = \{y \mid (A_y, y) \subset K\}$ are disjoint closed sets whose union is Y. However, Y is connected, so this is a contradiction. •

Next, it will be useful to extend the notion of the graph of one function, $G(f)$, to the graph of a finite sequence of functions. If for $1 \leq i \leq n$, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c., we define $G(f_1, f_2, \ldots, f_n) = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \le i \le n\}$.

Theorem 4.4.3. *Suppose $X_1, X_2, \ldots, X_{n+1}$ is a finite collection of Hausdorff continua and $f_1, f_2, \ldots, f_n$ is a finite collection of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for $1 \le i \le n$. If $f_i(x)$ is connected for each x in X_{i+1} and each $i, 1 \le i \le n$, then $G(f_1, f_2, \ldots, f_n)$ is connected.*

Proof: We will use induction on the number of spaces, n. For the base case, suppose X_1, X_2 are Hausdorff continua, $f_1 : X_2 \to 2^{X_1}$ is an upper-semi continuous function, and $f_1(x)$ is connected for each x in X_2. Then $G(f_1)$ is connected by Theorem 4.4.2.

Now suppose the theorem is true for a graph on n spaces; we must show that the theorem also holds for $n+1$ spaces. That is, we must show $G(f_1, f_2, \ldots, f_n)$ is connected.

By the inductive hypothesis, the graph $G(f_2, f_3, \ldots, f_n)$ is connected. Define an upper semi-continuous function $f^* : G(f_2, f_3, \ldots, f_n) \to 2^{X_1}$ by $f^*(x_2, x_3, \ldots, x_{n+1}) = f_1(x_2)$. To show that f^* is indeed upper semi-continuous, let $(x_2, x_3, \ldots, x_{n+1})$ be in $G(f_2, f_3, \ldots, f_n)$, so that $f^*(x_2, x_3, \ldots, x_{n+1}) = f_1(x_2)$. Let V be an open set in X_1 that contains $f_1(x_2)$. We need to find an open set in $G(f_2, f_3, \ldots, f_n)$ containing x whose image lies in V. Since f_1 is u.s.c., there exists some open U in X_2 (with $x_2 \in V$) so that $f_1(U)$ is a subset of V. Thus, $O = (U \times X_3 \times X_4 \times \ldots \times X_{n+1}) \cap G(f_2, f_3, \ldots, f_n)$ is an open set in $G(f_2, f_3, \ldots, f_n)$ containing $(x_2, x_3, \ldots, x_{n+1})$ such that $f^*(O) \subseteq V$.

Thus, f^* is u.s.c. Moreover, $f^*(x_2, x_3, \ldots, x_{n+1})$ is connected for all $(x_2, x_3, \ldots, x_{n+1}) \in G(f_2, f_3, \ldots, f_n)$. (For, $f^*(x_2, x_3, \ldots, x_{n+1}) = f_1(x_2)$, which was assumed to be connected.) Thus, by Theorem 4.4.2, the graph of f^* is connected. However, the graph of f^* is precisely the set of all ordered pairs $(x_1, x_2, x_3, \ldots, x_{n+1})$ with $x_i \in f_i(x_{i+1})$ for each i. This set is in fact $G(f_1, f_2, \ldots, f_n)$, so $G(f_1, f_2, \ldots, f_n)$ has been shown to be connected and the proof is complete. •

Theorem 4.4.4. *Suppose that X_i is a Hausdorff continuum for each i and $f_i(x)$ is connected for each $x \in X_{i+1}$. Then G_n is connected for each positive integer n.*

Proof: We note that $G_n = G(f_1, f_2, ..., f_n) \times \prod_{i=n+2}^{\infty} X_i$. Since $G(f_1, f_2, ..., f_n)$ is connected (by Theorem 4.4.3) and $\prod_{i=n+2}^{\infty} X_i$ is connected as well, G_n is connected. •

Theorem 4.4.5. *Suppose $X_1, X_2, \ldots, X_{n+1}$ is a finite collection of Hausdorff continua and $f_1, f_2, \ldots, f_n$ is a finite collection of u.s.c. functions such that $f_i : X_{i+1} \to 2^{X_i}$ for $1 \leq i \leq n$. If for each i, $1 \leq i \leq n$ and each $y \in X_i$, $\{x \in X_{i+1} | y \in f_i(x)\}$ is a non-empty, connected set, then $G(f_1, f_2, ..., f_n)$ is connected.*

Proof: To get this result, we shall adjust Mahavier's proof of Theorem 4.4.3. By Theorem 4.4.2, the theorem is true for only one bonding function f_1.

Assume the inductive hypothesis. That is, assume that if for all i and for all $y \in X_i$, $\{x \in X_{i+1} \mid y \in f_i(x)\}$ is a non-empty, connected set, then the graph on $< n+1$ u.s.c. functions is connected.

Need: $G(f_1, f_2, ..., f_n, f_{n+1})$ is connected.

By hypothesis, $G(f_1, f_2, ..., f_n)$ is connected. Assume by way of contradiction that H and K are mutually separated non-empty point sets with $H \cup K = G(f_1, f_2, ..., f_n, f_{n+1})$. Since the graph is closed, we know that H and K are in fact disjoint closed sets.

Let $h : G(f_1, f_2, ..., f_n, f_{n+1}) \rightarrow G(f_1, f_2, ..., f_n)$ be the continuous map defined by $h(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n)$. Since $h(G(f_1, f_2, ..., f_n, f_{n+1})) = h(H \cup K) = G(f_1, f_2, ..., f_n)$ is connected, there is a point $p = (p_1, p_2, \ldots, p_n, p_{n+1})$ belonging to $h(H)$ and $h(K)$. Note that both $h(H)$ and $h(K)$ are compact and hence, closed.

Thus, $\{(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}) \in G(f_1, f_2, ..., f_n, f_{n+1}) |\ x_i = p_i$ for $1 \leq i \leq n+1,\ x_{n+2} \in \{z \in X_{n+2} \mid p_{n+1} \in f_{n+2}(z)\}\}$ is a connected set, because it is a product of connected sets. But this set intersects both H and K, so H and K could not have been mutually separated. So, we have a contradiction and the proof is complete. •

Theorem 4.4.6. *Let X_i be a Hausdorff continuum for each positive integer i. Suppose $f_i : X_{i+1} \rightarrow 2^{X_i}$ is u.s.c. and for each $x_i \in X_i$, $\{y \in X_{i+1} \mid x_i \in f_i(y)\}$ is a non-empty, connected set. Then for each positive integer n, G_n is connected.*

Proof: By Theorem 4.4.5, $G(f_1, f_2, \ldots, f_{n+1})$ is connected. We note that $G_n = G(f_1, f_2, \ldots, f_{n+1}) \times \prod_{i=n+2}^{\infty} X_i$. Since X_i is a continuum for each integer $i \geq n+2$, we have that $\prod_{i=n+2}^{\infty} X_i$ is a continuum. Thus, G_n is a product of two connected sets, and hence, is connected. •

Theorem 4.4.7. *Suppose that for each positive integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each x in X_{i+1}, $f_i(x)$ is connected. Then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

Proof: By Theorems 4.3.1 and 4.4.4, for each positive integer n, G_n is a non-empty, compact, connected set; that is, each G_n is a (Hausdorff) continuum. Moreover, since $G_{n+1} \subseteq G_n$ for each n, $\{G_n\}_{n=1}^{\infty}$ is a monotonic collection of Hausdorff continua. That means $\bigcap_{n=1}^{\infty} G_n$ is a Hausdorff continuum. But $\varprojlim \mathbf{f} = \bigcap_{n=1}^{\infty} G_n$, so the result is proven. •

Theorem 4.4.8. *Suppose that for each positive integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each $x \in X_i$, $\{y \in X_{i+1} \mid x \in f_i(y)\}$ is a non-empty, connected set. Then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

Proof: By Theorems 4.3.1 and 4.4.6, for each positive integer n, G_n is a Hausdorff continuum. The rest of the proof is the same as the proof of Theorem 4.4.7. •

Next, Mahavier and Ingram give a generalized version of the "space-skipping" theorem seen in Chapter 3. However, we must first define the notion of composition of u.s.c. functions. Let X, Y, and Z be compact Hausdorff spaces, and suppose $f : X \to 2^Y$ and $g : Y \to 2^Z$ are u.s.c. functions. Then $g \circ f : X \to 2^Z$ is defined by $(g \circ f)(x) = \{z \in Z|$ there exists $y \in Y$ such that $y \in f(x)$ and $z \in g(y)\}$.

Theorem 4.5.1. *Suppose $X_1, X_2, \ldots,$ is a sequence of compact Hausdorff spaces and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. for each positive integer i. If $n_1, n_2, \ldots,$ is an increasing sequence of positive integers, let $g_1, g_2, \ldots$ be the sequence of functions with the property that $g_i = f_{n_i} \circ f_{n_i+1} \circ \cdots \circ f_{n_{i+1}-1}$ for each i. If $F : \prod_{i>0} X_i \to \prod_{i>0} X_{n_i}$ is given by $F(x_1, x_2, x_3, \ldots) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$, then $F|\varprojlim \mathbf{f}$ is a continuous transformation from $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{g}$.*

Proof: Let $O = (\prod_{i=1}^{\infty} O_{n_i}) \cap \varprojlim \mathbf{g}$ be basic open in $\varprojlim \mathbf{g}$ (where O_{n_i} is open in X_{n_i}, and for some positive integer k, $O_{n_i} = X_{n_i}$ for $i \geq k$). Then $(F|\varprojlim \mathbf{f})^{-1}(O) = (\prod_{j=1}^{\infty} O_j) \cap \varprojlim \mathbf{f}$, where if $j = n_i$ for some i, $O_j = O_{n_i}$, and for all other j, $O_j = X_j$. Since $(F|\varprojlim \mathbf{f})^{-1}(O)$ is open in $\varprojlim \mathbf{f}$, $F|\varprojlim \mathbf{f}$ is continuous. The fact that $F|\varprojlim \mathbf{f}$ maps onto $\varprojlim \mathbf{g}$ is clear. •

Theorem 4.5.2. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be sequences of compact Hausdorff spaces and, for each positive integer i, let $f_i : X_{i+1} \to 2^{X_i}$ and $g_i : Y_{i+1} \to 2^{Y_i}$ be u.s.c. functions. Suppose further that, for each positive integer i, $\varphi_i : X_i \to Y_i$ is a mapping such that $\varphi_i \circ f_i = g_i \circ \varphi_{i+1}$. Then the function $\varphi : \varprojlim \mathbf{f} \to \varprojlim \mathbf{g}$ given by $\varphi(x) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \ldots)$ is continuous. Moreover, φ is 1-1 (and surjective) if each φ_i is 1-1 (and surjective).*

Proof: First, we must show that φ maps into $\varprojlim \mathbf{g}$. Let $p = (p_1, p_2, \ldots) \in \varprojlim \mathbf{f}$; we need to show that $\varphi(p) \in \varprojlim \mathbf{g}$. That is, we need to show that for any i, $\varphi_i(p_i) \in g_i(\varphi_{i+1}(p_{i+1}))$. By hypothesis, $g_i(\varphi_{i+1}(p_{i+1})) = \varphi_i(f_i(p_{i+1}))$; however, $p_i \in f_i(p_{i+1})$, so $\varphi_i(p_i) \in \varphi_i(f_i(p_{i+1})) = g_i(\varphi_{i+1}(p_{i+1}))$. Thus, φ does map into $\varprojlim \mathbf{g}$; it remains to show that φ is continuous.

Let $x = (x_1, x_2, \ldots) \in \varprojlim \mathbf{f}$, and let $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{g}$ be basic open in $\varprojlim \mathbf{g}$, with O containing $\varphi(x) = (\varphi_1(x_1), \varphi_2(x_2), \ldots)$. We need to show there exists an open set U in $\varprojlim \mathbf{f}$ containing x so that $\varphi(U) \subseteq O$.

We note that, since $\prod_{i=1}^{\infty} O_i$ is basic open in $\prod_{i=1}^{\infty} Y_i$, O_i is open in Y_i for each i; also, for some positive integer k, if $i > k$, $O_i = Y_i$. Now, since each φ_i is continuous, for all i the set $U_i = \varphi_i^{-1}(O_i)$ is open in X_i and contains x_i. Hence, the open set $U = \bigcap_{i=1}^{k} \overleftarrow{U_i}$ contains $(x_1, x_2, \ldots)$. To show that $\varphi(U) \subseteq O$, let us assume $p = (p_1, p_2, \ldots) \in U$ and show that $\varphi(p) \in O$. For $i \leq k$, $p_i \in U_i = \varphi_i^{-1}(O_i)$, so we have that $\varphi_i(p_i) \in O_i$. For $i > k$, since $O_i = Y_i$, $\varphi_i(p_i) \in O_i$ automatically. So, since $p \in \varprojlim \mathbf{f}$, $\varphi(p) \in \varprojlim \mathbf{g}$ and $\varphi(p) \in O$. Thus, $\varphi(U) \subseteq O$, and we have shown that φ is continuous.

Finally, we will show that if each φ_i is 1-1 (and surjective), then φ is 1-1 (and surjective). First, suppose $\varphi(x) = \varphi(y)$, so that $\varphi_i(x_i) = \varphi_i(y_i)$ for each i. Since each φ_i is 1-1, $x_i = y_i$ for each i. Thus, $x = y$ and φ is 1-1. Now suppose that each φ_i is also surjective, and let $y = (y_1, y_2, y_3, \ldots) \in \varprojlim \mathbf{g}$. Again, because φ_i is surjective for each i, it follows that (for each i) there exists some $x_i \in X_i$ with $\varphi_i(x_i) = y_i$. Then $\varphi(x_1, x_2, x_3, \ldots) = (y_1, y_2, y_3, \ldots)$, but we must verify that $(x_1, x_2, x_3, \ldots) \in \varprojlim \mathbf{f}$; i.e., we must show that $x_i \in f_i(x_{i+1})$ for each i.

We note that $y_i \in g_i(y_{i+1}) = g_i(\varphi_{i+1}(x_{i+1})) = \varphi_i(f_i(x_{i+1}))$; thus, $\varphi_i^{-1}(y_i) \in f_i(x_{i+1})$. However, since φ_i was 1-1, $\varphi_i^{-1}(y_i) = x_i$, so $x_i \in f_i(x_{i+1})$, as desired. Thus, $x \in \varprojlim \mathbf{f}$ and φ is surjective. •

Given that X is a compact Hausdorff space and $f : X \to 2^X$ and $g : X \to 2^X$ are u.s.c. functions, we say f and g are *topologically conjugate* if there exists a homeomorphism h with $h(X) = X$ and $h \circ f = g \circ h$.

Theorem 4.5.3. *Suppose X is a compact Hausdorff space. If $f : X \to 2^X$ and $g : X \to 2^X$ are topologically conjugate u.s.c. functions, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$.*

Proof: Since f and g are topologically conjugate, there is a homeomorphism h with $h(X) = X$ and $h \circ f = g \circ h$. Let $\varphi : \varprojlim \mathbf{f} \to \varprojlim \mathbf{g}$ be defined by $\varphi(x_1, x_2, \ldots) = (\varphi_1(x_1), \varphi_2(x_2), \ldots)$, where $\varphi_i = h$ for all i. Because $h \circ f = g \circ h$, each φ_i satisfies the hypothesis of Theorem 4.5.2; thus, φ is continuous. Moreover, since h is 1-1 and surjective, φ is 1-1 and surjective. Therefore, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$. •

Examples and Counterexamples

According to Theorem 4.4.7, if (1) each X_i is a Hausdorff continuum, (2) $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and (3) for each x in X_{i+1}, $f_i(x)$ is connected, then $\varprojlim \mathbf{f}$ is a Hausdorff continuum. However, the following example shows that if condition (3) is omitted, $\varprojlim \mathbf{f}$ need not be connected.

Example 1: For each positive integer i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be defined by the graph consisting of straight line segments connecting the points $(0,0)$ to $(\frac{1}{4}, \frac{1}{4})$, $(0,0)$ to $(1,0)$, $(1,0)$ to $(1,1)$, and $(\frac{3}{4}, \frac{1}{4})$ to $(1,1)$.

Then $\varprojlim \mathbf{f}$ is not connected because the space contains an isolated point, namely, $p = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 1, 1, 1, \ldots)$. To see that p is isolated, we will find an open set containing p and no other point in $\varprojlim \mathbf{f}$. Let $O_1 \subset X_1$ be $(\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon)$, let $O_2 \subset X_2$ be $(\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon)$, let $O_3 \subset X_3$ be $(\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$, and let $O_4 \subset X_4$ be $(1 - \epsilon, 1]$, where ϵ is chosen small enough so that $0 \notin O_1$ or O_2, $\frac{3}{4} \notin O_1$ or O_2, $\frac{1}{4} \notin O_3$, $\frac{11}{12} \notin O_3$, and $\frac{11}{12} \notin O_4$. Then $p = \overleftarrow{O_1} \cap \overleftarrow{O_2} \cap \overleftarrow{O_3} \cap \overleftarrow{O_4}$, which is open. $\bullet$

Next, it is worth noting that the conclusion of Theorem 4.5.2 is *only* that the function $F|\varprojlim \mathbf{f}$ be a continuous transformation, rather than a full-fledged homeomorphism. Indeed, the following example shows that even if the hypotheses of Theorem 4.5.1 apply, $F|\varprojlim \mathbf{f}$ need *not* be a homeomorphism.

Example 2: For each i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be defined by the graph consisting of the straight line segments joining the points $(0,1)$ to $(\frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{2})$ to $(1, \frac{1}{2})$, and $(1,0)$ to $(1, \frac{1}{2})$. Then it follows that $f \circ f : [0,1] \to 2^{[0,1]}$ is the graph consisting of the straight line segments joining $(0,0)$ to $(0, \frac{1}{2})$, $(0, \frac{1}{2})$ to $(1, \frac{1}{2})$, and $(1, \frac{1}{2})$ to $(1,1)$. (We will abbreviate $f \circ f$ by f^2.)

$\varprojlim \mathbf{f}$ is *not* homeomorphic to $\varprojlim \mathbf{f}^2$ because one space contains a triod while the other space is an arc.

Justification: $\varprojlim \mathbf{f}$ contains a triod. For, let A_1 be the subset of $\varprojlim \mathbf{f}$ consisting of all points of form $(x, 1-x, 1, 0, 1, 0, \ldots)$, where $x \in (\frac{1}{2}, 1]$. Let A_2 be the subset of $\varprojlim \mathbf{f}$ consisting of all points of form $(\frac{1}{2}, \frac{1}{2}, x, 1-x, 1, 0, 1, \ldots)$, where $x \in (\frac{1}{2}, 1]$. Finally, let A_3 be the subset of $\varprojlim \mathbf{f}$ consisting of all points of form $(\frac{1}{2}, x, 1, 0, 1, 0, \ldots)$, where $x \in [0, \frac{1}{2})$. Because $\bar{A}_1$, $\bar{A}_2$ and $\bar{A}_3$ are all arcs with exactly one point, $(\frac{1}{2}, \frac{1}{2}, 1, 0, 1, 0, \ldots)$, in common, $\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$ is a triod.

However, $\varprojlim \mathbf{f}^2$ is an arc. For, by Theorem 4.4.7, $\varprojlim \mathbf{f}^2$ is a Hausdorff continuum, and it is easily seen that every point in this space is a cut point except for $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$. So, this continuum has exactly two cut points and is therefore an arc. •

Ingram and Mahavier give the following example to show "the variety of continua that can be produced" using inverse limits with u.s.c. bonding functions.

Example 3: For each positive integer i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be the graph consisting of the straight line segments joining $(0,0)$ to $(1,0)$ and $(0,0)$ to $(1,1)$. Then $\varprojlim \mathbf{f}$ is a fan.

Justification: For each positive integer n, let K_n be the set of all points of form $(0,0,\ldots,0,x,x,x,\ldots)$, where the first $n-1$ entries are 0, and $x \in [0,1]$. Then each K_n is an arc, $\bigcup_{n=1}^{\infty} K_n = \varprojlim \mathbf{f}$, and $\bigcap_{n=1}^{\infty} K_n = (0,0,0,\ldots)$, a single point. So $\varprojlim \mathbf{f}$ is indeed a fan. •

Finally, in the case where $X_i = [0,1]$ and $f_i : [0,1] \to 2^{[0,1]}$ is the same u.s.c. bonding function for all positive integers i, $\varprojlim \mathbf{f}$ may be not only 1-dimensional or infinite-dimensional, but n-dimensional for any positive integer n. Mahavier and Ingram give a two-dimensional example that is easily generalized:

Example 4: Again, for each positive integer i let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be the graph consisting of the straight line segments joining $(0,0)$ to $(0,\frac{1}{2})$, $(0,\frac{1}{2})$ to $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$ to $(\frac{1}{2},1)$, and $(\frac{1}{2},1)$ to $(1,1)$. Then $\varprojlim \mathbf{f}$ consists precisely of all points of form

i) $(1,\ldots,1,1,x,\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},y,0,0,\ldots)$, where $x \in [\frac{1}{2},1]$ and $y \in [0,\frac{1}{2}]$,

ii) $(1,\ldots,1,1,x,\frac{1}{2},\frac{1}{2},\ldots)$, where $x \in [\frac{1}{2},1]$,

iii) $(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2},y,0,0,\ldots)$, where $y \in [0,\frac{1}{2}]$,

iv) $(1,\ldots,1,1,1,\ldots)$,

v) $(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots)$,

vi) $(0,\ldots,0,0,0,\ldots)$.

Thus, $\varprojlim \mathbf{f}$ is the union of countably many 2-cells, 1-cells, and single points. It follows that $\varprojlim \mathbf{f}$ is 2-dimensional.

The bonding function with two "stair-steps" gives rise to a two-dimensional inverse limit; an argument similar to the one given in Example 4 shows that a bonding function with n "stair-steps" gives rise to an n-dimensional inverse limit. For each positive integer i let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be the graph consisting of the straight line segments joining $(\frac{j}{n}, \frac{j}{n})$ to $(\frac{j}{n}, \frac{j+1}{n})$ and joining $(\frac{j}{n}, \frac{j+1}{n})$ to $(\frac{j+1}{n}, \frac{j+1}{n})$ for $0 \le j \le n-1$. Then $\varprojlim \mathbf{f}$ contains all points of form

$$\begin{aligned}(1, \ldots, 1, x_n, \tfrac{n-1}{n}, \ldots, \tfrac{n-1}{n}, \ldots, x_{n-1}, \tfrac{n-2}{n}, \ldots\ldots,\\ \tfrac{i}{n}, x_i, \tfrac{i-1}{n}, \ldots, x_{i-1}, \tfrac{i-2}{n}, \ldots\ldots, \tfrac{1}{n}, x_1, 0, \ldots)\end{aligned}$$

where $x_i \in [\frac{i-1}{n}, \frac{i}{n}]$ for $1 \le i \le n$. Thus, $\varprojlim \mathbf{f}$ contains countably many n-cells. Since $\varprojlim \mathbf{f}$ in fact consists of these countably many n-cells and also countably many j-cells where $j < n$, it follows that $\varprojlim \mathbf{f}$ is n-dimensional.

Chapter 5

An Extension of the Inverse Limit with u.s.c. Bonding Functions

We now expand on the results of Ingram and Mahavier by introducing yet another generalization of an inverse limit. Suppose that for each integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Then we define $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ to be the inverse limit space consisting of all points of form $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots)$, where $x_i \in f_i(x_{i+1})$ for each integer i, and a basis for the topology on the space is

$\{O \cap \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}} | \; O$ is basic open in $\prod_{i\in\mathbb{Z}} X_i\}$.

We will often call this space a "two-sided" inverse limit.

If each f_i is a continuous function, then the two-sided inverse limit is clearly homeomorphic to the standard one. However, if each f_i is u.s.c., the two-sided inverse limit, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, may be different from $\varprojlim\{X_i, f_i\}_{i>0}$.** We will provide some examples below, but first we prove some basic theorems analogous to the theorems seen in Chapter 4.

Theorem $\widehat{4.3.2}$. *Suppose that, for each integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Then $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is non-empty and compact.*

Proof: For each integer $z = 0, -1, -2, \ldots$, the space $\varprojlim\{X_i, f_i\}_{i>z}$ is non-empty and compact, by Theorem 4.3.2. Thus, for each such integer z, the set $\cdots \times X_{z-2} \times X_{z-1} \times X_z \times \varprojlim\{X_i, f_i\}_{i>z}$ is a compact subset of $\prod_{i\in\mathbb{Z}} X_i$. (For convenience, let $K_z = \cdots \times X_{z-2} \times X_{z-1} \times X_z \times \varprojlim\{X_i, f_i\}_{i>z}$.) We note that if w and z are both integers with $w < z$, $K_w \subseteq K_z$. That means that $\{K_z\}_{z\leq 0}$ is a monotonic collection of compact (hence, closed) subsets of $\prod_{i\in\mathbb{Z}} X_i$, a compact space. Thus, $\bigcap_{z\leq 0} K_z$ is non-empty and compact. But $\bigcap_{z\leq 0} K_z = \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, so the proof is complete. •

Theorem $\widehat{4.4.7}$. *Suppose that for each integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each x in X_{i+1}, $f_i(x)$ is connected. Then $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is a Hausdorff continuum.*

Proof: For each integer $z = 0, -1, -2, \ldots$, the space $\varprojlim\{X_i, f_i\}_{i>z}$ is a Hausdorff continuum, by Theorem 4.4.7. Again, we define $K_z = \cdots \times X_{z-2} \times X_{z-1} \times X_z \times \varprojlim\{X_i, f_i\}_{i>z}$. Since each X_i is a Hausdorff continuum, as is $\varprojlim\{X_i, f_i\}_{i>z}$ for each z, it follows that K_z is a Hausdorff continuum for $z = 0, -1, -2, \ldots$ As before, if $w < z$, then $K_w \subseteq K_z$, so that $\{K_z\}_{z\leq 0}$ is a monotonic collection of Hausdorff continua. It follows that $\bigcap_{z\leq 0} K_z$ is a Hausdorff continuum. Since $\bigcap_{z\leq 0} K_z = \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, the proof is complete. •

We now present an example to demonstrate how the two-sided inverse limit, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, may be different from the standard inverse limit, $\varprojlim\{X_i, f_i\}_{i>0}$.

For each integer i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be the graph consisting of the straight line segments joining $(0,0)$ to $(1,0)$ and $(0,0)$ to $(1,1)$. (This bonding function is the same as in Example 3 in Chapter 4.) If we let A_z be the set of all points of form $(\ldots, 0, 0, x, x, \ldots)$, with 0's up to the $(z-1)$th slot and $x \in [0,1]$, then A_z is an arc for each integer z. Let $A = \{(\ldots, x, x, x, \ldots)|\ x \in [0,1]\}$, so that A is also an arc. Thus, $(\bigcup_{z\in\mathbb{Z}} A_z) \cup (A) = \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, and $(\bigcap_{z\in\mathbb{Z}} A_z) \cap (A) = (\ldots, 0, 0, 0, \ldots)$, a single point. Thus, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is a fan.

However, this fan is not homeomorphic to the fan given by $\varprojlim\{X_i, f_i\}_{i>0}$ in Example 3. For, as we will show, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ contains a limit arc while $\varprojlim\{X_i, f_i\}_{i>0}$ does not.

Consider the arc A in $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ given by $\{(\ldots, x, x, x, \ldots)|\ x \in [0,1]\}$. We will prove that A consists entirely of limit points of $(\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}) \setminus A$. To that end, let $O = (\prod_{i\in\mathbb{Z}} O_i) \cap \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ be a basic open set containing some point $(\ldots, x, x, x, \ldots)$, where $x \in [0,1]$. If for each i, $O_i = X_i$, then clearly O contains points not in A. So suppose O is a proper subset of the space. Since O is open, there must be some least integer i for which $O_i \subsetneq X_i$, and some greatest integer j for which $O_j \subsetneq X_j$. If $x \neq 0$, and $\bar{x}$ lies in the ith slot, clearly $(\ldots, 0, 0, \ldots, 0, \bar{x}, x, x, \ldots) \in O$. If $x = 0$, and $\bar{0}$ lies in the jth slot, then $(\ldots, 0, 0, \ldots, \bar{0}, 1, 1, \ldots) \in O$. Either way, O must contain a point in $(\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}) \setminus A$, and thus, A is a limit arc.

On the other hand, the space $\varprojlim\{X_i, f_i\}_{i>0}$ has no limit arc. To see this, consider a general point $(0, 0, \ldots, 0, \bar{x}, x, x, \ldots)$ lying in an arc $\widehat{A} = \{(0, 0, \ldots, 0, \bar{x}, x, x, \ldots)|\ x \in [0,1]\}$, where $\bar{x}$ lies in the ith slot. If $x \neq 0$, the point $(0, 0, \ldots, 0, \bar{x}, x, x, \ldots)$ cannot be a limit point of $(\varprojlim\{X_i, f_i\}_{i>0}) \setminus \widehat{A}$ for the following reason:

Let $O_1 = [0, \frac{x}{2}) \subset X_1$, $O_2 = [0, \frac{x}{2}) \subset X_2$, $\cdots$, $O_{i-1} = [0, \frac{x}{2}) \subset X_{i-1}$, and $O_i = (\frac{x}{2}, 1] \subset X_i$. Then $\overleftarrow{O_1} \cap \overleftarrow{O_2} \cap \cdots \cap \overleftarrow{O_{i-1}} \cap \overleftarrow{O_i}$ is open in $\varprojlim\{X_i, f_i\}_{i>0}$, contains $(0, 0, \ldots, 0, \bar{x}, x, x, \ldots)$, but misses $(\varprojlim\{X_i, f_i\}_{i>0}) \setminus \widehat{A}$ entirely. $\bullet$

Chapter 6

An Indecomposable Continuum Produced by an Inverse Limit on u.s.c. Functions

The Knaster continuum described in example 2 at the end of Chapter 3 is a famous example of an *indecomposable continuum*, i.e., a continuum that is not the union of two proper subcontinua. We conclude this paper with an original example of an inverse limit on u.s.c. bonding functions that turns out to be an indecomposable continuum.

Example: For each positive integer i, let $X_i = [0, 1]$ and let $f_i : X_{i+1} \to 2^{X_i}$ be defined by the graph consisting of the following straight line segments:

1. For each even integer $n \geq 0$, the segment joining the points $(\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$ and $(\frac{1}{2^n}, 1)$.
2. For each odd integer $n \geq 1$, the segment joining the points $(\frac{1}{2^n}, \frac{1}{2^n})$ and $(\frac{1}{2^{n+1}}, 1)$.
3. The vertical line segment joining the points $(0, 0)$ and $(0, 1)$.

Then $\varprojlim \mathbf{f}$ is an indecomposable continuum.

Proof: By Theorem 4.4.7, $\varprojlim \mathbf{f}$ is a continuum. It remains to show that $\varprojlim \mathbf{f}$ is indecomposable.

Claim: If H is a proper subcontinuum of $\varprojlim \mathbf{f}$, then there exists some positive integer N so that if $n \geq N$, $\pi_n(H) \neq X_n$.

Justification: Suppose not, i.e., suppose H is proper but for every positive integer N, there exists some $n > N$ such that $\pi_n(H) = X_n$. By the way the graph of f is defined, it is clear that if $\pi_i(H) = X_i$, then $\pi_{i-1}(H) = X_{i-1}$; thus, we may as well assume that $\pi_n(H) = X_n$ for each positive integer n. Since H is proper, there is some point $p = (p_i)_{i=1}^{\infty} = (p_1, p_2, p_3, \ldots)$ such that $p \in \varprojlim \mathbf{f} \setminus H$. We will show that p is in fact a point of H.

Case 1: Suppose $p_i \neq 0$ for all positive integers i. Since, for a given positive integer k, $p_k \in \pi_k(H)$, there exists some point in H of form

$(x_1, x_2, \ldots, x_{k-1}, p_k, \ldots)$.

However, since $p_k \neq 0$, by the way the graph of f is defined, $f_{k-1}(p_k)$ is a unique non-zero number in X_{k-1}. That means $f_{k-1}(p_k) = p_{k-1}$. In a similar way, each of $x_1, \ldots, x_{k-1}$ is uniquely determined, and that forces $x_i = p_i$ for $1 \leq i \leq k$. Thus, a point of form $(p_1, p_2, \ldots, p_k, \ldots)$ is in H. Indeed, for each positive integer j, a point of form $(p_1, p_2, \ldots, p_j, \ldots)$ is in H. The point $(p_i)_{i=1}^{\infty}$ is therefore a limit point of the sequence of points in H that we just described; hence, because H is closed, $(p_i)_{i=1}^{\infty} \in H$.

Case 2: Suppose for some least integer i, $p_i = 0$. Then, since the only possible preimage of 0 via f_i is 0, $p_n = 0$ for each integer $n \geq i$.

Suppose $p_1 = 0$. By the above argument, since $f_1^{-1}(p_1) = p_2 = 0$, and $f_i^{-1}(p_i) = p_{i+1} = 0$ for $i \geq 1$, the only way that $p_1 = 0 \in \pi_1(H)$ is possible is if $(0, 0, 0, \ldots) = (p_i)_{i=1}^{\infty} \in H$. That would be a contradiction.

So, suppose instead that $p_i = 0$ for some least integer $i > 1$. Because H is compact, the projection of H onto the graph of f_{i-1} is closed. Thus, since $\pi_i(H) = X_i$, by the way the graph is defined, each ordered pair $(0, x)$ in $G(f_{i-1})$ is a limit point of the projection of H onto $G(f_{i-1})$. Thus, the ordered pair $(0, p_{i-1})$ is in that projection. Since $p_{i-1} \neq 0$ by assumption, the image of p_{i-1} via f_{i-2} is the unique non-zero number p_{i-2}; the image of p_{i-2} via f_{i-3} is the unique non-zero number p_{i-3}, and so forth. Now because $\pi_i(H) = X_i$, we know $p_i = 0 \in \pi_i(H)$; since the only possible preimage of 0 is 0, H must therefore contain a point of form $(h_1, h_2, \ldots, h_{i-1}, 0, 0, \ldots)$. However, as we noted, the projection of H onto $G(f_{i-1})$ contains the ordered pair $(0, p_{i-1})$. That is, H contains some point $(h_1, h_2, \ldots, h_{i-1}, 0, 0, \ldots)$ where $h_{i-1} = p_{i-1}$. But, as previously argued, $h_{i-1} = p_{i-1}$ would force $h_k = p_k$ for $k \leq i - 1$. That is, $(p_1, p_2, \ldots, p_{i-1}, 0, 0, \ldots) = (p_i)_{i=1}^{\infty} \in H$.

In either case, a contradiction has been reached. Thus, if H is a proper subcontinuum, there exists some positive integer N so that if $n \geq N$, $\pi_n(H) \neq X_n$.

Now, suppose by way of contradiction that $\varprojlim \mathbf{f} = H \cup K$, a union of two proper subcontinua. By the above argument, there exists some least positive integer N so that for all $n \geq N$, $\pi_n(H) \neq X_n$ and $\pi_n(K) \neq X_n$. So, we may assume without loss of generality that $0 \in \pi_N(H)$ and $0 \notin \pi_N(K)$. Since the unique preimage of 0 via f_N is 0, $\pi_{N+1}(H)$ must contain 0. Thus, because $\pi_{N+1}(H)$ is a proper subcontinuum of $[0, 1]$ containing 0, but $\pi_{N+1}(K) \neq [0, 1]$, it follows that $\pi_{N+1}(H)$ is some interval of form $[0, a]$ where $0 < a < 1$. However, by the way the graph of f_N is defined, there is some $x \in (0, a]$ (indeed, infinitely many such x) with $f_N(x) = 1$. That means $1 \in \pi_N(H)$. Since $\pi_N(H)$ is a subcontinuum of $[0,1]$ that contains both 0 and 1, it follows that $\pi_N(H) = [0,1] = X_N$. This is a contradiction, for we assumed $\pi_N(H) \neq X_N$. So $\varprojlim \mathbf{f}$ is indecomposable. •

We note that the continuum in this example ($\varprojlim \mathbf{f}$) contains the proper subcontinuum $\{(x, y, 0, 0, \ldots)|\ y \in [0,1], x \in f_1(y)\}$, a copy of the topologist's sine curve. Therefore, $\varprojlim \mathbf{f}$ is clearly not homeomorphic to the Knaster continuum (whose proper subcontinua are all arcs). However, if $\varprojlim \mathbf{f}$ is homeomorphic to any known space, it remains an open question what that space is.

Another open question is the following: if for each positive integer i, $X_i = [0,1]$ and $f = f_i : X_{i+1} \to 2^{X_i}$ is u.s.c., are there some necessary conditions the graph of f must satisfy in order for $\varprojlim \mathbf{f}$ to be indecomposable? Are there sufficient conditions?

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Appendix

* In general, for an inverse limit space $\varprojlim \mathbf{f}$ with upper semi-continuous bonding functions, the collection $B = \{\overleftarrow{O} \mid O$ is open in some $X_i\}$ is **not** a basis for $\varprojlim \mathbf{f}$. Consider the case where for all positive integers i, $X_i = [0,1]$ and $f_i : X_{i+1} \to 2^{X_i}$ is given by the graph in $[0,1] \times [0,1]$ consisting of the line segments joining $(0,0)$ to $(1,0)$ and joining $(0,1)$ to $(1,1)$. Then the open set $G = (\frac{1}{2}, 1] \times [0, \frac{1}{2}) \times [0,1] \times [0,1] \times \ldots$ does not contain a member of B containing $(1,0,0,0,\ldots)$. For, any such member b of B would have to contain an open set of form (i) $\overleftarrow{[0,a)}, a < 1$ or (ii) $\overleftarrow{(a,1]}, a > 0$. In case (i), b would contain $(0,0,\ldots)$; in case (ii), b would contain $(1,1,\ldots)$. In either case, b would fail to be a subset of G, so B cannot be a basis.

** Thanks to Dr. Stewart Baldwin for suggesting this possibility and encouraging me to explore it.