A Study of Regularity and Normality

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Mitchell John Jaeger, son of Richard & Vicke Jaeger, was born in Green Bay, Wisconsin on June 13th, 1984. After graduating from Oconto Falls High School in 2002, he entered the University of Wisconsin - Madison as a Nuclear Engineering major. In 2006, he graduated with a dual degree in Nuclear Engineering and Mathematics. He graduated with the distinction of distinguished scholar. He then enrolled in Auburn University to pursue a Master Degree in Mathematics.

Thesis Abstract

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The paper is a broad overview of the separation axioms, commonly known as the Hausdorff, Regular and Normal axioms. After a review of basic definitions and lemmas, we proceed to certain examples and theorems dealing with Paracompact and Lindelöf spaces. We then progress to metric spaces, separable spaces and countability before we end with several examples of spaces that satisfy certain separation axioms and not others, and the properties that these spaces satisfy.

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Chapter 1

Introduction

Topology is one of the major areas of Abstract Mathematics. In essence, it is the study of geometric spaces and continuity. The Euclidean plane is a topological space, but in fact any set X may be a topological space if it satisfies certain requirements. Namely, if there exists a collection T of subsets of X such that (1) both the empty set and X are in T, (2) the union of the elements of any subcollection of T is in T, and (3) the intersection of the elements of any finite subcollection of T is in T. If these three criteria are met, then T is called a topology on X, and X together with T form a topological space.

These topological spaces allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics and are a central unifying notion. In this thesis we will take a closer look at some properties held by topological spaces, namely Regularity and Normality, and the types of spaces that have them, and those that don't.

Chapter 2

Background Definitions

If X is a set, a *basis* for a topology on X is a collection B of subsets of X (called *basis elements*) such that: (1) For each $x \in X$, there is at least one basis element B containing x and (2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset (B_1 \cap B_2)$. If B satisfies these two conditions, then we define the *topology T generated by B* as follows: A subset U of X is said to be open in X (or to be an element of T) if for each $x \in U$, there is a basis element $B \in B$ such that $x \in B$ and $B \subset U$. Then B is a *basis* for the topology on X and each member of B is called a *basic open set of X*.

A topological space X is called a *Hausdorff space* if for each pair a,b of distinct points of X, there exist disjoint open sets containing a and b respectively. We sometimes refer to this as the Hausdorff axiom, one of the separation axioms.

A space X is said to be *regular* if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be *normal* if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B respectively. These are the other two separation axioms to be dealt with in this thesis.

Suppose that X is a space, $M \subset X$ and G is a collection of open sets which covers the set M. Then the collection G' is said to be a *refinement* of G covering M if and only if for each point $x \in M$ there is an element of G' that contains x and lies in some element of G.

Suppose that X is a space and G is a collection of subsets of X. Then G is said to be *locally finite* if and only if for each point $x \in X$ there is an open set U containing x so that U intersects only finitely many elements of G.

The topological space X is said to be *paracompact* if and only if it is Hausdorff and every collection of open sets that covers X has a locally finite refinement that covers X.

The topological space X is said to be *Lindelöf* if and only if for every collection G of open sets that covers X, there is a countable subcollection $G' \subset G$ that covers X.

A space X is said to be *completely separable* if it has a countable basis.

A subset A of X is said to be *dense* in X if any open set of X intersects A.

A space X is said to be *separable* if it has a countable dense subset.

A *metric* on a set X is a function $d : X \times X \longrightarrow \mathbb{R}$ having the properties:

$d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

$d(x, y) = d(y, x)$ for all $x, y \in X$.

(Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given $r > 0$, the *r-Ball* is defined as $B(x, r) = \{y : d(x, y) < r\}$. If these r-balls form a basis for a topology over X, then we define this topology to be the *metric topology*.

A space X is *first countable* if it has a countable basis for each point $x \in X$.

Let W be the collection of all half-open real intervals of the form $[a, b) = \{x : a \leq x < b\}$, where $a < b$. Then the topology generated by W on the real line is called the *Sorgenfrey line.*

Let S be the sorgenfry line. Let $X = S \times S$ be the product space of the Sorgenfry line crossed with itself (called the *Sorgenfrey Plane*) in which a basic open set containing a point (x, y) is a square containing only the border below and the border to the left of the rectangle with the point (x, y) as its lower left vertex. Thus we will denote a basic open square of length ϵ containing a point a to be $S(a, \epsilon)$.

Lemma 1.1 (Pasting Lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Example 1.2 Let $X = \{(x, y)|y \geq 0\}$ be the space given by the upper half plane including the x-axis. Define all basic open sets containing a point (x, y) above the x-axis to be the product of all open intervals $U = (a, b)$ (such that $x \in U$) and all open $V = (c, d), c > 0$ (such that $y \in V$). Define all basic open sets containing a point $(x, 0)$ on the x-axis to be the union of the point $(x, 0)$ with the product of all open intervals $U = (a, b)$ (such that $x \in U$) and all open intervals $V = (0, d)$. We define this space to be the *tangent rectangle topology.*

Example 1.3 Let $X = \{(x, y) : x, y \in R, y > 0\}$ be the upper half-plane with the usual Euclidean topology T_x. Let Y be the real line that represents the x-axis. Let T_y be the topology generated by all sets of the form $\{x\} \cup D$ where $x \in Y$ and D is an open disc in X that is tangent to Y at x. Let $Z = X \cup Y$ and let the topology T_z on Z be the topology generated by T_x together with the topology generated by T_y. We shall refer to this topological space as the *Tangent Disc Topology.*

A space X is *perfectly normal* if X is normal and every closed subset of X is the common part of countably many open sets.

Suppose that X is a metric space with metric d. Then the sequence $\{p_i\}$ is said to be a *Cauchy Sequence* if and only if for each $\epsilon > 0$ there exists and integer N so that if n and m are integers larger than N, then $d(p_n, p_m) < \epsilon$.

Suppose that X is a metric space with metric d. Then the metric is said to be a *complete metric* for X if and only if every Cauchy sequence (with respect to the metric d) in X converges to a point in X. A space is said to be a *complete metric space* if and only if there is a complete metric for the topology of X.

A *Baire space* is a topological space in which every countable intersection of dense open sets is non-empty. Often an equivalent definition is used that states a topological space is a *Baire Space* if given any countable collection of nowhere dense closed sets, their union is not the whole space.

Chapter 3

Paracompact and Lindelöf Spaces

Theorem 3.1 If M is a closed subspace of the paracompact space X then it is also paracompact.

Proof: Let M be a closed subspace of the paracompact space X. Let A be a covering of M by sets open in M. For each $\alpha \in A$, we pick an open set α' of X such that $\alpha' \cap M = \alpha$. Then we cover X by the open sets α' and the open set $X - M$. We then let B be a locally finite open refinement of this covering that covers X. Then $R = \{\beta \cap M : \beta \in B\}$ is a locally finite open refinement of A, and thus M is paracompact. •

Theorem 3.2 Every paracompact Hausdorff space is regular.

Proof: Let X be a paracompact space. Also let $x \in X$ and let $H \subseteq X$ be closed such that $x \notin H$. Take an $h \in H$, then there exists an A_h with $h \in A_h$ and a B_h with $x \in B_h$ such that $A_h \cap B_h = \emptyset$. We can then see that $\{A_h : h \in H\} \cup \{X - H\}$ covers X. Then, by paracompactness, there exists a locally finite refinement R which covers X. We define a new set C of elements of R such that $C = \{S \in R : S \cap H \neq \emptyset\}$. If $U = \bigcup_{S \in C} S$, then obviously U is open as the union of open sets that contains H. Now, since R is locally finite, there exists an open O with $x \in O$ that only meets a finite number of sets in R. We call these sets $A_1, A_2, ..., A_n$. Each A_i is contained in some A_{hi} or $X - H$. If we define $B := O \bigcap (\bigcap_{i=1}^{n} B_{hi})$ then B is open with $x \in B$ and $U \cap B = \emptyset$. •

Theorem 3.3 Every paracompact Hausdorff space is normal.

Proof: Let X be a paracompact space. From the previous theorem we know that X is regular. So let A and B be closed sets of X such that $A \cap B = \emptyset$. Since X is regular, we can choose, for each $b \in B$ an open set U_b containing b and an V_b containing A such that $U_b \cap V_b = \emptyset$. We then cover X with the open sets U_b and the open set $X - B$. Since X is paracompact, we take a locally finite open refinement C that covers X. We then form the subcollection D of C that consists of all the elements of C that intersect B, that is $D = \{S \in C : S \cap B \neq \emptyset\}$. Then D covers B. We then see that for every $a \in A$ there exists some open O_a containing a that intersects only finitely many elements of the locally finite refinement C. Let these elements that O_a intersects be denoted by $S_{a1}, ..., S_{an}$. Each of these S_{ai} is contained in some element U_{ba_i} of the original cover of X. Each of these U_{ba_i} has a corresponding V_{ba_i} that contains a but misses U_{ba_i}. Therefore, $O_a \cap V_{ba_i}$ contains a but misses U_{ba_i} and thus misses S_{ai}. So, we see that the open set $W_a = O_a \cap V_{ba_1} \cap ... \cap V_{ba_n}$ contains a but misses $\cup_{b \in B} U_b$ the open set containing B. Such a open W_a can be found for every $a \in A$. So $\cup_{a \in A} W_a$ is an open set that contains A and misses $\cup_{b \in B} U_b$, the open set containing B and thus X is normal. •

Theorem 3.4 A completely separable space is Lindelöf.

Proof: Let X be a completely separable space and let $B = \{B_n\}_{n=1}^{\infty}$ be a countable basis of X and C and open cover of X. For each natural number n, let $c_n \in C$ be an element of C that contains B_n if one exists. If one does not exist, then define $c_n = \emptyset$. We then define $C' = \{c_n : c_n \neq \emptyset\}$. Notice that C' is countable. We then claim that this C' covers X. We show this by letting $x \in X$. Then there is some $c \in C$ that contains x since C covers X. Then there exists a $B_m \in B$ with $x \in B_m \subseteq c$. Thus we see there is some $c_n \in C'$ with $B_m \subseteq c_n$. Therefore $x \in c_n$ which implies C' covers X, and thus X is Lindelöf. •

Theorem 3.5 A separable paracompact space is Lindelöf.

Proof: Let X be a separable paracompact space. Let I be an index set such that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of X. By the paracompactness of X, we can find a locally finite open refinement $\{V_\beta\}_{\beta\in J}$ for some index set J, of $\{U_\alpha\}_{\alpha\in I}$ where we assume $V_\beta \neq \emptyset\ \forall\ \beta$. Since X is separable, it has a countable dense subset $\{x_i : i \in \mathbb{Z}\}$, each V_β contains at least one x_i. We suppose towards a contradiction that $\{V_\beta\}$ is an uncountable collection. Then there exists some x_i contained in uncountably many V_β. But this contradicts the local finiteness of $\{V_\beta\}$. Thus we conclude that $\{V_\beta\}$ is in fact a countable collection. So, for each V_β, we choose some $U_{\alpha(\beta)} \supseteq V_\beta$. Then $\left\{U_{\alpha(\beta)}\right\}$ is a countable subcover of $\{U_\alpha\}$, and thus X is Lindelöf. •

Chapter 4

Metric Spaces

Theorem 4.1 A metric space is normal

Proof: Let X be a metric space and let d be the metric on X. Let A and B be two disjoint closed sets in X. We need to show there are two disjoint open sets U and V containing A and B respectively in X. For each element $a \in A$, choose an r_a such that the open ball $B(a, r_a)$ around a with radius r_a does not intersect the closed set B. Similarly, for each $b \in B$, we choose an r_b such that $B(b, r_b)$ does not intersect A. Then we define the open sets U and V to be $U = \cup_{a \in A} B(a, r_a/2)$ and $V = \cup_{b \in B} B(b, r_b/2)$. It is clear that U contains the closed set A and that V contains the closed set B. We claim that $U \cap V = \emptyset$. We assume towards a contradiction that this is not the case. Then there is some element $x \in U \cap V$. This would imply that for some $a \in A$ and some $b \in B$, $x \in B(a, r_a/2) \cap B(b, r_b/2)$. Then the triangle inequality tells us that $d(a, b) < (r_a + r_b)/2$. So if $r_a \leq r_b$, then $d(a, b) < r_b$ we see that $B(b, r_b)$ contains $a \in A$. But this is not possible by the definition of $B(b, r_b)$. On the other hand, if $r_b \leq r_a$, then $d(a, b) < r_a$. This then implies that $B(a, r_a)$ conains $b \in B$. But again, this is not possible by the definition of $B(a, r_a)$. Therefore, we see that U and V are disjoint, and therefore X is normal. •

Let A be an index set. For every $a \in A$ let $I_a = (0, 1] \times \{a\}$. Then define the space X to be $X = \bigcup_{a \in A} I_a \cup (0, 0)$. Furthermore, define a metric on X to be:

$d((x, a), (y, b)) = |x - y|$ if $a = b$

$d((x, a), (y, b)) = |x| + |y|$ if $a \neq b$

Notice that if m and n are two elements in X, then $d(m,n) \geq 0$, and if $l, m, n \in X$ then $d(l,m) + d(m,n) = |l - m| + |m - n| \geq |l - n| = d(l,n)$. Thus $d((x,a),(y,b))$ is a metric on X. The space X together with this metric d is known as the *hedgehog space based on the index set* A.

Theorem 4.2 The hedgehog space based on the uncountable index set A is a non-separable metric space.

Proof: Let H be the hedgehog space which we know to be metric, and let D be any dense set in H. Define U_x to be the open subset $U_x = (1/3, 2/3) \times \{x\}$ of one of the "spines". Since D is dense in H, there exists some $d \in D$ such that $d \in U_x$. However, since there is a spine I_x for every $a \in A$, there exists uncountably many disjoint open sets U_x which must each contain distinct $d_x \in D$. Then, since the reals are uncountable, D must also be uncountable. Therefore, there can not exist a countable dense subset of H, and thus H is non-separable. •

Theorem 4.3 A separable metric space is completely separable.

Proof: Let X be a separable metric space, and let $A = \{a_0, a_1, ...\}$ be its countably dense subset. Then take the open balls centered at each a_i with radius $\epsilon = d(a_i, a_j)$ for $j \neq i$ so that there are countably many balls centered at countably many points, thus leaving countably many open sets we claim to be a basis for X. First we show these open sets cover X. It is clear that these open sets cover A, so we must only look at some element $p \in X - A$ and show that it is contained in one of the open sets. Fix some $a_i \in A$ and let $r = d(p, a_i)/2$. Then let $U = B(p, r)$ be an open ball of radius r centered at p. Since A is dense, this open U must contain some $a_j \in A$. We then notice that $d(p, a_j) < d(p, a_i)/2$ which implies that $2d(p, a_j) < d(p, a_i)$. Moreover, by the triangle inequality, we see that $d(p, a_j) + d(a_j, a_i) \geq d(p, a_i)$. This implies that $d(a_j, a_i) \geq d(p, a_i) - d(p, a_j) > 2d(p, a_j) - d(p, a_j) = d(p, a_j)$ since $2d(p, a_j) < d(p, a_i)$. Thus the open ball $B(a_j, d(a_i, a_j))$ contains the point p, and thus the open sets described above over X. To show that this cover is a basis of X, suppose we have an open set O containing some $x \in X$. Then there exists some r 0 such that $B(x,r) \subset O$. Then suppose we take $B(x, r/8)$ which must contain some a_n from the countable dense subset. Also, we can look at the open set $B(x, 3r/8)$ minus the closure of $B(x, r/4)$. This is an open set, being an open set minus a closed set. Since this open set is disjoint from $B(x, r/8)$, it must contain another distinct element a_m of the countable dense subset. Now, by construction, $r/8 < d(a_n, a_m) < r/2$. This then implies that $B(a_n, d(a_n, a_m))$ contains x and lies completely in $B(x,r) \subset O$. Thus the constructed cover is a basis for the topology on X. •

Baire Category Theorem (4.4) If X is a complete metric space, then X is a Baire Space.

Proof: Let X be a complete metric space with metric d and let $\{U_n\}$ be a countable collection of dense open sets in X. Let $x_o \in X$ and $\epsilon_o > 0$ be given. Since U_1 is dense and open, we can choose $\epsilon_1 > 0$ and $x_1 \in U_1$ such that $d(x_o, x_1) < \epsilon_o/2$, $\epsilon_1 < \epsilon_o/2$ and $B(x_1, \epsilon_1) \subset U_1$. Then we can choose $\epsilon_2 > 0$ and $x_2 \in U_2$ such that $d(x_1, x_2) < \epsilon_1/2$, $\epsilon_2 < \epsilon_1/2$ and $B(x_2, \epsilon_2) \subset U_1$. By induction, we can construct a sequence x_n with $x_n \in U_n$ for each n and a sequence ϵ_n such that $d(x_{n-1}, x_n) < \epsilon_{n-1}/2$, $\epsilon_n < \epsilon_{n-1}/2$ and $B(x_n, \epsilon_n) \subset U_n$. Therefore for some positive integer j less than n, we have

$$d(x_j, x_n) < \epsilon_j(1/2+...+1/2^{n-j}) < \epsilon_j \leq \epsilon_o/2^j.$$

Thus the sequence x_n, $n = 1, 2, 3, ...$ is a cauchy sequence and converges by hypothesis to some $x \in X$. We also know that for every n, $d(x, x_n) \leq \epsilon_n$. Moreover, it follows that

$$d(x, x_n) \leq d(x, x_{n+1}) + d(x_n, x_{n+1}) < \epsilon_{n+1} + \epsilon_n/2 < \epsilon_n.$$

This then implies that $d(x, x_n) < \epsilon_n$ for every n, implying $d(x, x_o) < \epsilon_o$. Thus $x \in U_n$ for all $n = 1, 2, 3...$, and X is therefore a Baire space. •

Chapter 5

Separability and Countability

Example 5.1 The Sorgenfrey Line is separable. This can be easily seen since the rational numbers are dense in the topology generated by the Sorgenfrey Line. Let W be the Sorgenfrey Line, and let r be any real number. Then $[r, \infty)$ is an open set containing r. Thus, any basis for the space W must contain some basic open set U_r that contains r that is contained in $[r, \infty)$. Also note that if $r_1 < r_2$ then r_1 cannot lie in U_{r_2}. Since this is true for all real numbers $r \in R$, we see that there must at least be uncountably many basic open sets for any basis of W. •

Example 5.2 The Sorgenfrey line is first countable. Let X be the Sorgenfrey line and let $x \in X$. Then the set of all basis elements of the form $[x, x + 1/n)$ for all natural numbers n is a basis for each point x in X. And since this set has countably many elements, X is first countable. •

Example 5.3 The Sorgenfrey line is not completely separable. Let S be the Sorgenfrey line. Let B be a basis for S. Choose for each $x \in S$ an element B_{xn} of B that contains x such that $B_{xn} \subset [x, x + 1/n)$. Thus $x = inf(B_{xn} | n \in \mathbb{N})$, and $B_{xn} = B_{yn}$ implies $x = y$. For any y such that $x < y$, B_{xn} does not contain y. Thus for each $x \in S$ we must have a local basis. Since S has uncountably many elements, we see that B must be uncountable.
•

Example 5.4 The Tangent Rectangle Topology is separable since the subset of the rationals crossed with the positive rationals are countable and dense in X.•

Example 5.5 The Tangent Rectangle Topology X is not completely separable. Let L be the x-axis and let $x \in L$. Clearly $L \subset X$. If we choose a basic open set U in the Tangent Rectangle Topology containing x, then $L \cap U = x$. Since L countains uncountably many elements, this implies that there would necessarily have to be uncountably many basic open sets to cover all of L, and thus X can not be completely separable. •

Chapter 6

REGULARITY

Theorem 6.1 A space X is regular if and only if given a point $x \in X$ and an open set U containing x, there is an open set $V \subset U$ such that $x \in V$ and the closure of V is contained in U.

Proof: First suppose X is a regular space and that $x \in X$. Suppose there is an open set $U \subset X$ such that $x \in U$. Let $A = X - U$. Then A is a closed subset of X and $x \notin A$. Thus, since X is regular, we can find two disjoint open sets V and W such that $x \in V$ and $A \subset W$. Now we see that the closure of V misses A, since if this were not true A would contain a limit point of V. If $y \in A$, then W is open and contains y but misses V. Thus y is not a limit point of V, and thus the closure of V misses A and is contained in U.

Conversely, suppose there exists a point $x \in X$ and a closed set $A \subset X$ such that $x \notin A$. Let $U = X - A$. Thus U is open and contains x. By hypothesis, we can find an open V that contains x and whose closure is contained in U. If we let W be the complement in X of the closure of V, then W is open and contains A since the closure of V is a proper subset of U and thus completely misses A. Therefore, $V \cap W = \emptyset$ and $x \in V$, $A \subset W$, implying that X is regular. •

Theorem 6.2 A metric space is regular.

Proof: By Theorem 4.1, we know that a metric space is normal. And since normality implies regularity, we know that a metric space is regular. •

Example 6.3 The Tangent Rectangle Topology is Hausdorff but not regular.

Proof: First we show that X is Hausdorff. Let $p, q \in X$ such that $p \neq q$. Then $p = (x_p, y_p)$, $q = (x_q, y_q)$. If both $y_p, y_q > 0$, then we have two choices. Since $p \neq q$, we know that either $x_p \neq x_q$ or $y_p \neq y_q$. Suppose first that $x_p \neq x_q$, then we can find two disjoint open intervals U_p and U_q on the x-axis that correspond to basic open sets in X such that $x_p \in U_p$, $x_q \in U_q$ and $U_p \cap U_q = \emptyset$ since the real line is Hausdorff. If $x_p = x_q$, then we know that $y_p \neq y_q$. In this case we can find disjoint open intervals V_p and V_q that correspond to basic open sets in X on the positive y-axis such that $y_p \in V_p$ and $y_q \in V_q$. If $y_p = 0$ and $y_q > 0$, then we can find two disjoint open intervals on the y-axis $V_p = (0, a)$ and $V_q = (b, c)$ such that $b > a$ that correspond to disjoint basic open sets in X containing p and q respectively. A similar argument follows if $y_q = 0$ and $y_p > 0$. Now if both $y_p = y_q = 0$, then we know that $x_p \neq x_q$. We choose two disjoint open intervals on the x-axis U_p and U_q that correspond to disjoint basic open sets in X containing p and q respectively. Therefore, we conclude that X is Hausdorff.

To show that X is not regular, we take a point p on the horizontal axis. Then we take a basic open set U in the tangent rectangle topology containing p. We know that this open set is of the form $\{(a, b) \times (0, c)\} \cup \{p\}$ where $a < p < b$. Now we choose another basic open set V containing p such that $V \subset U$. Then V is of the form $(d, e) \times (0, f) \cup p$, where $a < d < p < e < b$ and $f < c$. But now the closure of V in not fully contained in U. This is because the closure of V contains all points on the horizontal axis between $(d, 0)$ and $(e, 0)$, which are explicitly not contained in U. And thus X can not be regular. •

Example 6.4 The Tangent Disc Topology is regular. Suppose we have a point x in Z. There are two possibilities for x. Either it is in X (the open upper half plane) or it is in Y (the horizontal axis). First let us suppose $x \in X$. We can then find an $r > 0$ and the basic open ball $B(x, r)$. If we take the open ball $B(x, r/2)$ which contains x and whose closure is properly contained in $B(x, r)$, then we have regularity by the Theorem 6.1. In the case where x is in Y, we can then find an $r > 0$ and the basic open disc tangent to the horizontal axis at x of diameter r. We then consider the open disc tangent to x of diameter $r/2$. The closure of this disc is properly contained in the disc of radius r, and hence again by theorem 6.1, we have regularity of Z. •

Chapter 7

Normality

Example 7.1 The Sorgenfrey Plane is not normal. We first look at the diagonal across the space with negative slope, that is $A = \{(x, y) : y = -x\}$. Clearly A is closed in X and is discrete since we can find open neighborhoods $S(a, \epsilon)$ of each point $a \in A$ such that $S(a, \epsilon) \cap A = \{a\}$. Now let Q be the rationals and let $B = \{(b, -b) : b \notin Q\}$. It is also easy to see that B and $A - B$ are closed in X. We claim that we cannot find disjoint open sets in X containing B and $A - B$ respectively. Let U and V be disjoint opens sets such that $B \subset U$ and $A - B \subset V$. For each irrational real number b, there exists some open square $S(b, r_b)$ with $(b, -b)$ as its lower left vertex lying in U. We then define the subset $B_n \subset B$ to be $B_n = \{(b, -b) \in B : r_b > 1/n\}$. Thus we form the countable collection of subsets $\{B_n\}$. Since $A - B$ is clearly countable (as it contains all of the elements $(q, -q)$, where q is a rational number), and since $\{B_n\}$ contains all $(b, -b)$, b irrational, we form a countable covering of A in the usual Euclidean topology. Now since A is nonmeager in the usual Euclidean topology (it is not the countable union of nowhere dense sets), we see that at least one of the B_n fails to be nowhere dense. Thus there must be some interval J in A such that $\{(x, -x) \in A : r_x > 1/m\}$ is dense in the usual Euclidean topology. This means that any open set containing some $(q, -q)$, q rational, in the interval J must intersect the open set V. Thus U and V cannot be disjoint, and therefore X is not normal. •

Theorem 7.2 If X is normal, then for each closed set A in X and any open set O containing A, there exists and open set U such that $A \subset U \subset \bar{U} \subset O$.

Proof: Let X be a normal space and let A be a closed subset of X. Suppose O is open in X such that $A \subset O$. Then O^c is closed and $O^c \cap A = \emptyset$. Thus there exists disjoint open sets U and V such that $A \subset U$ and $O^c \subset V$. Since $U \cap V = \emptyset$, $U \subset V^c$ which in turn implies that $\bar{U} \subset V^c$ because V is open. But we know that $O^c \subset V$ which implies that $V^c \subset O$ and so we have $A \subset U \subset \bar{U} \subset O$. And thus for each closed set A and any open set O containing A, there exists an open set U such that $A \subset U \subset \bar{U} \subset O$. •

Theorem 7.3 X is normal if and only if for each pair of disjoint closed sets A and B, there exists a continuous function from X into the unit interval $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof: Let $f : X \longrightarrow [0, 1]$ be a continuous function from X into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Let U be an open subset of $[0, 1]$. Since f is continuous, the inverse image of U under f is open. If we let $U = [0, a)$ and $V = (b, 1]$ be two open sets of $[0, 1]$ such that $a < b$, then $U \cap V = \emptyset$. Also, since U and V are open, we know that $f^{-1}(U)$ and $f^{-1}(V)$ are open, and because $0 \in U$ and $1 \in V$, we have $f^{-1}(0) \in f^{-1}(A)$ and $f^{-1}(1) \in f^{-1}(V)$. Furtermore, since $U \cap V = \emptyset$, we get $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and X is thus normal.

Conversely, let A and B be disjoint closed subsets of a normal space X. Set $U_1 = B^c$. Then U_1 is an open set that contains A. Now since X is normal, we can use Theorem 7.2 to find an open set $U_{1/2}$ such that $A \subset U_{1/2} \subset \bar{U_{1/2}} \subset U_1$. Similarly, by normality of X, there exists open sets $U_{1/4}$ and $U_{3/4}$ such that $A \subset U_{1/4} \subset \bar{U_{1/4}} \subset U_{1/2} \subset \bar{U_{1/2}} \subset U_{3/4} \subset \bar{U_{3/4}} \subset U_1$. Continuing in this way, we define an open set U_d for every d in the set $D = \{m/(2^n) : m, n \in Z; m \leq 2^n\}$ such that $A \subset U_d \subset \bar{U_d} \subset U_1$ and $\bar{U_d} \subset U_c$ if $d < c$. Now, if we define the function f to be:

$$f(x) = inf\, \{d : x \in U_d, d \in D\}$$

$f(x) = 1$ if $x \notin U_d$ if $d \in D$

Then it is clear that $f(A) = 0$ and $f(B) = 1$, and we need only check that f is continuous. Take some basic open set V in $[0,1]$. Then $V = (a,b)$ for some $a,b \in [0,1], a < b$. This also means that $V = [0,b) \cap (a,1]$, the intersection of two open subsets in $[0,1]$. Then $f^{-1}(V) = f^{-1}([0,b) \cap (a,1]) = \{x \in X : 0 \leq f(x) < b$ and $a < f(x) \leq 1 = \{x \in X : 0 \leq f(x) < b\} \cap \{a < f(x) \leq 1\} = f^{-1}([0,b)) \cap f^{-1}((a,1])$.

However, $f^{-1}([0,b)) = \{x \in X : 0 \leq f(x) < b\} = \cup_{d<b} \{U_d : d \in D\}$ and thus is open being the union of open sets. Furthermore, for $a \in [0,1)$, $f(x) > a$ if and only if $x \notin \bar{U_d}$ for some $d > a$. Then $\{x : f(x) > b\} = \cup_{d>a} \left\{\bar{U_d}^c : d \in D\right\}$ is again open. Thus the inverse image of an open set under f is again open, implying f is continuous, thereby finishing the proof. •

Example 7.4 The set X of all countable ordinals together with the first uncountable ordinal is normal, but it is not perfectly normal. X is clearly normal. With the usual notation, if we let the first uncountable ordinal to be ω_1, then the set $\{\omega_1\}$ is closed in X, but it cannot be the countable common part of open sets in X. To see this, take any open set that contains ω_1. Since all the other elements in the set are countable, the compliment of the open set must have only countably many elements in it. This leaves uncountably many elements in the open set containing ω_1, and this will be true for any open set containing ω_1. And thus we would need $\{\omega_1\}$ to be the common part of at least uncountably many open sets, and thus X is not perfectly normal. •

Theorem 7.5 X is perfectly normal if and only if for each pair A, B of disjoint closed subsets of X, there exists a continuous function f from X into [0,1] such that the inverse image of 0 under f is A and the inverse image of 1 under f is B.

Proof: First suppose there is a continuous function $f(x)$ such that $f(x) = 0$ if and only if $x \in A$ for some closed set $A \subset X$. Let $G_n = \{x \in X : f(x) < (1/n)\}$. Then $\bigcap_{n=1}^{\infty}$ $\{x \in X : f(x) < (1/n)\} = \{x \in X : f(x) = 0\} = A$, and thus the closed set A is the common part of countably many open sets.

Now suppose the space X is perfectly normal. That is, suppose X is normal and that every closed subset of X is the common part of countably many open sets. Let A and B be two disjoint closed subsets of X. We need to define a continuous function $h : X \to [0, 1]$ satisfying the conditions set forth in the theorem. Since X is perfectly normal, we know that the closed set A can be written as $G_1 \cap G_2 \cap G_3 \cap ...$ where each G_n is open. Assume $G_n \cap B = \emptyset$, otherwise replace G_n with $G_n - B$. Since X is normal, we know that for each n there exists a continuous function $f_n : X \to [0, 1]$ such that $f_n(x) = 0$ for $x \in A$ and $f_n(x) = 1$ for $x \in X - G_n$. We then define $f(x) = \sum_{n=1}^{\infty}(1/2^n)f_n(x)$. We claim that this function is continuous. We first notice that $f(x)$ is the limit of the partial sums of the infinite summation. Each of these finite partial sums (call them $g_N = \sum_{k=1}^{N}(1/2^k)f_k(x)$) is clearly continuous being the finite sum of continuous functions. Since each of the $f_n(x)$ maps all of X into the unit interval, and since each $f_n(x)$ is multiplied by $1/2^n$, the sequence of the partial sums is a uniformly convergent sequence. Thus we let $\epsilon > 0$ and let x_o be a given point of X. Since the sequence is uniformly convergent, we know there exists a M such that for any x, we have $|g_M(x) - f(x)| < \epsilon/3$, and since $g_M(x)$ is continuous at all $x \in X$, it is also continuous at x_o. Thus there is an open set O containing x_o such that $|g_M(x) - g_M(x_o)| < \epsilon/3$ for each $x \in O$, implying that $|f(x) - f(x_o)| < \epsilon$. And thus we see that x_o is a point of continuity of f, and since the choice of x_o was arbitrary, we see that the function f is continuous at all $x \in X$, that is f is continuous.

Now notice that for each n , $B \subset X - G_n$. This implies that $f_n(x) = 1$ for $x \in B$. So we know there exists a continuous $f : X \to [0, 1]$ such that $x \in A \Leftrightarrow f(x) = 0$ (since if $x \notin A$ and $x \notin G_m$ for some m, then $x \in X - G_m$ implying $f_m(x) = 1$) and $x \in B$ implies $f(x) = 1$. Now let $D = \{x \in X : f(x) < 1/2\}$, $E = \{x \in X : f(x) = 1/2\}$, and $F = \{x \in X : f(x) > 1/2\}$. Then we see that $D \cup E$ and $E \cup F$ are closed and that $(D \cup E) \cap B = \emptyset$. Thus we have two closed disjoint sets in X. Now by a similar argument as above, we can find a new continous function $g : X \to [1/2, 1]$ such that $x \in (D \cup E)$ implies that $g(x) = 1/2$ and $x \in B$ if and only if $g(x) = 1$. So if we define a new function $h : X \to [0, 1]$ to be such that $h(x) = f(x)$ for $x \in D \cup E$ and $h(x) = g(x)$ for $E \cup F$, the we see that h is continuous (by the pasting lemma) since $(D \cup E) \cap (E \cup F) = E$ and for $x \in E$, $f(x) = 1/2 = g(x)$ and $(D \cup E) \cup (E \cup F) = X$. •

Example 7.6 The tangent open disc topological space is not normal. First it should be noted that the horizontal axis Y is closed in Z. This is because each basic open set containing some point $z \in Y$ contains at most only that one point of Y. Therefore the rationals Q and the irrationals I in Y are disjoint closed subsets of Z. Suppose U and V are open sets in Z such that $Q \subset U$ and $I \subset V$. Then for each $z \in I$ there exists some open disc $D_z \subset V$ of radius r_z which is tangent to Y at z. Now we define the subset I_n of I to be $I_n = \{z \in I : r_z > 1/n\}$. Then we form the countable collection of sets $\{I_n\}$. Since Q is countable and $\{I_n\}$ clearly contains all the irrationals, they form a countable covering of Y under the usual Euclidean topology. We know that in the usual Euclidean topology, the real line is not the countable union of nowhere dense sets. We thus claim that at least one of the I_n fails to be nowhere dense in the usual Euclidean topology. We know that each element of Q is nowhere dense, and thus we know for some fixed positive integer m, there is an interval $[e, f]$ in Y such that the set $\{z \in I : r_z > 1/m\}$ is dense in $[e, f]$. This means that every open set containing an element of the rationals in the interval (e, f) must intersect the open set V. But this in turn means that U and V cannot be disjoint. And thus we have two closed sets that can not be contained in two disjoint open sets, and thus the tangent open disc topological space is not normal. •

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