Enclosings of Small Cycle Systems

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Enclosings of Small Cycle Systems

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Vita

Nicholas Newman, son of Larry and Myong Newman, was born on October 7, 1981 on MacDill Air Force Base in Florida. He has three older brothers Tony, Ray, and Brady. He graduated from Carroll High School in 2000. He then attended Troy State University and graduated with a Bachelor of Science degree in Mathematics in May 2004. He enrolled in Graduate School at Auburn University the following August.

Dissertation Abstract

Enclosings of Small Cycle Systems

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In 2003 Hurd and others considered the problem of enclosing a triple system TS(v, λ) in a triple system TS$(v + s, \lambda + m)$ [13, 14], focusing on smallest possible enclosings. In the second chapter, their result is generalized using a new proof based on a graph-theoretic technique.

Four constructions are presented; they are exhaustive in the sense that, for each possible congruence of the parameters v or s and m, at least one construction can be applied to obtain an enclosing. In each construction, the value of v or s is restricted.

This naturally led to the question of whether or not a λ-fold 4-cycle system could be enclosed for all possible values. In the third chapter, we completely solve the enclosing problem by construction for λ-fold 4-cycle systems for $u \geq 2$.

Acknowledgments

I would first like to thank Dr. Chris Rodger for his invaluable knowledge, guidance, and patience. I am indebted to him. I would also like to thank Drs. Dean Hoffman and Pete Johnson and the rest of the Auburn faculty for guiding me throughout my graduate career.

I would also like to thank Dr. Ken Roblee whose undergraduate research project led me to this point, and Holly for being particularly understanding during the writing of this dissertation. Finally, I would like to thank my family -Larry, Myong, Tony, Ray, and Brady, and my friends for their love and support throughout my life.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

1.1 Definitions

A complete graph on v vertices, denoted by K_v, is a simple graph in which there is an edge between every pair of its vertices. A λ-fold complete graph on v vertices, denoted λK_v, is a multi-graph in which there are λ edges between every pair of its vertices.

Figure 1.1: K_5 and $2K_5$

A k-factor of a graph G is a spanning k-regular subgraph of G. This means that the k-factor is incident to each vertex of G and that each vertex has the same degree k in the subgraph. In particular, a 1-factor of a graph G would be a spanning subgraph of G of independent edges.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. A *k-cycle* $(v_0, v_1, \ldots, v_{k-1})$ is a graph with vertex set $\{v_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{v_i, v_{i+1}\} \mid i \in \mathbb{Z}_n\}$ (reducing the subscript modulo n). A *k-cycle system* of a multi-graph G is an ordered pair (V, C) where V is the vertex set of G and C is a set of k-cycles, the edges of which partition the edges of G. A k-cycle system of K_n is known in the literature as a *k-cycle system of order n.*

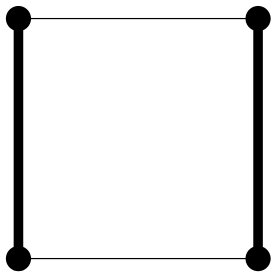

Figure 1.2: 1-factor of K_4 (in bold)

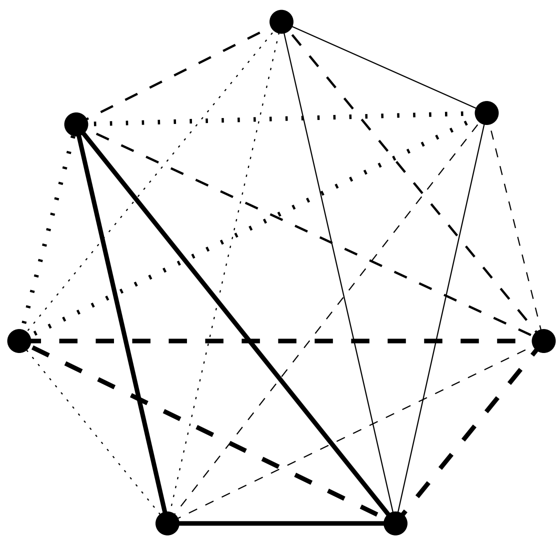

Figure 1.3: Steiner Triple System of order 7

A k-cycle system of λK_v is conveniently denoted by $kCS(v, \lambda)$. A $3CS(v, 1)$ is more commonly known as a Steiner triple system, or STS($|V|$) for short. It is clear that if there exists a kCS(v, λ) then:

(i) $\lambda(v-1)$ is even,

(ii) k divides $\lambda v(v-1)/2$, and

(iii) $v \geq k$.

For each k, v is said to be λ-admissible if (i)-(iii) are satisfied.

A $kCS(v, \lambda)$, (V, C), is said to be *enclosed* in a $kCS(w, \lambda + m)$, (W, P), if $V \subset W$, $C \subset P$, and $m \geq 1$. In the related situation when the number of vertices increases but the index does not (i.e. $m = 0$), then the $kCS(v, \lambda)$ is said to be *embedded* in the larger system.

All other definitions used can be found in Lindner and Rodger's, "Design Theory," [17] or West's "Introduction to Graph Theory" [22].

1.2 History

Doyen and Wilson [7] solved the embedding problem for Steiner triple systems, answering the question: for which values of w can a STS(v) always be embedded in a STS(w)? The generalization of this question has also been answered for cycles of lengths greater than 3 in many cases [3, 12]. These have become known as Doyen-Wilson problems.

A similar problem occurs when discussing a partial k-cycle system. A *partial* k-cycle system of order n and index λ is a subgraph of λK_n the edges of which are partitioned into k-cycles. Numerous results on embeddings and enclosings of k-cycle systems and partial k-cycle systems can be seen in various papers including Treash [21] who produced a finite embedding of a partial triple system. Further efforts were done by Lindner [15], and by Andersen, Hilton, and Mendelsohn [1] improving the bounds until recently Bryant and Horsley constructed best possible embeddings of partial STS(v)'s into STS(w)'s for all admissible $w \geq 2v+1$ [4]. The enclosing problem for 3-cycle systems is yet to be completely solved. We address this subject in this dissertation being motivated by the work done by Munson, Hurd and Sarvate [13, 14]; see Chapter 2. Work on embedding 4-cycle systems and partial 4-cycle systems have been addressed by Horak and Lindner [11], and Horton, Lindner, and Rodger [12] among others. Chapter 3 addresses the enclosings of λ-fold 4-cycle systems.

Chapter 2

Enclosings of 3-Cycle Systems

We begin with constructions that will enclose λ-fold triple systems. Four constructions are presented; they are exhaustive in the sense that, for each possible congruence of the parameters v or s and m, at least one construction can be applied to obtain an enclosing. In each construction, the value of v or s is restricted.

We begin with some definitions that are more specific to enclosing λ-fold triple systems.

2.1 Preliminaries

A (partial) balanced incomplete block design, or (partial) BIBD(v, k, λ), is an ordered pair (V, B) where B is a collection of subsets of a set V of order v, each subset being called a block, such that all blocks have size k and each pair of symbols in V appears together in (at most) λ blocks. When $k = 3$, a BIBD$(v, 3, \lambda)$ is often called a triple system, a TS(v, λ).

In this section, we allow repeated elements in all sets, except for blocks, and use $X \subseteq Y$ to denote the fact that each element occurs at least as many times in Y as it does in X. The TS(v, λ), (V_1, B_1), is said to be enclosed in the TS$(v + s, \lambda + m)$, (V_2, B_2), if $V_1 \subseteq V_2$ and $B_1 \subseteq B_2$.

Over the past five years, Hurd, Munson, and Sarvate [13, 14] have addressed the subject of minimal enclosings of triple systems proving two results:

Theorem 2.1. *[13] Each TS(v, λ) can be minimally enclosed into a TS$(v + 1, \lambda + m)$ in the case where $m > 0$ is as small as possible.*

In the second paper, they presented partial results for the enclosing of a TS(v, λ) into a TS$(v + s, \lambda + 1)$ where s is as small as possible [14]. Theorem 2.1 is an immediate corollary of our approach (see Corollary 2.1). Their latter results head in a related but different direction, considering the problem when the index is increased by exactly 1. They considered the problem related to the existence of enclosings of triple systems for any v, with $1 \leq \lambda \leq 6$, of BIBD$(v, 3, \lambda)$ into a BIBD$(v+s, 3, \lambda+1)$ for minimal positive s. The non-existence of enclosings for otherwise suitable parameters is proved, and the difficult cases for even λ were considered. They completely solve the case for $\lambda \leq 3$ and $\lambda = 5$, and partially complete the cases $\lambda = 4$ and 6. In some cases a 1-factorization of a complete graph or complete n-partite graph is used to obtain the minimal enclosing. A list of open cases for $\lambda = 4$ and $\lambda = 6$ was also included [14].

We should make note of a new necessary condition found in [13] regarding enclosings of block designs. Of particular interest in this dissertation is the application to triple systems.

Theorem 2.2. *[13] A necessary condition for enclosing* $X = BIBD(v, 3, \lambda)$ *into* $Y = BIBD(v+s, 3, \lambda+m)$ *is that*

$$s \leq \frac{1+v}{2} - \frac{\sqrt{(1+v)^2(\lambda+m)^2 - 4mv(v-1)(\lambda+m)}}{2(\lambda+m)} \quad \textit{or}$$

$$s \geq \frac{1+v}{2} + \frac{\sqrt{(1+v)^2(\lambda+m)^2 - 4mv(v-1)(\lambda+m)}}{2(\lambda+m)}.$$

This theorem is presented in the context that the following theorems in Chapter 3 provide constructions for enclosings that do not encompass all possible values of v, s, λ, and m due to additional restrictions placed on the bounds presented above.

Here, we present a new construction of enclosings making extensive use of a graph-theoretic result. A partial triple system of index λ is said to be *equitable* if for each pair of symbols v and w, $|t(v) - t(w)| \leq 1$, where $t(v)$ is the number of triples containing v. If $T = (V, B)$ is a partial triple system of index λ, then let $G(T)$ be the graph with the vertex set V in which the vertices $x, y \in V$ are joined by z edges if and only if the pair $\{x, y\}$ occurs in $\lambda - z$ triples in B. The edges in $G(T)$ represent the pairs that need to be placed in triples to boost T to a TS(v, λ). Fu and Rodger [8] found necessary and sufficient conditions for the existence of an equitable triple system T of index λ for which $G(T)$ has, for example, a 1-factorization, proving the following results.

Theorem 2.3. *[8] Suppose that* $\mu \geq 1$ *and* $\nu \geq 3$. *Then*

i) there exists an x*-regular graph* H *on* ν *vertices and of multiplicity at most* μ *whose edges can be partitioned into triples, such that*

ii) $\mu K_\nu - E(H)$ *has a 1-factorization*

if and only if

(a) $0 \leq x \leq \mu(\nu - 1)$,

(b) if $x >$ *0 then* 3 *divides* $x\nu$,

(c) if $x < \mu(\nu - 1)$ *then* 2 *divides* ν, *and*

(d) 2 *divides* x.

They also proved the following companion result.

Theorem 2.4. *[8] Suppose that* $\mu, \nu \geq 1$:

(a) $0 \leq x \leq \mu(\nu - 1)$,

(b) 3 *divides* $x\nu$, *and*

(c) $\mu(\nu - 1)$ *and* x *are even.*

Then there exists an x*-regular multigraph* H *of maximum multiplicity no greater than* μ *with* ν *vertices whose edges can be partitioned into triples, such that* $\mu K_\nu - E(H)$ *has a 2-factorization.*

We discuss some terminology and history before moving on to the enclosings.

Table 2.1 [17] below provides the necessary and sufficient conditions for the existence of λ-fold triple systems. An integer v is said to be λ-admissible if:

(i) $v \neq 2$

(ii) 3 divides $\lambda v(v-1)/2$, and

(iii) $\lambda(v-1)$ is even.

This definition is made in the context of the existence of triple systems, conditions (i)-(iii) being obvious necessary conditions for their existence. An interpretation of Table 2.1 [17] is that there exists a TS(v, λ) if and only if v is λ-admissible.

λ	Restrictions on v
0 (mod 6)	$v \neq 2$
1, 5 (mod 6)	1, 3 (mod 6)
2, 4 (mod 6)	0, 1 (mod 3)
3 (mod 6)	All odd v

Table 2.1

Using Table 2.1 as a guideline, we construct enclosings of a TS(v, λ) in a TS(v+s, λ+m), ensuring that for all congruence classes (mod 6) of v, s, λ and m at least one construction is applicable (see Tables 2.2 and 2.3).

2.2 Enclosings when $|S|$ is $(\lambda + m)$-admissible

In this section we provide our first sufficient conditions for the existence of an enclosing.

Theorem 2.5. *Let $v, \lambda, m,$ and s be positive integers. Then every TS(v, λ) can be enclosed in a TS($v + s, \lambda + m$), if:*

(a) $s \leq m(v-1)/(\lambda+m)$, and

(b) both $v + s$ and s are $(\lambda + m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We now add s new vertices in $S = \{n_1, n_2, ..., n_s\}$ to form an enclosing $(Z_v \cup S, B')$ of T as follows. Let (S, B_2) be a TS($s, \lambda + m$) (this exists by assumption (b)). The remaining edges not yet occurring in triples are therefore the edges in mK_v, and the $\lambda + m$ edges joining each vertex in Z_v to each vertex in S. Then there are $mv(v-1)/2 + vs(\lambda + m)$ remaining edges so, since $v + s$ is $(\lambda + m)$-admissible, it must be that the number of remaining edges is divisible by 3:

$$mv(v-1)/2 + vs(\lambda + m) \equiv 0 \pmod 3, \text{ so}$$

$$mv(v-1)+2vs(\lambda+m)\equiv 0 \text{ (mod 3), and}$$

$$mv(v-1)-vs(\lambda+m)\equiv 0 \text{ (mod 3),}$$

and so 3 divides xv, where $x=(m(v-1)-s(\lambda+m))$. We intend to apply Theorems 2.2 and 2.3, when v is even and odd respectively, with $x=(m(v-1)-s(\lambda+m))$, $\mu=m$, and $\nu=v$ in both cases. Note that by assumption (a), $x\geq 0$. We have just checked that condition (b) in each of Theorems 2.2 and 2.3 holds, and condition (a) clearly holds by assumption (a). We now check the remaining conditions considering the cases where v is even and odd in turn.

Case 1: Suppose v is even. Condition (c) of Theorem 2.3 clearly holds, so it remains to show that 2 divides x. Since v is even and λ-admissible, by (iii), it must be the case that λ is even. If m is even, then $m(v-1)-s(\lambda+m)$ is clearly even. If m is odd, then $(\lambda+m)$ is odd which, by (iii), implies $(v+s)$ is odd, since it is $(\lambda+m)$-admissible, and thus $m(v-1)-s(\lambda+m)$ is even.

So by Theorem 2.3 there exists a set of triples B_3 which induces an $x=(m(v-1)-s(\lambda+m))$-regular subgraph on the vertex set Z_v whose complement in mK_v has a 1-factorization into the $s(\lambda+m)$ 1-factors in $F=\{F_1,F_2,...F_{s(\lambda+m)}\}$.

Let $B_4=\{\{n_j,a,b\}\mid 1\leq j\leq s,\ i\equiv j \text{ (mod } s\text{), and } \{a,b\}\in E(F_i)\}$. Then each of the remaining edges clearly occurs in a triple in B_4.

Therefore $(Z_v\cup S,\ B_1\cup B_2\cup B_3\cup B_4=B')$ is clearly a $\mathrm{TS}(v+s,\lambda+m)$ containing T.

Case 2: Suppose v is odd. Checking the remaining conditions of Theorem 2.4, clearly $m(v-1)$ is even. Therefore x is clearly even unless $s(\lambda+m)$ is odd; but then $v+s$ is even and $(\lambda+m)$ is odd, contradicting $v+s$ being $(\lambda+m)$-admissible. So, condition (c) of Theorem 2.4 is met.

By Theorem 2.4, there exists a set B_3 of triples on the vertex set Z_v which induces an $x=(m(v-1)-s(\lambda+m))$-regular subgraph whose complement on Z_v has a 2-factorization consisting of the $s(\lambda+m)/2$ 2-factors in $F=\{F_1,F_2,...F_{s(\lambda+m)/2}\}$. So define $B_4=\{\{n_j,a,b\}\mid 1\leq j\leq s,\ i\equiv j \text{ (mod } s\text{), and } \{a,b\}\in E(F_i)\}$.

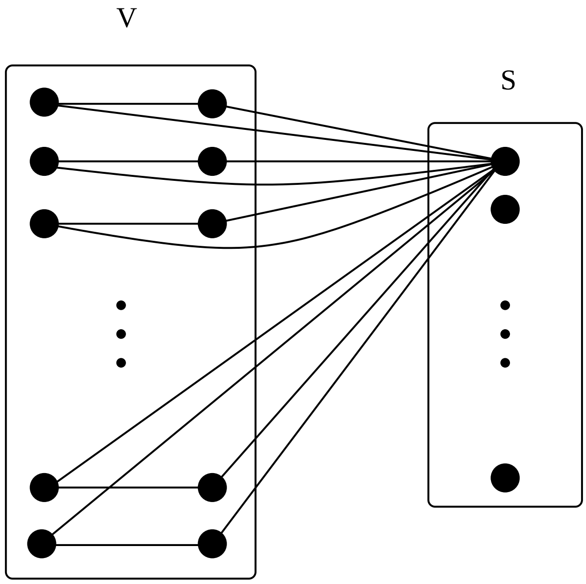

Figure 2.1: Using the 1-factor in Theorem 2.5 Case 1

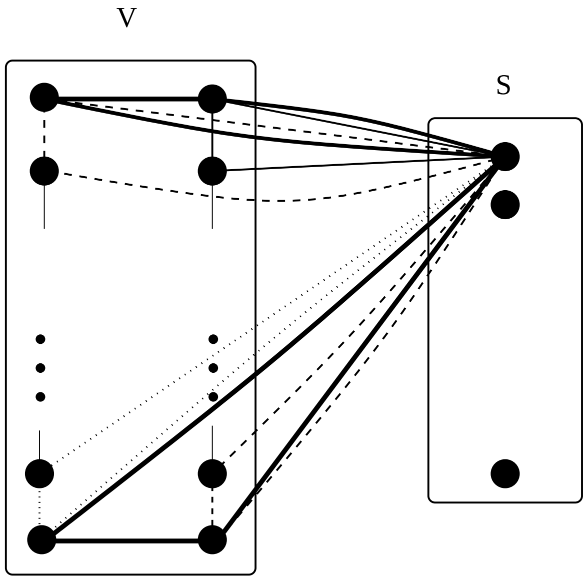

Figure 2.2: Using the 2-factor (not necessarily connected) in Theorem 2.5 Case 2

Then $(Z_v \cup S,\ B_1 \cup B_2 \cup B_3 \cup B_4 = B')$ is clearly a TS$(v+s, \lambda+m)$ containing T. $\square$

We now easily obtain the result of Hurd, Munson, and Sarvate [13] in the following corollary.

Corollary 2.1. *There exists an enclosing of every TS(v,λ) in a TS$(v+1,\lambda+m)$ if and only if* $1 \leq m(v-1)/(\lambda+m)$ *and* $v+1$ *is* $(\lambda+m)$*-admissible.*

Proof First, suppose that there exists an enclosing of a TS(v,λ), (V, B_1) in a TS$(v+1,\lambda+m)$, $(V \cup S, B')$. Then $S = \{n_1\}$ and n_1 has degree $v(\lambda+m)$. Each of the $v(\lambda+m)/2$ triples containing n_1 in B' contains an edge in mK_v. Therefore $mv(v-1)/2 \geq v(\lambda+m)/2$, so $1 \leq m(v-1)/(\lambda+m)$.

Next, suppose that there exists a TS(v,λ), (V, B_1), such that $1 \leq m(v-1)/(\lambda+m)$, and that $(v+1)$ is $(\lambda+m)$-admissible. Clearly $s = 1$ is $(\lambda+m)$-admissible. Therefore, by Theorem 2.5, there exists an enclosing TS$(v+1,\lambda+m)$, $(V \cup S, B')$ of (V, B_1). $\square$

2.3 Enclosings when $|S|$+1 is $(\lambda+m)$-admissible

In this section we investigate the enclosing of a TS(v,λ) in a TS$(v+s,\lambda+m)$ when $|S|$+1 is $(\lambda+m)$-admissible. We essentially borrow a vertex from Z_v and repeat the process from Theorem 2.5.

Theorem 2.6. *Let* v, λ, m, *and* s *be positive integers. Then every TS(v,λ) can be enclosed in a TS$(v+s,\lambda+m)$ if:*

(a) $s \leq (m(v-2)-m)/(\lambda+m)$,

(b) both $v+s$ *and* $s+1$ *are* $(\lambda+m)$*-admissible.*

Proof Let $T = (Z_v, B_1)$ be a TS(v,λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_2, ..., n_s\}$ and adjoin vertex $0 \in Z_v$ to the set S creating $S' = \{n_1, n_2, ..., n_s, 0\}$. Let (S', B_2) be a TS$(s+1,\lambda+m)$ (this exists by assumption (b)). The remaining edges not yet occurring in triples are therefore the edges in mK_{v-1}, the $\lambda+m$ edges joining each vertex in Z_{v-1} to each vertex in S, and the m edges joining 0 to each vertex in Z_{v-1}.

Following the proof of Theorem 2.5, since $v+s$ is $(\lambda+m)$-admissible, it must be that the number of remaining edges is divisible by 3:

$$m(v-1)(v-2)/2+(v-1)s(\lambda+m)+m(v-1)\equiv 0 \pmod 3, \text{ so}$$

$$(v-1)(m(v-2)+2s(\lambda+m)+2m)\equiv 0 \pmod 3, \text{ and}$$

$$(v-1)(m(v-2)-s(\lambda+m)-m)\equiv 0 \pmod 3.$$

Therefore, either

$$(v-1)\equiv 0 \pmod 3 \text{ or}$$

$$m(v-2)-s(\lambda+m)-m\equiv 0 \pmod 3.$$

In either case, it follows that 3 divides $x(v-1)$, where $x=(m(v-2)-s(\lambda+m)-m)$. By assumption (a), $x\geq 0$. Therefore, with $\mu=m$ and $\nu=v-1$ in each of Theorems 2.2 and 2.3 it can be seen that condition (b) and condition (a) clearly hold in each theorem. We now examine the cases where v is even and odd in turn.

Case 1: Suppose v is even. Then by (iii), λ must be even since v is λ-admissible. Since $v+s\not\equiv s+1 \pmod 2$, using condition (b), the only way (iii) can hold in both cases is if $\lambda+m$ is even. So m is even. Let $x=(m(v-2)-s(\lambda+m)-m)$. Then x is even, and $m(v-2)$ is even, so condition (c) of Theorem 2.4 holds (with $\mu=m$ and $\nu=v-1$). We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 2 of Theorem 2.5.

Case 2: Next, suppose v is odd. Let $x=(m(v-2)-s(\lambda+m)-m)$. 2 clearly divides $(v-1)$, so condition (c) of Theorem 2.3 is satisfied. Since $v+s$ and $s+1$ are $(\lambda+m)$-admissible, $s(\lambda+m)$ and $(v+s-1)(\lambda+m)$ are even; since v is λ-admissible, $(v-1)\lambda$ is even. So by (iii), $(v+s-1)(\lambda+m)-(v-1)\lambda=m(v-1)+s(\lambda+m)\equiv x \pmod 2$ is even. So condition (d) of Theorem 2.3 is met (with $\mu=m$ and $\nu=v-1$). We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.5.

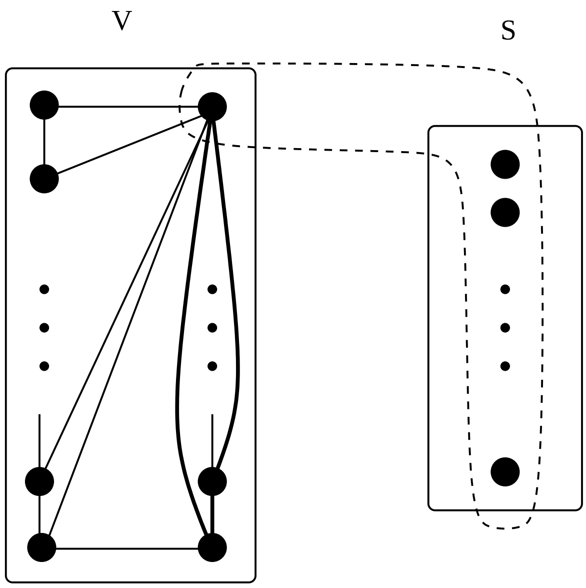

Figure 2.3: Idea of the construction of Theorem 2.6 Case 2

□

2.4 Enclosings when $|S|$+3 is $(\lambda + m)$-admissible

In this section we investigate the enclosing of a TS(v, λ) where $|S|$+3 is $(\lambda + m)$-admissible.

Theorem 2.7. *Let $v, \lambda, m,$ and s be positive integers. Then every TS(v, λ) can be enclosed in a TS($v + s, \lambda + m$) if:*

(a) $s \leq (m(v-4) - 3m)/(\lambda + m)$,

(b) both $v + s$ and $s + 3$ are $(\lambda + m)$-admissible, and

(c) $s \equiv 0$ or 4 (mod 6) or $s \geq 7$.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_2, ..., n_s\}$ and adjoin vertices 0, 1, 2 $\in Z_v$ to the set S creating the set $S' = \{n_1, n_2, ..., n_s, 0, 1, 2\}$.

If $s \equiv 0$ or 4 (mod 6) then $s+3 \equiv 1$ or 3 (mod 6). Let (S', B_2') be a TS $(s+3, \lambda+m)$ that consists of $(\lambda+m)$ copies of TS$(s+3, 1)$ each of which contains the triple $\{0, 1, 2\}$ (this exists by assumptions (b) and (c)), removing λ copies of the triple $\{0, 1, 2\}$ from B_2' we let this set of triples be B_2. Then let (S', B_2) be a partial triples system on S'.

If $s \geq 7$ we need to enclose λK_3 into $(\lambda+m)K_{s+3}$ where K_3 is λ copies of the triple $\{0, 1, 2\}$. We let $v' = 3$. By assumption (b), $s+3$ is $(\lambda+m)$-admissible, and 3 is of course m-admissible. If $(\lambda+m)v's \leq (\lambda+m)s(s-1)/2$ we can apply Theorem 2.3. Since $v' = 3$, we have

$$3 \leq (s-1)/2, \text{ and}$$

$$s \geq 7.$$

By assumption (c), this bound holds. We then let (S', B_2) be the partial triple system given by the above construction and Theorem 2.5 ignoring the triples of λK_3.

Since $v+s$ is $(\lambda+m)$-admissible, it must be that the number of remaining edges is divisible by 3:

$$m(v-3)(v-4)/2 + (\lambda+m)(v-3)s + m3(v-3) \equiv 0 \pmod 3, \text{ so}$$

$$(v-3)(m(v-4) + (\lambda+m)2s + 6m) \equiv 0 \pmod 3, \text{ and}$$

$$(v-3)(m(v-4) - s(\lambda+m) - 3m) \equiv 0 \pmod 3.$$

Therefore, either

$$(v-3) \equiv 0 \pmod 3 \text{ or}$$

$$m(v-4) - s(\lambda+m) - 3m \equiv 0 \pmod 3.$$

In either case, it follows that 3 divides $x(v-3)$, where $x = (m(v-4) - (\lambda+m)s - 3m)$. By assumption (a), $x \geq 0$. Therefore, with $\mu = m$ and $\nu = v-3$ in each of Theorems 2.2

and 2.3 it can be seen that conditions (a) and (b) clearly hold in each theorem. We now examine the cases where v is even and odd in turn, using the value of x in both cases.

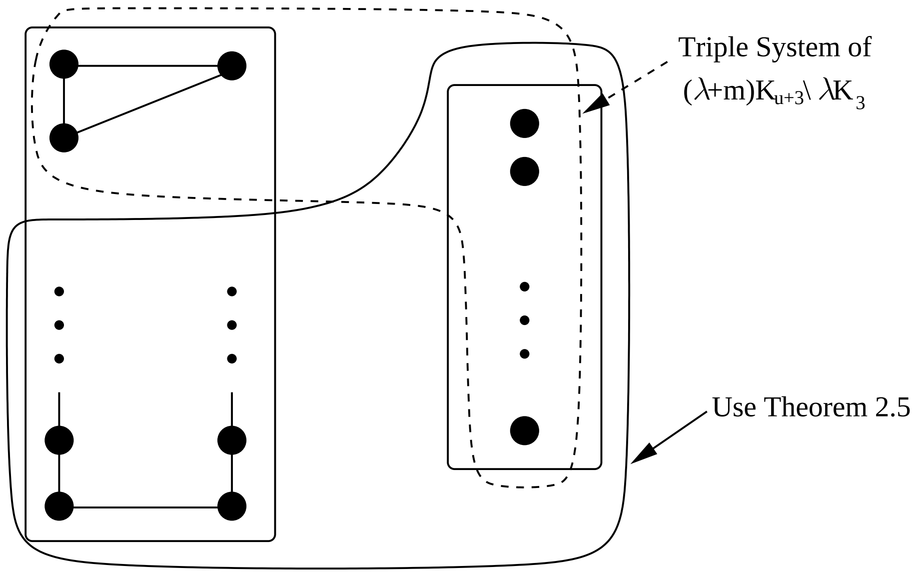

Figure 2.4: Idea of the construction of Theorem 2.7

Case 1: Suppose v is even. Then by (iii), λ is even since v is λ-admissible. Since $v + s \not\equiv s + 3 \pmod 2$, the only way condition (b) of Theorem 2.4 and (iii) can hold is if $\lambda + m$ is even. So m is even, $x = (m(v-4) - (\lambda+m)s - 3m)$ is even, and $m(v-4)$ is even, so condition (c) of Theorem 2.4 holds (with $\mu = m$ and $\nu = v - 3$). We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 2 of Theorem 2.5.

Case 2: Next, suppose v is odd. Clearly 2 divides $(v-3)$, so condition (c) of Theorem 2.3 is satisfied. Since $v+s$ and $s+3$ are $(\lambda+m)$-admissible, $(\lambda+m)(s+2)$ and $(\lambda+m)(v+s-1)$ are even and thus, $(\lambda+m)s$ is even as well; since v is λ-admissible, $\lambda(v-1)$ is even. So, by (iii), $(\lambda+m)(v+s-1) - \lambda(v-1) = m(v-4) + (\lambda+m)s + 3m \equiv x \pmod 2$ is even. So condition (d) of Theorem 2.3 is met (with $\mu = m$ and $\nu = v - 3$). We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.5.

□

2.5 Other Enclosings

The preceding constructions have addressed the existence of enclosings for all values of s and $\lambda + m$ (restricted by the bounds on s), except when $s \equiv 5 \pmod 6$ and $(\lambda + m) \equiv 1$ or $5 \pmod 6$. We now address this case to give a comprehensive list of constructions for all s and $\lambda + m$ within the tolerance of our bounds.

Theorem 2.8. *Let v, λ, m, and s be positive integers with $s \equiv 5 \pmod 6$. Let v be even and $(\lambda+m) \equiv 1$ or $5 \pmod 6$, $m > 1$. Then every TS(v, λ) can be enclosed in a TS($v+s, \lambda+m$), if:*

(a) $s \leq (m-1)(v-1)/(\lambda+m-1)$ if $(\lambda+m) \equiv 1 \pmod 6$,

(b) $s \leq (m-1)(v-3)/(\lambda+m-1)$ if $(\lambda+m) \equiv 5 \pmod 6$, and

(c) $v + s$ is $(\lambda+m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_2, ..., n_s\}$.

Case 1: Let v be even and $(\lambda + m) \equiv 1 \pmod 6$. By Table 2.1, assumption (c) implies that $(v + s) \equiv 1$ or $3 \pmod 6$. So let $(Z_v \cup S, B_2)$ be a TS($v + s$, 1). Since $\lambda + m - 1 \equiv 0 \pmod 6$, $s \equiv 5 \pmod 6$ is $(\lambda + m - 1)$-admissible. Let (S, B_3) be a TS($s, \lambda + m - 1$) .

We now intend to apply Theorem 2.3 as in Theorem 2.5 since $s \leq (m-1)(v-1)/(\lambda + m-1)$ with $x = ((m-1)(v-1)-(\lambda+m-1)s) \geq 0$ (by assumption (a)), $\mu = m-1$, and $\nu = v$. Condition (a) of Theorem 2.3 clearly holds since $x \geq 0$. Since $v + s$ is $(\lambda + m)$-admissible, the number of remaining edges must be divisible by 3:

$$(m-1)v(v-1)/2 + (\lambda + m - 1)vs \equiv 0 \pmod 3, \text{ so}$$

$$v((m-1)(v-1) - (\lambda + m - 1)s \equiv 0 \pmod 3,$$

and so 3 divides xv, satisfying condition (b) of Theorem 2.3. Condition (c) of Theorem 2.3 is clear as well since $\nu = v$ is even. λ must be even (by (iii)) and $\mu = m - 1$ must be even so condition (d) of Theorem 2.3 is satisfied. We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.5.

Case 2: Let v be even and $(\lambda + m) \equiv 5 \pmod 6$. Again, since $(v + s) \equiv 1$ or $3 \pmod 6$ we let $(Z_v \cup S, B_2)$ be a TS$(v + s,\ 1)$. We adjoin vertex $0 \in Z_v$ to the set S creating $S' = \{n_1, n_2, ..., n_s, 0\}$. Since $s + 1 \equiv 0 \pmod 6$ and $(\lambda + m - 1) \equiv 4 \pmod 6$, let (S', B_3) be a TS$(s + 1, \lambda + m - 1)$.

We now apply Theorem 2.4, as in Theorem 2.6, with $x = (m-1)(v-3)-(\lambda+m-1)s \geq 0$ (by assumption (b)), $\mu = m - 1$, and $\nu = v - 1$. Condition (a) of Theorem 2.4 clearly holds since $x \geq 0$. We now check the remaining conditions of Theorem 2.4. Since $v + s$ is $(\lambda + m)$-admissible, the number of remaining edges must be divisible by 3:

$$(m-1)(v-1)(v-2)/2 + (\lambda + m - 1)(v-1)s + (m-1)(v-1) \equiv 0 \pmod 3, \text{ so}$$

$$(v-1)((m-1)(v-3) - (\lambda + m - 1)s \equiv 0 \pmod 3,$$

and so 3 divides $x(v - 1)$, satisfying condition (b) of Theorem 2.4. Since $\mu = m - 1$ and $(\lambda+m-1)$ must be even, condition (c) of Theorem 2.4 is satisfied. We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.6. □

Theorem 2.9. *Let $v, \lambda, m,$ and s be positive integers with $s \equiv 5 \pmod 6$. Let $v \equiv 2$ or $4 \pmod 6$, $(\lambda + m) \equiv 1$ or $5 \pmod 6$, and $m = 1$. Then every TS(v, λ) can be enclosed in a TS$(v + s, \lambda + 1)$ if:*

(a) $s \geq v + 1$, and

(b) $v + s$ is $(\lambda + m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_2, ..., n_s\}$.

Case 1: Let $v \equiv 2 \pmod 6$. By Table 2.1, assumption (b) implies that $\lambda \equiv 0 \pmod 6$. $(v+s) \equiv 1 \pmod 6$ and $m = 1$ so $(\lambda+m) \equiv 1 \pmod 6$. Let $(Z_v \cup S, B_2)$ be a TS$(v+s,\ 1)$.

The remaining edges not yet occurring in triples are therefore the edges in λK_s and the λ edges joining each vertex in Z_v to each vertex in S. Since

$$\lambda s(s-1)/2 \geq \lambda vs/2, \text{ we have that}$$

$$s \geq v+1.$$

We can now apply Theorem 2.4, as in Case 2 of Theorem 2.5, and choose S to be Z_v with $x = \lambda(s-1) - \lambda v \geq 0$ (by assumption (a)), $\mu = \lambda$, and $\nu = s$. Conditions (a) and (c) of Theorem 2.4 are clear. Since $v+s$ is $(\lambda+m)$-admissible, the number of remaining edges must be divisible by 3:

$$\lambda s(s-1)/2 + \lambda vs \equiv 0 \pmod 3, \text{ so}$$

$$s(\lambda(s-1) - \lambda v) \equiv 0 \pmod 3,$$

and so 3 divides xs. Therefore, condition (b) is satisfied and we therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 2 of Theorem 2.5.

Case 2: Let $v \equiv 4 \pmod 6$. By (iii), assumption (b) implies that λ is even. $(v+s) \equiv 3 \pmod 6$, $m = 1$, and $(\lambda + m) \equiv 1$ or $5 \pmod 6$. Let $(Z_v \cup S, B_2)$ be a TS$(v+s, 1)$. The remaining edges not yet occurring in triples are therefore the edges in λK_s and the λ edges joining each vertex in Z_v to each vertex in S. Since

$$\lambda s(s-1)/2 \geq \lambda vs/2, \text{ we have that}$$

$$s \geq v+1.$$

We therefore obtain the desired enclosing by proceeding in the same manner as Case 1 with $x = \lambda(s-1) - \lambda v$, $\mu = \lambda$, and $\nu = s$. □

2.6 Large Enclosings

We have looked at enclosings involving 1-factorizations and 2-factorizations in the graph of mK_v. We now switch our construction, and apply Theorems 2.2 and 2.3 to the added vertices in the graph of $(\lambda + m)K_s$ to give us enclosings involving values of $s > v$.

Theorem 2.10. *Let v, λ, m, and s be positive integers. Then every TS(v, λ) can be enclosed in a TS($v + s, \lambda + m$) if:*

(a) $s \geq v + 1$, and

(b) v is m-admissible and $v + s$ is $(\lambda + m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_1, ..., n_s\}$ to form an enclosing $(Z_v \cup S, B')$ of T as follows. Let (Z_v, B_2) be a TS(v, m) (this exists by assumption (b)). The remaining edges not yet occurring in triples are therefore the edges in $(\lambda + m)K_s$, and the $(\lambda + m)$ edges joining each vertex in Z_v to each vertex in S. Then there are $(\lambda + m)s(s - 1)/2 + vs(\lambda + m)$ remaining edges so, since $v + s$ is $(\lambda + m)$-admissible, it must be the that the remaining edges is divisible by 3:

$$(\lambda + m)s(s - 1)/2 + (\lambda + m)vs \equiv 0 \pmod{3}, \text{ so}$$

$$(\lambda + m)s(s - 1) - (\lambda + m)vs \equiv 0 \pmod{3}, \text{ and}$$

$$(\lambda + m)s(s - v - 1) \equiv 0 \pmod{3},$$

and so 3 divides xs, where $x = (\lambda + m)(s - v - 1)$. We have just checked that condition (b) in each of Theorems 2.2 and 2.3 holds, and condition (a) clearly holds by assumption (a). We intend to apply Theorems 2.2 and 2.3, where s is even and odd in turn, with $x = (\lambda + m)(s - v - 1)$, $\mu = \lambda + m$, and $\nu = s$ in both cases.

Case 1: Suppose s is even. Condition (c) of Theorem 2.3 clearly holds, so it remains to show that 2 divides x. If $(\lambda + m)$ is even then x is clearly even. If $(\lambda + m)$ is odd, condition (iii) implies $(v + s)$ is odd (by assumption (b)), and thus $x = (\lambda + m)(s - v - 1)$ is even.

So condition (d) of Theorem 2.3 is satisfied. We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.5.

Case 2: Suppose s is odd. Checking the remaining conditions of Theorem 2.4, $(\lambda + m)(s-1)$ is clearly even. Therefore x is even unless $(\lambda+m)v$ is odd; but, by (iii), if $(\lambda+m)v$ is odd then $(\lambda + m)(s-1)$ is odd, contradicting s being odd. So, condition (c) of Theorem 2.4 is met. We then get the desired enclosing by proceeding in the same manner as in the proof of case 2 of Theorem 2.5. □

Theorem 2.11. *Let v, λ, m, and s be positive integers. Then every TS(v, λ) can be enclosed in a TS$(v+s, \lambda+m)$ if:*

(a) $1 \leq m(v-1)/(\lambda+m)$ and $v+1$ is $(\lambda+m)$-admissible, and

(b) $s \geq v+3$ and $v+s$ is $(\lambda+m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_1, ..., n_s\}$ and adjoin vertex $n_s \in S$ to the set Z_v creating $Z_v' = \{0, 1, 2, ..., v, n_s\}$. Let (Z_v', B_2) be a TS$(v+1, \lambda+m)$ (this exists by assumption (a) and Corollary 2.5). Since $v+s$ is $(\lambda+m)$-admissible, it must be the case that the number of remaining edges is divisible by 3:

$$(\lambda+m)(v+1)(s-1) + (\lambda+m)(s-1)(s-2)/2 \equiv 0 \pmod 3, \text{ so}$$

$$(\lambda+m)(s-1)(-(v-1)+s-2) \pmod 3, \text{ and}$$

$$(\lambda+m)(s-1)(s-v-3) \pmod 3$$

and so 3 divides $x(s-1)$ where $x = (\lambda+m)(s-v-3) \geq 0$. We intend to apply Theorems 2.2 and 2.3, when s is odd and even, respectively, with $x = (\lambda+m)(s-v-3)$, $\mu = \lambda+m$, and $\nu = s-1$.

Case 1: Suppose s is odd, then $s-1$ is even. Condition (c) of Theorem 2.3 clearly holds, so it remains to show that 2 divides x. If $(\lambda+m)$ is even the x is clearly even. If $(\lambda+m)$ is odd, condition (iii) implies $v+s-1$ is even and thus $s-v-3$ is even. Therefore x

is even, satisfying condition (d) of Theorem 2.3. We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 1 of Theorem 2.5 with the above values for x, μ, and ν.

Case 2: Suppose s is even, then $s-1$ is odd. Clearly 2 divides $\nu - 1$. If $(\lambda + m)$ is even, x is clearly even. If $(\lambda + m)$ is odd, then condition (iii) and assumption (b) imply that $v + s - 1$ must be even. Thus $s - v - 3$ is even and therefore x is even. So, condition (c) of Theorem 2.4 holds. We therefore obtain the desired enclosing by proceeding in the same manner as in the proof of Case 2 of Theorem 2.5. □

Theorem 2.12. *Let v, λ, m, and s be positive integers with $s \equiv 1$ or $5 \pmod 6$. Then every TS(v, λ) can be enclosed in a TS($v + s, \lambda + m$) if:*

(a) $s \geq v+1$ and $(\lambda+m) \equiv 1 \pmod 6$ or $(\lambda+m) \equiv 5 \pmod 6$ and $v \equiv 0$, 1, 3, or, $4 \pmod 6$, or

(b) $s \geq v+3$, $1 \leq (m-1)(v-1)(\lambda+m-1)$ and $v+1$ is $(\lambda+m)$-admissible if $(\lambda+m) \equiv 5 \pmod 6$ and $v \equiv 2$ or $5 \pmod 6$, and

(c) $v + s$ is $(\lambda + m)$-admissible.

Proof Let $T = (Z_v, B_1)$ be a TS(v, λ); so v is λ-admissible. We add the s new vertices in $S = \{n_1, n_1, ..., n_s\}$. Let $(Z_v \cup S, B_2)$ be a TS$(v + s, 1)$ (this exists by Table 2.1 since $(\lambda + m) \equiv 1$ or 5 (mod 6), and assumption (c)). The necessary conditions of the following cases are checked by following the proof of Theorem 2.8.

Case 1: If $s \geq v + 1$ and $(\lambda + m) \equiv 1$ (mod 6), or $(\lambda + m) \equiv 5$ and $v \equiv 0, 1, 3,$ or 4 (mod 6) then $\lambda + m - 1 \equiv 0$ (mod 6) and v is $(\lambda + m - 1)$-admissible. We then proceed in the same manner as Theorem 2.10 with $x = (\lambda + m - 1)(s - v - 1)$, $\mu = \lambda + m - 1$, and $\nu = s$.

Case 2: First, assume assumption (b). Then $\lambda + m - 1 \equiv 4$ (mod 6). We adjoin vertex $n_s \in S$ to the set Z_v creating $Z'_v = \{0, 1, 2, ..., v, n_s\}$. Let (Z'_v, B_2) be a TS$(v+1, \lambda+m-1)$ (this exists by assumption (b) and Corollary 2.5). We then proceed in the same manner as Theorem 2.10 with $x = (\lambda+m-1)(s-v-3) \geq 0$ (by assumption (b)), $\mu = \lambda+m-1$, and $\nu = s - 1$.

2.7 Conclusion

The constructions presented here are exhaustive in the sense that for each possible congruence of s or v (mod 6) and each possible congruence of $(\lambda + m)$ or m (mod 6), at least one theorem can be applied to obtain an enclosing, as described in the tables below. Of course, not all enclosings have been found, since each result places restrictions on s or v, given the other parameters.

Notice that Theorem 2.2 contains a bound that one can view as being quadratic in s, given all other parameters of our enclosings. As an example of how one would use the results in the previous section, suppose that we attempt to enclose a $\mathrm{TS}(82, 8)$ in a triple system that is near the bounds of Theorem 2.2. In order to do so we will examine the case when $m = 1$ and vary s. The necessary condition requires that

$$s \leq \frac{83}{2} - \frac{\sqrt{(83^2)81 - 4(82)81(9)}}{2(9)} \approx 10.13 \text{ or}$$

$$s \geq \frac{83}{2} + \frac{\sqrt{(83^2)81 - 4(82)81(9)}}{2(9)} \approx 72.87 \ .$$

It can be seen that the necessary condition creates a "gap" of values that is unusual in these types of designs. This gap is created by he increasing need to use each edge with two v vertices, or two s vertices, with two "mixed" edges (those having a v and s vertex). We will see this phenomenon in the following example. We will use $s = 73$ for this example, looking at a value bordering the necessary condition. If we simply use the theorems in this chapter we can enclose a TS(82, 5) in a TS(155, 9) if $s \leq \frac{m(v-1)}{(\lambda+m)}$ (Theorem 2.5) giving us $s \leq \frac{1(81)}{9} = 9$ which gets us very close to the first bound. Or, $s \geq 83$ (Theorem 2.10) which is not nearly as close as we would like. It is expected that the constructions in this chapter could be used to greater effect as the following suggests.

From Table 2.1, we see that it is possible for a $\mathrm{TS}(155, 9)$ to exist. Then Let B_1 consist of the triples of a $\mathrm{TS}(82, 8)$ then V is a $1K_{82}$ and S is a $9K_{73}$ with a $9K_{82,73}$ between the two sets of vertices. We will attempt to use our first construction presented. Table 2.1

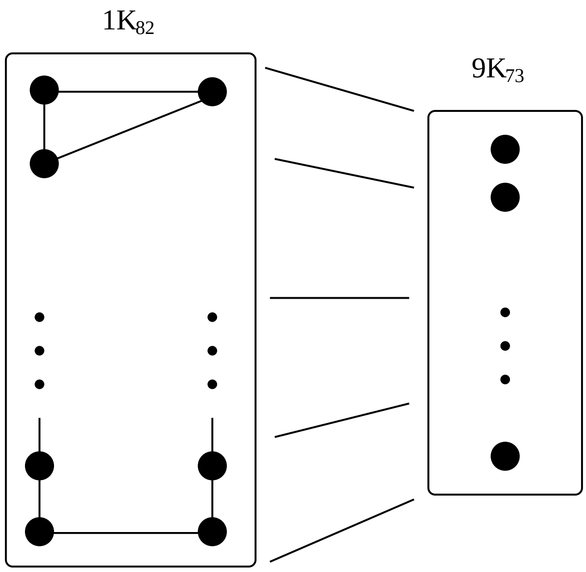

Figure 2.5: Example bordering the bounds of Theorem 2.3

says that a TS$(73, 9)$ exists so let these triples make up the set B_2. There are now 3321 edges in $1K_{82}$ which consists of 81 1-factors each of which contains 41 edges. Each 1-factor uses 82 edges in $9K_{82,73}$. Using every edge in $1K_{82}$ in a triple with 2 vertices in V and one in S leaves $53874 - 82(81) = 47232$ edges that need to be put in triples, but there is no possibility of putting these edges into triples given the aforementioned partitions. So, this construction has "failed" to enclose the TS$(82, 8)$ into a TS$(155, 9)$.

We will next attempt a construction by "switching" our construction by using a modification of Theorem 2.7 which would allow us to get closer to the necessary bound rather than with the bound in Theorem 2.10. This idea was more commonly seen in the section Large Enclosings, but is applicable in our point here. We will use the construction that "borrows" 3 vertices. Of course, we will let B_1 be the triples of a TS$(82, 8)$. Next, we will create the set V' by adjoining the vertices $n_1, n_2, n_3 \in S$ to V, and let the remaining vertices of S be the set S'. By Table 2.1, a TS$(85, 1)$, $(V \cup \{n_1, n_2, n_3\}, B_2)$, exists and let the 24 unused edges between n_1, n_2 and n_3 form a set of 8 triples, B_3. Then there are 35 edges in each of the 621 1-factors that comprise the 21735 edges in $9K_{70}$ (the induced graph on S'). We need $9(85) = 765$ 1-factors which requires 26775 edges in S'. Using all edges possible, there are still $53874 - 2(21735) = 10404$ edges still to be put in triples which cannot be done without "breaking up" some of the already formed triples since all edges not used in triples are "mixed". That is, they connect V vertices with S vertices which will not allow for triples to be formed without some connected edge (those already in triples). Essentially, since a bipartite graph has no odd cycles, and what remains is a bipartite graph, we cannot form a triple. So this construction has "failed" as well.

We use the term "failed" loosely and in the sense that each construction leaves some edges not partitioned into triples. But, with a lot of well chosen triple deconstruction, we could construct triples involving the remaining edges between the sets V and S and the edges of the deconstructed triples to finish the enclosing. With values close to the bounds given by Theorem 2.3, the constructions would require nearly every edge in V and S to use two edges between V and S.

We have seen the usefulness of the theorems provided by Fu and Rodger [8]. In the majority of our constructions, we have dealt mainly with triples containing a symbol of S and two symbols in V. Section 2.5 is the first instance where we have extensively used "mixed" triples (Those having one symbol in V and two symbols in S and vice versa.). It is easily seen that this type of construction relaxes the bounds given in the previous four sections. It is the hopes of the authors that the flexibility of using "mixed" triples will constitute a lowering of the restrictive bounds presented, creating a larger family of enclosings.

$s \pmod 6$	Restrictions on $\lambda + m \pmod 6$	Theorem Construction
0	0, 2, 4	2.5, 2.6, 2.7
0	1, 3, 5	2.6, 2.7
1	0	2.5, 2.6, 2.7
1	2, 4	2.5, 2.7
1	1, 3, 5	2.5
2	0	2.5, 2.6, 2.7
2	1, 2, 4, 5	2.6
2	3	2.6, 2.7
3	0, 2, 4	2.5, 2.6, 2.7
3	1, 3, 5	2.5
4	0	2.5, 2.6, 2.7
4	1, 5	2.7
4	2, 4	2.5, 2.7
4	3	2.6, 2.7
5	0	2.5, 2.6, 2.7
5	1, 5	2.8, 2.9
5	2, 4	2.6
5	3	2.5

Table 2.2

$v \pmod 6$	Restrictions on $m \pmod 6$	Theorem Construction
0	0, 2, 4	2.10
0	1, 3, 5	2.11
1	0, 1, 2, 3, 4, 5	2.10
2	0	2.10
2	1, 2, 3, 4, 5	2.11
3	0, 1, 2, 3, 4, 5	2.10
4	0, 2, 4	2.10
4	1, 5	2.12
4	3	2.11
5	0, 3	2.10
5	1, 5	2.12
5	2, 4	2.11

Table 2.3

Chapter 3

Enclosings of 4-Cycle Systems

3.1 Preliminaries

We now investigate the enclosings of λ-fold 4-cycle systems. In this section we completely solve the enclosing problem for λ-fold 4-cycle systems when $u \geq 2$ proving the following theorem.

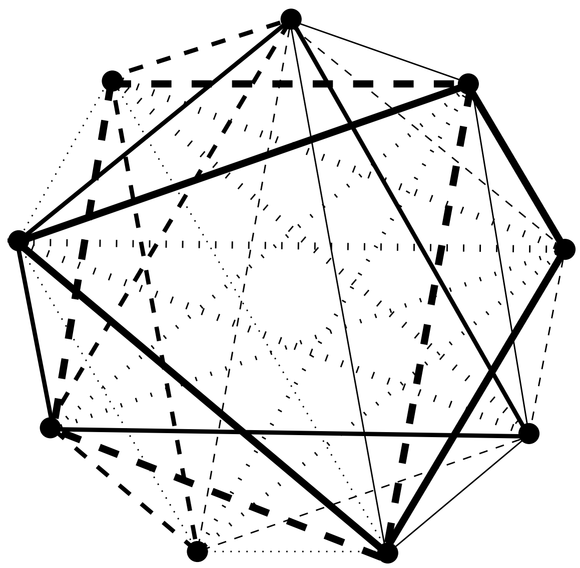

Figure 3.1: 4-cycle decomposition of K_9

Theorem 3.1. *Let $u > 1$. Every 4-cycle system of λK_v can be enclosed in a 4-cycle system of $(\lambda + m)K_{v+u}$ iff*

(a) $(v + u - 1)(\lambda + m) \equiv 0$ (mod 2), and

(b) $u(u - 1)(\lambda + m)/2 + mv(v - 1)/2 + vu(\lambda + m) \equiv 0$ (mod 4).

Throughout, we will use standard graph theoretic terminology which, if not defined here, can be found in [17, 22]. If G and H are two graphs then let $G \cup H$ be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then let $G \vee H$ be the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{\{g, h\} \mid g \in V(G), h \in V(H)\}$. If H is a subgraph of G, let $G - H$ be the subgraph of G containing precisely those edges of G which are not in H. Let $\lambda K(X, Y)$ be the bipartite graph with vertex set $X \cup Y$ on which each $x \in X$ is joined to each $y \in Y$ with λ edges. Throughout this section let $v = |V|$, $u = |U|$, and $w = |W|$.

We begin by proving the necessity of conditions (a)-(c) in Theorem 3.1 which clearly follows by the following lemma.

Lemma 3.1. *Suppose there exists a 4-cycle system of λK_v. Then conditions (a)-(c) of Theorem 3.1 hold iff in $(\lambda + m)K_{v+u}$:*

i) each vertex has even degree, and

ii) the number of edges is divisible by 4.

Proof

First, assume that conditions (a) and (b) hold. Then $(\lambda + m)(v + u - 1)$, the degree of each vertex in $(\lambda + m)K_{v+u}$, is even, thus proving (i). By (b), $u(u-1)(\lambda+m)/2 + mv(v-1)/2 + vu(\lambda + m) \equiv 0 \pmod 4$, adding in $\lambda v(v-1)/2$, which is the number of edges in a λK_v (which must be divisible by 4), we have: $u(u-1)(\lambda + m)/2 + mv(v-1)/2 + vu(\lambda + m) + \lambda v(v-1) \equiv 0 \pmod 4$ which is the number of edges in $(\lambda + m)K_{v+u}$, proving (ii).

Now assume that conditions (i) and (ii) hold. Then $(\lambda+m)(v+u-1)$, the degree of a vertex in $(\lambda+m)K_{v+u}$ is even, proving (a) holds. $(\lambda+m)(v+u-1) = (\lambda+m)(u)+m(v-1) + \lambda(v-1)$ this implies that $(\lambda + m)(u) + m(v-1)$ is even, (because a 4-cycle system of λK_v is postulated to exist) $\lambda(v-1)$ must be even, proving (b). The number of edges of $(\lambda + m)K_{v+u}$ is $(\lambda + m)v(v-1)/2 + (\lambda + m)u(u-1)/2 + (\lambda + m)uv$ which is divisible by 4. Since there exists a 4-cycle system of λK_v, $\lambda v(v-1)/2$ is also divisible by 4. So $(\lambda + m)v(v-1)/2 + (\lambda + m)u(u-1)/2 + (\lambda + m)uv - \lambda v(v-1)/2 = u(u-1)(\lambda + m)/2 + mv(v-1)/2 + vu(\lambda + m) \equiv 0 \pmod 4$, giving us condition (b).

□

Table 3.1 [2] below summarizes these necessary as well as the sufficient conditions for the existence of 4-cycle systems of lK_w. An integer w is said to be l-admissible if conditions (i) and (ii) of Lemma 3.2 are satisfied for some index l. This definition is made in the context of the existence of 4-cycle systems, conditions (a)-(c) of Theorem 3.1 being obvious necessary conditions for their existence. An interpretation of Table 3.1 is that there exists a 4-cycle system (W, l) if and only if w is l-admissible.

l	Restrictions on w
1 (mod 4)	$w \equiv 1$ (mod 8)
2 (mod 4)	$w \equiv 0$ or 1 (mod 4)
3 (mod 4)	$w \equiv 1$ (mod 8)
0 (mod 4)	$w \neq 2$ or 3

Table 3.1

Necessary and sufficient conditions for the existence of 4CS(w, l).

The following two results will be used extensively to partition the edges of $K_{u,v}$ and mK_v into 4-cycles, respectively.

Theorem 3.2. *[20] There exists a 4-cycle system of $\lambda K_{x,y}$ if and only if*

(1) $x, y \geq 2$

(2) $\lambda xy \equiv 0$ *(mod 4)*

(3) $\lambda x \equiv \lambda y \equiv 0$ *(mod 2).*

The following table found in [9] will be useful in discussing our next lemma. Table 3.2 is a list of the leaves of maximum packings of λK_v with 4-cycles. F is a 1-factor; C_n is a cycle of length n; E_6 is the set of graphs on n vertices with 6 edges in which each vertex has even degree, D is a doubled edge; and F_2 can be chosen to be the set of graphs on n vertices in which all vertices have degree 1 except for either one vertex that has degree 5, or two vertices that have degree 3, or the graph with vertex set $\{0, 1, 2, 3, 4, 5\}$ and edge set $\{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 4\}, \{1, 5\}\}$; in this dissertation we assume F_2 is the latter two whenever possible.

$\lambda \pmod 4$ / $v \pmod 8$	0	1	2	3	4	5	6	7
1	F	ϕ	F	C_3	F	E_6	F	C_5
2	ϕ	ϕ	E_6, D if $v > 3$	E_6, D if $v > 3$	ϕ	ϕ	E_6, D	E_6, D
3	F	ϕ	F_2 if $v > 2$	C_5 if $v > 3$	F	E_6, D	F_2	C_3
4	ϕ	ϕ	ϕ if $v > 2$	ϕ if $v > 3$	ϕ	ϕ	ϕ	ϕ

Table 3.2 Maximum Packings; Use E_6 if simple leaves are required, and D otherwise.

The following lemma will be useful in the construction when the number of new vertices is 2.

Lemma 3.2. *If there exists a partial decomposition of λK_v into t 4-cycles, then there exists an equitable, partial decomposition of λK_v into t 4-cycles.*

Proof Let $L(\lambda, v)$ denote the number of edges in the leave of a maximum packing of λK_v with 4-cycles (see Table 3.2, and use D as the leave when available). Since all leaves can be chosen to be equitable, in view of Table 3.2, we can assume that $t < (\lambda \frac{v(v-1)}{2} - L(\lambda, v))/4$. We will proceed by induction on the index λ. Bryant et al [5] proved the case when $\lambda = 1$. Assume that the hypothesis is true for $\lambda \leq k-1$. In most cases the result will immediately follow by taking the union of two graphs with the same vertex set, namely $(\lambda - z)K_v$ and zK_v, with $z \in \{1, 2\}$, to form the graph λK_v.

Case 1: Suppose $t \leq t^* = ((\lambda-1)\frac{v(v-1)}{2} - L(\lambda-1, v))/4$. An equitable, partial 4-cycle decomposition of $(\lambda-1)K_v$ then exists by the induction hypothesis; this is also an equitable, partial decomposition of λK_v into 4-cycles.

Case 2: Suppose $t^* < t \leq t^{**} = (\lambda\frac{v(v-1)}{2} - \min_{z\in\{1,2\}}\{L(\lambda - z, v) + L(z, v)\})/4$. Let (V, B_1) and (V, B_2) be maximum packings of $(\lambda - z)K_v$ and zK_v, respectively with $z \in \{1, 2\}$. Let $G(B_1)$ and $G(B_2)$ be the graphs induced by (V, B_1) and (V, B_2) respectively, naming the vertices so that $d_{G(B_1)}(i) \leq d_{G(B_1)}(j)$ and $d_{G(B_2)}(i) \geq d_{G(B_1)}(j)$ for $0 \leq i < j \leq v-1$. Let G be the union of $G(B_1)$ and $G(B_2)$. Then clearly each vertex in G has degree in $\{d_{G(B_1)}(0) + d_{G(B_2)}(v-1) + d \mid d \in \{0, 2\}\}$ or $\{d_{G(B_1)}(0) + d_{G(B_2)}(v-1) + d \mid d \in \{2, 4\}\}$. In either case, it follows that $(V, B_1 \cup B_2)$ is the desired equitable partial 4-cycle system.

Case 3: Now suppose that $t^{**} < t < (\lambda\frac{v(v-1)}{2} - L(\lambda, v))/4$. This is nearly a maximum packing of λK_v. We use the same approach as in Case 2, starting with up to two maximum packings (V, B_1) of $(\lambda - z)K_v$ and (V, B_2) of zK_v, except that we may need to align the respective leaves to create more 4-cycles. It suffices to consider one choice of z for each of the possible values of λ and v. These are chosen so that the union of the leaves $L_1 \cup L_2$ has as many 4-cycles as possible. Exactly $t - t^{**}$ of these 4-cycles are then added to $B_1 \cup B_2$ to obtain the desired equitable 4-cycle system; the following argument checks this is possible. Using Table 3.2, we consider each case in turn.

Subcase 1: Suppose first that the leaves of the two maximum packings are both 1-factors, L_1 and L_2. Let the leave L_1 of B_1 have edge set $\{\{2x, 2x+2\}, \{2x+1, 2x+3\} \mid x \in \{0, 1, 2, \ldots, v-1\}\}$ and the leave L_2 of B_2 have edge set $\{\{0,1\}, \{2,3\}, \ldots, \{v-2, v-1\}\}$. The additional 4-cycles are those in $B_3 = \{(y, y+1, y+3, y+2) \mid 0 \leq y < t - t^{**}\}$. Then $(V, B_1 \cup B_2 \cup B_3)$ produces the required equitable partial 4-cycle system.

Subcase 2: Let the leaves of the two maximum packings be C_5's with $L_1 = (0, 1, 2, 3, 4)$ and $L_2 = (0, 2, 5, 3, 6)$. These edges can be taken as the 2 cycles $(0, 2, 3, 4)$ and $(0, 1, 2, 5, 3, 6)$ adding the first to $B_1 \cup B_2$ to produce the required equitable partial 4-cycle system with leave E_6 (or simply use E_6 in Table 3.2).

Subcase 3: Let the leaves of the two maximum packings be a C_5 with $L_1 = (0, 1, 2, 3, 4)$ and a C_3 with $L_2 = (1, 3, 4)$. Add the 4-cycle $(0, 1, 3, 4)$ to $B_1 \cup B_2$ to obtain the required equitable partial 4-cycle system (or, we can simply think of removing a 4-cycle from a 4-cycle system).

The subcases take into account all relevant combinations of $(\lambda - z)K_v$ and zK_v to get the desired partial equitable 4-cycle decomposition of λK_v.

□

3.2 Enclosings when $u \geq 3$

In this section we provide our first sufficient conditions for the existence of an enclosing of a 4-cycle system of λK_v, proving that Theorem 3.1 is true when $u \geq 3$.

Proposition 3.1. *Let v, λ, m, and u be positive integers with $v \geq 4$, $u \geq 3$ and $\lambda \equiv 1 \pmod 4$. Then every $4CS(v, \lambda)$ can be enclosed in a $4CS(v+u, \lambda+m)$, if $v+u$ is $(\lambda+m)$-admissible.*

Proof We will proceed case by case based on the congruence of $m \pmod 4$ and the possible values of u. We can assume $V = \mathbb{Z}_v$

Let $C = (\mathbb{Z}_v, C_1)$ be a 4CS(v, λ). Since v is λ-admissible, by Table 3.1 we see that $|\mathbb{Z}_v| = v \equiv 1 \pmod 8$. Let $U = \{n_1, n_2, ..., n_u\}$ with $U \cap \mathbb{Z}_v = \emptyset$ and form an enclosing 4CS$(\mathbb{Z}_v \cup U, C')$ of C as follows.

Case 1: Suppose $m \equiv 1 \pmod 4$. Then $(\lambda + m) \equiv 2 \pmod 4$. Therefore, since $v + u$ is $(\lambda+m)$-admissible, $(\lambda+m) \equiv 1+1 = 2 \pmod 4$ and, since $v \equiv 1 \pmod 8$, Table 3.1 implies that $u \equiv 0$ or $3 \pmod 4$. Since $\lambda \equiv m \pmod 4$, there exists a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$.

(i) If $u \equiv 0 \pmod 4$, then u is $(\lambda + m)$-admissible so there exists a $4CS(u, \lambda + m)$, say (U, C_3); this exists by Table 3.1 since $u \equiv 0 \pmod 4$. This leaves the edges of $(\lambda + m)K(\mathbb{Z}_v, U)$ remaining. Clearly, the degree of each vertex in $(\lambda + m)K(\mathbb{Z}_v, U)$ is even and, as $u + v$ is $(\lambda + m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \cup U, C_4)$ be a 4CS of $(\lambda + m)K(\mathbb{Z}_v, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

(ii) If $u \equiv 3 \pmod 4$, adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$. Let $(U \cup \{0\}, C_3)$ be a 4CS$(u+1, \lambda+m)$ (this exists by Table 3.1). This leaves the edges of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ remaining. Clearly, the degree of each vertex in $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even and, as $u + v$ is $(\lambda + m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda + m)K(\mathbb{Z}_v \setminus \{0\}), U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

Case 2: $m \equiv 2 \pmod 4$. Then $(\lambda + m) \equiv 3 \pmod 4$. Therefore, by Table 3.1, $u \equiv 0 \pmod 8$. Since $m \equiv 2 \pmod 4$, and $v \equiv 1 \pmod 8$ there exists an m-fold 4CS(v, m), say $(\mathbb{Z}_v, C_2)$. By adjoining vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$, there also exists a 4CS$(u+1, \lambda+m)$, (U', C_3), (see Table 3.1). The remaining edges are those edges in $(\lambda+m)K(\mathbb{Z}_v, U)$. By Theorem 3.2, there exists a 4CS $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ of $(\lambda+m)K(\mathbb{Z}_v, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

Case 3: $m \equiv 3 \pmod 4$. Then $(\lambda+m) \equiv 0 \pmod 4$. Therefore, by Table 3.1, $u \in \mathbb{N}$, and there exists a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$.

(i) Let u be even. u is $(\lambda+m)$-admissible so there exists a 4CS$(u, \lambda+m)$, say (U, C_3); this exists by Table 3.1. This leaves the edges of $K(V, U)$ remaining. Clearly, the degree of each vertex is even and, as $u+v$ is $(\lambda+m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \cup U, C_4)$ be a 4CS of $(\lambda+m)K(V, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

(ii) Let u be odd. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$. Let (U', C_3) be a 4CS$(u+1, \lambda+m)$ (see Table 3.1). This leaves the edges of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ remaining. Clearly, the degree of each vertex in $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even, $u+v$ is $(\lambda+m)$-admissible, and the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

Case 4: $m \equiv 0 \pmod 4$. Then $(\lambda+m) \equiv 1 \pmod 4$. Therefore, by Table 3.1, $u \equiv 0 \pmod 8$, and there exists a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$. Let (U', C_3) be a 4CS$(u+1, \lambda+m)$ (see Table 3.1). This leaves the edges of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ remaining. Clearly, the degree of each vertex in $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even and, as $u+v$ is $(\lambda+m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

□

We will refer to the proof of Proposition 3.1 often when the constructions are the same, while still keeping track of the changing parameters of v, u, and m.

Proposition 3.2. *Let v, λ, m, and u be positive integers with $v \geq 4$, $u \geq 3$, and $\lambda \equiv 2 \pmod 4$. Then every $4CS(v, \lambda)$ can be enclosed in a $4CS(v + u, \lambda + m)$, if $v + u$ is $(\lambda + m)$-admissible.*

Proof We will proceed case by case based on the congruence of $m \pmod 4$ and the possible values of u. Again, we assume $V = \mathbb{Z}_v$.

Let $C = (\mathbb{Z}_v, C_1)$ be a 4CS(v, λ). Since v is λ-admissible, by Table 3.1 we see that $|\mathbb{Z}_v| = v \equiv 0$, 1, 4, or 5 (mod 8). Let $U = \{n_1, n_2, ..., n_u\}$ with $U \cap \mathbb{Z}_v = \emptyset$ and form an enclosing 4CS$(v + u, C')$ of C as follows.

Case 1: Suppose $m \equiv 1 \pmod 4$. Then $(\lambda + m) \equiv 3 \pmod 4$. And, for each value of v, $u \equiv 1$, 0, 5, or 4 (mod 8), respectively.

Since, in all cases, $u + v$ is 1-admissible, let $(\mathbb{Z}_v \cup U, C_2)$ be a 4CS$(v + u, 1)$. The remaining edges of $(\lambda + m)K_u$ can be decomposed into a 4CS$(u, \lambda + m - 1)$, (U, C_3). This exists by Table 3.1, since $u \equiv 0$ or 1 (mod 4), exactly one edge between each pair of vertices in U has been used in C_2. Let (V, C_4) be a 4CS$(v, m - 1)$ of $(m - 1)K_v$ under the same reasoning. Clearly, the degree of each vertex in $(\lambda + m - 1)K(V, U)$ must be even and the number of edges divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \cup U, C_5)$ be a 4CS of $(\lambda + m - 1)K(V, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

Case 2: Suppose $m \equiv 2 \pmod 4$. Then $(\lambda + m) \equiv 0 \pmod 4$. First, suppose $u \geq 4$. In each case, v is m-admissible. Let (V, C_2) be a 4CS(v, m) of mK_v. And, for $u \geq 4$, let (U, C_3) be a 4CS$(u, \lambda + m)$ of $(\lambda + m)K_u$. These exist by Table 3.1. As $(\lambda + m) \equiv 0$ (mod 4), Theorem 3.2 applies to $K(V, U)$. Let $(\mathbb{Z}_v \cup U, C_4)$ be a 4CS of $(\lambda + m)K(V, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

If $u = 3$, adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, n_3, 0\}$. Let (U', C_3) be a 4CS$(u + 1 = 4, \lambda + m)$. Clearly, the degree of each vertex in $(\lambda + m)K(\mathbb{Z}_v \setminus \{0\}, U)$

is even and, as $u+v$ is $(\lambda+m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

Case 3: Suppose $m \equiv 3 \pmod 4$. Then $(\lambda+m) \equiv 1 \pmod 4$. And for $u \equiv 0$, 1, 4, or 5 (mod 8), $u \equiv 1$, 0, 5, or 4 (mod 8), respectively. Let $(V \cup U, C_5)$ be a 4CS$(u+v, 1)$ (this exists by Table 3.1). We then proceed in the same manner as Case 2.

Case 4: Suppose $m \equiv 0 \pmod 4$. Then $(\lambda+m) \equiv 2 \pmod 4$.

If $v \equiv 0$ or 4 (mod 8), then $u \equiv 0$, 1, 4, or 5 (mod 8) (not respectively). In all cases, v is m-admissible and u is $(\lambda+m)$-admissible. Let (V, C_2) be a 4CS(v, m) and (U, C_3) be a 4CS$(u, \lambda+m)$. And, by Theorem 3.2, let $(V \cup U, C_4)$ be a 4CS of $(\lambda+m)K(V, U)$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

If $v \equiv 1$ or 5 (mod 8), then $u \equiv 0$, 3, 4, or 7 (mod 8) (not respectively). Let $(V \cup U, C_3)$ be a 4CS$(v+u, 2)$ (this exists by Table 3.1). Then, as in Case 2, there exists an enclosing $(\mathbb{Z}_v \cup U, C_4)$ of $(\lambda+m-2)K_{v+u}$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

□

We will continue in much the same fashion as Proposition 3.2 using the same constructions found in Proposition 3.1 with $\lambda \equiv 3 \pmod 4$

Proposition 3.3. *Let $v, \lambda, m,$ and u be positive integers with $v \geq 4$, $u \geq 3$, and $\lambda \equiv 3 \pmod 4$. Then every 4CS(v, λ) can be enclosed in a 4CS$(v+u, \lambda+m)$, if $v+u$ is $(\lambda+m)$-admissible.*

Proof We will proceed case by case based on the congruence of $m \pmod 4$ and the possible values of u. Again, we assume $V = \mathbb{Z}_v$.

Let $C = (\mathbb{Z}_v, C_1)$ be a 4CS(v, λ). Since v is λ-admissible, by Table 3.1 we see that $|\mathbb{Z}_v| = v \equiv 1 \pmod 8$. Let $U = \{n_1, n_2, ..., n_u\}$ with $U \cap \mathbb{Z}_v = \emptyset$ and form an enclosing 4CS$(v+u, C')$ of C as follows.

Case 1: Suppose $m \equiv 1 \pmod 4$. Then $(\lambda + m) \equiv 0 \pmod 4$.

(i) Let u be even. By Table 3.1 there exists a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$. Since $u \geq 4$ (being even), there exists a 4CS$(u, \lambda+m)$, (U, C_3), and a 4CS$(v+u, C_4)$ of $(\lambda+m)K(V, U)$. Therefore, we get an enclosing in a similar fashion as in Proposition 3.1 Case 1 (i).

(ii) Let u be odd. Then there exists a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$. We then continue to construct our enclosing just as in Proposition 3.1 Case 1 (ii).

Case 2: Suppose $m \equiv 0$ or $2 \pmod 4$. Then $(\lambda + m) \equiv 1$ or $3 \pmod 4$, respectively, and it must be that $u \equiv 0 \pmod 8$ (by Table 3.1). So there exists a 4CS$(u+1, \lambda+m)$, say (U', C_3), and let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda + m)K(\mathbb{Z}_v \setminus \{0\}, U)$. We then construct our enclosing just as in Proposition 3.1 Case 1 (ii).

Case 3: Suppose $m \equiv 3 \pmod 4$. Then $(\lambda + m) \equiv 2 \pmod 4$, and it must be that $u \equiv 0$ or 3 $(mod\ 4)$.

(i) If $u \equiv 0$ $(mod\ 4)$, then we proceed as in Proposition 3.1 Case 1 (i) with a 4CS(v, m), say $(\mathbb{Z}_v, C_2)$, a 4CS$(u, \lambda + m)$, say (U, C_3), and finally, $(\mathbb{Z}_v \cup U, C_4)$ being a 4CS of $(\lambda + m)K(V, U)$ giving us our desired enclosing.

(ii) If $u \equiv 3$ $(mod\ 4)$, then we proceed as in Proposition 3.1 Case 1 (ii) by adjoining vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, ..., n_u, 0\}$. Let (U', C_3) be a 4CS$(u + 1, \lambda + m)$ (see Table 3.1). This leaves the edges of $(\lambda + m)K(\mathbb{Z}_v \setminus \{0\}, U)$ remaining. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda + m)K(\mathbb{Z}_v \setminus \{0\}, U)$. Then it is clear that $(\mathbb{Z}_v \cup U, C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

□

Proposition 3.4. *Let $v, \lambda, m,$ and u be positive integers with $v \geq 4$, $u \geq 3$, and $\lambda \equiv 0$ (mod 4). Then every 4CS(v, λ) can be enclosed in a 4CS$(v + u, \lambda + m)$, if $v + u$ is $(\lambda + m)$-admissible.*

Proof We will proceed case by case based on the congruence of $m \pmod 4$ and the possible values of u. Again, we assume $V = \mathbb{Z}_v$.

Let $C = (\mathbb{Z}_v, C_1)$ be a 4CS(v, λ). Since v is λ-admissible, by Table 3.1 we see that $|\mathbb{Z}_v| = v$ can take on all values ≥ 4. Let $U = \{n_1, n_2, ..., n_u\}$ with $U \cap \mathbb{Z}_v = \emptyset$ and form an enclosing 4CS$(v + u, C')$ of C as follows.

Case 1: Suppose $m \equiv 1 \pmod 4$. Then $(\lambda + m) \equiv 1 \pmod 4$. And, for each value of $v \equiv$0, 1, 2, 3, 4, 5, 6, or 7 (mod 8), $u \equiv$ 1, 0, 7, 6, 5, 4, 3, or 2 (mod 8), respectively. Let $(U \cup V, C_5)$ be a 4CS$(v + u, 1)$. Then let (V, C_2) be a 4CS$(v, m - 1)$, if there are any edges left and, if $u \geq 4$, then let (U, C_3) be a 4CS$(u, \lambda + m - 1)$ (these exist by Table 3.1). By Theorem 3.2, there exists a 4CS of $(\lambda + m - 1)K(U, V)$, say $(V \cup U, C_4)$.

If $u = 3$, then we proceed as in Proposition 3.2 Case 2. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, n_3, 0\}$. Let (U', C_3) be a 4CS$(u + 1 = 4, \lambda + m - 1)$. Clearly, the degree of each vertex in $(\lambda + m - 1)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even and, as $u + v$ is $(\lambda + m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_5)$ be a 4CS of $(\lambda + m - 1)K(\mathbb{Z}_v \setminus \{0\}, U)$.

Then $(\mathbb{Z}_v \cup U,\, C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

Case 2: Suppose $m \equiv 2 \pmod 4$. Then $(\lambda + m) \equiv 2 \pmod 4$. And, for each value of $v \equiv$0 (mod 4), $u \equiv$ 0, 1, 4, or 5(mod 8). If $v \equiv$1 (mod 4), $u \equiv$ 0, 3, 4, or 7(mod 8). If $v \equiv$2 (mod 4), $u \equiv$ 2, 3, 6, or 7(mod 8). And, if $v \equiv$3 (mod 4), $u \equiv$ 1, 2, 5, or 6(mod 8).

In all cases except when $u = 3$ we can construct our enclosing in the following way. Let $(U \cup V, C_5)$ be a 4CS$(v + u, 2)$. Then let (V, C_2) be a 4CS$(v, m - 2)$, if there are any edges left, and (U, C_3) be a 4CS$(u, \lambda + m - 2)$ (these exist by Table 3.1). By Theorem 3.2, there exists a 4CS of $(\lambda + m - 2)K(U, V)$, which we denote by $(V \cup U, C_4)$.

If $u = 3$, we proceed as in Case 1. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, n_3, 0\}$. Let (U', C_3) be a 4CS$(u+1 = 4, \lambda+m-2)$. Clearly, the degree of each vertex in $(\lambda+m-2)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even and, as $u+v$ is $(\lambda+m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_5)$ be a 4CS of $(\lambda+m-2)K(\mathbb{Z}_v \setminus \{0\}, U)$.

In any case, $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

Case 3: Suppose $m \equiv 3 \pmod 4$. Then $(\lambda + m) \equiv 3 \pmod 4$. And, for each value of $v \equiv 0$, 1, 2, 3, 4, 5, 6, or 7 (mod 8), $u \equiv 1$, 0, 7, 6, 5, 4, 3, or 2 (mod 8), respectively. Let $(U \cup V, C_5)$ be a 4CS$(v+u, 3)$. Then let (V, C_2) be a 4CS$(v, m-3)$, if there are any edges left, and (U, C_3) be a 4CS$(u, \lambda+m-3)$ (these exist by Table 3.1). By Theorem 3.2, there exists a 4CS of $(\lambda+m-3)K(U,V)$, which we denote by $(V \cup U, C_4)$.

If $u = 3$, we have to proceed as in Case 1. Adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, n_3, 0\}$. Let (U', C_3) be a 4CS$(u+1 = 4, \lambda+m-3)$. Clearly, the degree of each vertex in $(\lambda+m-3)K(\mathbb{Z}_v \setminus \{0\}, U)$ is even and, as $u+v$ is $(\lambda+m)$-admissible, the number of edges remaining is divisible by 4. Therefore, by Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_5)$ be a 4CS of $(\lambda+m-3)K(\mathbb{Z}_v \setminus \{0\}, U)$.

In any case, $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

Case 4: Suppose $m \equiv 0 \pmod 4$. Then $(\lambda + m) \equiv 0 \pmod 4$. So u is unrestricted by the necessary conditions.

Then let (V, C_2) be a 4CS(v, m), and, if $u \geq 4$, let (U, C_3) be a 4CS$(u, \lambda+m)$. By Theorem 3.2, there exists a 4CS of $(\lambda+m)K(U,V)$, which we denote by$(V \cup U, C_4)$.

If $u = 3$, we proceed in a similar manner as in Case 2. Let (V, C_2) be a 4CS(v, m). We then adjoin vertex $0 \in \mathbb{Z}_v$ to the set U creating $U' = \{n_1, n_2, n_3, 0\}$. Let (U', C_3) be a 4CS$(u+1 = 4, \lambda+m)$. By Theorem 3.2, the remaining edges can be decomposed into 4-cycles. Let $(\mathbb{Z}_v \setminus \{0\} \cup U, C_4)$ be a 4CS of $(\lambda+m)K(\mathbb{Z}_v \setminus \{0\}, U)$.

In any case, $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v+u, \lambda+m)$ containing C.

□

3.3 Enclosings when $u = 2$

In this section we prove that Theorem 3.1 is true in the case where $u = 2$.

Proposition 3.5. *Let v, λ, m, and u be positive integers with $v \geq 4$ and $u = 2$. Then every $4CS(v, \lambda)$ can be enclosed in a $4CS(v + u, \lambda + m)$, if $v + u$ is $(\lambda + m)$-admissible.*

Proof Let $C = (\mathbb{Z}_v, C_1)$ be a 4CS(v, λ). Since v is λ-admissible, by Table 3.1 we see that $|\mathbb{Z}_v| = v \equiv 7 \pmod 8$ when $(\lambda + m) \equiv 1$ or $3 \pmod 4$. $v \equiv 2$, 3, 6, or 7 if $(\lambda + m) \equiv 2 \pmod 4$. Finally, v can take on all values ≥ 4 if $(\lambda + m) \equiv 0 \pmod 4$. Let $U = \{u_H, u_T\}$ with $U \cap \mathbb{Z}_v = \emptyset$ and form an enclosing 4CS$(v + u, C')$ of C as follows.

Case 1: Suppose $(\lambda + m) \equiv 1$ or $3 \pmod 4$. Since $|U| = 2$ it must be that $v \equiv 7 \pmod 8$. Thus $\lambda \equiv 0 \pmod 4$ and $m \equiv 1$ or $3 \pmod 4$. Let (V, C_2) be an equitable, partial 4CS(v, m) containing $mv(v - 1)/2 - (\lambda + m)$ edges. This exists by Lemma 3.1. Let H be the complement of (V, C_2). H is clearly an even graph though possibly not connected. Apply an orientation to each edge in each component to have a directed Eulerian circuit. For each directed edge, take the following edges to create a 4-cycle: the directed edge, the edge connected to the head vertex and u_H, the edge connected to the tail vertex and u_T, and the edge between u_H and u_T. Let $(U \cup V, C_3)$ be 4CS of the aforementioned edges.

The remaining edges of $(\lambda + m)K(\mathbb{Z}_v, U)$ consist of those between $\mathbb{Z}_v$ and U. Let G^* be the graph on $\mathbb{Z}_v \cup U$ that is the complement of the edges in C_1, C_2, and C_3. $|E(G^*)| = 2(\lambda + m)v - 2(\lambda + m) = 2(\lambda + m)(v - 1)$. Thus, 4 divides $|E(G^*)|$. u_H and u_T each have degree $(\lambda + m)(v - 1)$ which is even. And, each vertex in $\mathbb{Z}_v$ has degree either $2(\lambda + m)$ or $2(\lambda + m) - d$ where d is the degree of the vertex in H. As H was an even graph, these vertices have even degree. Therefore, Theorem 3.2 applies. Let $(\mathbb{Z}_v \cup U, C_4)$ be a 4CS of $(\lambda + m)K(\mathbb{Z}_v, U) - H$.

Then $(\mathbb{Z}_v \cup U,\ C_1 \cup C_2 \cup C_3 \cup C_4 = C')$ is clearly a 4CS$(v + u, \lambda + m)$ containing C.

Case 2: Suppose $(\lambda + m) \equiv 2 \pmod 4$. Since $|U| = 2$, it must be that $v \equiv 2$, 3, 6, or 7 (mod 8). Thus $\lambda \equiv 0 \pmod 4$. We then proceed as in Case 1 taking note that $2(\lambda + m)(v - 1)$ is divisible by 4, and the degrees of G^* for each vertex is divisible by 2.

Case 3: Suppose $(\lambda + m) \equiv 0 \pmod 4$. v can therefore take on all values ≥ 4. When $v \equiv 2$, 3, 6, or 7 (mod 8), $\lambda \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 4$. If $v \equiv 0$, 4, or 5 (mod 8), $\lambda \equiv 0$ or $2 \pmod 4$; therefore, $m \equiv 0$ or $2 \pmod 4$, respectively. If $v \equiv 1 \pmod 8$, $\lambda \equiv 0$, 1, 2, or 3 (mod 4). In any case, we proceed as in Case 1 noting that 4 divides $|E(G^*)|$, and 2 divides the degrees of G^*.

□

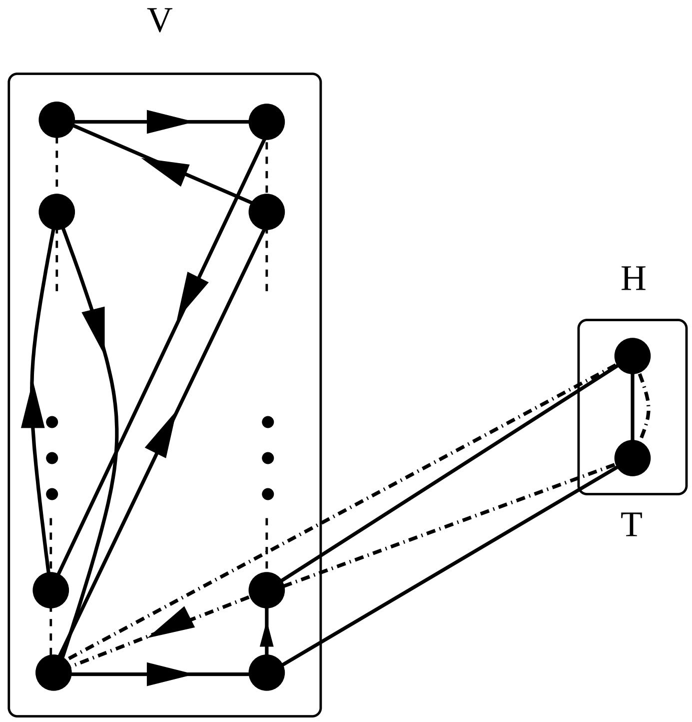

Figure 3.2: Connecting the Euler Tour (2 4-cycles constructed as an example)

3.4 Conclusion

We have provided constructions for all possible enclosings for $u \geq 2$, providing the sufficiency of Theorem 3.1 (Through Propositions 3.1-3.5). The case when $u = 1$ looks to be particularly difficult. Since 3 vertices must be in V, a decomposition of mK_V into 2-paths (denoted P_2) and 4-cycles must be obtained. This concept is not difficult on its own, but the difficulty arises, in that, the end of each P_2 must be connected to the u vertex, requiring each v vertex to be at the end of $(\lambda + m)$ P_2's while the remaining edges would still need to be decomposable into 4-cycles. Thus we need an equalized P_2 with a 4-cycle decomposition where the ends of each P_2 are evenly distributed among the v vertices. The figure below illustrates this problem. Work will be continued on this problem in order to completely solve the enclosings of 4-cycle systems.

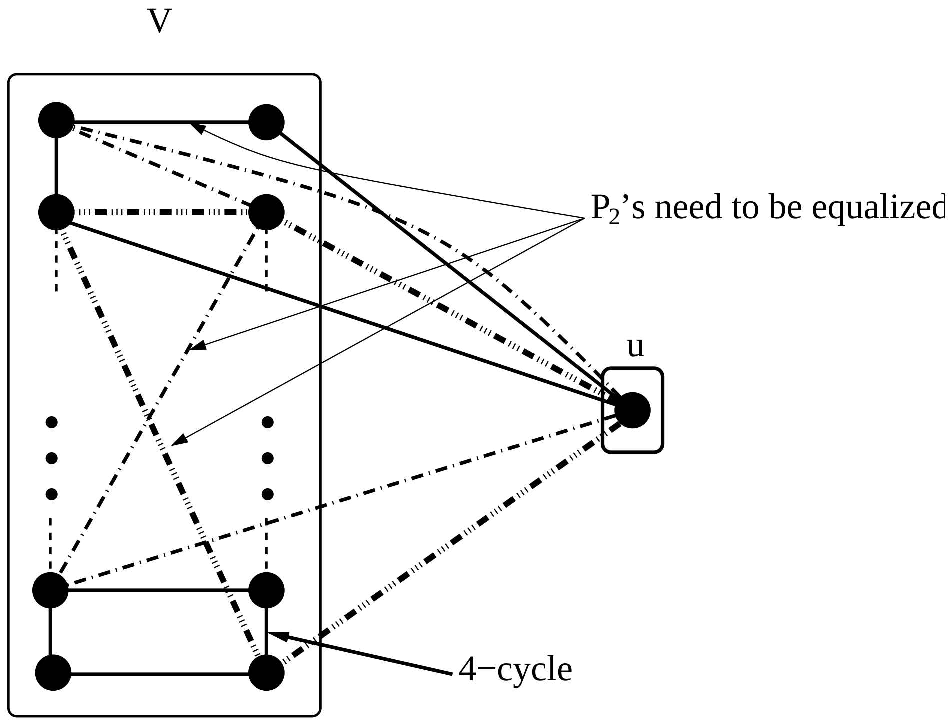

Figure 3.3: Constructing 4-cycles from P_2's

Chapter 4
Conclusions

It is disconcerting to the author that, despite much work, neither case has been completely solved. Due to the complexity of the quadratic necessary bounds found by Hurd et al [13], the enclosings of λ-fold Triples Systems is incomplete. It is worthy to note that the constructions presented in Chapter 2 are fairly comprehensive where the parameters are concerned. And, it is the belief of the author that the bounds on the remaining enclosings will be difficult to pare down. The author believes that a modification of the constructions presented will fill the remaining "holes". The difficulty arises in the scale on which this must be done, ranging over four parameters while manipulating multiple construction techniques will not be a simple endeavor.

In the case of the enclosings of the λ-fold 4-cycle systems, the only situation left unaddressed is when $u = 1$. A new approach may be necessary in this case, but it is the hope of the author that this will have a constructive proof similar to those presented in Chapter 3.

These enclosings naturally lead to the question of enclosing larger λ-fold cycle systems (k-cycle systems with $k \geq 5$). In particular, embeddings for partial cycle systems have been shown to exist [11, 16], and the author believes that at least the generalization of enclosing λ-fold even-cycle systems can be proved in a similar fashion as the enclosings of λ-fold 4-cycle systems presented in Chapter 3.

Another natural question is: Can a non-proper subsystem be enclosed in a larger $(\lambda + m)$-fold k-cycle system? That is, for what values of $\lambda, v, u,$ and m can the edges of $(\lambda + m)K_{v+u}$ with those in a particular copy of λK_v removed be partitioned into 4-cycles. In other words, does there exist a k-cycle system of $(\lambda + m)K_{v+u} \setminus \lambda K_v$? Notice that if there exists a 4-cycle system of λK_v, then this question is addressed in this dissertation. This leads to the following conjecture:

Conjecture 4.1. *Let $u \geq 1$. There exists a 4-cycle system of $(\lambda+m)K_{v+u} \setminus \lambda K_v$ iff*

(*a*) $(v+u-1)(\lambda+m) \equiv 0$ *(mod 2), and*

(*b*) $u(u-1)(\lambda+m)/2 + mv(v-1)/2 + vu(\lambda+m) \equiv 0$ *(mod 4).*

(*c*) $u(\lambda+m) + m(v-1) \equiv 0$ *(mod 2)*

Proof (of Necessary Conditions)

Conditions (a) and (b) follow from Theorem 3.1. The graph contains vertices of degree $u(\lambda+m)+m(v-1)$. Since each degree must be even, the necessity of condition (c) follows.

□

The author believes that many of the constructions presented can be reused for the question at hand and that it is mostly an exercise in narrowing down the admissible parameters.

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