Discrete sets, free sequences and cardinal properties of topological spaces

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Discrete sets, free sequences and cardinal properties of topological spaces

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A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
August 10, 2009

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Vita

Santi Spadaro, son of Agatino Natale Spadaro and Maria Luisa Ceccio, was born on June 18, 1981, in Messina, Italy, on the northeastern tip of Sicily, which according to somebody is the actual birthplace of The Bard. After two years as a Physics major in Messina he moved to the University of Catania, where he graduated in Mathematics with highest honors in April of 2005. He entered the Graduate Program at Auburn University in August of 2006. Besides Mathematics, he enjoys Music, Literature and Cold Weather.

Dissertation Abstract

Discrete sets, free sequences and cardinal properties of topological spaces

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Doctor of Philosophy, August 10, 2009
(Laurea, University of Catania, Catania, Italy, 2005)

55 Typed Pages

Directed by Gary Gruenhage

We study the influence of discrete sets and free sequences on cardinal properties of topological spaces. We focus mainly on the minimum number of discrete sets needed to cover a space X (denoted by $dis(X)$) and on reflection of cardinality by discrete sets, free sequences and their closures. In particular, we offer several classes of spaces such that the minimum number of discrete sets required to cover them is always bounded below by the dispersion character (i.e., minimum cardinality of a non-empty open set). Two of them are Baire generalized metric spaces, and the rest are classes of compacta. These latter classes offer several partial positive answers to a question of Juhász and Szentmiklóssy. In some cases we can weaken compactness to the Baire property plus some other good property. However, we construct a Baire hereditarily paracompact linearly ordered topological space such that the gap between $dis(X)$ and the dispersion character can be made arbitrarily big. We show that our results about generalized metric spaces are sharp by constructing examples of good Baire generalized metric spaces whose dispersion character exceeds the minimum number of discrete sets required to cover them. With regard to discrete reflection of cardinality we offer a series of improvements to results of Alan Dow and Ofelia Alas. We introduce a rather weak cardinal function, the breadth, defined as the supremum of cardinalities of closures of free sequences in a space, and prove some instances where it

manages to reflect cardinality. We finish with a common generalization of Arhangel'skii Theorem and De Groot's inequality and its increasing chain version.

Acknowledgments

First of all I'd like to thank my advisor, Dr. Gary Gruenhage, for his great patience and his constant encouragement and support. Thanks are also due to the committee members: Dr. Stewart Baldwin, for teaching two interesting courses in Set Theory and providing helpful feedback on preliminary versions of the results from this dissertation and Dr. Michel Smith, for his useful comments at the seminar, his wise advice regarding teaching matters, his patience and availability. I would also like to thank Dr. Georg Hetzer, whom I had the pleasure of having both as a teacher and former committee member, but was eventually replaced due to schedule conflicts. Last but not least, I would like to thank Carolyn, Gwen, Lori and Maritha (and Debby): the friendliest and most efficient secretaries I have ever met.

Style manual or journal used Journal of Approximation Theory (together with the style known as "auphd"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `auphd.sty`.

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Chapter 1

Introduction

The world is discrete. It is made up of quanta, quarks, atoms, elements, separate entities. It should come as no surprise then, that discrete sets play an important role even in an eminently continuous area of mathematics like Topology. In Topology a set is called discrete if each of its points can be separated from the others by an open set. This definition matches with the meaning of the word *discrete* in any other discipline. The *spread* of a space is the supremum of the cardinalities of its discrete sets. So, if a space has countable spread, each of its discrete sets is at most countable. A classical result of De Groot says that the cardinality of a space is bounded above by the power of the power of its spread. So a space of countable spread has cardinality at most 2^{2^ω}. This is only one of many results showing the great influence that discrete sets have on cardinal properties of topological spaces.

This dissertation deals with two very natural problems involving discrete sets. How many discrete sets are needed to cover a *good* space? When do closures of discrete sets *reflect* the cardinality of a space? The latter question goes back to an old paper of Arhangel'skii, where the author asks: is it true that in every compact space there is a discrete set whose closure has the cardinality of the whole space? Although the answer is known to be no, at least consistently, reflection properties of discrete sets have became an active area of research in Topology, as shown by the papers [1], [2], [6] and [21]. The *depth* of X (indicated with $g(X)$) is the supremum of cardinalities of closures of discrete sets in X. This is usually bigger than the spread, and closer to the cardinality: for example, the unit interval has countable spread but depth continuum. Alan Dow [5] proved that if X is a compact space of countable tightness where $g(X) \leq \mathfrak{c}$ then $|X| \leq \mathfrak{c}$. Ofelia Alas [1] proved that if X is a compact space where discrete sets have all size less than continuum and $g(X) \leq \mathfrak{c}$, then, under Martin's Axiom, $|X| \leq \mathfrak{c}$. These two results are partial answers to a special case of Arhangel'skii's Problem that was studied by Alas, Tkachuk and Wilson in [2]. In the final chapter of this dissertation we prove a common generalization of Alas and Dow's theorems that removes compactness from their assumption. Compactness was essential in both Alas and Dow's result. This suggests that the role of compactness in discrete reflection might be less crucial than previously thought. Moreover, we give a series of other improvements to their results, and from there ask a couple of natural questions. It looks like the construction of counterexamples to these questions would require completely new methods than those used for Arhangel'skii's original problem.

The cardinal function $dis(X)$ was introduced by Juhász and Van Mill in the paper [19], as the minimum number of discrete sets required to cover a space X. The authors were especially interested in its behavior on compact spaces. In particular, they asked: is it true that $dis(X) \geq \mathfrak{c}$ for every compact space X without isolated points. This is true for the unit interval, since it has countable spread and size continuum, so their question appears like a very natural and fundamental one. Juhász and Van Mill proved it to be true for compact hereditarily normal spaces and had some other partial answers that showed a counterexample to their question must have been a very weird compact space. In fact, the answer to their question was positive, as proved by Gruenhage in [14]. By exploiting a Lemma of Gruenhage but using a completely different approach, Juhász and Szentmiklóssy [20] proved that in every compact space X where every point has character at least κ, $dis(X) \geq 2^\kappa$. This generalizes both Gruenhage's result and the classical Čech-Pospišil Theorem.

One of the main questions in [20] is the following. When dealing with a space X, call a cardinal *small* if it is less than the cardinality of every non-empty open set in X. Is it true that no compact space can be covered by a small number of discrete sets? A positive answer would generalize their theorem, since in a compact space where every point has character at least κ, every open set has cardinality at least 2^κ. However, a solution to their question would seem to require completely different methods than those used to study $dis(X)$ so far, since there is no direct reference to character, so one cannot lean on Čech-Pospišil-like tecniques. Here we provide several partial positive answers to their question, that suggest a possible counterexample would be a rather pathological compact space. Moreover we obtain some results outside of the compact realm: for example we determine the least number of discrete sets required to cover a Σ-product, or in some cases we can replace compactness with a much weaker property, like the Baire property. Finally we give a systematic study of the Juhász and Szentmiklóssy's problem on two classes of Baire generalized metric spaces, inspired by our new result that no Baire metric space can be covered by a small number of discrete sets. Our study leads to two examples of very good Baire spaces that are very close to metric and yet can be covered by a small number of discrete sets. Also, we show a family of nice looking Baire linearly ordered topological spaces that can be covered by a *really small* number of discrete sets (see chapter 3 for a precise definition of really small). This shows that compactness cannot be relaxed to the Baire property in our result about compact LOTS.

In the final chapter we prove a common generalization of two basic theorems in the theory of cardinal functions -Arhangel'skii Theorem and De Groot's inequality- that involves the size of free sequences. The theorem has been proved by Juhász independently in 2003, but this is the first time it appears in print. Moreover, using and streamlining some ideas of Juhász from [18], we prove the increasing version of our theorem.

Chapter 2

Notation and background

The cardinality of a countable set is indicated with ω or $\aleph_0$. The Greek letter ω also stands for the set of all non-negative integers. The symbol $\aleph_1$ stands for the first uncountable cardinal, and $\mathfrak{c}$ stands for the cardinality of the continuum. For a cardinal κ, the symbol κ^+ indicates the least cardinal bigger than κ. If S is a set then $\mathcal{P}(S)$ stands for the power set of S. If S is a set and κ is a cardinal we set $[S]^\kappa = \{A \subset S : |A| = \kappa\}$ and $[S]^{\leq\kappa} = \{A \subset S : |A| \leq \kappa\}$. The *continuum hypothesis*, or CH, is the statement that $\mathfrak{c} = \aleph_1$. The *generalized continuum hypothesis*, or GCH, is the statement that $2^\kappa = \kappa^+$ for every cardinal κ.

A space is called *crowded* if it has no isolated points. The letter I will denote the closed unit interval. A G_κ*-set* in a space X is an intersection of κ many open sets. G_ω-sets are more commonly known as G_δ-sets. We will need several classical cardinal functions, whose definitions are recalled below.

Definition 2.1. *The* <u>*character of the point x in X*</u> *($\chi(x, X)$) is the least cardinality of a local base at x. The* <u>*character of the space X*</u> *is defined as $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. A space of countable character is also called* <u>*first-countable*</u>*.*

Definition 2.2. *The* <u>*spread of X*</u> *is defined as $s(X) = \sup\{|D| : D \subset X$ and D is discrete $\}$. We also define a related cardinal function as $\hat{s}(X) = \min\{\kappa :$ if $A \subset X$ and $|A| = \kappa$ then A is not discrete $\}$.*

Definition 2.3. *The* <u>*tightness of X*</u> *($t(X)$) is defined as the least cardinal number κ such that for every $A \subset X$ and $x \in \overline{A} \setminus A$ there is $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$.*

A set $\{x_\alpha : \alpha < \kappa\}$ is called a *free sequence* if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset$ for every $\beta < \kappa$. Every free sequence is a discrete set.

Definition 2.4. *We set* $F(X) = \sup\{|F| : F \subset X$ *and* F *is a free sequence* $\}$*. Also* $\hat{F}(X) = \min\{\kappa :$ *if* $A \subset X$ *and* $|A| = \kappa$ *then* A *is not a free sequence* $\}$*.*

The above cardinal function allows an elegant characterization of the tightness of a compact space.

Theorem 2.5. *(Arhangel'skii) Let* X *be a compact Hausdorff space. Then* $t(X) = F(X)$*.*

Proof. See [18], 3.12. □

A *cellular family* is a family of pairwise disjoint open sets.

Definition 2.6. *The cellularity of* X *is defined as* $c(X) = \sup\{|\mathcal{C}| : \mathcal{C}$ *is a cellular family of open subsets of* $X\}$*.*

Definition 2.7. *The weight of* X $(w(X))$ *is the least cardinality of a base for* X*.*

Since in a discrete set any point can be separated from the others by a basis element, it is clear that $s(X) \leq w(X)$.

Definition 2.8. *The dispersion character of* X *is defined as the least cardinality of a non-empty open set in* X*.*

Definition 2.9. *A family of open sets* $\mathcal{U}$ *is said to be a local* π*-base at the point* $x \in X$ *if for every open set* $V \subset X$ *such that* $x \in V$ *there is a set* $U \in \mathcal{U}$ *such that* $U \subset V$*. The* π*-character of the point* x *in* X $(\pi\chi(x, X))$ *is the least cardinality of a local* π*-base at* x*. The* π*-character of the space* X *is defined as* $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$*.*

The π-character plays a fundamental role in Set-theoretic Topology, since it characterizes the compact spaces that can be mapped onto Tychonoff cubes.

Theorem 2.10. *(Shapirovskii's Theorem on maps onto Tychonoff Cubes) Let* X *be a compact space. Then* X *can be mapped onto* I^κ *if and only if there is a closed set* $F \subset X$ *such that* $\pi\chi(p, F) \geq \kappa$ *for every* $p \in F$*.*

Definition 2.11. *A set $G \subset X$ is called* <u>regular open</u> *if $Int(\overline{G}) = G$. The number of regular open sets in X is indicated with $\rho(X)$.*

Regular open sets generate the topology of a regular space. A simple, yet very effective lower bound on the number of regular open sets is due to F. Burton Jones.

Lemma 2.12. *(Jones' Lemma) If X is a hereditarily normal space and $D \subset X$ is a discrete set then $\rho(X) \geq 2^{|D|}$.*

Proof. See [18], 3.1 b) □

One of the nicest consequences of Shapirovskii's Theorem on maps onto Tychonoff cubes is the following upper bound on the number of regular open sets.

Theorem 2.13. *(Shapirovskii) Let X be a compact hereditarily normal space. Then $\rho(X) \leq 2^{c(X)}$.*

Theorem 2.14. *(Čech-Pospišil) Let X be a compact space such that $\chi(x, X) \geq \kappa$ for every $x \in X$ then $|X| \geq 2^{\kappa}$.*

Proof. see [18], 3.16. □

Definition 2.15. *A map between topological spaces is called* <u>perfect</u> *if it is closed and has compact point inverses.*

Definition 2.16. *A space is called* <u>Lindelöf</u> *if every open cover has a countable subcover.*

Recall that a *refinement* $\mathcal{V}$ for a cover $\mathcal{U}$ of a space X is a family of subsets of X such that for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subset V$ and $\mathcal{U}$ still covers X. A family of subsets of a space X is called *point-countable* if every point of X is in at most countably many members of that family.

Definition 2.17. *A space X is called* <u>meta-Lindelöf</u> *if every open cover for X has a point-countable open refinement.*

Definition 2.18. *A space is called* <u>*collectionwise Hausdorff*</u> *if every closed discrete set expands to a disjoint family of open sets.*

Definition 2.19. *A space is called* <u>*Baire*</u> *if every intersection of countably many dense open sets is dense.*

Some of the most fruitful strengthenings of Baire involve topological games.

Definition 2.20. *Let X be a non-empty topological space. The* <u>*strong Choquet game*</u> *is defined as follows. Player I chooses an open set U_0 and a point $x_0 \in U_0$. Player II chooses an open set $V_0 \subset U_0$ such that $x_0 \in V_0$. Then player one chooses an open set $U_1 \subset V_0$ and a point $x_1 \in U_1$. Player II proceeds as before. Player II wins the game if $\bigcap_{n\in\omega} V_n \neq \emptyset$.*

Definition 2.21. *A space X is called* <u>*strong Choquet*</u> *if player II has a winning strategy in the strong Choquet game for X.*

We think that the meaning of *winning strategy* is rather intuitive. See [23], page 43, for a more precise definition.

Theorem 2.22. *Every strong Choquet space is Baire.*

Proof. See for example [23], Theorem 8.11. □

Definition 2.23. *Let (X, τ) be a space. Then the* <u>*dispersion character of X*</u> *is defined as* $\Delta(X) = \min\{|U| : U \in \tau \setminus \{\emptyset\}\}$.

Other more specialized notions and results will be recalled as the need arises.

Chapter 3

Covering Baire generalized metric and linearly ordered spaces by discrete sets

Our interest in the cardinal function $dis(X)$ was sparked by the discovery that $dis(X) \geq \Delta(X)$ was true for Baire metric spaces. This suggested it might be interesting to look at Juhász and Szentmiklóssy's question in the class of generalized metric spaces. Generalized metric spaces can be described as spaces that resemble metric spaces in some sense, yet can deviate from them a lot. For example, σ-spaces are a popular generalized metric class inspired by the Bing metrization theorem, that even contains spaces that fail to be first-countable. One such space is one of the main counterexamples in this dissertation, being a Baire σ-space for which $dis(X) < \Delta(X)$. The question of whether a first-countable example having all those features exists remains open in ZFC, while we do have a consistent first countable σ-space for which $dis(X) < \Delta(X)$. Such space is even a normal Moore space, and normal Moore spaces are known to be metric in some models of set theory. So it can be described as our strongest example, even if it relies on additional axioms. We also prove that $dis(X) \geq \Delta(X)$ is true for two Baire generalized metric classes satisfying a mild covering-type property.

The first class of compact spaces for which we found Juhász and Szentmiklóssy's conjecture to be true is that of compact linearly ordered spaces. This made us wonder whether compact could be replaced by Baire. We were able to construct a family of hereditarily paracompact Baire linearly ordered spaces for which $dis(X) < \Delta(X)$ and the gap between $dis(X)$ and $\Delta(X)$ can be made arbitrarily large.

3.1 Generalized metric spaces

Given a collection $\mathcal{G}$ of subsets of X, set $st(x, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : x \in G\}$ and $ord(x, \mathcal{G}) = |\{G \in \mathcal{G} : x \in G\}|$. Recall that a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers of X is said to be a *development* if $\{st(x, \mathcal{G}_n) : n \in \omega\}$ is a local base at x for every $x \in X$. A space is called *developable* if it admits a development. A regular developable space is called a *Moore space*.

We say that a set $A \subset X$ *expands to* a collection $\mathcal{C} \subset \mathcal{P}(X)$ if for every $x \in A$ there is $C \in \mathcal{C}$ such that $x \in C$.

Definition 3.1. *Let κ be a cardinal. We call a space* <u>*κ-expandable*</u> *if every closed discrete set expands to a collection of open sets $\mathcal{G}$ such that $ord(x, \mathcal{G}) \leq \kappa$ for every $x \in X$.*

The following theorem is new even for all complete metric spaces.

Theorem 3.2. *Let X be a Baire ω_1-expandable developable space. Then $dis(X) \geq \Delta(X)$.*

Proof. Fix a development $\{\mathcal{G}_n : n \in \omega\}$ for X and suppose by contradiction that $\tau = dis(X) < \Delta(X)$. Since the inequality $dis(X) \geq \omega_1$ is true for every crowded Baire space X we can assume that $\tau \geq \omega_1$. Set $X = \bigcup_{\alpha<\tau} D_\alpha$, where each D_α is discrete. Define $D_{\alpha,n} = \{x \in D_\alpha : st(x, \mathcal{G}_n) \cap D_\alpha = \{x\}\}$ and set $X_n = \bigcup_{\alpha\in\tau} D_{\alpha,n}$.

Claim: For every $x \in X_k$ there is a neighbourhood G of x such that $|G \cap X_k| \leq \tau$.

Proof of Claim. Let $G \in \mathcal{G}_k$ be such that $x \in G$. Then G hits each $D_{\alpha,k}$ in at most one point: indeed, if $y, z \in G \cap D_{\alpha,k}$ with $y \neq z$, we'd have both $st(y, \mathcal{G}_k) \cap D_{\alpha,k} = \{y\}$ and $z \in st(y, \mathcal{G}_k) \cap D_{\alpha,k}$, which is a contradiction. △

Now $X = \bigcup_{n\in\omega} X_n$, so, by the Baire property of X, there is $k \in \omega$ such that $U \subset \overline{X_k}$ for some non-empty open set U. By the claim we can assume that $|U \cap X_k| \leq \tau$. So $|U \cap (\overline{X_k} \setminus X_k) \cap D_{\alpha,j}| > \tau$ for some $\alpha < \tau$ and $j \in \omega$.

Notice that the set $D_{\alpha,j}$ is actually closed discrete: indeed suppose $y \notin D_{\alpha,j}$ were some limit point. Let $V \in \mathcal{G}_j$ be a neighbourhood of y and pick two points $z, w \in V \cap D_{\alpha,j}$. By definition of $D_{\alpha,j}$ we have $st(z, \mathcal{G}_j) \cap D_{\alpha,j} = \{z\}$. But $w \in V \subset st(z, \mathcal{G}_j)$, which leads to a contradiction.

Observe now that also $S := U \cap (\overline{X_k} \setminus X_k) \cap D_{\alpha,j}$ is closed discrete and hence we can expand it to a collection $\mathcal{U} = \{U_x : x \in S\}$ of open sets such that $ord(y, \mathcal{U}) \leq \omega_1$ for every $y \in X$. Set $V_x = U_x \cap st(x, \mathcal{G}_j) \cap U$ and observe that $V_x \neq V_y$ whenever $x \neq y$ and if we put $\mathcal{V} = \{V_x : x \in S\}$ then we also have that $ord(y, \mathcal{V}) \leq \omega_1$ for every $y \in X$. For every $x \in S$ pick $f(x) \in V_x \cap X_k$: the mapping f has domain of cardinality $> \tau$, range of cardinality $\leq \tau$ and fibers of cardinality $\leq \omega_1$, which is a contradiction. □

Corollary 3.3. *$dis(X) \geq \Delta(X)$, for every Baire collectionwise Hausdorff (or meta-Lindelöf) developable space X.*

Corollary 3.4. *$dis(X) \geq \Delta(X)$, for every Baire metric space X.*

Recall that a *network* is a collection $\mathcal{N}$ of subsets of a topological space such that for every open set $U \subset X$ and every $x \in U$ there is $N \in \mathcal{N}$ with $x \in N \subset U$. A *σ-space* is a space having a σ-discrete network.

Our next aim is proving that $dis(X) \geq \Delta(X)$ for every regular Baire ω_1-expandable σ-space. We could give a more direct proof, but we feel that the real explanation for that is the following probably folklore fact, a proof of which can be found in [4].

Lemma 3.5. *Every regular Baire σ-space has a dense metrizable G_δ-subspace.*

Call $dis^*(X)$ the least number of *closed discrete* sets required to cover X. Clearly $dis(X) \leq dis^*(X)$. In a σ-space, one can use a σ-discrete network to split every discrete set into a countable union of closed discrete sets. So the following lemma is clear.

Lemma 3.6. *If X is a crowded σ-space then $dis(X) = dis^*(X)$.*

The next lemma and its proof are essentially due to the anonymous referee of [28].

Lemma 3.7. *Let X be an ω_1-expandable crowded Baire space such that $dis^*(X) \leq \kappa$, and $A \subset X$ with $|A| \leq \kappa$. Then $|\overline{A}| \leq \kappa$.*

Proof. Since X is Baire crowded we can assume that $\kappa \geq \omega_1$. Let $X = \bigcup_{\alpha<\kappa} D_\alpha$, where each D_α is closed discrete. Let $B_\alpha = \overline{A} \cap D_\alpha$. Then B_α is closed discrete, so we may expand it to a family of open sets $\mathcal{U}_\alpha$ such that $ord(x, \mathcal{U}_\alpha) \leq \omega_1$ for every $x \in X$. Then $|\mathcal{U}_\alpha| = |B_\alpha|$ and for all $U \in \mathcal{U}_\alpha$, $U \cap A \neq \emptyset$. Fix some well-ordering of A and define a function $f : \mathcal{U}_\alpha \to A$ by:

$$f(U) = \min\{a \in A : a \in U\}.$$

We have that $|f^{-1}(a)| \leq \aleph_1$ for every $a \in A$, and therefore $|B_\alpha| = |\mathcal{U}_\alpha| \leq |A| \cdot \aleph_1 \leq \kappa$. Since $\overline{A} = \bigcup_{\alpha\in\kappa} B_\alpha$ it follows that $|\overline{A}| \leq \kappa$. □

The statement of the next theorem is due to the anonymous referee, and improves our original theorem where X was assumed to be paracompact.

Theorem 3.8. *Let X be a regular ω_1-expandable Baire σ-space. Then $dis(X) \geq \Delta(X)$.*

Proof. Fix some dense metrizable G_δ-subspace $M \subset X$ and suppose by contradiction that $dis^*(X) = dis(X) < \Delta(X)$. Then Lemma 3.7 implies that $\Delta(M) \geq \Delta(X)$ and, since M is Baire metric, by Corollary 3.4 we have $dis(X) \geq dis(M) \geq \Delta(M)$. So $dis(X) \geq \Delta(X)$, and we are done. □

Corollary 3.9. *For every paracompact Baire σ-space X (in particular, for every stratifiable Baire space), we have $dis(X) \geq \Delta(X)$.*

Notice that in the proofs of Theorems 3.2 and 3.8 all one needs is that X be $dis(X)$-expandable.

Also, while we didn't use any separation other than Hausdorff in Theorem 3.2, regularity seems to be essential in Theorem 3.8, since one needs a σ-discrete network consisting of closed sets to prove Lemma 3.5. This suggests the following question.

Question 3.10. *Is there a collectionwise Hausdorff or meta-Lindelöf (non regular) Baire σ-space X such that $dis(X) < \Delta(X)$?*

3.2 Good spaces with bad covers

We now offer two examples to show that ω_1-expandability is essential in Theorem 3.8. The first one is a modification of an example of Bailey and Gruenhage [3]. We will need the following combinatorial fact which slightly generalizes Lemma 9.23 of [17]. It must be well-known, but we include a proof anyway since we couldn't find a reference to it.

Lemma 3.11. *Let κ be any infinite cardinal. There is a family $\mathcal{A} \subset [\kappa]^{cf(\kappa)}$ of cardinality κ^+ such that $|A \cap B| < cf(\kappa)$ for every $A, B \in \mathcal{A}$.*

Proof. We begin by showing that there is a family $\mathcal{F}$ of functions from $cf(\kappa)$ to κ such that $|\mathcal{F}| = \kappa^+$ and $|\{\alpha \in cf(\kappa) : f(\alpha) = g(\alpha)\}| < cf(\kappa)$, for any $f, g \in \mathcal{F}$. Indeed, suppose we have constructed $\{f_\alpha : \alpha < \kappa\}$ with the stated property. Let $\kappa = \sup_{\alpha < cf(\kappa)} \kappa_\alpha$. Define $f : cf(\kappa) \to \kappa$ in such a way that $f(\tau) \neq f_\alpha(\tau)$, for every $\alpha < \kappa_\tau$ and $\tau \in cf(\kappa)$. Fix $\alpha \in \kappa$: if $\tau < cf(\kappa)$ is such that $f(\tau) = f_\alpha(\tau)$ we must have $\kappa_\tau \leq \alpha < \kappa$. Hence $|\{\tau \in cf(\kappa) : f(\tau) = f_\alpha(\tau)\}| < cf(\kappa)$.

Now for $\mathcal{A}$ we can take (on $cf(\kappa) \times \kappa$) the family of graphs of functions in $\mathcal{F}$. □

Example 3.12. *(ZFC) A regular Baire σ-space P for which $dis(P) < \Delta(P)$.*

Proof. Fix an almost disjoint family $\mathcal{A} \subset [\mathfrak{c}]^{cf(\mathfrak{c})}$ such that $|\mathcal{A}| = \mathfrak{c}^+$. For every partial function $\sigma \in \mathfrak{c}^{<\omega}$ such that $dom(\sigma) = k$ for some $k \in \omega$ let $\mathcal{L}_\sigma = \{f_{\sigma,A} : A \in \mathcal{A}\}$ where $f_{\sigma,A} : cf(\mathfrak{c}) \to \mathfrak{c}^{<\omega}$ is defined as follows: $dom(f_{\sigma,A}(\alpha)) = k+1$, $f_{\sigma,A}(\alpha) \restriction k = \sigma$ for every $\alpha \in cf(\mathfrak{c})$ and $\{f_{\sigma,A}(\alpha)(k) : \alpha \in cf(\mathfrak{c})\}$ is a faithful enumeration of A.

When $f \in \mathcal{L}_\sigma$ we will refer to $\rho_f = \sigma$ as the *root* of f, and set $k_f = dom(\sigma)$.

Let now $L = \bigcup_{\sigma \in \mathfrak{c}^{<\omega}} \mathcal{L}_\sigma$ and $B = \mathfrak{c}^\omega$. We are going to define a topology on $P = B \cup L$ that induces on B its natural topology. For every $\sigma \in \mathfrak{c}^{<\omega}$, let $[\sigma] = \{g \in B : g \supset \sigma\}$ and

$$B(\sigma) = [\sigma] \cup \{f \in L : \rho_f \supseteq \sigma\}.$$

Let $\{A_n : n \in \omega\}$ be a partition of $\mathfrak{c}$ into sets of cardinality $\mathfrak{c}$.

For $f \in L$, $\delta \in cf(\mathfrak{c})$ and $k \in \omega$ let

$$B_{\delta,k}(f) = \{f\} \cup \bigcup_{\gamma>\delta} \left\{ B(f(\gamma)) : f(\gamma)(k_f) \in \bigcup_{n>k} A_n \right\}.$$

The set $\mathcal{B} = \{B(\sigma), B_{\delta,k}(f) : \sigma \in \mathfrak{c}^{<\omega}, \delta \in cf(\mathfrak{c}), k \in \omega\}$ is a base for a topology on P, as items (2) and (3) in the following list of claims show.

1. For $\sigma_1, \sigma_2 \in \mathfrak{c}^{<\omega}$, $B(\sigma_1) \cap B(\sigma_2) = \emptyset$ if and only if σ_1 and σ_2 are incompatible.

2. Suppose $B(\sigma) \cap B_{\delta,k}(f) \neq \emptyset$. Then $\sigma \subseteq \rho_f$ or $\rho_f \subseteq \sigma$. If $\sigma \subseteq \rho_f$ then $B(\sigma) \cap B_{\delta,k}(f) = B_{\delta,k}(f)$. If $\sigma \supsetneq \rho_f$, then the intersection is $B(\sigma)$.

3. If $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g \subsetneq \rho_f$ then the intersection is either $B_{\delta,j}(f)$ or a set of the form $B(\sigma)$, for some $\sigma \in \{f(\gamma), g(\gamma') : \gamma > \delta, \gamma' > \delta'\}$.

4. If $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g = \rho_f$ then the intersection is a union of less than $cf(\mathfrak{c})$ sets of the form $B(\sigma)$ where $\sigma \in ran(f) \cap ran(g)$.

Proof of items (1)-(4). Item (1) is easy. For item (2), observe that $B_{\delta,k}(f) \subseteq B(\rho_f)$, so $B(\rho_f) \cap B(\sigma) \neq \emptyset$ which implies that ρ_f and σ are compatible. If $\sigma \subseteq \rho_f$ then for each $\gamma > \delta$ we have $\sigma \subseteq f(\gamma)$ and $f \in B(\sigma)$, so $B_{\delta,k}(f) \subseteq B(\sigma)$.

If $\sigma \supsetneq \rho_f$ then let $\gamma > \delta$ be the unique ordinal such that $B(\sigma) \cap B(f(\gamma)) \neq \emptyset$. Since σ and $f(\gamma)$ are compatible we must have $f(\gamma) \subset \sigma$, from which $B(\sigma) \subset B(f(\gamma))$ follows, and hence the claim.

To prove item (3) observe that if $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g \subsetneq \rho_f$ then $g \notin B_{\delta,j}(f)$ and, as the range of f consists of pairwise incompatible elements we have that $[g(\tau)] \cap [\rho_f] \neq \emptyset$ for at most one $\tau \in cf(\mathfrak{c})$. Therefore, $B_{\delta,j}(f) \cap B_{\delta',k}(g) = B(g(\tau)) \cap B_{\delta,j}(f)$, and the rest follows from item (2).

Item (4) follows from *almost-disjointness* of the ranges. □

Claim 1: The base $\mathcal{B}$ consists of clopen sets.

Proof of Claim 1. To see that $B_{\delta,j}(f)$ is closed pick $g \in L \setminus B_{\delta,j}(f)$ and let γ be large enough so that $f \notin B_{\gamma,j}(g)$. Suppose that $B_{\delta,j}(f) \cap B_{\gamma,j}(g) \neq \emptyset$. Then there are $\alpha > \delta$ and $\beta > \gamma$

such that $f(\alpha)$ and $g(\beta)$ are compatible. Now we must have $\rho_g = \rho_f$ or otherwise we would have either $\rho_f \supset g(\beta)$ and hence $f \in B_{\gamma,j}(g)$, or $\rho_g \supset f(\alpha)$, which would imply $g \in B_{\delta,j}(f)$. So, by item (4) we have $B_{\delta,j}(f) \cap B_{\gamma,j}(g) = \bigcup_{\tau \in C} B(g(\tau))$ where $|C| < cf(\mathfrak{c})$ and hence, if we let $\theta > \sup(C)$, then $B_{\theta,j}(g) \cap B_{\delta,j}(f) = \emptyset$.

Now, let $p \in B \setminus B_{\delta,j}(f)$ and $i = k_f + 2$. We claim that $B(p \restriction i) \cap B_{\delta,j}(f) = \emptyset$. Indeed, if that were not the case then $f(\gamma)$ and $p \restriction i$ would be compatible, for some γ. So $f(\gamma) \subset p \restriction i \subset p$, which implies $p \in B_{\delta,j}(f)$, contradicting the choice of p.

To see that $B(\sigma)$ is clopen, observe that B is dense in P and the subspace base is clopen, so we can restrict our attention to limit points of $B(\sigma)$ in L. Suppose that $f \in L \setminus B(\sigma)$ is some limit point, then, for all $\delta \in cf(\mathfrak{c})$ and all $j \in \omega$ we have $B_{\delta,j}(f) \cap B(\sigma) \neq \emptyset$. So ρ_f and σ are compatible; moreover $\rho_f \subsetneq \sigma$ or otherwise $f \in B(\sigma)$. Now there is at most one δ' such that $f(\delta')$ and σ are compatible, whence the absurd statement $B_{\delta'+1,0}(f) \cap B(\sigma) = \emptyset$.

$\triangle$

Claim 2: P is a σ-space.

Proof of Claim 2. For each $\sigma \in \mathfrak{c}^{<\omega}$ let $h(\sigma) \in \omega^{<\omega}$ be defined by $\sigma(i) \in A_j$ iff $h(\sigma)(i) = j$. For every $s \in \omega^{<\omega}$ put $\mathcal{B}_s = \{B(\sigma) : h(\sigma) = s\}$. We claim that $\mathcal{B}_s$ is a discrete collection of open sets. Notice that the elements of $\mathcal{B}_s$ are all disjoint. Now if $x \in B \setminus \bigcup \mathcal{B}_s$, let $j = dom(s)$; then either $x \restriction (j+1)$ extends (at most) one σ such that $h(\sigma) = s$ or $x \restriction (j+1)$ is incompatible with every such σ. So $B(x \restriction (j+1))$ will hit at most one element of $\mathcal{B}_s$. If $f \in L$ then let $l = \max(ran(s))$: we claim that $B_{0,l}(f)$ hits at most one element of $\mathcal{B}_s$. Indeed, for fixed α such that $f(\alpha)(k_f) \in \bigcup_{n>l} A_n$ either $f(\alpha)$ is incompatible with every σ such that $h(\sigma) = s$ or there is exactly one such σ which is compatible with $f(\alpha)$. In the latter case we can't have $\sigma \supset \rho_f$ because $f(\alpha)(k_f) \notin ran(s)$, hence we have $\sigma \subset \rho_f$, which implies $B_{0,l}(f) \subset B(\sigma)$.

Now we claim that L is a σ-closed discrete set. Indeed, for every $s \in \omega^{<\omega}$, set $L_s = \{f \in L : h(\rho_f) = s\}$. If $g \in L_s$ then every fundamental neighbourhood of g hits L_s in the single point g. If $g \notin L_s$ then either ρ_g is incompatible with every ρ_f such that $f \in L_s$, in which case every fundamental neighbourhood of g misses L_s, or there is $f \in L_s$ such that ρ_g and ρ_f are compatible. If $\rho_g \subsetneq \rho_f$ then let $l = s(k_g)$: we have $B_{0,l}(g) \cap L_s = \emptyset$. If $\rho_f \subset \rho_g$, then the root of every function of L which is in a fundamental neighbourhood of g has domain strictly larger than $dom(s)$ and hence every fundamental neighbourhood of g misses L_s. △

Observe now that P is Baire, because $B \subset P$ is a dense Baire subset. Also, $dis(P) = \mathfrak{c} < \mathfrak{c}^+ = \Delta(P)$ □

One of the properties of Bailey and Gruenhage's example that was lost in the modification is first-countability. This suggests the following question.

Question 3.13. *Is there in ZFC a first-countable regular σ-space X for which $dis(X) < \Delta(X)$?*

The reason why we insist on a ZFC example is that we already have a consistent answer to the previous question. In fact, the space we are now going to exhibit is first-countable, normal and shows that ω_1-expandability cannot be weakened to ω_2-expandability in Theorem 3.2. Our original motivation for constructing this example was showing that paracompactness could not be weakened to normality in Corollary 3.9.

Recall that a *Q-set* is an uncountable subset of a Polish space whose every subset is a relative F_σ, and a *Luzin set* is an uncountable subset of a Polish space P which meets every first category set of P in a countable set. The existence of Q-sets and Luzin sets in the reals is known to be independent of ZFC (see, for example, [25]). Fleissner and Miller [8] constructed a model of ZFC where there are a Q-set of the reals of cardinality $\aleph_2$ and a Luzin set of the reals of cardinality $\aleph_1$.

Lemma 3.14. *Let C be some Polish space having a base $\mathcal{B} = \{B_n : n \in \omega\}$ such that B_n is homeomorphic to C for every $n \in \omega$. Given a Q-set of cardinality $\aleph_2$ in C, there is one which is dense and has dispersion character $\aleph_2$. Given a Luzin set in C, there is one which is locally uncountable and dense.*

Proof. Let X be a Q-set in C. Let $\mathcal{B}' = \{B \in \mathcal{B} : |B \cap X| < \aleph_2\}$. Then $Y = X \setminus \bigcup \mathcal{B}'$ is a Q-set such that $\Delta(Y) = \aleph_2$. Set $n_0 = 0$ and let Z_0 be a homeomorphic copy of Y inside B_{n_0}. Set $Z = Z_0$ and let n_1 be the least integer such that $B_{n_1} \cap Z = \emptyset$: clearly $n_1 > n_0$. Now let $Z_1 \subset B_{n_1}$ be a homeomorphic copy of Y and set $Z = Z_0 \cup Z_1$. Now suppose you have constructed a Q-set Z such that $Z \cap B_i \neq 0$ for every $1 \leq i \leq n_{k-1}$ and let n_k be the least integer such that $Z \cap B_{n_k} = \emptyset$; let $Z_k \subset B_{n_k}$ be a homeomorphic copy of Y into B_{n_k}. At the end of the induction let $Z = \bigcup_{n \in \omega} Z_n$, then Z is a Q-set with the stated properties. The second statement is proved in a similar way. □

Example 3.15. *A normal Baire Moore space X for which $dis(X) < \Delta(X)$.*

Proof. Take a model of ZFC where there are a Luzin set $L' \subset \mathbb{R}$ and a Q-set $Z \subset \mathbb{R}$ with the properties stated in Lemma 3.14. Let f be any homeomorphism from the irrationals onto their square. Then $L = f(L' \setminus \mathbb{Q})$ is a Luzin subset of $(\mathbb{R} \setminus \mathbb{Q})^2$, and by Lemma 3.14 we can assume that it is locally uncountable and dense. Let $\mathbb{Q} = \{q_n : n \in \omega\}$ be an enumeration and set $Z_n = Z \times \{q_n\}$. Set $T = \bigcup_{n \in \omega} Z_n$ and define a topology on $X = L \cup T$ as follows: points of L have neighbourhoods just as in the Euclidean topology on the plane, while a neighbourhood of a point of $x \in Z_n$ is a disk tangent at x to Z_n, and lying in the upper half plane relative to that line. To see that X is Baire, observe that if $X = \bigcup_{n \in \omega} N_n$, where N_n is nowhere dense in X, then $L = \bigcup_{n \in \omega} L \cap N_n$. From the fact that L is dense in X it follows that $L \cap N_n$ is nowhere dense in L. From the fact that L is dense in the plane it follows that $L \cap N_n$ is nowhere dense in the plane. Since $L \cap N_n \subset L$ we have that $L \cap N_n$ is countable. So the uncountable set L would be covered by countably many countable sets, which is a contradiction.

Now observe that $\Delta(X) = \aleph_2 > \aleph_1 = dis(X)$.

To prove that X is normal let H and K be disjoint closed sets. It will be enough to show that H has a countable open cover, such that the closure of every member of it misses K (see Lemma 1.1.15 of [7]). Fix $n \in \omega$. We have $H \cap Z_n = \bigcup_{j \in \omega} H_j$, where H_j is closed in the Euclidean topology on Z_n for every $j \in \omega$. Fix $j \in \omega$. For each $x \in H_j$ let $D(x, r_x)$

be a disk tangent to Z_n at x such that $D(x, r_x) \cap K = \emptyset$ and $r_x = \frac{1}{k}$ for some $k \in \omega$. Let $U = \bigcup_{x \in H_j} D(x, r_x)$. First of all, we claim that no point of $K \cap Z_n$ is in $\overline{U}$: indeed if $x \in K \cap Z_n$ then let I_x be an interval containing x and missing H_j, then the closest that a point of H_j can come to x is one of the endpoints of I_x so there is room enough to separate x from U by a tangent disk.

Now $U = \bigcup_{n \in \omega} U_n$, where $U_n = \bigcup \{D(x, r_x) : r_x = \frac{1}{n}\}$. Let $V_n = \bigcup \{D(x, \frac{r_x}{2}) : r_x = \frac{1}{n}\}$. We claim that $\overline{V}_n \cap K \setminus Z_n = \emptyset$: indeed, if some point $x \in K \setminus Z_n$ were limit for V_n then we would have a sequence of disks of radius $\frac{1}{2n}$ clustering to it. But then $x \in U_n$, which contradicts $U \cap K = \emptyset$.

To separate points of $H \setminus T$ from K just choose for each such point an open set whose closure misses K and use second countabiliy of L. That shows how to define the required countable open cover of H.

Finally, a development for X is provided by $\mathcal{G}_n = \{D(x, n) : x \in X\}$ where $D(x, n) = B(x, \frac{1}{n}) \setminus \bigcup_{i<n} Z_i$ if $x \in L$, while if $x \notin L$, $D(x, n)$ is a tangent disk of radius less than $\frac{1}{n}$ which misses $\bigcup \{Z_i : i < n \text{ and } x \notin Z_i\}$. □

The cardinal $\aleph_2$ can be replaced by any cardinal not greater than $\mathfrak{c}$, under proper set theoretic assumptions (see [8]). So the previous example shows that the gap between $dis(X)$ and $\Delta(X)$ for normal Baire Moore spaces can be as big as the gap between the first uncountable cardinal and the continuum.

Since normal Moore spaces are, consistently, metrizable, there is no chance of getting in ZFC a space with all the properties of Example 3.15. Nevertheless, the following question remains open.

Question 3.16. *Is there in ZFC a normal Baire σ-space X for which $dis(X) < \Delta(X)$?*

Using a Q-set on a tangent disk space to get normality is an old trick (see for example [30]). Also, to get a regular Baire Moore space X for which $dis(X) < \Delta(X)$ it actually suffices to assume the negation of CH along with the existence of a Luzin set.

A potential way of weakening the set theoretic assumption in Example 3.15 would be to replace *Luzin set* with *Baire subset of cardinality* $\aleph_1$, but even such an object would be inconsistent with MA+ $\neg$ CH, while the presence of CH would make the whole construction worthless, so we have no clue even about the following.

Question 3.17. *Is there, at least under MA+ $\neg$ CH or under CH, a normal Baire σ-space X for which $dis(X) < \Delta(X)$?*

Also, notice that no regular Baire σ-space X for which $dis(X) < \Delta(X)$ can be separable under CH. That is because any regular separable space with points G_δ has cardinality $\leq \mathfrak{c}$ (fix any dense countable set D, then the map taking any regular open set to its intersection with D is 1-to-1. So there are no more than $\mathfrak{c}$ many regular open sets in the space, but every point in a regular space with G_δ-points is the intersection of countably many regular open sets). Thus $dis(X) = \aleph_1 \geq \Delta(X)$ if CH holds.

3.3 Linearly ordered spaces

Recall that a space is called a *GO space* if it embeds in a LOTS. We denote by $m(X)$ the minimum number of metrizable spaces needed to cover X. The following result is due to Ismail and Szymanski.

Lemma 3.18. *[16] Let X be a locally compact Lindelöf GO space. Then $w(X) \leq \omega \cdot m(X)$*

Theorem 3.19. *Let X be a locally compact Lindelöf GO space. Then $dis(X) = |X|$.*

Proof. Suppose by contradiction that there exists $\lambda < |X|$ such that $X = \bigcup\{D_\alpha : \alpha \in \lambda\}$ where each D_α is discrete. Then $|D_\alpha| \leq w(X) \leq \omega \cdot m(X) \leq \lambda$, for every $\alpha \in \lambda$. So $|X| \leq \sup\{|D_\alpha| : \alpha \in \lambda\} \cdot \lambda \leq \lambda < |X|$. □

Corollary 3.20. *Let X be a locally compact paracompact GO space. Then $dis(X) \geq \Delta(X)$.*

Proof. Every locally compact paracompact space contains a non-empty open set with the Lindelöf property (see [7], 5.1.27). Fix one such $U \subset X$. Then U is a locally compact Lindelöf GO space and hence $dis(X) \geq dis(U) \geq |U| \geq \Delta(X)$. □

In the previous corollary we cannot weaken locally compact paracompact to Baire paracompact, as the following example shows. Recall that a space is called *non-archimedean* if it has a base such that any two elements are either disjoint or one is contained in the other. Every non-archimedean space has a base which is a tree under reverse inclusion (see [26]), and from this it is easy to see that it is (hereditarily) paracompact.

Example 3.21. *There is a Baire non-archimedean (and hence hereditarily paracompact) LOTS X such that $dis(X) < \Delta(X)$.*

Proof. Let κ and λ be infinite cardinals such that $cf(\kappa) \leq \lambda$ but $\lambda < \kappa$. Let $\mathbb{W} = \{-1\} \cup \kappa$. Define an order on $\mathbb{W}$ by declaring -1 to be less than every ordinal. Let $X = \{f \in \mathbb{W}^{\lambda^+} : supp(f) < \lambda^+\}$, where $supp(f) = \min\{\gamma < \lambda^+ : f(\alpha) = 0 \text{ for every } \alpha \geq \gamma\}$. Now take the topology induced on X by the lexicographic order.

Claim 1: X is a strong Choquet space (and hence Baire).

Proof of Claim 1. We are going to describe a winning strategy for player II in the strong Choquet game. In his first move player I chooses any open set B_1 and a point $f_1 \in B_1$. Player II then chooses points $a_1, b_1 \in X$ such that $f_1 \in (a_1, b_1) \subset B_1$. Let now $\alpha_1 = \max\{supp(f_1), supp(a_1), supp(b_1)\}$ and $\overline{f_{\alpha_1}} = (f_1(\gamma) : 0 \leq \gamma < \alpha_1)$. Define $f_1^- = \overline{f_{\alpha_1}}^\frown(-1, 0, \ldots 0)$ and $f_1^+ = \overline{f_{\alpha_1}}^\frown(1, 0, \ldots, 0)$.

Clearly $a_1 < f_1^- < f_1 < f_1^+ < b_1$. Now in her first move player II chooses the open set $A_1 = (f_1^-, f_1^+)$.

Player I responds by choosing any open set $B_2 \subset A_1$ and a point $f_2 \in B_2$. Player II proceeds as before. Notice that f_{n+1} thus constructed agrees with f_n up to α_n and that the point $h = \left(\bigcup \overline{f_{\alpha_n}}\right)^\frown (0, 0, \ldots, 0)$ is in $\bigcap_{n\geq 1} A_n$. So II has a winning strategy. △

Claim 2: X is the union of λ^+ many discrete sets.

Proof of Claim 2. For every $\alpha \in \lambda^+$, let $D_\alpha = \{f \in X : supp(f) = \alpha\}$. Then $X = \bigcup_{\alpha \in \lambda^+} D_\alpha$ and each D_α is discrete. Indeed, let $f \in D_\alpha$ and define:

$$f^-(\beta) = \begin{cases} f(\beta) & \text{If } \beta < \alpha \\ -1 & \text{If } \beta = \alpha \\ 0 & \text{If } \beta > \alpha \end{cases} \tag{3.1}$$

Similarly define:

$$f^+(\beta) = \begin{cases} f(\beta) & \text{If } \beta < \alpha \\ 1 & \text{If } \beta = \alpha \\ 0 & \text{If } \beta > \alpha \end{cases} \tag{3.2}$$

Then $(f^-, f^+) \cap D_\alpha = \{f\}$. △

Claim 3: X is non-archimedean.

Proof of Claim 3. Let $\mathcal{B} = \{[\sigma] : \sigma \in \mathbb{W}^\alpha \text{ for some } \alpha \in \lambda^+\}$, where $[\sigma] = \{f \in X : \sigma \subset f\}$. Then $\mathcal{B}$ is a basis for our space. Every element of $\mathcal{B}$ is open: indeed, if $f \in [\sigma]$ then let $\alpha = \max\{dom(\sigma), supp(f)\}$ and f^+ and f^- be defined as in the proof of Claim 2. Then $f \in (f^-, f^+) \subset [\sigma]$.

Now let $c \in (a, b)$. Then there are ordinals α and β such that $a(\alpha) < c(\alpha)$, $c(\beta) < b(\beta)$, while $a(\gamma) = c(\gamma)$ and $c(\tau) = b(\tau)$ for every $\gamma < \alpha$ and every $\tau < \beta$. Set $\theta = \max\{\alpha, \beta\} + 1$. We have that $[c \restriction \theta] \subset (a, b)$.

Now given two elements of $\mathcal{B}$, either one is contained in the other, or they are disjoint. Therefore X is non-archimedean. △

To complete the proof observe that $\Delta(X) \geq \kappa^\lambda > \kappa > \lambda^+ \geq dis(X)$. □

Since for fixed λ there are arbitrarily big cardinals κ having cofinality λ, the former example shows that the gap between $dis(X)$ and $\Delta(X)$ can be arbitrarily big for hereditarily paracompact Baire LOTS.

Notice that the Lindelöf number of the previous space is $\geq \kappa$, in particular X is never Lindelöf.

Question 3.22. *Is $dis(X) \geq \Delta(X)$ true for every (Lindelöf, hereditarily paracompact) Čech complete LOTS X?*

Chapter 4

Covering compact spaces by discrete sets

Besides inspiring our study of the inequality $dis(X) \geq \Delta(X)$ for generalized metric spaces, Corollary 3.4 allowed us to prove a lemma that was crucial to many of our partial positive answers to Juhász and Szentmiklóssy's original question about compact spaces.

4.1 Hereditary separation

Testing a conjecture about compact spaces on compact hereditarily normal spaces is quite a natural thing to try, and indeed, Juhász and Van Mill already did that for the inequality $dis(X) \geq \mathfrak{c}$, before Gruenhage proved it to be true for every compact Hausdorff space.

Theorem 4.1. *([14]) Let $f : X \to Y$ be a perfect map. Then $dis(X) \geq dis(Y)$.*

Let κ^ω be the product of countably many copies of the discrete space κ.

A *cellular* family is a family of pairwise disjoint open sets in X. The following lemma is crucial to most of our results.

Lemma 4.2. *Let X be a compact space whose every open set contains a cellular family of cardinality κ. Then $dis(X) \geq \kappa^\omega$.*

Proof. Use regularity of X to find a cellular family $\{U_\alpha : \alpha < \kappa\}$ such that the closures of its members are pairwise disjoint. Suppose you have constructed open sets $\{U_\sigma : \sigma \in \kappa^{<n}\}$. Then let $\{U_{\sigma^\frown\alpha} : \alpha \in \kappa\}$ be a cellular family inside U_σ such that the closures of its members are pairwise disjoint and contained in U_σ.

For each $f \in \kappa^\omega$ let $F_f = \bigcap_{n \in \omega} \overline{U_{f \upharpoonright n}}$, which is a non-empty set because of compactness, and set $Z = \bigcup_{f \in \kappa^\omega} F_f$.

We are now going to show a perfect map Φ from Z onto κ^ω. Note that κ^ω is a complete (and hence Baire) metric space and $\Delta(\kappa^\omega) = \kappa^\omega$. So, by Theorem 4.1 and Corollary 3.4 we will get that $dis(X) \geq \kappa^\omega$.

Define Φ simply as $\Phi(x) = f$ whenever $x \in F_f$. It is easy to see that the F_fs are pairwise disjoint, so Φ is well-defined. Moreover, Φ is clearly continuous, onto and has compact fibers.

The following characterization of closed maps is well-known (see [7], Theorem 1.4.13)

Fact 4.3. *A mapping $f : X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists in Y a neighbourhood V of the point y such that $f^{-1}(V) \subset U$.*

Let now $f \in \kappa^\omega$, and U be an open set in Z such that $\Phi^{-1}(f) = F_f = \bigcap_{n\in\omega} \overline{U_{f\restriction n}} \subset U$. By compactness, we can find an increasing sequence of integers $\{j_k : 1 \leq k \leq n\}$ such that $\overline{U_{f\restriction j_n}} = \bigcap_{1\leq k\leq n} \overline{U_{f\restriction j_k}} \subset U$.

So let $B(f \restriction j_n)$ be the basic neighbourhood in κ^ω determined by $f \restriction j_n$. Then $\Phi^{-1}(B(f \restriction j_n)) \subset U_{f\restriction j_n} \subset U$, which proves Φ is closed. □

Theorem 4.4. *Let X be a hereditarily collectionwise Hausdorff compact space. Then $dis(X) \geq \Delta(X)$.*

Proof. Recall that cellularity and spread coincide for hereditarily collectionwise Hausdorff spaces (see [18], 2.23 a)). So if $c(G) < \Delta(X)$, for some open set $G \subset X$ we also have $s(G) < \Delta(X) \leq \Delta(G)$. Hence $dis(X) \geq \Delta(X)$.

Suppose now that $c(G) \geq \Delta(X)$ for every open set $G \subset X$. If $\Delta(X)$ is a successor cardinal then every open set contains a cellular family of size $\Delta(X)$, and hence, in view of Lemma 4.2 we have $dis(X) \geq \Delta(X)$.

If $\Delta(X)$ is a limit cardinal then, again by Lemma 4.2, every open set contains a cellular family of size κ for every $\kappa < \Delta(X)$. Hence $dis(X) \geq \kappa$ for every $\kappa < \Delta(X)$, which implies $dis(X) \geq \Delta(X)$ again. □

The following corollary also follows from Theorem 3.19.

Corollary 4.5. *For every compact LOTS X, $dis(X) \geq \Delta(X)$.*

Proof. Compact LOTS are monotonically normal, and monotone normality is hereditary (see [12]). $\square$

From Theorem 4.4 it also follows that, under V=L, $dis(X) \geq \Delta(X)$ for every compact hereditarily normal space X. Indeed, Stephen Watson [33] proved that compact hereditarily normal spaces are hereditarily collectionwise Hausdorff in the constructible universe. We can do better, and prove that $dis(X) \geq \Delta(X)$ for X compact hereditarily normal under a slight weakening of GCH.

Theorem 4.6. *(for every cardinal κ, $2^\kappa < 2^{\kappa^+}$) Let X be a compact T_5 space. Then $dis(X) \geq \Delta(X)$.*

Proof. Suppose first that $c(G) < \Delta(X)$ for some open set G. Since $c(G) = c(\overline{G})$ and $\overline{G}$ is compact T_5 we can assume that $X = \overline{G}$.

Let $\kappa = c(X)$. By Shapirovskii's bound on the number of regular open sets (see [18], 3.21) we have $\rho(X) \leq 2^\kappa$. Note that $\kappa^+ \leq \Delta(X)$. If $dis(X) < \Delta(X)$ then we would have $s(X) \geq \Delta(X)$ and hence we could find a discrete $D \subset X$ such that $|D| \geq \kappa^+$. By Jones' Lemma, $\rho(X) \geq 2^{\kappa^+} > 2^\kappa$, which contradicts our upper bound for the number of regular open sets.

If $c(G) \geq \Delta(X)$ for every open set G, then reasoning as in the last few lines of the proof of Theorem 4.4 we can conclude that $dis(X) \geq \Delta(X)$. $\square$

Question 4.7. *Is it true in ZFC that $dis(X) \geq \Delta(X)$ for every compact T_5 space?*

4.2 The shadow of a metric space

A trivial observation is that all compact metrizable spaces satisfy $dis(X) \geq \Delta(X)$.

The two most popular generalizations of compact metrizable spaces are dyadic compacta and Eberlein compacta. In fact, they are two somewhat opposite classes, as their intersection is precisely the class of compact metrizable spaces (see Arhangelskii).

This made us wonder whether $dis(X) \geq \Delta(X)$ was true for them. In fact, we are able to prove that for the weaker classes of *polyadic* and *Gul'ko* compacta. To achieve that we first need to prove that $dis(X)$ is always bounded below by the tightness. Recall that a space is called initially κ-compact if every set of cardinality $\leq \kappa$ has a complete accumulation point.

Lemma 4.8. *([11]) Let X be an initially κ-compact space such that $dis(X) \leq \kappa$. Then X is compact.*

Lemma 4.9. *If X is compact then $dis(X) \geq t(X)$.*

Proof. Suppose by contradiction that $\kappa = dis(X) < t(X)$. Let $A \subset X$ be a non-closed set, and $[A]_\kappa$ be its κ-closure, that is, the union of the closures of its subsets of cardinality κ. If we could prove that this last set is closed then we would have $t(X) \leq \kappa$, which is what we want.

If $[A]_\kappa$ is not closed then it cannot be initially κ-compact, or otherwise, since $dis([A]_\kappa) \leq \kappa$, it would be compact by Lemma 4.8. So there is $B \subset [A]_\kappa$ such that $|B| \leq \kappa$ and B has no point of complete accumulation in $[A]_\kappa$; then, by compactness, there is a point $x \notin [A]_\kappa$ that is of complete accumulation for B. But this contradicts the well-known and easy to prove fact that $[[A]_\kappa]_\kappa = [A]_\kappa$.

□

A compactum is called *polyadic* if it is the continuous image of some power of the one-point compactification of some discrete set.

The following lemmas are due to Gerlits.

Lemma 4.10. *[9] Let X be polyadic and $A \subset X$. Then there is a polyadic $P \subset X$ such that $A \subset P$ and $c(P) \leq c(A)$.*

Lemma 4.11. *[10] If X is polyadic then $w(X) = t(X) \cdot c(X)$.*

Theorem 4.12. *For a polyadic compactum X we have $dis(X) \geq \Delta(X)$.*

Proof. If $c(U) \geq \Delta(X)$ for any open set $U \subset X$ then we are done by Lemma 4.2. If there exists some open U such that $c(U) < \Delta(X)$, then let P be a polyadic space such that $U \subset P$ and $c(P) \leq c(U)$. Assume $dis(P) < \Delta(X)$. Then $t(P) < \Delta(X)$, which implies $s(P) \leq w(P) < \Delta(X)$, and we are done, since $|P| \geq \Delta(X)$. □

Recall that an *Eberlein compactum* is a compact space which embeds in $C_p(Y)$ for some compact Y. Equivalently, a space is an Eberlein compactum if and only if it is a weakly compact subspace of a Banach space. A *Gul'ko compactum* is a compact space X such that $C_p(X)$ is a Lindelöf Σ-space. A *Corson compactum* is a compact space with embeds in a Σ-product of lines. The following chain of implications holds:

$$\text{Eberlein} \Rightarrow \text{Gul'ko} \Rightarrow \text{Corson}$$

Lemma 4.13. *Let X be a hereditarily meta-Lindelöf space such that $dis(X) \leq \kappa$. If $A \subset X$ is such that $|A| \leq \kappa$ then $|\overline{A}| \leq \kappa$.*

Proof. If $\kappa < \omega$ then the statement is obviously true. Assume that κ is infinite, and let $X = \bigcup_{\alpha<\kappa} D_\alpha$, where each D_α is discrete. Let $B_\alpha = \overline{A} \cap D_\alpha$. For every $x \in B_\alpha$, let U_x be an open set such that $U_x \cap B_\alpha = \{x\}$. Then $\bigcup_{x \in B_\alpha} U_x$ is meta-Lindelöf, and hence $\{U_x : x \in B_\alpha\}$ has a point-countable open refinement $\mathcal{V}_\alpha$. Now for every $x \in B_\alpha$ choose $V_x \in \mathcal{V}_\alpha$ such that $x \in V_x$ and let $\mathcal{U}_\alpha = \{V_x : x \in B_\alpha\}$. Clearly $|\mathcal{U}_\alpha| = |B_\alpha|$ and for all $U \in \mathcal{U}_\alpha$, $U \cap A \neq \emptyset$. Fix some well-ordering of A and define a function $f : \mathcal{U}_\alpha \to A$ by:

$$f(U) = \min\{a \in A : a \in U\}.$$

Point-countability of $\mathcal{U}_\alpha$ implies that $|f^{-1}(a)| \leq \aleph_0$ for every $a \in A$, and therefore $|B_\alpha| = |\mathcal{U}_\alpha| \leq |A| \cdot \aleph_0 \leq \kappa$.

Since $\overline{A} = \bigcup_{\alpha \in \kappa} B_\alpha$ it follows that $|\overline{A}| \leq \kappa$. □

Theorem 4.14. *Let X be a hereditarily meta-Lindelöf space containing a dense Baire metrizable subset. Then $dis(X) \geq \Delta(X)$.*

Proof. Let $M \subset X$ be a dense metrizable subset and suppose by contradiction that $dis(X) < \Delta(X)$. Then, by the previous lemma we have $\Delta(M) = \Delta(X)$. So $dis(X) \geq dis(M) \geq \Delta(M) = \Delta(X)$, which is a contradiction. □

Corollary 4.15. *For every Gul'ko compactum X we have $dis(X) \geq \Delta(X)$.*

Proof. Yakovlev ([34]) proved that every Corson compactum is hereditarily meta-Lindelöf and Gruenhage ([13]) proved that every Gul'ko compactum contains a dense Baire metrizable subset. □

We are sorry to admit that we haven't been able to answer the following two questions.

Question 4.16. *Is $dis(X) \geq \Delta(X)$ for every Corson compact X?*

Question 4.17. *Is $dis(X) \geq \Delta(X)$ for every compact space with a (Baire) dense metrizable subset?*

As an application of the results in this section we are now going to determine how many discrete sets are needed to cover the Σ-product of a Cantor cube.

Theorem 4.18. $dis(\Sigma(2^\kappa)) = \kappa^\omega$.

To prove that we will embed in $\Sigma(2^\kappa)$ an Eberlein compactum X for which $\Delta(X) = \kappa^\omega$.

Recall that a family $\mathcal{A}$ of subsets of a set T is called *adequate* if:

1. For every $A \in \mathcal{A}$, $\mathcal{P}(A) \subset \mathcal{A}$.
2. If $[A]^{<\omega} \subset \mathcal{A}$ then $A \in \mathcal{A}$.

It is easy to see that $\mathcal{A}$ with the topology inherited from the product space 2^T is closed, and hence compact. Such a space is called an *adequate compactum*. Adequate families are one of the most useful tools for constructing Corson compacta: especially handy is the adequate family of all chains of a partial order. If the partial order has no uncountable chains, then the corresponding adequate compactum is Corson.

Leiderman and Sokolov characterized all adequate Eberlein compacta. For a point $x \in 2^T$ define the *support of* x as $supp(x) = \{a \in T : x(a) = 1\}$.

Theorem 4.19. *([24]) Let X be an adequate compact embedded in 2^T. Then X is an Eberlein compact if and only if there is a partition $T = \bigcup_{i\in\omega} T_i$ such that $|supp(x) \cap T_i| < \aleph_0$ for each $x \in X$ and $i \in \omega$.*

The next example is a modification of an example due to Leiderman and Sokolov. Their original space was a strong Eberlein compactum (a weakly compact subset of a Hilbert space), and hence scattered. Our space is far from being scattered.

Example 4.20. *Let κ be any infinite cardinal. There is an Eberlein compactum, embedded in 2^κ, such that $\Delta(X) = \kappa^\omega$.*

Proof. Let $W_0 = Lim(\kappa)$ and let $\{x_\alpha : \alpha \in \kappa\}$ be an increasing enumeration of W_0. Let $W_i = \{x_\alpha + i : \alpha \in \kappa\}$. Now let $T = \bigcup_{i\in\omega} W_i \times (W_i \cup \{-i\})$. Define an order on T as follows : $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if and only if $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$. Then every chain in T is countable, so the adequate compact X constructed from the adequate family consisting of all chains in T is Corson. Moreover, the partition in the definition of T, along with Theorem 4.19 shows that X is Eberlein. It remains to check that $\Delta(X) = \kappa^\omega$. To see that, let U be any basic open set. Then U is the set of all chains containing some fixed finite chain $\{(\alpha_i, \beta_i) : i \leq k\}$, enumerated in increasing order, and missing a fixed finite number of elements $\{(\gamma_j, \delta_j) : j \leq r\}$. Let t be an integer such that $\{\alpha_i : i \leq k\} \cup \{\gamma_j : j \leq r\} \subset \bigcup_{s<t} W_s$. Now, for every chain of the form $\{\alpha_s : s \geq t\}$ with $\alpha_s \in W_s$ for every $s \geq t$ and $\alpha_t > \alpha_k$ we have that $\{(\alpha_i, \beta_i) : i \leq k\} \cup \{(\alpha_s, -s) : s \geq t\} \in U$. Now the set of all such chains has cardinality κ^ω, since there is a natural bijection between that set and the set of all countable increasing sequences in κ. □

Every Σ-product of compact spaces is countably compact, which reminds us of the following question.

Question 4.21. *Is $dis(X) \geq \mathfrak{c}$ for X countably compact crowded?*

4.3 Homogeneity and beyond

The starting point for our next pair of results is the following easy observation.

Theorem 4.22. *Let X be a homogeneous compactum. Then $dis(X) \geq \Delta(X)$.*

Proof. Combining Arhangel'skii's theorem with the Juhász-Szentmiklóssy's result cited in the introduction we get $dis(X) \geq 2^{\chi(X)} \geq \Delta(X)$. □

A space is *homogeneous with respect to character* if $\chi(x, X) = \chi(y, X)$ for any $x, y \in X$. A space X is *power homogeneous* if X^κ is homogeneous for some κ.

The following lemma is due to Juhász and Van Mill.

Lemma 4.23. *([19]) Every infinite compactum contains a point x with $\chi(x, X) < dis(X)$.*

We are also going to need a couple of results from Guit Jan Ridderbos' PhD Thesis.

Lemma 4.24. *([27]) Let X be power homogeneous. If the set of all points of π-character κ is dense in X, then $\pi\chi(X) \leq \kappa$.*

Let $\pi\kappa\chi(X) = \sup\{\pi\kappa\chi(x, X) : x \in X\}$, where $\pi\kappa\chi(x, X)$ is the least cardinality of a π-network at x consisting of G_κ-sets.

Lemma 4.25. *([27]) Let X be a power-homogeneous space of pointwise countable type such that $\pi\kappa\chi(X) \leq \kappa$. Then either $\chi(X) \leq \kappa$ or X is homogeneous with respect to character.*

Theorem 4.26. *(CH) Let X be a power-homogeneous compactum. Then the minimum number of discrete sets required to cover X is at least* $\min\{\Delta(X), \omega_3\}$.

Proof. Suppose that $\beta\omega$ does not embed in X, then X does not map onto I^{ω_1} (see the proof of [18], 3.22) and hence, as a consequence of Shapirovskii's Theorem on maps onto Tychonoff Cubes, the set of all points of countable π-character is dense in X. Therefore, by Lemma 4.24, $\pi\chi(X) \leq \omega$. If $\chi(X) \leq \omega$, then $|X| \leq \omega_1$, by Arhangel'skii's theorem, and since $dis(X) \geq \omega_1$ holds for every compactum, we are done. Otherwise, X is homogeneous with respect to character, and hence $|X| \leq 2^{\chi(X)} \leq dis(X)$, by Juhász and Szentmiklóssy's result.

If $\beta\omega$ embeds in X then $dis(X) \geq 2^{\omega_1}$. Suppose that $dis(X) < \omega_3$, that is $dis(X) \leq \omega_2$. Then, by Lemma 4.23, X contains a dense set of G_{ω_1} points. If $\chi(X) \leq \omega_1$, then $\Delta(X) \leq 2^{\omega_1}$ and we are done. Otherwise, X is homogeneous with respect to character, and $dis(X) \geq \Delta(X)$ is true again. □

Corollary 4.27. *(CH) If X is a power-homogeneous compactum such that $|X| \leq \omega_3$ then $dis(X) \geq \Delta(X)$.*

Question 4.28. *Is $dis(X) \geq \Delta(X)$ true for every power-homogeneous compactum?*

The following proposition at least says that the gap between $\Delta(X)$ and $dis(X)$ can't be too big for power-homogeneous compacta.

Proposition 4.29. *Let X be a power homogeneous compactum. Then $\Delta(X) \leq 2^{dis(X)}$.*

Proof. Suppose by way of contradiction that $dis(X) \leq \kappa$ but $|U| > 2^\kappa$ for every open $U \subset X$. Then by Lemma 4.23 the set of all points of character less than κ is dense in X, which implies $\pi\chi(X) \leq \kappa$. Thus, in particular, $\pi\kappa\chi(X) \leq \kappa$. If $\chi(X) \leq \kappa$, then, by Arhangel'skii's Theorem, $|X| \leq 2^\kappa$, which contradicts our initial assumption. Otherwise $\chi(X) \geq \kappa^+$ and X is homogeneous with respect to character, which even implies $dis(X) \geq 2^{\kappa^+}$, again a contradiction. □

Chapter 5

Closures of discrete sets, closures of free sequences and cardinality

5.1 A crash course on elementary submodels

In this and the next chapter of our dissertation we will make use of a technique from Model Theory, that is gradually becoming a standard tool in Set-theoretic Topology. Here we provide some basics on elementary submodels and their applications to Topology, that will make this chapter self-contained. None of the results cited in this section is our own, we refer the reader to [5] for more information as well as the missing proofs.

Given a formula $\phi(x_1, x_2, \dots, x_n)$ of Set Theory, having free variables $\{x_1, x_2, \dots x_n\}$ and a set M we write $M \models \phi(x_1, x_2, \dots x_n)$ if the formula $\phi(x_1, x_2, \dots x_n)$ is true when you restrict all quantifiers to M. For example if $\phi(X) = (\exists x)(x \in X)$ then $M \models \phi(X)$ if and only if $(\exists x \in M)(x \in X)$, that is, $X \cap M$ is non-empty.

Definition 5.1. *If $\{a_1, a_2, \dots, a_n\} \subset M \subset N$ we say that the formula $\phi(a_1, a_2, \dots, a_n)$ is* <u>*absolute*</u> *for M and N if $M \models \phi(a_1, a_2, \dots, a_n)$ if and only if $N \models \phi(a_1, a_2, \dots a_n)$.*

Definition 5.2. *We will say that* <u>*M is an elementary submodel of N*</u> *and write $M \prec N$ if for all $n < \omega$ and for all formulae ϕ with at most n free variables and for all $\{a_1, a_2, \dots, a_n\} \subset M$ we have that ϕ is absolute for M and N.*

In practice M can take the place of N as long as the formulae we are taking up have all free variables in M, or, in other words, all the objects we are dealing with in our proof lie in M. It would be nice if N could be taken to be the whole set-theoretic universe. This is not feasible; however, before writing a proof, we already know the size of the largest object we will be considering. Say this is θ. Then we can take for N the set $H(\theta^+)$, consisting of all *hereditary sets of size* $\leq \theta$, which is a portion of the set-theoretic universe that is known to satisfy all set-theoretic axioms that we are going to need and contains all objects we are going to take up.

Most proofs of cardinal inequalities by elementary submodels follow a common plan. Suppose you want to prove the size of a *good enough* topological space is no more 2^κ.

1. Start with an elementary submodel of size 2^κ having as elements X, the topology on X, all cardinals you will be dealing with in your proof and a few other things. Theorem 5.3 below tells you there always exists such an elementary submodel.

2. Assume that there is a point $p \in X \setminus M$. Get a contradiction. Then $X \subset M$ and hence $|X| \leq |M| \leq 2^\kappa$.

3. Sometimes you are going to need your elementary submodel to be κ-closed (that is, each of its subsets of size κ is an element of it). This is helpful, for example, if you know that the size of a certain item in your space is at most κ. So if you are trying to get a contradiction by inductively constructing such an item of size κ^+ inside M, you know you can always continue because the inductive step is always an element of M. For instance, if you know the spread of your space is at most κ, you could try and get a contradiction by constructing a discrete set of size κ^+ inside M. Theorem 5.5 below says that you can always get a κ-closed elementary submodel of size 2^κ.

We now list three theorems that are the backbones of the use of elementary submodels in Topology.

Theorem 5.3. *For any set H and $A \subset H$ there is an elementary submodel $M \prec H$ such that $A \subset M$ and $|M| \leq |A| \cdot \omega$.*

Theorem 5.4. *If $M \prec H(\theta)$, where θ is a regular cardinal and $\kappa \in M$ is a cardinal such that $\kappa \subset M$ then for all $A \in M$ with $|A| \leq \kappa$ we have $A \subset M$. In particular, each countable element of M is a subset of M.*

Theorem 5.5. *For any regular $\theta \geq 2^\kappa$ and for any $A \subset H(\theta)$ with $|A| \leq 2^\kappa$ there is an $M \prec H(\theta)$ so that $A \subset M$, $|M| = 2^\kappa$ and $M^\kappa \subset M$.*

One of the reasons why elementary submodels are so useful in topology is that they make ugly-looking arguments involving transfinite induction transparent. And their ability to eat up a transfinite induction in a single bite is an outproduct of their nice behaviour with respect to chains.

If $\prec$ defines a linear order on $\mathcal{M}$ then $\mathcal{M}$ is called an *elementary chain.*

Theorem 5.6. *Let $\mathcal{M}$ be an elementary chain. Then $M \prec \bigcup \mathcal{M}$ whenever $M \in \mathcal{M}$.*

Corollary 5.7. *A chain under inclusion of elementary submodels of H is an elementary chain. Moreover its union is an elementary submodel of H.*

Proof. Let $M, N \in \mathcal{M}$ and suppose without loss that $M \subset N \prec H$. Fix $n \in \omega$, let ϕ be a formula with at most n free variables and $\{a_1, a_2, \ldots, a_n\} \subset M$. Suppose $N \models \phi(a_1, a_2, \ldots, a_n)$. Then $H \models \phi(a_1, a_2, \ldots, a_n)$. Since $M \prec H$ we have $M \models \phi(a_1, a_2, \ldots, a_n)$. So $M \prec N$, so $\mathcal{M}$ is an elementary chain. Let now $\{a_1, a_2, \ldots, a_n\} \subset \bigcup \mathcal{M}$. Then there is $M \in \mathcal{M}$ such that $\{a_1, a_2, \ldots, a_n\} \subset M \prec \bigcup \mathcal{M}$. Now $M \prec H$ and $H \models \phi(a_1, a_2, \ldots, a_n)$ imply that $M \models \phi(a_1, a_2, \ldots, a_n)$. By $M \prec \bigcup \mathcal{M}$ we have $\bigcup \mathcal{M} \models \phi(a_1, a_2, \ldots, a_n)$. So $\bigcup \mathcal{M} \prec H$. □

5.2 Depth, spread, free sequences and cardinality

Alas, Tkachuk and Wilson [2] asked whether a compact space in which the closure of every discrete set has size $\leq \mathfrak{c}$ must have size $\leq \mathfrak{c}$.

In [1] Ofelia Alas proves the following theorem, by way of a partial positive answer.

Theorem 5.8. *(MA) Let X be a Lindelöf regular weakly discretely generated space such that $\hat{s}(X) \leq \mathfrak{c}$ and $|\overline{D}| \leq \mathfrak{c}$ for every discrete $D \subset X$. Then $|X| \leq \mathfrak{c}$.*

We are going to prove that regular, Lindelöf and weakly discretely generated can all be dropped from the above theorem. But, first of all let's define two cardinal functions that will be handy in our study of this and related problems. Recall that a set $\{x_\alpha : \alpha < \kappa\}$ is called a *free sequence* if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset$ for every $\beta < \kappa$. Every free sequence is a discrete set.

Definition 5.9. *Set* $g(X) = \sup\{|\overline{D}| : D \subset X \text{ is discrete }\}$ *(the* *depth of* X*) and* $b(X) = \sup\{|\overline{F}| : F \subset X \text{ is a free sequence }\}$ *(the* *breadth of* X*).*

The condition $g(X) \leq \kappa$ appears to be a lot stronger than $b(X) \leq \kappa$. In fact, while the former implies that $|X| \leq 2^\kappa$ (simply observe that the hereditarily Lindelöf number is discretely reflexive [2] and use De Groot's inequality $|X| \leq 2^{hL(X)}$), the latter alone does not put any bound on the cardinality of X. For example, the one-point compactification of a discrete set of arbitrary cardinality satisfies $b(X) = \omega$.

Before proving our first theorem, we need an old lemma of Shapirovskii, and a lemma about elementary submodels, which must be well-known, although we could not find a direct reference to it.

Lemma 5.10. *Suppose* $\mathfrak{c}$ *is a regular cardinal. Let* $\theta \geq (2^{<\mathfrak{c}})^+$ *be a regular cardinal and* $A \subset H(\theta)$ *be a set of size* $\leq 2^{<\mathfrak{c}}$*. Then there is an elementary submodel* $M \prec H(\theta)$ *such that* $A \subset M$, $|M| = 2^{<\mathfrak{c}}$ *and* M *is* λ*-closed for every* $\lambda < \mathfrak{c}$*.*

Proof. It follows from regularity of the cardinal $\mathfrak{c}$ that $(2^{<\mathfrak{c}})^{|\alpha|} = 2^{<\mathfrak{c}}$ for every $\alpha < \mathfrak{c}$. Let now $M_0 \prec H(\theta)$ be such that $A \subset M_0$ and $|M_0| \leq 2^{<\mathfrak{c}}$. Suppose we have constructed $\{M_\alpha : \alpha < \beta\}$ such that for every $\alpha < \beta$ we have $M_\alpha \prec H(\theta)$, $|M_\alpha| \leq 2^{<\mathfrak{c}}$. Then let $M_\alpha \prec H(\theta)$ be such that $M_\beta \cup [M_\beta]^{|\alpha|} \subset M_\alpha$ for every $\beta < \alpha$ and $|M_\alpha| \leq 2^{<\mathfrak{c}}$. Then $\{M_\alpha : \alpha < \mathfrak{c}\}$ is a chain under containment of elementary submodels of $H(\theta)$ and hence it is also an elementary chain, from which it follows that $M = \bigcup_{\alpha<\mathfrak{c}} M_\alpha$ is an elementary submodel of $H(\theta)$.

To see that M is $< \mathfrak{c}$-closed let $\lambda < \mathfrak{c}$ and $\{x_\alpha : \alpha < \lambda\} \subset M$. Then, by regularity of $\mathfrak{c}$ there is $\tau < \mathfrak{c}$ such that $\{x_\alpha : \alpha < \lambda\} \subset M_\tau$. We can certainly assume $\tau > \lambda$. But $[M_\tau]^{|\lambda|} \subset M_{\tau+1}$ and therefore $\{x_\alpha : \alpha < \lambda\} \in M_{\tau+1} \subset M$. □

Lemma 5.11. *(Shapirovskii, see [18], 2.13) Let* $\mathcal{U}$ *be an open cover for some space* X*. Then there is a discrete* $D \subset X$ *and a subcover* $\mathcal{W} \subset \mathcal{U}$ *such that* $|\mathcal{W}| = |D|$ *and* $X = \overline{D} \cup \bigcup \mathcal{W}$*.*

Theorem 5.12. *(*$2^{<\mathfrak{c}} = \mathfrak{c}$*) Let* X *be a space such that* $\hat{s}(X) \cdot g(X) \leq \mathfrak{c}$*. Then* $|X| \leq \mathfrak{c}$*.*

Proof. Let M be an elementary submodel of a large enough fraction of the universe such that $\{X, \tau\} \subset M$, $\mathfrak{c} \cup \{\mathfrak{c}\} \subset M$, $|M| \leq \mathfrak{c}$ and M is λ-closed for every $\lambda < \mathfrak{c}$.

We claim that $X \subset M$. Suppose not and fix $p \in X \setminus M$. We claim that for every $x \in X \cap M$ we can choose an open $U \in M$ such that $x \in U$ and $p \notin U$. Indeed, fix $x \in X \cap M$ and let $\mathcal{V} \in M$ be the set of all open sets $V \subset X$ such that $x \notin \overline{V}$. Then $\mathcal{V}$ covers $X \setminus \{x\}$, so by Shapirovskii's Lemma we can find a discrete $D \in M$ and a subfamily $\mathcal{W} \subset \mathcal{V}$ such that $\mathcal{W} \in M$, $|\mathcal{W}| = |D| \leq \mathfrak{c}$ and $X \setminus \{x\} \subset \overline{D} \cup \bigcup \mathcal{W}$. Now $\mathcal{W} \in M$ and $|\mathcal{W}| \leq \mathfrak{c}$ imply that $\mathcal{W} \subset M$. Notice that, since $D \in M$, also $\overline{D} \in M$ which implies $\overline{D} \subset M$, since $|\overline{D}| \leq \mathfrak{c}$. So $p \notin \overline{D}$ and hence there is $W \in \mathcal{W}$ such that $p \in W$. Let $U = X \setminus \overline{W}$. Then $U \in M$ is a neighbourhood of x such that $p \notin U$.

So for every $x \in X \cap M$ choose $U_x \in M$ such that $p \notin U$. The family $\mathcal{U} = \{U_x : x \in X \cap M\}$ covers $X \cap M$, so, by Shapirovskii's Lemma there is a discrete set $D \subset X \cap M$ and a set $\mathcal{W} \subset \mathcal{U}$ such that $|\mathcal{W}| = |D| < \mathfrak{c}$ with $X \cap M \subset \overline{D} \cup \bigcup \mathcal{W}$. Since M is $< \mathfrak{c}$-closed we have that $D \in M$ and $\mathcal{W} \in M$, and hence $M \models X \subset \overline{D} \cup \bigcup \mathcal{W}$. Now $p \notin W$ for any $W \in \mathcal{W}$ and $p \notin \overline{D}$, since $\overline{D} \subset X \cap M$, by the same reason as before. But that's a contradiction. □

Can we switch discrete sets with free sequences in the previous theorem? Clearly not, and the one-point compactification of a discrete set is a counterexample. However there are some cases where we can. Let's start by proving a kind of free-sequence version of Shapirovskii's Lemma.

Lemma 5.13. *Let X be a space such that the closure of every free sequence is Lindelöf and $\mathcal{U}$ be an open cover for X. Then there is a free sequence $F \subset X$ and a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| = |F|$ and $X = \overline{F} \cup \bigcup \mathcal{V}$.*

Proof. Suppose you have constructed, for some ordinal β, a free sequence $\{x_\alpha : \alpha < \beta\}$ and countable subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ such that $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup_{\alpha \leq \gamma} \bigcup \mathcal{U}_\alpha$ for every $\gamma < \beta$.

Let $\mathcal{U}_\beta$ be a countable subcollection of $\mathcal{U}$ covering the Lindelöf subspace $\overline{\{x_\alpha : \alpha < \beta\}}$ and pick a point $x_\beta \in X \setminus \bigcup_{\alpha \leq \beta} \bigcup \mathcal{U}_\beta$. Let κ be the least ordinal such that

$$\overline{\{x_\alpha : \alpha < \kappa\}} \cup \bigcup_{\alpha<\kappa} \bigcup \mathcal{U}_\alpha = X.$$

Then $\{x_\alpha : \alpha < \kappa\}$ is a free sequence and for $\mathcal{V} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ we have $|\mathcal{V}| = \kappa$. □

Theorem 5.14. *($2^{<\mathfrak{c}} = \mathfrak{c}$) Let X be a Lindelöf space such that $\psi(X) \leq \mathfrak{c}$ and $\hat{F}(X) \cdot b(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let M be a $< \mathfrak{c}$-closed elementary submodel such that $\mathfrak{c} \cup \{\mathfrak{c}\} \subset M$ and $\{X, \tau\} \subset M$.
Claim: The closure of every free sequence in $X \cap M$ is Lindelöf.

Proof of Claim. Let $F \subset X \cap M$ be a free sequence in $X \cap M$ well-ordered in type κ (where $\kappa \leq \mathfrak{c}$ because $|M| \leq \mathfrak{c}$). We claim that F is also a free sequence in X. Denote by F_β the initial segment of F determined by its βth element. Let $\alpha = \sup\{\beta < \alpha : F_\beta$ is a free sequence in X by the same well-ordering of $F\}$. Then F_α is a free sequence in X. If not, there would be some $\beta < \alpha$ such that $x \in \overline{F_\beta} \cap \overline{F_\alpha \setminus F_\beta}$ and $x \notin M$. But F_β is a free sequence in X and therefore $|F_\beta| < \mathfrak{c}$. Thus $F_\beta \in M$, and hence $\overline{F_\beta} \in M$, which along with $|\overline{F_\beta}| \leq \mathfrak{c}$ implies that $\overline{F_\beta} \subset M$. So $x \in M$, which is a contradiction. But now $F_{\alpha+1}$ is also a free sequence in X, because you can't spoil freeness by adding a single isolated point. Therefore $\alpha = \kappa$, which proves that F is a free sequence in X. Proceeding as before we get that $\overline{F} \subset X \cap M$, which proves our claim, since closed subspaces of Lindelöf spaces are Lindelöf. △

We claim that $X \subset M$. Suppose not, and let $p \in X \setminus M$. For every $x \in X \cap M$ use $\psi(X) \leq \mathfrak{c}$ to pick a neighbourhood $U_x \in M$ of x such that $p \notin U_x$. Let $\mathcal{U} = \{U_x : x \in X \cap M\}$. By Lemma 5.13, there are a free sequence $F \subset X \cap M$ and a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|F| = |\mathcal{V}| < \mathfrak{c}$ with $X \cap M \subset \overline{F} \cup \bigcup \mathcal{V}$. Now $|F| < \mathfrak{c}$, so $F \in M$ and hence $\overline{F} \in M$, which, along with $|\overline{F}| \leq \mathfrak{c}$ implies that $\overline{F} \subset M$. Also, $\mathcal{V} \subset M$ and $|\mathcal{V}| < \mathfrak{c}$ imply that

$\mathcal{V} \in M$. Therefore $M \models X \subset \overline{F} \cup \bigcup \mathcal{V}$ and hence there is $V \in \mathcal{V}$ such that $p \in V$, which is a contradiction. □

Pseudocharacter $\leq \kappa$ is not discretely reflexive, unless the space is compact (see [2]). The following lemma shows that the pseudocharacter of a space never exceeds its depth.

Lemma 5.15. *Let κ be an infinite cardinal and X be a space where $|\overline{D}| \leq \kappa$ for every discrete $D \subset X$. Then $\psi(X) \leq \kappa$. If in addition X is regular then $\psi(F, X) \leq \kappa$, for every closed $F \subset X$ such that $|F| \leq \kappa$.*

Proof. Let $F \subset X$ be a κ-sized closed set (or a point, if X is not regular). Now let $\mathcal{V} = \{V \subset X : V \text{ is open and } \overline{V} \cap F = \emptyset\}$. Then $\mathcal{V}$ covers $X \setminus F$ and hence we can find a discrete $D \subset X \setminus F$ and a subcollection $\mathcal{U} \subset \mathcal{V}$ with $|\mathcal{U}| = |D|$ such that $X \setminus F \subset \bigcup \mathcal{U} \cup \overline{D}$. So $(\bigcap_{x \in \overline{D} \setminus F} X \setminus \{x\}) \cap (\bigcap_{U \in \mathcal{U}} X \setminus \overline{U}) = F$, which implies that $\psi(F, X) \leq \kappa$. □

The following corollary is another improvement of Alas' Theorem.

Corollary 5.16. *($2^{<\mathfrak{c}} = \mathfrak{c}$) Let X be a Lindelöf space such that $\hat{F}(X) \cdot g(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$.*

Proof. This follows from Lemma 5.15 and Theorem 5.14. □

In the above corollary Lindelöfness can be removed, if one assumes the space to be regular.

Theorem 5.17. *($2^{<\mathfrak{c}} = \mathfrak{c}$) Let X be a regular space such that $\hat{F}(X) \leq \mathfrak{c}$ and $|\overline{D}| \leq \mathfrak{c}$ for every discrete $D \subset X$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let M be an elementary submodel as before. By Lemma 5.15 every $\mathfrak{c}$-sized closed subset of X has pseudocharacter $\leq \mathfrak{c}$.

We claim that $X \subset M$. Suppose not and fix $p \in X \setminus M$ and suppose that for some $\beta < \mathfrak{c}$ we have constructed a free sequence $\{x_\alpha : \alpha < \beta\} \subset M$ and open sets $\{U_\alpha : \alpha < \beta\} \subset M$. We have $p \notin \overline{\{x_\alpha : \alpha < \beta\}}$. Now use the claim to choose a sequence $\mathcal{G} \in M$ of open sets such that $|\mathcal{G}| \leq \mathfrak{c}$ and $\overline{\{x_\alpha : \alpha < \beta\}} = \bigcap \mathcal{G}$. We have $\mathcal{G} \subset M$, so we can choose an open set

$U_\beta \in M$ with $p \notin U_\beta$ and $\overline{\{x_\alpha : \alpha < \beta\}} \subset U_\beta$. Now use $< \mathfrak{c}$-closed and elementarity to pick $x_\beta \in (X \setminus \bigcup_{\alpha \leq \beta} U_\alpha) \cap M$. Thus $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ is a $\mathfrak{c}$-sized free sequence in X, which is a contradiction. □

In Theorem 5.17 one can safely work in ZFC if free sequences are assumed to be countable. So we have a common framework for Alas' Theorem and Dow's result about compact spaces of countable tightness mentioned in the introduction. We have only one case left to exhaust all relationships between the four cardinal functions we have defined and cardinality.

Theorem 5.18. *($2^{<\mathfrak{c}} = \mathfrak{c}$) Let X be a regular space such that $\hat{s}(X) \cdot b(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let $F \subset X$. We claim that $\psi(F, X) \leq \mathfrak{c}$. Indeed, for every $x \notin F$ use regularity to choose an open neighbourhood V_x of x such that $\overline{V_x} \cap F = \emptyset$. Then $\{V_x : x \notin F\}$ covers $X \setminus F$, so we can choose a discrete $D \subset X \setminus F$ such that $X \setminus F \subset \bigcup\{\overline{V}_x : x \in D\} \cup \overline{D}$. Now we claim that for every $p \in \overline{D} \setminus F$ we can choose an $E \subset D$ such that $p \in \overline{E}$ and $\overline{E} \cap F = \emptyset$. Indeed, simply use regularity to find an open neighbourhood U of p such that $\overline{U} \cap F = \emptyset$ and set $E = U \cap D$. So $F = \bigcap\{X \setminus \overline{E} : E \subset D \text{ and } \overline{E} \cap F = \emptyset\} \cap \bigcap\{V_x : x \in D\}$. This implies that $\psi(F, X) \leq \mathfrak{c}$ since $|D| < \mathfrak{c}$ and hence $2^{|D|} \leq \mathfrak{c}$, by the set-theoretic assumption. Now, an argument similar to the proof of Theorem 5.17 will finish the proof. □

Regularity can be replaced by Lindelöfness. We leave the details to the reader.

Question 5.19. *Is there in ZFC a Hausdorff non-regular space such that free sequences are countable (discrete sets are countable), $|\overline{D}| \leq \mathfrak{c}$ for every discrete $D \subset X$ (for every free sequence $F \subset X$) and yet $|X| > \mathfrak{c}$?*

Question 5.20. *Is there, in some model of set theory, some (compact) regular space X such that every discrete set has size $< \mathfrak{c}$, the closure of every discrete set has size $\leq \mathfrak{c}$ and yet the space has size $> \mathfrak{c}$.*

To find a Hausdorff counterexample to the above question, take a model of $\omega_1 < \mathfrak{c} < 2^{\omega_1}$ and let $X = 2^{\omega_1}$. Let $\tau = \{U \setminus C : U$ is open in the usual topology on 2^{ω_1} and $|C| \leq \omega_1\}$. Then every discrete set in (X, τ) is closed and has size $\omega_1 < \mathfrak{c}$.

Chapter 6

Arhangel'skii, De Groot, free sequences and increasing chains

6.1 Introduction

In 1968 A.V. Arhangel'skii proved his famous theorem saying that the cardinality of a compact first-countable Hausdorff space does not exceed the continuum. This solved a long-standing question of Alexandroff and boosted an active line of research investigating generalizations of it. Here are two highlights.

Theorem 6.1. *(Arhangel'skii-Shapirovskii) Let X be Hausdorff space. Then:*

$$|X| \leq 2^{t(X) \cdot L(X) \cdot \psi(X)}$$

Theorem 6.2. *(Bell-Ginsburgh-Woods) Let X be normal weakly Lindelöf first-countable space. Then $|X| \leq \mathfrak{c}$.*

Here a space is *weakly Lindelöf* if every open cover has a countable subcollection whose union is dense in the space. The question asking whether normal can be replaced with regular in this last result is certainly one of the most interesting in this area. A good survey of Arhangel'skii Theorem and its offsprings is Hodel's ([15]).

An important tool in Arhangel'skii's proof of his theorem is the notion of a free sequence. We have already seen that $t(X) = F(X)$ in compact T_2 spaces. If X is Lindelöf, this is not true anymore. Indeed, assume CH and take a Luzin subspace of the real line with the density topology. Then the tightness is uncountable, since every countable set is closed discrete, but free sequences are countable because the space is hereditarily Lindelöf. However, we always have $F(X) \leq L(X) \cdot t(X)$ for every Hausdorff space X. Here we prove that if X is Hausdorff then $|X| \leq 2^{\psi(X) \cdot F(X) \cdot L(X)}$. This is a generalization of Theorem 6.1

in view of what we just said. Also, we prove the increasing strengthening of our theorem, and the proof we give seems to be shorter and simpler than even the proof of the increasing strengthening of Arhangel'skii's theorem as given by Juhász (see [18], 6.11), although it still relies on some of his ideas.

6.2 A common generalization of Arhangel'skii's Theorem and De Groot's inequality

István Juhász has kindly informed us that he independently proved Theorem 6.4 and presented it along with other results in a series of talks in Jerusalem in 2003, but never got around to publish it.

Define $\Phi(X) = \sup\{L(X \setminus \{x\}) : x \in X\}$.

Lemma 6.3. *$\Phi(X) = L(X) \cdot \psi(X)$.*

Proof. Obviously $L(X) \leq \Phi(X)$. Also, if $L(X \setminus \{x\}) \leq \kappa$ then for every $y \neq x$ select U_y such that $x \notin \overline{U_y}$. Then $\mathcal{U} = \{U_y : y \neq x\}$ covers $X \setminus \{x\}$ and hence we can find a subcover $\mathcal{V}$ having cardinality $\leq \kappa$. Then $\bigcap\{X \setminus \overline{U} : U \in \mathcal{U}\} = \{x\}$, which proves that $\psi(x, X) \leq \kappa$. So, taking sups we have that $\psi(X) \leq \Phi(X)$, and hence $\psi(X) \cdot L(X) \leq \Phi(X)$.

To prove the other direction suppose that $L(X) \cdot \psi(X) = \kappa$ and let $\mathcal{U}$ be an open collection such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$. Then $X \setminus \{x\} = \bigcup\{X \setminus U : U \in \mathcal{U}\}$ and $L(X \setminus U) \leq \kappa$ for every $U \in \mathcal{U}$. Thus $L(X \setminus \{x\}) \leq \kappa$. $\square$

The following generalizes both Theorem 6.1 and De Groot's inequality saying that the cardinality of every hereditarily Lindelöf space does not exceed the continuum.

Theorem 6.4. *If X is T_2 then $|X| \leq 2^{\psi(X) \cdot L(X) \cdot F(X)}$.*

Proof. Let $\kappa = \psi(X) \cdot L(X) \cdot F(X)$, θ be a large enough regular cardinal and $M \prec H(\theta)$ be κ-closed, $|M| = 2^\kappa$ and $2^\kappa \cup \{X, \tau, 2^\kappa\} \subset M$. We claim that $X \subset M$. Suppose not and choose $p \in X \setminus M$. Let $x \in X \cap M$. Since $\psi(x, X) \leq \kappa$ there is a family $\mathcal{U} \in M$ of open sets such that $\bigcap \mathcal{U} = \{x\}$ and $|\mathcal{U}| \leq \kappa$. Now every 2^κ-sized element of M is also a subset of M, so $\mathcal{U} \subset M$ and hence we can choose an open set $U \in M$ such that $x \in U$ and $p \notin U$.

Let $\mathcal{U}$ be the set of all open $U \in M$ such that $p \notin U$. Then $\mathcal{U}$ covers $X \cap M$. Let $\mathcal{U}_0$ be any subcollection of $\mathcal{U}$ having cardinality $\leq \kappa$. Since $p \in X \setminus \bigcup \mathcal{U}_0$, by elementarity we can choose $x_0 \in X \cap M \setminus \bigcup \mathcal{U}_0$. Now suppose that for some $\beta \in \kappa^+$ we have constructed a set $\{x_\alpha : \alpha < \beta\}$ and subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ such that $|\mathcal{U}_\alpha| \leq \kappa$ for every $\alpha < \beta$ and $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup \bigcup_{\alpha \leq \gamma} \mathcal{U}_\alpha$ and let $\mathcal{U}_\beta$ be a subcollection of $\mathcal{U}$ having cardinality $\leq \kappa$ such that $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{U}_\beta$ and pick a point $x_\beta \in X \cap M \setminus \bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$. If the induction didn't stop before reaching κ^+ then $\{x_\alpha : \alpha < \kappa^+\}$ would be a free sequence of size κ^+ in X. So there is a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $X \cap M \subset \bigcup \mathcal{V}$. Therefore $M \models X \subset \bigcup \mathcal{V}$ and hence $H(\theta) \models X \subset \bigcup \mathcal{V}$. So there is $V \in \mathcal{V}$ such that $p \in V$, which is a contradiction. □

6.3 The increasing strenghtening

Suppose $X = \bigcup_{\alpha < \lambda} X_\alpha$ where $X_\alpha \subset X_\beta$ whenever $\alpha < \beta$ and we know that $f(X_\alpha) \leq \kappa$ for every $\alpha < \lambda$ for some cardinal function f. What can we conclude about $f(X)$? This general question has been the object of systematic study by Juhász, who dedicated the whole chapter 6 of his book [18] to it, Juhász and Szentmiklossy [22] and Tkachenko [31] [32]. In particular, we talk of an *increasing strengthening* of a cardinal inequality when we can extend a cardinal inequality from a single space to an increasing chain of spaces of any length. Increasing strengthenings of cardinal inequalities often involve rather technical and complicated arguments. This is the case with the increasing strengthening of Arhangel'skii's Theorem [18] and that of the Bell-Ginsburgh-Woods Theorem [22]. We are now going to prove the increasing strengthening of Theorem 6.1.

Lemma 6.5. *([18], 6.11) If X is T_2, Y is a subspace of X with $L(Y) \leq \kappa$ and $p \in Y$, then for every open set U in X containing p there is a family $\mathcal{R}$ of regular closed neighbourhoods* *of p in X such that $|\mathcal{R}| \leq \kappa$ and*

$$U \cap Y \supset \bigcap \mathcal{R} \cap Y$$

.

Theorem 6.6. *Let $X = \bigcup_{\alpha<\lambda} X_\alpha$, where $X_\alpha \subset X_\beta$ whenever $\alpha < \beta$ and suppose that $F(X_\alpha) \cdot \psi(X_\alpha) \cdot L(X_\alpha) \leq \kappa$ for every $\alpha < \lambda$. Then $|X| \leq 2^\kappa$.*

Proof. If $\lambda \leq 2^\kappa$ then we are done by Theorem 6.4, so we can assume that $\lambda = (2^\kappa)^+$. Call a set $A \subset X$ *bounded* if $|A| \leq 2^\kappa$. The following claim is contained in [18], 6.11 but we include its proof for completeness.
Claim: If $A \in [X]^{\leq\kappa}$ then $\overline{A}$ is bounded.

Proof of claim. Let $A \subset X$ be bounded. Since $\rho(\overline{A}) \leq 2^{|A|} \leq 2^\kappa$, by [18], 2.6 d), it will suffice to prove that if F is closed unbounded then $\rho(F) > 2^\kappa$. Let $F_\alpha = F \cap X_\alpha$. Then $L(F_\alpha) \leq \kappa$ for every $\alpha < \lambda$. Fix $x \in F$, then there is $\alpha_0 < \lambda$ such that $x \in F_{\alpha_0}$. We have $x \in F_\alpha$ for every $\alpha \in \lambda \setminus \alpha_0$, so $\psi(x, F_\alpha) \leq \kappa$ and hence we can find families $\{\mathcal{U}_\alpha : \alpha_0 < \alpha < \kappa\}$ of open sets such that $|\mathcal{U}_\alpha| \leq \kappa$ and $\bigcap \mathcal{U}_\alpha \cap F_\alpha = \{x\}$ for every $\alpha > \alpha_0$. Fix now $\alpha > \alpha_0$. For every $U \in \mathcal{U}_\alpha$ use the Lemma to select a family $\mathcal{R}_U$ of regular closed sets such that $|\mathcal{R}_U| \leq \kappa$ and $\bigcap \mathcal{R}_U \cap F_\alpha \subset U \cap F_\alpha$. Let $\mathcal{R}_\alpha = \bigcup \{\mathcal{R}_U : U \in \mathcal{U}_\alpha\}$. Then $\bigcap \mathcal{R}_\alpha \cap F_\alpha = \{x\}$. Suppose by contradiction that $\rho(F) \leq 2^\kappa$, then, since $\lambda > 2^\kappa$ we can find a κ-sized family $\mathcal{R}_x$ consisting of regular closed sets and a set $a \in [\lambda]^\lambda$ such that $\mathcal{R}_\alpha = \mathcal{R}_x$ for every $\alpha \in a$. So $\bigcap R_x \cap X_\alpha = \{x\}$ for cofinally many α's, which can only be if $\bigcap \mathcal{R}_x = \{x\}$. Hence we have found an injection from F into the family of all families of size $\leq \kappa$ consisting of regular closed sets, which implies $|F| \leq \rho(F)^\kappa \leq 2^\kappa$. But that contradicts the fact that F is unbounded. △

Let θ be a large enough regular cardinal and $M \prec H(\theta)$ be κ-closed, $|M| = 2^\kappa$ and $2^\kappa \cup \{X, \tau, 2^\kappa\} \subset M$. We claim that $X \subset M$. Suppose not and choose $p \in X \setminus M$. We claim that for every $x \in X \cap M$ we can choose a neighbourhood $U \in M$ of x such that $p \notin U$.

Indeed, fix $x \in X \cap M$ and let $\mathcal{V}$ be the set of all open sets V such that $x \notin \overline{V}$. Note that $\mathcal{V}$ covers $X \setminus \{x\}$. Suppose we have constructed subcollections $\{\mathcal{V}_\alpha : \alpha < \beta\}$ of $\mathcal{V}$ such that $|\mathcal{V}_\alpha| \leq \kappa$ for every $\alpha < \beta$ and a set $\{x_\alpha : \alpha < \beta\}$ such that $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup\bigcup_{\alpha<\gamma} \mathcal{V}_\alpha$ for every $\gamma < \beta$, where the closure is meant in $X \setminus \{x\}$. By the Claim, the set $\overline{\{x_\alpha : \alpha < \beta\}}$ is bounded and hence there is $\lambda_\beta < \lambda$ such that $\overline{\{x_\alpha : \alpha < \beta\}} \subset X_{\lambda_\beta}$. Hence $L(\overline{\{x_\alpha : \alpha < \beta\}}) \leq \kappa$, so there is a subcollection $\mathcal{V}_\beta$ of $\mathcal{V}$ such that $|\mathcal{V}_\beta| \leq \kappa$ and $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{V}_\beta$. If the induction didn't stop before reaching κ^+ then $F = \{x_\alpha : \alpha < \kappa^+\}$ would be a free sequence of length κ^+ in $X \setminus \{x\}$. Now F cannot converge to x, because, since $|F| \geq \kappa^+$ and $L(X \setminus \{x\}) \leq \kappa$, the set F has a complete accumulation point in $X \setminus \{x\}$. Therefore, there is an open neighbourhood G of x which misses κ^+ many points of F and $F \setminus G$ is a free sequence in X of cardinality κ^+. Now $F \setminus G$ is bounded and hence there is $\tau < \lambda$ such that $F \setminus G \subset X_\tau$, but that contradicts $F(X_\tau) \leq \kappa$.

So there is a subcollection $\mathcal{W} \subset \mathcal{V}$ such that $|\mathcal{W}| \leq \kappa$ and $X \setminus \{x\} \subset \bigcup \mathcal{W}$. By elementarity we can take $\mathcal{W} \in M$ and hence $\mathcal{W} \subset M$, since $|\mathcal{W}| \leq \kappa$. Let $W \in \mathcal{W}$ such that $p \in W$. Then the set $U = X \setminus \overline{W} \in M$ is an open neighbourhood of x such that $p \notin U$.

Now let $\mathcal{U}$ be the set of all open sets $U \in M$ such that $p \notin U$. Then $\mathcal{U}$ covers $X \cap M$. Let $\mathcal{U}_0$ be any subcollection of $\mathcal{U}$ having cardinality $\leq \kappa$. Since there is a point (namely p) in $X \setminus \bigcup \mathcal{U}_0$, by elementarity we can pick $x_0 \in X \cap M \setminus \bigcup \mathcal{U}_0$. Suppose that for some $\beta \in \kappa^+$ we have constructed a set $\{x_\alpha : \alpha < \beta\}$ and subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ such that $|\mathcal{U}_\alpha| \leq \kappa$ for every $\alpha < \beta$ and $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup\bigcup_{\alpha \leq \gamma} \mathcal{U}_\alpha$. Since $\overline{\{x_\alpha : \alpha < \beta\}}$ is bounded we have $L(\overline{\{x_\alpha : \alpha < \beta\}} \leq \kappa$ and hence we can find a subcollection $\mathcal{U}_\beta$ of $\mathcal{U}$ having cardinality $\leq \kappa$ such that $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{U}_\beta$. If $\mathcal{U}_\beta$ is not a cover of X, as before, we can pick a point $x_\beta \in X \cap M \setminus \bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$. If we didn't stop then $\{x_\alpha : \alpha < \kappa^+\}$ would be a free sequence of size κ^+ in X. But that can't be since $\{x_\alpha : \alpha < \kappa^+\}$ is bounded. So there is a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ such that $X \cap M \subset \bigcup \mathcal{V}$. Therefore $M \models X \subset \bigcup \mathcal{V}$ and hence $H(\theta) \models X \subset \bigcup \mathcal{V}$. Thus there is $V \in \mathcal{V}$ such that $p \in V$, which is a contradiction. $\square$

Bibliography

[1] O. Alas, *On closures of discrete subsets*, Q and A in General Topology, Vol. 20 (2002), 85-89.

[2] O. Alas, V. Tkachuk, R. Wilson, *Closures of discrete sets often reflect global properties*, Topology Proc. 25 Spring (2000).

[3] B. Bailey and G. Gruenhage, *On a question concerning sharp bases*, Topology Appl. 153 (2005), 90–96.

[4] E. van Douwen, *An unbaireable stratifiable space*, Proc. Amer. Math. Soc. 67 (1977), 324–326.

[5] A. Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. (1988), 13(1), 17-72.

[6] A. Dow, *Closures of discrete sets in compact spaces*, Studia Math. Sci. Hung. 42 (2005), 227–234.

[7] R. Engelking, *General Topology*, second ed., Sigma Series in Pure Mathematics, no.6, Heldermann Verlag, Berlin, 1989.

[8] W. Fleissner and A. Miller, *On Q-sets*, Proc. Amer. Math. Soc. 78 (1980), 280–284.

[9] J. Gerlits, *On a problem of S. Mrówka*, Period. Math. Hungar. 4 (1973), 71–80.

[10] J. Gerlits, *On a generalization of dyadicity*, Studia Sci. Math. Hungar. 13 (1978), 1–17.

[11] J. Gerlits, I. Juhász and Z. Szentmiklóssy, *Two improvements on Tkachenko's addition theorem*, Comment. Math. Univ. Carolinae, 46, 4 (2005), 705-710.

[12] G. Gruenhage, *Generalized metric spaces* in Handbook of Set Theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Amsterdam 1984.

[13] G. Gruenhage, *A note on Gul'ko compact spaces*, Proc. Amer. Math. Soc. 100 (1987), 371–376.

[14] G. Gruenhage, *Covering compacta by discrete and other separated sets*, preprint.

[15] R. Hodel, *Arhangel skii's Solution to Alexandroff's Problem: A Survey*, Topology Appl. 153, 2199–2217.

[16] M. Ismail and A. Szymanski, *On the metrizability number and related invariants of spaces. II*, Topology Appl. 71 (1996), 179–191.

[17] T. Jech, *Set Theory: The Third Millennium Edition, revised and expanded*, Springer, Berlin 2002.

[18] I. Juhász, *Cardinal Function in Topology - Ten Years Later*, Math. Centre Tracts 123, 1980 Amsterdam.

[19] I. Juhász, J. van Mill, *Covering compacta by discrete subspaces*, Topology Appl. 154 (2007), 283–286.

[20] I. Juhász, Z. Szentmiklóssy, *A strengthening of the Čech-Pospišil theorem*, Topology Appl. 155 (2008), 2102–2104 .

[21] I. Juhász, Z. Szentmiklóssy, *Discrete subspaces of countably tight compacta*, Annals of Pure and Applied Logic 140 (2006), 72–74.

[22] I. Juhász, Z. Szentmiklóssy, *Increasing strengthenings of cardinal function inequalities*, Fund. Math. 126 (1986), no. 3, 209–216.

[23] A. Kechris, *Classical descriptive set theory*, Springer, New York, 1994.

[24] A. Leiderman, G. Sokolov, *Adequate families of sets and Corson compacts*, Comment. Math. Univ. Carolin. 25 (1984), no. 2, 233–246.

[25] A. Miller, *Special subsets of the real line* in Handbook of Set Theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Amsterdam 1984.

[26] P.J. Nyikos, *On some non-archimedean spaces of Alexandroff and Urysohn*, Topology Appl. 91 (1999), 1–23.

[27] G.J. Ridderbos, *Power homogeneity in Topology*, Doctoral Thesis, Vrije Universiteit, Amsterdam (2007).

[28] S. Spadaro, *Covering by discrete and closed discrete sets*, Topology Appl. 156 (2009), 721–727.

[29] S. Spadaro, *A note on discrete sets*, submitted.

[30] F.D. Tall, *Normality versus Collectionwise Normality* in Handbook of Set Theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Amsterdam 1984.

[31] M.G. Tkachenko, *Chains and cardinals*, Soviet Math. Dokl. 19 (1978), no. 2, 382–385.

[32] M.G. Tkachenko, *The behavior of cardinal invariants under the union operation for a chain of spaces*, Moscow Univ. Math. Bull. 33 (1978), no. 4, 39–46.

[33] S. Watson, *Locally compact normal spaces in the constructible universe*, Can. J. Math., vol. XXXIV, n.5, 1982, 1091–1096.

[34] N. Yakovlev, *On bicompacta in Σ-products and related spaces.*, Comment. Math. Univ. Carolin. 21 (1980), no. 2, 263–283.