Constructing Cubic Splines on the Sphere

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Thesis Abstract

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A method to approximate functions defined on a sphere using tensor product cubic B-splines is presented here. The method is based on decomposing the sphere into six identical patches obtained by radially projecting the six faces of a circumscribed cube onto the spherical surface. The theory of univariate splines has been generalized in different forms to functions of several variables. Among these extensions the tensor product splines are the easiest to handle. Although the tensor product splines are restricted to rectangular domains rendering their applicability limited they are extremely efficient compared to other surface approximation techniques which are far more complicated and hence computationally less attractive.

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CHAPTER 1

INTRODUCTION

1.1 Objective

Cubic splines on a spherical domain may be used for the approximation of functions defined on such domains which serve as a tool for modeling and analyzing of physical processes. However evaluation of these functions explicitly is sometimes difficult due to the limitations of the underlying processes. In such cases it becomes pertinent to form an approximation of the defined function. We also run into cases where the evaluation of the exact function is computationally expensive forcing us to use approximation techniques. The problem of functional approximation may be broadly classified into two categories. The first involving problems where the exact function is unknown and approximation technique is based on function value at certain set of discrete points. The second class of problems is related to physical process modeling. These usually involve operator equations. Our aim is to look into a suitable set of approximations $\mathcal{A}$ and develop means to select an appropriate approximation. Since we want the function to be approximated by some member of the approximation set $\mathcal{A}$ we need to device a method to select this member. Usually this is done by choosing an approximation member such that the error is within a certain factor of the least error that can be achieved. Functions arising from physical processes are generally smooth implying an obvious need for the approximants to be sufficiently smooth. Functional approximations on the sphere are of research interest since many geophysical applications including oceanography, climate modeling and modeling of earth's gravitational potential involve large amount of data on the surface of the earth (basic model is a sphere) or on the satellite orbit (approximately a spherical manifold).

1.1.1 Spline approximation and its significance

It appears that an obvious choice for function approximation is the polynomial $p_m \in \mathbb{P}_m$ because of its relative smoothness and easy manipulation on a digital computer. However it turns out that interpolating polynomials do not always converge to the function being interpolated [3]. The following theorem justifies this.

Theorem 1.1. *Let $[a, b]$ be fixed and suppose that for each $k \geqslant 1, t_{k_1}, t_{k_2}, \ldots, t_{k_k}$ is a collection of points in $[a, b]$. Then there exists a function $f \in \mathbb{C}[a, b]$ such that*

$$||(f - \mathcal{L}_m f)||_\infty \longrightarrow \infty \quad \text{as } m \to \infty \tag{1.1}$$

where $\mathcal{L}_m f$ is the unique polynomial of order k interpolating f at $t_{k_1}, t_{k_2}, \ldots, t_{k_k}$.

This leads to the approximation using smooth piecewise polynomials i.e splines.

1.2 Spline Theory

We address here the one dimensional case of approximating a given function using univariate splines. Mathematically we may represent it as

$$f(x_i) = s_m(x_i),$$

where x_i denote the nodal points. That is to say that the constructed spline agrees with the function values at a certain set of points termed here as the nodal points. We would want this spline to possess a certain degree of smoothness. We then represent the spline on any partition say Δ as a linear combination of the basis elements of the linear space to which it belongs. These basis elements are generally called the B-splines.

Definition 1.1 (**Piecewise Polynomials** [3]). *Let* $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$*, and write* $\Delta = \{x_i\}_0^n$*. The set* Δ *partitions the interval* $[a, b]$ *into* n *subintervals,* $I_i = [x_i, x_{i+1}),\ i = 0, 1, \ldots, n-2,$ *and* $I_{n-1} = [x_{n-1}, x_n]$*. Given a positive integer* m*, let*

$$\mathbb{PP}_m(\Delta) = \{f : \text{there exists polynomials } p_0, p_1, \ldots, p_{n-1} \text{ in } \mathbb{P}_m \text{ with } f(x) = p_i(x) \text{ for } x \in I_i, i = 0, 1, \ldots, n-1\},$$

where

$$\mathbb{P}_m = \{p(x) : p(x) = \sum_{i=1}^{m} c_i x^{i-1},\ \ c_1, c_2, \ldots, c_m,\ x \text{ real}\}.$$

We call $\mathbb{PP}_m(\Delta)$ *the space of piecewise polynomials of order* m *with knots* $x_1, x_2, \ldots, x_{n-1}$.

Switching from the approximation of a given function by a polynomial to approximation using piecewise polynomials provides us with a degree of flexibility. However piecewise polynomials are not necessarily smooth. To maintain flexibility and at the same time allow a certain degree of global smoothness we now define a class of functions known as polynomial splines.

Definition 1.2. *Let* Δ *be a partition of the interval* $[a, b]$ *as in Definition 1.1, and let* m *be a positive integer. Let*

$$\mathbb{S}_m(\Delta) = \mathbb{PP}_m(\Delta) \cap \mathbb{C}^{m-2}[a, b],$$

where $\mathbb{PP}_m(\Delta)$ *is the space of piecewise polynomials defined in (1.1). We call* $\mathbb{S}_m(\Delta)$ *the space of polynomial splines of order* m *(degree* $m-1$*) with respect to* Δ*.*

Polynomial splines spaces are finite dimensional linear spaces. The dimension of this space of splines is $dim(\mathbb{S}_m(\Delta)) = n + m - 1$. The above definition clearly implies that any polynomial on Δ of degree $\leqslant m-1$ is a spline function of degree $m-1$ on

Δ. In general a spline of degree $m-1$ is represented by different polynomials in each interval $I_i,\ i=0,1,\ldots,n-1$. This may give rise to dicontinuities in its $(m-1)-th$ derivatives at the internal nodes $x_1, x_2, \ldots, x_{n-1}$. The nodes for which this actually happens are called *active nodes*.

Let $s_m \in \mathbb{S}_m(\Delta)$ be a spline of degree $m-1$ defined on the partition Δ. Let us denote the restriction of this spline function as $s_{m|[x_i,x_{i+1}]}$ where

$$s_{m|[x_i,x_{i+1}]} = \sum_{j=0}^{m-1} s_{ji}(x-x_i)^j, \quad \text{if } x \in [x_i, x_{i+1}]$$

so we have mn coefficients to determine. Again we have the continuity conditions at the internal nodes. Each internal node has $m-1$ continuity conditions which amounts to $(n-1)(m-1)$ conditions. We therefore have $mn-(n-1)(m-1) = m+n-1$ coefficients to determine. Since we are talking about an interpolatory spline we have

$$s_{m|[x_i,x_{i+1}]}(x_i) = f(x_i) \equiv f_i \quad \text{for } i = 0,\ldots,n$$

where the $n+1$ function values are known. We now have $(m+n-1)-(n+1) = m-2$ coefficients still unaccounted for. To be more precise we still need $m-2$ conditions to determine the spline completely. This leads to imposing further constraints i.e conditions for periodicity or conditions for the spline to be natural.

Mathematically they are represented as

1. Periodic splines, if

$$s_m^l(a) = s_m^l(b), \quad l = 0,\ldots,m-2. \tag{1.2}$$

2. Natural splines, if for $m = 2p-1$, with $l \geqslant 2$

$$s_m^{p+j}(a) = s_m^{p+j}(b) = 0, \quad j = 0,1\ldots,p-2 \tag{1.3}$$

We will be dealing with cubic periodic splines throughout our work and unless otherwise mentioned $m = 4$.

1.2.1 B-spline Representation

Before we define the *B-Spline representation* for a spline $s_m \in \mathbb{S}_m(\Delta)$ and what the B-splines themselves are we define the concept of divided difference since B-splines can be defined in terms of the divided difference.

Definition 1.3 (**Divided Difference** [2]). *The n-th divided difference of a function f at the points $x_0, x_1, \ldots, x_n$ (which are assumed to be distinct) is the leading coefficient (i.e the coefficient of x^n) of the unique polynomial $p_{n+1}(x)$ of degree n which satisfies $p_{n+1}(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.$*

Mathematically the n-th divided difference is denoted as

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'_{n+1}(x_i)}, \tag{1.4}$$

where

$$\omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i).$$

We now define the B-splines in terms of divided difference

Definition 1.4 (**Normalized B-splines**). *The normalized B-splines of degree $m-1$ relative to the distinct nodes $x_i, x_{i+1}, \ldots, x_{i+m}$ is defined as*

$$\mathbb{B}_{i,m}(x) = (x_{i+m} - x_i) g[x_i, \ldots, x_{i+m}]. \tag{1.5}$$

where

$$g(t) = (t - x)_+^{m-1} = \begin{cases} (t - x)^{m-1} & \text{if } x \leqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

To find an explicit expression for the normalized B-splines we state here the the Uniqueness Theorem for an interpolating polynomial which forms a basis for for this explicit expression.

Theorem 1.2. *Given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ and $n+1$ corresponding values $f(x_0), f(x_1), \ldots, f(x_n)$ there exists a unique polynomial $p_{n+1} \in \mathbb{P}_{n+1}$ such that $p_{n+1}(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.*

The uniqueness of the interpolating polynomial provides for the comparison between the Lagrange's form of the interpolating polynomial and the Newton's divided difference formula. This comparison along with the notion that the divided difference is the coefficient of x^n in the interpolating polynomial yields the explicit representation for the n-th divided difference as defined in equation (1.4). Using equation (1.5) in the expression for normalized B-splines we arrive at the following explicit representation for B-splines

$$\mathbb{B}_{i,m}(x) = (x_{i+m} - x_i) \sum_{j=0}^{m} \frac{(x_{j+i} - x)_+^{m-1}}{\prod\limits_{\substack{l=0 \\ l \neq j}}^{m} (x_{j+i} - x_{l+i})}. \tag{1.6}$$

We note here that the m-th order normalized B-spline have *active nodes* $x_i, x_{i+1}, \ldots, x_m$ and vanish outside the interval $[x_i, x_{i+m}]$. The term normalized for B-splines has been introduced since B-splines can have varied sizes depending on the location of the nodal points, for example

$$\mathbb{Q}_{i,1}(x) = \begin{cases} \frac{1}{x_{i+1} - x_i}, & x_i \leqslant x < x_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{1.7}$$

is a B-spline which is not normalized and can be extremely large or extremely small depending on the step size of the nodal points. The direct evaluation of the B-spline, $\mathbb{B}_{i,m}(x)$ from its Definition 1.5 in terms of divided difference may result in considerable loss of accuracy during computation of the various difference quotients. The B-spline admits a recurrence relation [7] which requires no special treatment in the case of repeated nodal points and which does not result in loss of accuracy.

$$\mathbb{B}_{i,m}(x) = \frac{x - x_i}{x_{i+m-1} - x_i}\mathbb{B}_{i,m-1}(x) + \frac{x_{i+m} - x}{x_{i+m} - x_{i+1}}\mathbb{B}_{i+1,m-1}(x) \tag{1.8}$$

$$\mathbb{B}_{i,1}(x) = \begin{cases} 1 & \text{if } x_i \leqslant x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

We state here an important property of the B-splines which explains why they are termed as normalized B-splines.

Theorem 1.3. *The B-splines form a partition of unity; that is*

$$\sum_{i=j+1-m}^{j} \mathbb{B}_{i,m}(x) = 1 \quad \forall\ x_j \leqslant x < x_{j+1} \tag{1.10}$$

Definition 1.5 **(Extended partition).** *Let* Δ *be partition of the interval* $[a, b]$ *as in Definition 1.1. Suppose we have nodes* $x_{-m+1}, x_{-m+2}, \ldots, x_{-1}$ *and* $x_{n+1}, x_{n+2}, \ldots, x_{n+m-1}$ *such that*

$$x_{-m+1} \leqslant x_{-m+2} \leqslant \cdots \leqslant x_{-1} \leqslant x_0 \leqslant x_1 \cdots \leqslant x_n \leqslant x_{n+1} \leqslant x_{n+2} \leqslant \cdots x_{n+m-1} \tag{1.11}$$

then we call $\bar{\Delta} = \{x_i\}_{-m+1}^{n+m-1}$ *an extended partition associated with* $\mathbb{S}_m(\Delta)$.

Theorem 1.4. *Let* $\bar{\Delta} = \{x_i\}_{-m+1}^{n+m-1}$ *be an extended partition associated with* $\mathbb{S}_m(\Delta)$ *and suppose* $x_n < x_{n+m-1}$. *For* $i = -m+1, -m+2, \ldots, 1, 2, \ldots, n-1$, *let*

$$\mathbb{B}_{i,m}(x) = (x_{i+m} - x_i) \sum_{t=0}^{m} \frac{(x_{j+i} - x)_+^{m-1}}{\prod\limits_{\substack{l=0 \\ l \neq j}}^{m} (x_{j+i} - x_{l+i})}, \quad x_0 \leqslant x \leqslant x_n. \tag{1.12}$$

Then $\{\mathbb{B}_{i,m}\}_{-m+1}^{n-1}$ *forms a basis for* $\mathbb{S}_m(\Delta)$ *with*

$$\mathbb{B}_{i,m} = 0 \quad \textit{for } x \notin [x_i, x_{i+m}]$$

and

$$\mathbb{B}_{i,m}(x) > 0 \quad \textit{for } x \in (x_i, x_{i+m}).$$

In view of Theorem 1.4 we can now define any spline $s_m(x) \in \mathbb{S}_m(\Delta)$ uniquely as a linear combination of these basis elements i.e

$$s_m(x) = \sum_{-m+1}^{n-1} c_i \mathbb{B}_{i,m}(x). \tag{1.13}$$

The real numbers c_i are called the B-spline coefficients of s_m. The nodes in equation (1.11) are generally chosen as periodic or coincident. For periodicity we must have

$$\begin{aligned} x_{-i} &= x_{n-i} - b + a, \\ x_{n+i} &= x_i + b - a. \end{aligned} \tag{1.14}$$

Using equations (1.14) and (1.6) we have the following condition for periodicity

$$\mathbb{B}_{-i,m}(x) = \mathbb{B}_{n-i,m}(x + b - a), \quad i = 1, \ldots, m - 1 \tag{1.15}$$

Since the B-splines vanish outside their local support i.e $[x_i, x_{i+m}]$ the condition defined in equation (1.15) will be satisfied if and only if

$$c_{-i} = c_{n-i}, \quad i = 1, \ldots, m - 1. \tag{1.16}$$

1.2.2 Tensor Product Splines

In this section we introduce the *Tensor Product Splines* as an extension of the univariate B-spline representation. We require here only a few concepts concerning tensor products of vector spaces to define the tensor product polynomial spline. Mathematically we denote the tensor product of two vector spaces U and V as $U \otimes V$. Let us fix a field say $\mathbb{C}$ and let U and V be vector spaces defined over this field. Then we can have

Definition 1.6 (**Bilinear Mapping**). *A function f from $U \times V$ to the vector space P is said to be bilinear if it is linear in each of the two variables when the other is kept fixed.*

The space of all such bilinear functions forms a vector space over the field $\mathbb{C}$ under addition and scalar multiplication.

If U is a linear space of functions defined on some set X and V, a linear space of functions defined on some set Y into $\mathbb{R}$ then for each $u \in U$ and $v \in V$ the rule

$$w(x, y) = u(x)v(y), \quad \forall (x, y) \in X \times Y \tag{1.17}$$

defines a function on $X \times Y$ called the tensor product of u with v [1] and is denoted as $u \otimes v$. Further the set of all finite linear combinations of the functions on $X \times Y$ of the form $u \otimes v$ is called the tensor product of U with V. Hence

$$\begin{aligned} U \otimes V \quad = \quad & \left\{ \sum_{i=1}^{n} \alpha_i (u_i \otimes v_i) : \alpha_i \in \mathbb{R}, u_i \in U, v_i \in V, \right. && (1.18) \\ & \left. i = 1, \ldots, n; \text{ for some } n \right\} && (1.19) \end{aligned}$$

It can be verified that the tensor product defined above is bilinear, i.e., the map

$$U \times V \longrightarrow U \otimes V : (u, v) \longmapsto u \otimes v \tag{1.20}$$

is linear in each argument.

$$
\begin{aligned}
(\alpha_1 u_1 + \alpha_2 u_2) \otimes v &= \alpha_1 (u_1 \otimes v) + \alpha_2 (u_2 \otimes v) \\
u \otimes (\beta_1 v_1 + \beta_2 v_2) &= \beta_1 (u \otimes v_1) + \beta_2 (u \otimes v_2).
\end{aligned} \tag{1.21}
$$

Remark 1.1. *The tensor product discussed above forms a linear space of functions defined on $X \times Y$ and its dimension is given by the following proposition.*

Proposition 1.1 (Tensor Product Splines). *If U and V are some vector spaces defined over a field $\mathbb{C}$, then*

$$dim(U \otimes V) = (dimU)(dimV)$$

As mentioned earlier the tensor product we use here is simply an extension from the univariate case to the bivariate case and hence we will be considering two sets of partitions one along the horizontal and one along the vertical.

Definition 1.7 (Tensor product Splines). *Consider the strictly increasing sequences*

$$a = x_0 < x_1 < \cdots < x_n = b \tag{1.22}$$

and

$$c = y_0 < y_1 < \cdots < y_p = d \tag{1.23}$$

then the function $s(x, y)$ is called a bivariate(tensor product) spline on $R = [a, b] \times [c, d]$, of degree $m - 1 > 0$ in x and $l - 1 > 0$ in y with respect to the partition along the horizontal and the vertical as defined in equations (1.22)-(1.23) *if the following conditions are satisfied*

1. *On each subrectangle* $R_{i,j} = [x_i, x_{i+1}] \times [y_i, y_{i+1}], s(x, y)$ *is given by a polynomial of degree* $m-1$ *in* x *and* $l-1$ *in* y.

$$s_{|R_{i,j}} \in \mathbb{P}_m \otimes \mathbb{P}_l, \quad i = 0, 1, \ldots, n-1; \quad j = 0, 1, \ldots, p-1.$$

2. *The function* $s(x, y)$ *and all its partial derivatives*

$$\frac{\partial^{i+j} s(x, y)}{\partial x^i \partial y^j} \in \mathbb{C}(R), \quad i = 0, 1, \ldots, m-2; \quad j = 0, 1, \ldots, l-2.$$

For our particular case $m = l = 4$ and the dimension of the vector space $\mathbb{S}(\Delta_1, \Delta_2)$ of all functions satisfying the above two conditions is $dim(\mathbb{S}(\Delta_1, \Delta_2)) = (m + n - 1) \times (l + p - 1)$ where $\Delta_i,\ i = 1, 2$ are the partitions along the horizontal and the vertical and are defined as $\Delta_1 = \{x_i\}_0^n$ and $\Delta_2 = \{y_j\}_0^p$ respectively. We can define an extended partition as in Definition 1.5 and because of the tensor product nature of the vector space it is possible to work with the one dimensional B-spline basis. This gives us the unique tensor product representation of a spline $s(x, y)$ in terms of its basis functions

$$s(x, y) = \sum_{i=-m+1}^{n-1} \sum_{j=-l+1}^{p-1} c_{i,j} \mathbb{B}_{i,m}(x) \mathbb{B}_{j,l}(y) \tag{1.24}$$

where $\mathbb{B}_{i,m}(x)$ and $\mathbb{B}_{j,l}(y)$ are the normalized B-splines defined on the partitions Δ_1 and Δ_2 respectively. For the approximating function to be periodic in both $x(horizontal)$ and $y(vertical)$ direction we have

$$\begin{aligned} \frac{\partial^i s(a, y)}{\partial x^i} &= \frac{\partial^i s(b, y)}{\partial x^i}, \quad i = 0, \ldots, m-2; \quad c \leqslant y \leqslant d \\ \frac{\partial^j s(x, c)}{\partial y^j} &= \frac{\partial^i s(x, d)}{\partial y^j}, \quad j = 0, \ldots, l-2; \quad a \leqslant x \leqslant b. \end{aligned} \tag{1.25}$$

1.2.3 Error Estimates

We present here a priori error bounds for the interpolation procedures introduced in Sections 1.2.1 and 1.2.2. The derivation of the interpolation error, $f - \mathcal{S}_f$ and its derivatives in L_2 norm and the L_∞ norm are given. Here $\mathcal{S}_f$ denotes the approximation to the function f. It is observed that if the function f is sufficiently smooth, $\mathcal{S}_f$ is a fourth order approximation to f in both L_2 and L_∞.

Theorem 1.5 (**Variational Problem**). *Let Δ and $f \equiv \{f_0, f_1, \ldots, f_n, f_a^{'}, f_b^{'}\}$ be given and $V \equiv \{w \in \mathbb{PC}_2^2(I) \mid w(x_i) = f(i), \quad 0 \leqslant i \leqslant n$ and $\mathcal{D}w(x_i) = f_i^{'}, \quad i = 0$ and $n\}$. The variational problem of finding the functions $p \in V$ which minimize $||\mathcal{D}^2 w||_2^2$ over all $w \in V$ has the unique solution $\mathcal{S}_f$.*

The function $p \in V$ is a solution of the variational problem if and only if

$$\left(\mathcal{D}^2 p, \mathcal{D}^2 \delta\right)_2 = 0$$

for all $\delta \in V_0 \equiv \{w \in \mathbb{PC}_2^2(I) \mid w(x_i) = 0, 0 \leqslant i \leqslant n, \text{and } \mathcal{D}w(x_i) = 0, \ i = 0 \text{ and } n\}$. By definition we have

$$||\mathcal{D}^2 p + \mathcal{D}^2 \delta||_2^2 = (\mathcal{D}^2 p, \mathcal{D}^2 p)_2 + 2(\mathcal{D}^2 p, \mathcal{D}^2 \delta)_2 + (\mathcal{D}^2 \delta, \mathcal{D}^2 \delta)_2. \tag{1.26}$$

By the orthogonality condition we have

$$||\mathcal{D}^2 p + \mathcal{D}^2 \delta||_2^2 = (\mathcal{D}^2 p, \mathcal{D}^2 p)_2 + (\mathcal{D}^2 \delta, \mathcal{D}^2 \delta)_2, \tag{1.27}$$

and this would gives us the following corollary

Corollary 1.1 (**First integral relation**). *If $f \in \mathbb{PC}_2^2(I)$, then*

$$||\mathcal{D}^2 f||_2^2 = ||\mathcal{D}^2 \mathcal{S}_f||_2^2 + ||\mathcal{D}^2 \mathcal{S}_f - \mathcal{D}^2 f||_2^2. \tag{1.28}$$

Theorem 1.6 (**Preliminary Result**). *If* $f \in \mathbb{PC}_2^2(I)$, *then*

$$\begin{aligned} ||\mathcal{D}^2(f - \mathcal{S}_f)||_2 &\leqslant ||\mathcal{D}^2 f||_2 && (1.29) \\ ||\mathcal{D}(f - \mathcal{S}_f)||_2 &\leqslant 2\pi^{-1} h ||\mathcal{D}^2 f||_2 && (1.30) \\ ||(f - \mathcal{S}_f)||_2 &\leqslant 2\pi^{-2} h^2 ||\mathcal{D}^2 f||_2 && (1.31) \end{aligned}$$

Inequality (1.29) follows directly from Corollary 1.1. We can see from the above theorem that $\mathcal{S}_f$ is a second-order approximation to f. Intuitively we may assume that a smoother function will result in a higher order of convergence and hence we have the following theorem.

Theorem 1.7. *If* $f \in \mathbb{PC}_4^\infty$, *then*

$$||f - \mathcal{S}_f||_\infty \leqslant \frac{5}{584} h^4 ||\mathcal{D}^4 f||_\infty. \tag{1.32}$$

Moreover if $f \in \mathbb{C}^5(I)$ *and* Δ *is a uniform partition, then*

$$||f - \mathcal{S}_f||_\infty \leqslant h^4 (\frac{1}{384} ||\mathcal{D}^4 f||_\infty + \frac{1}{240} h ||\mathcal{D}^5 f||_\infty) \tag{1.33}$$

Proof. Since $f - \mathcal{S}_f = f - \mathcal{S}_{\hbar} + \mathcal{S}_{\hbar} - \mathcal{S}_f$ where $\mathcal{S}_{\hbar}$ is the cubic Hermite interpolate of f, we have

$$||f - \mathcal{S}_f||_\infty \leqslant ||f - \mathcal{S}_{\hbar}||_\infty + ||\mathcal{S}_{\hbar} - \mathcal{S}_f||_\infty. \tag{1.34}$$

Now

$$||f - \mathcal{S}_{\hbar}||_\infty \leqslant \frac{1}{384} h^4 ||\mathcal{D}^4 f||_\infty \tag{1.35}$$

and

$$\begin{aligned} \mathcal{S}_{\hbar} - \mathcal{S}_f &= \sum_{i=1}^{n} e_i^{'} h_i^{'}(x), \quad \text{hence} \\ ||\mathcal{S}_{\hbar} - \mathcal{S}_f||_\infty &= ||\sum_{i=1}^{n} e_i^{'} h_i^{'}(x)||_\infty \\ &\leqslant ||\mathbf{e}^{'}||_\infty \Big\{ \sum_{i=1}^{n} h_i^{'}(x) \Big\}, \end{aligned} \tag{1.36}$$

where

$$||\mathbf{e}^{'}||_\infty \equiv \max_{1 \leqslant i \leqslant n} |e_i^{'}| \leqslant \frac{1}{24} h^3 ||\mathcal{D}^4 f||_\infty. \tag{1.37}$$

Now, since

$$|h_i^{'}| + |h_{i+1}^{'}| \leqslant \frac{h}{4} \text{ for all } x \in [x_i, x_{i+1}] \text{ and } 0 \leqslant i \leqslant n, \tag{1.38}$$

we have from equation (1.37)

$$||\sum_{i=1}^{n} e_i^{'} h_i^{'}(x)||_\infty \leqslant \frac{1}{96} h^4 ||\mathcal{D}^4 f||_\infty, \tag{1.39}$$

which gives us the required result

$$||f - \mathcal{S}_f||_\infty \leqslant \frac{1}{384} h^4 ||\mathcal{D}^4 f||_\infty + \frac{1}{96} h^4 ||\mathcal{D}^4 f||_\infty = \frac{5}{584} h^4 ||\mathcal{D}^4 f||_\infty. \tag{1.40}$$

□

We proceed now to the error bounds of the bivariate interpolation procedure and find that as in case of the the univariate splines the approximation $\mathcal{S}_f$ for a sufficiently smooth function is fourth order accurate in both the L_2 and the L_∞-norm.

Theorem 1.8. *If* $f \in \mathbb{PC}_4^2(U)$, *then*

$$||f - \mathcal{S}_f||_2 \leqslant 4\pi^{-4} \Big(h^4 ||\mathcal{D}_x^4 f||_2 + h^2 k^2 ||\mathcal{D}_x^2 \mathcal{D}_y^2 f||_2 + k^4 ||\mathcal{D}_y^4 f||_2 \Big). \tag{1.41}$$

The above theorem gives us the error bound in L_2-norm. For the error bound in the L_∞-norm we have the following theorem

Theorem 1.9. *If* $f \in \mathbb{PC}_4^\infty(U)$, *then*

$$||f - \mathcal{S}_f||_\infty \leqslant \frac{5}{384}h^4||\mathcal{D}_x^4 f||_\infty + \frac{4}{9}h^2k^2||\mathcal{D}_x^2\mathcal{D}_y^2 f||_\infty + \frac{5}{384}k^4||\mathcal{D}_y^4 f||_\infty, \tag{1.42}$$

which clearly shows that the approximating spline is fourth order accurate.

Chapter 2

Radial Projection

The radial projection method which we describe here (see [4], p. 24) is a method to radially project the points on the surface of the cube onto the sphere. Since the tensor product B-splines are restricted to rectangular domains all calculations will essentially be done on the cube.

The terminology used here will run as follows: The surface of the cube will be termed as the box $\mathcal{B}_d$ centered at the origin and of side length $2d$. We will denote the sphere centered at the origin with radius r as $\mathcal{S}_r$. Mathematically we may represent the box and the sphere as follows

$$\begin{aligned} \mathcal{B}_d &= \{x \mid x(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, ||x||_\infty = d\} \\ \mathcal{S}_r &= \{a \mid a(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n, ||a||_2 = r\} \end{aligned}$$

The radial projection from the box to the sphere is defined as a mapping

$$\mathcal{P} : \mathcal{B}_d \longrightarrow \mathcal{S}_r$$

given by

$$\mathcal{P}(x) = r\frac{x}{||x||} = a,$$

where as the inverse mapping from the sphere to the box is given by

$$\mathcal{P}^{-1}(a) = d\frac{a}{||a||_\infty} = x.$$

2.1 Characteristics

The radial projection $\mathcal{P}$ is a one-one mapping from the box $\mathcal{B}_d$ to the sphere $\mathcal{S}_r$. We mention below some related properties. The following lemma shows that the mapping $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$ are both Lipschitz continuous.

Lemma 2.1. *The radial projection $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$ satisfy the inequalities*

$$||\mathcal{P}(x) - \mathcal{P}(y)|| \leqslant \frac{2r}{||x||}\Big(||x-y||\Big), \tag{2.1}$$

$$||\mathcal{P}^{-1}(a) - \mathcal{P}^{-1}(b)||_\infty \leqslant \frac{2d}{||a||_\infty}\Big(||a-b||_\infty\Big). \tag{2.2}$$

Proof.

$$\begin{aligned}
||\mathcal{P}(x) - \mathcal{P}(y)|| &= ||\frac{r}{||x||}x - \frac{r}{||y||}y|| \\
&= \frac{r}{||x||\ ||y||}\Big\{||(||y||x - ||x||y)||\Big\} \\
&= \frac{r}{||x||\ ||y||}\Big\{||\Big(||y||(x-y) - y(||y|| - ||x||)\Big)||\Big\} \\
&\leqslant \frac{r}{||x||\ ||y||}\Big\{||y||\ ||x-y|| + ||y||\Big|||y|| - ||x||\Big|\Big\} \\
&\leqslant \frac{2r}{||x||}\Big\{||x-y||\Big\}
\end{aligned}$$

In a similiar way we can prove inequality (2.2) □

Corollary 2.1. *The radial projection $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$, are globally Lipscitz continuous mappings, that is,*

$$||\mathcal{P}(x) - \mathcal{P}(y)|| \leqslant \frac{2r}{d}||x-y||, \tag{2.3}$$

and

$$||\mathcal{P}^{-1}(a) - \mathcal{P}^{-1}(b)||_\infty \leqslant \frac{2dn}{r}||a-b||_\infty. \tag{2.4}$$

This follows directly from the lemma 2.1 by observing that for any $x \in \mathbb{R}^n$, $||x||_\infty \leqslant x \leqslant \sqrt{n}||x||_\infty$ and that for any $a \in \mathcal{S}_r$, $\frac{r}{\sqrt{n}} \leqslant ||a||_\infty \leqslant r$. For any $x \in \mathcal{B}_d$, we have $x \in \mathcal{B}_d$, $d \leqslant x \leqslant d\sqrt{n}$.

Chapter 3

Analysis

In this chapter we discuss the construction of *Tensor Product Splines* as a natural extension of the B-spline representation of a spline. We also estimate the difference $(f - s_m(x))$ where f is the function defined at the nodal points and $s_m(x)$ is the approximating spline. Univariate spline representation is analyzed in Section 3.1. In Section 3.2 we analyze the B-spline representation of a spline and in Section 3.3 extend the construction to Bivariate splines and analyze it.

3.1 Univariate Cubic Spline Interpolation

Let us consider a partition Δ of an interval $[a, b]$ as defined in Definition 1.1 with $a \equiv x_0 = x_n \equiv b$ and the corresponding function evaluations at the nodal points $f_i,\ i = 0, 1, \ldots, n-1$. Our aim here will be to develop an efficient method to construct a periodic cubic spline interpolating the function values at the distinct nodal points [5]. Since the degree of the spline is $m - 1 = 3$ the spline must be twice continuously differentiable i.e the second order derivative must be continuous. We introduce here the following notations

$$f_i = s_4(x_i), \quad m = s_4^{'}(x_i), \ \text{ and } M = s_4^{''}(x_i), \quad i = 0, 1, \ldots, n-1.$$

Due to the periodic consideration we have $f_{n+j} = f_j$ and $M_{n+j} = M_j$ for $j = 0, 1$. Since $s_{4,i-1} \in \mathbb{P}_4,\ s_{4,i-1}^{''}$ is linear and

$$s_{4,i-1}^{''}(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i} \quad \text{for } x \in [x_{i-1}, x_i] \tag{3.1}$$

where $h_i = x_i - x_{i-1},\ i = 1, \ldots, n$. Integrating (3.1) twice we get

$$s_{4,i-1}(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + C_{i-1}(x - x_{i-1}) + \tilde{C}_{i-1} \qquad (3.2)$$

and the constants C_{i-1} and $\tilde{C}_{i-1}$ are determined by imposing the end point values $s_4(x_{i-1}) = f_{i-1}$ and $s_4(x_i) = f_i$.This gives us, for $i = 1, \ldots, n$

$$\tilde{C}_{i-1} = f_{i-1} - M_{i-1}\frac{h_i^2}{6}, \quad C_{i-1} = \frac{f_i - f_{i-1}}{h_i} - \frac{h_i}{6}(M_i - M_{i-1}). \qquad (3.3)$$

Imposing the continuity of the first derivatives at x_i, we get

$$\begin{aligned} s_4^{'}(x_i^-) = & \quad \frac{h_i}{6}M_{i-1} + \frac{h_i}{3}M_i + \frac{f_i - f_{i-1}}{h_i} \\ = & \quad -\frac{h_{i+1}}{3}M_i - \frac{h_{i+1}}{6}M_{i+1} + \frac{f_{i+1} - f_i}{h_{i+1}} \\ = & \quad s_4^{'}(x_i^+), \end{aligned}$$

where

$$s_4^{'}(x_i^-) = \lim_{t \to 0} s_4^{'}(x \pm t).$$

This gives us the following linear system also know as the M-continuity system

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, \ldots, n \qquad (3.4)$$

where

$$\begin{aligned} \mu_i &= \frac{h_i}{h_i + h_{i+1}}, \quad \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} \\ d_i &= \frac{6}{h_i + h_{i+1}}\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right) \quad i = 1, \ldots, n. \end{aligned}$$

It is clear from (3.2) that the only unknows are $M_0, M_1, \ldots, M_{n-1}$. So our task of finding a periodic cubic spline representation interpolating the given function values now reduces to solving the linear system (3.4) of n equations and n unknowns. This construction of the spline produces a system tridiagonal in nature. In matrix notation it is represented as

$$\begin{pmatrix} 2 & \mu_{n-1} & 0 & \dots & \lambda_{n-1} \\ \lambda_{n-2} & 2 & \mu_{n-2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \lambda_1 & 2 & \mu_1 \\ \mu_n & 0 & 0 & \lambda_n & 2 \end{pmatrix} \begin{pmatrix} M_{n-1} \\ M_{n-2} \\ \vdots \\ M_1 \\ M_0 \end{pmatrix} = \begin{pmatrix} d_{n-1} \\ d_{n-2} \\ \vdots \\ d_1 \\ d_0 \end{pmatrix} \tag{3.5}$$

and can be easily solved on a computer using existing techniques.

3.1.1 Radial Projection: The One Dimensional Case

In the above discussion we have considered the spline to be periodic in nature as we want to apply the construction on a circle C with radius $r = 1$ such that the first and last nodal point coincide i.e $x_0 = x_n$. The radial projection of the nodal points on the square onto a unit circle involves basic geometry. Let the four corners, of the square under consideration be $P_1(1, 1), P_2(-1, 1), P_3(-1, -1), P_4(1, -1)$ in this order. Let $P(x, y)$ be any point on the side of the square, say the side joining P_1 and P_2. Let $P^{'}(x^{'}, y^{'})$ be the radial projection of the point $P(x, y)$ on the circle. Then we have the relation

$$\tan\theta = \frac{y^{'}}{x^{'}} = \frac{y}{x} \tag{3.6}$$

where θ is the angle between the X-axis and the vector joining the point P to the origin.

Since the point $P^{'}$ lies on the circle we have

$$y^{'2} + x^{'2} = 1. \tag{3.7}$$

Solving for $x^{'}$ and $y^{'}$ we have

$$\begin{aligned} x^{'} &= \frac{x}{\sqrt{x^2+y^2}} \\ y^{'} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}$$

Taking points systematically anticlockwise on the square starting at P_1 and then projecting them onto the circle we have points on the circle which we now label as $x_i,\ i = 0, 1, \ldots, n-1$. This provides us with the setting required to apply the univariate spline constructed in Section 3.1. However we are still required to find the interval lengths $h_i,\ i = 0, 1, \ldots, n-1$ between the nodal points. To do this we make use of the inner product of two vectors. Let $\vec{x}_i$ and $\vec{x}_{i-1}$ be two vectors then

$$\cos\alpha_i = \frac{< \vec{x}_i, \vec{x}_{i-1} >}{||\vec{x}_i||\ ||\vec{x}_{i-1}||}, \tag{3.8}$$

where α_i as the angle between two vectors labelled here as $\vec{x}_i$ and $\vec{x}_{i-1}$. Hence the arc length or the interval length $h_i = r\alpha_i$.

Table 3.1: The following table shows the error and the observed rate of convergence for various step sizes for the function $f(\theta) = \sin(\theta)$.

Step Size	Error	Order of Convergence
$2.243994752564133e-01$	$1.521985704066554e-04$	
$1.047197551196593e-01$	$7.326907390286181e-06$	$3.980413480161499e+00$
$5.067084925144792e-02$	$4.029381312631907e-07$	$3.995561255766796e+00$
$2.493327502848618e-02$	$2.703779530789965e-08$	$3.809570269779505e+00$
$1.236847501412775e-02$	$1.903539716356657e-09$	$3.785053304137925e+00$

The experimentally observed order of convergence is defined as

$$p = \frac{\ln\left(\frac{Er_{j+1}}{Er_j}\right)}{\ln\left(\frac{h_{j+1}}{h_j}\right)}\ ,$$

where Er_j denotes the error at the j^{th} refinement and is defined as

$$Er_j = ||h_x(f - s_4(x))||_2 \ ,$$

and h_x represents the step size for the intermediate points taken to test the spline function developed.

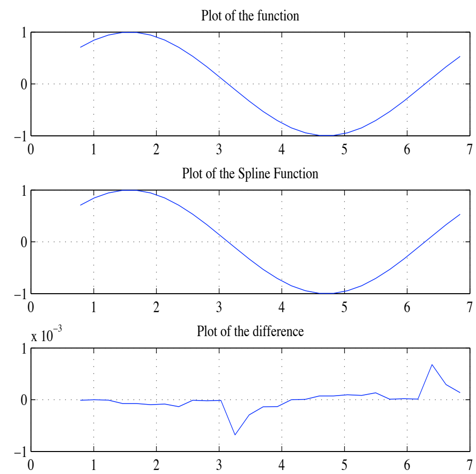

Figure 3.1: Approximation of the function $f(\theta) = \sin\theta, N = 28$

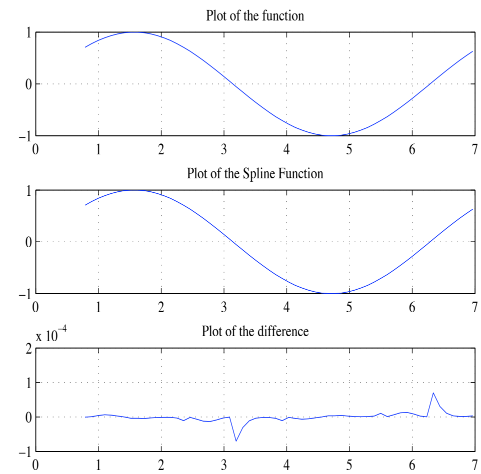

Figure 3.2: Approximation of the function $f(\theta) = \sin\theta, N = 60$

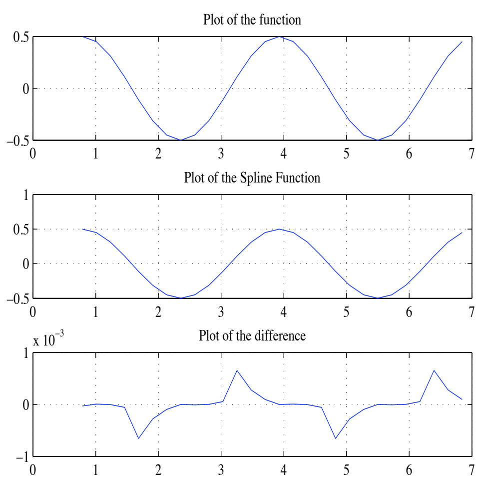

Figure 3.3: Approximation of the function $f(\theta) = \sin\theta\cos\theta, N = 28$

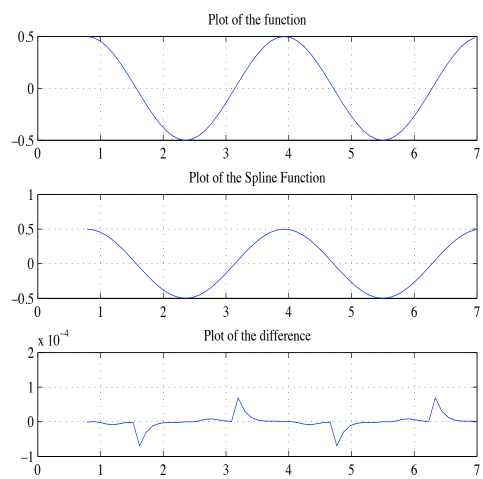

Figure 3.4: Approximation of the function $f(\theta) = \sin\theta\cos\theta, N = 60$

3.1.2 Periodic Splines on a Square

Defining periodic splines on a square is based again on the construction given in Section 3.1. However in this case the nodal points on the square are not projected on a circle. Hence with equally spaced nodes on the edges of the square i.e $h_i = h_{i+1}$ we have from equation (3.4)

$$\begin{aligned} \mu_i &= \frac{1}{2}, \quad \lambda_i = \frac{1}{2} \\ d_i &= \frac{3}{h_i^2}(f_{i+1} - 2f_i + f_{i-1}), \quad i = 1, \ldots, n. \end{aligned}$$

Table 3.2: The following table shows the error and the observed rate of convergence for various step sizes for the function $f(\theta) = \sin^3(\theta)$.

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$2.993908237928666e-02$	
$5.000000000000000e-01$	$3.738564612692465e-03$	$3.001473631942863e+00$
$2.500000000000000e-01$	$1.044464521796866e-04$	$5.161649073884993e+00$
$1.250000000000000e-01$	$4.952085825195140e-06$	$4.398583359488427e+00$
$6.250000000000000e-02$	$2.868804616210702e-07$	$4.109514698407482e+00$
$3.125000000000000e-02$	$1.758372110067624e-08$	$4.028137400984083e+00$
$1.562500000000000e-02$	$1.093598905987019e-09$	$4.007084798744635e+00$

Table 3.3: The following table shows the error and the observed rate of convergence for various step sizes for a function $f \notin \mathbb{C}^1[a,b]$

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$8.728715607973290e-03$	
$5.000000000000000e-01$	$8.728715607973281e-03$	$1.441541926716714e-15$
$2.500000000000000e-01$	$2.822924332662890e-03$	$1.628578924802774e+00$
$1.250000000000000e-01$	$9.724452119338235e-04$	$1.537501583136462e+00$
$6.250000000000000e-02$	$3.437050954186293e-04$	$1.500445730420506e+00$
$3.125000000000000e-02$	$1.215162841762690e-04$	$1.500021580057984e+00$
$1.562500000000000e-02$	$4.295235806309275e-05$	$1.500340417834547e+00$

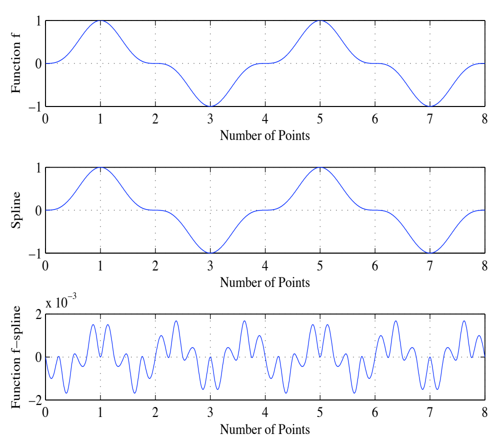

Figure 3.5: Approximation of the function $f(\theta) = \sin^3\theta, h = 2.5 \times 10^{-1}$

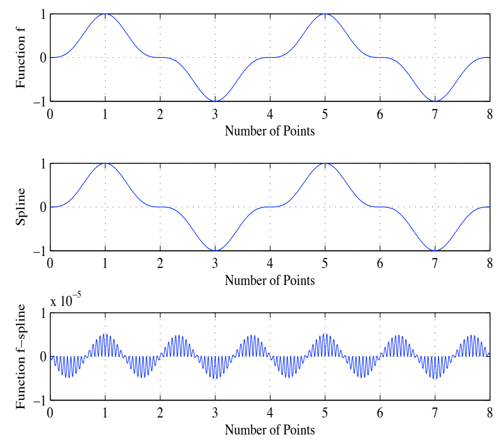

Figure 3.6: Approximation of the function $f(\theta) = \sin^3\theta, h = 6.25 \times 10^{-2}$

It is evident from the Figure 3.8 below that the function $f \notin \mathbb{C}^1[a, b]$ and hence the observed order of convergence. In view of the approximation power of splines [3] we may expect that the order of approximation attainable will increase with the smoothness of the class of functions $\mathbb{F}$ being approximated. However this is true only up to a limit. In fact if $\mathbb{F} \cap \mathbb{P}_m = \emptyset$, then the maximal order of convergence possible for the class $\mathbb{F}$ is Δ^m, no matter how smooth $\mathbb{F}$ is assumed. In our case $m = 4$. This is known as the saturation result.

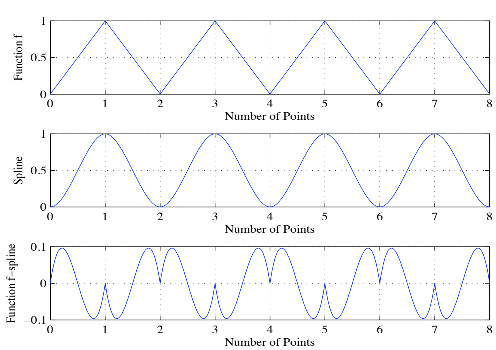

Figure 3.7: Approximation of a function $f \notin \mathbb{C}^1[a, b], h = 1$

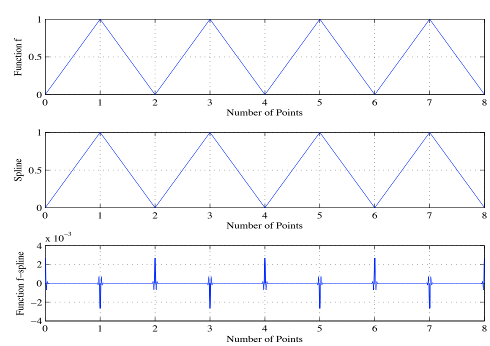

Figure 3.8: Approximation of a function $f \notin \mathbb{C}^1[a, b], h = 1.5625 \times 10^{-2}$

3.2 B-spline representation on a Square

Our aim here is to construct a cubic spline which is represented as a linear combination of the $B-splines$ $\mathbb{B}_{i,m}(x)$ as in Section 1.4. We consider the extended partition $\bar{\Delta} = \{x_i\}_{-m+1}^{n+m-1}$ of the interval $[a, b]$ as defined in the Definition 1.5. Using the periodicity condition available to us through the equations (1.14),(1.15) and (1.16) we have a system of linear equations available to us which we can represent in a matrix form. We denote the matrix of all basis functions as B. Using the properties of the B-spline and noting that the spline function must agree with the function values at the nodal points $x_i,\ i = 0, 1, \ldots, n-1$ we arrive at the following matrix representation of the linear system,

$$BC = F,$$

which implies $C = B^{-1}F$.

$$B = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \mathbb{B}^0_{-3} & \mathbb{B}^0_{-2} & \mathbb{B}^0_{-1} \\ \mathbb{B}^1_0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \mathbb{B}^1_{-2} & \mathbb{B}^1_{-1} \\ \mathbb{B}^2_0 & \mathbb{B}^2_1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \mathbb{B}^2_{-1} \\ \mathbb{B}^3_0 & \mathbb{B}^3_1 & \mathbb{B}^3_2 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \mathbb{B}^4_1 & \mathbb{B}^4_2 & \mathbb{B}^4_3 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \mathbb{B}^{n-3}_{n-6} & \mathbb{B}^{n-3}_{n-5} & \mathbb{B}^{n-3}_{n-4} & 0 & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & * & \mathbb{B}^{n-2}_{n-4} & \mathbb{B}^{n-2}_{n-3} & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & * & \mathbb{B}^{n-1}_{n-3} & \mathbb{B}^{n-1}_{n-2} & 0 \end{pmatrix} \qquad (3.9)$$

The matrix B has as its entries B-splines evaluated at the nodal points where $\mathbb{B}^j_i$ represents $\mathbb{B}_{i,m}(x_j),\quad i = -m+1, -m+2, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$. We have the zero entries in the above matrix because the B-spline $\mathbb{B}_{i,m}(x)$ vanishes outside its local support $[x_i, x_{i+m}]$.

The vector of all coefficients is represented as C and the vector of all function values at the nodal points is represented as F, then

$$C = \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ \vdots \\ C_{n-3} \\ C_{n-2} \\ C_{n-1} \end{pmatrix} \text{ and } F = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \vdots \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \end{pmatrix} \tag{3.10}$$

The problem of representing the cubic spline as a linear combination of the basis functions is numerically equivalent to the problem of finding the B-spline coefficients, i.e., the vector C. Since the matrix B and the vector F is completely known we can easily solve for C on a digital computer.

Table 3.4: The following table shows the error and the observed rate of convergence for various step sizes for the function $f(\theta) = \sin(\frac{\pi}{4}\theta)$.

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$3.412826951770349e-03$	
$5.000000000000000e-01$	$9.534612985020590e-05$	$5.161649073886882e+00$
$2.500000000000000e-01$	$4.520615188600750e-06$	$4.398583359486799e+00$
$1.250000000000000e-01$	$2.618848335825963e-07$	$4.109514698295172e+00$
$6.250000000000000e-02$	$1.605166782410018e-08$	$4.028137400685653e+00$
$3.125000000000000e-02$	$9.983146507216004e-10$	$4.007084797322440e+00$

As we are approximating a function defined on a square of edge length 2, the approximating spline will have a periodicity of $b-a$ which in our case is 8. Hence the factor of $\frac{\pi}{4}$ in the function $f(\theta) = \sin\frac{\pi}{4}\theta$.

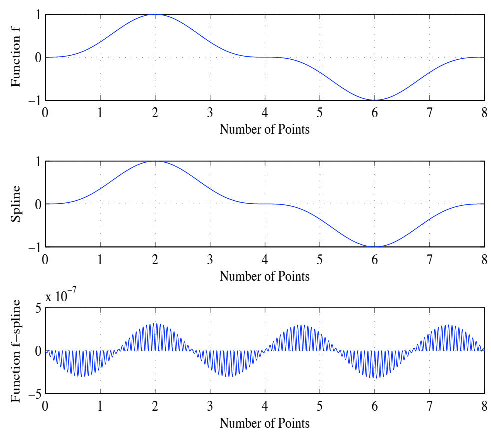

Figure 3.9: Approximation of a function $f(\theta) = \sin\frac{\pi}{4}\theta, h = 6.25 \times 10^{-2}$

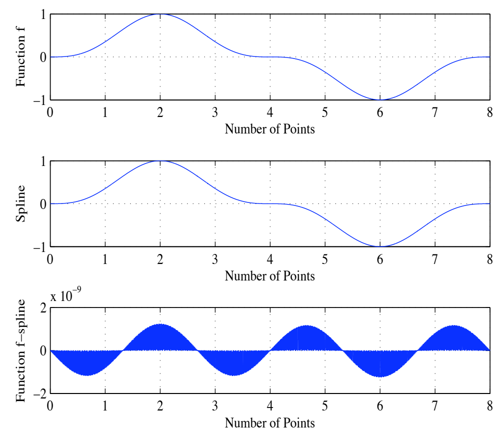

Figure 3.10: Approximation of a function $f(\theta) = \sin\frac{\pi}{4}\theta, h = 1.56\times10^{-2}$

Table 3.5: The following table shows the error and the observed rate of convergence for various step sizes for the function $f \notin \mathbb{C}^1[a, b]$

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$7.968190728250351e-03$	
$5.000000000000000e-01$	$7.968190728250389e-03$	$-7.047538308392802e-15$
$2.500000000000000e-01$	$2.576965563447752e-03$	$1.628578922185994e+00$
$1.250000000000000e-01$	$8.877169929217066e-04$	$1.537501539184818e+00$
$6.250000000000000e-02$	$3.137585531721901e-04$	$1.500445025990139e+00$
$3.125000000000000e-02$	$1.109295998711029e-04$	$1.500010409963090e+00$
$1.562500000000000e-02$	$3.921497851505102e-05$	$1.500167662913683e+00$

3.3 Tensor Product Splines

In this section we will construct the tensor product spline first on a square patch and then extend this notion to the cube treating each face of the cube as a square patch. This will give us a method to approximate any function posed on the sphere with the help of the mapping $\mathcal{P}$ and $\mathcal{P}^{-1}$ from the surface of the cube to the surface of the sphere and back respectively.

From equation (1.24) we have the unique tensor product representation of a spline defined over a rectangle $[a, b] \times [c, d]$ with respect to the extended partition $\bar{\Delta}_1$ and $\bar{\Delta}_2$ given by Definition 1.5 in the horizontal and vertical directions respectively. Keeping in mind that we do not want periodicity on a single patch but rather across the four faces of the cube we do not implement the periodic conditions for the single patch which we create. The cube in question is the box $\mathcal{B}_d$ as defined in Chapter 2. It is centered at the origin with $d = 1$.

It is clear from the above setup that that the total number of coefficients that need to be determined are $(n+3) \times (p+3)$ since $m = l = 4$. However we only have the function values at the nodal points $(x_i, y_j),\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, p$. Hence we still need $2n + 2p + 8$ conditions to have a unique spline representation on the patch. These extra conditions are given as restrictions on the derivatives of the spline at the boundary and the corner points of the grid formed by the partitions Δ_1 and Δ_2.

We give here a brief list of the boundary conditions generally associated with the tensor product cubic splines.

Boundary conditions of the first type

$$\begin{aligned}
\frac{\partial s}{\partial x}(x_i, y_j) &= f^x_{ij}, \quad i = 0, n; \quad j = 0, 1, \ldots, p \\
\frac{\partial s}{\partial y}(x_i, y_j) &= f^y_{ij}, \quad i = 0, 1, \ldots, n; \quad j = 0, p \\
\frac{\partial^2 s}{\partial x \partial y}(x_i, y_j) &= f^{xy}_{ij}, \quad i = 0, n; \quad j = 0, p
\end{aligned} \tag{3.11}$$

The total number of first boundary conditions here is $2n + 2p + 8$.

Boundary conditions of the second type

$$
\begin{aligned}
\frac{\partial^2 s}{\partial x^2}(x_i, y_j) &= f_{ij}^x, \quad i = 0, n; \quad j = 0, 1, \ldots, p \\
\frac{\partial^2 s}{\partial y^2}(x_i, y_j) &= f_{ij}^y, \quad i = 0, 1, \ldots, n; \quad j = 0, p \\
\frac{\partial^4 s}{\partial x^2 \partial y^2}(x_i, y_j) &= f_{ij}^{xy}, \quad i = 0, n; \quad j = 0, p
\end{aligned} \tag{3.12}
$$

The total number of second boundary conditions is again $2n + 2p + 8$.

Boundary conditions of the third type

Boundary conditions of the third type are called periodic boundary conditions. Periodicity with respect to the horizontal variable must be $P_x = b - a$ and with respect to the vertical variable must be $P_y = d - c$. For our particular case we must have $P_x = P_y = 8$

$$
\begin{aligned}
s(x_0, y_j) &= s(x_n, y_j), \quad j = 0, 1, \ldots, p \\
s(x_i, y_0) &= s(x_i, y_p), \quad i = 0, 1, \ldots, n \\
\frac{\partial^k s}{\partial x^k}(x_0, y_j) &= \frac{\partial^k s}{\partial x^k}(x_n, y_j), \quad j = 0, 1, \ldots, p, \quad k = 1, 2 \\
\frac{\partial^l s}{\partial y^l}(x_i, y_0) &= \frac{\partial^l s}{\partial y^l}(x_i, y_p), \quad i = 0, 1, \ldots, n, \quad l = 1, 2 \\
\frac{\partial^{2k} s}{\partial x^k \partial y^k}(x_0, y_j) &= \frac{\partial^k s}{\partial x^k \partial y^k}(x_n, y_j), \quad j = 0, 1, \ldots, p, \quad k = 1, 2 \\
\frac{\partial^{2k} s}{\partial x^k \partial y^k}(x_i, y_0) &= \frac{\partial^{2k} s}{\partial x^k \partial y^k}(x_i, y_p), \quad i = 0, 1, \ldots, n, \quad k = 1, 2
\end{aligned} \tag{3.13}
$$

Applying the boundary conditions of the first type in order to determine the coefficients we end up with a system of linear equations which in matrix form may be represented as

$$
\begin{aligned}
MCN &= F \\
C &= M^{-1} F N^{-1}
\end{aligned} \tag{3.14}
$$

where $M_{(p+3)\times(p+3)}$ is the matrix of B-splines in the vertical direction, $N_{(n+3)\times(n+3)}$ is the matrix of B-splines in the horizontal direction, $C_{(p+3)\times(n+3)}$ is the coefficient matrix and $F_{(p+3)\times(n+3)}$ is the matrix of all function values at the nodal points in the grid along with its derivatives at the boundary and corner points.

Table 3.6: The following table shows the error and the observed rate of convergence for various step sizes for the function $f(x,y) = x^6y^6$

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$7.060256060377759e+00$	
$5.000000000000000e-01$	$9.550664958381095e-01$	$2.886047419513821e+00$
$2.500000000000000e-01$	$5.683782321068801e-02$	$4.070677975228285e+00$
$1.250000000000000e-01$	$3.408501271897639e-03$	$4.059641876435491e+00$
$6.250000000000000e-02$	$2.095067852178469e-04$	$4.024068647573170e+00$

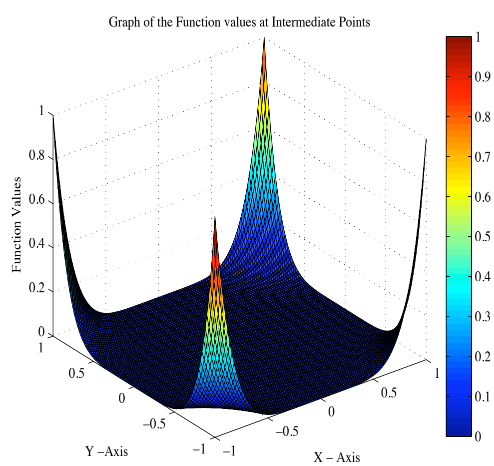

Figure 3.11: Function $f(x,y) = x^6y^6$

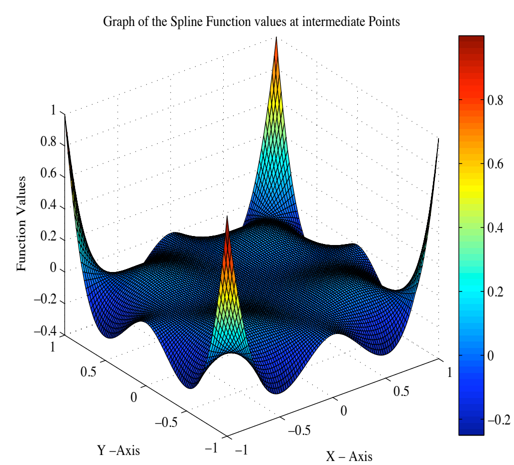

Figure 3.12: Approximation of a function $f(x,y) = x^6y^6, h = 1$

To obtain the the tensor product spline representation on the cube we note that spline function must be periodic with periodicity along x-direction P_x, y-direction P_y and z-direction P_z and $P_x = P_y = P_z = 8$. Even though the tensor products are defined on individual patches, the spline function must be periodic across the four adjacent faces of the cube. Let us consider here the tensor product defined on a

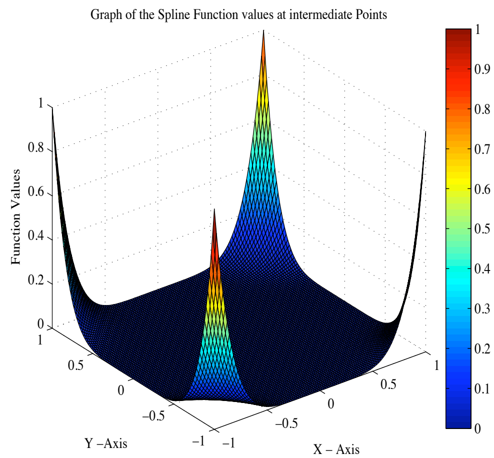

Figure 3.13: Approximation of a function $f(x,y) = x^6y^6, h = 1.25 \times 10^{-1}$

single patch. Then to obtain periodicity along the horizontal we need to wrap the fictitious nodes $u_{-i}, i = -m+1, \ldots, -1$ and $u_{n+i}, i = 1, \ldots, m-1$ along the horizontal so that they coincide with the nodal points on the adjacent sides. For periodicity along the vertical we do the same, i.e., wrap the nodes $v_{-i}, i = -m+1, \ldots, -1$ and $v_{p+i}, i = 1, \ldots, m-1$ along the vertical so that they coincide with nodal points on the adjacent sides. To do this we must have the following

$$\begin{aligned} u_{-i} &= u_i - hw, \quad i = 1,2,3 \\ u_{n+i} &= u_n + hw, \quad i = 1,2,3 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} v_{-i} &= v_i - hw, \quad i = 1,2,3 \\ v_{p+i} &= v_p + hw, \quad i = 1,2,3 \end{aligned} \tag{3.16}$$

where u_i denotes the nodal points along the horizontal, v_j denotes the nodal points along the vertical and hw is the step size of the adjacent side.

Table 3.7: The following table shows the error and the observed rate of convergence for various step sizes for the function $f(x, y, z) = \sin(xyz)$

Step Size	Error	Order of Convergence
$1.000000000000000e+00$	$1.670920218139921e-01$	
$5.000000000000000e-01$	$7.961222685614932e-03$	$4.391509023007923e+00$
$2.500000000000000e-01$	$4.648371361131741e-04$	$4.098192780859205e+00$
$1.250000000000000e-01$	$2.836252157206563e-05$	$4.034667624804363e+00$
$6.250000000000000e-02$	$1.759114473732576e-06$	$4.011064527357291e+00$
$3.125000000000000e-02$	$1.097374360545641e-07$	$4.002721689943840e+00$

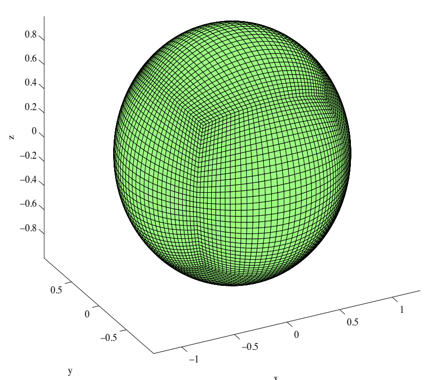

Figure 3.14: Mesh on the sphere $\mathcal{S}_r$

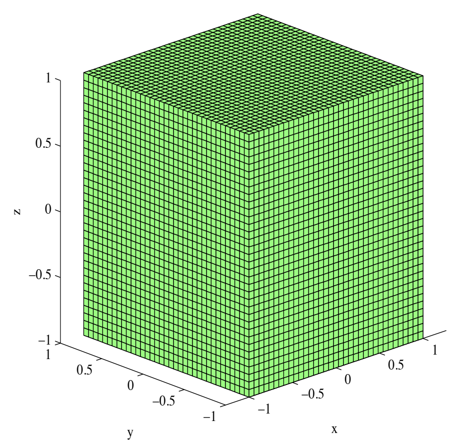

Figure 3.15: Mesh on the cube $\mathcal{B}_d$

Chapter 4

Conclusion

We developed and analyzed a method to approximate functions posed on the sphere. The method described here is based on tensor product splines and because of the tensor nature of the resulting space many algebraic properties of univariate polynomial splines can easily be carried over. Although the tensor product splines have a restricted use they are computationally advantageous due to their easy implementation on a digital computer.

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Appendices

Appendix A

Notations

A.0.1 One dimensional case

$$I \equiv [a, b] \equiv \{x \mid a \leqslant x \leqslant b\}.$$

For each nonnegative integer t and for each $p,\ 1 \leqslant p \leqslant \infty,\ \mathbb{PC}_t^p(I)$ will denote the set of all real valued functions $f(x)$ such that

1. $f(x)$ is $t-1$ times continuously differentiable,

2. there exists $x_i, \quad 0 \leqslant i \leqslant n-1$, with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

such that on each open subinterval $(x_i, x_{i+1}),\ 0 \leqslant i \leqslant n-1,\ \mathcal{D}^{t-1}(f)$ is continuously differentiable, and

3. the L_p norm of $\mathcal{D}^t(f)$ is finite i.e.,

$$||\mathcal{D}^t(f)||_p \equiv \Big(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |\mathcal{D}^t(f(x))|^p dx\Big)^{\frac{1}{p}} < \infty.$$

For the special case of $p = \infty$, we must have

$$||\mathcal{D}^t(f)||_\infty \equiv \max_{0 \leqslant i \leqslant n-1} \sup_{x \in (x_i, x_{i+1})} |\mathcal{D}^t(f(x))| < \infty$$

A.0.2 Bivariate case

$$U \equiv [a, b] \times [c, d] \equiv \{(x, y) \mid a \leqslant x \leqslant b \text{ and } c \leqslant y \leqslant d\}.$$

As in the univariate case, for each nonnegative integer t and for each $p,\ 1 \leqslant p \leqslant \infty,\ \mathbb{PC}_t^p(U)$ will denote the set of all real valued functions $f(x, y)$ such that

1. $f(x, y)$ is $t-1$ times continuously differentiable, i.e.,

$$\mathcal{D}_x^l \mathcal{D}_y^k f(x, y), \quad 0 \leqslant l + k \leqslant t - 1.$$

exists and is continuous,

2. there exists $x_i,\ 0 \leqslant i \leqslant n-1,$ and $y_j,\ 0 \leqslant j \leqslant p-1$, with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and

$$c = y_0 < y_1 < y_2 < \cdots < y_{p-1} < y_p = d$$

such that on each open subrectangle,

$$(x_i, x_{i+1}) \times (y_j, y_{j+1}), \quad 0 \leqslant i \leqslant n-1, \quad 0 \leqslant j \leqslant p-1$$

we have

$$\mathcal{D}_x^l \mathcal{D}_y^k f(x,y), \quad 0 \leqslant l+k \leqslant t-1.$$

continuously differentibale, and

3. for all $0 \leqslant l+k \leqslant t$, the $L_p - norm$ of $\mathcal{D}_x^l \mathcal{D}_y^k f(x,y)$ is finite i.e.,

$$||\mathcal{D}_x^l \mathcal{D}_y^k f(x,y)||_p \equiv \Big(\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{i+1}} |\mathcal{D}_x^l \mathcal{D}_y^k f(x,y)|^p dy\ dx \Big)^{1/p} < \infty$$

For the special case of $p = \infty$ we must have

$$||\mathcal{D}_x^l \mathcal{D}_y^k f(x,y)||_\infty = \max_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant p-1}} \sup_{(x,y) \in (x_i, x_{i+1}) \times (y_i, y_{i+1})} |\mathcal{D}_x^l \mathcal{D}_y^k f(x,y)| < \infty$$