Indecomposable and Chainable Continua

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Indecomposable and Chainable Continua

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Vita

Frank Herman Sturm was born October 8, 1983 to Alvin and Rachel Sturm in Pasadena, TX. Growing up in Pasadena, he graduated from Sam Rayburn High School in 2002 and then attended San Jacinto Community College for a year before enrolling in the University of Houston.

In the summer of 2006, Frank graduated from the University of Houston with a BSc. in Mathematics. After spending a year as a tennis instructor in the Houston area, he left the big and beautiful Lone Star State for the smaller (yet still beautiful) state of Alabama to enroll as a graduate student in Auburn University, where he continues his studies in mathematics.

Thesis Abstract

Indecomposable and Chainable Continua

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This thesis covers topics in continuum theory related to indecomposable continua and chainable continua. Theorems are presented to characterize indecomposable continua and then chainability is explored in its connection to inverse limit spaces. The end result is to provide an example of an hereditarily indecomposable continuum.

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Chapter 0

Background Definitions and Theorems

The following definitions and theorems have been compiled from my first graduate topology course with Dr. Gruenhage [5], the notes Dr. Michel Smith uses in his topology course [4], as well as the master's thesis of Scott Varagona [6]. Theorems are stated without proof, however, a reader unfamiliar with these theorems should be able to find proofs of equivalent theorems in [1], [2], and [3]

Definition 0.1. Suppose X is a set and $\mathcal{T}$ is a collection of subsets of X such that

1. $X \in \mathcal{T}$;

2. $\emptyset \in \mathcal{T}$;

3. If $\mathcal{U} \subset \mathcal{T}$, then $\cup\mathcal{U} \in \mathcal{T}$;

4. If $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ is finite, then $\cap\mathcal{U} \in \mathcal{T}$;

then the pair $(X, \mathcal{T})$ is called a **topological space** with **topology** $\mathcal{T}$. Such a topological space will often be referred to simply as X when the associated topology $\mathcal{T}$ is understood. The members of $\mathcal{T}$ are called **open sets**. If $K \subset X$ and $X \setminus K$ is open, then K is called a **closed** subset of X.

Unless otherwise stated, in this chapter $(X, \mathcal{T})$ is presumed to be a topological space.

Definition 0.2. Suppose $M \subset X$. The **closure of** M (denoted $\overline{M}$) is the intersection of all closed subsets of X that contain M.

Definition 0.3. Suppose M is a subset of a topological space X. A point $p \in X$ is a **limit point** of M if every open set containing p contains a point in M different from p.

Theorem 0.4. *If $M \subset X$, then $\overline{M} = M \cup \{p : p \text{ is a limit point of } M\}$.*

Definition 0.5. Suppose $D \subset X$. D is **dense in** X, means that each nonempty open subset of X contains a point of D. To say that D is **somewhere dense in** X means that there is U, a nonempty open subset of X such that $D \cap U$ is dense in the subspace U. Lastly, to say that D is **nowhere dense in** X, means that D is not somewhere dense in X.

Theorem 0.6. *If $D \subset X$ and D is dense in X, then $\overline{D} = X$.*

Theorem 0.7. *Suppose that $S \subset X$ and $S \neq \emptyset$. If $\mathcal{T}_S = \{S \cap O : O \in \mathcal{T}\}$, then $\mathcal{T}_S$ forms a topology on the set S.*

Definition 0.8. With regards to the topological space $(S, \mathcal{T}_S)$ described in the previous theorem, the topology $\mathcal{T}_S$ is called the **subspace topology on** S **with respect** X and S is refered to as a **subspace of** X.

Definition 0.9. Suppose $\mathcal{B}$ is a collection of open subsets of X with the property that if $x \in X$ and O is an open subset of X containing x, then there is $B \in \mathcal{B}$ such that $x \in B \subset O$. $\mathcal{B}$ is called a **basis** for the topology on X and an element of $\mathcal{B}$ is called a **basic open set in** X.

Theorem 0.10. *Suppose $\mathcal{B}$ is a collection of subsets of X such that*

1. *If $x \in X$, there exists some $B \in \mathcal{B}$ with $x \in B$.*

2. *If B_1 and B_2 are in $\mathcal{B}$ and $x \in X$ such that $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset (B_1 \cap B_2)$.*

Then $\mathcal{T}' = \{\bigcup \mathcal{B}' | \mathcal{B}' \subset \mathcal{B}\}$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}'$.

In the above theorem, a topology such as $\mathcal{T}'$ is said to be **generated** by the basis $\mathcal{B}$.

Definition 0.11. X is **Hausdorff** means that if $p, q \in X$ are distinct points in X, then there are disjoint open sets O_p and O_q containing p and q, respectively.

X is **regular** means that if $H \subset X$ and $p \in X$ such that $p \notin H$, then there exist disjoint open sets O_H and O_p such that $H \subset O_H$ and $p \in O_p$.

X is **normal**, means that if H and K are disjoint closed subsets of X, then there are disjoint open sets O_H and O_K containing H and K, respectively.

Definition 0.12. Suppose A and B are sets and f is a function from A to B (denoted $f : A \to B$). If $C \subset A$, we define $f(C) = \{f(c)|\ c \in C\}$ and call $f(C)$ the **image of** C under f.

If $b \in B$, then the **preimage of** b (written as $f^{-1}(b)$) is the collection $\{a \in A|\ f(a) = b\}$. Similarly, if $D \subset B$, then $f^{-1}(D) = \{a \in A : f(a) \in D\} = \cup\{f^{-1}(d) : d \in D\}$ and $f^{-1}(D)$ is called the **preimage of** D.

Definition 0.13. Suppose A and B are sets and $f : A \to B$.

1. f is **onto** means that if $b \in B$, then there is $a \in A$ such that $f(a) = b$.

2. f is **one-to-one** means that if a and a' are two distinct points in A, then $f(a) \neq f(a')$.

Often, the **inverse** of f is denoted as f^{-1} and explicity stated to be a function to avoid confusion with the preimage of a point or set.

Theorem 0.14. *Suppose A and B are sets and $f : A \to B$ is a function that is one-to-one and onto. Then $g : B \to A$ defined as $g(f(a)) = a$ is well defined. The function g is called the* ***inverse of*** *f.*

Definition 0.15. Suppose each of X and Y is a topological space, $f : X \to Y$ is a function, and $x \in X$). To say that f **continuous at** x means that if V is an open set in Y containing $f(x)$, there is U an open subset of X such that $x \in U$ and $f(U) \subseteq V$. If f is continuous at each point in X, then f is said to be **continuous**.

Theorem 0.16. *Suppose each of X and Y is a topological space and $f : X \to Y$. The following are equivalent:*

1. *f is continuous.*

2. *If V is an open subset of Y, then the preimage of V, $f^{-1}(V)$ is open in X.*

3. *If K is a closed subset of Y, then $f^{-1}(K)$ is a closed subset of X.*

Definition 0.17. Suppose each of X and Y is a topological space and $f : X \to Y$. f is an **open function** means that if U is open in X, then the image of U, $f(U)$, is open in Y. It is often simply stated that **the function f is open**.

Theorem 0.18. *Suppose that each of X, Y, and Z is a topological space and each of f, g, and h is a function such that $f : X \to Y$, $g : Y \to Z$, $h : X \to Z$, and $h = g \circ f$. For each of the properties listed below, if both f and g have the given property, then h has this property.*

1. *continuous*

2. *open*

3. *one-to-one*

4. *onto*

Definition 0.19. If X and Y are topological spaces and $f : X \to Y$ is a function that is one-to-one, onto, continous, and open, then f is called a **homeomorphism** and the spaces X and Y are said to be **homeomorphic**.

Definition 0.20. Suppose A is a set.

- A is **countable**, means there is a function $f : A \to \mathbb{N}$ that is one-to-one;

- To say that A **is finite**, means that there is a positive integer n and a function $f : A \to \{1, 2, \ldots, 3\}$ such that f is one-to-one;

- If A is not finite, then A is said to be **infinite**;

- If A is not countable, then A is said to be **uncountable**

Definition 0.21. Suppose X is a topological space and $x \in X$. If $\mathcal{B}_x$ is a collection of open subsets of X, then $\mathcal{B}_x$ is called a **local basis** at x if

1. for each $B \in \mathcal{B}$, $x \in B$;

2. if O is an open set in X and $x \in O$, then there is of $B \in \mathcal{B}_x$ such that $x \in B \subseteq O$.

Definition 0.22. The space X is called **first countable** if for each $x \in X$, there exists a countable local basis at x. A space X is called **second countable** if X has a basis that is countable.

Definition 0.23. Suppose $M \subset X$. If $\mathcal{U}$ is a collection of subsets of X such that $M \subset \cup\mathcal{U}$, then $\mathcal{U}$ is said to be a **cover** of M; if each element of $\mathcal{U}$ is an open subset of X, then $\mathcal{U}$ is said to be an **open cover** of M. Lastly, if $\mathcal{U}$ is an open cover of M and $\mathcal{F} \subset \mathcal{U}$ such that $\mathcal{F}$ covers M, then $\mathcal{F}$ is called a **subcover of** M **from** $\mathcal{U}$; if $\mathcal{F}$ is finite, then $\mathcal{F}$ is called a **finite subcover of** M **from** $\mathcal{U}$.

Definition 0.24. The space X is **compact** means that if $\mathcal{U}$ is an open cover of X, then there is a finite subcover of X from $\mathcal{U}$.

Theorem 0.25. *The interval [0,1], as a subspace of* $\mathbb{R}$ *is compact.*

Theorem 0.26. *Suppose that each of* X *and* Y *is a topological space and* $f : X \to Y$ *is continuous. If* X *is compact, then* $f(X)$ *is a compact subspace of* Y*.*

Definition 0.27. Suppose that S is a collection of subsets of X. S has the **finite intersection propert (f.i.p.) in** X means that if $\mathcal{F}$ is a finite nonempty subset of S, then $\cap\mathcal{F} \neq \emptyset$.

Theorem 0.28. *Suppose that* X *is a compact Hausdorff space.*

1. X *is normal.*

2. *If* K *is a closed subset of* X*, then the subspace* K *is compact.*

3. If $\mathcal{K}$ is a collection of closed subsets of X with the finite intersection property, then $\cap\mathcal{K} \neq \emptyset$.

Theorem 0.29. *Suppose X is compact and Hausdorff. If $\mathcal{U}$ is a countable collection of dense open subsets of X, then $\cap\mathcal{U}$ is dense in X.*

Corollary 0.30. *Suppose X is compact and Hausdorff. If $\mathcal{C}$ is a countable collection of nowhere dense closed subsets of X, then $\cup\mathcal{C} \neq X$*

Definition 0.31. If $\mathcal{I}$ is a nonempty set and for each $i \in \mathcal{I}$, A_i is a nonempty set, then the **product of** $\{A_i : i \in \mathcal{I}\}$, denoted $\prod_{i\in\mathcal{I}} A_i$, is the collection of functions from $\mathcal{I}$ into $\cup_{i\in\mathcal{I}} A_i$, to which the function γ belongs if and only if $\gamma(i) \in A_i$ for each $i \in \mathcal{I}$ (Note: If $\mathcal{I}$ is infinite, the existence of $\prod_{i\in\mathcal{I}} A_i$ depends on the Axiom of Choice.)

If $\mathcal{I}$ is finite (respectively countable), it is usually assumed that $\mathcal{I} = \{1, 2, \ldots, |\mathcal{I}|\}$, where $|\mathcal{I}|$ is the cardinality of $\mathcal{I}$ (respectively, $\mathcal{I} = \mathbb{N}$). In such a case $\prod_{i\in\mathcal{I}} A_i$ may be denoted

$$A_1 \times \cdots \times A_n, \text{ with } \ n = |\mathcal{I}|,$$

and the elements of this set are considered as ordered $n-$tuples. In the case that $\mathcal{I}$ is countable, the elements $\prod_{i\in\mathcal{I}} A_i$ can be thought of as infinite sequences whose i^{th} term is in A_i.

Theorem 0.32. *Suppose n is an integer greater than 1 and for each integer i, $1 \leq i \leq n$, suppose X_i is a topological space. Let*

$$\boldsymbol{X} = \prod_{i=1}^{n} X_i,$$

and let $\mathcal{B}$ denote the set

$$\{\prod_{i=1}^{n} O_i : O_i \neq \emptyset, \text{and } O_i \text{ is open in } X_i\}$$

$\mathcal{B}$ *forms a basis for a topology on* $\boldsymbol{X}$.

Definition 0.33. A space such as $\mathbf{X}$, as described in the previous theorem, together with the topology formed by the basis $\mathcal{B}$ (also described previously) is called a **finite product space** and its topology is called the **finite product topology**

Theorem 0.34. *Suppose that* $\mathcal{I}$ *is a nonempty set and for each* $i \in \mathcal{I}$, X_i *is a topological space. Let* $\boldsymbol{X} = \prod_{i \in \mathcal{I}} X_i$ *and let* $\mathcal{B}$ *be the set to which* $\prod_{i \in \mathcal{I}} O_i$ *belongs if and only if*

1. *for each* $i \in \mathcal{I}$, $O_i \neq \emptyset$ *and* O_i *is an open subset of* X_i;

2. *there is* $\mathcal{F}$ *a finite subset of* $\mathcal{I}$ *such that* $O_i = X_i$ *unless* $i \in \mathcal{F}$;

$\mathcal{B}$ *forms the basis for a topology on* $\boldsymbol{X}$.

Definition 0.35. $\mathbf{X}$ in the above theorem is called a **product space** and the topology formed from the basis $\mathcal{B}$ is called the **product topology** on $\mathbf{X}$. In general, if $\mathcal{I}$ is a nonempty set, a **product space with index** $\mathcal{I}$ refers to a space formed in the manner of $\mathbf{X}$ previously described.

Definition 0.36. Suppose $\mathbf{X}$ is a product space with index $\mathcal{I}$. If $i \in \mathcal{I}$, then $\pi_i : \mathbf{X} \to X_i$ such that if $\mathbf{x} \in \mathbf{X}$, then $\pi_i(\mathbf{x}) = \mathbf{x}(i)$ (recall that $\mathbf{x}$ is a function with domain $\mathcal{I}$). The function π_i is called the **projection of X onto** X_i.

Theorem 0.37. *If* $\boldsymbol{X}$ *is a product space with index* $\mathcal{I}$ *and* $i \in \mathcal{I}$, *then the function* π_i *is continuous and onto.*

Definition 0.38. Suppose X is a topological space and $d : X \times X \to \mathbb{R}$ is a function such that if each of x, y, and z is in X, then

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

2. $d(x, y) = d(y, x)$, and

3. $d(x, z) \leq d(x, y) + d(y, z)$.

The function d is said to be a **metric** on X, and the ordered pair X, d is called a **metric space**. If $p \in X$ and $\epsilon > 0$, $B(p, \epsilon)$ denotes the set $\{x \in X \ : \ d(x, p) < \epsilon\}$ and is called the **open ball of radius** ϵ **centered at** p.

Theorem 0.39. *Suppose* (X, d) *is a metric space, then the collection* $\mathcal{B}$ *defined as* $\{B(p, \epsilon) : p \in X \text{ and } \epsilon > 0\}$ *forms a basis for a topology on* X*.*

Definition 0.40. If (X, d) is a metric space, then the topology formed from the basis of open balls centered at points of X is called the **metric topology generated by** d. If $(X, \mathcal{T})$ is a topology and there is a metric d such that the metric topology generated by d is the same as $\mathcal{T}$, then $\mathcal{T}$ is said to be metrizable.

Theorem 0.41. *If* (X, d) *is a metric space then* X *is a normal space*

Definition 0.42. Suppose (X, d) is a metric space, $x \in X$, and $A, B \subset X$. The **minimum distance between** x **and** A is equal to $inf(\{d(x, a) \ : \ a \ \in \ A\})$, and the **minimum distance between** A **and** B is equal to $inf(\{d(x, B) : x \in A\})$.

Definition 0.43. A topological space is **connected** if it is not the union of two nonempty disjoint open sets. If $C \subset X$ and C as a subspace of X is connected, then C is also refered to as a **connected subset of** X.

Theorem 0.44. *The interval [0,1] as a subspace of* $\mathbb{R}$*, is connected.*

Theorem 0.45. *Suppose* C *is a connected subset of* X*. If* $A \subset X$ *and* $C \subset A \subset \overline{C}$*, then* A *is also a connected subset of* X*.*

Definition 0.46. Suppose each of H and K is a subset of the space X. H and K are called **mutually separated** if $\overline{H} \cap K = H \cap \overline{K} = \emptyset$.

Theorem 0.47. *Suppose* $M \subset X$*.* M *is not a connected subset of* X *if and only if* M *is the union of two nonempty mutually separated subsets of* X*.*

Theorem 0.48. *If* $M \subset X$ *and* M *is connected in* X*, then* $\overline{M}$ *is also connectd in* X*.*

Theorem 0.49. *Suppose $\mathcal{C}$ is a collection of connected subsets of X and K is a connected subset of X such that if $C \in \mathcal{C}$, then K and C are not mutually separated. The set $K \cup (\cup \mathcal{C})$ is a connected subset of X.*

Theorem 0.50. *Suppose each of X and Y is topological space and $f : X \to Y$ is a continuous function. If X is connected, then $f(X)$, the image of X under f, is a connected subset of Y.*

Definition 0.51. If $p \in X$, then the **component of X containing** p is the union of all connected sets in X that contain p. This set is sometimes denoted C_p.

Definition 0.52. A subset of X that is both closed and open in X is called a **clopen** subset of X. Notice that the empty set and X are each clopen subsets of X.

Theorem 0.53. *The space X contains a proper subset that is clopen if and only if X is not connected.*

Corollary 0.54. *Suppose U and V are nonempty disjoint subsets of X such that $U \cup V = X$. If both U and V are open, or if both U and V are closed, then X is not connected.*

Definition 0.55. If $p \in X$, the **quasicomponent** of X is the intersection of all clopen subsets of X that contain p. This set is sometimes refered to as Q_p.

Definition 0.56. A topological space X is called a **continuum** if it is non-empty, Hausdorff, compact, and connected. If X is a continuum and the topology of X can be generated by a metric, then X is called a **metric continuum**.

Corollary 0.57. *The interval [0,1], as a subspace of $\mathbb{R}$, is a continuum.*

Definition 0.58. If X is a continuum, $A \subset X$, and the subspace A is a continuum, then A is called a **subcontinuum** of X. If A is a proper subset of X, then A is a **proper subcontinuum**.

Chapter 1

Indecomposable Continua

In this chapter some characterizations of indecomposable continua will be developed. This developement will begin with a useful theorem called the Boundary Bouncing Theorem.

Unless otherwise stated, in this chapter $(X, \mathcal{T})$ is assumed to be a topological space such that X is a continuum.

Lemma 1.1. *If $x \in X$, C_x is the component of X that contains x, and Q_x is the quasi-component of X that contains x, then $C_x \subset Q_x$*

Proof. If K is a clopen subuset of X and $x \in K$, then $C_x \subset K$, else the subspace C_x is the union of two nonempty disjoint clopen subsets of C_x, namely $C_x \cap K$ and $C_x \cap (X \setminus K)$. Thus,

$$C_x \subset \cap\{K : K \text{ is a clopen subset of } X \textit{ and } x \in K\} = Q_x.$$

□

Lemma 1.2. *Suppose $\boldsymbol{X}$ is a compact Hausdorff space and $\mathcal{K}$ is a collection of closed subsets of X. If $C = \cap\mathcal{K}$ and U is an open set containing C, then there is $\{F_1, F_2, \ldots, F_n\}$, a finite subset of $\mathcal{K}$, such that $\cap_{i=1}^{n} F_i \subset U$.*

Proof. Let $\mathcal{K}$ and C be as defined in the Lemma. Choose an open set O containing C. $X \setminus O$ is a closed subset of X and therefore it is compact. For each $F \in \mathcal{K}$, $X \setminus F$ is open. Because $\cap\mathcal{K} = C$, $\cup\{(X \setminus F : F \in \mathcal{K}) = X \setminus \cap\mathcal{K} = X \setminus C$. $C \subset O$ implies that $X \setminus O \subset X \setminus C$, and so it follows that the collection $\{X \setminus F\} : F \in \mathcal{K}\}$ is an open cover of $X \setminus O$. Because $X \setminus O$ is compact, there is a finite subcovering of $X \setminus O$, $\{X \setminus F_1, X \setminus F_2, \ldots, X \setminus F_n\}$. Because $X \setminus O \subset \cup_1^n (X \setminus F_i)$ it follows that $\cap_1^n F_i \subset O$. □

Lemma 1.3. *If X is compact and Hausdorff and $x \in X$, then $Q_x = C_x$.*

Proof. By Lemma 1.1, $C_x \subset Q_x$. In order to prove the converse relationship it will suffice to show that Q_x is connected.

Suppose that A and B are mutually separated sets such that $A \cup B = Q_x$; it will be shown that either A or B must be empty. Without loss of generality, assume $x \in A$. Because $Q_x = A \cup B$ and Q_x is closed, $\overline{A} \subset A \cup B$ and $\overline{B} \subset A \cup B$. Since A and B are mutually separated it must be that $\overline{A} \subset A$ and $\overline{B} \subset B$, hence each of A and B is closed.

X is compact and Hausdorff, therefore it is normal. Because A and B are mutually separated, they are disjoint; thus U and V may be chosen to be disjoint open subsets of X such that $A \subset U$ and $B \subset V$.

Q_x is the intersection of clopen sets, and so by Lemma 1.2 the finite collection $\{F_1, F_2, \ldots, F_n\}$ may be chosen so that $\cap_{i=1}^{n} F_i \subset U \cup V$. If we define $\mathbf{F}$ as $\cap_{i=1}^{n} F_i$, then $\mathbf{F}$ is the interesection of clopen sets and therefore $\mathbf{F}$ is clopen. Define $U' = \mathbf{F} \cap U$. $A \subset U \cap \mathbf{F}$ and $x \in A$, therefore $x \in U'$. U and $\mathbf{F}$ are open, so U' must be open, and because V is open and $\mathbf{F}$ is closed, $\mathbf{F} \setminus V = U$ is closed as well. It follows that U' is clopen and contains x, hence $Q_x \subset U'$. This means $V \cap Q_x = \emptyset$. $A \subset V \cap Q_x$, hence, $A = \emptyset$.

From the preceeding argument it follows that Q_x is not the union of two nonempty mutually separated subsets, hence Q_x is connected. Since $x \in Q_x$, $Q_x \subset C_x$. Combining this with the result of Lemma 1.1 yields $Q_x = C_x$. □

Theorem 1.4 (Boundary Bouncing Theorem)**.** *Suppose X is a continuum, $a \in X$, and O is a nonempty open subset of X such that $a \notin O$. There is C, a connected subset of X, such that $a \in C$, $C \cap O = \emptyset$, and $C \cap \overline{O} \neq \emptyset$.*

Proof. Suppose $a \in X$ and O is a nonempty open subset of X such that $a \notin O$. Let $Y = X \setminus O$. Because X is connected $Y \cap \overline{O} \neq \emptyset$, else $\overline{O}$ would be open meaning $\overline{O} = O$ and it follows that Y and O are two disjoint clopen sets whose union is X, which would mean X is not connected and not a continuum. As a subspace Y is a closed subset of X, thus, as a subspace Y is compact and Hausdorff. Because $a \notin O$, $a \in Y$. Now suppose U is a clopen subset of Y. It will be shown that $U \cap \overline{O} \neq \emptyset$.

Case 1: If $U = Y$, then $U \cap \overline{O} = Y \cap \overline{O} \neq \emptyset$.

Case 2: Suppose $U \neq Y$. Let $V = Y \setminus U$; it follows that V is a clopen set such that $U \cap V \neq \emptyset$ and $U \cup V = Y$. Thus, $X = O \cup Y = O \cup U \cup V$. Since Y is closed in X and each of U and V is closed in Y, each of U and V is closed in X. Thus U and $V \cup \overline{O}$ are closed subsets of X whose union is X. Because X is a continuum, X is connected, which means $U \cap (V \cup \overline{O}) \neq \emptyset$. Since $U \cap V = \emptyset$ it follows that $U \cap \overline{O} \neq \emptyset$. Continuing with the main proof, let $\mathcal{Q}_a = \{Q : a \in Q$ and Q is clopen in $Y\}$ and let $\mathcal{K} = \{Q \cap \overline{O} : Q \in \mathcal{Q}_a$ (note that $\cap \mathcal{Q}_a$ is the quasicomponent of Y that contains a). It will now be shown that $\mathcal{K}$ has the f.i.p.

If $\mathcal{F}$ is a finite subset of $\mathcal{K}$, then there is $\mathcal{F}'$ a finite subset of $\mathcal{Q}_a$ such that $(\cap F') \cap \overline{O} = \cap \mathcal{F}$; $\cap \mathcal{F}'$ is clopen and contains a, therefore $(\cap \mathcal{F}') \cap \overline{O} \neq \emptyset$ and so $\cap F \neq \emptyset$. The result of the above facet is that $\cap \mathcal{K} \neq \emptyset$, since Y is compact and Hausdorff. Since $\cap \mathcal{K} = (\cap \mathcal{Q}_a) \cap \overline{O}$, it means that the quasicomponent of Y that contains a intersects $\overline{O}$. Again the fact that Y is compact and Hausdorff means that the quasicomponent containing a is the connected component of y that contains a, thus this component is a connected subset of Y (and thus a connected subset of X) that contains a and intersects $\overline{O}$. □

Definition 1.5. Suppose the space X is a continuum. X is **indecomposable** means that if H and K are proper subcontinua of X, then $H \cup K \neq X$.

Many familiar continua are not indecomposable, which is one reason indecomposable continua are interesting. For instance, the closed interval [0,1] with the subspace topology derived from $\mathbb{R}$ is not indecomposable because $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are both proper subcontinua and $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1] = [0, 1]$. The following theorem is a further characterization of an indecomposable continuum.

Theorem 1.6. *X is an indecomposable continuum if and only if each proper subcontinuum of X is nowhere dense in X.*

Proof. (⇒) By way of contrapositive it will be shown that if X is a continuum and C is a proper subcontinuum of X that is somwhere dense in X (*ie* not nowhere dense), then X is not indecomposable.

Let U be a nonempty open set in X such that every nonempty open subset of U intersects C. This means that C is dense in U. Because C is closed, $U \subset C$.

Let S be the subspace $X \setminus U$. S is a closed subset of C, thus S is compact. Let $a \in S \setminus C$ and let C_a be the component of S containing a. Because S is closed, C_a is closed in X, and thus C_a is a p.s.c. of X.

If $(X \setminus C) \subset C_a$, then $X = C_a \cup C$ and thus X is not indecomposable.

If $X \setminus C$ is not a subset of C_a, let $b \in X$ that is not in $C \cup C_a$. $S \setminus \{a\}$ is open in S, so by the Boundary Bouncing Theorem, we may choose Q to be a subset of $S \setminus \{a\}$ such that $C_a \subset Q$ and Q is clopen in S.

Let $C_1 = Q \cup C$ and $C_2 = (S \setminus Q) \cup C$. Both C_1 and C_2 are connected for if $x \in Q$ (respectively $S \setminus Q$), then C_x, the component of x in S, is a subset of Q (resp. $S \setminus Q$). Because $C_x \cap \overline{U} \neq \emptyset$, $C_x \cap C \neq \emptyset$; thus C_1 (resp C_2) is the union of a collection of connected sets, whose intersection is nonempty. It follows that both C_1 and C_2 are connected subsets of X.

Because Q and $S \setminus Q$ are closed subsets of a subspace that is closed in X, Q and $S \setminus Q$ are closed in X, thus C_1 and C_2 are closed in X; from this it follows that each of C_1 and C_2 is a subcontinua of X.

The point $a \in C_1$ is not in $(S \setminus Q) \cup C = C_1$, and the point $b \in C_2$ is not in $Q \cup C = C_2$, which means C_1 and C_2 are proper subcontinua of X. Lastly,

$$C_1 \cup C_2 = (Q \cup C) \cup ((S \setminus Q) \cup C) = (S \cup C) = X$$

and so we have that X is not indecomposable.

(⇒) To prove the converse, suppose that every proper subcontinuum of X is nowhere dense and that C_1 and C_2 are proper subcontinuum of X. C_1 is nowhere dense, which means U, a nonempty open subset of U may be chosen so that $U \cap C_1 = \emptyset$. C_2 is nowhere dense, which means there is a nonempty open subset of U that does not intersect C_2. Call such a set U'. $U' \cap (C_1 \cup C_2) = \emptyset$, thus $C_1 \cup C_2 \neq X$. Because C_1 and C_2 are chosen arbitrarily, if follows that X is indecomposable. □

Definition 1.7. If the space X is a continuum and $p \in X$, then the **compossant** of X that contains p is the union of all proper subcontinuum of X that contain p.

Theorem 1.8. *If X is an indecomposable continuum and each of C and D is a compossant of X, then $C = D$ or $C \cap D = \emptyset$.*

Proof. Suppose that X is an indecomposable continuum, p_1 and p_2 are points in X, K_1 is the compossant of X at p_1, and K_2 is the compossant of X at p_2. Let Q_1 be the collection to which C belongs if and only if C is a proper subcontinuum of X containing p_1; define Q_2 similarly with respect to p_2.

If $C_1 \in Q_1$, $C_2 \in Q_2$, and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is a proper subcontinuum containing p_1 and p_2; hence, $K_1 = K_2$, for if $x \in K_i$, where i is 1 or 2, and B is a proper subcontinuum containing x and p_i, then let $C' = B \cup (C_1 \cup C_2)$. C' is a proper subcontinuum because it is connected and is the finite union of proper subcontinua, thus $x \in K_1 \cap K_2$.

From the above paragraph, if $K_1 \neq K_2$, then no element of Q_1 intersects an element of Q_2. Therefore, $\cup Q_1 \bigcap C = \emptyset$ for every $C \in Q_2$; thus, $\cup Q_1 \bigcap \cup Q_2 = \emptyset$. Because $K_1 = \cup Q_1$ and $K_2 = \cup Q_2$, $K_1 \cap K_2 = \emptyset$. □

Definition 1.9. The continuum X has the **Countable Compossant Property (CCP)** if each compossant of X is the union of countably many subcontinua of X

Theorem 1.10. *If X is an indecomposable continuum and every compossant of X can be written as a countable collection of proper subcontinua, then X has an uncountable number of compossants.*

Proof. It will be shown by way of contrapositive, that if X has at most countable many compossants and each compossant of X is a union of countably many subcontinua of X, then X is not indecomposable.

Suppose that X is an indecomposable continuum and $K_1, K_2, \ldots$ are distinct compossants of X. If $n \in \mathbb{N}$, define $Q_n = \{K_{n,1}, K_{n,2}, \ldots\}$, where $K_{n,i}$ is a proper subcontinuum, such that $\cup Q_n = K_n$. Because Q_n is countable for each $n \in \mathbb{N}$ it follows that $\mathbf{Q} = \bigcup_{i=1}^{\infty} Q_n$ is countable. Let $\{C_1, C_2, \ldots\}$ be an enumeration of $\mathbf{Q}$. Because X is indecomposable, if $n \in \mathbb{N}$, then C_i is nowhere dense (this follows from Theorem 1.6).

Define $U_n = X \setminus C_i$ for each $n \in \mathbb{N}$. It follows that U_n is open and dense in X. By Baire's theorem, the intersection of a countable collection of dense open subsets of a compact Hausdorff space is dense; thus $\cap_{i=1}^{\infty} U_i \neq \emptyset$ and I can conclude that

$$\bigcup_{i=1}^{\infty} C_i = X \setminus \bigcap_{i=1}^{\infty} U_i \neq X.$$

Because $\bigcup_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} (\cup Q_i) = \bigcup_{i=1}^{\infty} C_i$, there must be a point of X not contained in any of the listed compossants. Thus the number of compossants cannot be countable. □

Theorem 1.11. *Suppose X is a metric continuum, with metric d. If X is nondegenerate and indecomposable, then each compossant of X is the union of a countable collection of subcontinua of X.*

Proof. Let $p \in X$ and let A be a dense countable subset of X. If $a \in A$ and $i \in \mathbb{N}$, let $\mathcal{P}_i(a)$ be the set to which C belongs, if and only C is a subcontinuum of X, $p \in C$, and $C \cap B(a, \frac{1}{i}) = \emptyset$. Now define $\mathbf{P}_i(a)$ as the set $\cup \mathcal{P}_i(a)$.

$\mathbf{P}_i(a)$ is a union of connected sets containing p, hence $\mathbf{P}_i(a)$ is connected and thus $\overline{\mathbf{P}_i(a)}$ is connected. $\mathbf{P}_i(a) \subset X \setminus B(a, \frac{1}{i})$, thus $\overline{\mathbf{P}_i(a)} \subset X \setminus B(a, \frac{1}{i})$, which means $\mathbf{P}_i(a) \in \mathcal{P}_i(a)$, thus $\mathbf{P}_i(a)$ is a proper subcontinuum of X.

Let $\mathcal{C}_p = \{\mathbf{P}_i(a) : i \in \mathbb{N},\ a \in A\}$. It will be shown that $\cup \mathcal{C}_p$ is the compossant of X that contains p. First note that $\cup \mathcal{C}_p$ is the union of proper subcontinua of X, each of which contains p (*ie* $\cup \mathcal{C}_p$ is a subset of the compossant of X containing p). Now suppose

C is a proper subcontinuum of X and $p \in C$. C is closed and $C \neq X$, hence $X \setminus C$ is a nonempty open set. A is dense in X so $a \in A$ may be chosen so that $a \in X \setminus C$ (*ie* $a \notin C$). Because $a \notin C$, and C is closed $d(a, C) > 0$. Choose $i \in \mathbb{N}$ so that $\frac{1}{i} < d(a, C)$. It follows that $C \cap B(a, \frac{1}{i}) = \emptyset$, hence $C \in \mathcal{P}_i(a)$, meaning $C \subset \mathbf{P}_i(a) \subset \cup \mathcal{C}_p$. Thus, $\cup \mathcal{C}_p$ is the compossant of X that contains p.

□

Corollary 1.12. *If X is a metric continua, then X has an uncountable number of compossants.*

Theorem 1.13. *If $p \in X$, then the compossant of X containing p is dense in X*

Proof. Let $p \in X$ and suppose U is a nonempty open subset of X.

Case 1: If $p \in U$, then $\{p\} \in U \cap K_p$

Case 2: Suppose $p \notin U$, and let $q \in U$. X is regular, so U' may be chosen such that $q \in U'$ and $\overline{U'} \subset U$. By Theorem 1.4, C may be chosen to be a connected subset of X such that $p \in C$, $C \cap U' = \emptyset$ and $C \cap \overline{U'} \neq \emptyset$. It follows that $\overline{C}$ is connected and $q \notin \overline{C}$; hence, $\overline{C}$ is a proper subcontinuum of X that contains p. $\overline{C} \cap \overline{U'} \neq \emptyset$ and $\overline{U'} \subset U$, so it also follows that $\overline{C} \cap U \neq \emptyset$. □

Definition 1.14. Suppose X is a continuum and a and b are two points in X. X is **irreducible between** a **and** b means that no proper subcontinuum of X contains both a and b.

Theorem 1.15. *Suppose X has the CCP. X is indecomposable if and only if there are three points a, b, and c in X such that X is irreducible between each pair in $\{a, b, c\}$.*

Proof. ($\Leftarrow$) Suppose $\{a, b, c\} \subset X$ and X is irreducible between any pair in $\{a, b, c\}$, and suppose A and B are proper subcontinuums of X and $A \cup B = X$.

This means $a \in A \cup B$; let's assume that $a \in A$. $b \in A \cup B$ and $b \notin A$ because A is a *p.s.c* containing a; let's assume $b \in B$. Of course this means that c is not contained in A

or B because each is a *p.s.c.* one contains a and the other contains b. Thus $c \notin A \cup B$ and $A \cup B \neq X$.

($\Rightarrow$) Supposing now that X is indecomposable. By the previous theorem, X has an uncountable number of disjoint compossants and so I may choose three nonempty disjoint compossants A, B, C and choose $a \in A,\ b \in B,\ c \in C$ and X is irreducible between each pair in $\{a, b, c\}$. □

Definition 1.16. Suppose X is a continuum and $p \in X$. $End(p, X)$ is defined as the collection

$$End(p, X) = \{q \in X : X \text{ is irreducible between } p \text{ and } q\}.$$

Theorem 1.17. *Suppose X is a nondegenerate indecomposable continuum and $p \in X$. If $End(p, X) \neq \emptyset$, then $End(p, X)$ is dense in X.*

Proof. Let $q \in End(p, X)$. Let K_p and K_q denote the compossant of X containing p and the compossant of X containing q, respectively. Because X is irreducible between p and q, $q \notin K_p$; thus $K_p \neq K_q$, so by Theorem 1.8 $K_p \cap K_q = \emptyset$. □

Theorem 1.18. *If a space X has a connected dense subset, then X is connected.*

Proof. Let X be a space and let K be a connected dense subset of X. K is connected therefore $\overline{K}$ is connected, by 0.45. K is dense, therefore $\overline{K} = X$ (by 0.6); hence, X is connected. □

Theorem 1.19. *Suppose X is a nondegenerate indecomposable continuum and X has the CCP. If K is the union of countably many proper subcontinua of X, then $X \setminus K$ is connected.*

Proof. From Theorem 1.10, X has an uncountable number of compossants, hence, a compossant C may be chosen such that $C \cap K = \emptyset$, where K is as described in the theorem. A compossant such as C is connected and dense, hence it follows that any subspace of X that contains C is connected. Thus, $X \setminus K$ is connected. □

Theorem 1.20. *Suppose $\mathcal{C}$ a collection of subcontinua of X such that if F is a finite nonempty subset of $\mathcal{C}$, then there is $C \in F$, such that $C \subset \cap F$. $\cap\mathcal{C}$ is a nonempty subcontinuum of X*

Proof. Notice that $\mathcal{C}$ is a collection of closed subsets of a compact space such that the intersection of a finite (and nonempty) subset of $\mathcal{C}$ is nonempty; thus, $\cap\mathcal{C} \neq \emptyset$. Let $V = \cap\mathcal{C}$.

To show V is connected, suppose V is not connected. If V is a closed set that is not connected, then V is the union of two disjoint closed sets (proved previously). Call two such sets A and B. Because A and B are each disjoint closed subsets of the compact Hausdorff space X, there are disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $A \subset \mathcal{O}_1$ and $B \subset \mathcal{O}_2$. If $C \in \mathcal{C}$, define $C' = C \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$; $C' \neq \emptyset$, because C is a nonempty connected set that intersects A and B. Let $C' = \{C' : C \in \mathcal{C}\}$; $\mathcal{C}'$ will also have the finite intersection property, for if $F' \subset \mathcal{C}'$ is finite and $F \subset \mathcal{C}$ corresponds to F', then there is $C \in \mathcal{C}$ such that $C \subset \cap F$, thus $C' \subset F'$. Let $V' = \cap\mathcal{C}'$. V' is nonempty and $V' \subset V$, thus $V = A \cup B$ is not a subset of $\mathcal{O}_1 \cup \mathcal{O}_2$, which contradicts an implication of the assumption.

□

Theorem 1.21. *Suppose that X is a continuum, $p, q \in X$, X is irreducible between p and q, and each nondegenerate subcontinuum containing q is not indecomposable. $End(p, X)$ is a continuum.*

If K is a proper subcontinuum of X containing p, define $\mathcal{O}_K$ to be $X \setminus K$.

(i) *$\mathcal{O}_K$ is connected.*

Proof. By way of contrapositive, suppose that $\mathcal{O}_K$ is not connected. Let U and V be nonempty disjoint sets open in $\mathcal{O}_K$ such that $U \cup V = \mathcal{O}_K$; notice that U and V are open in X as well because $\mathcal{O}_K$ is open in X. WLOG, assume $q \in U$ and let C be the component of $X \setminus V$ containing q. By the boundary bouncing theorem, C intersects $Bd(V)$. Because $Bd(V) \subset K$, $C \cap K \neq \emptyset \Rightarrow C \cup K$ is a subcontinuum of X. Because

V is nonempty, it must be that $C \cup K$ is a proper subcontinuum of X containing p and q, which is against the original hypothesis. $\square$

From the above argument, it follows that if K is a proper subcontinuum of X containing p, then $\overline{\mathcal{O}_K}$ is subcontinuum of X that intersects K at its boundary. Define $\mathcal{C}$ as

$$\mathcal{C} = \{\overline{\mathcal{O}_K} : K \text{ is a proper subcontinuum of X containing } p\}.$$

(ii) *If $\mathcal{F}$ is a nonempty finite subset of C, then there is $C \in \mathcal{C}$ such that $C \subset \cap\mathcal{F} \neq \emptyset$.*

Proof. Let F' be a finite nonempty subset of $\mathcal{C}$; call the elements of F' $1', 2', 3', ..., n'$ (ie $F' = \{1', 2', 3', \ldots, n'\}$). If $i' \in F'$ let i be a proper subcontinuum of X containing p such that $\overline{\mathcal{O}_i} = i'$; define $F = \{i : i' \in F'\}$. If $i \in F$, then i is a proper subcontinuum containing p and not q; thus $\cup F$ is a proper subcontinuum of X. Let $K = \cup F$.

If $i \in F$, then $i \subset K \Rightarrow \mathcal{O}_K \subset \mathcal{O}_i$; thus

$$\overline{\mathcal{O}_K} \subset \overline{\cap\{\mathcal{O}_i : i \in F\}} \subset \cap\{\overline{\mathcal{O}_i} : i \in F\} = \cap F'.$$

$\square$

With (ii), it follows from Theorem 1.20 that $\cap\mathcal{C}$ is a subcontinuum of X. Let $V = \cap\mathcal{C}$.

Notice that if K is a proper subcontinuum containing p, then $End(p, X) \subset \overline{\mathcal{O}_K}$; thus $End(p, X) \subset V$.

If we are fortunate enough that $End(p, X) = V$ then the theorem is proved. But suppose $End(p, X) \neq V$. To begin, we know that V is not indecomposable because it contains q. Let A and B be proper subcontinuum of V such that $A \cup B = V$; assume that $q \in A$.

(iii) $A \subset End(p, X)$

Proof. By way of contradiction, suppose A is not a subset of $End(p, X)$ and that t is an element of A such that $t \notin End(p, X)$. Let K be a proper subcontinuum of X containing p and t.

If there is $b \in \mathcal{O}_K$ such that $b \notin A$, then it follows that $A \cup K$ is a proper subcontinuum of X that contains p and q and so X is not irreducible between the two points; thus, we may assume $\mathcal{O}_K \subset A$. A is closed $\Rightarrow \overline{\mathcal{O}_K} \subset A$, and because $V \subset \overline{\mathcal{O}_K}$ it must be that $V \subset A$; however, this means $V = A$, which conflicts with our assumption that A is a proper subcontinuum of V. □

(iv) $End(p, X) \subset A$

Proof. By way of contradiction, suppose $b \in V$ and $b \notin A$. This necessitates that $b \in B$.

Because we are under the assumption that $End(p, X) \neq V$ we will let $t \in V$ such that $t \notin End(p, X)$, and we will let K be a proper subcontinuum of X that contains t and p. The set $K \cup B$ is the union to two intersecting subcontinua of X, thus $K \cup B$ is subcontinuum of X. Recall that B is a proper subcontinuum of V, so we may let $a \in A$ such that $a \notin B$. From (iii) above, a is necessarly in $End(p, X)$ and so $a \notin K$; hence, $K \cup B$ is a proper subcontinuum of X containg p and $b \Rightarrow b \notin End(p, X)$. □

From (iii) and (iv) above we have that $End(p, X) = A$. Because A is a subcontinuum of a subcontinuum of X, $End(p, X)$ is a subcontinuum of X.

Chapter 2

Forming Chains

Unless otherwise stated, in the chapter it is assumed that $(X, \mathcal{T})$ is a topological space.

Definition 2.1. Suppose $n \in \mathbb{N}$, and $\mathcal{C}$ is a collection formed from n subsets of X. To say that $\mathcal{C}$ **forms a chain**, means that $\mathcal{C}$ can be enumerated by the integers $1, 2, \ldots, n$ (that is $\mathcal{C} = \{C_1, C_2, \ldots C_n\}$) so that if $i, j \leq n$, then $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. If the enumeration of $\mathcal{C}$ is defined (or understood), then it is said that $\mathcal{C}$ **is a chain in** X. The **length** of $\mathcal{C}$ is the number of elements in $\mathcal{C}$ and is denoted by $|\mathcal{C}|$. The elements of the chain $\mathcal{C}$ are called **links of** $\mathcal{C}$, where the **first** and **last** links of $\mathcal{C}$ are C_1 and C_2, respectively. An **interior link** of $\mathcal{C}$ is a link that is not a first or last link of $\mathcal{C}$. If each of C and D is a link in $\mathcal{C}$ and $C \cap D \neq \emptyset$, then C and D are called **adjacent links in** C.

Note: Although the definition of a chain is general enough to allow for links of a chain to be any nonempty subset of X, it will be more convenient for the purposes of this paper to assume (unless stated otherwise) that links of a chain in X are open subsets of X.

The following is a list a of conventions used when speaking of chains:

1. Chains are denoted with capital script letters.
2. Links of a chain are denoted with plain capital letters (usually the same letter used to denote the chain).
3. Unless stated specifically, if it is said that $\mathcal{C}$ is a chain of length n, then it is assumed that $\{C_1, C_2, \ldots C_n\}$ is the enumeration of $\mathcal{C}$.
4. A sequence of chains is indexed with superscripts (*ie* $\{\mathcal{C}^i\}_{i=1}^{\infty}$).

The next theorem will be given with out proof.

Theorem 2.2. *If $\mathcal{C}$ is a chain in X of length n, and $\mathcal{D}$ is the enumerated collection $\{D_1, D_2, \ldots, D_n\}$, where $D_i = C_{n-i+1}$, then $\mathcal{D}$ is a chain in X.*

Definition 2.3. Suppose the chains $\mathcal{C}$ and $\mathcal{D}$ are as described in the previous theorem. $\mathcal{D}$ is called**the reverse of** $\mathcal{C}$, and will be denoted as $-\mathcal{C} = \{-C_1, -C_2, \ldots, -C_n\}$.

Theorem 2.4. *If $\mathcal{C}$ is a chain in X and $\mathcal{C}^1$ and $\mathcal{C}^2$ are disjoint nonempty subsets of $\mathcal{C}$ such that $\mathcal{C}^1 \cup \mathcal{C}^2 = \mathcal{C}$, then $(\cup\mathcal{C}^1) \cap (\cup\mathcal{C}^2) \neq \emptyset$.*

Proof. Suppose $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a chain and each of $\mathcal{C}^1$ and $\mathcal{C}^2$ are nonempty disjoint subsets of $\mathcal{C}$ such that $\mathcal{C}^1 \cup \mathcal{C}^2 = \mathcal{C}$. Without loss of generality, assume that C_1, the first link of $\mathcal{C}$, is in $\mathcal{C}^1$. Let l be the least index from $\mathcal{C}$ such that $C_l \in \mathcal{C}^2$. $l > 1$, which means $l-1$ is an index of a link in $\mathcal{C}$; furthermore $C_{l-1} \in \mathcal{C}^1$ since l is the least index such that $C_l \in \mathcal{C}^2$. C_{l-1} and C_l are adjacent which means $C_{l-1} \cap C_l \neq \emptyset$, hence $(\cup\mathcal{C}^1) \cap (\cup\mathcal{C}^2) \neq \emptyset$. □

Definition 2.5. If X is a topological space and each of $\mathcal{V}$ and $\mathcal{U}$ is a collection of open sets, then to say that $\mathcal{V}$ **refines** (respectively **properly refines**) $\mathcal{U}$, means that if $V \in \mathcal{V}$ then there is $U \in \mathcal{U}$ such that $V \subset U$ (respectively $\overline{V} \subset U$).

Lemma 2.6. *Suppose that $\mathcal{B}$ is a base for $\mathbf{X}$, $M \subset X$, and $\mathcal{U}$ is an open cover of M. There is $\mathcal{V}$ an open cover of X that refines $\mathcal{U}$ such that $\mathcal{V} \subset \mathcal{B}$.*

Proof. For each $x \in M$, let $U_x \in \mathcal{U}$ such that $x \in U_x$, and let $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x$. Let $\mathcal{V} = \{B_x : x \in M\}$.

For each $x \in X$, B_x is defined; thus $\cup\mathcal{V} = X$, so $\mathcal{V}$ covers X.

If $V \in \mathcal{V}$, there is $x \in X$ such that $V = B_x$. Thus there is $U_x \in \mathcal{U}$ such that $B_x \subset U_x \in \mathcal{U}$. Hence, $\mathcal{V}$ refines $\mathcal{U}$. □

Definition 2.7. Suppose that $n \in \mathbb{N}$ and $\mathcal{C}$ is a chain of length n in X. If $l, m \in \mathbb{N}$ (with $l, m \leq n$), then the **segment of** $\mathcal{C}$ **from** l **to** m is the collection $\{C_i : i \in \mathbb{N},\ min(l,m) \leq i \leq max(l,m)\}$, and is denoted $\mathcal{C}(l,m)$.

The following lemma is also intuitive and will be given without proof.

Lemma 2.8. *If $\mathcal{C}$ is a chain in the space X, then each segment of $\mathcal{C}$ forms a chain in X.*

Definition 2.9. Suppose now that $\mathcal{C}(h,j)$ is a segment from the chain $\mathcal{C}$ and $\mathcal{D}(k,m)$ is a segment from the chain $\mathcal{D}$. To say that $\mathcal{D}(k,m)$ is **anchored** in $\mathcal{C}(h,j)$ means that $D_k \subset C_h$ and $D_m \subset C_j$.

To say that $\mathcal{D}$ is **anchored in** $\mathcal{C}$ means that the first link of $\mathcal{D}$ is a subset of the first link of $\mathcal{C}$ and the last link of $\mathcal{D}$ is a subset of the last link of $\mathcal{C}$.

Theorem 2.10. *Suppose that $\mathcal{C}$ is a chain in X and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. Suppose also that $\mathcal{C}(h,j)$ is a segment from $\mathcal{C}$ and $\mathcal{D}(k,m)$ is a segment from $\mathcal{D}$ such that $\mathcal{D}(k,m)$ is anchored in $\mathcal{C}(h,j)$. If C_i is an interior link in $\mathcal{C}(h,j)$, then there is $D_l \in D(k,m)$ such that C_i is the only link in $\mathcal{C}$ such that $D_l \subset C_i$.*

Proof. Let $\mathcal{D}^1 = \{D \in \mathcal{D}(k,m) : D \subset \cup\mathcal{C}(1, i-1)$ and $\mathcal{D}^2 = \{D \in \mathcal{D}(k,m) : D \subset \cup\mathcal{C}(i+1, |\mathcal{C}|)\}$. $\mathcal{D}^1$ and $\mathcal{D}^2$ are nonempty $D_k \subset C_h$ and $D_m \subset C_j$ and i is between h and j. $\mathcal{D}^1$ and $\mathcal{D}^2$ are disjoint, for if $D \in \mathcal{D}^1$ and $C_a \in \mathcal{C}(1, i-1)$ such that $D \subset C_a$, if $C_b \in \mathcal{C}$ such that $D \subset C_b$ then $|a-b| \leq 1$, which means $b \leq a+1 \leq i$, and so $C_b \notin C(i+1, |\mathcal{C}|)$, and therefore $D \notin \mathcal{D}^2$.

It follows that $\mathcal{D}^1$ and $\mathcal{D}^2$ are nonempty disjoint subsets of $\mathcal{D}(k,m)$, which means $\mathcal{D}^1 \cup \mathcal{D}^2 \neq \mathcal{D}(k,m)$, since $\mathcal{D}(k,m)$ forms a chain. Thus, there is $D_l \in \mathcal{D}(k,m)$ such that $D_l \notin \mathcal{D}^1 \cup \mathcal{D}^2$. Because $\mathcal{D}$ refines $\mathcal{C}$, there is a link in $\mathcal{C}$ that contains D_l, thus C_i is the only link in $\mathcal{C}$ such that $D_l \subset C_i$. □

Theorem 2.11. *Suppose $\mathcal{C}$ is a chain in the the space X and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. Suppose also that $\mathcal{D}(k,m)$ is a segment from $\mathcal{D}$. The collection $\mathcal{C}' = \{C \in \mathcal{C} : C \cap (\cup\mathcal{D}(k,m))\}$ is a segment in $\mathcal{C}$.*

Proof. Let $C_h, C_j \in \mathcal{C}'$ such that $h < j$ and if $C_i \in \mathcal{C}'$, then $h < i < j$; thus $\mathcal{C}' \subset \mathcal{C}(h,j)$. It will now be shown that if $C_i \in \mathcal{C}(h,j)$, then $C_i \in \mathcal{C}'$. Let $D_a, D_b \in D(k,m)$ such that $D_a \cap C_h \neq \emptyset$ and $D_b \cap C_j \neq \emptyset$. Let $C_{h'} \in \mathcal{C}'$ such that $D_a \subset C_{h'}$, similarly, choose $C_{j'}$ so that $D_b \subset C_{j'}$. Because $C_{h'}$ contains D_a and $D_a \cap C_h \neq \emptyset$, $C_{h'} \cap C_h \neq \emptyset$, thus $|h-h'| \leq 1$, which means $h' = h$ or $h' = h + 1$. In a similar manner, it may be concluded that $j' = j$ or $j' = j - 1$.

Thus, $\mathcal{D}(a, b)$ is a segment in $\mathcal{D}$ anchored in $\mathcal{C}(h', j')$. Because C_h, C'_h, C_j, and C'_j each intersect D_a or D_b, these links are necessarily in $\mathcal{C}'$. Suppose that $C_i \in \mathcal{C}(h, j)$ and $h' < i < j'$. C_i is therefore an interior link of $\mathcal{C}(h', j')$, so by 2.10 there is $D_l \in \mathcal{D}(a, b)$ such that $D_l \subset C_i$. Because $D_a, D_b \in \mathcal{D}(k, m)$, $\mathcal{D}(a, b) \subset \mathcal{D}(k, m)$, thus $D_l \in \mathcal{D}(k, m)$ such that $D_l \cap C_i \neq \emptyset$; hence, $C_i \in \mathcal{C}'$.

It follows that $\mathcal{C}(h, j) \subset \mathcal{C}'$, thus $\mathcal{C}'$ is the segment $\mathcal{C}(h, j)$. □

Theorem 2.12. *Suppose $n \in \mathbb{N}$ and $\mathcal{C}$ is a chain of length n in X. Also suppose that U is an open subset of X. If $i \in \mathbb{N}$ (with $i \leq n$) and for each $C \in \mathcal{C}$, $U \cap C \neq \emptyset$ if and only if $C_i \cap C \neq \emptyset$, then $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$, where $D_i = U$ and $D_j = C_j$ (if $j \neq i$), is a chain in X.*

Proof. Suppose $j \in \mathbb{N}$ (with $j \leq n$ and $j \neq i$). $D_j \cap D_i = C_j \cap U$ and $C_j \cap U \neq \emptyset$ if and only if $C_j \cap C_i \neq \emptyset$ if and only if $|j - i| \leq 1$; thus, $D_j \cap D_i \neq \emptyset$ if and only if if and only if $|j - i| \leq 1$. If $k \in \mathbb{N}$ (with $k \leq n$ and $k \neq i$), then $D_j \cap D_k = C_j \cap C_k$; thus $C_j \cap D_k \neq \emptyset$ if and only if $|j - k| \leq 1$. □

Definition 2.13. In the previous theorem, the chain $\mathcal{D}$ is called **the chain formed from $\mathcal{C}$ by replacing link C_i with U**, and is denoted $\mathcal{C}(C_i, U)$.

Definition 2.14. Suppose $\mathcal{C}$ is a chain in X. To say that $\mathcal{C}$ is a spaced chain means that if $C, D \in \mathcal{C}$, then $\overline{C} \cap \overline{D} = \emptyset$ if and only if $C \cap D = \emptyset$. In other words, the closure of two links in $\mathcal{C}$ intersect if and only if the two links are adjacent. If X is a metric space with metric d, then **spacing of $\mathcal{C}$** is defined as

$$S(\mathcal{C}) = min(\{d(\overline{C}, \overline{D}) : C, D \in \mathcal{C} \text{ and } C \cap D = \emptyset\}.$$

For the following chapter it will be useful to note that when the metric space X is compact or each link in the chain $\mathcal{C}$ is bounded, that $S(C) > 0$.

Theorem 2.15. *Suppose X is normal, A is a closed subset of X, and $\mathcal{C} = \{C_1 \cdots C_{|\mathcal{C}'|}\}$ is a chain in X that covers A. There is $\mathcal{D}$, a spaced chain of length n, such that $\mathcal{D}$ properly refines $\mathcal{C}$ and if $D_i \in \mathcal{D}$, then $D_i \subset C_i$.*

Proof. Begin by choosing $b_2, b_3, \ldots, b_{|\mathcal{C}|}$ such that, if $i \in \mathbb{N}$ (with $2 \leq i \leq |\mathcal{C}|$), then $b_i \in C_{i-1} \cap C_i$ (this is possible since $|(i-1) - i| \leq 1$).

If $j \in \mathbb{N}$, and $j \leq |\mathcal{C}|$, let $\mathcal{U}_j = \mathcal{C} \setminus \{C_j\}$, and define B_j as $B_j = \{b_i : C_i \cap C_j \neq \emptyset$. Thus B_j intersects each link in $\mathcal{C}$ that is adjacent to C_j; finally, define A_j as $A_j = B_j \cup (A \setminus \cup \mathcal{U})$.

Thus, for each $j \in \mathbb{N}$ (with $j \leq |\mathcal{C}|$) $\cup \mathcal{U}_j$ is open and B_j is finite and is threfore closed, which means A_j is the union of two closed subsets of C_j; hence, A_j is a nonempty closed subset of C_j. Because X is normal, D_j may be chosen to be an open set such that $A_j \subset D_j$ and $\overline{D_j} \subset C_j$.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_{|\mathcal{C}|}\}$. By the selection of each element of $\mathcal{D}$, it is hopefully clear that if $D_i \in \mathcal{D}$, then $\overline{D_i} \subset C_i$, which will also yield that $\mathcal{D}$ is a proper refinement of $\mathcal{C}$. To show that $\mathcal{D}$ is a chain, suppose that $D_k, D_l \in \mathcal{D}$. Without loss of generality, assume that $k \leq l$.

If $|k - l| \leq 1$, then $l = k$ or $l - 1 = k$. If $k = l$, then $D_k \cap D_l \neq \emptyset$, since $D_k \neq \emptyset$. If $k = l - 1$, then $b_l \in C_k \cap C_l$. C_k and C_l are each adjacent to C_l, thus

$$b_l \in B_k \cap B_l \subset A_k \cap A_l \subset D_k \cap D_l.$$

If $|k - l| > 1$, then $C_k \cap C_l = \emptyset$. Because $D_k \subset C_k$ and $D_l \subset C_l$, $D_k \cap D_l = \emptyset$ as well.

Thus, $\mathcal{D}$ is a chain. □

Chapter 3

Chainable Continua

With the notions developed in the previous chapter regarding chains, the relationship between chains and continuum will follow in this chapter. First, it must be stated what is meant for a subset of a topological space to be chainable.

Definition 3.1. Suppose that X is a topological space and that $M \subset X$. To say that M is chainable, means that if $\mathcal{U}$ is an open cover of $\mathcal{M}$, then there is a chain $\mathcal{C}$ that refines $\mathcal{U}$ and covers M.

Before showing that the interval [0,1] is chianable a useful lemma will be proven.

Lemma 3.2. *Suppose that (X, d) is a metric space. If C is a compact subset of X and $\mathcal{U}$ is a open cover of C, then there is $\epsilon > 0$ such that if $p \in C$, then there is $U \in \mathcal{U}$ such that $B(p, \epsilon) \subset U$.*

Proof. Suppose C is a compact subset of X and $\mathcal{U}$ is an open cover of C. If $\delta > 0$ let C_δ denote the subset of C to which p belongs if and only if no element of $\mathcal{U}$ contains $B(p, \delta)$. Notice that if $0 < a < b$ then $C_b \subset C_a$

Thus, $\{\overline{C_{\frac{1}{n}}} : n \in \mathbb{N}\}$ is a nested collection of closed subsets of C. $\cap\{\overline{C_c} : c > 0\} = \emptyset$ because for each $p \in C$ there is $c > 0$ such that $B(p, c)$ is contained in some $U \in \mathcal{U}$; thus, there is $n \in \mathbb{N}$, such that $\overline{C_{\frac{1}{n}}} = \emptyset$ since C is compact. It follows that if $p \in C$, then $B(p, \frac{1}{n})$ is contained in some element of $\mathcal{U}$. □

Theorem 3.3. *The interval [0,1], with the subspace topology from $\mathbb{R}$ is chainable.*

Proof. It is assumed that the metric on $[0, 1]$ is the conventional abosolute difference metric (*ie* $d(x, y) = max(x - y, y - x)$).

Suppose $\mathcal{U}$ is an open cover of $[0,1]$. $[0,1]$ is compact, so by Theorem 3.2, $\epsilon > 0$ may be chose so that if $x \in [0,1]$, then then $B(x,\epsilon)$ is contained in an element of $\mathcal{U}$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. If $i \in \mathbb{N}$, and $1 \leq i \leq n+1$, let $F_i = B(\frac{i-1}{n}, \frac{1}{n})$. Thus, $\mathcal{F} = \{F_1, F_2, \ldots, F_{n+1}\}$ is a refinement of $\mathcal{U}$.

As intervals, $F_1, F_2, \ldots, F_{n+1}$ may be writtens as follows:

$F_1 = [0, \frac{1}{n})$, $F_{n+1} = (\frac{n-1}{n}, 1]$, and if $i \in \mathbb{N}$ and $1 < i < n$,

$$F_i = (\frac{i-1}{n} - \frac{1}{n}, \frac{i-1}{n} + \frac{1}{n}) = (\frac{i-2}{n}, \frac{i}{n}).$$

Note that if $i \in \mathbb{N}$ and $i \leq n$, then the right endpoint of F_i is $\frac{i}{n}$ and the left endpoint of F_{i+1} is $\frac{i-1}{n}$.

To show $\mathcal{F}$ is a chain, choose $j, k \in \mathbb{N}$ such that $j \leq k \leq n+1$.

If $|j-k| \leq 1$, then it follows that $j = k$ or $j+1 = k$. If $j = k$, then $\mathcal{F}_j \cap F_k = F_j \neq \emptyset$ and if $j+1 = k$, then

$$F_j \cap F_k \supset \left[\frac{i-1}{n}, \frac{i}{n}\right) \cap \left(\frac{i-1}{n}, \frac{i}{n}\right] = \left(\frac{i-1}{n}, \frac{i}{n}\right) \neq \emptyset.$$

If $|j-k| > 1$, then $j+2 \leq k$, which means

$$F_j \cap F_k \subset \left[0, \frac{j}{n}\right) \cap \left(\frac{(j+2)-2}{n}, 1\right] = \left[0, \frac{j}{n}\right) \cap \left(\frac{j}{n}, 1\right] = \emptyset.$$

Thus, $F_j \cap F_k \neq \emptyset$ if and only if $|j-k| \leq 1$. □

Corollary 3.4. *If X is a nonempty subcontinuum of $[0,1]$, then X is chainable.*

Proof. Suppose X is a subcontinuum of $[0,1]$. X is either a singleton or X is homeomorphic to $[0,1]$. If X is a singleton, then X is chainable, for if $\mathcal{U}$ is an open cover of X, you can pick $U \in \mathcal{U}$ such that $X \subset U$, and $\{U\}$ will form a chain of length one that covers X and refines $\mathcal{U}$.

If X is homeomorphic to [0,1], pick $h : [0,1] \to X$ such that h is a homeomorphism. Suppose $\mathcal{U}$ is an open cover of X. Let $\mathcal{U}' = \{h^{-1}(U) : U \in \mathcal{U}\}$. h is continuous, hence each element of $\mathcal{U}'$ is open in $[0,1]$; X is range of X and $\cup\mathcal{U} = X$, hence, $[0,1] = \cup\mathcal{U}'$. It follows that $\mathcal{U}'$ is an open cover of $[0,1]$.

$[0,1]$ is chainable, so $\mathcal{C}'$, a chain covering $[0,1]$ and refining $\mathcal{U}'$ may be chosen. Let $\mathcal{C} = \{h(C'_i) : C'_i \in \mathcal{C}'\}$. Since h is open, $\mathcal{C}$ is an open collection. Since h is onto and $\cup\mathcal{U}' = [0,1]$, $X = \cup\mathcal{U}$. Lastly, to show $\mathcal{C}$ to show that $\mathcal{C}$ is a chain, if $C'_i \in \mathcal{C}'$, let $C_i = h(C'_i)$. This provides an enumeration of $\mathcal{C}$. If $C_i, C_j \in \mathcal{C}$, then because h is one-to-one and onto, $C_i \cap C_j \neq \emptyset$ if and only if $h^{-1}(C_i) \cap h^{-1}(C_J) \neq \emptyset$ if and only if $|i - j| \leq 1$; thus $\mathcal{C}$ is a chain covering X.

$\mathcal{C}$ will refine $\mathcal{U}$, for if $C_i \in \mathcal{C}$, then $h^{-1}(C_i) \in \mathcal{C}'$. There is $U' \in \mathcal{U}'$ such that $h^{-1}(C_i) \subset U'$, thus $h(U') \in \mathcal{U}$ such that $C_i \subset h(U')$. □

Definition 3.5. If (X, d) is a metric space and $\mathcal{C}$ is a chain in X, then the mesh of $\mathcal{C}$ is defined as $mesh(\mathcal{C}) = min(\{diam(C) : C \in \mathcal{C}\})$, where $diam(C) = sup(\{d(x, y) : x, y \in \mathcal{C}\})$ for each $C \subset X$. If each link in $\mathcal{C}$ is bounded and $s > 0$ such that $s \geq mesh(\mathcal{C})$, then $\mathcal{C}$ is called an $s - chain$.

Theorem 3.6. *Suppose that X is a compact metric space with metric d, and $\{\mathcal{C}^i\}_{i=1}^{\infty}$ is a sequence of chains of open sets such that*

1. *For each i the chain $\mathcal{C}^{i+1}$ is a proper refinement of $\mathcal{C}^i$, and*

2. *For each i, $\mathcal{C}^i$ is an $\frac{1}{i} - chain$.*

The set $\cap_{i=1}^{\infty}(\cup\mathcal{C}^i)$ is chainable and a subcontinuum of X.

Proof. For ease, let $M = \cap_{i=1}^{\infty} \cup C^i$. We may assume without of generality, that if $n \in \mathbb{N}$ and C is a first or last link in the chain $\mathcal{C}^n$, then $C \cap M \neq \emptyset$, and $C \cap M$ is not a subset of another link in C^n (else, $C^n \setminus \{C\}$ can be enumerated to form a chain that covers M).

Note that for each positive integer i, if $C \in \mathcal{C}^{i+1}$, then $\overline{C} \subset \cup\mathcal{C}^i$, since $\mathcal{C}^{i+1}$ properly refines $\mathcal{C}^i$; thus

$$\overline{\cup\mathcal{C}^{i+1}} = \bigcup_{C\in\mathcal{C}^{i+1}} \overline{C} \ \subset \cup\mathcal{C}^i,$$

and therefore $M = \cap_{i=1}^{\infty}\overline{\cup\mathcal{C}^i}$ is a nonempty closed set. Hence M is compact.

Secondly, M is connected. If M were not connected, then because M is compact, there would be disjoint, nonempty, closed sets H and K, such that $M = H \cup K$. X is normal, thus we can choose disjoint open sets U and V, such that $H \subset U$ and $K \subset V$. Without losing generality, we may assume $\overline{U} \cap \overline{V} = \emptyset$. Let $\epsilon = min(D(H,U), d(K,V), (\overline{U}, \overline{V}))$.

Notice that if $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, then a link in $\mathcal{C}^n$ cannot intersect U and V, for if $x \in U$ and $y \in V$, then $d(x,y) > e(\overline{U}, \overline{V}) \geq \epsilon > \frac{1}{n}$. This means there is a link in $\mathcal{C}^n$ that is not a subset of $U \cup V$; thus $\overline{\cup\mathcal{C}^n} \setminus (U \cup V)$ is a nonempty closed set. Because

$$\overline{\cup\mathcal{C}^{n+1}} \subset \overline{\cup\mathcal{C}^n},$$

it follows that

$$\left(\cap_{n>\frac{1}{\epsilon}}\overline{\cup\mathcal{C}^n}\right) \setminus (U \cup V) \neq \emptyset.$$

Of course the above set is also a subset of M and is therefore a subset of U and V (contradiction).

Before showing chainability notice that each link in each chain must intersect M, for if $\mathcal{C}^n = \{C_1, \ldots, C_N\}$ and $j \in \mathbb{N}$ (with $j \leq N$) such that $C_j \cap M = \emptyset$, then $1 < j < n$ (by one of the initial assumptions) and $U = \cup_{i=1}^{j-1} C_i$ and $V = \cup_{i=j+1}^{N} C_i$ are disjoint open sets, each intersecting M, whose union contains M, hence M is not connected.

To show chainability, suppose $\mathcal{U}$ is an open cover of C. Since X is a compact metric space and M is a closed subset of X, Theorem 3.7 states that $\epsilon > 0$ may be chosen such that if $p \in M$, then $\{B(p,\epsilon)\} \subset U$ for some $U \in \mathcal{U}$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. If C is a link in C^n, then by the previous argument, $C \cap M \neq \emptyset$. Let $p' \in C \cap M$. Because $mesh(\mathcal{C}^n) < \frac{1}{n} < \frac{\epsilon}{2}$, $diam(C) < \frac{\epsilon}{2}$; thus $d(p', c) < 2\frac{\epsilon}{2} = \epsilon$ for each $c \in C$, which means

$C \subset B(p', \epsilon)$. By our choice of ϵ there is $U \in \mathcal{U}$ such that $B(p, \epsilon) \subset U$; hence, there is $U \in \mathcal{U}$ such that $C \subset U$. Therefore $\mathcal{C}^n$ refines $\mathcal{U}$ and M is chainable. □

Lemma 3.7. *Suppose (X, d) is a metric space, $M \subset X$, $\mathcal{U}$ is an open cover of M, and $\epsilon > 0$. Let $\mathcal{B}(\epsilon) = \{B(x, \delta) : x \in M$ and $\delta \leq \epsilon\}$. There is $\mathcal{V}$, an open cover of M that refines $\mathcal{U}$ such that $\mathcal{V} \subset \mathcal{B}$.*

Proof. The proof is similar to that of 2.6. It will be shown that each element of x is contained in an element of $\mathcal{B}(\epsilon)$, which is contained in an element of $\mathcal{U}$. Choose $x \in M$ and $U \in \mathcal{U}$, such that $x \in U$. There is $\delta > 0$ such that $B(x, \delta) \subset U$, since U is open. Thus $B(x, min(\delta, \epsilon)) \in \mathcal{B}(\epsilon)$ and $B(x, min(\delta, \epsilon)) \subset B(x, \delta) \subset U$. □

Theorem 3.8. *Suppose M is a chainable continuum lying in the metric space (X, d) and $\{a_i\}_{i=1}^{\infty}$ is a decreasing sequence converging to zero. There exists a sequence of spaced chains $\{C^n\}_{n=1}^{\infty}$, each of which covers M, such that if $i \in \mathbb{N}$, then*

1. *$\mathcal{C}^{i+1}$ properly refines C^i,*

2. *$mesh(\mathcal{C}^i) \leq a_i$,*

3. *$M = \cap_{n=1}^{\infty}(\cup \mathcal{C}^n)$.*

Proof. To begin, let $\mathcal{U}_1 = \{B(x, 1) : x \in M\}$. $\mathcal{U}_1$ is an open cover of M, so $\mathcal{D}_1$ may be chosen to be a chain that refines U_1 and covers M. Since metric spaces are normal and M is chainable and thus closed, by 2.15, C^1 may be a spaced chain that properly refines $\mathcal{D}^1$.

The remaining chains will be defined inductively. Suppose $\mathcal{C}^i$ is defined as spaced chain covering M. By 3.7, $\mathcal{U}_{i+1}$ may be chosen to be a refinement of $\mathcal{C}^i$, such that $\mathcal{U}_{i+1} \subset \mathcal{B}(a_{i+1})$, where $\mathcal{B}(a_{i+1} = \{B(x, a_{i+1}) : x \in M\}$. Let $\mathcal{D}^{i+1}$ be a chain refining $\mathcal{U}_{i+1}$ and covering M. By 2.15, $\mathcal{C}^{i+1}$ may be chosen to be a spaced chain that properly refines $\mathcal{D}^{i+1}$.

The sequence of spaced chains $\{C^i\}_{i=1}^{\infty}$ is now defined and each chain in the sequence does cover M. It remains to show that properties 1, 2, and 3 are satisfied by this sequence.

Suppose $i \in \mathbb{N}$. $\mathcal{D}^{i+1}$ refines $\mathcal{C}^i$ and $\mathcal{C}^{i+1}$ properly refines $\mathcal{D}^{i+1}$, thus $\mathcal{C}^{i+1}$ properly refines $\mathcal{C}^i$; hence, 1 is satisfied.

Because $\mathcal{U}_i \subset \mathcal{B}(a_i)$, if $D \in \mathcal{D}^i$, $diam(D) \leq a_i$; thus $mesh(\mathcal{D}^i) \leq a_i$, meaning $\mathcal{D}^i$ is an a_i-chain (satisfying 2).

Lastly, each chain covers M, so $M \subset \cap_{i=1}^{\infty}(\cup \mathcal{C}^i)$. To prove the converse, suppose $x \in \cap_{i=1}^{\infty}$. It will be shown that x is a limit point of M.

Suppose O is open and $x \in O$. $\epsilon > 0$ may be chosen so that $B(x, \epsilon) \subset O$ (since $\{B(p, \delta) : p \in X,\ and\ \delta > 0\}$ is a base for X). Because $\{a_i\}_{i=1}^{\infty}$ converges to zero, $n \in \mathbb{N}$ may be chosen such that $a_n < \epsilon$. Let $C_k^n \in \mathcal{C}^n$ such that $x \in C_k^n$. C_k^n is contain is contained in some element in $\mathcal{D}^n$, which is contained in some element in $\mathcal{U}_n$; so $B(p, \epsilon') \in \mathcal{U}_n$ may be chosen such that $C_k^n \subset B(p, \epsilon')$. By construction of $\mathcal{U}_n$, $p \in M$ and $\epsilon' < a_n < \epsilon$. Thus $p \in B(x, \epsilon)$. We have that x is a limit point of M. M is a continuum and thus closed in X. This means it contains all of its limit points, yeilding that $\cap_{i=1}^{\infty}(\cup \mathcal{C}^i) \subset M$. $\square$

Chapter 4

Inverse Limit Spaces and Chainable Continua

This chapter will offer a further characterization of chainable subsets of metric spaces. It is shown in the previous chapter that a chainable subset of a metric space is a continuum. In this chapter it will be shown that a subset of a metric space is chainable if and only if it is homemorphic a specific type of inverse limit space. The notion of an inverse limit space will now be developed.

Suppose that if $i \in \mathbb{N}$, then X_i is a topological space. Suppose also that f_i is a continuous function from X_{i+1} to X_i. Let $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ denote the subset of $\prod_{i=1}^{\infty} X_i$, to which the sequence $\{x_i\}_{i=1}^{\infty}$ belongs if and only if $x_i = f_i(x_{i+1})$, for each $i \in \mathbb{N}$.

If $i \in \mathbb{N}$ and $O_i \subset X_i$, let $\overleftarrow{O_i}$ denote the collection $\{x \in X|\ x_i \in O_i\}$; thus, $\overleftarrow{O_i} \subset \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

Theorem 4.1. *The collection $\mathcal{B}$, defined as*

$$\mathcal{B} = \{\overleftarrow{O_i} : i \in \mathbb{N} \textit{ and } O_i \text{ is an open subset of } X_i\},$$

is a basis for a topology on $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

The proof to the above theorem can be found in [6].

Definition 4.2. The space $\mathbf{X} = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ with topology generated by $\mathcal{B}$ (as defined in 4.1) is called an **inverse limit space**. If $i \in \mathbb{N}$, X_i is called the i^{th} **factor spaces** and f_i is called the i^{th} **bonding map**. Furthermore, if $O_i \subset X_i$ and $x \in \overleftarrow{O_i}$, then x is said to **pass through the set O_i in X_i**.

Inverse limit spaces are a valuable commodity in topology and are dealt with thouroughly in [6]. For the purposes of this chapter, the only inverse limit spaces that will be considered are those whose factor spaces are $[0,1]$. For this reason, necessary theorems, whose proofs can be found in [6], will be stated (without proof), and afterwards, it will be shown (with proof) that each chainable continua is in fact homeomorphic to an inverse limit space whose factor spaces are each $[0,1]$.

Theorem 4.3. *Suppose that if $i \in \mathbb{N}$, X_i is a topological space and $f_i : X_{i+1} \to X_i$ is a continuous function.*

1. $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ *is Hausdorff if X_i is Hausdorff for each $i \in \mathbb{N}$.*
2. $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ *is compact if X_i is compact for each $i \in \mathbb{N}$.*

The proof of 1 and 2 may be found respectively in 3.1 and 3.4 in [6].

When considering an inverse limit space $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, it is useful to look at compositions of the bonding maps. In such cases, the following convention will be used:
If $i, j \in \mathbb{N}$ and $i < j$, $f_i^j : X_j \to X_i$, such that

$$f_i^j = f_i \circ f_{i+1} \circ \cdots \circ f_{j-2} \circ f_{j-1}.$$

Notice that with this notation, $f_i^{i+1} = f_i$. Furthermore, f_i^j is a composition of continuous functions, hence f_i^j is continuous by an extension of 0.18.

For the following lemmas and theorems, suppose that if $i \in \mathbb{N}$, then $X_i = [0,1]$ and if $f_i : X_{i+1} \to X_i$ is a continuous function, let $\mathbf{X}$ denote the space $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

Lemma 4.4. *Suppose $n \in \mathbb{N}$, O_n is an open subset of X_n. If $i \in \mathbb{N}$ and $O_{n+i} = (f_n^{n+i})^{-1}(O_n)$ (ie O_{n+i} is the preimage of O_n under f_n^{n+i}), then $\overleftarrow{O_n} = \overleftarrow{O_{n+i}}$.*

Proof. Because $O_{n+i} = (f_n^{n+i})^{-1}(O_n)$, it follows that if $x \in \mathbf{X}$, then $x_n \in O_n$ if and only if $x_{n+i} \in O_{n+i}$. Hence $x\overleftarrow{O_n}$ if and only if $x \in \overleftarrow{O_{n+i}}$, meaning $\overleftarrow{O_n} = \overleftarrow{O_{n+i}}$. $\square$

The next lemma could be stated as a corollary to theorems in [6], however, due to the general nature of [6], it is felt that proving the following theorem for this specific instance is beneficial.

Lemma 4.5. *If* $n \in \mathbb{N}$*, then* $\pi_n(\boldsymbol{X})$ *is a subcontinuum of* $[0,1]$*.*

Proof. Suppose $n \in \mathbb{N}$. Let $K = \cap_{i=1}^{\infty} f_n^{n+i}([0,1])$. Because f_n^{n+j} is continuous for each $j \in \mathbb{N}$, and $X_{n+j} = [0,1]$ is a continuum, it follows that $f_n^{n+j}(X_{n+j})$ is a subcontinuum of X_n. Since $f_{n+j}^{n+j+1}(X_{n+j+1}]) \subset X_{n+j}$, for each $j \in \mathbb{N}$,

$$[0,1] \supset f_n^{n+1}(X_{n+1}) \supset f_n^{n+2}(X_{n+2}) \supset \cdots,$$

and since $f_n^{n+j}(X_{n+j}) \neq \emptyset$ for each $j \in \mathbb{N}$, it follows that K is in fact the intersection of a nested collection of nonempty subcontinua of X_n. By 1.20, K must be a subcontinuum of $X_n = [0,1]$.

It will now be shown that $\pi_n(\mathbf{X}) = K$. First note that if $x \in \mathbf{X}$ (*ie* $x_n \in \pi_n(\mathbf{X})$), then $f_n^{n+i}(x_{n+i}) \in K$ for each $i \in \mathbb{N}$; thus $x_n \in K$. It follows that $\pi_n(\mathbf{X}) \subset K$.

To show $K \subset \pi_n(\mathbf{X})$, a less straightforward proof is recquired. If $i \in \mathbb{N}$, define K_i as

$$K_i = \cap_{j=1}^{\infty} f_{n+i}^{n+i+j}(X_{n+i+j}).$$

For the same reasons that K is a nonempty subcontinuum of X_n, K_i is a nonempty subcontinuum of X_{n+i}. Now suppose that $y_n \in K$. Since y_n is in the image of f_n^{n+i} for each $i \in \mathbb{N}$, and $f_n^{n+i} = f_n \circ f_{n+i}^{n+i}$, it must be that $(f_n)^{-1}(y_n) \cap f_{n+1}^{n+i+1}(X_{n+i+1}) \neq \emptyset$ for each $i \in \mathbb{N}$. Because

$$(f_n)^{-1}(y_n) \cap K_1 = \cap_{j=1}^{\infty} (f_n)^{-1}(y_n) \cap (f_{n+1}^{n+j+1}(X_{n+j+1})),$$

$(f_n)^{-1}(y_n) \cap K_1$ is the intersection of a decreasing collection of nonempty compact sets, thus $(f_n)^{-1}(y_n) \cap K_1 \neq \emptyset$. Choose $y_{n+1} \in (f_n)^{-1}(y_n) \cap K_1$.

For each $i \in \mathbb{N}$, if y_{n+i} is chosen, then for reasons similar to those above, $(f_{n+i})^{-1}(y_{n+i}) \cap K_{i+1}$ is also the intersection of a decreasing sequence of nonempty compact subsets of X_{n+i+1}, meaning $(f_{n+i})^{-1}(y_{n+i}) \cap K_{i+1} \neq \emptyset$. Choose $y_{n+i+1} \in (f_{n+i})^{-1}(y_{n+i}) \cap K_{i+1}$.

Consider the sequence $\{x_i\}_{i=1}^{\infty}$, where

(a) $x_i = f_i^n(y_n)$, if $i < n$,

(b) $x_i = y_i$, if $i \geq n$ (where y_i is as chosen previously)

To show the sequence is in $\mathbf{X}$, pick $i \in \mathbb{N}$

(a) If $i < n$, then $x_i = f_i^n(y_n) = f_i(f_{i+1}^n(y_n)) = f_i(x_{i+1})$.

(b) If $i \geq n$, then $x_i = y_i$. $y_{i+1} \in (f_i)^{-1}y_i$, thus $x_i = y_i = f_i(y_{i+1}) = f_i(x_{i+1})$.

Because $x_n = y_n$, it follows that $K \subset \pi_n(\mathbf{X})$. Thus $\pi_n(\mathbf{X}) = K$, which means $\pi_n(\mathbf{X})$ is a continuum. □

Theorem 4.6. ***X*** *is chainable*

Proof. Suppose $\mathcal{U}$ is an open cover of $\mathbf{X}$; by Lemma 2.6, $\mathcal{V}$, an open cover of $\mathbf{X}$ may be chosen so that $\mathcal{V}$ refines $\mathcal{U}$, and $\mathcal{V} \subset \mathcal{B}$, where $\mathcal{B}$ is as defined in 4.1.

By Theorem 4.3 part 2, $\mathbf{X}$ is compact, thus $\mathcal{F}$ may be chosen as a finite subcover of $\mathbf{X}$ from $\mathcal{V}$. If $F \in \mathcal{F}$, then F is a basic open subset of $\mathbf{X}$, therefore a positive integer n_F and O_{n_f}, an open subset of $[0,1]$, may be chosen so that $F = \overleftarrow{O_{n_F}}$.

Let $N = max(n_F : F \in \mathcal{F})$. If $F \in \mathcal{F}$, let $O_F = (f_{n_F}^N)^{-1}(O_{n_f}))$; by Lemma 4.4, $F = \overleftarrow{O_{n_f}} = \overleftarrow{O_N^F}$. Let $X_N = \pi_N(\mathbf{X})$; X_N is a subcontinuum of $[0,1]$ by 4.5. Let $\mathcal{F}_N = \{O_N^F \cap [0,1] : F \in \mathcal{F}\}$. Since $\mathcal{F}$ covers $\mathbf{X}$, $\mathcal{F}_N$ covers X_N. Because projection mappings are open, and the image of π_N (restricted to $\mathbf{X}$) is X_N, it follows that $\pi_N(F)$ is an open subset of X_N for each $F \in \mathcal{F}$, thus $\mathcal{F}_N$ is an open cover of X_N. The corollary to 3.3 yields that X_N is chainable, and that $\mathcal{C}^N$ may be chosen to be a chain coverring X_N that refines $\mathcal{F}_N$.

If C_i^N is a link in $\mathcal{C}^N$, let $C_i = \overleftarrow{C_i^N}$. It will first be shown that if $C_i^N \in \mathcal{C}^N$, then C_i is open in $\mathbf{X}$. Suppose $C_i^N \in \mathcal{C}^N$. C_i^N is open in X_N and X_N is a subspace of $[0,1]$, thus there is O_N open in $[0,1]$ such that $O_N \cap X_N = C_i^N$. $C_i^N \subset O$, hence $C_i = \overleftarrow{C_i^N} \subset \overleftarrow{O_N}$. If $x \in \overleftarrow{O_N}$, then $x_n \in O_n \cap \pi_N \mathbf{X}$, which means $x_n \in C_i^N$, thus $\overleftarrow{O_N} \subset \overleftarrow{C_i^N} = C_i$. It follows that $\mathcal{C}$ is a collection of basic open subsets of $\mathbf{X}$.

Let $\mathcal{C}$ denote the collection $\{C_i : C_i^N \in \mathcal{C}^N\}$. It will now be shown that $\mathcal{C}$ is a chain, $\mathcal{C}$ covers $\mathbf{X}$, and $\mathcal{C}$ refines $\mathcal{U}$.

Suppose $C_i, C_j \in \mathcal{C}$ and $C_i \cap C_j \neq \emptyset$. $x \in C_i \cap C_j$ if and only if $x_n \in C_i^N \cap C_j^N$ if and only if $|i - j| \leq 1$; thus $\mathcal{C}$ is a chain.

If $x \in \mathbf{X}$, then $x_N \in \pi_N(\mathbf{X}) = X_N$, which means $C_i^N \in \mathcal{C}^N$ may be chosen so that $x_N \in C_i^N$. It follows that $x \in \overleftarrow{C_i^N} = C_i$. Thus, $\cup\mathcal{C} = \mathbf{X}$ and therefore $\mathcal{C}$ covers $\mathbf{X}$.

Lastly, $\mathcal{C}$ refines $\mathcal{U}$. Suppose $C_i \in \mathcal{C}$ and let C_i^N be the corresponding link in $\mathcal{C}^N$. $\mathcal{C}^N$ refines $\mathcal{F}_N = \{O_N^F \cap [0,1] : F \in \mathcal{F}\}$, so $F' \in \mathcal{F}$ may be chosen so that $C_i^N \subset O_N^{F'}$; this means that

$$C_i = \overleftarrow{C_i^N} \subset \overleftarrow{O_N^{F'}} = F.$$

$\mathcal{F} \subset \mathcal{V}$, so $V' \in \mathcal{V}$ may be chosen so that $F' \subset V'$. $\mathcal{V}$ refines $\mathcal{U}$, so $U' \in \mathcal{U}$ may be chosen so that $V' \subset U'$. Thus, $F' \subset U' \in \mathcal{U}$. It may be concluded that $\mathcal{C}$ refines the open cover $\mathcal{U}$ picked originally.

Hence, if $\mathcal{U}$ is an open cover of $\mathbf{X}$, there is a chain $\mathcal{C}$ that covers $\mathbf{X}$ and refines $\mathcal{U}$.

$\square$

It has now been shown that that an inverse limit space whose factor spaces are each $[0,1]$ is a chainable continuum. The rest of the chapter is devoted to showing that each chainable subset of a metric space is homeomorphic to an inverse limit whose factor spaces are each $[0,1]$.

Definition 4.7. If $\mathcal{C}$ is a chain, the **indexing set for** $\mathcal{C}$ is the collection $\mathcal{I} = \{1, 2, \ldots, |\mathcal{C}|\}$. Suppose $\mathcal{C}$ is a chain and $\mathcal{I}$ is the indexing set for $\mathcal{C}$. If $A \subset \mathcal{I}$, and $j, k \in A$ (with $j < k$), then to say that j and k are consecutive elements in A, means that if $l \in A$, then $l \leq j$ or $l \geq k$; C_j and C_k may be referred to as **consecutive links in terms of** A.

Definition 4.8. If $n \in \mathbb{N}$ and $\mathcal{K} = \{[\frac{i-1}{n}, \frac{i}{n}] : i \in \mathbb{N},\ 1 \leq i \leq n\}$, then $\mathcal{K}$ is called the **rusted chain of length n** and if $j \in \mathbb{N}$ and $j \leq n$ then $K_j = [\frac{j-1}{n}, \frac{j}{n}]$ is the j^{th} **rusty-link** of K.

$\mathcal{K}$ is a chain in the general sense, however, it is unlike the chains used previously, because each link in $\mathcal{K}$ is not an open subset of $[0,1]$. The term "rusted " came to mind defining such chains, because unlike chains in normal spaces whose links are open subsets, a link in a rusted chain cannot be replaced by a proper subset of the link and still cover $[0,1]$; thus, there is less flexibility.

The following construction will help describe how to "refine" a rusted chain with another rusted chain.

Suppose $\mathcal{C}$ is a chain and $\mathcal{D}$ is a chain that refines $\mathcal{C}$ such that the union of two adjacent links in $\mathcal{D}$ does not intersect more than two links in $\mathcal{C}$. Let $\mathcal{I}$ be the indexing set of $\mathcal{C}$, and let $\mathcal{J}$ be the indexing set of $\mathcal{D}$. Define T as

$$T = \{j \in \mathcal{J} : D_j \text{ intersects two links in } \mathcal{C}\},$$

and define U as

$$U = \{j \in \mathcal{J} : j + 1 \in T\}.$$

Let $F = \{\frac{j}{n} : j \in T \cup U\} \cup \{0, 1\}$. For each $j \in \mathcal{J}$, let $i(j)$ be the least element of $\mathcal{I}$ such that $D_j \cap C_{i(j)} \neq \emptyset$. Define $\overline{f} : F \to [0,1]$ defined as follows:

if $j \in T$, then $\overline{f}(\frac{j}{n}) = \frac{i(j)}{m}$, where $m = |\mathcal{C}|$;

if $j \in U \setminus T$, then $\overline{f}(\frac{j}{n}) = f(\frac{j+1}{n})$.

If $0 \notin T \cup U$, let $j_0 = min(T)$ and let $\overline{f}(0) = \overline{f}(\frac{j_0}{n})$.

If $1 \notin T \cup U$, let $j_1 = max(T \cup U)$ and let $\overline{f}(1) = \overline{f}(\frac{j_1}{n})$.

Let f be the piecewise linear expansion of $\overline{f}$.

Definition 4.9. In the previous construction,f is called the **bending function** relative to $\mathcal{D}$ in $\mathcal{C}$. The collection F used to define $\overline{f}$ is called the **defining set for** f.

Let K be the rusted chain of length $n = |\mathcal{D}|$. Because F is a subset of the endpoints of links in K it follows that if $j \in \mathcal{J}$, f is linear over K_j.

By the initial condition, that the union of two adjacent links in $\mathcal{D}$ intersect at most two links in $\mathcal{C}$, if $j \in T \cap U$, then D_j and D_{j+1} intersect the same two links in $\mathcal{C}$ (else $D_j \cup D_{j+1}$ intersects more than two links in $\mathcal{C}$); thus $f(\frac{j}{n}) = \frac{i(j)}{m}$ for every $j \in T$. Notice that this also implies that if $j+1 \in T$, then then $f(\frac{j}{n}) = f(\frac{j+1}{n})$ regardless of whether $j \in T$ or $j \in U \setminus T$.

Lemma 4.10. *Suppose (X, d) is a metric space, $\mathcal{C}$ is a spaced chain in X, and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. Suppose also that $\mathcal{D}(k, l)$ is a segment in $\mathcal{D}$. If $mesh(\mathcal{D}) < \frac{S(\mathcal{C})}{4}$, where $S(\mathcal{C})$ is defined in 2.14, and $|k - l| \leq 2$, then $\cup\mathcal{D}(k, l)$ intersects at most two links in $\mathcal{C}$.*

Proof. If $|k - l| \leq 2$, then there are at most three links in $\mathcal{D}(k, l)$. If $\mathcal{D}(k, l)$ has no more than two links, then because $D_k \cap D_l \neq \emptyset$,

$$diam(\cup\mathcal{D}(k, l) \leq diam(D_k) + diam(D_l) \leq \frac{S(\mathcal{C})}{2}) < S(\mathcal{C}).$$

If $\mathcal{D}(k, l)$ contains three links, let D_j denote the link that is not D_k or D_l. Thus, D_j intersects both D_k and D_l, and

$$diam(\cup\mathcal{D}(k, l)) \leq diam(D_k \cup D_j) + diam(D_l) \leq \frac{S(\mathcal{C})}{2} + \frac{S(\mathcal{C})}{4} < S(\mathcal{C}).$$

If $C_i, C_j \in \mathcal{C}$ and each intersects $\cup\mathcal{D}(k, l)$, then $d(C_i, C_j) < S(\mathcal{C})$. If $d(C_i, C_j) < S(\mathcal{C})$, then C_i and C_j must be adjacent. □

Lemma 4.11. *Suppose (X, d) is a metric space, $\mathcal{C}$ is a spaced chain, $\mathcal{D}$ is a chain that refines $\mathcal{C}$, and $mesh(\mathcal{D}) < \frac{S(\mathcal{C})}{4}$. If $\mathcal{C}(h, j)$ is a segment of $\mathcal{C}$, $\mathcal{D}(k, m)$ is a segment of $\mathcal{D}$ that is anchored in $\mathcal{C}(h, j)$, and C_i is an interior link of $\mathcal{C}(h, j)$, then there is $D_l \in \mathcal{D}(k, m)$ such that D_l only intersects C_i.*

Proof. Let $\mathcal{D}^1 = \{D \in \mathcal{D}(k, m) : D \cap (\cup\mathcal{C}(1, i-1)) \neq \emptyset\}$, and let $\mathcal{D}^2 = \{D \in \mathcal{D}(k, m) : D \cap (\cup\mathcal{C}(i+1, |\mathcal{C}|)) \neq \emptyset\}$. It will be shown that $\mathcal{D}^1$ and $\mathcal{D}^2$ are disjoint. Suppose $D \in \mathcal{D}^1$, and $C_a \in \mathcal{C}(1, i-1)$ such that $D \cap C_a \neq \emptyset$. From 4.10, D cannot intersect a link in $\mathcal{C}$ that is not adjacent to C_a; hence, if $C_b \in \mathcal{C}$ and $D \cap C_b \neq \emptyset$, then $b \leq a+1 \leq i$, meaning $C_b \notin \mathcal{C}(i+1, |\mathcal{C}|)$.

By 2.11, $\mathcal{D}^1 \cup \mathcal{D}^2$ cannot contain every link in $\mathcal{D}(k, m)$. Pick $D_l \in \mathcal{D}(k, m)$, such that $D_l \notin \mathcal{D}^1 \cup \mathcal{D}^2$. It follows that D_l cannot intersect a link in $\mathcal{C}(1, i-1) \cup \mathcal{C}(i+1, |\mathcal{C}|)$, thus D_l only intersects C_i. □

For Theorem 10 through Theorem 13, the following are assumed:

(1) $m \in \mathbb{N}$ and $\mathcal{C}$ is a spaced chain of length m

(2) $n \in \mathbb{N}$ and $\mathcal{D}$ is a spaced chain of length n that properly refines $\mathcal{C}$ such that $mesh(\mathcal{D}) < \frac{S(\mathcal{C})}{4}$.

(3) $\mathcal{I}$ and $\mathcal{J}$ are the respective indexing sets of $\mathcal{C}$ and D.

(4) $T \subset \mathcal{J}$ to which j belongs if and only if D_j intersects two links in $\mathcal{C}$.

(5) $U \subset \mathcal{J}$ defined as $U = \{j : j+1 \in T\}$.

(6) $F = T \cup U \cup \{0, 1\}$

(7) f is the bending function relative to $\mathcal{D}$ laying in $\mathcal{C}$.

(8) $K^{\mathcal{C}}$ is the rusted chain of length m and $K^{\mathcal{D}}$ is a rusted chain of length n.

Notice that by assumption (2) and Theorem 4.10, a link in $\mathcal{D}$ cannot intersect more than two links in $\mathcal{C}$, so the function f in assumption (7) is definable.

Theorem 4.12. *If $j, k \in F$ (with $j < k$)are consecutive indices of F, then $|f(\frac{j}{n}) - f(\frac{k}{n})| \leq \frac{1}{m}$; furthermore, if $l \in \mathcal{J}$ (with $j < l < k$) and $i \in \mathcal{I}$ such that $D_l \cap C_i \neq \emptyset$, then $D_{l+1} \cap C_i \neq \emptyset$.*

Proof. First note that the hypothesis is true when $f(\frac{j}{n}) = f(\frac{k}{n})$ and that this occurs if

(a) $k \in T$ and $j + 1 = k$, or

(b) $j = 0$ and $0 \notin T \cup U$, or

(c) $k = n$ and $n \notin T \cup U$.

The final case to consider is when the negation of (a), (b), and (c) occur. Not (a) implies that $k \in U \setminus T$ (else, $j < k-1 < k$ and $k-1 \in U \subset F$ is between j and k). Not (b) implies $j > 0$, and therefore $j \in T \cup U$. $j \notin U$ (else, $j+1 \in T \subset F$ and $j < j+1 < k$), which means $j \in T$. Let h be the least index in $\mathcal{I}$ such that $C_h \cap D_j \neq \emptyset$. Because $k \in U \setminus T$, $k \neq n$, and therefore $k+1 \in T$; let i be the least index of $\mathcal{I}$ such that $C_i \cap D_{k+1} \neq \emptyset$.

By induction, it will be shown that $D_{j+1}, D_{j+2}, \ldots D_k$ are each contained in the same link in $\mathcal{C}$. Since $j+1 \notin F$, D_{j+1} intersects exactly one link in $\mathcal{C}$; let g be the index of the link in $\mathcal{C}$ that intersects D_{j+1}. Let $l = k - j$. If $p \in \mathbb{N}$, $1 \leq p < l$, and $D_{j+p} \subset C_g$, then $j < j+p+1 \leq k$ implies that $j+p+1 \notin T$, and so D_{j+p+1} intersects only one link in $\mathcal{C}$; D_{j+p+1} intersects D_{j+p} and $D_{j+p} \subset C_g$, hence, $D_{j+p+1} \subset C_g$. Thus, $D_{j+1}, \ldots, D_k$ (equivalently $D_{j+1}, \ldots, D_{j+l} = D_k$) are each a subset of C_g.

$D_{k+1} \cap D_k \neq \emptyset$ and $D_k \subset C_g$, hence $D_{k+1} \cap C_g \neq \emptyset$. If $h = i$, then $f(\frac{j}{n}) = \frac{h}{m} = \frac{i}{m} = f(\frac{k}{n})$ and the hypothesis of the theorem is true.

Suppose now that $h < i$; this means $h < h+1 \leq i$, and by Theorem 2.10 , there is $q \in \mathcal{J}$ such that $j < q < k+1$ and $D_q \subset C_{h+1}$. Because $D_{j+1}, \ldots D_k$ are each contained in C_g, $g = h+1$, and therefore $h+1$ is the least index in $\mathcal{I}$ such that $D_{k+1} \cap C_{h+1} \neq \emptyset$; thus, $f(\frac{k}{n}) = f(\frac{k+1}{n}) = \frac{h+1}{m}$ and $|f(\frac{j}{n}) - f(\frac{k}{n})| = |\frac{h}{m} - \frac{h+1}{m}| = \frac{1}{m}$.

Lastly, suppose that $i < h$. $D_j \cap C_{h+1} = \emptyset$, or else D_{k+1} intersects C_{h+1} and C_i, which is not possible by 4.10 because C_i and C_{h+1} are disjoint (nonadjacent). Therefore, $D_j \cap C_h \neq \emptyset$ (else D_j would intersect three links in $\mathcal{C}$), meaning $g = h$. It follows that $i = h - 1$ and $|f(\frac{j}{n}) - f(\frac{k}{n})| = |\frac{h}{m} - \frac{h-1}{m}| = \frac{1}{m}$.

□

Corollary 4.13. *$diam(f(K_j^{\mathcal{D}})) \leq \frac{1}{2m}$ for each $j \in \mathcal{J}$.*

Proof. Suppose $l \in \mathcal{J}$. Let j and k be two elements of F such that $j \leq l \leq k$ and if $q \in F$, $q \leq j$ or $q \geq k$; f is defined to be linear between consecutive points in F, thus $diam(f([\frac{j}{n},\frac{k}{n}])) = |f(\frac{j}{n}) - f(\frac{k}{n})|$. If $f(\frac{j}{n}) = f(\frac{k}{n})$, then the diameter of $f(K_l^n)$ is 0 because $K_l^{\mathcal{D}} \subset [\frac{j}{n},\frac{k}{n}]$. If $f(\frac{j}{n}) \neq f(\frac{k}{n})$, then let $h \in \mathcal{I}$ such that C_h is the least link in $\mathcal{C}$ intersecting D_j. As in the proof of the previous theorem we can conclude that $j \in T$ and $k \in U \setminus T$; let $i \in \mathcal{I}$ such that C_i is the least link in $\mathcal{C}$ intersecting D_{k+1}. Because the mesh of $\mathcal{D}$ is less than $\frac{S(\mathcal{C})}{4}$, there are at least three links in $\mathcal{D}$ between D_j and D_{k+1}, thus there is at least two links of $\mathcal{D}$ between D_j and D_k. Because f is defined to be linear between $\frac{j}{n}$ and $\frac{k}{n}$, if $x, y \in [\frac{j}{n},\frac{k}{n}]$, then $|f(x) - f(y)| \leq \frac{1}{m}\frac{|x-y|}{|\frac{j}{n}-\frac{k}{n}|} \leq \frac{n|x-y|}{2m}$. Thus, $diam(f(K_l^{\mathcal{D}})) = |f(\frac{l}{n}) - f(\frac{l-1}{n})| \leq \frac{n}{2m}\frac{1}{n} = \frac{1}{2m}$. □

Theorem 4.14. *If $j \in \mathcal{J}$, $i \in \mathcal{I}$, and $D_j \cap C_i \neq \emptyset$, then $f(K_j^n) \subset K_i^m$.*

Proof. First suppose that $j \in T$. Let h be the least index of $\mathcal{I}$ such that $C_h \cap D_j \neq \emptyset$. $f(\frac{j}{n}) = \frac{h}{m}$ and because $j - 1 \in U$, $f(\frac{j-1}{n}) = f(\frac{j}{n}) = \frac{h}{m}$. Because f is defined to be linear between $\frac{j-1}{n}$ and $\frac{j}{n}$, f must be constant over $K_j^n = [\frac{j-1}{n},\frac{j}{n}]$; thus $f(K_j^n) \subset \{\frac{h}{m}\}$. j is assumed to be in T, meaning D_j intersects two link in $\mathcal{C}$; furthermore, these two links must be adjacent. C_h is the least such link, meaning D_j must also intersect C_{h+1}. $\frac{h}{m} \subset K_h^m \cap K_{h+1}^m$, therefore $f(K_j^n) \subset K_h^m \cap K_{h+1}^m$.

The remainder of the proof will follow by induction. Beginning by showing the hypothesis holds if $j = 1$.

If $1 \in T$, then the hypothesis holds by the previous argument. If $1 \notin T$, let l be the least element of T and let $i \in \mathcal{I}$ such that D_l intersects C_i and C_{i+1}. $D_1 \subset C_i$ or $D_1 \subset C_{i+1}$, for if not, $D_1 \subset C_{i-1}$ (or $D_1 \subset C_{i+2}$), which means there is an index l', with $1 \leq l' < l$, such that $D_{l'}$ intersects C_{i-1} and C_i (or $D_{l'}$ intersects C_{i+1} and C_{i+2}). Thus, $l' \in T$ and $l' \neq l$ because no link in $\mathcal{D}$ intersects more than one link in $\mathcal{C}$, which means $l' < l$ and therefore l is not the least element of T.

By definition, $f(0) = f(\frac{l}{n}) = f(\frac{l-1}{n})$; thus, f is constant over $[0, \frac{l-1}{n}]$ (note: $l - 1 \in U$ and is therefore also defined as $f(\frac{l}{n})$). It follows that $f(K_1^n) \subset K_i^m \cap K_{i+1}^m \subset K_i^m$.

Suppose now that $j \in \mathcal{J}$, and for each $k \in \mathcal{J}$ (with $k < j$), if $i \in \mathcal{I}$ and $D_k \subset C_i \neq \emptyset$, then $f(K_k^n) \subset K_i^{\mathcal{C}}$.

If $j \in T$, then the hypothesis of the theorem holds by the initial argument of the theorem. If $j \notin T$, let k and l be consecutive links in F such that $k \leq j+1 \leq l$. If $k \leq j+1 \leq l$, then $h \in \mathcal{I}$ may be chosen so that $D_{j+1} \cap C_h \neq \emptyset$, and let l be the least index in F that is greater than $j+1$. Because D_{j+1} only intersects the i^{th} link in $\mathcal{C}$, D_l must intersect a link adjacent to C_h; thus, D_l intersects two links in $\mathcal{C}$ and C_h is one, which means $f(\frac{l}{n}) \in f(K_l^n) \subset K_h^m$. $D_j \cap C_h \neq \emptyset$ and thus, $f(\frac{j}{n}) \in f(K_j^n) \subset K_h^m$. It follows that the left most and right most point of K_{j+1}^n are each in K_h^m, therefor $f(K_{j+1}^n) \subset K_h^m$, because f is linearly defined over K_{j+1}^n.

If $j \notin T$, let k and l be two consecutive indices in F such that $k \leq j \leq l$. Let $i \in \mathcal{I}$ such that $D_j \cap C_i \neq \emptyset$. By Theorem 4.10, $D_k \cap C_i \neq \emptyset$ and $D_l \subset C_i$; thus, by the inductive hypothesis $f(K_k^n) \subset K_i^m$. If $l \in U$, $l+1 \in T$ and D_{k+1} intersects C_i, which means that $f(\frac{l}{n}) = f(\frac{l+1}{n}) \in K_i^m$; if $l \notin U$, then $l = n$ and therefore $f(\frac{l}{n}) = f(1) = f(\frac{k}{n}) \in K_i^m$. Regardless of the case, both $f(\frac{k}{n})$ and $f(\frac{l}{n})$ are in K_i^m, which means that $f(K_j^n) \subset K_i^m$ because $K_i^m \subset [\frac{k}{n}, \frac{l}{n}]$ and f is linear over this interval.

□

For Theorem 4.15 through Corollary 4.19, suppose the following

(1) X is a metric space and M is a chainable subset of X;

(2) $\mathcal{C}^1, \mathcal{C}^2, \ldots$ is a sequence of spaced chains in X, with indexing sets respective $\mathcal{I}(1), \mathcal{I}(2), \ldots$, and respective lengths $n(1), n(2), \ldots$ such that

   (a) $\mathcal{C}^{i+1}$ properly refines $\mathcal{C}^i$;

   (b) $mesh(\mathcal{C}^{i+1}) < min(\frac{1}{i+1}, \frac{S(\mathcal{C}^i)}{4})$,

   (c) $\cap_{i=1}^{\infty}(\cup\mathcal{C}^i) = M$;

(3) K^i is a rusted chain of length $n(i)$

(4) $f_i : I \to I$ is the bending function relative to $\mathcal{C}^{i+1}$ inside $\mathcal{C}^i$;

(5) F_i is the defining set of f_i;

(6) If $g \in M$, $i(g)$ is the least index in $\mathcal{I}(i)$ such that the $i(g)^{th}$ link in $\mathcal{C}^i$ contains g. For ease, the $i(g)^{th}$ link in C^i will be referred to as $C(g,i)$ (*ie* $C(g,i)$ is the first link in $\mathcal{C}^i$ that contains g), and the $i(g)^{th}$ link in K^i, $(K^i_{i(g)})$, will be referred to as $K(g,j)$.

For each $g \in M$, let $h_i(g) = \cap_{j>i} f_i^j(K(g,i))$.

Theorem 4.15. *If $g \in M$, and $i \in \mathbb{N}$, $h_i(g)$ is a singleton.*

Proof. For each $i \in \mathbb{N}$, $\mathcal{C}^{i+1}$ properly refines $\mathcal{C}^i$, so by Theorem 4.10 $f_i(K(g,i+1)) \subset K(g,i)$ because $C(g,i+1) \cap C(g,i) \supset \{g\} \neq \emptyset$. Hence,

$$K(g,i) \supset f_i^{i+1}(K(g,i)) \supset f_i^{i+2}(K(g,i+2)) \supset \cdots,$$

which implies that $h_i(g)$ is the intersection of a nested collection of nonempty closed subsets of $[0,1]$, meaning $h_i(g) \neq \emptyset$.

To show that $h_i(g)$ is in fact a singleton, for each $i \in \mathbb{N}$, the case for $h_1(g)$ will be made and then generalized. First note that $f_1 = f_1^2$ is a function that is linear over $K(g,2)$ and that $diam(f_1(K(g,2))) \leq \frac{1}{2n(1)} = (\frac{1}{2})^{2-1} \cdot \frac{1}{n(1)}$.

If $j \in \mathbb{N}$ (with $j \geq 2$), and f_1^j is linear over $K(g,j)$ such that

$$diam(f_1^j(K(g,j))) \leq \left(\frac{1}{2}\right)^{j-1} \cdot \frac{1}{n(1)},$$

then $f_1^{j+1} = f_1^j \circ f_j$ is linear over $K(g,j+1)$ because f_j is linear over $K(g,j+1)$ and $f(K(g,j+1)) \subset K(g,j)$, and

$$\begin{aligned} diam(f_1^{j+1}(K(g,j+1))) &\leq \tfrac{diam(f_j(K(g,j+1)))}{diam(K(g,j))} \cdot diam(f_1^j(K(g,j))) \\ &\leq \tfrac{1}{2} \cdot \left(\tfrac{1}{2}\right)^{j-1} \cdot \tfrac{1}{n(1)} \\ &\leq \left(\tfrac{1}{2}\right)^{j} \cdot \tfrac{1}{n(1)} \end{aligned}$$

because $diam(f_j(K(g,j+1))) \leq \frac{1}{2n(j)} = \frac{1}{2} \cdot diam(K(g,j))$. It follows that

$$\lim_{j\to\infty} diam(f_1^j(K(g,j))) = \lim_{j\to\infty} \left(\frac{1}{2}\right)^{j-1} \left(\frac{1}{n(1)}\right) = 0,$$

meaning $h_1(g)$ is a singleton.

In general, note that $f_i = f_i^{i+1}$ is a function that is linear over $K(g,i+1)$ and that $diam(f_i(K(g,i+1))) \leq \frac{1}{2} \cdot \frac{1}{n(i)} = (\frac{1}{2})^1 \frac{1}{n(i)}$.

If $j \in \mathbb{N}$ (with $j \geq i+1$), and f_i^j is linear over $K(g,j)$ such that

$$diam(f_i^j(K(g,j))) \leq \left(\frac{1}{2}\right)^{j-i} \cdot \frac{1}{n(i)},$$

then $f_i^{j+1} = f_i^j \circ f_j$ is linear over $K(g,j+1)$ because f_j is linear over $K(g,j+1)$ and $f_j(K(g,j+1)) \subset K(g,j)$, and

$$\begin{aligned} diam(f_i^{j+1}(K(g,j+1))) &\leq \tfrac{diam(f_j(K(g,j+1)))}{diam(K(g,j))} \cdot diam(f_i^j(K(g,j+1))) \\ &\leq \tfrac{1}{2} \cdot \left(\tfrac{1}{2}\right)^{j-i} \cdot \tfrac{1}{n(i)} \\ &\leq \left(\tfrac{1}{2}\right)^{j} \cdot \tfrac{1}{n(i)} \end{aligned}$$

because $diam(f_j(K(g,j+1))) \leq \frac{1}{2n(j)} = \frac{1}{2} \cdot diam(K(g,j))$. It follows that

$$\lim_{j\to\infty} diam(f_i^j(K(g,j))) = \lim_{j\to\infty} \left(\frac{1}{2}\right)^{j-i} \left(\frac{1}{n(i)}\right) = 0,$$

meaning $h_i(g)$ is a singleton. $\square$

Theorem 4.16. *Suppose $g \in M$. For each $i \in \mathbb{N}$, let $a_i \in h_i(g)$. $\{a_j\}_{j=1}^{\infty}$ is in the inverse limit space $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, where $X_i = [0,1]$ and f_i is the bending function described prior to 4.15.*

Proof. Let $g \in M$ and $\{a_j\}_{j=1}^{\infty}$ be as suggested in the theorem. If $i \in \mathbb{N}$, then for each $k \in \mathbb{N}$ (with $k \geq i+1$),

$$f_i(a_{i+1}) = f_i(h_{i+1}(g)) \subset f_i(f_{i+1}^{j}(K(g,j))) \subset f_i^{j}(K(g,j)),$$

thus $f_i(a_{i+1}) \subset \cap_{j=1}^{\infty} f_i^{i+j}(K(g,i+j)) = h_i(g)$, so by definition $f_i(a_{i+1}) = a_i$. □

By the previous two theorems, it follows that for each $i \in \mathbb{N}$, h_i can be thought of as a function from M into I. and $h : M \to \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, defined as $h(g) = \{h_i\}_{i=1}^{\infty}$, is a function from M into $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

Theorem 4.17. *The function h as defined above is continuous and one-to-one.*

Proof. h is one-to-one, for if $p, q \in M$ and $p \neq q$, then there is $i \in \mathbb{N}$ such that $\frac{1}{i} < \frac{d(p,q)}{2}$, which means if $C, D \in \mathcal{C}^i$, $p \in C$ and $q \in D$, then there is one link of $\mathcal{C}^i$ between C and D, meaning $K(p,i) \cap K(q,i) = \emptyset$ and $h_i(p) \neq h_i(q)$.

To show continuity, suppose $p \in M$, and U is open in $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ such that $h(p) \in U$. Without losing generality, it may be assumed that U is a basic open set. Let $i \in \mathbb{N}$, and let U_i be an open subset of X_i such that $U = \overleftarrow{U_i}$.

By the topological nature of $[0,1]$, $\epsilon > 0$ may be chosen so that $X_i \cap (h_i(p) - \epsilon, h_i(p) + \epsilon) \subset U_i$, and $j \in \mathbb{N}$ may be chosen so that $\left(\frac{1}{2}\right)^{j-1} < \epsilon$. $C(p, i+j) \cap M$ is an open subset of M and if $x \in C(p, i+j)$, then $C(x, i+1) \cap C(p, i+j) \neq \emptyset$; thus, $K(x, i+j) \cap K(p, i+j) \neq \emptyset$ and $f_i^{i+j}(K(x, i+j) \cap f_i^{j+1}(K(p, i+j)) \neq \emptyset$. The diameter of the f_i^{i+j} - image of a rusty link in $K^{n(i+j)}$ is not greater than $\left(\frac{1}{2}\right)^{j}$, meaning

$$|h_i(x) - h_i(p)| \leq \left(\frac{1}{2}\right)^{j} + \left(\frac{1}{2}\right)^{j} = \left(\frac{1}{2}\right)^{j-1} < \epsilon.$$

□

Theorem 4.18. *If $\{a_j\}_{j=1}^{\infty} \in \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, there is $g \in M$ such that $h_j(g) = a_j$ for each $j \in \mathbb{N}$.*

Proof. Let $a = \{a_j\}_{j=1}^{\infty} \in \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. For each $j \in \mathbb{N}$, let $K(a,j)$ be the lowest indexed rusted link in K^j that contains a_j; let $C(a,j)$, be the link in $\mathcal{C}^j$ corresponding to $K(a,j)$. Let $V(j) = \{K_i^j : K_i^j \text{ is adjacent to } K(a,j)\}$.

It will now be shown that for each $i \in \mathbb{N}$, $f_i(\cup V(i+1)) \subset \cup V(i)$ and each rusty link in K^{i+1} is only contained in a link in $V(i)$. Let $\mathbf{K}$ be the union of $K(a,i+1)$ and another rusted link in $V(i+1)$. Thus $\mathbf{K}$ is connected and contains a_{i+1}. By Corollary 3.5, the diameter of the image of a link in K^{i+1} under f_i does not exceed $\frac{1}{2(n(i))}$; because $\mathbf{K}$ is connected and f_i is continuous, $f_i(\mathbf{K})$ is connected, and

$$diam(f_i(\mathbf{K})) \leq \frac{1}{2n(i)} + \frac{1}{2n(i)} = \frac{1}{n(i)}.$$

Because $f(\mathbf{K}) \cap K(a,i)$ contains $f_i(a_{i+1}) = a_i$, and the diameter of $f_i(\mathbf{K})$ does not exceed $\frac{1}{n(i)}$, each point in $f_i(\mathbf{K})$ must be in $K(a,i)$ or a rusted link in K^i that is adjacent to $K(a,i)$; hence $f_i(\mathbf{K}) \subset \cup V(i)$. Because the choice of the adjacent link used to form $\mathbf{K}$ is arbitrary, it follows that $f_i(V(i+1)) \subset V_i$. Furthermore, because $a_i \in K(a,i)$, a_i is at least $\frac{1}{n(i)}$ from the boundary of $\cup V(i)$ we know that if $L \in K^{i+1}$, $f_i(L)$ is not a subset of $Bd(\cup V(i))$ and therefore $f_i(L)$ is not a subset of any link in K^i that is not in $V(i)$.

Analogous to $V(j)$ above, if $j \in \mathbb{N}$, let $W(j) = \{C_i^j : C_i^j \text{ is adjacent to } C(a,j)\}$; in other words, each link in $W(j)$ corresponds to a link in $V(j)$. We now want to show that $\cup W(i+1) \subset \cup W(i)$ for each $i \in \mathbb{N}$.

From Theorem 3.6, if $D \in \mathcal{C}^{i+1}$ and $C \in \mathcal{C}^i$ such that $D \cap C \neq \emptyset$, then if K_D is the link in K^{i+1} corresponding to D and K_C is the link in K^i corresponding to C, then $f_i(K_D) \subset K_C$. By the previous argument, each link of $V(i+1)$ is only a subset of a link in $V(i)$, thus a link in $\mathcal{C}^{i+1}$ can only intersect a link (or links) in $W(i)$; hence, $\cup W(i+1) \subset \cup W(i)$ for each $i \in \mathbb{N}$.

From the above argument, $\{\cup W(i) : i \in \mathbb{N}\}$ is a decreasing collection of nonempty sets and $\cap_{i=1}^{\infty}\overline{(\cup W(i))} \neq \emptyset$ and we can choose $g \in \cap_{i=1}^{\infty}\overline{(\cup W(i))}$. Note that $W(i)$ is a segment in

$\mathcal{C}^i$ and $diam(\cup W(i)) \leq \frac{3}{n(i)}$; this means

$$\lim_{i\to\infty} diam(\cup W(i)) = 0$$

and g is the only element in the intersection.

The current claim is that $h(g) = \{a_j\}_{j=1}^{\infty}$. We know that $g \in \overline{\cup W(i)}$, however $C(g,i)$ may not be a link in $W(i)$; if $i \in \mathbb{N}$, let $W'(i) = \{C(g,i)\} \cup W(i)$, and let $V'(i) = \{K(g,i)\} \cup V(i)$. Because each link in $W'(i)$ corresponds to a link in $V'(i)$ and $W'(i)$ is segment in C^i with length at most four, it follows that $V'(i)$ is a segment of rusty links in K^i with length at most four. Furthermore, $f_i(\cup V(i+1)) \subset V(i)$ and $f_i(V(g,i+1)) \subset V(g,i)$, which means that $f_i(\cup V'(i+1)) \subset V'(i)$.

We now have that for each $i \in \mathbb{N}$, $h_i(x) \in \cup V'(i)$. By Corollary 3.5, for each $j \in \mathbb{N}$ (with $j > i$),

$$diam(f_i^j(\cup V'(j))) \leq 4 \cdot (\frac{1}{2})^j \cdot \frac{1}{n(i)},$$

thus, $\{f_i^j(\cup V'(j)) : j \in \mathbb{N},\ j > i\}$ is a decreasing sequence of nonempty closed connected sets whose diameters converge to 0 and $\cap_{j>i} f_i^j(\cup V'(j))$ contains exactly one point. By our construction $h_i(g)$ must be in this intersection because $K(g,j) \subset \cup V'(j)$ for each $j > i$, and a_i is in this intersection because $K(a,j) \in V(j) \subset V'(j)$ for each $j > i$; thus $h_i(g) = a_i$. □

Corollary 4.19. $h : M \to \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$*, defined previously is a homeomorphism from* M *onto* $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$.

Chapter 5

An Hereditarily Indecomposable Continuum

Definition 5.1. A continuum X is said to be hereditarily indecomposable, if each subcontinuum of X is indecomposable.

In this section an hereditarily indecomposable continuum will be constructed using the notions of chainability.

Lemma 5.2. *If $\mathcal{C}$ is a chain that covers the continuum K, and $\mathcal{C}' \subset \mathcal{C}$ containing exactly those links in $\mathcal{C}$ that intersect K, then $\mathcal{C}'$ is a segment in $\mathcal{C}$.*

Proof. Let $K, \mathcal{C}$ and $\mathcal{C}'$, be as described. Let m denote the minimum index for a link in $\mathcal{C}'$ and let M denote the maximum index for a link in $\mathcal{C}'$. If $\mathcal{C}'$ is not a segment, then there is $j \in \mathbb{N}$, $(m < j < M)$ such that $C_j \notin \mathcal{C} \setminus \mathcal{C}'$; it would then follow, that if $U = \cup_{i=m}^{j-1} C_i$ and $V = \cup_{i=j+1}^{M} C_i$, then U and V are disjoint open sets, such that each intersect K and $K \subset \cup\mathcal{C}' \subset U \cup V$. Hence, K could not be connected. □

Lemma 5.3. *If M is chainable, then each subcontinuum of M is chainable.*

The above lemma is given without proof, however, the following reasoning is provided. Suppose $\{\mathcal{C}^n\}_{n=1}^{\infty}$ is a sequence of chains covering M, as described in 3.8, and K is a subcontinuum of M. If $\tilde{\mathcal{C}^n}$ is described as the collection of links in $\mathcal{C}^n$ that intersect K, then by 5.2, $\tilde{\mathcal{C}^n}$ is a segment from $\mathcal{C}^n$, and can therefore be thought of as a chain as well. The sequence $\{\tilde{\mathcal{C}^n}\}_{n=1}^{\infty}$, will have all the properties necessary in 3.6 to ensure that K is chainable.

Definition 5.4. Suppose $\mathcal{C}$ is a spaced chain with $|\mathcal{C}| \geq 6$ and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. To say that $\mathcal{D}$ is **doubly coiled** in the interior of $\mathcal{C}$ means that if C_g and C_h are links of

$\mathcal{C}$, $3 \leq g, h \leq |C_g| - 2$, and $|g - h| \geq 2$, then there are links D_i, D_j, and D_k in $\mathcal{D}$ such that $i < j < k$, $(D_i \cup D_k) \subset C_g$ and $D_j \subset C_h$.

Lemma 5.5. *Suppose $\mathcal{C}$ is a spaced chain with $m = |\mathcal{C}| \geq 6$ and $\mathcal{D}$ is chain refining $\mathcal{C}$. $\mathcal{D}$ is doubly coiled in the interior of $\mathcal{C}$ if and only if there are links D_t, D_u, D_v, and D_w in $\mathcal{D}$ such that $t < u < v < w$ and one of the following holds:*

(a) $D_t \cup D_v \subset C_3$ and $D_u \cup D_w \subset C_{m-2}$, where $m = |\mathcal{C}|$, or

(b) $D_t \cup D_v \subset C_{m-2}$ and $D_u \cup D_w \subset C_3$.

Proof. Suppose that $\mathcal{C}$ is a spaced chain (with $|\mathcal{C}| \geq 6$), $\mathcal{D}$ is a chain that refines $\mathcal{C}$; let $m = |\mathcal{C}|$.

($\Rightarrow$) If $\mathcal{D}$ is doubly coiled in the interior of $\mathcal{C}$, then because C_3 and C_{m-2} (the second and second to last links of $\mathcal{C}$ respectively) are interior links of $\mathcal{C}$, and $|(m-2) - 2| = m - 4 \geq 2$, there are links D_i, D_j, and D_k in $\mathcal{D}$ such that $i < j < k$, $D_i \cup D_k \subset C_3$, and $D_j \subset C_{m-2}$. Similarly there are links $D_{i'}, D_{j'}$, and $D_{k'}$ in $\mathcal{D}$ such that $i' < j' < k'$, $D_{i'} \cup D_{k'} \subset C_{m-2}$ and $D_{j'} \subset C_3$. Let $t = min(i, i')$.

If $t = i$, then let $u = min(i', j)$, let $v = min(j', k)$ and let $w = max(k', j)$. $i < j < k$ and $i < i' < j' < k'$, so it follows that $t = i < u < v$. If $v = j'$, then $v < k' \leq w$, and similarly if $v = k$, then $v \leq j' < k' \leq w$; hence, $t < u < v < w$. Because $D_{i'} \cup D_j \subset C_{m-1}$, $D_{j'} \cup D_k \subset C_3$ and $D_{k'} \cup D_j \subset C_{m-2}$, it follows that $D_t \cup D_v \subset C_3$ and $D_u \cup D_w \subset C_{m-2}$.

If $t = i'$, a similar argument may be used by letting $u = min(i, j')$, $v = min(j, k')$, and $w = max(k, j')$, and showing that $t < u < v < w$ and that $D_t \cup D_v \subset C_{m-2}$ and $D_u \cup D_w \subset C_3$.

($\Leftarrow$) Suppose that there are links D_t, D_u, D_v and D_w in $\mathcal{D}$ such that $t < u < v < w$, $D_t \cup D_v \subset C_3$ and $D_u \cup D_w \subset C_{m-2}$ (as in part (a) of the theorem). Suppose C_g and C_h are links in $\mathcal{C}$ such that $2 \geq g, h \leq m - 2$ and $|g - h| \geq 2$. Without loss of generality, suppose that $g < h$; notice that this means $3 \leq g < h \leq m - 2$. By Theorem 2.10, because $\mathcal{C}$ is spaced, $3 \leq g \leq m - 2$, $t < u$, $D_t \subset C_3$, and $D_u \subset C_{m-2}$, there is a link D_i in $\mathcal{D}$ such that

$D_i \subset C_g$ and $t \leq i \leq u$. Because $|g-h| \geq 2$, $|g-(m-2)| \geq 2$; thus, C_g and C_{m-2} are not adjacent, meaning $i \neq u$ and therefore $i < u$.

A similar argument may be used to show that, there is D_j and D_k in $\mathcal{D}$ such that $i \leq j \leq u$, $u \leq k \leq v$, $D_j \subset C_h$, and $D_k \subset C_g$. It is now shown that there is D_i, D_j, and D_k in $\mathcal{D}$ such that $D_i \cup D_k \subset C_g$ and $D_j \subset C_h$. To conclude the argument for this case, now pick D_l such that $v \leq l \leq w$ and $D_l \subset C_h$; it follows that D_j, D_k, and D_l are links in $\mathcal{D}$ such that $j < k < l$, $D_j \cup D_l \subset C_h$ and $D_k \subset C_g$.

In the case that there are links D_t, D_u, D_v, and D_w in $\mathcal{D}$ such that $t < u < v < w$, $D_t \cup D_v \subset C_{m-1}$ and $D_u \cup D_w \subset C_2$ (as in case (b) of the theorem) an argument similar to the previous one will show that $\mathcal{D}$ is doubly coiled in the interior of $\mathcal{C}$.

□

Theorem 5.6. *Suppose* $\{\mathcal{C}_n\}_{n=1}^{\infty}$ *is a sequence of chains such that if* $n \in \mathbb{N}$*, then*

(i) $\mathcal{C}_{n+1}$ *properly refines* $\mathcal{C}_n$*,*

(ii) $\mathcal{C}_n$ *is a* $\frac{1}{n}-$*chain, and*

(iii) $\mathcal{C}_{n+1}$ *is doubly coiled in the interior of* $\mathcal{C}_n$*.*

If $M = \cap_{n=1}^{\infty}(\cup \mathcal{C}_n)$*, then* M *is indecomposable.*

Proof. Suppose K is a proper subcontinuum of M. Let $\tilde{\mathcal{C}_i} \subset \mathcal{C}_i$ containing exactly those links in $\mathcal{C}_i$ that intersect K. It will be shown that no $\epsilon - ball$ centered at a point in K is contained in K; hence, K has no interior.

Let $p \in K$ and $\epsilon > 0$. If $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{3}$, then if C_a^n, C_b^n, and C_c^n are three consecutive links in $\mathcal{C}_n$ and one contains p, then $C_a^n \cup C_b^n \cup C_c^n \subset B(p, \epsilon)$ (the open ball of radius ϵ centered at p). Let $q \in M \setminus K$ and choose N to be a positive integer such that $\frac{1}{N} < min(\frac{\epsilon}{3}, \frac{d(q,K)}{4})$. Because $mesh(\mathcal{C}_N) < \frac{1}{N} < \frac{d(p,q)}{7}$, the union of six adjacent links in $\mathcal{C}_{N+1}$ cannot contain both p and q, and a link in $\mathcal{C}_N$ containing q will not intersect K. Let w be the index of a link in $\mathcal{C}_N$ that contains p and let z be the index of a link in $\mathcal{C}_N$ that contains q. It is possible that w or z is not in $\{3, \dots, |\mathcal{C}^N| - 2\}$, so choose x and y to be indices of links in $\mathcal{C}_N$ such that $|w - x| \leq 2$, $|z - y| \leq 2$, and $w, z \in \{3, \dots, (|\mathcal{C}^N| - 2)\}$.

Because no six adjacent links in $\mathcal{C}_N$ cover both p and q, $|w - z| \geq 6$. Since $|x - w| \leq 2$ and $|y - z| \leq 2$, $|x - y| > |w - z| - |x - w| - |y - z| \geq 6 - 4 = 2$ and so by the initial assumptions $r, s, t \in \mathbb{N}$ (with $r < s < t$) may be chosen so that each is an index of a link in $\mathcal{C}_{N+1}$, $C_r^{N+1} \cup C_t^{n+1} \subset C_x^N$, and $C_s^{N+1} \subset C_y^N$. Recall that the union of four adjacent links in $\mathcal{C}_N$ cannot contain q and cover K; because $|y - z| \leq 2$ and C_y contains q, C_y^N cannot intersect K, and because $C_s^{N+1} \subset C_y^N$, $C_s^{N+1} \cap K = \emptyset$ as well. It follows that $C_s^{N+1} \notin \mathcal{C}'_{N+1}$ (where $\mathcal{C}'_{N+1}$ is the segment from $\mathcal{C}_{N+1}$ containing exactly those links intersecting K); this means that C_r^{N+1} or C_t^{N+1} is not an element of $\mathcal{C}'_{N+1}$ since $\mathcal{C}'_{N+1}$ is a segment. Because $|w - x| \leq 2$ and the diameters of each link in $\mathcal{C}_N$ is less than $\frac{\epsilon}{3}$, the diameter of $C_w^N \cup C_x^N$ is less than ϵ. $p \in C_w^N$ so $C_w^N \cup C_x^N \subset B(p, \epsilon)$. Both C_r^{N+1} and C_t^{N+1} are disjoint subsets of C_x^N, and therefore each is a subset of $B(p, \epsilon)$. Because C_r^{N+1} or C_t^{N+1} is not in $\mathcal{C}'_{N+1}$, it follows that $B(p, \epsilon)$ contains an open set that is not a subset of K.

Hence, a proper subcontinuum of M must be nowhere dense in M, meaning M is indecomposable.

□

Corollary 5.7. *Suppose $\{\mathcal{C}_n\}_{n=1}^{\infty}$ is a sequence of spaced chains such that for each $n \in \mathbb{N}$,*

(i) $\mathcal{C}_n$ is a $\frac{1}{n}$−chain,

(ii) $\mathcal{C}_{n+1}$ properly refines $\mathcal{C}_n$, and

(iii) if $m = |\mathcal{C}_n|$, then there are integers t, u, v, w such that $1 < t < u < v < w < |\mathcal{C}_{n+1}|$ such that $C_t^{n+1} \cup C_v^{n+1} \subset C_2^n$ and $C_u^{n+1} \cup C_w^{n+1} \subset C_{m-1}^n$, or $C_t^{n+1} \cup C_v^{n+1} \subset C_{m-1}^n$ and $C_u^{n+1} \cup C_w^{n+1} \subset C_2^n$;

then $M = \cap_{i=1}^{\infty}(\cup \mathcal{C}_n)$ is an indecomposable continuum.

Proof. By Lemma 4.6, property (iii) in the Corollary is equivalent to property (3) in Theorem 4.7, thus M is indecomposable. □

Definition 5.8. Suppose $\mathcal{C}$ is a chain with length greater than five, and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. To say that $\mathcal{D}$ is **very crooked** in $\mathcal{C}$ means that if $C_r, C_s \in \mathcal{C}$ (with $|r-s| \geq 5$), and t and w are each indices from $\mathcal{D}$, such that $D_t \subset C_r$ and $D_w \subset C_s$, then there are indices u and v such that $t < u < v < w$, and

1. if $r < s$, then $D_u \subset C_{s-1}$ and $D_v \subset C_{r+1}$;

2. if $r > s$, then $D_u \subset C_{s+1}$ and $D_v \subset C_{r-1}$.

Theorem 5.9. *Suppose that $\{\mathcal{C}_n\}_{n=1}^{\infty}$, is a sequence of chains with respective lengths $\{l_n\}_{n=1}^{\infty}$, such that $l_1 = |\mathcal{C}_1| \geq 6$ and for each $n \in \mathbb{N}$,*

(a) $\mathcal{C}_n$ is a $\frac{1}{n}$−chain,

(b) $\mathcal{C}_{n+1}$ properly refines $\mathcal{C}_n$,

(c) $\mathcal{C}_{n+1}$ is very crooked in $\mathcal{C}_n$, and

(d) $C_1^{n+1} \subset C_1^n$ and $C_{l_{n+1}}^{n+1} \subset C_{l_n}^n$.

If $M = \cap_{i=1}^{\infty}(\cup\mathcal{C}_i)$ and K is a subcontinuum of M, then K is indecomposable.

Proof. Let $M = \cap_{i=1}^{\infty}(\cup\mathcal{C}_i)$ and suppose K is a proper subcontinuum of M. If K is a singleton, then there do not exist two nonempty proper subcontinuums of K; hence K is indecomposable. Suppose then, that K is not a singleton.

For each $i \in \mathbb{N}$, let $\tilde{\mathcal{C}_i}$ denote the collection of links in $\mathcal{C}_i$ that intersect K. It follows from 5.2 that $\tilde{\mathcal{C}_i}$ is a segment in $\mathcal{C}_i$.

Let $\mathcal{K}_i$ denote the chain formed by reenumerating the links in $\tilde{\mathcal{C}_i}$. It follows that $\cap_{i=1}^{\infty}(\cup\mathcal{K}_i) = K$, for if x is in the intersection, then $d(x, K) < \frac{1}{i}$ for each $i \in \mathbb{N}$, thus x is a limit point of $K \Rightarrow x \in K$. $\mathcal{K}_n$ is spaced because $\mathcal{C}_n$ is spaced. Lastly, $\mathcal{K}_{n+1}$ is a refinement of $\mathcal{K}_n$, for if K_i^{n+1} is a link in $\mathcal{K}_{n+1}$ and $C_{i'}^{n+1}$ is the link in $\mathcal{C}_{n+1}$ corresponding to K_i^{n+1}, then there is a link $C_{j'}^n \in \mathcal{C}_n$ that contains $C_{i'}^{n+1}$, but this means that $C_{j'}^n$ intersects K, and so there is a link $K_j^n \in \mathcal{K}_n$ that contains $C_{i'}^{n+1} = K_i^{n+1}$, therefore, $\mathcal{K}_{n+1}$ is a refinement of $\mathcal{K}_n$.

Let $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \frac{diam(K)}{6}$; thus, for each integer $n \geq N$, $mesh(\mathcal{K}_n) \leq \frac{diam(K)}{6}$, meaning $|\mathcal{K}_n| \geq 6$ in order for $\mathcal{K}_n$ to cover K.

Now suppose n is an integer and $n \geq N$, and let $m = |\mathcal{K}_n|$. K_2^n and K_{m-1}^n correspond to links in in $\tilde{\mathcal{C}_n}$ (and thus $\mathcal{C}^n$ as well), call these corresponding links $C_{g'}^n$ and $C_{h'}^n$, respectively. It follows that $K_1^n = C_{g'-1}^n$ and $K_m^n = C_{h'+1}^n$, since $\mathcal{K}^n$ corresponds to a segment from $\mathcal{C}^n$; further, it is established that $g' < h'$. Because $\mathcal{K}^n$ is a spaced chain that covers the continuum K and because K_2^n and K_{m-1}^n are interior links of $\mathcal{K}^n$, points p and q in K can be chosen, such that K_2^n is the only link in $\mathcal{K}_n$ that covers p and K_{m-1}^n is the only link in $\mathcal{K}_n$ that covers q. p and q are each covered by links in $\mathcal{K}_{n+1}$, so let $\mathcal{K}_s^{n+1}$ and K_x^{n+1} be links in $\mathcal{K}^{n+1}$ containing p and q respectively. Since $p \in K_s^{n+1}$, K_s^{n+1} contains a point contained in exactly one link in $\mathcal{K}_n$ (*ie* K_2^n); for this reason and the fact that $\mathcal{K}^{n+1}$ refines $\mathcal{K}^n$, it follows that $K_s^{n+1} \subset K_2^n$. By a similar argument, $K_x^{n+1} \subset K_{m-1}^n$ because $q \in K_x^{n+1}$.

Letting $C_{s'}^{n+1}$ and $C_{x'}^{n+1}$ each be the links in $\mathcal{C}^{n+1}$ corresponding to K_s^{n+1} and K_x^{n+1}, it follows that $C_{s'}^{n+1} \subset C_g^n$ and $C_{x'}^{n+1} \subset C_h^n$.

Case 1: Suppose $s' < x'$...., then by the initial assumptions, there is $C_{u'}^{n+1}$ and $C_{v'}^{n+1}$ in $\mathcal{C}^{n+1}$ such that $s' < u' < v' < x'$, $C_{u'}^{n+1} \subset C_{h'-1}^n$, and $C_{v'}^{n+1} \subset C_{g'+1}^n$ (remember that $g' < h'$). Let K_u^{n+1} and K_v^{n+1} be links in $\mathcal{K}^{n+1}$ corresponding to $C_{u'}^{n+1}$ and $C_{v'}^{n+1}$. Thus, $s < u < v < x$, $K_u^{n+1} \subset K_{m-2}^n$ and $K_v^{n+1} \subset K_3^n$. Since $s < u$, $K_s^{n+1} \subset K_2^n$ and $K_u^{n+1} \subset K_{m-2}^n$, there is K_t^{n+1} such that $s < t < u$, $K_t^{n+1} \subset K_3^n$; similarly, there is K_w^{n+1} such that $v < w < x$ and $K_w^{n+1} \subset K_{m-2}^n$. It follows that $K_t^{n+1}, K_u^{n+1}, K_v^{n+1}$ and K_w^{n+1} are links in $\mathcal{K}^{n+1}$ such that $t < u < v < w$, $K_t^{n+1} \cup K_v^{n+1} \subset K_3^n$ and $K_u^{n+1} \cup K_w^{n+1} \subset K_{m-2}^n$.

Case 2: If $x' < s'$, an argument similar to that of Case 1, will choose links K_w^{n+1}, K_v^{n+1}, K_u^{n+1} and K_t^{n+1} in $\mathcal{K}^{n+1}$ such that $w < v < u < t$, $K_w^{n+1} \cup K_u^{n+1} \subset K_2^n$ and $K_v^{n+1} \cup K_t^{n+1} \subset K_{m-2}^n$.

Because the above argument holds for each integer n such that $n \geq N$, it follows that $\{\mathcal{K}^n\}_{n=N}^{\infty}$ is a collection of spaced chains satisfying conditions (i), (ii), and (iii) in Corollary 4.8; thus, $M = \cap_{n=N}^{\infty}(\cup\mathcal{K}^n)$ is indecomposable. □

From the above theorem, if a chainable continuum is formed from a sequence of chains as described in the theorem, then such a continuum is hereditarily indecomposable. The final step is to show that such a continuum exists.

Some modifications to previous terms will come in handy.

Definition 5.10. Suppose $\mathcal{C}$ is a chain in $\mathbb{R}^2$. To say that $\mathcal{C}$ is a **rectangular chain**, means that if $C \in \mathcal{C}$, then there are real numbers a, b, c, and d such that $C = (a, b) \times (c, d)$. To say that the rectangular chain $\mathcal{C}$ is a **straight rectangular chain** means that there are numbers c, d and if $C_i \in \mathcal{C}$ and $C_i = (a_i, b_i) \times (c_i, d_i)$, then $c_i = c$ and $d_i = d$.

Definition 5.11. Suppose $\mathcal{C}$ is a chain and $\mathcal{D}$ is a chain that refines $\mathcal{C}$. To say that $\mathcal{D}$ is a **snug refinement** of $\mathcal{C}$ means that each link of $\mathcal{D}$ is a subset of exactly one link in $\mathcal{C}$.

Definition 5.12. If $\mathcal{C}$ is a chain and $\mathcal{D}$ is a chain that is anchored in $\mathcal{C}$, then to say that $\mathcal{D}$ is **securely anchored in** $\mathcal{C}$, means that the first link of $\mathcal{D}$ is only contained in the first link of $\mathcal{C}$, the last link of $\mathcal{D}$ is only contained in the last link in $\mathcal{C}$, and an interior link of $\mathcal{D}$, is only contained in an interior link in $\mathcal{C}$.

Definition 5.13. Suppose $\mathcal{C}$ is a spaced rectangular chain. $\mathcal{C}$ is **straight**, means that there is $c, d \in \mathbb{R}$ (with $c < d$) so that if $C_i \in \mathcal{C}$ and $C_i = (a_i, b_i) \times (c_i, d_i)$, then $c_i = c$ and $d_i = d$.

Theorem 5.14. *If $\mathcal{C}$ is a spaced rectangular chain that is straight, then there is a rectangular chain $\mathcal{D}$, such that $\mathcal{D}$ is securely anchored in $\mathcal{C}$, and $\mathcal{D}$ is very crooked in $\mathcal{C}$.*

Proof. First note, that for a very crooked chain in $\mathcal{C}$ to have any novel qualities, that there must be two links in $\mathcal{C}$ whose indices differ by five. If $|\mathcal{C}| \leq 5$, then the fact that $\mathcal{C}$ is very crooked in $\mathcal{C}$ is vacuously true. Thus, if $n \leq 5$ and $\mathcal{C}$ is a spaced rectangular chain that is straight, then $\mathcal{C}$ is refined by a rectangular chain that is securely anchored in $\mathcal{C}$ and very crooked in $\mathcal{C}$.

The remainder of the proof will be done inductively. Suppose that $n \in \mathbb{N}$ (with $n > 5$) and for each $m \in \mathbb{N}$, $m < n$, it is known that if $\mathcal{C}'$ is a spaced rectangular chain that is straight and $|\mathcal{C}'| = m$, then there is a rectangular chain that refines $\mathcal{C}'$ that is securley anchored in $\mathcal{C}'$ and very crooked in $\mathcal{C}'$.

It will know be shown that if $\mathcal{C}$ is a spaced, rectangular chain that is straight and $|\mathcal{C}| = n$, then there is a rectangular chain that is securely anchored in $\mathcal{C}$ and very crooked in $\mathcal{C}$).

With a brief slight of hand, the author now focuses the audience's attention to the specific case of the chain $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, where

$$C_i = (i - \frac{2}{3}, i + \frac{2}{3}) \times (0, 3).$$

It is hoped that the reader will accept a validation of this specific case to cary over to all other spaced rectangular chains that are straight.

Define the collections $\mathcal{C}^a, \mathcal{C}^b$, and $\mathcal{C}^c$ as follows:

(a) $\mathcal{C}^a = \{C_i^a = C_i \cap (\mathbb{R} \times (2,3)) : 1 \leq i \leq n-1\}$

(b) $\mathcal{C}^b = \{C_i^b = C_{i+1} \cap (\mathbb{R} \times (1,2)) : 1 \leq i \leq n-2\}$

(c) $\mathcal{C}^c = \{C_i^c = C_{i+1} \cap (\mathbb{R} \times (0,1)) : 1 \leq i \leq n-1\}$.

It will be taken for granted that each of $\mathcal{C}^a, \mathcal{C}^b$, and $\mathcal{C}^c$ forms a chain that is spaced, rectangular, straight and with length less than n. By the induction hypothesis, $\mathcal{D}^a$, $\mathcal{D}^b$, and $\mathcal{D}^c$, may be chosen to be rectangular chains that are very crooked and securely anchored in the respective chains $\mathcal{C}^a, \mathcal{C}^b$, and $\mathcal{C}^c$.

Let $x = |\mathcal{D}^a|, y = |\mathcal{D}^b|$, and $z = |\mathcal{D}^c|$, and construct the chain $\mathcal{D}$ as follows:

(a) if $1 \leq i \leq x$, let $D_i = \mathcal{D}_i^a$;

(b) let $D_{x+1} = (n - 1 - \frac{1}{3}, n - 1 + \frac{1}{3}) \times (1, 3)$;

(c) if $x + 2 \leq i \leq x + y + 1$, let $D_i = -D^b_{i-(x+1)}$;

(d) let $D_{x+y+2} = (2 - \frac{1}{3}, 2 + \frac{1}{3}) \times (0, 2)$;

(e) if $x + y + 3 \leq i \leq x + y + z + 2$, let $D_i = D^b_{i-(x+y+2)}$.

Notice that $D_{x+1} \subset C_{n-2} \neq \emptyset$, and that $D_{x+y+2} \cap C_3 \neq \emptyset$; thus D_{x+1} and D_{x+y+2} cannot be subsets of C_1 or C_2. Because $\mathcal{D}^a$ and $\mathcal{D}^b$ are securely anchored in $\mathcal{C}^a$ and $\mathcal{C}^b$, respectively, and because D_{x+1} only intersects $C_{n-} \cap (2, 3) \times \mathbb{R}$, D_{x+1} only intersects the last links of $\mathcal{D}^a$ and $\mathcal{D}^b$. Similarly, D_{x+y+2} only intersects the first of $\mathcal{D}^b$ and the first link of $\mathcal{D}^c$.

$\mathcal{D}$ is securely anchored in $\mathcal{C}$. To show this, first note that D_{x+1} and D_{x+y+2} only intersect C_{n-2} and C_3, respectively, so neither can be a subset of D_1 or $D_{x+y+z+2}$. D^a is the only defining chain for $\mathcal{D}$ that intersects C_1, thus a link of $\mathcal{D}$ that is contained in $\mathcal{C}^1$ must be from $\mathcal{D}^a$. $\mathcal{D}^a$ is securely anchored in $\mathcal{C}^a$, so D^a_1 is the only link contained in C^a_1. By the construction of C^a_1, if a link in $\mathcal{D}^a$ does not lie inside C^a_1, then it will not lie inside C_1; hence $D_1 = D^a_1 \subset C^a_1 \subset C_1$ and D_1 is the only link in $\mathcal{D}$ that is a subset of C_1.

In a similar fashion it can be shown that $\mathcal{D}_{x+y+z+2} \subset C_n$ and $\mathcal{D}_{x+y+z+2}$ is the only link of $\mathcal{D}$ that is a subset of C_n.

The final step is to prove that $\mathcal{D}$ is very crooked. Suppose $C_r.C_s \in \mathbb{N}$ such that $|r - s| \geq 5$, and that $D_t, D_w \in \mathcal{D}$ such that $D_t \subset C_r$ and $D_w \subset C_s$. It will be shown that if $r < s$), then there is $D_u, D_v \in \mathcal{D}$, such that $t < u < v < w$ and $D_u \subset C_{s-1}$ and $D_v \subset C_{r+1}$; the case when $D_t \subset C_s$ and $D_w \subset C_r$ can be proven in a similar manner.

For the moment, suppose that $t \notin \{x + 1, x + y + 2\}$ and let $q \in \{a, b, c\}$ such that D_t is chosen from the defining chain $\mathcal{D}^q$.

If D_w is also defined from a link in $\mathcal{D}^q$, then it follows that appropriate links D_u and D_v exist, since $\mathcal{D}^q$ is very crooked in $\mathcal{C}^q$, and $\mathcal{C}^q$ refines $\mathcal{C}$.

If C_s contains a link in $\mathcal{D}^q$, then $D_{w'}$ may be chosen to be a link defined from a link in $\mathcal{D}^q$ such that $D_{w'} \subset C_s$; furthermore, because the last link of $\mathcal{D}^q$ is contained in a link in $\mathcal{C}$ with index greater than or equal to s, it may be assumed that $t < w'$. From the argument in the previous paragraph, links D_u and D_v may be chosen, so that $D_u \subset C_{s-1}$ and $D_v \subset C_{r+1}$.

Lastly, if C_s does not contain a link in $\mathcal{D}^q$, then it follows that $s = n$ and $q \neq c$. Only the last link of $\mathcal{D}$ is contained in $\mathcal{C}_n$, thus $w = x + y + z + 2$. If $t = 1$, then $r = 1$, and D_{x+1} and D_{x+y+2} are links in $\mathcal{D}$ such that $t < x + 1 < x + y + 2 < w$, $D_{x+1} \subset C_{n-1}$ and $D_{x+y+2} \subset C_2$. If $t > 1$, then $r > 1$ and C_r contains a link in $\mathcal{D}^c$. Let $D^c_{z'}$ be the first link of $\mathcal{D}^c$ that is contained in C_r, and define t' as $t' = x + y + 2 + z'$. Because $D_{t'}$ and D_w are both defined from links in $\mathcal{D}^c$ and $\mathcal{D}^c$ is very crooked in $\mathcal{C}$, there are links D_u and D_v such that $D_u \subset C_{n-1}$ and $D_v \subset D_{r+1}$.

Earlier, t was excused from being equal to $x + 1$ or $x + y + 2$; these cases shall now be unexcused. $t \neq x + 1$, since this would mean $r \geq n - 3$ and thus s would have to be greater than $r + 5 = n + 2$, meaning $s > n$. If $t = x + y + 2$, then let $t' = x + y + 3$. $D_{t'}$ is defined by the first link in $\mathcal{D}^c$, which is very crooked in $\mathcal{C}^c$; thus, there is D_u and D_v such that $t' < u < v < w$, $D_u \subset C_{n-1}$ and $D_v \subset C_3$. Since $t < t'$, and D_t only intersects C_2, it follows that D_u and D_v are appropriate choices for t as well.

It is now concluded that if $n \in \mathbb{N}$ and $\mathcal{C}$ is a spaced rectangular chain of length n that is straight, then there is a rectangular chain that is very crooked in $\mathcal{C}$. □

The following theorem is not so much a corollary, as it is a theorem that would have been preferable to prove using a technique similar to the previous proof. A sketch of an argument will be given, but a solid proof requires further development of the properties of $\mathbb{R}^2$.

Corollary 5.15. *If $\mathcal{C}$ is a chain of convex open subsets of $\mathbb{R}^2$, then there is $\mathcal{D}$, a chain of convex open subsets of $\mathbb{R}^2$, such that $\mathcal{D}$ is anchored in $\mathcal{C}$ and $\mathcal{D}$ is very crooked in $\mathcal{C}$.*

Sketch: Because each link in $\mathcal{C}$ is convex $\cup\mathcal{C}$ is path connected and there is an arc contained in $\cup\mathcal{C}$ that begins in the first link in $\mathcal{C}$ and ends in the last link in $\mathcal{C}$. This arc is "thickened" so that it remains inside of $\mathcal{C}$. Let A_i denote the intersection of this thickened arc with $C_i \in \mathcal{C}$, and let $\mathcal{A} = \{A_i : C_i \in \mathcal{C}\}$. $\mathcal{A}$ is a chain and there is a homeomorphism $h : \cup\mathcal{R}^{|\mathcal{A}|} \to \cup\mathcal{A}$, where $\mathcal{R}^{|\mathcal{A}|}$ is a spaced rectangular chain of length $|\mathcal{A}|$. From the prior theorem, there is $n \in \mathbb{N}$ such that $\mathcal{R}^{|\mathcal{A}|}$ can be refined by a rectangular chain of length n

that is very crooked in $\mathcal{R}^{|\mathcal{D}|}$; denote such a chain as $\mathcal{R}^n$. For each $R_i^n \in \mathcal{R}^n$, define B_i as $B_i = h^{-1}(R_i)$ and let $\mathcal{B} = \{B_i : R_i^n \in \mathcal{R}^n\}$. $\mathcal{B}$ is a chain that refines $\mathcal{C}$ and is very crooked in $\mathcal{C}$. Although the links in $\mathcal{B}$ may not be convex, $\cup\mathcal{B}$ is path connected and there is a an arc contained in $\cup\mathcal{B}$ that begins in the first link of $\mathcal{B}$ and ends in the last link of $\mathcal{B}$. This arc may be covered by $\mathcal{D}$ a chain of convex open subsets of $\mathbb{R}^2$ such that $\mathcal{D}$ refines $\mathcal{B}$; hence, $\mathcal{D}$ is very crooked in $\mathcal{C}$.

With the above "corollary" in mind, the following sequence of chains may be defined.

Let $\mathcal{C}^1$ be a chain of length six, whose links are open balls with radius $\frac{1}{2}$. For each $i \in \mathbb{N}$, if $\mathcal{C}^i$ is defined as a chain whose links are open convex subsets of $\mathbb{R}^2$, let $\mathcal{D}^i$ be a chain properly refines $\mathcal{C}^i$ such that each link in $\mathcal{D}^i$ is convex and open, and $mesh(\mathcal{D}^i) < \frac{1}{i+1}$. Let $\mathcal{C}^{i+1}$ be a chain of convex open subsets of $\mathbb{R}^2$, such that $\mathcal{C}^{i+1}$ is very crooked in $\mathcal{D}^i$.

For each $i \in \mathbb{N}$,

1. $\mathcal{D}^i$ properly refines $\mathcal{C}^i$ and $\mathcal{C}^{i+1}$ refines $\mathcal{D}^i$, therefore $\mathcal{C}^{i+1}$ properly refines $\mathcal{C}^i$;

2. $mesh(\mathcal{C}^1) = 1$ and $mesh(\mathcal{C}^{i+1}) \leq mesh(\mathcal{D}^i) < \frac{1}{i+1}$;

3. $\mathcal{D}^i$ refines $\mathcal{C}^i$ and $\mathcal{C}^{i+1}$ is very crooked in $\mathcal{D}^i$, therefore $\mathcal{C}^{i+1}$ is very crooked in $\mathcal{C}^i$.

Thus, by 5.9, $\cap_{i=1}^{\infty}(\cup\mathcal{C}^i)$ is an hereditarily indecomposable continuum.

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