Properties of Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs

Except where reference is made to the work of others, the work described in this dissertation is my own or was done in collaboration with my advisory committee. This dissertation does not include proprietary or classified information.

Regina Greiwe Jackson

Certificate of Approval:

Stewart Baldwin
Professor
Mathematics and Statistics

Michel Smith, Chair
Professor
Mathematics and Statistics

Gary Gruenhage
Professor
Mathematics and Statistics

Jo Heath
Professor Emerita
Mathematics and Statistics

George Flowers
Dean
Graduate School

Properties of Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs

Regina Greiwe Jackson

A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
December 18, 2009

Properties of Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs

Regina Greiwe Jackson

Signature of Author

Date of Graduation

Vita

Regina Marie Greiwe, daughter of Michael Greiwe and Karen Broome, was born February 4, 1982. She attended Tri-County High School in Buena Vista, Georgia where she graduated fourth in her class in 1999. She then entered Columbus State University under an Honors scholarship where she graduated Magna cum Laude in June, 2003. She was awarded with a Bachelor of Science degree in Applied Mathematics. While at Columbus State University, she was president of the local chapter of the MAA and was awarded the Mathematics Award in 2003. The following year she entered Auburn University to pursue a Master of Science degree in Mathematics. While at Auburn University, she served as president of the local chapter of SIAM and has been involved in Science Olympiad and mathematics education. In 2006, she was awarded a Master of Science in Mathematics. She is the wife of Raymond Jackson, Jr., and she is the mother of Laura Jeanette Amilia Greiwe, and Caroline Elizabeth Jackson.

Dissertation Abstract

Properties of Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs

Regina Greiwe Jackson

Doctor of Philosophy, December 18, 2009
(M.A., Auburn University, 2006)
(B.S., Columbus State University, 2003)

41 Typed Pages

Directed by Michel Smith

Nonmetric hereditarily indecomposable subcontinua of finite products of Lexicographic arcs are examined. It is shown that these subcontinua cannot intersect certain subsets of the products. Then nonmetric hereditarily indecomposable subcontinua of these same products cross the Hilbert cubes are examined and are shown to not intersect certain subsets. In conclusion, it is shown that all hereditarily indecomposable subcontinua of the product of three Lexicographic arcs are metric.

Acknowledgments

I would like to thank my family for helping me take care of lifes little bumps so that I could concentrate on my work. I would also like to thank my topology professors for providing me with inspiration along the way. I would like to thank Jo Heath for pushing me to work with Dr. Michel Smith. She knew, better than I did, where I would excel. Thank you Dr. Smith for being patient with me, providing me with inspiration (and an interesting problem), and showing so much excitement for what I've done. I would also like to thank you, the reader, for taking the time to look at my thoughts and work.

Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliography follows van Leunen’s *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

Table of Contents

List of Figures

Chapter 1

Introduction and Background

A continuum is traditionally defined as a compact, connected, metric space. One of the simplest examples is an arc, which is a space homeomorphic to the unit interval. Basically it is a continuum with two endpoints, one of which is mapped to 0, and the other which is mapped to 1. In this paper we are concerned with one of many nonmetric analogs to the arc, the Lexicographic arc. Throughout we will define a **continuum** to be a compact, connected Hausdorff space. If a continuum is metric, it will be stated.

In 1951, Bing showed that higher dimensional metric continua must contain nondegenerate hereditarily indecomposable subcontinua [1]. He also is responsible for showing that the set of all pseudo-arcs is a dense G_δ set in R^n [2]. In a sense this means that the set of all hereditarily indecomposable subcontinua of Euclidean space is large. Recently, Michel Smith has been studying how hereditarily indecomposable continua sit inside nonmetric continua [6] [7] [8] [9] [10] [11]. Surprisingly, it seems that metrizability and hereditary indecomposability could possibly be linked. He has shown that all hereditarily indecomposable subcontinua of the inverse limit of both Souslin arcs and Lexicographic arcs are metric [8] [9]. Also, he has shown that the product of two Souslin arcs contain only metric hereditarily indecomposable continua [10] [11]. The purpose of this paper is to investigate the existence of hereditarily indecomposable continua in finite products of Lexicographic arcs. Michel Smith, in conjunction with Jennifer Stone, showed that every hereditarily indecomposable subcontinuum of the product of two Lexicographic arcs is metric [11]. The author will develop a technique to restrict the existence of nonmetric hereditarily indecomposable subcontinua in finite products, and in finite products cross the Hilbert cube. Most of the techniques used can be extended to a countable product of Lexicographic arcs.

The goal is to show that any hereditarily indecomposable subcontinuum of a finite product of Lexicographic arcs is metrizable. In this paper, the author will prove it for the product of three of these arcs.

Let us start by describing the Lexicographic arc and by introducing some notation that will be used throughout the paper. The Lexicographic arc is the set $[0,1] \times [0,1]$ given an order topology based on the following order. We say that $(a,b) < (c,d)$ provided that either $a < c$, or $a = c$ and $b < d$. This produces a compact, connected, Hausdorff space with endpoints $(0,0)$ and $(1,1)$. We will denote the Lexicographic arc with L, and we will introduce a notation of a point in $[0,1] \times [0,1]$ to simplify the interval notation and points in higher dimensional products. Let the point $(x,y) = x_y$. Further research into the Lexicographic arc can be found in the author's Masters Thesis [13].

One way to think of the lexicographic arc is to think of the unit square as an arc which has each point replaced with another arc. Refer to Figure 1.1.

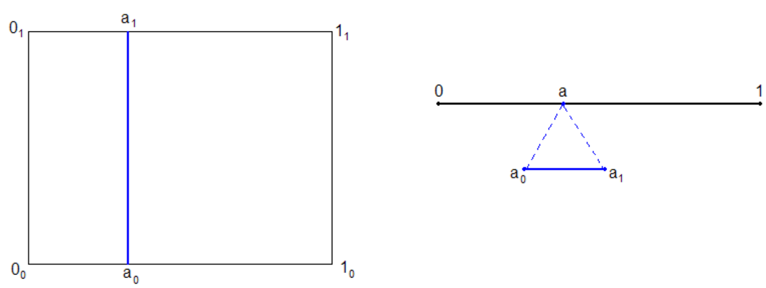

Figure 1.1: The Lexicographic Arc

In turn, products of n Lexicographic arcs can be thought of as the unit n-cube with each point replaced with an n-cube. Figure 1.2 depicts the product of two and three said arcs. We will denote the product of n Lexicographic arcs as $\prod_{i=1}^{n} L = L^n$.

During our exploration we will have need of two more spaces, the Hilbert cube and the compact Double Arrow Space. The Hilbert cube is defined as the countable product of unit intervals, and it will be denoted by $[0,1]^\infty = \prod_{i=1}^{\infty}[0,1]$ with the usual topology on $[0,1]$.

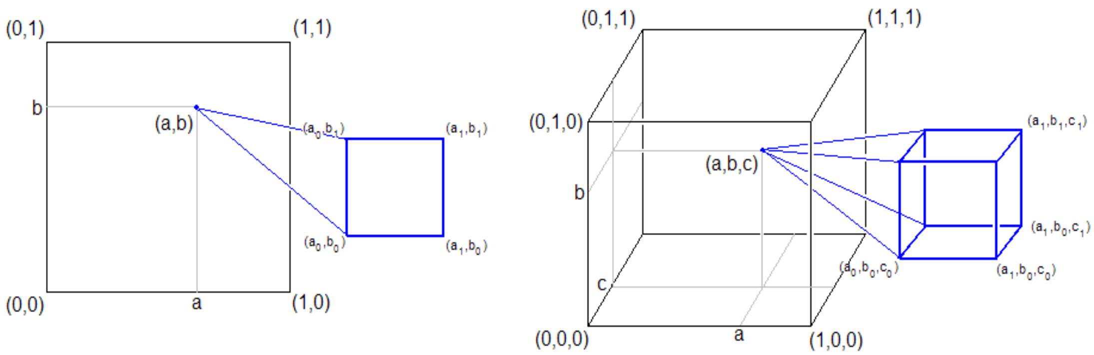

Figure 1.2: L^2 and L^3

The Double Arrow space is the order topology induced by the lexicographic order described at the beginning of this chapter on the set $[0,1] \times \{0,1\}$.

The following definitions and theorems will be used extensively throughout the paper.

Definition 1.1. *A space is said to be* ***separable*** *if it contains a countable, dense subset.*

Definition 1.2. *A space is said to be* ***completely separable*** *given it has a countable basis.*

Definition 1.3. *A continuum is* ***indecomposable*** *if it cannot be written as the union of two proper subcontinuum.*

Definition 1.4. *A continuum is* ***hereditarily indecomposable*** *if every subcontinuum is indecomposable.*

Definition 1.5. *Let A be a subset of a space X. The* ***component*** *of $x \in A$ is the union of all connected subsets of A containing x.*

Definition 1.6. *Let A and B be subsets of a space X. X can be* ***separated over*** *A* ***and*** *B provided that there exist disjoint open sets, U and V, such that $U \cup V = X$, $A \subset U$, and $B \subset V$.*

Definition 1.7. *A point $x \in X$ is a* ***cut point of*** *X if $X - \{x\}$ is not connected.*

Definition 1.8. *An* ***arc*** *is a continuum having exactly two non-cut points.*

Further exploration of the properties of a metric arc was done by Nadler [4].

Definition 1.9. *A subset of a space is* ***nowhere dense*** *in the space if it has empty interior.*

Definition 1.10. *Let X be a topological space. Then the* ***hyperspace*** *of X denoted by 2^X is the space of nonempty compact subsets of X. Let $\{U_1, U_2, U_3, ..., U_n\}$ be a finite collection of open subsets of X. Then the collection of sets of the form $\{K \in 2^X | K \in \bigcup_{i=1}^{n} U_i$ and for each $1 \leq i \leq n$, $U_i \cap K \neq \emptyset\}$ constitutes a basis for 2^X.*

Definition 1.11. *Let X be a topological space. Then we denote the subspace of 2^X known as the* ***hyperspace of continua*** *as $C(X) = \{K \in 2^x | K$ is a continuum$\}$.*

For a more in depth look at hyperspaces refer to [5].

Definition 1.12. *Let x be a point in a continuum X. Define a partial order on $C(X)$ using inclusion. Then an* ***order arc of*** *x* ***in*** *X is an arc in $C(X)$ with endpoints $\{x\}$ and X.*

Theorem 1.1. *If A is a proper subcontinuum of an indecomposable continuum X, then A is nowhere dense in X.*

Theorem 1.2. *If a space X is compact and completely separable, then X is metrizable.*

Theorem 1.3. *If an arc is separable, then it is completely separable.*

Corollary 1.1. *If an arc is separable, then it is metrizable.*

Theorem 1.4. *Let X be a compact Hausdorff space, and let A and B be disjoint closed subsets of X such that no component intersects both A and B. Then X can be separated over A and B.*

Theorem 1.5. *Let X be a hereditarily indecomposable continuum. If E and F are disjoint, closed subsets of X contained in open sets U and V respectively, then there exist closed sets A, B, and C such that $X = A \cup B \cup C$, $E \subset A$, $F \subset C$, $A \cap B \subset V - F$, $B \cap C \subset U - E$, and $A \cap C = \emptyset$.*

A proof of the metric case can be found in [3].

Theorem 1.6. *The order arc generated by a hereditarily indecomposable subcontinuum is unique.*

Theorem 1.7. *Let X be a hereditarily indecomposable continuum, and let $\alpha(p)$ denote the order arc of some point $p \in X$. If there exists an open neighborhood, U of p, containing a countable collection of open subsets, $\{G_i\}_{i=1}^{\infty}$, having the properties that for each $H \subset K \in \alpha(p)$ such that $H \neq K$, there is an $i < \infty$ such that $G_i \cap (K - H) \neq \emptyset$ and $\bar{G}_i \cap H = \emptyset$, then $\alpha(p)$ is separable.*

Proof. Let X be a hereditarily indecomposable continuum, and choose $p \in X$. Let U be an open neighborhood, U of p, containing a countable collection of open subsets, $\{G_i\}_{i=1}^{\infty}$, having the properties that for each $H \subset K \in \alpha(p)$ such that $H \neq K$, there is an $i < \infty$ such that $G_i \cap (K - H) \neq \emptyset$ and $\bar{G}_i \cap H = \emptyset$. Let X_i be the irreducible subcontinuum of X between p and $\bar{G}_i$. Then $\{X_i\}_{i=1}^{\infty}$ is countable.

We claim that $\{X_i\}_{i=1}^{\infty}$ is dense in $\alpha(p)$. Let $H \subset K \in \alpha(p)$ such that $H \neq K$. Then we must show that there exists $i < \infty$ such that $H \subset X_i \subset K$ and $H \neq X_i \neq K$. By the hypothesis, there exists $i < \infty$ such that $G_i \cap (K - H) \neq \emptyset$ and $\bar{G}_i \cap H = \emptyset$. Since $\bar{G}_i \cap H = \emptyset$, we have that $H \subset X_i$ and $H \neq X_i$. Since $X_i \cap G_i = \emptyset$, by virtue of irreducibility, and $G_i \cap (K - H) \neq \emptyset$, $X_i \subset K$ and $X_i \neq K$. Thus $\{X_i\}_{i=1}^{\infty}$ is dense in $\alpha(p)$. Therefore $\alpha(p)$ is separable. □

Chapter 2

Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs

In this chapter we will restrict the subsets that a nonmetric hereditarily indecomposable subcontinuum of L^n can intersect. We will start with a metric subset and add in a dimension of L at a time. We will define some notation for the subsets we will be using. Fix $a^i \in [0,1]$ for each $1 \leq i \leq n$.

1. Let $S = \prod_{i=1}^{n}(a_0^i, a_1^i)$. We will refer to this set as the interior of a **metric cube**.
2. Let $S_0^j = \prod_{1\leq i<j}(a_0^i, a_1^i) \times \{a_0^j\} \times \prod_{j<i\leq n}(a_0^i, a_1^i)$ be known as the **"jth" lower metric face**.
3. Let $S_1^j = \prod_{1\leq i<j}(a_0^i, a_1^i) \times \{a_1^j\} \times \prod_{j<i\leq n}(a_0^i, a_1^i)$ be known as the **"jth" upper metric face**.
4. Let $P^j = \prod_{1\leq i<j}(a_0^i, a_1^i) \times L \times \prod_{j<i\leq n}(a_0^i, a_1^i)$. We will refer to this set as the **"jth" tube**.
5. Let $P_0^j = \prod_{1\leq i<j}(a_0^i, a_1^i) \times [0_0, a_0^j) \times \prod_{j<i\leq n}(a_0^i, a_1^i)$ be the **"jth" lower tube**.
6. Let $P_1^j = \prod_{1\leq i<j}(a_0^i, a_1^i) \times (a_1^j, 1_1] \times \prod_{j<i\leq n}(a_0^i, a_1^i)$ be known as the **"jth" upper tube**.

First we will show that if M is a nonmetric hereditarily indecomposable subcontinuum of L^n, then it cannot intersect the interior of a metric cube. Please note that the following theorems are also true for a countable product of lexicographic arcs.

For the following propositions fix $a^i \in [0,1]$ for each $1 \leq i \leq n$.

Proposition 2.1. *If M is a nonmetric hereditarily indecomposable subcontinuum of L^n, then $M \cap S = \emptyset$.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n. Since M is nonmetric, we may assume wlog that $\pi_1(M) = L$. Suppose $M \cap S \neq \emptyset$. Then there exists a point, $\vec{a} \in M \cap S$. Notice that the boundary of S is homeomorphic to a sphere which is metrizable. Let $\mathcal{B}$ be a countable basis for $bd(S)$, and let $\{D_j\}_{j=1}^{\infty}$ be the collection of all finite unions of elements of $\mathcal{B}$.

Let us assume that $a^1 < 1$, and let K be an irreducible subcontinuum of M from $\vec{a}$ to $\{1_1\} \times \prod_{i=2}^{n} L$. Then we will let $\{K_\alpha\}_{\alpha \in \Gamma}$ denote the components of K in $L^n - S$ indexed by the set Γ. Then for each $\alpha \in \Gamma$, let $\hat{K}_\alpha = K_\alpha \cap bd(S)$. Notice that $\hat{K}_\alpha$ is a nonempty closed subset of $bd(S)$. For each $j < \infty$ let $K_j = \bigcup_{\hat{K}_\alpha \subset D_j} K_\alpha$. Finally, let $x_j = lub\{\pi_1(K_j)\}$. So we have used K to construct $\{x_j\}_{j<\infty}$, a countable subset of L.

We will now use Theorem 1.5 and the metrizability of $bd(S)$ to show that $\{x_j\}_{j<\infty}$ is uncountable, contradicting its construction. Let $b \in (a^1, 1)$. We claim that there exists $j < \infty$ such that $x_j \in (b_0, b_1)$. Let $E = M \cap ([b_1, 1_1] \times \prod_{i=2}^{n} L)$, $F = \{\vec{a}\}$, $U = (b_0, 1_1] \times \prod_{i=2}^{n} L$, and $V = S$. Then by Theorem 1.5, there exists closed subsets A, B, and C of L^n such that $M = A \cup B \cup C$, $E \subset A$, $F \subset C$, $A \cap B \subset (V - F)$, $B \cap C \subset (U - E)$, and $A \cap C = \emptyset$. Refer to Figure 2.1.

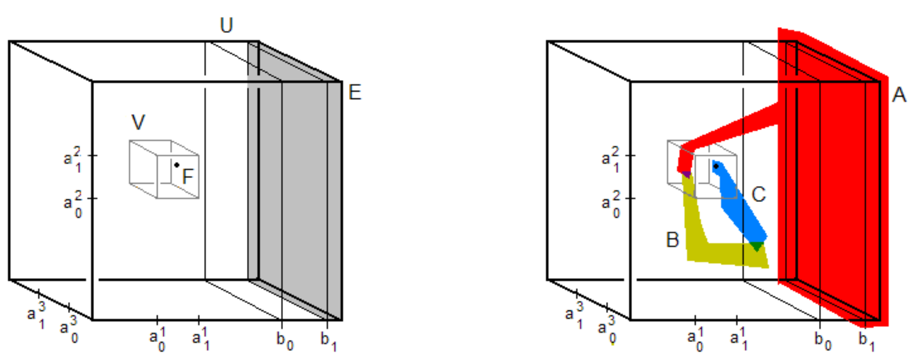

Figure 2.1: Proposition 2.1: A, B, and C

Let $\hat{A} = A \cap bd(S)$, $\hat{B} = B \cap bd(S)$, and $\hat{C} = C \cap bd(S)$. Then $\hat{B}$ and $\hat{A} \cup \hat{C}$ are disjoint closed subsets of $bd(S)$. Hence by normality there exists disjoint open subsets of $bd(S)$, O and W, such that $\hat{B} \subset O$ and $\hat{A} \cup \hat{C} \subset W$. By compactness, there exists $j < \infty$ such that $\hat{B} \subset D_j \subset O$. Then $\hat{B} \subset D_j$ and $(\hat{A} \cup \hat{C}) \cap D_j = \emptyset$. By the Theorem 1.4, there exists a component, I, of B intersecting both $A \cap B$ and $B \cap C$.

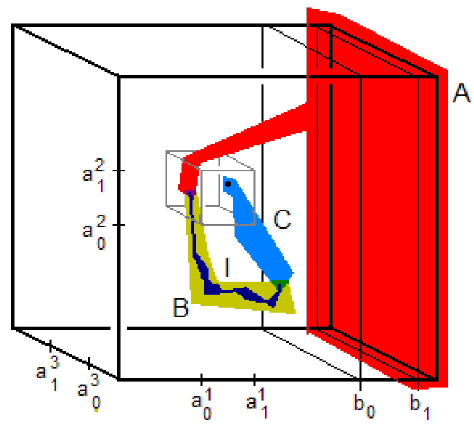

Figure 2.2: Proposition 2.1: The Component I

Since $I \subset B$, $I \cap bd(S) \subset \hat{B} \subset D_j$. So $I - S$ is a subset of K_j. Now $B \cap C \subset (U - E) = (b_0, b_1) \times \prod_{i=2}^{n} L$, implying that $x_j > b_0$. Also since $\hat{A} \cap D_j = \emptyset$, we have $\hat{A} \cap K_j = \emptyset$ and $A \cap K_j = \emptyset$. So $E \cap K_j = \emptyset$ which implies that $x_j < b_1$. So for each $b \in (a^1, 1)$ there is a $j < \infty$ such that $x_j \in (b_0, b_1)$. Thus $\{x_j\}_{j<\infty}$ is uncountable, a contradiction. Therefore $M \cap S = \emptyset$. $\square$

Next we will note that M cannot intersect an isolated metric face. Recall that a metric face is defined as $S^j_0 = \prod_{i<j}(a^i_0, a^i_1) \times \{a^j_0\} \times \prod_{i>j}(a^i_0, a^i_1)$ or $S^j_1 = \prod_{i<j}(a^i_0, a^i_1) \times \{a^j_1\} \times \prod_{i>j}(a^i_0, a^i_1)$. The proof follows a similar argument to the proof of Proposition 2.1.

Proposition 2.2. *Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and choose $1 \leq j \leq n$ such that $a^j \notin \{0, 1\}$. If $\vec{a} \in M \cap S^j_0$, then there is a sequence in*

$M \cap P_0^j$ *converging to* $\vec{a}$*. Similarly if* $\vec{a} \in M \cap S_1^j$*, then there exists a sequence in* $M \cap P_1^j$ *converging to* $\vec{a}$*.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and wlog choose $j = 1$ to simplify notation. Suppose that $a^1 \notin \{0, 1\}$ and that $\pi_1(M) = L$. Let $\vec{a} \in M \cap S_0^1$. Suppose that there is no sequence in $M \cap P_0^1$ converging to $\vec{a}$. Then there is an open neighborhood of $\vec{a}$, $O \subset P_0^1$ such that $M \cap O \subset S_0^1$. Hence $O \cap M$ is metrizable.

Let $\mathcal{B}$ be a countable basis for $bd(S)$, and let $\{D_k\}_{k=1}^{\infty}$ be the collection of finite unions of elements of $\mathcal{B}$. Notice that $a^1 \neq 1$. Let K be an irreducible subcontinuum of M from $\vec{a}$ to $\{1_1\} \times \prod_{i=2}^{n} L$. Let $\{K_\alpha\}_{\alpha \in \Gamma}$ denote the components of K in $L^n - O$ indexed by Γ. Then for each $\alpha > 0$, let $\hat{K}_\alpha = K_\alpha \cap bd(S)$. Notice that $\hat{K}_\alpha$ is a nonempty closed subset of $bd(S)$. Using compactness, for each $k < \infty$, let $K_k = \bigcup_{\hat{K}_\alpha \subset D_k} K_\alpha$. Finally, let $x_k = lub\{\pi_1(K_k)\}$.

We will now use Theorem 1.5 and the metrizability of $bd(S)$ to show that $\{x_k\}_{j<\infty}$ is uncountable, a contradiction. Let $b \in (a^1, 1)$. We claim that there exists $j < \infty$ such that $x_k \in (b_0, b_1)$. Let $E = M \cap ([b_1, 1_1] \times \prod_{i=2}^{n} L)$, $F = \{\vec{a}\}$, $U = (b_0, 1_1] \times \prod_{i=2}^{n} L$, and $V = O$. Then by Theorem 1.5, there exists A, B, and C, closed subsets of L^n such that $M = A \cup B \cup C$, $E \subset A$, $F \subset C$, $A \cap B \subset (V - F)$, $B \cap C \subset (U - E)$, and $A \cap C = \emptyset$. Notice that $A \cap V$, $B \cap V$, and $C \cap V$ are each subsets of S_0^1. Let $\hat{A} = A \cap bd(S)$, $\hat{B} = B \cap bd(S)$, and $\hat{C} = C \cap bd(S)$. Refer to Figure 2.3.

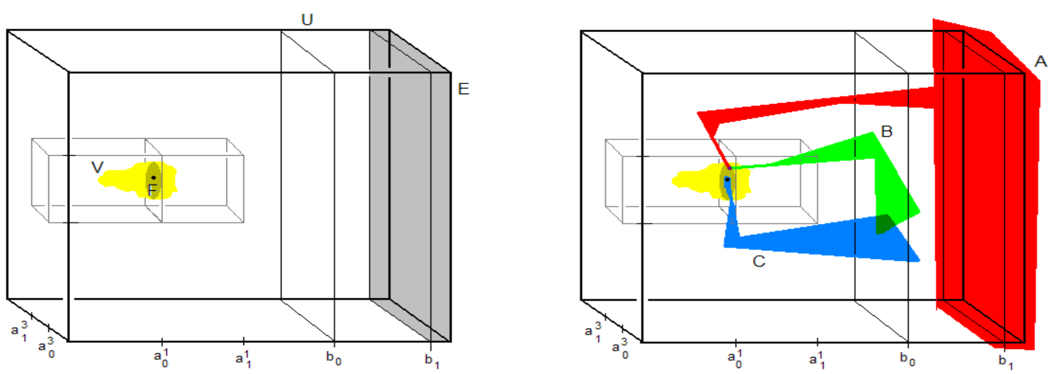

Figure 2.3: Proposition 2.2: Convergent Sequence in P_0^1

Notice that $\hat{C}$ and $\hat{A} \cup \hat{B}$ are disjoint closed subsets of $bd(S)$. By normality, there exists disjoint open sets W_1 and W_2 such that $\hat{C} \subset W_1$ and $(\hat{A} \cup \hat{B}) \subset W_2$. By compactness, we may assume that there exists $k < \infty$ such that $W_1 = D_k$. Then $\hat{C} \subset D_k$ and $(\hat{A} \cup \hat{B}) \subset D_k = \emptyset$. By Theorem 1.4, there exists a component of C intersecting both $\{\vec{a}\}$ and $B \cap C$ as depicted in Figure 2.4.

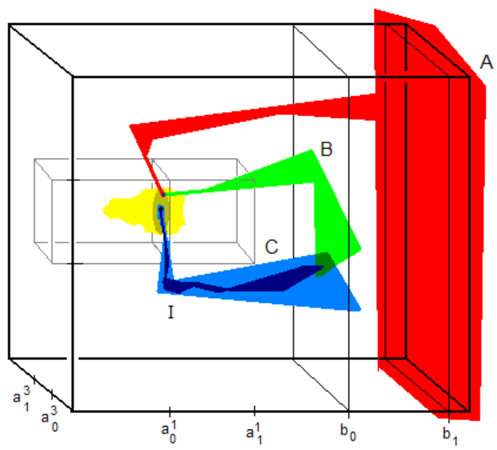

Figure 2.4: Proposition 2.2: Convergent Sequence and the component I

Since $I \subset C$, $I \cap (L^n - O)$ is a subset of K_k. Since $B \cap C \subset (b_0, b_1) \times \prod_{i=2}^{n} L$, we have that $x_k > b_0$. Now $K_k \cap \hat{A} = \emptyset$, implying that $K_k \cap E = \emptyset$. Thus $x_k < b_1$. Thus $x_k \in (b_0, b_1)$. So for each $b \in (a^1, 1)$ there is a $k < \infty$ such that $x_k \in (b_0, b_1)$. Thus $\{x_k\}_{k<\infty}$ is uncountable, a contradiction. Therefore there exists a sequence in $M \cap P^1_0$ converging to $\vec{a}$.

Similarly, if $\vec{a} \in M \cap S^1_1$, then we can use K irreducible from $\vec{a}$ to $\{0_0\} \times \prod_{i=2}^{n} L$ to show that $(0, a^1)$ is countable, a contradiction. Therefore there exists a sequence in $M \cap P^1_1$ converging to $\vec{a}$. $\square$

Proposition 2.1 states that if $\vec{z} \in M$, then there is at least one coordinate, say $x^i_{y^i}$, for some $1 \leq i \leq n$ such that $y^i \in \{0, 1\}$. In other words, M is restricted to metric faces. Proposition 2.2 states that these faces cannot be isolated.

For the following corollary, note that the subspace of L restricted to $x_y \in L$ such that $y \in \{0,1\}$ is homeomorphic to the Double Arrow Space, Z.

Corollary 2.1. *If M is a nonmetric hereditarily indecomposable subcontinuum of L^n, then $M \cap P^j$ is embeddable in $Z \times \prod_{i=2}^{n}(0,1)$.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and choose $j = 1$ to simplify notation. We claim that $f : (M \cap P^1) \to (Z \times \prod_{i=2}^{n}(0,1))$ defined by $f(a^1_{y^1}, a^2_{y^2}, \cdots, a^n_{y^n}) = (a^1_{y^1}, y^2, \cdots, y^n)$ is an embedding. Since $P^1 = L \times \prod_{i=2}^{n}(a^i_0, a^i_1)$, we have that f is an embedding if it is well-defined. By Proposition 2.1, since $y^i \notin \{0,1\}$ for $1 < i \leq n$, $y^1 \in \{0,1\}$. Hence $a^1_{y^1} \in Z$, and f is well-defined. □

We will now use Proposition 2.1 along with Theorem 1.4 to produce a separable order arc in $C(M)$. We will then use this order arc to show that M cannot intersect a tube. Recall that a tube is $P^j = \prod_{i<j}(a^i_0, a^i_1) \times L \times \prod_{i>j}(a^i_0, a^i_1)$.

Proposition 2.3. *If M is a nonmetric hereditarily indecomposable subcontinuum of L^n, and $p \in M \cap P^j$, then $\alpha(p)$ is separable.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and choose $j = 1$ to simplify notation. Choose $p \in M \cap P^1$. Let $U = M \cap P^1$. Then U is an open neighborhood of p in M. Let $\mathcal{B}$ be a countable basis for $\prod_{i=2}^{n}(a^i_0, a^i_1)$, $\mathcal{B}_0$ be a countable basis for $[0_0, 0_1) \times \prod_{i=2}^{n}(a^i_0, a^i_1)$, and $\mathcal{B}_1$ be a countable basis for $(1_0, 1_1] \times \prod_{i=2}^{n}(a^i_0, a^i_1)$. Define $\mathcal{G} = \{(q_0, r_1) \times B | q < r,\ q, r \in \mathbf{Q} \text{ and } B \in \mathcal{B}\} \cup \{L \times B | B \in \mathcal{B}\} \cup \mathcal{B}_0 \cup \mathcal{B}_1$. Notice that $\mathcal{G}$ is a countable collection of open subsets of U. To use Theorem 1.7, we will need to show that for each $H \subset K \in \alpha(p)$ such that $H \neq K$, there is a $G \in \mathcal{G}$ such that $\bar{G} \cap H = \emptyset$ and $G \cap (K - H) \neq \emptyset$.

Let $H \subset K \in \alpha(p)$ such that $H \neq K$. Now H and K are subcontinua of M implying that they are each hereditarily indecomposable. Hence H is nowhere dense in K. Thus there exists $k \in U \cap (K - H)$ and there exists V, an open neighborhood of k in U, such that $V \cap H = \emptyset$. By Proposition 2.1, k cannot be contained in a metric cube, thus k is contained in a face. Now we will use a basis for P^1 to find a $G \in \mathcal{G}$ satisfying the desired properties.

Notice that either K is metric or nonmetric. If K is metric, then by Proposition 2.1 K cannot intersect any metric cube. So $K \subset \{x_y\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$ where $y \in \{0, 1\}$. Hence there exists $B \in \mathcal{B}$ such that $k \in B \times \{x_y\} \subset \bar{B} \times \{x_y\} \subset V$. Let $G = B \times L$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

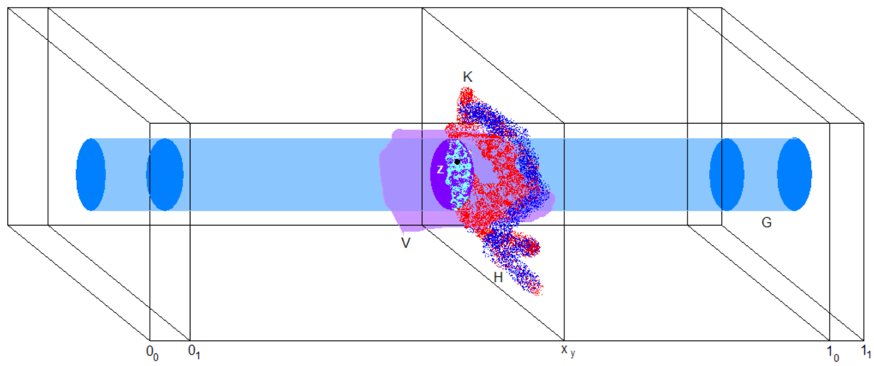

Figure 2.5: Proposition 3.3: K is metric.

Now suppose that K is nonmetric. Again, by Proposition 2.1, $k \in \{x_y\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$ for $x \in [0, 1]$ and $y \in \{0, 1\}$. This gives rise to three cases:

1. $k \in \{0_0\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$,
2. $k \in \{1_1\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$, or
3. $k \in [0_1, 1_0] \times \prod_{i=2}^{n}(a_0^i, a_1^i)$.

Case 1

Suppose $k \in \{0_0\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$.

Then $k \in V \cap [0_0, 0_1) \times \prod_{i=2}^{n}(a_0^i, a_1^i)$. So there exists $B \in \mathcal{B}_0$ such that $k \in B \subset \bar{B} \subset V \cap [0_0, 0_1) \times \prod_{i=2}^{n}(a_0^i, a_1^i)$. Let $G = B$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

Case 2

Suppose $k \in \{1_1\} \times \prod_{i=2}^{n}(a_0^i, a_1^i)$.

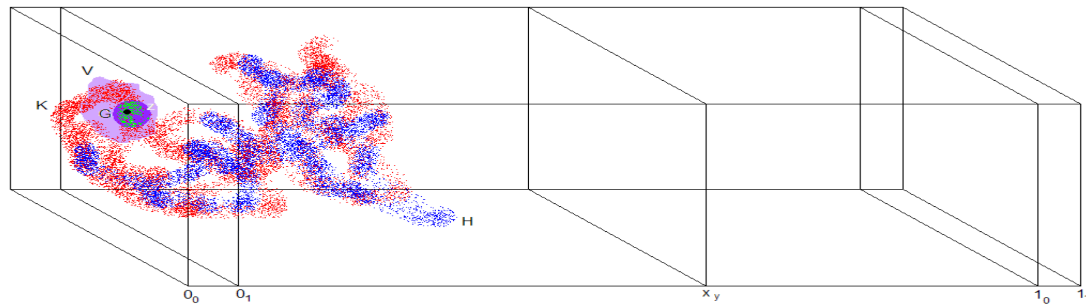

Figure 2.6: Proposition 3.3: Case 1

Similar to Case 1, there is a $B \in \mathcal{B}_1$ such that $k \in B \subset \bar{B} \subset V \cap (1_0, 1_1] \times \prod_{i=2}^{n}(a_0^i, a_1^i)$. Let $G = B$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

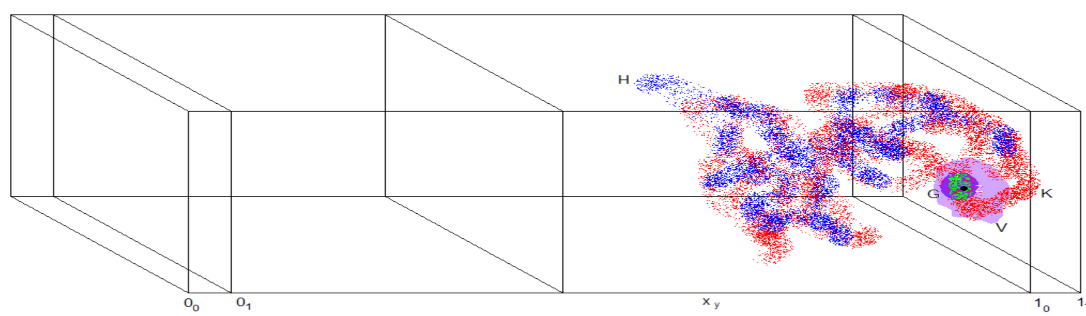

Figure 2.7: Proposition 3.3: Case 2

Case 3

Suppose $k \in [0_1, 1_0] \times \prod_{i=2}^{n}(a_0^i, a_1^i)$.

Then k is in a metric face, and V is a nonmetric open subset of U. Since $P^1 = L \times \prod_{i=2}^{n}(a_0^i, a_1^i)$, we have a_b, $c_d \in L$ such that $a < c$ and W open in $\prod_{i=2}^{n}(a_0^i, a_1^i)$ such that $k \in (a_b, c_d) \times W \subset V$. By propositions 1 and 2, k is on an unisolated face. So there exists $\hat{k} \in (a_1, c_0) \times W \subset V$. Now there exists $B \in \mathcal{B}$ and q, $r \in \mathbf{Q}$ such that $\bar{B} \subset W$ and $a < q < r < c$. Let $G = (q_0, r_1) \times B$. Then $G \in \mathcal{G}$, $\hat{k} \in G \cap (K - H)$, and by proposition 3 $\bar{G} = [q_1, r_0] \times \bar{B} = (q_0, r_1) \times \bar{B} \subset V$. Hence $\bar{G} \cap H = \emptyset$.

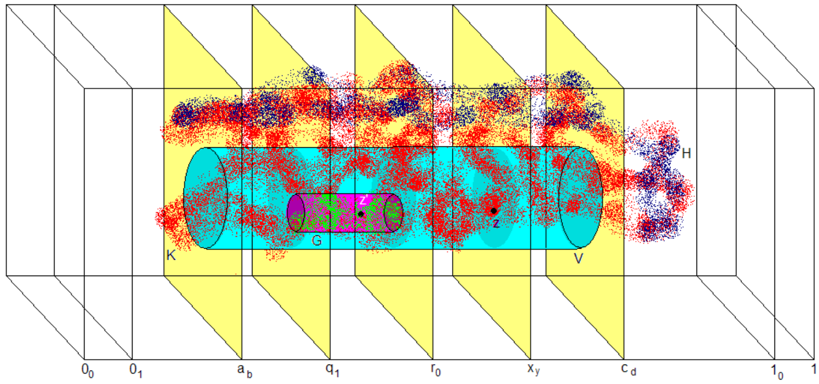

Figure 2.8: Proposition 3.3: Case 3

Hence we have satisfied the hypothesis for Theorem 1.7. Therefore $\alpha(p)$ is separable. $\square$

We will now show that M cannot intersect a tube, by using Proposition 2.3 to contradict the uncountability of an interval in L.

Proposition 2.4. *If M is a nonmetric hereditarily indecomposable subcontinuum of L^n, then $M \cap P^j = \emptyset$ for each $1 \leq j \leq n$.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and choose $j = 1$ to simplify notation. Suppose that $p \in M \cap P^1$. Then by Proposition 2.3 $\alpha(p)$ is separable. Let $\mathcal{D} = \{D_i\}_{i=1}^{\infty}$ be a countable dense subset of $\alpha(p)$, and wlog assume $\pi_1(M) = L$. So $\pi_1(p) = x_y$ for some $x_y \in L$. By Proposition 2.1, $y \in \{0, 1\}$. We can assume that $x < 1$.

Let $y_i = lub\{\pi_1(D_i)\}$ for each $i < \infty$, and let $\mathcal{Y} = \{y_i\}_{i=1}^{\infty}$. We claim that $\mathcal{Y}$ is dense in $(x_y, 1_1]$. We must show that for $a_b < c_d \in (x_y, 1_1]$ there exists $i < \infty$ such that $y_i \in (a_b, c_d)$. Let e_f, $g_h \in (a_b, c_d)$ such that $e_f < g_h$. Let H be an irreducible subcontinuum of M from p to $\{e_f\} \times \prod_{i=2}^{n} L$, and let K be an irreducible subcontinuum of M from p to $\{g_h\} \times \prod_{i=2}^{n} L$. Then $H \subset K \in \alpha(p)$. Hence there exists $i < \infty$ such that $D_i \in (H, K) \subset \alpha(p)$. Since $D_i \subset K$, we have that $D_i \cap (g_h, 1_1] \times \prod_{i=2}^{n} L = \emptyset$. So $y_i \leq g_h$. Since $H \subset D_i$, we have that

$D_i \cap [e_f, 1_1] \times \prod_{i=2}^{n} L \neq \emptyset$. so $y_i \geq e_f$. Then $y_i \in [e_f, g_h] \subset (a_b, c_d)$. Thus $\mathcal{Y}$ is dense in $(x_y, 1_1]$ implying that $(x_y, 1_1]$ is separable, a contradiction. Therefore $M \cap P^1 = \emptyset$. $\square$

Chapter 3

Nonmetric Hereditarily Indecomposable Subcontinua of Finite Products of Lexicographic Arcs and Hilbert Cubes

Our investigation continues with an exploration of how hereditarily indecomposable continua behave in products of Lexicographic arcs and Hilbert cubes. The arguments in this chapter are generalizations of the arguments in the previous chapter. The first theorem follows from Theorem 1.5.

Theorem 3.1. *If M is a hereditarily indecomposable subcontinuum of $L \times [0,1]^\infty$, then M is metric.*

Proof. Let M be a hereditarily indecomposable subcontinuum of $L \times [0,1]^\infty$. Suppose that M is nonmetric. Then it can be assumed that $\pi_1(M) = L$. For each $0 < \epsilon < \frac{1}{4}$, let

$$E_\epsilon = M \cap ([0_0, \epsilon_0] \times [0,1]^\infty) \tag{3.1}$$

$$F_\epsilon = M \cap ([(1-\epsilon)_1, 1_1] \times [0,1]^\infty) \tag{3.2}$$

$$U_\epsilon = [0_0, \epsilon_1) \tag{3.3}$$

$$V_\epsilon = ((1-\epsilon)_0, 1_1] \tag{3.4}$$

Then E_ϵ, F_ϵ are closed subsets of M, and U_ϵ, V_ϵ are open subsets of $L \times [0,1]^\infty$ containing E_ϵ and F_ϵ respectively. By Theorem 1.5, there exists nonempty closed subsets, A_ϵ, B_ϵ, and C_ϵ such that

$$M = A_\epsilon \cup B_\epsilon \cup C_\epsilon \tag{3.5}$$

$$E_\epsilon \subset A_\epsilon \tag{3.6}$$

$$F_\epsilon \subset C_\epsilon \tag{3.7}$$

$$A_\epsilon \cap C_\epsilon = \emptyset \tag{3.8}$$

$$A_\epsilon \cap B_\epsilon \subset (V_\epsilon - F_\epsilon) \tag{3.9}$$

$$B_\epsilon \cap C_\epsilon \subset (U_\epsilon - E_\epsilon) \tag{3.10}$$

We will now focus on the subset of $L \times [0,1]^\infty$, $X = [\frac{1}{2}_0, \frac{1}{2}_1] \times [0,1]^\infty$. This subset is homeomorphic to the Hilbert cube, $[0,1]^\infty$, and thus it is metrizable. Refer to Figure 3.1

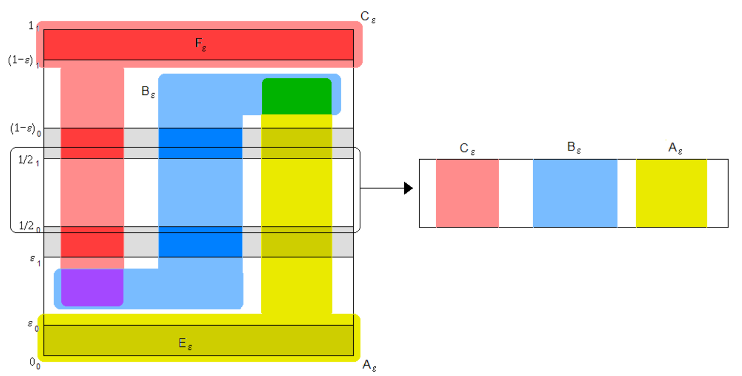

Figure 3.1: Theorem 2.1: A_ϵ, B_ϵ, and C_ϵ

Recall that these sets exist for each $0 < \epsilon < \frac{1}{4}$. Let $\hat{A}_\epsilon = X \cap (\bigcap_{\epsilon \leq \alpha < \frac{1}{4}} A_\alpha)$, and let $\hat{C}_\epsilon = X \cap (\bigcap_{\epsilon < \alpha < \frac{1}{4}} A_\alpha) \cap C_\epsilon$. We claim that $\hat{A}_\epsilon$ and $\hat{C}_\epsilon$ are nonempty. Note that each is an intersection of a collection of closed subsets of a compact space. We will show that the intersection of a finite collection of these sets is nonempty. By compactness, the finite intersection property applies. So let $\epsilon < \alpha^1 < \cdots < \alpha^m < \frac{1}{4}$. For each $1 \leq i < j \leq m$ we have that $U_{\alpha^i} \subset E_{\alpha^j} \subset A_{\alpha^j}$ and that $V_{\alpha^i} \subset F_{\alpha^j} \subset C_{\alpha^j}$ by (3.1), (3.4), (3.6), and (3.9). Now we have that $A_{\alpha^i} \cap A_{\alpha^j} \neq \emptyset$, $C_\epsilon \cap A_{\alpha^i} \neq \emptyset$, $A_{\alpha^i} \cap X \neq \emptyset$, and $C_{\alpha^i} \cap X \neq \emptyset$ as in Figure 3.2.

Thus $X \cap (A_{\alpha^1} \cap \cdots \cap A_{\alpha^m}) \cap A_\epsilon \neq \emptyset$ and $X \cap (A_{\alpha^1} \cap \cdots \cap A_{\alpha^m}) \cap C_\epsilon \neq \emptyset$. Hence by the finite intersection property, $\hat{A}_\epsilon \neq \emptyset$ and $\hat{C}_\epsilon \neq \emptyset$.

Now recall that $X = [\frac{1}{2}_0, \frac{1}{2}_1] \times [0,1]^\infty$ is homeomorphic to the Hilbert cube, and we have shown that $\hat{A}_\epsilon$ and $\hat{C}_\epsilon$ are nonempty closed subsets of X. Notice since A_ϵ and C_ϵ are

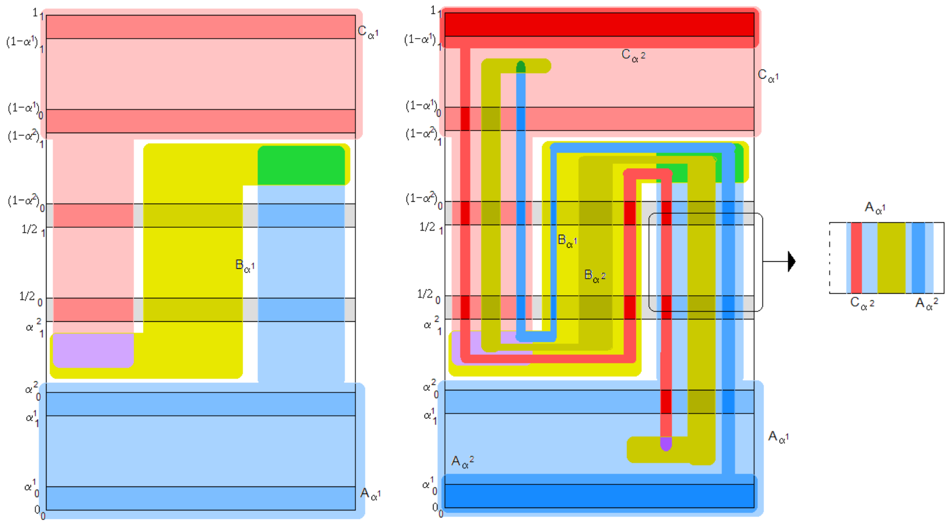

Figure 3.2: Theorem 2.1: $\epsilon \leq \alpha_1 \leq \alpha_2 \leq \frac{1}{4}$

disjoint, $\hat{A}_\epsilon$ and $\hat{C}_\epsilon$ are disjoint. Hence by normality, there exist disjoint open subsets, O_ϵ, W_ϵ, of X such that $\hat{A}_\epsilon \subset O_\epsilon$ and $\hat{C}_\epsilon \subset W_\epsilon$. Let $\mathcal{B}$ be a countable basis of X. Let $\mathcal{D}$ be the collection of all finite unions of elements of $\mathcal{B}$. By construction, $\mathcal{D}$ is countable. By compactness, it may be assumed that O_ϵ and W_ϵ are elements of $\mathcal{D}$. Thus we have that $O_\alpha = O_\beta$ for uncountably many $0 < \alpha < \frac{1}{4}$ and $0 < \beta < \frac{1}{4}$. Straight from this construction, we get that there exist α and $\beta \in (0, \frac{1}{4})$ such that $\alpha < \beta$ and $O = O_\alpha = O_\beta$. Since $\hat{A}_\alpha \subset \hat{A}_\beta$ and $\hat{C}_\alpha \subset \hat{A}_\beta$, we have that $\hat{C}_\alpha \subset O_\alpha \cap W_\alpha$. By normality, $O_\alpha \cap W_\alpha = \emptyset$, a contradiction. Therefore M is metrizable. $\square$

Now that we have shown that $L \times [0,1]^\infty$ does not contain a nonmetric continuum, let's explore the subsets of $L^n \times [0,1]^\infty$. We will start with a metric subset of $L^n \times [0,1]^\infty$, same as in the previous chapter, and add in a dimension of L at a time. Notation for the subsets we will be using is redefined. Fix $a^i \in [0,1]$ for each $1 \leq i \leq n$. Recall that $[0,1]^\infty = \prod_{i=1}^{\infty}[0,1]$, the space known as the Hilbert cube.

1. Let $S = \prod_{i=1}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$. We will refer to this set as the interior of a **metric cube**.

2. Let $S_0^j = \prod_{1\le i<j}(a_0^i, a_1^i) \times \{a_0^j\} \times \prod_{j<i\le n}(a_0^i, a_1^i) \times [0,1]^\infty$ be known as the **"jth" lower metric face**.

3. Let $S_1^j = \prod_{1\le i<j}(a_0^i, a_1^i) \times \{a_1^j\} \times \prod_{j<i\le n}(a_0^i, a_1^i) \times [0,1]^\infty$ be known as the **"jth" upper metric face**.

4. Let $P^j = \prod_{1\le i<j}(a_0^i, a_1^i) \times L \times \prod_{j<i\le n}(a_0^i, a_1^i) \times [0,1]^\infty$. We will refer to this set as the **"jth" tube**.

5. Let $P_0^j = \prod_{1\le i<j}(a_0^i, a_1^i) \times [0_0, a_0^j) \times \prod_{j<i\le n}(a_0^i, a_1^i) \times [0,1]^\infty$ be the **"jth" lower tube**.

6. Let $P_1^j = \prod_{1\le i<j}(a_0^i, a_1^i) \times (a_1^j, 1_1] \times \prod_{j<i\le n}(a_0^i, a_1^i) \times [0,1]^\infty$ be known as the **"jth" upper tube**.

For the following propositions, fix $a^i \in [0,1]$ for $1 \le i \le n$.

Proposition 3.1. *If M is a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, then $M \cap S = \emptyset$.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$. Since M is nonmetric, we may assume wlog that $\pi_1(M) = L$. Suppose $M \cap S \neq \emptyset$. Then there exists a point, $\vec{a} \in M \cap S$. Notice that the boundary of S is homeomorphic to a sphere which is metrizable. Let $\mathcal{B}$ be a countable basis for $bd(S)$, and let $\{D_j\}_{j=1}^\infty$ be the collection of all finite unions of elements of $\mathcal{B}$.

Let us assume that $a^1 < 1$, and let K be an irreducible subcontinuum of M from $\vec{a}$ to $\{1_1\} \times \prod_{i=2}^n L \times [0,1]^\infty$. Then we will let $\{K_\alpha\}_{\alpha\in\Gamma}$ denote the components of K in $(L^n \times [0,1]^\infty) - S$ indexed by the set Γ. Then for each $\alpha \in \Gamma$, let $\hat{K}_\alpha = K_\alpha \cap bd(S)$. Notice that $\hat{K}_\alpha$ is a nonempty closed subset of $bd(S)$. By compactness, there is a $j_\alpha < \infty$ such that $\hat{K}_\alpha \subset D_{j_\alpha}$. Let $K_j = \bigcup\{K_\alpha | j_\alpha = j\} = \bigcup_{\hat{K}_\alpha \subset D_j} K_\alpha$. Finally, let $x_j = lub\{\pi_1(K_j)\}$. So we have used K to construct $\{x_j\}_{j<\infty}$, a countable subset of L.

We will now use Theorem 1.5 and the metrizability of $bd(S)$ to show that $\{x_j\}_{j<\infty}$ is uncountable, contradicting its construction. Let $b \in (a^1, 1)$. We claim that there exists

$j < \infty$ such that $x_j \in (b_0, b_1)$. Let $E = M \cap ([b_1, 1_1] \times \prod_{i=2}^{n} L \times [0,1]^\infty)$, $F = \{\vec{a}\}$, $U = (b_0, 1_1] \times \prod_{i=2}^{n} L \times [0,1]^\infty$, and $V = S$. Then by Theorem 1.5, there exists closed subsets A, B, and C of $L^n \times [0,1]^\infty$ such that $M = A \cup B \cup C$, $E \subset A$, $F \subset C$, $A \cap B \subset (V - F)$, $B \cap C \subset (U - E)$, and $A \cap C = \emptyset$. Refer to Figure 3.3.

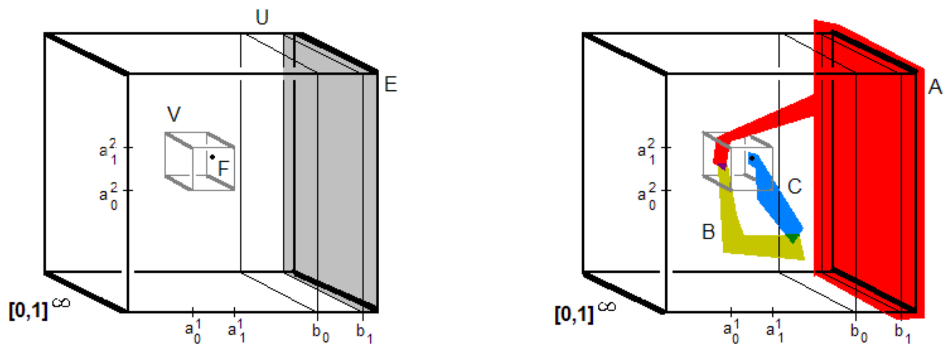

Figure 3.3: Proposition 3.1: A, B, and C

Let $\hat{A} = A \cap bd(S)$, $\hat{B} = B \cap bd(S)$, and $\hat{C} = C \cap bd(S)$. Then $\hat{B}$ and $\hat{A} \cup \hat{C}$ are disjoint closed subsets of $bd(S)$. Hence by normality there exists disjoint open subsets of $bd(S)$, O and W, such that $\hat{B} \subset O$ and $\hat{A} \cup \hat{C} \subset W$. By compactness, there exists $j < \infty$ such that $\hat{B} \subset D_j \subset O$. Then $\hat{B} \subset D_j$ and $(\hat{A} \cup \hat{C}) \cap D_j = \emptyset$. By Theorem 1.4, there exists a component, I, of B intersecting both $A \cap B$ and $B \cap C$.

Since $I \subset B$, $I \cap bd(S) \subset \hat{B} \subset D_j$. So $I - S$ is a subset of K_j. Now $B \cap C \subset (U - E) = (b_0, b_1) \times \prod_{i=2}^{n} L \times [0,1]^\infty$ implying that $x_j > b_0$. Also, since $\hat{A} \cap D_j = \emptyset$, we have $\hat{A} \cap K_j = \emptyset$ and $A \cap K_j = \emptyset$. So $E \cap K_j = \emptyset$, which implies that $x_j < b_1$. So for each $b \in (a^1, 1)$ there is a $j < \infty$ such that $x_j \in (b_0, b_1)$. Thus $\{x_j\}_{j<\infty}$ is uncountable, a contradiction. Therefore $M \cap S = \emptyset$. □

Now we will show that if a point of M lies on a face, then there is a sequence of points in M converging to it.

Proposition 3.2. *Let M be a nonmetric hereditarily indecomposable subcontinuum of L^n, and choose $1 \leq j \leq n$ such that $a^j \notin \{0,1\}$. If $\vec{a} \in M \cap S_0^j$, then there is a sequence in*

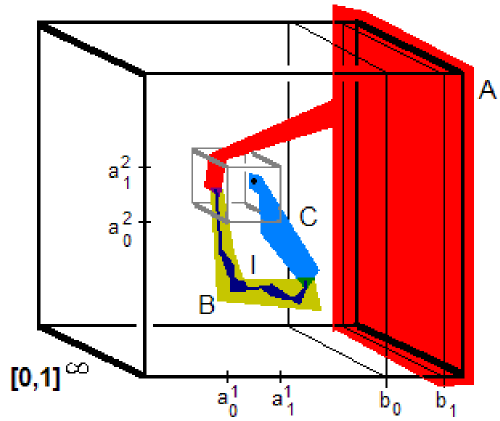

Figure 3.4: Proposition 3.1: The Component I

$M \cap P_0^j$ converging to $\vec{a}$. Similarly if $\vec{a} \in M \cap S_1^j$, then there exists a sequence in $M \cap P_1^j$ converging to $\vec{a}$.

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, and choose $j = 1$ to simplify notation. Suppose that $a^1 \notin \{0,1\}$ and that $\pi_1(M) = L$. Let $\vec{a} \in M \cap S_0^1$. Suppose that there is no sequence in $M \cap P_0^1$ converging to $\vec{a}$. Then there is an open neighborhood of $\vec{a}$, $O \subset P_0^1$ such that $M \cap O \subset S_0^1$. Hence $O \cap M$ is metrizable.

Let $\mathcal{B}$ be a countable basis for $bd(S)$, and let $\{D_k\}_{k=1}^\infty$ be the collection of finite unions of elements of $\mathcal{B}$. Notice that $a^1 \neq 1$. Let K be an irreducible subcontinuum of M from $\vec{a}$ to $\{1_1\} \times \prod_{i=2}^n L \times [0,1]^\infty$. Let $\{K_\alpha\}_{\alpha \in \Gamma}$ denote the components of K in $(L^n \times [0,1]^\infty) - O$ indexed by Γ. Then for each $\alpha > 0$, let $\hat{K}_\alpha = K_\alpha \cap bd(S)$. Notice that $\hat{K}_\alpha$ is a nonempty closed subset of $bd(S)$. By compactness, there is a $k_\alpha < \infty$ such that $\hat{K}_\alpha \subset D_{k_\alpha}$. Let $K_k = \bigcup\{K_\alpha | k_\alpha = k\} = \bigcup_{\hat{K}_\alpha \subset D_k} K_\alpha$. Finally, let $x_k = lub\{\pi_1(K_k)\}$.

We will now use Theorem 1.5 and the metrizability of $bd(S)$ to show that $\{x_k\}_{j<\infty}$ is uncountable, a contradiction. Let $b \in (a^1, 1)$. We claim that there exists $j < \infty$ such that $x_k \in (b_0, b_1)$. Let $E = M \cap ([b_1, 1_1] \times \prod_{i=2}^n L)$, $F = \{\vec{a}\}$, $U = (b_0, 1_1] \times \prod_{i=2}^n L$, and $V = O$. Then by Theorem 1.5, there exists A, B, and C, closed subsets of $L^n \times [0,1]^\infty$ such that

$M = A \cup B \cup C$, $E \subset A$, $F \subset C$, $A \cap B \subset (V - F)$, $B \cap C \subset (U - E)$, and $A \cap C = \emptyset$. Notice that $A \cap V$, $B \cap V$, and $C \cap V$ are each subsets of S_0^1. Let $\hat{A} = A \cap bd(S)$, $\hat{B} = B \cap bd(S)$, and $\hat{C} = C \cap bd(S)$. Refer to Figure 3.5.

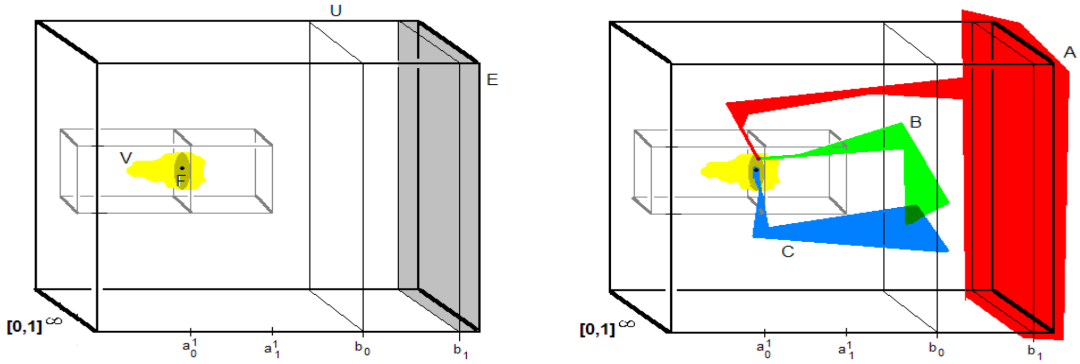

Figure 3.5: Proposition 3.2: Convergent Sequence in P_0^1

Notice that $\hat{C}$ and $(\hat{A} \cup \hat{B})$ are disjoint closed subsets of $bd(S)$. By normality there exists disjoint open sets W_1 and W_2 such that $\hat{C} \subset W_1$ and $(\hat{A} \cup \hat{B}) \subset W_2$. By compactness, we may assume that there exists $k < \infty$ such that $W_1 = D_k$. Then $\hat{C} \subset D_k$ and $(\hat{A} \cup \hat{B}) \subset D_k = \emptyset$. By Theorem 1.4, there exists a component of C intersecting both $\{\vec{a}\}$ and $B \cap C$ as depicted in Figure 2.4.

Since $I \subset C$, $I \cap [(L^n \times [0,1]^\infty) - O]$ is a subset of K_k. Since $B \cap C \subset (b_0, b_1) \times \prod_{i=2}^n L$, we have that $x_k > b_0$. Now $K_k \cap \hat{A} = \emptyset$ implying that $K_k \cap E = \emptyset$. Thus $x_k < b_1$. Thus $x_k \in (b_0, b_1)$. So for each $b \in (a^1, 1)$ there is a $k < \infty$ such that $x_k \in (b_0, b_1)$. Thus $\{x_k\}_{k<\infty}$ is uncountable, a contradiction. Therefore there exists a sequence in $M \cap P_0^1$ converging to $\vec{a}$.

Similarly, if $\vec{a} \in M \cap S_1^1$, then we can use K irreducible from $\vec{a}$ to $\{0_0\} \times \prod_{i=2}^n L \times [0,1]^\infty$ to show that $(0, a^1)$ is countable, a contradiction. Therefore there exists a sequence in $M \cap P_1^1$ converging to $\vec{a}$. $\square$

For the next proposition we will need a direct corollary of Proposition 3.1. Recall that the "jth" tube is defined as $P^j = \prod_{1 \le i < j} (a_0^i, a_1^i) \times L \times \prod_{j < i \le n} (a_0^i, a_1^i) \times [0,1]^\infty$.

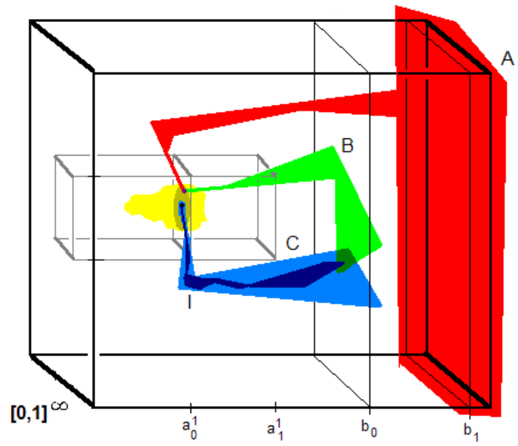

Figure 3.6: Proposition 3.2: Convergent Sequence and the Component I

Corollary 3.1. *If M is a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, then $M \cap P^j$ can be embedded in $Z \times [0,1]^\infty$.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, and choose $j = 1$ to simplify notation. We claim that $f : (M \cap P^1) \to (Z \times \prod_{i=2}^n (0,1))$ defined by $f(a^1_{y^1}, a^2_{y^2}, \cdots, a^n_{y^n}, \vec{x}) = (a^1_{y^1}, y^2, \cdots, y^n, \vec{x})$ is an embedding. Since $P^1 = L \times \prod_{i=2}^n (a^i_0, a^i_1) \times [0,1]^\infty$, we have that f is an embedding if it is well-defined. By Proposition 3.1, since $y^i \notin \{0,1\}$ for $1 < i \leq n$, we have $y^1 \in \{0,1\}$. Hence $a^1_{y^1} \in Z$, and f is well-defined. □

Now we can use Proposition 3.2 and Corollary 3.1 to restrict nonmetric hereditarily indecomposable subcontinua of $L^n \times [0,1]^\infty$ even further.

Proposition 3.3. *If M is a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, and $p \in M \cap P^j$, then $\alpha(p)$ is separable.*

Proof. Let M be a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, and choose $j = 1$ to simplify notation. Choose $p \in M \cap P^1$. Let $U = M \cap P^1$. Then U is an open neighborhood of p in M. Let $\mathcal{B}$ be a countable basis for $\prod_{i=2}^n (a^i_0, a^i_1) \times [0,1]^\infty$,

$\mathcal{B}_0$ be a countable basis for $[0_0, 0_1) \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$, and $\mathcal{B}_1$ be a countable basis for $(1_0, 1_1] \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$. Define $\mathcal{G} = \{(q_0, r_1) \times B | q < r,\ q, r \in \mathbf{Q}$ and $B \in \mathcal{B}\} \cup \{L \times B | B \in \mathcal{B}\} \cup \mathcal{B}_0 \cup \mathcal{B}_1$. Notice that $\mathcal{G}$ is a countable collection of open subsets of U. To use Theorem 1.7, we will need to show that for each $H \subset K \in \alpha(p)$ such that $H \neq K$, there is a $G \in \mathcal{G}$ such that $\bar{G} \cap H = \emptyset$ and $G \cap (K - H) \neq \emptyset$.

Let $H \subset K \in \alpha(p)$ such that $H \neq K$. Now H and K are subcontinua of M implying that they are each hereditarily indecomposable. Hence H is nowhere dense in K. Thus there exists $k \in U \cap (K - H)$ and there exists V, an open neighborhood of k in U, such that $V \cap H = \emptyset$. By Proposition 3.1, k cannot be contained in a metric cube, thus k is contained in a face. Now we will use a basis for P^1 to find a $G \in \mathcal{G}$ satisfying the desired properties.

Notice that either K is metric or nonmetric. If K is metric, then by Proposition 3.1 K cannot intersect any metric cube. So $K \subset \{x_y\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$ where $y \in \{0, 1\}$. Hence there exists $B \in \mathcal{B}$ such that $k \in B \times \{x_y\} \subset \bar{B} \times \{x_y\} \subset V$. Let $G = B \times L$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

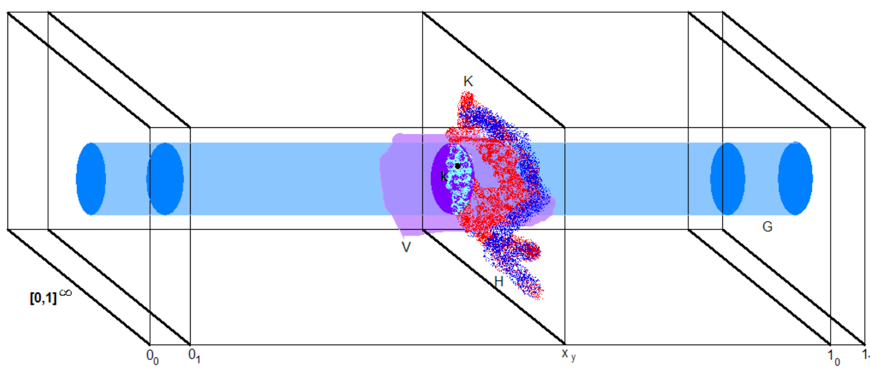

Figure 3.7: Proposition 2.3: K is metric.

Now suppose that K is nonmetric. Again, by Proposition 3.1, $k \in \{x_y\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$ for $x \in [0,1]$ and $y \in \{0,1\}$. This gives rise to three cases:

1. $k \in \{0_0\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$,

2. $k \in \{1_1\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$, or

3. $k \in [0_1, 1_0] \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$.

Case 1

Suppose $k \in \{0_0\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$.

Then $k \in V \cap ([0_0, 0_1) \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty)$. So there exists $B \in \mathcal{B}_0$ such that $k \in B \subset \bar{B} \subset V \cap ([0_0, 0_1) \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty)$. Let $G = B$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

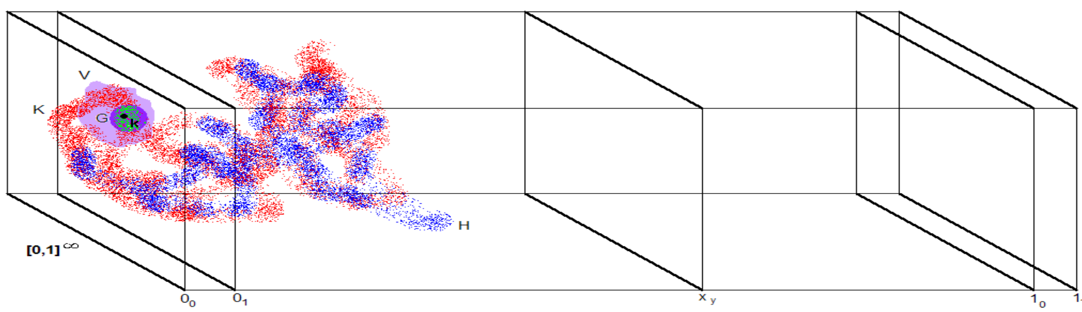

Figure 3.8: Proposition 2.3: Case 1

Case 2

Suppose $k \in \{1_1\} \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$.

Similar to Case 1, there is a $B \in \mathcal{B}_1$ such that $k \in B \subset \bar{B} \subset V \cap ((1_0, 1_1] \times \prod_{i=2}^{n}(a_0^i, a_1^i))$. Let $G = B$. Then $G \in \mathcal{G}$, $k \in G \cap (K - H)$, and $\bar{G} \cap H = \emptyset$.

Case 3

Suppose $k \in [0_1, 1_0] \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$.

Then k is in a metric face, and V is a nonmetric open subset of U. Since $P^1 = L \times \prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$, we have a_b, $c_d \in L$ such that $a < c$ and W open in $\prod_{i=2}^{n}(a_0^i, a_1^i) \times [0,1]^\infty$ such that $k \in (a_b, c_d) \times W \subset V$. By Propositions 3.1 and 3.2, k is on an unisolated face. So there exists $\hat{k} \in (a_1, c_0) \times W \subset V$. Now there exists $B \in \mathcal{B}$ and q, $r \in \mathbf{Q}$ such that $\bar{B} \subset W$

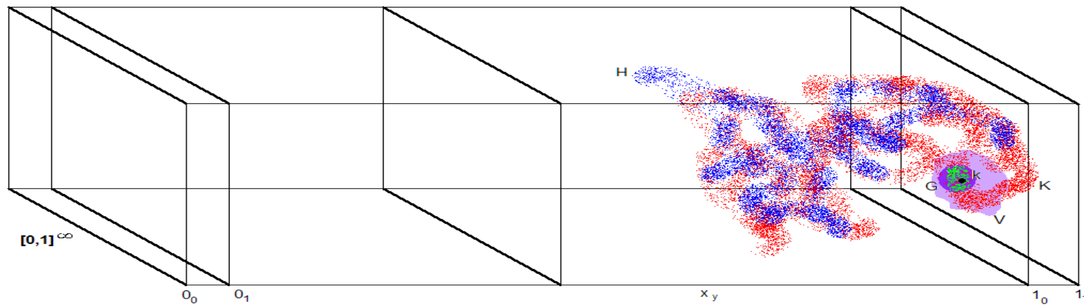

Figure 3.9: Proposition 2.3: Case 2

and $a < q < r < c$. Let $G = (q_0, r_1) \times B$. Then $G \in \mathcal{G}$, $\hat{k} \in G \cap (K - H)$, and by Corollary 3.1 $\bar{G} = [q_1, r_0] \times \bar{B} = (q_0, r_1) \times \bar{B} \subset V$. Hence $\bar{G} \cap H = \emptyset$.

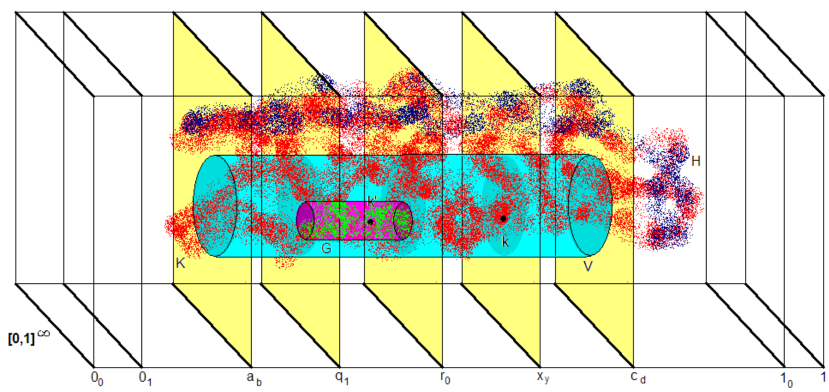

Figure 3.10: Proposition 2.3: Case 3

Hence we have satisfied the hypothesis for Theorem 1.7. Therefore $\alpha(p)$ is separable. □

In other words, if M is a nonmetric hereditarily indecomposable subcontinuum of $L^n \times [0,1]^\infty$, it must travel along the "edges" of a metric subspace, where an "edge" constitutes the subspace of $L^n \times [0,1]^\infty$ which restricts two of the Lexicographic coordinates to single points of the form x_y for $y \in \{0, 1\}$.

Chapter 4

Conclusions for the Product of Three Lexicographic Arcs and the Product of Two Lexicographic Arcs with the Hilbert Cube

Our first conclusion follows from the results of Chapter 2. There we discovered that adding nonmetrizability restricts a hereditarily indecomposable subcontinuum of L^n from intersecting metric cubes, or their faces. A nonmetric hereditarily indecomposable subcontinuum must then travel along the edges of the metric cubes. In the case of L^3, these edges are arcs. This contradicts the hereditary indecomposability.

Theorem 4.1. *If M is a hereditarily indecomposable subcontinuum of L^3, then M is metrizable.*

Proof. Let M be a hereditarily indecomposable subcontinuum of L^3. Suppose that M is nonmetric. We can assume that $pi_1(M) = L$. Then by Theorem 1.4, there is an irreducible subcontinuum of M from $\{0_0\} \times L \times L$ to $\{0_1\} \times L \times L$. We will denote this subcontinuum as K_0. Now either K_0 is metric or nonmetric.

Suppose that K_0 is metrizable. Then there exists a and $b \in [0, 1]$ such that K_0 is a subset of $[0_0, 0_1] \times [a_0, a_1] \times [b_0, b_1]$. By Proposition 2.3 $M \cap ((0_0, 0_1) \times (a_0, a_1) \times L) = \emptyset$, $M \cap ((0_0, 0_1) \times L \times (b_0, b_1)) = \emptyset$, and $M \cap (L \times (a_0, a_1) \times (b_0, b_1)) = \emptyset$. This implies that $K_0 \cap ((0_0, 0_1) \times (a_0, a_1) \times [b_0, b_1]) = \emptyset$, $M \cap ((0_0, 0_1) \times [a_0, a_1] \times (b_0, b_1)) = \emptyset$, and $M \cap ([0_0, 0_1] \times (a_0, a_1) \times (b_0, b_1)) = \emptyset$. In other words, each point of K_0 has two coordinates that are of the form x_0 or x_1. This implies, along with the fact that K_0 is irreducible from $\{0_0\} \times L \times L$ to $\{0_1\} \times L \times L$, that $\pi_1^{-1}([0_0, 0_1]) \cap K_0$ contains $[0_0, 0_1] \times \{a_y\} \times \{b_y\}$ where y, $z \in \{0, 1\}$. Hence K_0 contains an arc contradicting the hereditary indecomposability of M.

Therefore K_0 is nonmetric. Thus it can be assumed that $pi_2(K_0) = L$. By Theorem 1.4 there exists an irreducible subcontinuum of K_0 from $[0_0, 0_1] \times \{a_0\} \times L$ to $[0_0, 0_1] \times \{a_1\} \times L$ for some fixed $a \in [0, 1]$, say K_1. So K_1 is contained in $[0_0, 0_1] \times [a_0, a_1] \times L$ which by Theorem 3.1 means that K_1 is metrizable. Hence by the same argument as in the case that K_0 was metrizable, we have that K_1 contains an arc of the form $\{0_y\} \times [a_0, a_1] \times \{b_z\}$ for some fixed $b \in [0, 1]$ and y, $z \in \{0, 1\}$. This in turn again contradicts the hereditary indecomposability of M. Therefore M is metrizable. □

The generalization in Chapter 3 was that instead of Euclidean cubes, our points were replaced with Hilbert cubes. This means that each point is replaced with a space for which all metric continua are embedded. The same argument as in L^3 generalizes with a few changes to $L^2 \times [0, 1]^\infty$.

Theorem 4.2. *If M is a hereditarily indecomposable subcontinuum of $L^2 \times [0, 1]^\infty$, then M is metrizable.*

Proof. Let M be a hereditarily indecomposable subcontinuum of $L^2 \times [0, 1]^\infty$. Suppose that M is nonmetric. We can assume that $pi_1(M) = L$. Then by Theorem 1.4, there is an irreducible subcontinuum of M from $\{\frac{1}{2}_0\} \times L \times [0, 1]^\infty$ to $\{\frac{1}{2}_1\} \times L \times [0, 1]^\infty$. We will denote this subcontinuum as K_0. Now either K_0 is metric or nonmetric.

Suppose that K_0 is metrizable. Then there exists $a \in [0, 1]$ such that K_0 is a subset of $[\frac{1}{2}_0, \frac{1}{2}_1] \times [a_0, a_1] \times [0, 1]^\infty$. By Proposition 3.1 $M \cap ((\frac{1}{2}_0, \frac{1}{2}_1) \times (a_0, a_1) \times [0, 1]^\infty) = \emptyset$ which implies that $K_0 \cap ((\frac{1}{2}_0, \frac{1}{2}_1) \times (a_0, a_1) \times [0, 1]^\infty) = \emptyset$. By Proposition 3.3, $M \cap ((\frac{1}{2}_0, \frac{1}{2}_1) \times L \times [0, 1]^\infty) = \emptyset$ and $M \cap (L \times (a_0, a_1) \times [0, 1]^\infty) = \emptyset$. This implies that $K_0 \cap ((\frac{1}{2}_0, \frac{1}{2}_1) \times [a_0, a_1] \times [0, 1]^\infty) = \emptyset$, and $K_0 \cap ([\frac{1}{2}_0, \frac{1}{2}_1] \times (a_0, a_1) \times [0, 1]^\infty) = \emptyset$. This implies, along with the fact that K_0 is irreducible from $\{\frac{1}{2}_0\} \times L \times [0, 1]^\infty$ to $\{\frac{1}{2}_1\} \times L \times [0, 1]^\infty$, that $\pi_1^{-1}([\frac{1}{2}_0, \frac{1}{2}_1]) \cap K_0$ contains $[\frac{1}{2}_0, \frac{1}{2}_1] \times \{a_y\} \times \{\vec{z}\}$ where $y \in \{0, 1\}$ and $\vec{z} \in [0, 1]^\infty$. Hence K_0 contains an arc, contradicting the hereditary indecomposability of M.

Therefore K_0 is nonmetric. Thus it can be assumed that $pi_2(K_0) = L$. By Theorem 1.4 there exists an irreducible subcontinuum of K_0 from $[\frac{1}{2}_0, \frac{1}{2}_1] \times \{a_0\} \times [0, 1]^\infty$ to $[\frac{1}{2}_0, \frac{1}{2}_1] \times$

$\{a_1\} \times [0,1]^\infty$ for some fixed $a \in [0,1]$, say K_1. So K_1 is contained in $[\frac{1}{2}_0, \frac{1}{2}_1] \times [a_0, a_1] \times [0,1]^\infty$ which by Theorem 3.1 means that K_1 is metrizable. Hence by the same argument as in the case that K_0 was metrizable, we have that K_1 contains an arc of the form $\{\frac{1}{2}_y\} \times [a_0, a_1] \times \{\vec{z}\}$ for some fixed $\vec{z} \in [0,1]^\infty$ and $y \in \{0,1\}$. This in turn again contradicts the hereditary indecomposability of M. Therefore M is metrizable. $\square$

Theorem 4.2 is the beginning of the next step to showing that the product of four Lexicographic arcs contains only metric hereditarily indecomposable subcontinua. From there the author is working on establishing a inductive argument to show the same for all finite products of Lexicographic arcs.

Bibliography

[1] R. H. Bing, "A Homogeneous Indecomposable Plane Continuum," Duke Mathematics Journal 15 (1948), 729-742.

[2] R. H. Bing, "Higher Dimensional Hereditarily Indecomposable Continuum," Trans. Amer. Math. Soc. 71 (1951), 267-273.

[3] Krasinkiewicz and Minc, "Mappings onto Indecomposable Continua," Bulletin de L'Academie Polonaise des Sciences 25 (1977), 675-680.

[4] S. Nadler, "Continuum Theory," Marcel Dekker, Inc., New York, 1992.

[5] S. Nadler, "Hyperspaces of Sets," Marcel Dekker, Inc., New York, 1978.

[6] M. Smith, "Hausdorff Hypercubes which do not contain Arcless Continua," Pro AMS 95, No. 1 (1985), 109-114.

[7] M. Smith, "On Non-Metric Indecomposable and Hereditarily Indecomposable Subcontinua of Products of Long Hausdorff Arcs," Aportaciones Matematicas, special issue in honor of 60th birthday of David Bellamy (2006).

[8] M. Smith, "Hereditarily Indecomposable Subcontinua of the Inverse Limit of Lexicographic Arcs are Metric," Topology Appl. 154, No. 15 (2007), 2813-2816.

[9] M. Smith, "Hereditarily Indecomposable Subcontinua of Inverse Limits of Souslin Arcs are Metric.," in press, Topology Appl.

[10] M. Smith, "Hereditarily Indecomposable Subcontinua of the Product of Two Souslin Arcs are Metric," preprint (2009).

[11] M. Smith and J. Stone, "Every Non-Metric Indecomposable Subcontinuum of the Square of the Lexicographic Arc contains an Arc," presented at the Topology Conference, Rolla, NC (2006); submitted.

[12] K. Kuratowski, "Topology," vII, Academic Press, New York, 1968.

[13] R. Greiwe, "A Brief Exploration of the Sorgenfrey Line and the Lexicographic Order," Masters Thesis, Auburn University (2006).

APPENDICES

Appendix A

Notes on the style-file project

These style-files for use with LaTeX are maintained by Darrel Hankerson[1] and Ed Slaminka[2].

In 1990, department heads and other representatives met with Dean Doorenbos and Judy Bush-Crofton (then responsible for manuscript approval). This meeting was prompted by a memorandum[3] from members of the mathematics departments concerning the *Thesis and Dissertation Guide* and the approval process. There was wide agreement among the participants (including Dean Doorenbos) to support the basic recommendations outlined in the memorandum. The revised *Guide* reflected some (but not all) of the agreements of the meeting.

Ms Bush-Crofton was supportive of the plan to obtain "official approval" of these style files.[4] Unfortunately, Ms Bush-Crofton left the Graduate School before the process was completed. In 1994, we were revisiting some of the same problems which were resolved at the 1990 meeting.

In Summer 1994, I sent several memoranda to Ms Ilga Trend of the Graduate School, reminding her of the agreements made at the 1990 meeting. Professors A. Scottedward Hodel and Stan Reeves provided additional support. In short, it is essential that the Graduate School honor its commitments of the 1990 meeting. It should be emphasized that Dean Doorenbos is to thank for the success of that meeting.

Maintaining these LaTeX files has been more work than expected, in part due to continuing changes in requirement by the graduate school. The Graduate School occasionally has complete memory loss about the agreements of the 1990 meeting. If the Graduate School rejects your manuscript based on items controlled by the style-files, ask your advisor to contact the Graduate school (and copy to me) to urge cooperation.

Finally, there have been several requests for additions to the package (mostly formatting changes for figures, etc.). While such changes are not really part of the thesis-style package, it could be beneficial to collect these options and distribute with the package (making it easier on the next student). I'm especially interested in changes needed by various departments.

---

[1] Mathematics and Statistics, 221 Parker Hall, 844-3641, `hankedr@auburn.edu`

[2] Mathematics and Statistics, 218 Parker, `slamiee@auburn.edu`

[3] Originally, the memorandum was presented to Professor Larry Wit. A copy is available on request.

[4] Followup memoranda gave a definition of "official approval." Copies will be sent on request.