Decision Models for a Two-stage Supply Chain Planning under Uncertainty with Time-Sensitive Shortages and Real Option Approach

by

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Abstract

The primary objective of this research is to develop analytical models for typical supply chain situations to help inventory decision-makers. We also derive closed form solutions for each model and reveal several managerial insights from our models through numerical examples. Additionally, this research gives decision-makers insights on how to implement demand uncertainty and shortage into a mathematical model in a two-stage supply chain and shows them what differences these proposed analytical models make as opposed to the traditional models.

First, we model customer impatience in an inventory problem with stochastic demand and time-sensitive shortages. This research explores various backorder rate functions in a single period stochastic inventory problem in an effort to characterize a diversity of customer responses to shortages. We use concepts from utility theory to formally classify customers in terms of their willingness to wait for the supplier to replenish shortages. Additionally, we introduce the notion of expected value of risk profile information (EVRPI), and then conduct additional sensitivity analyses to determine the most and least opportune conditions for distinguishing between customer risk-behaviors.

Second, we optimize backorder lead-time (response time) in a two-stage system with time-dependent partial backlogging and stochastic demand. In this research, backorder cost is characterized as a function of backorder response time. We also regard backorder rate as a decreasing function of response time. We develop a representative expected cost function and closed form optimal solutions for several demand distributions.

Third, we adopt an option approach to improve inventory decisions in a supply chain. First of all, we apply a real option-pricing framework (e.g., straddle) for determining order

quantity under partial backlogging and uncertain demand situation. We establish an optimal condition for the required order quantity when a firm has an desirable fill rate. We develop a closed form solution for optimal order quantity to minimize the expected total cost.

Finally, we implement an option contract to hedge the risk of the demand uncertainty. We show that the option contract leads to an improvement in the overall supply chain profits and product availability in the two-stage supply chain system. This research considers a standard newsvendor problem with price dependent stochastic demand in a single manufacturer and retailer channel. We derive closed form solutions for the appropriate option prices set by the manufacturer as an incentive for the retailer to establish optimal pricing and order quantity decisions for coordinating the channel.

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Great things are done by a series of small things brought together.

\- Vincent Van Gogh

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Chapter 1

Introduction

1.1 Problem Statement

Carrying inventories becomes inevitable in most businesses because the production and consumption take place at different times in different locations and at different rates. In fact, inventory is one of the single largest investments made by most businesses. Given today's global financial crisis, the supply chain inventory management could be more crucial than before. The roughly 400 companies in the S&P Industrials have close to $ 500 billion invested in inventory. Inventory capital costs absorb a significant percentage of operating profits for a company. For automotive and consumer products, these costs absorb approximately 40% of operating profits. Therefore, inventory optimization is one of the significant topics in supply chain circles today considering the dynamic state of markets all over the world. As a growing number of organizations have proved, astute planning and management can wring 20-30 % out of current inventories, saving "several millions" in direct costs and achieving huge gains in operational performance, all the while maintaining or even improving product availability and customer service.

Indigent supply chain inventory management could spell disaster for any company. The higher the inventory investment as a percentage of total assets of a company, the higher the damage caused by poor inventory management. Consider the following situation recently faced by the Japanese game company Nintendo, Inc. (Schlaffer, 2007):

Wii Shortages Could Cost Nintendo Billions In Sales:

The Nintendo Wii was one of the most popular consoles in 2006 and it is the most

popular in 2007. Only there is a problem, the company is still having problems meeting demand. If you were hoping to provide your children, family or other loved one/friend with a Wii this Christmas, I say that it is far too late to do so. No store will have it in stock though you may be able to purchase one that has been jacked up in price on eBay, chances of receiving it in time are slim. It seems a parts shortage has struck Nintendo and it is going to hurt the company financially. Despite the shortage, the company says it is doing everything it can to meet demand. But that's not enough; it may end up losing just slightly over a billion dollars if it cannot fix these problems.

To optimize the deployment of inventory, you need to manage the uncertainties, constraints, and complexities across a multi-stage supply chain on a continuous basis. Therefore, many companies adopt inventory control systems, enabling them to handle many variables and continuously update in order to optimize their multi-stage supply chain systems.

However, in many cases inventory decision-makers need analytical models to grasp the big picture of supply chain inventory problems before making executive decisions or implementing inventory control systems. In fact, a reliable analytical model is important for practitioners (e.g., decision-maker) to make proper predictions of their field of interest. Especially, this research develops appropriate models in order to abstract the features of a supply chain system as a set of parameters or parameterized functions. These analytical models are simple but provide effectively an overall view of the supply chain system.

In general, there are five basic blocks for inventory management activities: (1) Demand forecasting or demand management, (2) Sales and operations planning, (3) Production planning, (4) Material requirements planning, (5) Inventory reduction and shortage management.

Every activity is critical, but as shown in the Nintendo shortage case, demand forecasting and shortage management play a key role in supply chain inventory management. Therefore, we focus our attention on the activities of demand forecasting and shortage management. In particular, this research focuses on how to implement the situation of demand uncertainty and shortage into a mathematical model properly based on a two-stage supply chain.

1.2 Research Objectives

The first objective of this study is to implement time-sensitive shortages into a conventional analytical inventory model. We consider the fact that shortages are partially backlogged; a fraction of shortages incur lost sales penalties while the remaining shortages are backlogged. Therefore, we examine the implications of incorporating the notion of time-dependent partial backlogging in the single period stochastic inventory problem. Moreover, we explore linear and nonlinear decreasing backorder rate functions with respect to lead-time in an inventory problem with time-dependent partial backlogging, demand uncertainty, and emergency replenishment in an effort to characterize diverse customer responses to shortages.

The second objective of this study is to consider demand uncertainty in an inventory decision model. The conventional approach, such as the newsvendor problem, of inventory-stocking decisions relies on a specific distribution of demand for the inventory item to implement demand uncertainty. In dealing with market uncertainty, option-pricing models have become a powerful tool in corporate finance. The literature that applies option-pricing models to capital budgeting - often referred to as real options - is extensive. We explore an option-pricing model to consider demand uncertainty in a supply chain.

The third objective of this research is to establish a coordination model of a two-stage supply chain with a real option framework (e.g.,option contract). Actions taken by the two parties in the supply chain often result in profits that are lower than what could be achieved if the supply chain were to coordinate its actions with a common objective of maximizing supply chain profits. This research develops an option contract in a newsvendor problem with price dependent stochastic demand and shows that the option contract could lead to coordination in the supply chain through improving the product availability and the overall supply chain profits.

1.3   Research Plan

Chapter 2 develops a decision model considering customer impatience with stochastic demand and time-sensitive shortages. We use concepts from utility theory to formally classify customers in terms of their willingness to wait for the supplier to replenish shortages. We conduct sensitivity analyses to determine the most and least opportune conditions for distinguishing between customer risk-behaviors.

Chapter 3 establishes an additional model to optimize backorder lead-time (response time) in a two-stage system with time-dependent partial backlogging and stochastic demand. We consider backorder cost as a function of response time. A representative expected cost function is derived and the closed form optimal solution is determined for a general demand distribution.

Chapter 4 develops an inventory decision model with an option framework in a supply chain. We apply a real option-pricing framework (e.g., straddle) for determining order quantity under partial backlogging and demand uncertainty. We compare the results between the traditional approach and the option approach with a numerical example.

Chapter 5 implements an option contract to improve an overall supply chain profits and product availability in a two-stage supply chain system. We derive closed form solutions for the appropriate option prices to coordinate a supply chain system. We illustrate our result with numerical examples to help decision making in a supply chain coordination.

Chapter 6 presents a brief conclusion along with some suggestions for future research.

Chapter 2

Modeling Customer Impatience in an Inventory Problem with Stochastic Demand and
Time-Sensitive Shortages

ABSTRACT

Customers across all stages of the supply chain often respond negatively to inventory short-
ages. One approach to modeling customer responses to shortages in the inventory control
literature is time-dependent partial backlogging. Partial backlogging refers to the case in
which a customer will backorder shortages with some probability, or will otherwise solicit
the supplier's competitors to fulfill outstanding shortages. If the backorder rate (i.e., the
probability that a customer elects to backorder shortages) is assumed to be dependent on
the supplier's backorder replenishment lead-time, then shortages are said to be represented
as time-dependent partial backlogging. This research explores various backorder rate func-
tions in a single period stochastic inventory problem in an effort to characterize a diversity
of customer responses to shortages. We use concepts from utility theory to formally classify
customers in terms of their willingness to wait for the supplier to replenish shortages. Under
assumptions, we verify the existence of a unique optimal solution that corresponds to each
customer type. Sensitivity analysis is conducted in order to compare the optimal actions
associated with each customer type under a variety of conditions. Additionally, we intro-
duce the notion of expected value of risk profile information (EVRPI), and then conduct
additional sensitivity analyses to determine the most and least opportune conditions for
distinguishing between customer risk-behaviors.

2.1 Introduction

Inventory shortages are often an indicator of suboptimal supply chain performance caused by a mismatch between supply and demand. In general, shortages are classified as either backorders or lost sales. Immediate consequences of backlogged shortages include increased administrative costs, the cost of delayed revenue, emergency transportation costs, and diminished customer perception (i.e., the loss of customer goodwill), while lost sales are characterized by the opportunity cost of lost revenue and diminished customer perception. In the long run, inventory shortages can compromise an organization's market share and negatively affect long term profitability.

Conventional stochastic inventory models such as the single period problem (or the newsvendor problem) and its many variants (e.g., Khouja 1999) often assume that shortages are either completely backlogged or that all sales are lost. Although this assumption is sometimes plausible in practice, there are situations in which an alternative approach to modeling shortages is appropriate. This research explores the implications of incorporating the notion of time-dependent partial backlogging in the single period stochastic inventory problem. When shortages are *partially backlogged*, a fraction of shortages incur lost sales penalties while the remaining shortages are backlogged. Therefore, *time-dependent partial backlogging* implies that the backorder rate (i.e., the fraction of shortages backlogged) depends on the time associated with replenishing the outstanding backorder. In many practical situations, customers are likely to fulfill shortages from a supplier's competitor who has inventory on hand if the backorder lead time is extensive. On the other hand, the supplier is more likely to retain the customer's business and possibly avoid the long-term consequences of shortages if the backorder lead-time is reasonably short. From this perspective, it is evident that the time-dependent partial backlogging approach to modeling shortages is particularly useful to firms who compete in time-sensitive markets and embrace service and responsiveness as a competitive strategy (e.g., Stalk and Hout 1990).

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Table 2.1: A summary of the related literature.

Literature	Demand uncertain	Emergency replenish	Pricing decision	Stock deteriorate	# backorder rate functions
Zhou et al.(2004)	x	x	x	x	1
Skouri et al.(2003)	x	x	x	o	1
Dye et al.(2006)	x	x	x	o	1
Dye(2007)	x	x	o	o	1
Abad(1996)	x	x	o	o	1
Lodree(2007)	o	o	x	x	1
My Model	o	o	x	x	3

Several variations of inventory models with time-dependent partial backlogging have been discussed in the research literature as shown in Table 2.1 including (i) models with time-varying demand (Zou et al. 2004); (ii) models with time-varying demand and stock deterioration (e.g., Skouri and Papachristos 2003; Dye et al. 2006); (iii) models with stock deterioration, ordering decisions, and pricing decisions (Dye 2007); and (iv) models with time-varying demand, ordering decisions, pricing decisions, and stock deterioration (Abad 1996). In general, the backorder rate is assumed to be a piecewise linear function of the backorder lead-time, except for Papachristos and Skouri (2000) and San José et al. (2006) who consider exponential backorder rate functions. Additionally, the majority of the literature involves continuous review inventory policies in which shortages are replenished at the time of the next scheduled delivery. However in practice, suppliers may attempt to replenish backlogged shortages before the next scheduled delivery by engaging an emergency replenishment process that involves emergency procurement of component parts, emergency production runs, overtime labor, and expedited delivery. The time-dependent backlogging literature also addresses various demand processes including time-varying, price-dependent, and stock dependent; but the majority ignore demand uncertainty. The latter two issues (demand uncertainty at the time of the inventory decision and emergency replenishment after demand realization) are addressed in Lodree (2007), where the time-dependent partial backlogging approach is used to model shortages in the newsvendor problem.

This study explores linear and nonlinear backorder rate functions in an inventory problem with time-dependent partial backlogging, demand uncertainty, and emergency replenishment in an effort to characterize a diversity of customer responses to shortages. Moreover, we find it convenient to use concepts from utility theory to characterize customer responses to shortage as either risk-averse, -neutral, or -seeking. We also compare the optimal inventory levels associated with each customer type through sensitivity analysis and identify the conditions in which there are minute or significant differences in the optimal levels. Finally, we use this framework to determine the benefit of understanding the dominant market characteristic in terms of being risk-averse, -neutral, or -seeking with respect to time sensitivity. In particular, if a firm is uncertain about the characteristics of the market it serves, we define the expected value of risk profile information (EVRPI) so that the firm can assess the value of a study or survey whose results reveal the true dominant market characteristic.

2.2   General Mathematical Model

In this section, we present a mathematical model for the newsvendor problem with time-dependent partial backlogging. To do so, let β represent the backorder rate, which can be interpreted as the fraction of shortages backlogged or the probability that a given customer will choose to backlog shortages. Since β is time-dependent, we have that $\beta : L \mapsto [0, 1]$, where $L \in [0, \infty)$ is the backorder lead-time (refer to Table 2.2 for a list of notations used repeatedly in this study). We assume that L is directly proportional to the magnitude of an observed shortage. In other words, we assume that L is a linear function in $x - Q$ for $x \geq Q$, where x is observed demand, Q is the inventory level before demand realization, and $\max\{x - Q, 0\}$ is the observed number of shortages. Without loss of generality, we assume that $L = x - Q$ so that the terms "backorder lead-time" and "magnitude of shortage" can be used interchangeably for the purposes of this study. Therefore, the backorder rate is expressed as the function $\beta(x - Q)$. Let M be the maximum allowable shortage in the sense that the probability of backlogging is zero if $x - Q \geq M$. Then assuming X is a continuous

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Table 2.2: List of notations.

Q:	Order quantity (the decision variable).
X:	Supplier's demand (the buyer's order), a continuous random variable.
x:	Actual value of demand.
$f(x)$:	Probability density function (pdf) of X.
$F(x)$:	Cumulative distribution function (cdf) of X.
c_O:	Unit ordering/production cost before demand realization.
c_H:	Unit cost of holding excess inventory at the of the season.
c_B:	Unit ordering/production cost after demand realization for backlogged shortages.
c_{LS}:	Unit cost of shortages that are lost sales.
$\beta(x-Q)$:	Fraction of shortages that are backlogged.
M:	Lost sales threshold.
$TC(Q)$:	Total expected cost function.
Q^*:	Optimal quantity that minimizes $TC(Q)$.

random variable that represents demand, the newsvendor problem with time-dependent partial backlogging can be expressed as follows:

$$
\begin{aligned}
TC(Q) \;=\; & \text{Order Cost + Expected Holding Cost + Expected Backorder Cost} \\
& + \text{ Expected Lost Sales Cost} \\
\;=\; & c_O Q + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x + c_B \int_Q^{Q+M} (x-Q)\beta(x-Q)f(x)\mathrm{d}x \\
& + \; c_{LS} \int_Q^\infty (x-Q)\left[1-\beta(x-Q)\right] f(x)\mathrm{d}x \tag{2.1}
\end{aligned}
$$

The backorder rate function should satisfy the following properties.

Property 2.1 $\dfrac{\mathrm{d}\beta(x-Q)}{\mathrm{d}(x-Q)} < 0 \;\; \forall \;\; x \in [Q, Q+M) \;\; and \; \dfrac{\mathrm{d}\beta(x-Q)}{\mathrm{d}(x-Q)} = 0 \;\; \forall \;\; x \in [Q+M, \infty).$

Property 2.2 $\displaystyle\lim_{(x-Q)\to 0^+} \beta(x-Q) = 1.$

Property 2.3 $\displaystyle\lim_{(x-Q)\to\infty} \beta(x-Q) = 0.$

Property 2.1 indicates that $\beta(x - Q)$ is a continuous and decreasing function in the number of observed shortages, which suggests that customers are more likely to wait for backorder replenishment if the backorder lead-time is short, and are less likely to wait if the lead-time is close to the lost sales threshold. The second part of Property 2.1 is a result of the fact that the probability of backlogging remains zero if $x - Q \geq M$. Properties 2.2 and 2.3 reiterate Property 2.1, but also ensure that $\beta(x - Q) \in [0, 1]$.

We want to determine the inventory level Q^* that minimizes $TC(Q)$ given by Eq. (2.1), provided an optimum exists. It turns out that the convexity of Eq. (2.1) cannot be guaranteed in general, but the following theorem identifies a sufficent (but not necessary) condition for the existence of a unique minimizer. Before presenting the theorem, we introduce the following assumption.

Assumption 2.1 $c_O < c_B < c_{LS}$

The inequalities $c_O < c_B$ and $c_O < c_{LS}$ are assumed in order to avoid trivial cases and are also reflective of practice. The inequality $c_B < c_{LS}$ is also representative of practice, although there are some situations where backorder costs exceed short term lost sale costs such as lost revenue. Additionally, the latter inequality will enable us to prove the next theorem as well as other results presented later.

Theorem 2.1 *A sufficient condition for $TC(Q)$ to be a convex function for all $Q \geq 0$ is*

$$(x - Q) \cdot \beta''(x - Q) \leq 2\beta'(x - Q) \tag{2.2}$$

$$\text{where } \beta'(x - Q) = \frac{\mathrm{d}\beta(x - Q)}{\mathrm{d}Q} \text{ and } \beta''(x - Q) = \frac{\mathrm{d}\beta'(x - Q)}{\mathrm{d}Q}$$

Proof: It is shown in Appendix A.1.
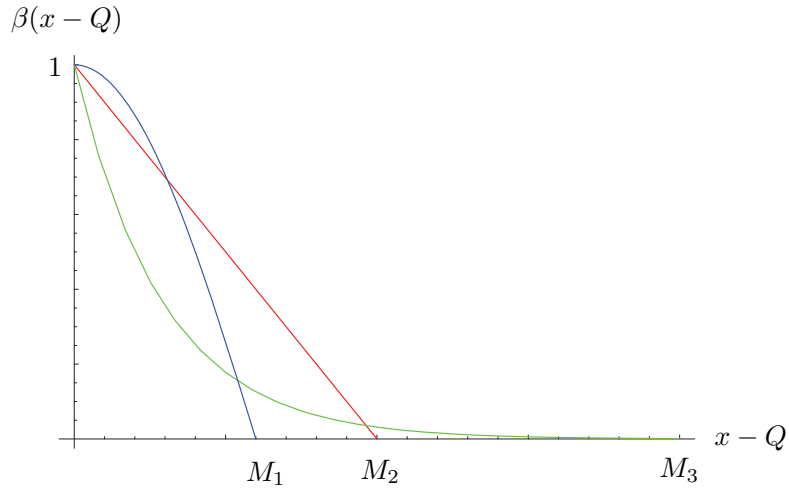
Figure 2.1: Customer impatience and lost sale thresholds.

2.3   A Classification of Customer Impatience

An obvious candidate for measuring customer impatience is the lost sales threshold, M. For example, if $M_1 < M_2$ (where M_1 and M_2 are lost sales thresholds for customer 1 and customer 2 respectively), then it seems reasonable to conclude that customer 1 is more impatient than customer 2. However, Figure 2.1 tells a different story. In particular, one can argue from Figure 2.1 that customer 3 is more impatient than customer 2 and that customer 2 is more impatient than customer 1, even though $M_1 < M_2 < M_3$. At the very least, Figure 2.1 reveals that M alone does not distinctively measure customer impatience and that the shape of $\beta(x - Q)$ should also be taken into account in distinguishing among varying degrees of customer impatience.

Consider the case in which $M_1 = M_2 = M_3 = M$ as shown in Figure 2.2. According to Figure 2.2, customer 3 is more impatient than customer 2, and customer 2 is more impatient than customer 1. Figure 2.2 also suggests that customer 1 is patient at first, but decreasingly patient because his rate of change in impatience is increasing. On the other hand, customer 3 is impatient at first, but decreasingly impatient. The relative impatience of these customers can be attributed to individual attitudes and personalities, but could

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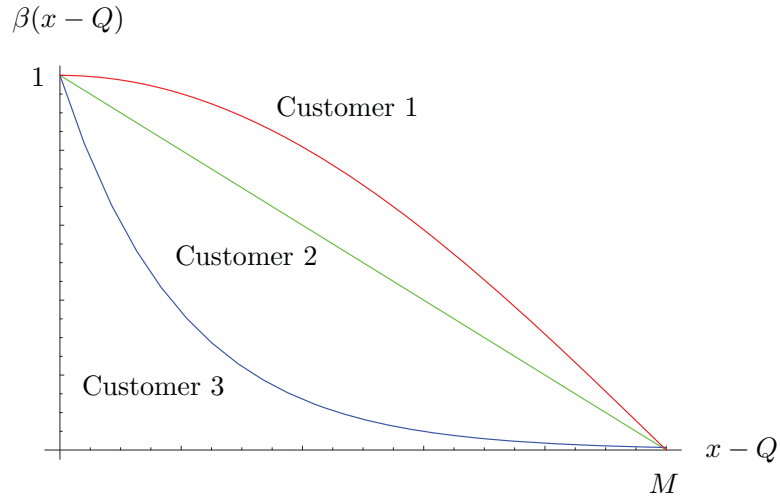
Figure 2.2: Customer impatience and backorder rates.

also be interpreted as one person's attitude with respect to waiting for different classes of products.

For instance, customer 3 could represent an individual's willingness to wait for a product, such as a dairy, party product or a computer mouse pad, in which a suitable substitute can be easily identified. In this case, the customer will most likely purchase the suitable substitute as opposed to waiting for backorder replenishment, even if the backorder lead-time is relatively short. Customer 1 would represent this same customer's willingness to wait for a different product type, such as automobile parts, classic furniture, or specialized computer software, in which a product of comparable utility cannot be as easily identified. In this case, the customer is more likely to wait for backorder replenishment, especially if the backorder lead-time is relatively short. These notions of impatience, increasing impatience, and decreasing impatience for describing customer behavior with respect to waiting for backorder replenishment can be defined more precisely using analogous concepts and terminology from utility theory.

Let $u(y)$ represent the utility of the value y. Then the following result is fundamental to utility theory (for example, see Winkler 2003):

Theorem 2.2 *Let $u(y)$ be a continuous and twice differentiable function. Then a decision-maker is*

- *Risk-averse if and only if $u''(y) < 0$.*

- *Risk-neutral if and only if $u''(y) = 0$.*

- *Risk-seeking if and only if $u''(y) > 0$.*

We will classify customers as risk-averse, risk-neutral, or risk-seeking based on the backorder rate function $\beta(x - Q)$ as opposed to some utility function $u(y)$. In order to develop our classification scheme, we introduce the following definitions all of which are special cases of classical utility theory.

Definition 2.1 *A **lottery with respect to waiting time**, $\mathcal{L} = (t_1, p_1; \ldots; t_n, p_n)$, consists of a set of waiting times $\{t_1, \ldots, t_n\}$ and a set of probabilities $\{p_1, \ldots, p_n\}$ such that a decision-maker waits for t_i time units with probability p_i, where $i = 1, \ldots, n$.*

Definition 2.2 *The **certainty equivalent with respect to waiting time** of a lottery $\mathcal{L} = (t_1, p_1; \ldots; t_n, p_n)$, denoted $CE_W(\mathcal{L})$, is the waiting time such that the decision-maker is indifferent between $\mathcal{L}$ and waiting for $CE_W(\mathcal{L})$ with certainty.*

Definition 2.3 *The **risk premium with respect to waiting time** of a lottery $\mathcal{L} = (t_1, p_1; \ldots; t_n, p_n)$, denoted $RP_W(\mathcal{L})$, is defined as*

$$RP_W(\mathcal{L}) = E_W[\mathcal{L}] - CE_W(\mathcal{L})$$

where $E_W[\mathcal{L}]$ is the expected value of the lottery $\mathcal{L}$.

Definition 2.4 *If $\mathcal{L} = (t_1, p_1; \ldots; t_n, p_n)$ is a lottery with respect to waiting time with $n > 1$, then a decision-maker is*

- ***Risk-averse with respect to waiting time*** *if and only if $RP_W(\mathcal{L}) < 0$.*

- ***Risk-neutral with respect to waiting time*** *if and only if $RP_W(\mathcal{L}) = 0$.*

- ***Risk-seeking with respect to waiting time*** *if and only if $RP_W(\mathcal{L}) > 0$.*

The difference in conventional utility theory and utility theory with respect to waiting time as described by definitions 2.1 $\sim$ 2.4 is noticeably observable in Definition 2.4. In particular, the definition of conventional risk aversion is $RP_W(L) > 0$ (see Winkler 2003, for example), but the definition of risk aversion with respect to waiting time is $RP_W(L) < 0$. Similarly, the inequalities are also reversed in the definitions of conventional risk-seeking and risk-seeking with respect to waiting time. In order to illustrate Definition 2.4, consider two decision-makers, DM_1 and DM_2, who are presented with a lottery with respect to waiting time related to the time they wait to be seated in a restaurant. In particular, the lottery is defined as $\mathcal{L} = (50\text{-min.}, 0.25; 10\text{-min.}, 0.75)$. Suppose DM_1 specifies $CE_{W1}(\mathcal{L}) = 12$ minutes and DM_2 specifies $CE_{W2}(\mathcal{L}) = 40$ minutes. Since $E_W[\mathcal{L}] = 20$ minutes, we have $RP_{W1}(\mathcal{L}) = 20 - 12 = 8$ minutes and $RP_{W2}(\mathcal{L}) = 20 - 40 = -20$ minutes. Since $CE_{W1}(\mathcal{L})$ is a shorter wait than $E_W[\mathcal{L}]$, DM_1 considers $\mathcal{L}$ to be $RP_{W1}(\mathcal{L}) = 8$ minutes better than its expected value, which suggests that DM_1 is influenced more by the possibility of the 10-minute wait than the risk of a 50-minute wait. Therefore, DM_1 is risk-seeking. On the other hand, since $CE_{W2}(\mathcal{L})$ is a longer wait than $E_W[\mathcal{L}]$, DM_2 considers $\mathcal{L}$ to be $RP_{W2}(\mathcal{L}) = -20$ minutes better (i.e., 20 minutes worse) than its expected value, which suggests that DM_2 is more influenced by the risk of a 50-minute wait than the chances of a 10-minute wait. Thus DM_2 prefers to avoid the lottery's risk and is therefore risk-averse. This example gives some insight into the logic behind Definition 2.4.

The following theorem relates the backorder rate function, $\beta(x - Q)$, to risk-averse, -neutral, and -seeking behavior with respect to waiting time.

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Theorem 2.3 *Let $\beta : y \mapsto [0,1]$, where $y \in [0,\infty)$, be a continuous and twice differentiable function. Then a decision-maker is*

- *Risk-averse with respect to waiting time if and only if $\beta''(y) < 0$.*

- *Risk-neutral with respect to waiting time if and only if $\beta''(y) = 0$.*

- *Risk-seeking with respect to waiting time if and only if $\beta''(y) > 0$.*

Proof: Please refer to Appendix A.2.

Based on Theorem 2.3, customers 1, 2, and 3 in Figure 2.2 are risk-averse, -neutral, and -seeking, respectively. Thus when customers all have the same lost sales threshold, M, the risk-averse customer is more patient than the risk-neutral customer, and the risk neutral customer is more patient than the risk-seeking customer since $\beta_1 > \beta_2 > \beta_3$ for all $x - Q \in [0, M]$, where β_i is the backorder rate for customer i.

2.3.1 Risk-Neutral Behavior

A backorder rate function $\beta(x - Q)$ that describes risk-neutral behavior with respect to waiting time (see definitions 2.1 $\sim$ 2.4) should satisfy properties 2.1 $\sim$ 2.3 as well as the second part of Theorem 2.3. The following proposition defines a representative backorder rate function that satisfies these conditions.

Proposition 2.1 *Suppose $x - Q \geq 0$. Then*

$$
\beta(x - Q) \quad = \quad
\begin{cases}
1 - \frac{x-Q}{M}, & x \in [Q, Q + M) \\
0, & x \in [Q + M, \infty)
\end{cases}
\tag{2.3}
$$

satisfies properties 2.1 $\sim$ 2.3 and the second part of Theorem 2.3.

Proof: Since

$$\frac{\mathrm{d}\beta(x-Q)}{\mathrm{d}(x-Q)} = \begin{cases} -\dfrac{1}{M} < 0, & x \in [Q, Q+M) \\ \\ 0, & x \in [Q+M, \infty) \end{cases}$$

Property 2.1 holds. Also,

$$\frac{\mathrm{d}^2\beta(x-Q)}{\mathrm{d}(x-Q)^2} = 0$$

shows that the second part of Theorem 2.3 holds. Now

$$\lim_{x-Q\to 0} \max\left\{1 - \frac{x-Q}{M}, 0\right\} = 1$$

shows that Property 2.2 holds, and

$$\lim_{x-Q\to\infty} \max\left\{1 - \frac{x-Q}{M}, 0\right\} = \max\{-\infty, 0\} = 0$$

shows that Property 2.3 holds. Q.E.D.

The next result indicates the existence of Q^* that minimizes $TC(Q)$ given by Eq. (2.1) when $\beta(x-Q)$ is defined as in Proposition 2.1.

Theorem 2.4 *Suppose $\beta(x-Q)$ is defined as in Proposition 2.1. Then $TC(Q)$ given by Eq. (2.1) is a convex function.*

Proof: If $x \in [Q, Q+M)$, then $\beta'(x-Q) = \frac{1}{M}$ and $\beta''(x-Q) = 0$. Thus the inequality $(x-Q)\beta''(x-Q) \le 2\beta'(x-Q)$ given by Eq. (2.2) reduces to $\frac{2}{M} \ge 0$. Since the last inequality always holds, it follows that $\beta(x-Q)$ satisfies the condition in Theorem 2.1. If $x \in [Q+M, \infty)$, then $\beta(x-Q) = 0$, which also satisfies the condition in Theorem 2.1.

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Therefore, Theorem 2.1 guarantees that $TC(Q)$ is a convex function for all $Q \geq 0$ whenever $\beta(x - Q)$ is defined as in Eq. (2.3).

This can be proven by showing the second derivation of $TC(Q)$ is positive. When $\beta(x - Q)$ is defined as the linear equation, $TC(Q)$ can be written as

$$
\begin{aligned}
TC(Q) & = c_O Q + c_H \int_0^Q (Q - x) f(x) \mathrm{d}x + c_{LS} \int_Q^\infty (x - Q) f(x) \mathrm{d}x \\
& + (c_B - c_{LS}) \int_Q^{Q+M} (x - Q) \left(1 - \frac{x - Q}{M} \right) f(x) \mathrm{d}x
\end{aligned}
\tag{2.4}
$$

Thus, the first order derivatives of $TC(Q)$ are calculated as follows

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} & = c_O + c_H \int_0^Q f(x) \mathrm{d}x - c_{LS} \int_Q^\infty f(x) \mathrm{d}x \\
& + (c_B - c_{LS}) \int_Q^{Q+M} \left[-1 + \frac{2(x - Q)}{M} \right] f(x) \mathrm{d}x
\end{aligned}
\tag{2.5}
$$

and second derivatives of $TC(Q)$ yields

$$
\begin{aligned}
\frac{\mathrm{d}^2 TC(Q)}{\mathrm{d}Q^2} & = (c_H + c_{LS}) f(Q) + (c_{LS} - c_B) f(Q) \\
& + \frac{2(c_{LS} - c_B)}{M} \int_Q^{Q+M} f(x) \mathrm{d}x
\end{aligned}
\tag{2.6}
$$

If Assumption 2.1 holds, it follows that $c_{LS} - c_B > 0$ so that all terms of $TC(Q)$ are definately non-negative and it means the convexity of $TC(Q)$ can be ascertained for a general demand distribution.Q.E.D.

2.3.2 Risk-Seeking Behavior

A backorder rate function $\beta(x - Q)$ that describes risk-seeking behavior with respect to waiting time should satisfy properties 2.1 $\sim$ 2.3 as well as the third part of Theorem 2.3. The following proposition defines a representative backorder rate function that satisfies these conditions.

Proposition 2.2 *Suppose $x - Q \geq 0$ and $a > 0$ is a constant. Then*

$$\beta(x - Q) = \begin{cases} e^{-a(x-Q)}, & x \in [Q, Q + M) \\ 0, & x \in [Q + M, \infty) \end{cases} \tag{2.7}$$

satisfies properties 2.1 $\sim$ 2.3 and the third part of Theorem 2.3.

Proof: The proof is similar to the proof of Proposition 2.1. Refer to Appendix A.3 for details.

Theorem 2.5 *Suppose $\beta(x - Q)$ is defined by Eq. (2.7) and $a < \frac{2}{M}$. Then $TC(Q)$ given by Eq. (2.1) is a convex function.*

Proof: Let $A = (0, \frac{2}{a})$ and $B = (M, \infty)$. It is straightforward to show that Eq. (2.7) reduces to $x - Q \in A$. Since $\beta(x - Q) = 0 \ \forall \ x - Q \in B$ and Eq. (2.7) holds, it follows from Theorem 2.1 that $TC(Q)$ is convex $\forall \ x - Q \in A \cup B$. In order to verify that $TC(Q)$ is convex $\forall \ Q \geq 0$, we need to show that convexity holds $\forall \ x - Q \in \mathbb{R}^+$, which actually reduces to showing that $A \cup B = \mathbb{R}^+$. Let $C = (\frac{2}{a}, M)$. Then $A \cup B \cup C = \mathbb{R}^+$. However, since the condition $a \leq \frac{2}{M}$ is given, we have that $C = \emptyset$ and $A \cup B \cup C = A \cup B = \mathbb{R}^+$.

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This can be proven by showing the second derivation of $TC(Q)$ is positive. When $\beta(x - Q)$ is defined as a exponential decreasing function, $TC(Q)$ can be written as

$$
\begin{aligned}
TC(Q) \;=\; & c_O Q + c_H \int_0^Q (Q - x) f(x) \mathrm{d}x + c_{LS} \int_Q^\infty (x - Q) f(x) \mathrm{d}x \\
& + (c_B - c_{LS}) \int_Q^{Q+M} (x - Q) \left(e^{-a(x-Q)} \right) f(x) \mathrm{d}x
\end{aligned}
\tag{2.8}
$$

Thus, the first order derivatives of $TC(Q)$ are calculated as follows

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} \;=\; & c_O + c_H \int_0^Q f(x) \mathrm{d}x - c_{LS} \int_Q^\infty f(x) \mathrm{d}x \\
& + (c_B - c_{LS}) \int_Q^{Q+M} \left[-e^{-a(x-Q)} + (x - Q) a e^{-a(x-Q)} \right] f(x) \mathrm{d}x
\end{aligned}
\tag{2.9}
$$

and second derivatives of $TC(Q)$ yields

$$
\begin{aligned}
\frac{\mathrm{d}^2 TC(Q)}{\mathrm{d}Q^2} \;=\; & (c_H + c_{LS}) f(Q) + (c_{LS} - c_B) f(Q) \\
& + (c_{LS} - c_B) \int_Q^{Q+M} \left[2a e^{-a(x-Q)} - (x - Q) a^2 e^{-a(x-Q)} \right] f(x) \mathrm{d}x
\end{aligned}
\tag{2.10}
$$

If Assumption 2.1 holds, it follows that $c_{LS} - c_B > 0$. We need one condition of $\left[2a e^{-a(x-Q)} - (x - Q) a^2 e^{-a(x-Q)} \right] > 0$. It can be reduced to $a(x - Q) < 2$ for $x \in [Q, Q+M)$. Therefore, all terms of $TC(Q)$ under the condition of $a < \frac{2}{M}$ are definately non-negative. It means the convexity of $TC(Q)$ can be ascertained for a general demand distribution. Q.E.D.

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Note that the parameter a controls the rate at which $\beta(x - Q)$ given by Eq. (2.7) decreases. In particular, $a_1 > a_2$ implies that customer 1 is more risk seeking (or more impatient) than customer 2. The condition $a < \frac{2}{M}$ suggests that Theorem 2.5 can only guarantee convexity if $\beta(x - Q)$ does not decrease too quickly as $x - Q$ approaches M.

2.3.3 Risk-Averse Behavior

A backorder rate function $\beta(x - Q)$ that describes risk-averse behavior with respect to waiting time should satisfy properties 2.1 $\sim$ 2.3 as well as the first part of Theorem 2.3. The following proposition defines a representative backorder rate function that satisfies these conditions.

Proposition 2.3 *Suppose $x - Q \geq 0$. Then*

$$
\beta(x - Q) \;=\; \begin{cases} \cos\left[\dfrac{(x - Q)\pi}{2M}\right], & x \in [Q, Q + M) \\[2mm] 0, & x \in [Q + M, \infty) \end{cases}
$$

$$(2.11)$$

satisfies properties 2.1 $\sim$ 2.3 and the first part of Theorem 2.3.

Proof: The proof is similar to the proof of Proposition 2.1. Refer to Appendix A.4 for details.

Theorem 2.6 *Suppose $\beta(x - Q)$ is defined by Eq. (2.11). Then $TC(Q)$ given by Eq. (2.1) is a convex function.*

Proof: If $x \in [Q, Q + M)$, then

$$
\begin{aligned}
\beta'(x - Q) &= \frac{\pi}{2M} \sin\left(\frac{x - Q}{2M}\pi\right) \\
\beta''(x - Q) &= -\frac{\pi^2}{4M^2} \cos\left(\frac{x - Q}{2M}\pi\right)
\end{aligned}
$$

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Thus the inequality given by Eq. (2.2) reduces to

$$-\frac{(x-Q)\pi}{4M}\cot\left(\frac{x-Q}{2M}\pi\right) \leq 1. \tag{2.12}$$

If we let $z = (\pi/2M)(x-Q)$, then Eq. (2.12) becomes $(-z/2)\cot(z) \leq 1$, where $z \in (0, \pi/2)$. Since $\cot(z) > 0$ and $-z/2 < 0$ for $z \in (0, \pi/2)$, we have $(-z/2)\cot(z) < 0 \leq 1$, which implies that the inequality given by Eq. (2.12) holds. Thus it follows that $\beta(x-Q)$ satisfies the condition in Theorem 2.1. If $x \in [Q+M, \infty)$, then $\beta(x-Q) = 0$, which also satisfies the condition in Theorem 2.1. Therefore, Theorem 2.1 guarantees that $TC(Q)$ is a convex function for all $Q \geq 0$ whenever $\beta(x-Q)$ is defined as in Eq. (2.11).

This can be proven by showing the second derivation of $TC(Q)$ is positive. When $\beta(x-Q)$ is defined as a smoothly decreasing function, $TC(Q)$ can be written as

$$
\begin{aligned}
TC(Q) &= c_O Q + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x + c_{LS}\int_Q^\infty (x-Q)f(x)\mathrm{d}x \\
&\quad + (c_B - c_{LS})\int_Q^{Q+M}(x-Q)\cos(\frac{x-Q}{2M}\pi)f(x)\mathrm{d}x
\end{aligned}
\tag{2.13}
$$

Thus, the first order derivatives of $TC(Q)$ are calculated as follows

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} &= c_O + c_H\int_0^Q f(x)\mathrm{d}x - c_{LS}\int_Q^\infty f(x)\mathrm{d}x + (c_B - c_{LS}) \\
&\quad \times \int_Q^{Q+M}\left[-\cos(\frac{x-Q}{2M}\pi) + (x-Q)\frac{\pi}{2M}\sin(\frac{x-Q}{2M}\pi)\right]f(x)\mathrm{d}x
\end{aligned}
\tag{2.14}
$$

and second derivatives of $TC(Q)$ yields

$$
\begin{aligned}
\frac{\mathrm{d}^2 TC(Q)}{\mathrm{d}Q^2} &= (c_H + c_{LS})f(Q) + (c_{LS} - c_B)f(Q) + (c_{LS} - c_B) \\
&\quad \times \int_Q^{Q+M}\left[\frac{\pi}{M}\sin(\frac{x-Q}{2M}\pi) + (x-Q)(\frac{\pi}{2M})^2\cos(\frac{x-Q}{2M}\pi)\right]f(x)\mathrm{d}x
\end{aligned}
$$

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$$(2.15)$$

If Assumption 2.1 holds, it follows that $c_{LS} - c_B > 0$. We need to verify $\frac{\pi}{M} \sin(\frac{x-Q}{2M}\pi) + (x-Q)(\frac{\pi}{2M})^2 \cos(\frac{x-Q}{2M}\pi) > 0$. It can be reduced to $-\frac{(x-Q)\pi}{4M} \cot\left(\frac{x-Q}{2M}\pi\right) < 1$ for $x \in [Q, Q+M)$. Therefore, all terms of $TC(Q)$ are definately non-negative and It means the convexity of $TC(Q)$ can be ascertained for a general demand distribution. Q.E.D.

2.4 Sensitivity Analysis and Management Insights

This section investigates the effects that various problem parameters have on optimal ordering decisions. The following example data was used for the analysis:

$$X \sim N(500, 500), c_O = 50, c_H = 20, c_B = 75, c_{LS} = 100, M = 1000 \qquad (2.16)$$

Based on the results shown in Figure 2.3 through 2.6, we observe the following:

Observation 2.1 *Let Q_A, Q_N, and Q_S be the optimal order quantity associated with the risk-averse, -neutral, and -seeking cases, respectively. Then $Q_S \geq Q_N \geq Q_A$.*

This is intuitive since according to Figure 2.2, the risk-averse customer is more patient than the risk-neutral customer, and the risk-neutral customer is more patient than the risk-seeking customer. Therefore, it is reasonable to expect the optimal order quantity to be an increasing function of customer impatience.

Observation 2.2 *Optimal order quantities are always non-decreasing in c_{LS} and c_B, and always non-increasing in c_H and M.*

These results are intuitive and consistent with the results reported in Lodree (2007).
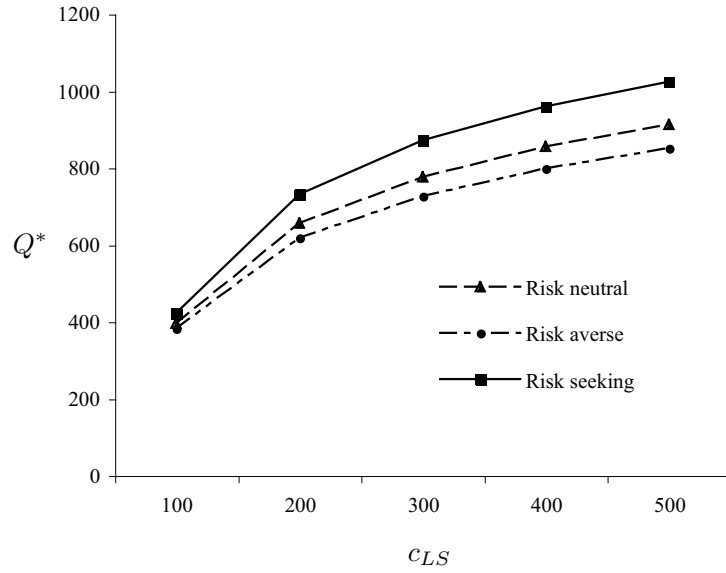
Figure 2.3: The effect of c_{LS} on the optimal order quantity.
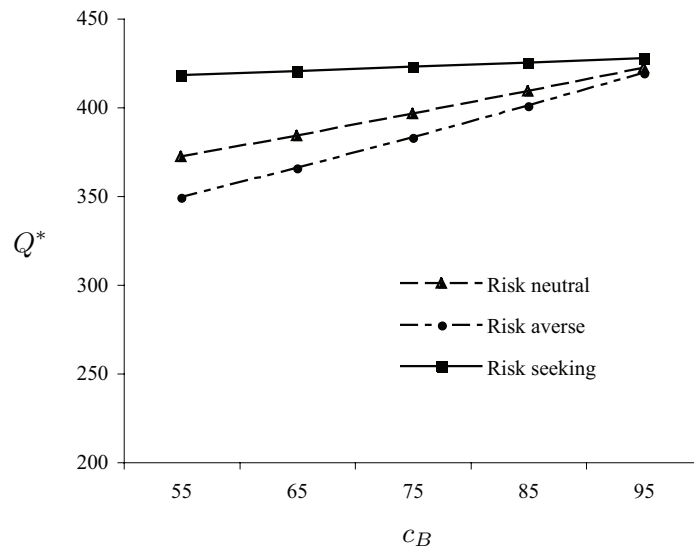
Figure 2.4: The effect of c_B on the optimal order quantity.
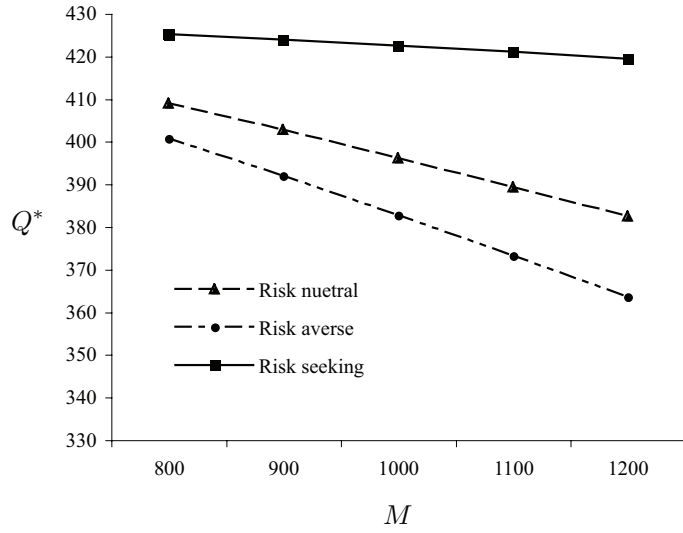
Figure 2.5: The effect of M on the optimal order quantity.
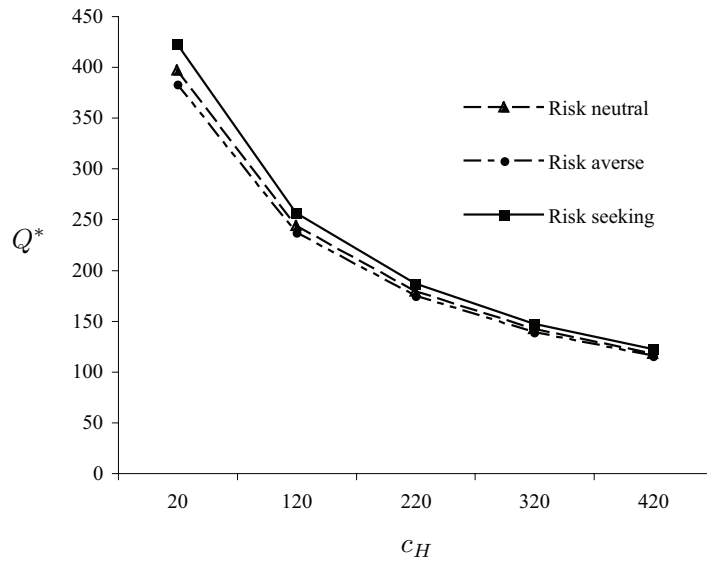
Figure 2.6: The effect of c_H on the optimal order quantity.

Observation 2.3 *According to Figure 2.3 and Figure 2.5, differences in optimal order quantities among the three customer types are increasing functions of c_{LS} and M.*

From a managerial perspective, this means that it is in the decision-maker's best interest to be astute with respect to customer risk profiles when (i) the lost sales cost is expensive (i.e., the product is expensive) and (ii) when the backorder / lost sales threshold is large. These effects can be explained mathematically by observing Eq. (A.1) from Appendix A. In particular, the effect of the backorder rate function $\beta(x - Q)$ on the total expected cost function $TC(Q)$ is magnified when either c_{LS} or M is increased. Thus it is reasonable to expect increasing differences in Q^* among the three cases as $\beta(x - Q)$ assumes a more dominant role in $TC(Q)$. Finally, note that of these two parameters, it can be observed from Figure 2.3 and 2.5 that the optimal decision is more sensitive to c_{LS}.

Observation 2.4 *According to Figure 2.4 and Figure 2.6, differences in optimal order quantities among the three customer types are decreasing functions of c_B and c_H.*

From a managerial perspective, this means that the decision-maker should keenly observe customer risk profiles when (i) the backorder cost is small relative to the lost sales cost and (ii) the holding cost is small relative to the ordering cost, backorder cost, and lost sales cost. These effects can also be explained mathematically by observing Eq. (A.1) from Appendix A. Since $c_B \leq c_{LS}$ based on Assumption 2.1, our analysis involves increasing c_B only up until it reaches c_{LS}. Thus as c_B approaches c_{LS}, the term involving $\beta(x - Q)$ in Eq. (A.1) approaches zero, and the effects of customer risk profiles become increasingly negligible (in fact, $TC(Q)$ becomes the newsvendor problem). As for c_H, it is increased beyond c_O, c_B, and c_{LS} for the purpose of our analysis, although this is not likely to occur in practice. However, the results suggest that differences in optimal order quantities among the three customer types become increasingly insignificant as holding costs become increasingly dominant in $TC(Q)$ (or equivalently, as $\beta(x - Q)$ becomes less dominant in the expected cost function).

Observation 2.5 *Differences in optimal order quantities between the risk-seeking and risk-neutral cases are always greater than the differences in optimal order quantities between the risk-averse and risk-neutral cases.*

This is necessarily a consequence of the specific backorder rate functions studied in this study for the risk-averse and risk-seeking cases. More specifically, the risk-seeking backorder rate function given by Eq. (2.7) is more different than the risk neutral case when compared to the risk-averse backorder rate function given by Eq.(2.11). To illustrate this point, consider the data in Eq. (2.16) and suppose $x - Q = 300$. Then using equations (2.3), (2.7), and (2.11), $\beta_A(300) - \beta_N(300) = 0.191$ and $\beta_N(300) - \beta_S(300) = 0.698$, where $\beta_N(x - Q)$, $\beta_S(x - Q)$, and $\beta_A(x - Q)$ are equations (2.3), (2.7), and (2.11), respectively. Since $\beta_N(300) - \beta_S(300) > \beta_A(300) - \beta_N(300)$, it is reasonable to expected that $Q_N - Q_S > Q_A - Q_N$.

2.5 Value of Risk Profile Information

Suppose a decision-maker can conduct a study to obtain more information about the risk profile of a customer or market segment. This section explores the expected value of conducting such a study. To carry out the analysis, let us first assume that the decision-maker currently orders Q_N, which is the optimal order quantity associated with the risk-neutral case. If the customer is actually risk-averse, then the expected value of a market study is

$$V = TC_A(Q_N) - TC_A(Q_A),$$

where $TC_A(\cdot)$ and Q_A are the expected cost function and optimal order quantity, respectively, for the risk-averse case. In general, let V_{ij} equal the value of the market study if the decision-maker orders Q_i and the customer risk profile is actually j, where $i, j \in \{A, N, S\}$ (Averse, Neutral, Seeking). Also, let $TC_i(\cdot)$ and Q_i represent the expected total cost function and optimal order quantity, respectively, for case $i \in \{A, N, S\}$. If $i = j$, then clearly

$V_{ij} = 0$ for all $i \in \{A, N, S\}$. Otherwise if $i \neq j$, then

$$
\begin{aligned}
V_{NA} &= TC_A(Q_N) - TC_A(Q_A) \\
V_{NS} &= TC_S(Q_N) - TC_S(Q_S) \\
V_{AN} &= TC_N(Q_A) - TC_N(Q_N) \\
V_{AS} &= TC_S(Q_A) - TC_S(Q_S) \\
V_{SA} &= TC_A(Q_S) - TC_A(Q_A) \\
V_{SN} &= TC_N(Q_S) - TC_N(Q_N).
\end{aligned}
$$

The example data from Eq. (2.16) was used to construct the graphs shown in Figure 2.7 through 2.15. The following insights can be interpreted from these figures:

Observation 2.6 *The expected value of risk profile information (EVRPI) is an increasing function of c_{LS} and M, and a decreasing function of c_B and c_H.*

This observation is intuitive since differences in optimal order quantities among the three customer types are increasing in c_{LS} and M, and decreasing in c_B and c_H (see Section 3.4.2).

Observation 2.7 *EVRPI is most valuable when c_{LS} is large and when the decision-maker assumes a risk-seeking profile, but the customer is actually risk-averse (see Figure 2.7: EVRPI is nearly 4,000). On the other hand, the risk profile information is least valuable whenever the value of c_B is very close to the value of c_{LS} (see Figure 2.7 through 2.12).*

More generally, the larger values of EVRPI occur when either the actual risk profile is risk-seeking and the decision-maker assumes risk-averse, or vice-versa.

Observation 2.8 *Based on the Observation 2.7, the decision that minimizes the worst possible outcome (a min-max criterion) assumes that the risk profile is risk-neutral (see Figure 2.7, 2.9, 2.10, 2.12, 2.13, and 2.15).*

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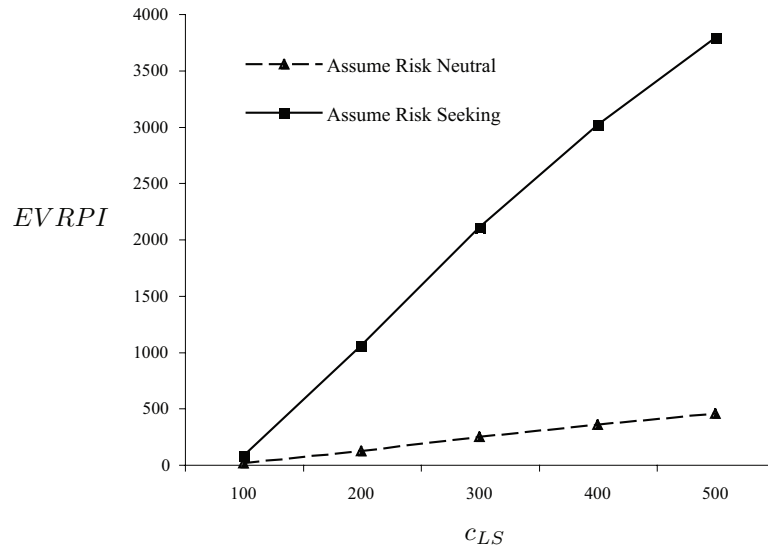
Figure 2.7: The effect of c_{LS} on EVRPI if customer is risk-averse.
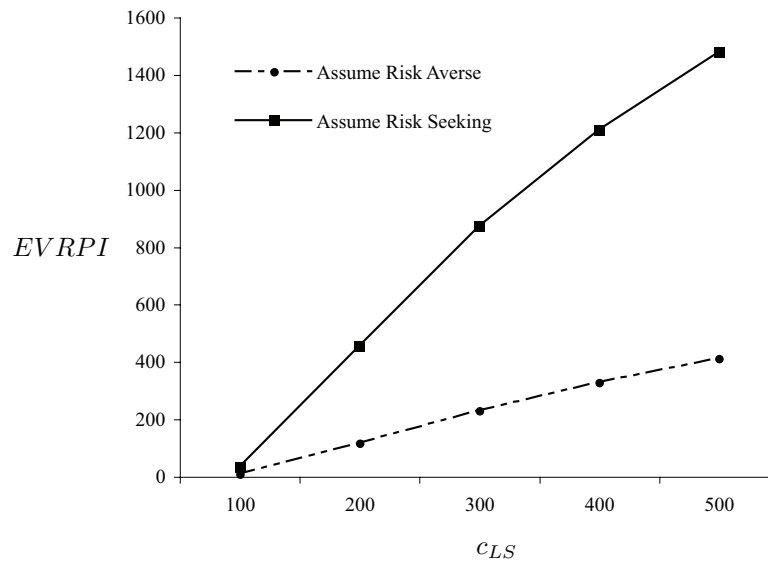
Figure 2.8: The effect of c_{LS} on EVRPI if customer is risk-neutral.

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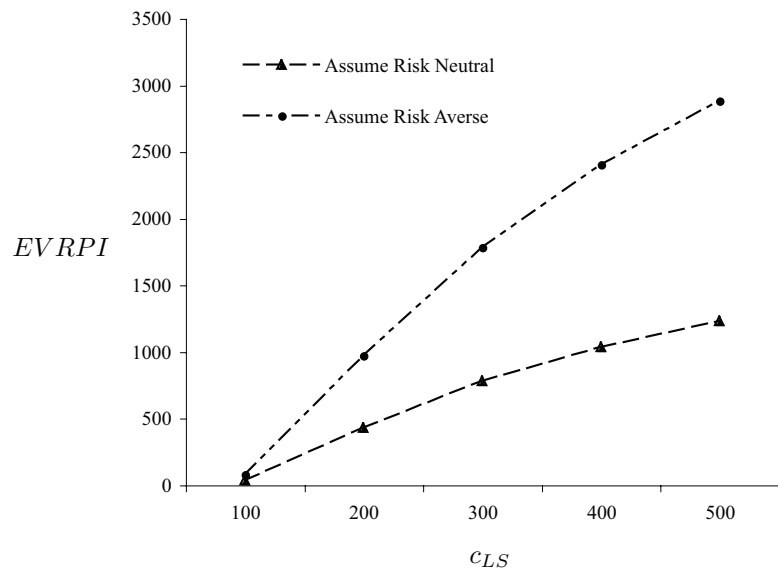
Figure 2.9: The effect of c_{LS} on EVRPI if customer is risk-seeking.
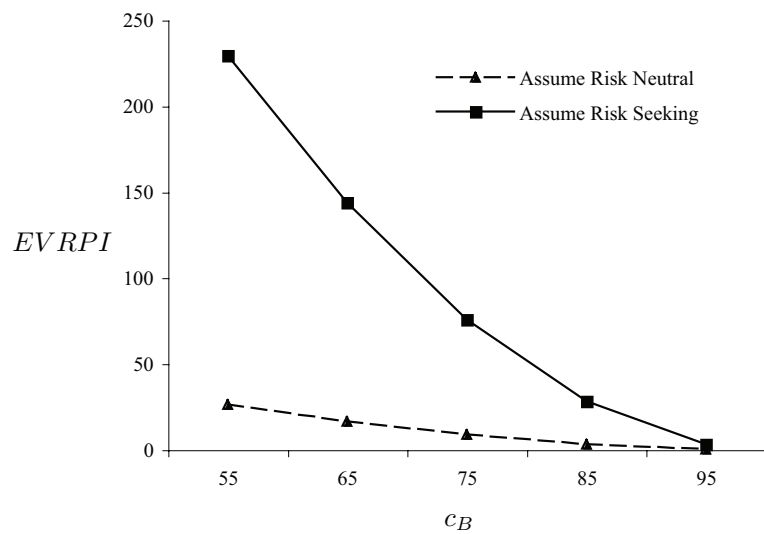
Figure 2.10: The effect of c_B on EVRPI if customer is risk-averse.
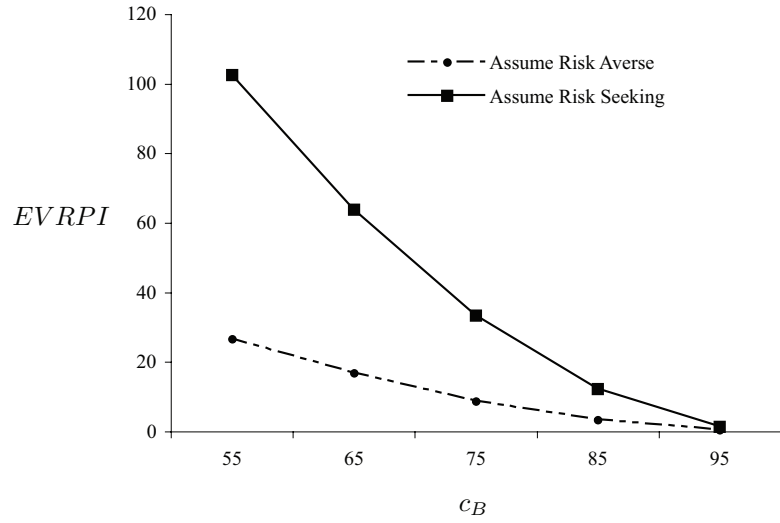
Figure 2.11: The effect of c_B on EVRPI if customer is risk-neutral.
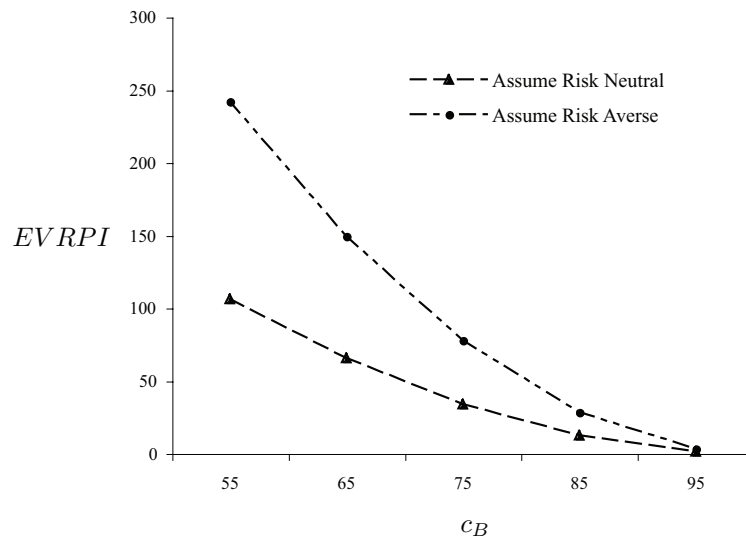
Figure 2.12: The effect of c_B on EVRPI if customer is risk-seeking.

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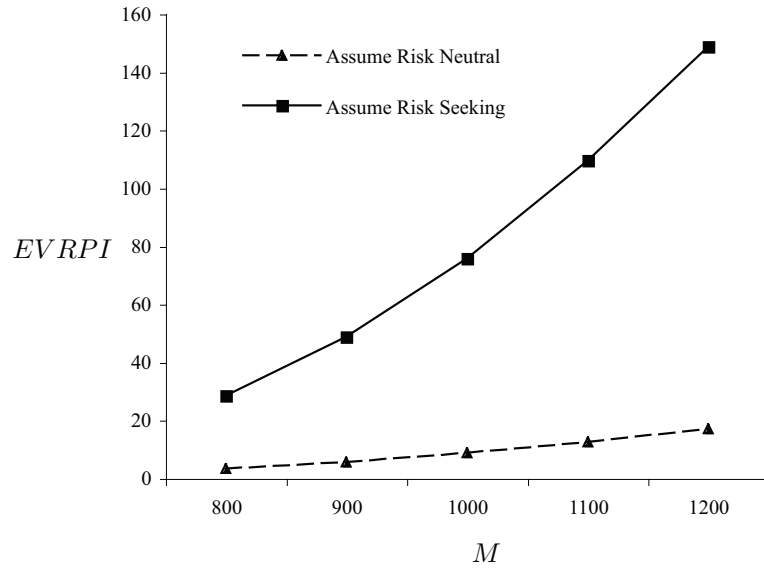
Figure 2.13: The effect of M on EVRPI if customer is risk-averse.
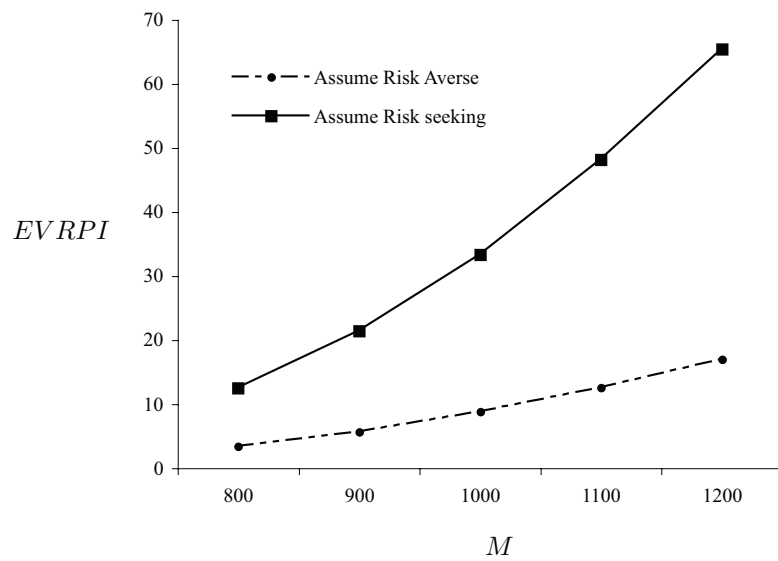
Figure 2.14: The effect of M on EVRPI if customer is risk-neutral.
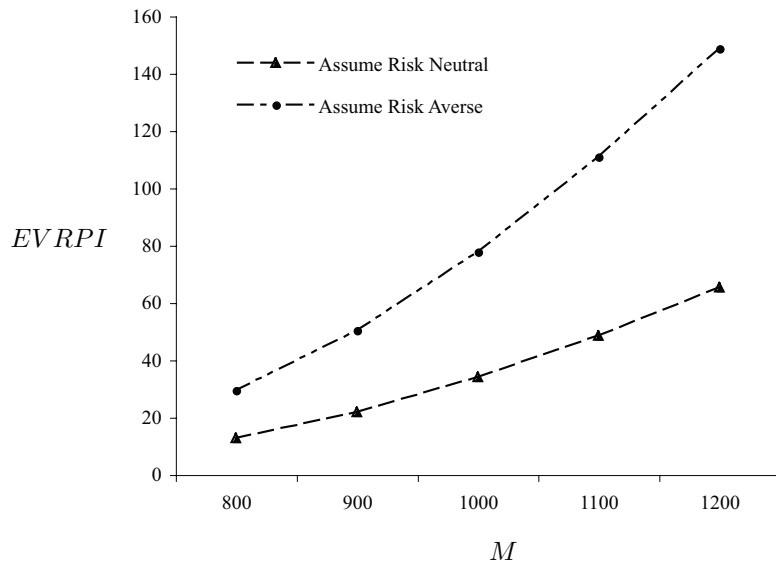
Figure 2.15: The effect of M on EVRPI if customer is risk-seeking.

On the other hand, if the customer's risk profile is not likely to be risk-seeking, then it is better to assume that the customer is risk-averse. To see this, consider that if we ignore the risk seeking case (Figure 2.9), then the penalty for assuming risk-neutral if the customer is actually risk-averse is slightly greater than the penalty for assuming risk-averse and the customer is actually risk-neutral (see Figure 2.7 and 2.8). Similar arguments can be used to determine optimal decisions if the customer is not likely to be risk-averse or -neutral.

Observation 2.9 *As previously mentioned, the largest values of EVRPI occur when c_{LS} is large. However, the rate of change in EVRPI is more sensitive to c_B (compare Figure 2.7 through 2.9 to Figure 2.10 through 2.12).*

2.6   Conclusion

Shortages are often represented as either backorders or lost sales in conventional inventory models and in practice. The time-dependent backlogging approach to characterizing

inventory shortages acknowledges that there is some probability associated with whether or not a customer will backorder a shortage, and that this probability (i.e., the backorder rate) is related to the lead-time associated with replenishing the outstanding backorder. While the majority of the research literature considers time-dependent backlogging within the context of continuous review models with deterministic demand, this research studies time-dependent partial backlogging in the single period inventory problem with stochastic demand.

The backorder rate function is used to classify customers as either risk-averse, -neutral, or -seeking with respect to their willingness to wait for the supplier to replenish shortages. Representative backorder rate functions are presented, and the existence of a unique order quantity that minimizes total expected costs is shown for each case. Sensitivity analysis experiments are conducted to examine the similarities and differences in optimal order quantities among the three customer types as a function of various problem parameters.

Additionally, the expected value of risk profile information (EVRPI) is defined in order to assess the expected benefit of knowing the risk behavior of a customer or market segment. Our results suggest that EVRPI is most significant when the difference between the unit lost sales cost c_{LS} and the unit backorder cost c_B is large. Our results also indicate that EVRPI increases as the difference $c_{LS} - c_B$ increases and as the lost sales threshold M increases.

This study can be extended in several ways. A useful extension would be to relax Assumption 1 such that $c_B > c_{LS}$ is a possibility. This cost structure is known to happen in practice based on one of the author's interactions with the production manager of a major manufacturing firm. In particular, this production manager's primary objective in the event of a shortage is to maintain a positive rapport with the client (especially a major client). He is much less concerned with the short term cost inefficiencies associated with emergency procurement, production, and delivery in light of the long term implications related to compromising a business relationshp. Therefore, this manager would replenish backorders at

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unit cost c_B, even in the event that c_B far exceeds c_{LS}. This manufacturing firm's strategic approach to managing shortages suggests another promising extension of this study, namely an extension that entails both short term and long term management of inventory shortages in a multi-period setting. Finally, consider that we have verified the existence of a unique optimal order quantity for each of a specific set of backorder rate functions, and also that some of the results presented in sections 2.4 and 2.5 are specific to these functions and their interrelationships. Thus the development of other backorder rate functions, analysis of their mathematical properties, and the validity of our management insight results based on these functions and their interrelationships also warrants exploration.

Chapter 3

Optimizing Backorder Lead-Time and Order Quantity in a Supply Chain

with Partial Backlogging and Stochastic Demand

ABSTRACT

We examine a backorder lead-time (i.e., response time) optimization in a two-stage system with time-dependent partial backlogging and stochastic demand. From the partial backlogging point of view, the objective of representative mathematical models is to minimize the expected total cost related to ordering, inventory holding, and shortages. Backordering cost is often regarded as a constant in literature. However, the backorder cost is characterized as a function of response time in a real situation. Moreover, the backorder rate (i.e., the probability that a customer elects to backorder shortages) is a decreasing function of supplier response time. We investigate a supply chain system with a time-dependent backorder cost and rate in an uncertain demand setting. A representative expected cost function and closed form solution are derived for specific demand distributions. We also illustrate our results (e.g., cost savings) with an example.

3.1 Introduction

A firm achieving strategic fit shows the right balance between responsiveness and efficiency. Therefore, their supply chain should be structured to provide responsiveness to customers while improving the overall efficiency (Chopra et al. 2007). Optimizing the response time is one of the critical factors when companies are competing with each other under a highly uncertain demand situation. The company works to optimize the response time not only to reduce the expected cost as a faster response requires a higher cost, but also to increase responsiveness to customers. Many companies often get opportunities to

improve customer satisfaction and reduce the expected total cost by optimizing the response time, which is evident in changes such as lean manufacturing as a philosophy of production that emphasizes minimizing the amount of all resources (including time) used in various enterprise activities.

To address this situation, we developed and solved newsvendor models involving stochastic demand, time-dependent backorder rate and cost. A tactical planning approach is taken by viewing a two-stage supply chain system consisting of one supplier and one retailer with time-dependent partial backlogging and backorder cost. The objective of the supplier is to optimize order quantity and response time to replenish backorders in order to minimize its expected total cost. The decision variables considered are the order quantity Q and the response time (backorder lead-time) t. It turns out that determining the response time is equivalent to deciding how much financial capital should be invested in the future backorders. Simply stated, our problem is to decide how many products should be ordered before the selling season and how much financial capital should be invested into backorders during mid-season so that both backorder cost and lost sales cost are minimized.

More specifically, we consider a single period model consisting of various backorder processing time modes. A supplier determines the number of items Q to order such that Q items are available at the beginning of the season. For the time being, assume that the optimal value Q^* is determined based on the information about the demand distribution X, where X is a random variable with the probability density function $f(x)$ and the cumulative distribution function $F(x)$. If Q is larger than the realized demand, then the supplier incurs a unit inventory holding cost of c_H for the season. If Q is less than the realized demand, both backorder cost $c_B(t)$ and/or lost sales cost c_{LS} are incurred. Backorder cost $c_B(t)$ is one of the convex and decreasing functions for $t > 0$. The cost incurred during the backorder process is based on the supplier's selected backorder lead-time t. This assumption regarding the form of $c_B(t)$ implies that the response time (i.e., backorder lead-time) t can be reduced by increasing the backorder processing cost.

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Applications of this problem can be found in the auto industry, seasonal fashion products, and school supplies. In each of these examples, a number of products are available at the beginning of the season where the inventory and other costs are incurred for ordering too much, and backorders are used to satisfy the unfulfilled demand, and some amount of unfulfilled demand becomes lost sales. Our objective as a supplier is to minimize the sum of inventory holding costs, ordering costs, backorder, and lost sales costs.

3.1.1 The Newsvendor Problem with Time-dependent Partial Backlogging

Recall the newsvendor problem with time-dependent partial backlogging of Lodree (2007). In this newsvendor problem, β represents the backorder rate, which can be interpreted as the fraction of shortages backlogged or the probability that a given customer will choose to backlog shortages. Since β is time-dependent, $\beta : L \mapsto [0, 1]$, where $L \in [0, \infty)$ is the backorder lead-time. L is directly proportional to the magnitude of an observed shortage. In other words, L is a linear function in $x - Q$ for $x \geq Q$, where x is the observed demand, Q is the inventory level before demand realization, and $\max\{x - Q, 0\}$ is the observed number of shortages. Without loss of generality, $L = x - Q$ so that the terms backorder lead-time and "magnitude of shortage" can be used interchangeably. Therefore, the backorder rate is expressed as a function $\beta(x - Q)$. Then X is a continuous random variable that represents demand. The newsvendor problem with a time-dependent partial backlogging can be expressed as follows:

$$
\begin{aligned}
TC(Q) \;=\; & \text{Order Cost} + \text{Expected Holding Cost} + \text{Expected Backorder Cost} \\
& + \text{Expected Lost Sales Cost} \\
=\; & c_O Q + c_H \int_0^Q (Q - x) f(x)\mathrm{d}x + c_B \int_Q^\infty (x - Q)\beta(x - Q)f(x)\mathrm{d}x \\
& + \; c_{LS} \int_Q^\infty (x - Q)\left[1 - \beta(x - Q)\right] f(x)\mathrm{d}x \qquad (3.1)
\end{aligned}
$$

$\beta(x - Q) \in [0, 1]$ is a continuous and decreasing function in the number of observed shortages, which suggests that the customers are more likely to wait for backorder replenishment if the backorder lead-time is short, and are less likely to wait if the lead-time is too long. It determines the inventory level Q^* that minimizes $TC(Q)$, provided an optimum exists. It turns out that the convexity is guaranteed under a sufficient (but not necessary) condition for the existence of a unique minimizer.

3.2 Literature Review

Our problem is a choice of approaches to integrate two variations of stochastic inventory models that have been studied in literature. These variations can be classified as (1) stochastic inventory models with variable lead-time, and (2) stochastic inventory models with time-dependent partial backlogging. We now briefly review research in each of the two aforementioned types of the stochastic inventory models. Table 3.1 summarizes some literature related to stochastic inventory models.

First, Liao and Shyu (1991) present a probabilistic model in which lead-time is a unique decision variable and the order quantity is predetermined to minimize the sum of the expected holding cost and the additional cost. Moreover, in their model, the demand follows a normal distribution and the lead-time consists of n components each having a different cost for reduced lead-time. In many practical situations, lead-time is controllable; that is, the lead-time can be shortened, at the expense of extra costs, so as to improve customer service, reduce inventory investment in the safety stocks, and improve system responsiveness. Ben-Daya and Raouf (1994) extend the Liao and Shyu (1991) model by allowing both the lead-time and the order quantity as decision variables where the shortages are ignored. Ouyang et al. (1996) assume that the shortages and stockout cost are allowed. In addition, a mixture of backorders and lost sales is considered to generalize Ben-Daya and Raouf's (1994) model, where the backorder rate is fixed.

Table 3.1: A summary of the related literature.

Stochastic Inventory Model	Literature
Variable Lead-time	Liao and Shyu (1991), Ben-Daya and Raouf (1994), Ouyang et al. (1996), Wu (2001), Lodree and Jang (2004)
Time-dependent Partial Backlogging	Abad (1996), Chang and Dye (1999), Papachristos and Skouri (2000), Abad (2001), Wang (2002), Papachristos and Skouri (2003), Lodree (2007)

Wu (2001) extends the model of Ouyang et al. (1996) by considering the mixtures of normal distribution and allowing shortages. Moreover, the total amount of stock-out is considered as a mixture of backorders and lost sales during the stock-out period. In a practical situation, because the demands of the different customers do not have identical lead-times, then they use the mixtures of normal distribution to describe the lead-time.

Lodree and Jang (2004) consider the customer response time minimization in a two-stage system facing stochastic demand. They investigate a supply chain system in an uncertain demand setting that encompasses the customer waiting costs as well as the traditional plant costs (i.e., production and inventory costs). In this study, they assume all shortages are backlogged and replenished within a single period.

Second, some literature tackles the partial backlogging situation in an inventory decision problem. In many real-life situations, customers are likely to fulfill the shortages from a supply competitor who has inventory on hand if the backorder lead-time is extensive. On the other hand, the supplier is more likely to retain the customer's business and possibly avoid the long-term consequences of the shortages if the backorder lead-time is reasonably short. Especially for fashion commodities and seasonal products, the willingness of a customer to wait for backlogging during a shortage period declines with the length of the waiting time. When shortages are partially backlogged, a fraction of shortages incurs lost sales penalties

Table 3.2: Differences between Lodree (2007) & Proposed Model.

	Lodree (2007)	Proposed Model
Backorder cost	c_B : Constant	$c_B(t)$: Fuction of t
Backorder rate	$\beta(x - Q) = 1 - \frac{x-Q}{M}$ Linear fuction of $x - Q$	$\beta(t) = e^{-at}$ Exponential function of t
Decision variable	Q	Q, t
Total cost fuction	$TC[Q]$	$TC[Q, t]$

while the remaining shortages are backlogged. Therefore, time-dependent partial backlogging implies that the backorder rate (i.e., the fraction of shortages backlogged) depends on the time associated with replenishing the outstanding backorder.

Abad (1996) investigates optimal ordering and pricing policies for a perishable product with a deterministic and time-varying demand. The model is examined through an example with constant demand rate, exponential backorder rate, and exponential stock decay. Chang and Dye (1999) develop an inventory model in which the proportion of customers willing to accept backlogging is the reciprocal of a linear function of the waiting time. They examine the effect of the backlogging rate on the economic order quantity decision. Papachristos and Skouri (2000) establish a continuous review inventory model over a finite-planning horizon with deterministic varying demand and constant deterioration rate. The model allows for shortages, which are partially backlogged at a rate which varies exponentially with time.

Later, several related articles are presented, dealing with such inventory problems. For example, Abad (2001), Papachristos and Skouri (2003), and Wang (2002). Recently Lodree (2007) investigates a supply chain system in which a supplier prepares for the selling season by building stock levels prior to the beginning of the season and the shortages realized at the beginning of the season are represented as mixtures of the backorders and the lost sales.

In this research, we consider the concept of variable lead-time and time-dependent partial backlogging simultaneously in a stochastic inventory model unlike literature. The backorder cost is regarded as a constant in Lodree (2007) recently. However, the backorder

cost is characterized as a function of backorder lead-time in many real situations. Table 3.2 summarizes differences between Lodree (2007) and our proposed model. Our proposed model extends the basic newsvendor problem to consider the feature of time-dependent partial backlogging when a backorder occurs. The supplier fulfils the remaining customer demand at an optimal backorder cost and backorder rate, even though some amount of remaining customer demand becomes lost sales as the response time gets delayed further.

3.3   Model Formulation

We consider backorder lead-time as a decision variable, and we incorporate the possibility of an emergency replenishment during the selling season in the name of backorder after demand is realized. We apply a time-dependent partial backlogging approach to optimize the backorder lead-time and the order quantity to minimize the expected total cost.

Assumption 3.1 *All items during the expedited back ordering process are shipped together in one shipment.*

Assumption 3.2 *All variables, parameters, and costs (except for $Q \geq 0$ and $f(x) \geq 0$) are strictly greater than zero.*

Assumption 3.3 $c_O < c_B(t^*) < c_{LS}$

Assumption 3.4 *The plant produces a single product type.*

The goal of our model is to determine the number of items Q to order during the off-season and the backorder lead-time t to be used to replenish the backorders during the season to minimize the expected total costs. The notation of Table 3.3 will be used throughout this research. We assume backorder cost is the unit backorder processing cost function

Table 3.3: List of notations.

c_O:	Unit ordering cost
c_H:	Unit cost of the holding excess inventory
$c_B(t)$:	Unit back order cost
	where $c_B(t)$ is strictly convex and decreasing for $t > 0$
c_{LS}:	Unit cost of shortages that are lost sales
$\beta(t)$:	Fraction of shortages that are backlogged
	where $\beta(t)$ is strictly convex and decreasing for $t > 0$
$K(y)$:	Setup cost after demand realization
	where $K(y)$ is increasing for $y > 0$
t:	the unit back order processing time (the decision variable)
Q:	Order quantity (the decision variable)
Ne^{-bt}:	Special case of $c_B(t)$ for $b > 0$
	where N is a maximum value of the back order processing cost
e^{-at}:	Special case of $\beta(t)$ for $a > 0$
αy:	Special case of $K(y)$
X:	A continuous random variable of demand
$f(x)$:	Probability density function (pdf) of X
$F(x)$:	Cumulative distribution function (cdf) of X
$TC(\cdot)$:	Total expected cost function

of the backorder lead-time t. Therefore, the backorder cost can be represented as a function of $c_B(t)$. Thus the expected cost function is given by

$$
\begin{aligned}
TC(Q,t) \;=\; & \text{Order(production) Cost} + \text{Expected Holding Cost} \\
& +\; \text{Expected Back order Cost} + \text{Expected Lost Sales Cost} \\
=\; & c_O Q + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x + c_B(t)\int_Q^\infty (x-Q)\beta(t)f(x)\mathrm{d}x \\
& +\; c_{LS}\int_Q^\infty (x-Q)\left[1-\beta(t)\right]f(x)\mathrm{d}x \\
=\; & c_O Q + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x + c_{LS}\int_Q^\infty (x-Q)f(x)\mathrm{d}x \\
& +\; (c_B(t)-c_{LS})\int_Q^\infty (x-Q)\beta(t)f(x)\mathrm{d}x.
\end{aligned}
\tag{3.2}
$$

Our goal is to derive the optimal solution (Q^*, t^*) that minimizes Eq.(3.2). Although the expected cost function Eq.(3.2) is not convex for all feasible values of Q and t, its marginal properties allow us to find the globally optimal solution. To derive the solution, consider the following derivatives of $TC(Q, t)$. If the total cost function is a convex function, then the unique minimizer Q^* can be obtained by equating the first derivative to zero. First of all, we verify whether the total cost function is a convex function or not. The convexity of the total cost function is ensured since the second derivation is always positive as follows:

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}Q} &= c_O + c_H \int_0^Q f(x)\mathrm{d}x - c_{LS} \int_Q^\infty f(x)\mathrm{d}x + (c_{LS} - c_B(t)) \int_Q^\infty \beta(t)f(x)\mathrm{d}x \\
\frac{\mathrm{d}^2 TC(Q,t)}{\mathrm{d}Q^2} &= \{c_H + \beta(t)c_B(t) + (1 - \beta(t))c_{LS}\}f(Q)
\end{aligned}
\tag{3.3}
$$

We have a convex objective function and we can get a feasible point, a unique minimizer of order quantity, at which the first-order conditions hold that first derivative equal to zero.

$$
Q^* = F^{-1}\left[\frac{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - c_O}{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} + c_H}\right]
\tag{3.4}
$$

The first and second order derivatives of $TC(Q, t)$ are then

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}t} &= \{c_B'(t)\beta(t) + c_B(t)\beta'(t) - c_{LS}\beta'(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x \\
\frac{\mathrm{d}^2 TC(Q,t)}{\mathrm{d}t^2} &= \{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) + c_B(t)\beta''(t) - c_{LS}\beta''(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x \\
&= \{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) + (c_B(t) - c_{LS})\beta''(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x
\end{aligned}
\tag{3.5}
$$

Consider the second part of Eq.(3.5). Clearly the second part is non-negative if the condition of Eq.(3.6) is satisfied. The detailed derivation is in Eq.(B.3). The sufficient condition is simplified as follows:

$$
\begin{aligned}
c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) \quad &+ \quad (c_B(t) - c_{LS})\beta''(t) \geq 0 \\
c_{LS} - c_B(t) \quad &\leq \quad \frac{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t)}{\beta''(t)} \\
c_{LS} - c_B(t) \quad &\leq \quad \frac{2ab + b^2}{a^2} N e^{-bt} \\
c_{LS} \quad &\leq \quad \left(\frac{a+b}{a}\right)^2 c_B(t)
\end{aligned}
$$

$$(3.6)$$

Thus, $TC(Q,t)$ is convex in t for any fixed Q where the condition of Eq.(3.6) is satisfied, and the optimal value t^* corresponding to a fixed value Q is obtained by setting the first part of Eq.(3.5) equal to zero and solving for t, which yields Eq.(3.7) The value t^* is unique because Eq.(3.5) shows $TC(Q,t)$ is strictly convex in t for any fixed Q.

$$
\begin{aligned}
c_B'(t)\beta(t) \quad &= \quad -c_B(t)\beta'(t) + c_{LS}\beta'(t) \\
\beta(t) \quad &= \quad \frac{\{c_{LS} - c_B(t)\}\beta'(t)}{c_B'(t)} \\
t^* \quad &= \quad ln\left(\frac{(a+b)N}{a \cdot c_{LS}}\right) b^{-1}
\end{aligned}
$$

$$(3.7)$$

We need to determine the optimal regular season production quantity Q^*, which can be accomplished by examining Eq.(3.2) as a function of Q only. Because the second part of Eq.(3.3) is non-negative, $TC(Q,t)$ is convex in Q so that the optimal value Q^* (for a fixed value t) satisfies the resulting equation when Eq.(3.3) is set to zero. Since t^* given by Eq.(3.7) is independent of Q, Q^* is also optimal for $t = t^*$. Therefore it follows that

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the global optimal solution (Q^*, t^*) satisfies the first order conditions. Now equating the first part of Eq.(3.3) to zero and solving, we attain the optimal value Q^* as a closed form. By observing Eq.(3.7), we see that the optimal backorder processing policy is such that optimal backorder lead-time increases as N, a maximum value of back order processing cost, increases and decreases as c_{LS} increases. According to Eq.(3.7), we define the relationship between $c_B(t^*)$ and c_{LS}. The details are in Eq.(B.5)

$$
\begin{aligned}
c_B(t^*) &= N e^{-b \cdot b^{-1} ln\left(\frac{(a+b)N}{a \cdot c_{LS}}\right)} \\
&= N e^{ln\left(\frac{a \cdot c_{LS}}{(a+b)N}\right)} \\
&= N \frac{a \cdot c_{LS}}{(a+b)N} \\
&= \frac{a \cdot c_{LS}}{(a+b)}
\end{aligned}
\tag{3.8}
$$

3.3.1 The Model with Backorder Setup Cost

The backorder setup cost is incurred in setting up a work center, assembly line, or shipment to switch from a regular order process to a backorder process. We add the backorder setup cost to the proposed expected total cost function. Thus the expected cost function is given as follows:

$$
\begin{aligned}
TC(Q, t) &= \text{Order(production) Cost} + \text{Expected Back order Cost} \\
&\quad + \text{Expected Lost Sales} + \text{Expected Holding Cost} \\
&= c_O Q + \int_Q^\infty \left[K((x-Q)\beta(t)) + c_B(t)(x-Q)\beta(t)\right] f(x) \mathrm{d}x \\
&\quad + c_{LS} \int_Q^\infty (x-Q)\left[1 - \beta(t)\right] f(x)\mathrm{d}x + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x \tag{3.9}
\end{aligned}
$$

We use a similar process to attain optimal decisions for this model as in the former model considering no backorder setup cost. Finally, we attain the optimal value t^* and Q^* as a closed form and define the relationship between $c_B(t^*)$ and c_{LS} in Appendix B.2.

3.3.2 Managerial Insights regarding the Response Time

By observing Eq.(3.7), we see that the optimal response policy is such that the response time decreases as the customer impatience factor(a), the backorder cost parameter(b), and the lost sales cost(c_{LS}) increase and response time increases as the maximum value of the backorder cost(N) increases. Eq.(3.8) implies that the optimal backorder cost is related to a proportion $\frac{a}{(a+b)}$ of the lost sales cost. It is the same as our intuition that the suppliers want to reduce the amount of lost sales in case of increasing the customer impatience factor(a), increasing the backorder cost parameter(b), and/or increasing the lost sales cost. Additionally, by observing Eq.(B.9), we see that the response time increases as the backorder setup cost parameter(α) increases. It is also reasonable that increasing the backorder setup cost parameter(α) causes a decrease in the backorder rate and it causes the optimal response time to increase consequently.

In subsequent sections, we determine the optimal order quantity for various demand distributions with numerical examples and illustrate how these solutions can be interpreted.

3.4 An Illusrative Example

An example problem will be constructed in this section using the set of data. The example will be given for each of the following demand distributions: exponential and normal. First, the optimal backorder lead-time will be determined using Eq.(3.7)and Eq.(B.9). Recall that the optimal backorder lead-time (t^*) is independent of the demand distribution and the order quantity Q. Thus for the example problems, the optimal lead-time (t^*) is computed once while the optimal order quantity(Q^*) is determined for each different demand distribution. The following data will be used for the example problems.

$$E(X) = 2000, \sigma_X = 2000, c_O = \$10, c_H = \$75, c_{LS} = \$200, a = 0.005, b = 0.02, N = \$1000$$

In this section, the results of the numerical examples are studied in further detail so as to validate the mathematical model. We examine our results and then compare this solution to the result of the prior newsvendor model. We now illustrate that the prior newsvendor models solution produces higher expected cost than our solutions when backorder lead-time is optimized. The illustration includes the cases of the normal and the exponential distribution of demand.

Based on the results shown in Table 3.4 and 3.5, we infer the following insights. With regular ordering cost to the backorder processing, the prior newsvendor model's optimal order quantities associated with normal distribution and exponential distribution case are 2,530 for normal and 1,853 for the exponential case. Detailed comparisons of the prior newsvendor and the proposed solution are shown in Table 3.4 and 3.5. Tables show the differences of the optimal order quantities and the total costs between both models. Thus a negative value indicates a reduction of optimal order quantity or total cost. A study of Table 3.4 and 3.5 reveals the following insights:

3.4.1 Insights from Table 3.4 and 3.5

The data used have satisfied the sufficient condition of Eq.(3.6) and the optimal backorder lead-time is determined using Eq.(3.7). The optimal backorder lead-time is 160.94 hours (6.7days). The backorder lead-time determines a backorder rate of 0.447 and a backorder cost of $40. When we assume the demand is normally distributed, the optimal fill rate is 58.2% and the optimal order quantity is 2,415 and the total cost is $137,634. The difference between the prior and the proposed model shows 115 items for the order quantity and $5,324 for the total cost. In the exponentially distributed demand case, the optimal fill rate is the same and the optimal order quantity is 1,743 and the total cost is $168,366. The

difference between the prior and the proposed model shows 110 items for order quantity and \$9,325 for total cost.

Table 3.4: Comparison of Optimal Order quantities.

Dist.	Model	Optimal Q*	Difference
Norm.	Prior	2,530	-115
	Proposed	2,415	
Expo.	Prior	1,853	-110
	Proposed	1,743	

Table 3.5: Comparison of Total costs.

Dist.	Model	Cost				Total cost	Difference
		Order	Holding	Backorder	Lostsales		
Norm.	Prior	\$25,300	\$39,203	\$1,773	\$76,682	\$142,958	-\$5,324
	Proposed	\$24,150	\$35,455	\$10,867	\$67,162	\$137,634	
Expo.	Prior	\$18,525	\$48,344	\$2,504	\$108,318	\$177,691	-\$9,325
	Proposed	\$17,432	\$43,485	\$14,964	\$92,485	\$168,366	

When we do not optimize the backorder lead-time, the prior newsvendor solution results in significant expected loss (over \$5,300 per season for normal demand and over \$9,300 per season for exponential demand).

3.4.2   Sensitivity Analysis and Management Insights

This section investigates the effects that various problem parameters have on the optimal order quantity or the backorder lead-time decisions. Based on the results shown in Figure 3.1 through Figure 3.5, we observe the following:

Observation 3.1 *According to Figure 3.1, optimal response times are non-increasing in the customer's impatience factor(a). As the customer's impatience factor(a) increases to infinity, optimal response time converges at a certain point.*

Increasing the customer's impatience factor(a) means that more lost sales occur in shortages. The suppliers can prevent the amount of lost sales by decreasing the response

time to backorder. Assuming the extreme case, when the customer's impatience factor(a) is relatively large, every shortage becomes lost sales. The case should be the same as an entire lost sales case. Otherwise, when the customer's impatience factor(a) is very small, every shortage becomes a backorder. This case is regarded as an entire backorder case.

Observation 3.2 *According to Figure 3.2, optimal response times are non-increasing in the backorder cost parameter(b). As the backorder cost parameter(b) increases, optimal response time converges at a certain point.*

Increasing the backorder cost parameter(b) means that the backorder cost is decreased more sharply as the response time increases. Assuming the extreme case, when the backorder cost parameter(b) is relatively large, the backorder cost is very sensitive (a large difference in costs either way) to the response time and to the supplier's efforts to decrease the response time as much as possible. Otherwise, when the backorder cost parameter(b) becomes very small (but not zero), the backorder cost is not sensitive (not much difference in the costs either way) to the response time and to the supplier's efforts to increase the response time as much as possible.

Observation 3.3 *According to Figure 3.3, optimal response times are non-decreasing in N, α and non increasing in c_{LS}.*

When the maximum value of backorder cost(N) increases, the supplier will increase the optimal response time to reduce the backorder cost. Similarly, as the backorder setup cost parameter(α) increases, the supplier will increase the optimal response time to reduce the backorder cost. Otherwise, when the lost sales cost(c_{LS}) increases, the supplier decreases the optimal response time to reduce the amount of shortages.

Observation 3.4 *According to Figure 3.4, optimal order quantities are non-decreasing in c_{LS} and non-increasing in c_O, c_H.*
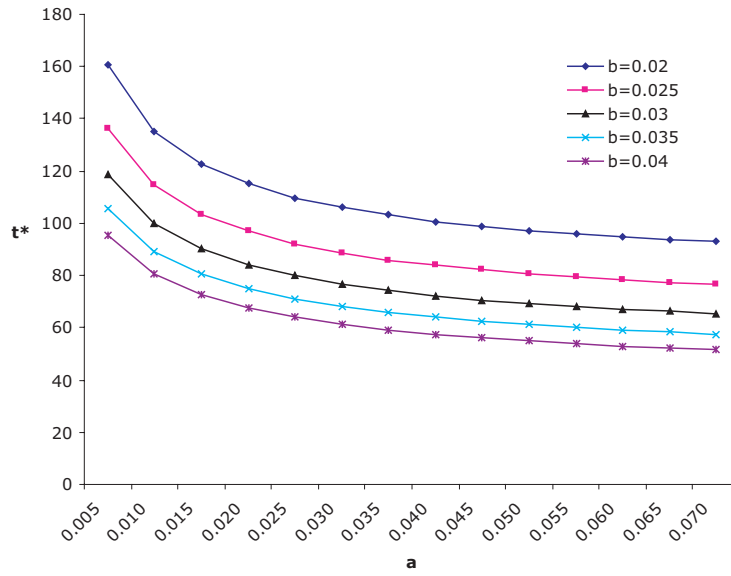
Figure 3.1: The effect of a on the optimal response time.
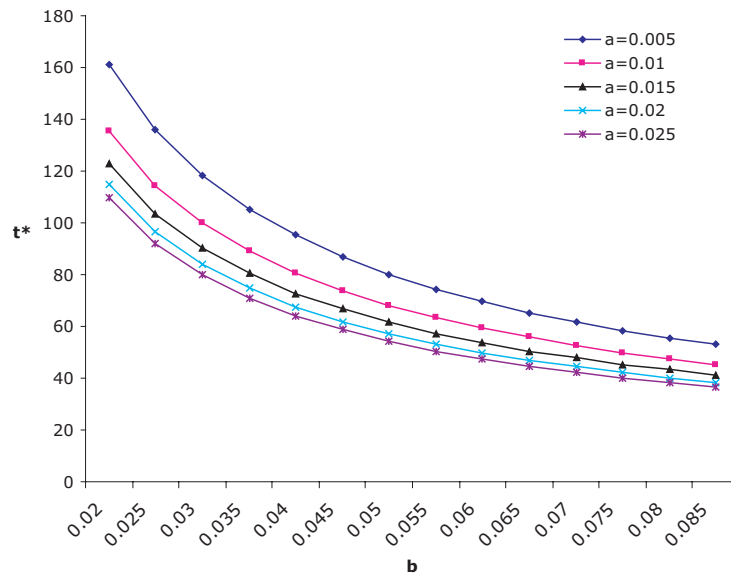
Figure 3.2: The effect of b on the optimal response time.

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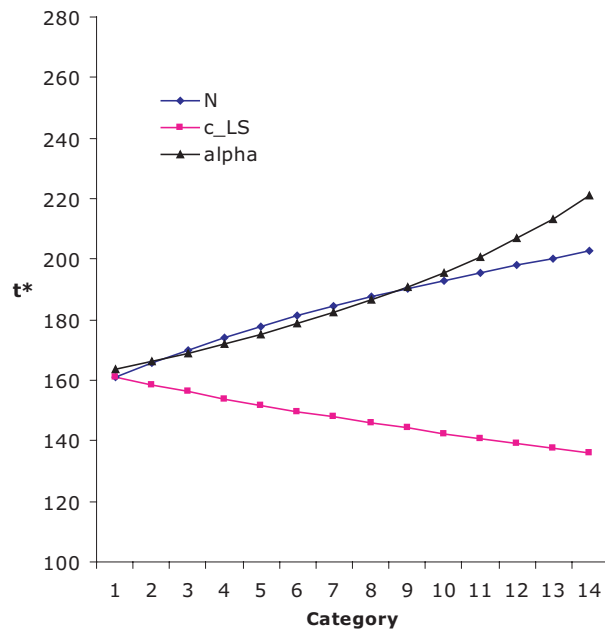
Figure 3.3: The effect of *parameters* on the optimal response time.
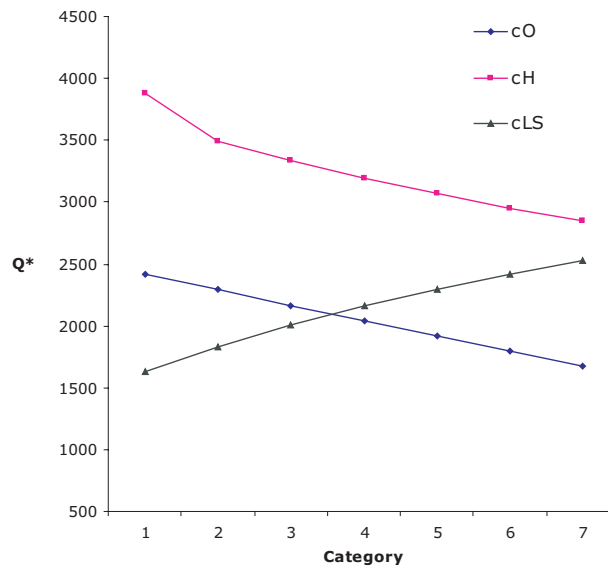
Figure 3.4: The effect of *parameters* on the optimal order quantity.

52
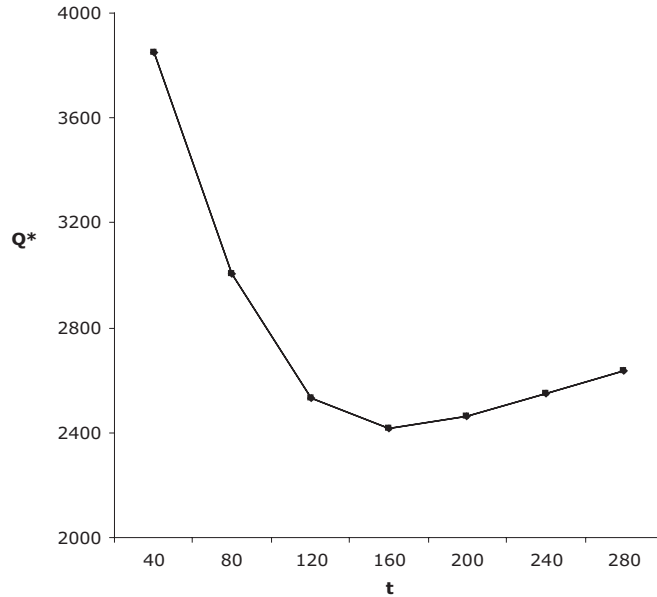
Figure 3.5: The effect of t on the optimal order quantity.

Optimal order quantity increases with the lost sales cost as the company is not willing to have lost sales when the lost sales cost increases. Otherwise, the ordering and the holding cost show an opposite phenomenon. These results are intuitive and consistent with the results reported in Lodree (2007).

Observation 3.5 *According to Figure 3.5, the optimal order quantity shows the minimum at the point of the optimal backorder lead-time.*

A fast backorder response costs so much that the company tries to reduce the shortages and increase the optimal order quantity. On the other hand, a delayed backorder response causes more lost sales and the company works to reduce the shortages by increasing the order quantity. At the optimal backorder lead-time, all trade offs between the backorders and the lost sales costs were mitigated and the optimal order quantity shows the minimum among other optimal order quantities.

3.5　Conclusion

The time-dependent backlogging approach is prevalent in characterizing inventory shortages in an inventory decision model. This approach acknowledges that there is some probability associated with whether or not a customer will backorder a shortage, and that this probability (i.e., the backorder rate) is related to the backorder lead-time associated with replenishing the outstanding backorder.

In this study, we consider the fact that the backorder cost is also related to the backorder lead-time on a stochastic inventory model for a two-stage supply chain system. We simultaneously consider the order quantity and the backorder lead-time as decision variables. Closed form optimal solutions are derived for the specific demand distribution, and intuitive insights were acknowledged by examining the solution for the normally distributed demand. Our example problem suggests that significant cost savings can be achieved by implementing the derived optimal solution as opposed to the prior model solution.

A limitation of the model arises from the selection of specific functions for (e.g., backorder cost function) which contain many cost behaviors as the parameters change, but not enough for the general case. For instance, the function $c_B(t)$ as Ne^{-bt} was selected because it logically represents the behavior of the time-cost trade-offs (i.e., the cost increases as the processing time decreases) and it exhibits analytic properties such that the analysis is tractable. However, a variety of the backorder cost functions should be explored. Practitioners should investigate these functions to implement this model into their specific supply chain situation. Even though the setup cost shows piece-wise increasing behaviors in practice, we approximate the backorder setup cost function as the linearly increasing continuous function in our model. Further research is needed to see how the results could change with a better approximation for the behavior of the setup cost.

Chapter 4

The Effects of an Option Approach to Stochastic Inventory Decisions

ABSTRACT

This study presents an option approach (e.g., straddle) for determining the order quantity under partial backlogging and demand uncertainty. First, we establish an optimal condition for the required order quantity when a firm has a desirable fill rate. Second, we develop a closed form solution for optimal order quantity to minimize the total cost. Finally, we compare the results between the traditional and the option approaches with a numerical example. The comparison could lead to several inventory decision implications and shows the advantages of an inventory decision with an option framework over the traditional inventory decision under high uncertainty.

4.1 Introduction

Almost every decision-maker needs to take a forecast. If he has an idea of what will happen in the future, he can make appropriate management decisions. He also needs to assess the effect of his present decisions on the future so that the right decisions are made today to create a desired condition in the future.

Inventory is capital intensive and, therefore, cost sensitive. The size of the investment in inventory is typically between 10% and 40% of the total assets of most large corporations (Stowe 1997). If we know how many items are likely to be demanded, we can improve the quality of decisions concerning production, procurement, placement and promotion. Consequently, we can minimize the expected total cost tied up in inventories, avoid running out of stock and, generally, increase the sales and improve the profits.
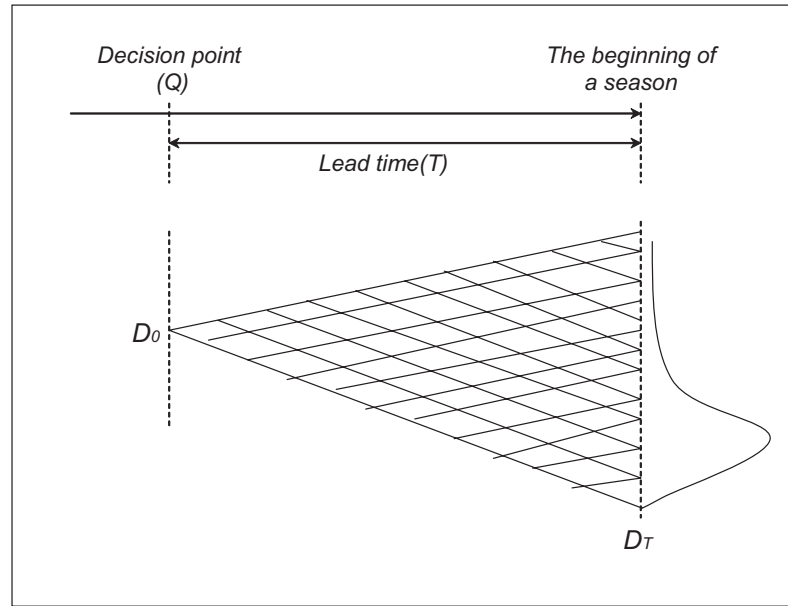
Figure 4.1: General problem description.

For decades, most studies on traditional inventory management under demand uncertainty analyze the detrimental impacts of demand distributions (i.e., normal distribution). Companies have typically estimated the demand using the normal distribution with its classic bell-shaped curve.

It is not appropriate to forecast demand as a normal distribution where negative demand is possible to generate in a highly uncertain situation. In addition, a traditional approach with demand distribution does not consider the information about a current demand that significantly affects the future demand.

Many companies receive orders continuously from its customers. At a decision point, A company could attain current demand information from their reservation systems (i.e., airlines industry). There will be additional customer orders for units or the cancellation of orders during the planning horizon after the decision point as shown in Figure 4.1. In a highly uncertain situation, the decision-makers are not sure whether the current demand

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will go up or down. This situation is closely related to the financial option-pricing frame-work. We adopt the framework of option pricing to implement a demand process in an uncertain situation.

This study develops a variant of the option pricing theory as applied to the inventory planning and reevaluates the basic newsvendor problem with partial backlogging to show that the tool of financial options can be used to make inventory decisions in supply chain management.

4.1.1 The Traditional Newsvendor Approach with Partial Backlogging

The newsvendor problem is not complex, but it can give sufficient information that enables us to compare between the traditional and the option approaches in a two-stage supply chain. First of all, we figure out how to decide the optimal order quantity in a traditional newsvendor approach. In this newsvendor problem, a unit inventory holding cost of c_H is incurred if Q is larger than the demand(x), and the unit back order cost $\beta \cdot c_B$ or/and unit lost sales cost $(1 - \beta)c_{LS}$ is incurred if Q is smaller than the demand(x). Then, X is a continuous random variable that represents the demand and β represents the backorder rate. The objective is to determine the quantity (Q) that minimizes the following expected cost function.

$$
\begin{aligned}
E[TC(Q)] &= \text{Order Cost} + \text{Expected Holding Cost} + \text{Expected Back order Cost} \\
&\quad + \text{Expected Lost Sales Cost} \\
&= c_O Q + c_H \int_0^Q (Q - x)f(x)\mathrm{d}x + c_B \int_Q^\infty (x - Q)\beta f(x)\mathrm{d}x \\
&\quad + c_{LS} \int_Q^\infty (x - Q)(1 - \beta)f(x)\mathrm{d}x \quad\quad\quad (4.1)
\end{aligned}
$$

If the total cost function is a convex function, then a unique minimizer (Q^*) can be obtained. And then, we determine whether the total cost function is a convex function.

The convexity of the total cost function is guaranteed since the second derivation is always positive. Refer to Appendix C.1 for details. Because we have a convex objective function, we can get an optimal order quantity at which the first-order condition is equal to zero.

$$Q^* \;=\; F^{-1}\left[\frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}\right] \tag{4.2}$$

Therefore, the optimal demand fill rate is represented by the following:

$$\rho^* = \frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H} \tag{4.3}$$

The traditional approach, such as the basic newsvendor problem, assumes demand as a specific distribution (e.g., normal distribution) to implement demand uncertainty as shown in this model. The optimal order quantity totally depends on the assumed specific distribution in the traditional approach.

4.2 Literature Review

There has been limited literature which applies financial theory to inventory decisions. Kim and Chung (1989) and Morris and Chang (1991) use the capital-asset pricing model (CAPM) as an alternative to the conventional profit-maximization approach. A key insight of this approach is that a demand beta (sensitivity of demand to the market index) is inversely related to the stocking level.

Several studies have applied the option-pricing approach to the inventory problem since 1990. Stowe and Gehr (1991) provide two financial approaches to handling the inventory problem. The first is a replicating portfolio approach comparing the value of a portfolio of securities to the inventory investment. The second approach applies an option-pricing

solution to the discrete sales problem. Chung (1990) also shows another option-pricing solution for inventory payoffs that is a linear function of the demand when the demand and the investors' discount rate are jointly log-normally distributed. Becker (1994) solves for the inventory reorder point using a binomial option-pricing approach.

Agrawal et al. (2000) show a single-period inventory model in which a risk-averse retailer faces uncertain customer demand and makes a purchasing-order quantity and a selling-price decision with the objective of maximizing the expected utility. Birge and Zhang (1999) derive an optimal policy with an option valuation model. Berling and Rosling (2005) apply a real options framework considering the stochastic demand and the purchase costs. They present the two inventory models, a single-period model of the newsboy type and an infinite-horizon model with a fixed set-up cost.

Our approach is an extension of the earlier studies. Our intuition is that if any inventory payoff function can be mapped onto an underlying state variable, the payoff can be replicated by a portfolio of options (e.g., straddle strategy). Straddle is one of the most popular options trading strategies in buying both a call and a put option at the same strike price and the same maturity date in the case of a volatile market. We also refer to the discrete time option pricing model of Cox et al. (1979) for the evaluation of the firm's inventory decision. The option approach presented here associates the demand for an inventory item with an underlying state variable. This approach is well chosen for the analysis of the firm's inventory decisions under uncertainty because the payoff from the firm's inventory decision is an option with its value depending on the uncertain future demands.

This study is organized as follows. The next section introduces the model and presents a total cost valuation using the option-pricing model. The following section then presents the optimality for the required order quantity when a firm has a given fill rate and yields a closed form solution for the optimal order quantity to minimize the total cost. The final section is a summary and has concluding remarks.

Current stock price S_0 $\Longrightarrow$ D_0 Current demand

Exercise price K $\Longrightarrow$ Q Order quantity

Time to maturity T $\Longrightarrow$ T Time to the end of season

Stock volatility σ $\Longrightarrow$ σ Demand volatility
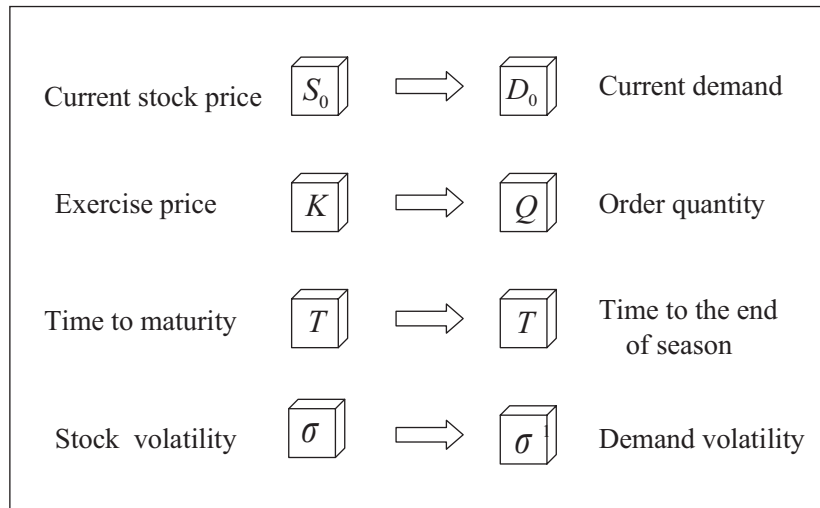
Figure 4.2: Analogy between financial option and inventory decision.

4.3 Model formulation

The underlying asset price can fluctuate continuously over time. A commonly used process to describe the evolution of the demand (asset price) is the lognormal diffusion process. A key feature of this process is that the demand (asset price) never drops below zero. Another feature of the lognormal diffusion process that is similar to the binomial is that the variance of the demand (asset price) grows with time. In fact, the variance grows proportionally to the length of the time horizon (Chopra 2007). We substitute the demand for the asset price in the option valuation because we assume that the characteristics of demand are the same as the asset price of the option in this study. The decision problem which the firm is facing at the decision point is to determine and to notify its supplier of the order quantity(Q), in order to meet the unknown accumulated orders (i.e., demand) of units from its customers at the end of the period. We consider the change of demand during a considerable lead-time to determine the order quantity at the decision point. Here we discuss a single period inventory problem. This approach is closely related to the principles of an asset pricing model, such as the Black-Scholes option-pricing model. Then we set up

Table 4.1: List of notations.

c_O :	Unit ordering cost
c_H :	Unit cost of the holding excess inventory
c_B:	Unit cost of the backlogged shortages
c_{LS}:	Unit cost of the shortages that are lost sales
β :	Fraction of shortages that are backlogged
r_f:	Risk free interest rate
T:	Lead-time
ρ:	Demand fill rate
Q :	Order quantity
Q_ρ :	Order quantity with a given demand fill rate
Q^*:	Optimal order quantity that minimizes the total cost
D_0:	Initial demand(i.e., number of reservations by the decision point)
D_T :	Demand (the customers order) at time T
X:	Random variable of demand
$f(x)$:	p.d.f. of the demand distribution
$F(x)$:	c.d.f. of the demand distribution
$TC(\cdot)$:	Total expected cost function

an analogy between the financial option and the inventory decision as shown in Figure 4.2. The notation of Table 4.1 will be used throughout this study.

4.3.1 Model Formulation with an Option Pricing Framework

The total cost may be represented as the order cost, holding cost, backorder cost, and lost sales cost. If the demand (D) is greater than the quantity ordered (Q), the cost is assumed to be the profit lost on unsatisfied demand. If the demand is less than the quantity ordered, unsold units incur a holding cost for storage at a warehouse. The total cost function can be represented as follows:

$$
TC(Q) = \begin{cases} Order\ cost + Backorder\ cost + Lost\ sales\ cost & \text{if } D \geq Q \\ Order\ cost + Holding\ cost & \text{if } D < Q \end{cases}
$$

(4.4)
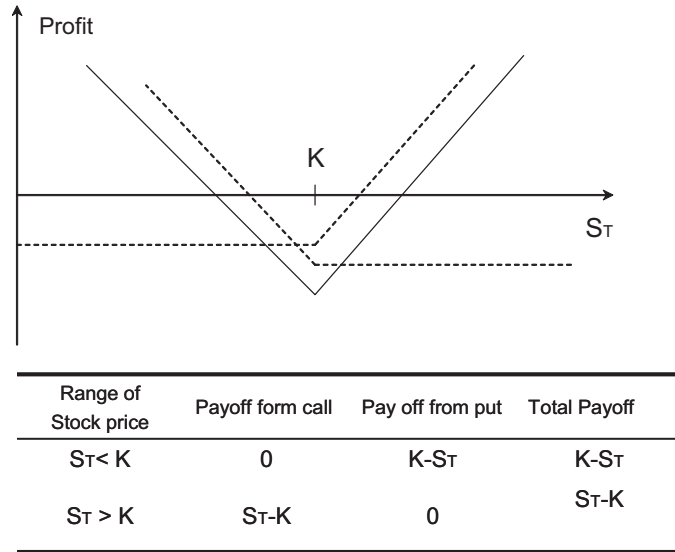
Figure 4.3: A Straddle quoted from John C.Hull (2002).

With the end-of-period convention, the cash flows at the end of the period will be $c_O Q + c_B \beta(D - Q) + c_{LS}(1 - \beta)(D - Q)$ if the demand is greater than the quantity ordered, or $c_O Q + c_H(Q - D)$ if the demand is less than the quantity ordered. To consider a general situation, we assume $c_B < c_{LS}$, although there are some situations where backorder costs exceed the short term lost sale costs.

$$TC(Q) = \begin{cases} c_O Q + c_B \beta(D - Q) + c_{LS}(1 - \beta)(D - Q) & \text{if } D \geq Q \\ c_O Q + c_H(Q - D) & \text{if } D < Q \end{cases}$$

(4.5)

This total cost combination closely imitates a straddle strategy in financial options as shown in Figure 4.3. Straddle is one of the most popular options trading strategies in buying both a call and a put option at the same strike price and on the same maturity date in the case of a volatile market. If you buy a straddle, you expect the price of the

underlying asset to move significantly, but you're not sure whether it will go up or down. Your risk in buying a straddle is limited to the premium you pay. The first part of Eq. (4.5) represents a call option situation and the second part of Eq. (4.5) represents a put option situation. Then the present total cost of cash flows can be expressed as the following conditional expectation. Refer to Appendix C.4 for details.

$$
\begin{aligned}
E[TC(Q)] &= e^{-r_f T} \cdot E[\{c_B \beta + (1-\beta)c_{LS}\}D \\
&\quad -\{c_B \beta + (1-\beta)c_{LS} - c_O\}Q | D \geq Q] \cdot P(D \geq Q) \\
&\quad +e^{-r_f T} \cdot E[(c_O + c_H)Q - c_H D | D < Q] \cdot P(D < Q)
\end{aligned}
$$

$$(4.6)$$

We convert the end of the period's conditional cash flow into a certain present value with a continuous discount factor. We can divide the former equation into two components.

$$
\begin{aligned}
E[TC(Q)] &= E[TC_a(Q)] + E[TC_b(Q)] \\
E[TC_a(Q)] &= e^{-r_f T} \cdot E[\{c_B \beta + (1-\beta)c_{LS}\}D \\
&\quad -\{c_B \beta + (1-\beta)c_{LS} - c_O\}Q | D \geq Q] \cdot P(D \geq Q) \\
E[TC_b(Q)] &= e^{-r_f T} \cdot E[(c_O + c_H)Q - c_H D | D < Q] \cdot P(D < Q)
\end{aligned}
\qquad (4.7)
$$

We denote the current demand by D_0 and the demand at the end of the lead-time by D_T. The ratio $\frac{D_T}{D_0}$ is a random variable with a lognormal distribution where $ln(\frac{D_i}{D_{i-1}}) = r_i$ is the rate of demand change in the i_{th} period and is a random variable with normal distribution under our assumptions. Let r_i have the expected value μ_r and variance σ_r^2 for each i. Then $r_1 + r_2 + r_3 + \cdots + r_T$ is a normal random variable with the expected

value and variance . Thus, we can define the expected value of $\frac{D_T}{D_0} = e^{r_1 + r_2 + r_3 + \cdots + r_T}$ as $E[\frac{D_T}{D_0}] = e^{T\mu_r + \frac{T\sigma_r^2}{2}}$.

The risk neutral approach to valuation, introduced by Cox et al. (1979), is based on the same arguments that underlie the option valuation. Under this assumption that the expected demand change for one period (e.g., $\frac{D_1}{D_0} = e^{r_f}$) is regarded as the exponential value of the risk free interest rate, the expected demand change rate $E[\frac{D_T}{D_0}]$ is assumed to be $e^{r_f \cdot T}$ where r_f is the risk free interest rate. In other words, since we assume that $\frac{D_T}{D_0}$ is log-normally distributed, we can define that the r_f as $\mu_r + \frac{\sigma_r^2}{2}$. Therefore, $ln(\frac{D_T}{D_0})$ is normally distributed using the relationship between the lognormal and the normal distribution. We can rewrite the first part of Eq. (4.7) as follows. Refer to Appendix C.6 for details.

$$
\begin{aligned}
E[TC_a(Q)] &= e^{-r_f T} \cdot D_0 \cdot \{c_B \beta + (1-\beta)c_{LS}\} \int_{ln(\frac{Q}{D_0})}^{\infty} r f(r) \mathrm{d}r \\
&\quad - e^{-r_f T} \cdot Q \{c_B \beta + (1-\beta)c_{LS} - c_O\} \int_{ln(\frac{Q}{D_0})}^{\infty} f(r) \mathrm{d}r \\
&= D_0\{c_B \beta + (1-\beta)c_{LS}\}N(d_1) - e^{-r_f T} \cdot Q\{c_B \beta + (1-\beta)c_{LS} - c_O\}N(d_2) \\
&\quad \text{where } \mathrm{d_1} = \frac{\ln(\frac{\mathrm{D_0}}{\mathrm{Q}}) + (\mathrm{r_f} + \frac{\sigma_r^2}{2})\mathrm{T}}{\sigma_\mathrm{r}\sqrt{\mathrm{T}}}, \quad \mathrm{d_2} = \frac{\ln(\frac{\mathrm{D_0}}{\mathrm{Q}}) + (\mathrm{r_f} - \frac{\sigma_r^2}{2})\mathrm{T}}{\sigma_\mathrm{r}\sqrt{\mathrm{T}}} \quad (4.8)
\end{aligned}
$$

We also rewrite the second part of Eq. (4.7) using similar procedures as follows. Refer to Appendix C.7 for details.

$$
\begin{aligned}
E[TC_b(Q)] &= e^{-r_f T} \cdot Q(c_O + c_H) \int_0^{ln(\frac{Q}{D_0})} f(r) \mathrm{d}r - e^{-r_f T} \cdot D_0 c_H \int_0^{ln(\frac{Q}{D_0})} r f(r) \mathrm{d}r \\
&= e^{-r_f T} \cdot Q(c_O + c_H)N(-d_2) - D_0 \cdot c_H N(-d_1) \\
&\quad \text{where } \mathrm{d_1} = \frac{\ln(\frac{\mathrm{D_0}}{\mathrm{Q}}) + (\mathrm{r_f} + \frac{\sigma_r^2}{2})\mathrm{T}}{\sigma_\mathrm{r}\sqrt{\mathrm{T}}}, \quad \mathrm{d_2} = \frac{\ln(\frac{\mathrm{D_0}}{\mathrm{Q}}) + (\mathrm{r_f} - \frac{\sigma_r^2}{2})\mathrm{T}}{\sigma_\mathrm{r}\sqrt{\mathrm{T}}} \quad (4.9)
\end{aligned}
$$

Then we can rewrite the Eq. (4.7) simply as follows where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

$$
\begin{aligned}
E[TC(Q)] &= E[TC_a(Q)] + E[TC_b(Q)]) \\
&= D_0\{c_B\beta + (1-\beta)c_{LS}\}N(d_1) - e^{-r_f T} \cdot Q\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2) \\
&\quad + e^{-r_f T} \cdot Q(c_O + c_H)N(-d_2) - D_0 \cdot c_H N(-d_1) \\
&\text{where } \ d_1 = \frac{\ln(\frac{D_0}{Q}) + (r_f + \frac{\sigma_r^2}{2})T}{\sigma_r\sqrt{T}}, \quad d_2 = \frac{\ln(\frac{D_0}{Q}) + (r_f - \frac{\sigma_r^2}{2})T}{\sigma_r\sqrt{T}} \qquad (4.10)
\end{aligned}
$$

4.4 Inventory Decisions

We develop two closed form solutions for the inventory problem with this model. First, we consider a firm that has an desirable fill rate to meet a certain level of customer satisfaction. Next, we consider a firm wants to minimize the expected total costs with an optimal fill rate.

4.4.1 Order Quantity with a Desirable Fill rate

Suppose that the decision-maker of the firm has a desirable fill rate, ρ. The firm desires to satisfy this level of fill rate. $N(-d_2)$ is interpreted as the fill rate, demand being less than the quantity ordered, as shown in the following equation.

$$
\begin{aligned}
P(D < Q) &= P(\frac{D}{D_0} < \frac{Q}{D_0}) \\
&= P(ln(\frac{D}{D_0}) < ln(\frac{Q}{D_0})) \\
&= 1 - P(r \geq ln(\frac{Q}{D_0})) \\
&= 1 - \int_{ln(\frac{Q}{D_0})}^{\infty} f(r)\mathrm{d}r = 1 - N(d_2) \\
&= N(-d_2)
\end{aligned}
$$

$$where \ \ ln(\frac{D_{t+1}}{D_t}) = r \sim Normal(\mu_r, \sigma_r) \qquad (4.11)$$

Then the minimum required order quantity can be obtained by letting $N(-d_2)$ equal to ρ. Finally, we develop the closed-form solution for the minimum required order quantity. Refer to Appendix C.10 for details.

$$\rho = N(-d_2)$$

$$Q_\rho = \left[\frac{D_0 e^{(r_f - \frac{\sigma_r^2}{2})T}}{e^{N^{-1}(1-\rho)\sigma_r \sqrt{T}}} \right] \qquad (4.12)$$

$N^{-1}(\cdot)$ denotes the inverse cumulative distribution function of the standard normal distribution.

4.4.2 Optimal Fill rate and Order Quantity

Now suppose that the objective is to minimize the present total cost cash flows. Then, we could attain optimal stock out probability and order quantity from $\frac{d[E[TC(Q)]]}{dQ} = 0$.

$$
\begin{aligned}
\frac{d[E[TC(Q)]]}{dQ} &= e^{-r_f T}(c_O + c_H)N(-d_2) - e^{-r_f T}\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2) = 0 \\
&\quad e^{-r_f T}(c_O + c_H)N(-d_2) = e^{-r_f T}\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2)
\end{aligned}
$$

$$(4.13)$$

Note that $c_B\beta + (1-\beta)c_{LS} - c_O$ represents the penalty cost of the shortage (i.e., lost profit per unit), and $N(d_2)$ represents the probability of stock-out. Thus the right-hand side of Eq. (4.13) measures the expected cost of shortages. Similarly, note that $c_O + c_H$ is the implicit cost of each unsold unit, and $N(-d_2)$ represents the fill rate. Thus the left-hand

side of Eq. (4.13) measures the expected cost of an overstock. The Eq. (4.13) can be simplified as follows:

$$\rho^* = N(-d_2) = \frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}$$

$$(4.14)$$

If we have a convex objective function and a feasible point at which the first-order conditions hold, then we know that we have found a unique minimizer. We need to verify the convexity of Eq. (4.10) as the convexity of the function guarantees the expected total cost function has a unique minimizer. The convexity of Eq. (4.10) is guaranteed since the second derivation is always positive as follows. Refer to Appendix C.9 for details.

$$\frac{\mathrm{d}[E[TC(Q)]]^2}{\mathrm{d}^2 Q} = e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}(Q\sigma_r\sqrt{T})^{-1}f(d_2) \qquad (4.15)$$

The optimal order quantity occurs when the expected shortage penalty cost equals the expected cost of the overstock. We derive the optimal order quantity (Q^*) as follows. Refer to Appendix C.10 for details.

$$N(-d_2) = \rho^*$$
$$Q^* = \frac{D_0 e^{(r_f - \frac{\sigma_r^2}{2})T}}{e^{\sigma_r\sqrt{T}N^{-1}(1-\rho^*)}} \qquad (4.16)$$

The value of ρ^* can be interpreted as the optimal fill rate to minimize the expected total cost. Then, we also obtain the closed-form solution for the optimal order quantity. The optimal fill rate is the same as the partial backlogging newsvendor problem. It is

reasonable to make the option framework approach applicable as the equation of optimal fill rate is composed with determined parameters.

4.5 An Illustrative Example

Comparisons between the traditional and option approaches will be given in this section using the same set of data. The following example data was used for the analysis:

$$c_O = 10, c_H = 5, c_B = 20, c_{LS} = 40, D_0 = 500, r_f = 0.05, \beta = 0.3, T = 1 \qquad (4.17)$$

We adopt the relationship proposed by Dixit et al. (1994) between the volatility of demand (σ_r) and the volatility of future demand at the end of the study period (σ_T) to compare both the approaches.

$$\sigma_r = \sqrt{\frac{\ln\left(\frac{\sigma_T^2}{D_0^2 e^{r_f T}} + 1\right)}{T}} \qquad (4.18)$$

Order Quantity with an Desirable Fill rate

The required order quantities of both the approaches with an desirable fill rate are attained according to the change of the volatility of demand (σ_r). We illustrate the differences between both the approaches with graphs to find some implications.

As shown in Figure 4.4, optimal order quantities are exponentially increased as the volatility of demand (σ_r) is increased with the fill rate greater than 50%. Otherwise, optimal order quantities are exponentially decreased as the volatility of demand (σ_r) is increased with the fill rate less than 50%. Optimal order quantities are the same regardless of the change of the volatility of demand (σ_r) with the fill rate 50%. However, the optimal order
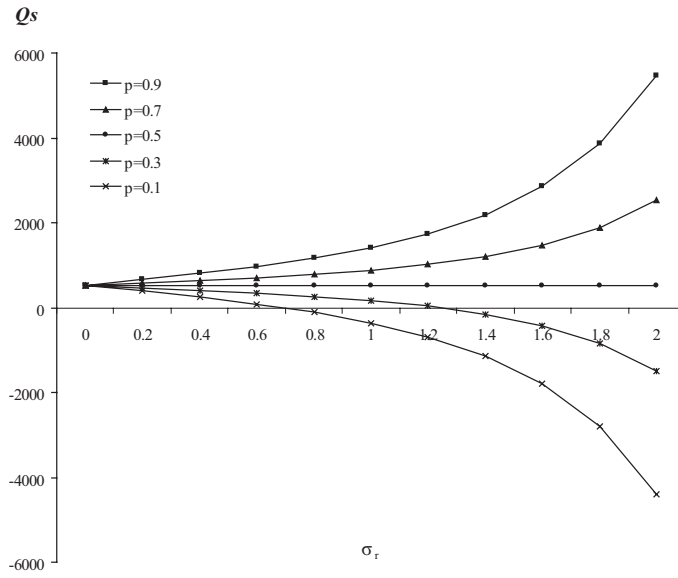
Figure 4.4: Order quantity with traditional approach.
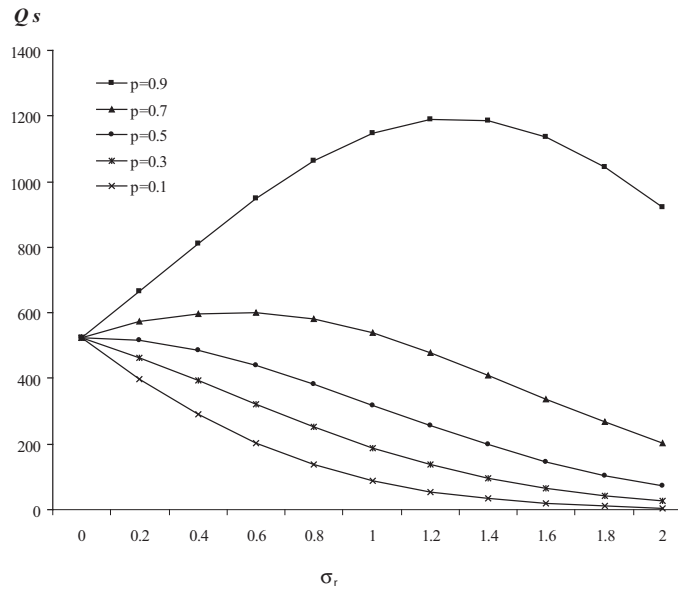
Figure 4.5: Order quantity with option approach.

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Table 4.2: Comparison Optimal order quantities and Total costs.

σ_r	Q*			TC(Q*)		
	Traditional	Option app.	Difference	Traditional	Option app.	Difference
0	526	526	-	$ 5,000	$ 5,000	-
0.2	557	546	(10)	$ 6,843	$ 6,525	($ 318)
0.4	590	546	(44)	$ 8,496	$ 8,078	($ 418)
0.6	627	524	(104)	$ 10,167	$ 9,598	($ 569)
0.8	672	483	(189)	$ 12,003	$ 11,029	($ 974)
1	728	428	(300)	$ 14,277	$ 12,322	($ 1,955)
1.2	802	364	(439)	$ 17,310	$ 13,445	($ 3,865)
1.4	906	297	(609)	$ 21,569	$ 14,383	($ 7,186)
1.6	1058	234	(825)	$ 27,824	$ 15,134	($ 12,689)
1.8	1289	176	(1,113)	$ 37,369	$ 15,714	($ 21,655)
2	1655	128	(1,527)	$ 52,491	$ 16,143	($ 36,348)

quantities show different shapes when using the option approach as shown in Figure 4.5. It shows that optimal order quantities are increasing in a nonlinear fashion to some point and then decreasing as the volatility of demand (σ_r) is increased with the fill rate greater than 50%. Otherwise, optimal order quantities are decreasing in a nonlinear fashion as the volatility of demand (σ_r) is increased with the fill rate less than or equal to 50%.

The traditional approach gives negative order quantities, which are not applicable in practice as shown in Figure 4.4. The optimal order quantities show a different pattern when we apply the option approach as the volatility of demand (σ_r) increase as shown in Figure 4.5. In the traditional approach with high volatility of demand (σ_r), the decision-maker can increase the order quantity beyond an acceptable amount. Consequently, the traditional approach does not address risk in an economically meaningful way under high uncertainty because it provides information of optimal order quantity as negative or beyond acceptable amount. The option approach shows that optimal order quantities are positive and have upper limits even in a highly uncertain situation.

Optimal Fill rate and Order Quantity

When we attain the optimal fill rate with data for an illustrative example, the optimal fill rate is determined by constant parameters. The optimal fill rate is the same, 61.54%, in this numerical example for both the approaches using Eq. (4.3) and Eq. (4.14). We check the optimal order quantities for both approaches as the volatility of demand (σ_r) increases. We change the volatility of demand (σ_r) within some intervals for both the approaches under the optimal fill rate (61.54%). Table 4.2 shows the comparison of the order quantities and total costs between the two approaches. As shown in Table 4.2, the optimal order quantities of the traditional approach are increased as the volatility of demand (σ_r) increases with the optimal fill rate. However, the optimal order quantities represent a different pattern when using the option approach. It shows that the pattern of optimal order quantities is definitely different from the traditional approach when the volatility of demand (σ_r) increases with the same optimal fill rate. In the option approach, the probability density function of demand under high volatility shows a more skewed shape towards the side of small demands. The option approach suggests us to reduce order quantities in a highly volatile demand situation unlike the traditional approach. The difference in optimal order quantities of both the approaches is not significant at a low volatility of demand (σ_r) but the difference can be significant at a high volatility of demand (σ_r). Table 4.2 shows us the difference of total costs is \$36,348 when the volatility of demand (σ_r) is equal to 2.

4.6   Conclusion

Our research presents an option approach to make a decision about the order quantity. This approach incorporates the economic principles of the asset pricing models, such as the Black-Scholes option-pricing model, to replace the expected total cost minimization sense of the traditional approach. Asset-pricing models have been incorporated into many of the other investing and financing pricing decisions of the firms, and we have shown that an option-pricing approach looks promising for determining the inventory decisions as well.

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This study presents an option-pricing framework to decide the order quantity under demand uncertainty during non-negligible lead-time. First, we establish an optimal condition for required order quantity when a firm has an desirable fill rate. Second, we develop a closed-form solution for optimal order quantity to minimize the total cost. Moreover, we compare the results between the traditional and option approaches with different values of volatility of the demand (σ_r). The comparison could lead to several inventory decision implications and shows the advantages (i.e., total cost reduction) of an inventory decision with a real option framework. The option approach addresses risk in an economically meaningful way. Specifically, it does not make any negative order quantity when the fill rate is less than 50% with a high volatility of demand (σ_r) but the traditional approach gives us a negative order quantity.

The usefulness of the option approach relies on the identification of financially traded assets that can be used to forecast the customer demand, allowing for the specification of payoff functions for the inventory problem. Another requirement for the application of the option approach is the willingness of decision-makers to adapt option-pricing models to specific inventory situations. If these requirements are met, there should be more opportunities for successfully applying the option approach to inventory decisions.

This work can be extended to various studies for inventory decisions in several ways. A useful extension would be to apply this approach to a multi-period problem. A multi-period problem is more likely to be applicable to practitioners. Another extension we suggest is to use a time-dependent backorder rate function and not a deterministic backorder rate (β) as used in this study. For example, the backorder rate is expressed as the function of the magnitude of the shortages (Lodree 2007).

Chapter 5

Coordinating a Two-Stage Supply Chain Based On Option Contract

ABSTRACT

Manufacturers and retailers are implementing option contracts to improve overall supply chain profits and product availability. This research considers a standard newsvendor problem with price dependent stochastic demand in a single manufacturer-retailer channel and shows the expected profits that each party receives under an option contract designed to eliminate the double marginalization problem. We derive closed form solutions for the appropriate option prices by the manufacturer as an incentive for the retailer to make optimal pricing and order quantity decisions for coordinating the channel. The option contract not only coordinates the supply chain; it also can divide the profit between the manufacturer and the retailer. The findings of this research are illustrated in a numerical example for a normal demand distribution.

5.1 Introduction

Recent years have seen growing interest in the area of supply chain coordination. One of the features of supply chain is the existence of the multiple decision-makers. Optimal supply chain performance requires the execution of a precise set of actions. Unfortunately, those actions are not always in the best interest of all the members of the supply chain. Actions taken by the two parties in the supply chain often result in profits that are lower than what could be achieved if the supply chain were to coordinate its actions with a common objective of maximizing supply chain profits. We consider a product whose demand is significantly affected by the retail price. And the retailer decides its price based on its margin. The retailer's margin is only a fraction of the supply chain margin, which could lead to a retail

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price that is higher than the optimal retail price and an actual order quantity that is lower than the optimal amount for the supply chain. To improve overall profits, the manufacturer designs a contract that encourages the retailer to purchase more and increase the level of product availability. This requires the manufacturer to share in some of the retailer's risk of the demand uncertainty.

We suggest an option contract to hedge the risk of the demand uncertainty. This can improve the product availability before the season and maximize the overall supply chain profits at the end of the season through coordination of the option prices and the order quantity. One of the most important benefits gained from trading in the option contract is that the retailer can order a wide range of items with a relatively small initial investment (e.g. option premium). When a retailer assumes a demand position in the future, he or she locks in a price for the commodity from the manufacturer. This fixed price is the option strike price at which the contract is bought or sold. Subsequently, as the demand for the commodity rises or falls, the option strike price follows suit, making or losing money. The option contract provides protection for each party (e.g.,manufacturer and retailer) against dangerous demand swings.

We use a newsvendor model to illustrate the benefits of an option contract in the supply chain. The newsvendor model is not complex, but it is sufficiently rich to offer many implications for the manufacturer and the retailer in supply chain coordination. In the newsvendor model the retailer orders a single product from the manufacturer well in advance of the selling season with a stochastic demand. The manufacturer produces the item after receiving the retailer's order and delivers their production to the retailer at the start of the selling season. The retailer has no additional replenishment opportunity. How much the retailer chooses to order depends on the terms of the trade, i.e., the contract, between the retailer and the manufacturer. Our model allows the retailer to choose his discounted retail price to increase the product demand. Coordination is more complex in

Table 5.1: A summary of the related literature.

Contract	Literature
Buyback	Padmanabhan and Png (1997), Emmons and Gilbert (1998), Donohue (2000), Taylor (2001), Lee, Padmanabhan, Taylor and Wang (2000)
Revenue sharing	Mortimer (2000), Dana and Spier (2001), Gerchak, Cho and Ray (2001)
Quantity flexibility	Pasternack (1985), Eppen and Iyer (1997), Tsay (1999), Cachon and Lariviere (2001), Lariviere (2002), Plambeck and Taylor (2002)
Rebates	Krishan, Kapuscinski and Butz (2001), Taylor (2002)
Quantity discounts	Jucker and Rosenblatt (1985), Pantumsinchai and Knowles (1991), Khouja (1996), Lin and Kroll (1997), Chen, Federgruen, and Zheng (2001), Weng (1995)
Option	Erkoc and Wu (2002), Barnes-Schuster, Bassok and Anupindi (2002), Cheng, Ettl, Lin, Schwarz and Yao (2003), Kleindorfer and Wu (2003), Kamrad and Siddique (2004),Martnez-de-Albenz and Simchi-Levi (2003), Ozer and Wei (2004), Burnetas and Ritchken (2005)

this setting because the incentives provided to align one action (e.g., order quantity) may cause distortions with the other action (e.g., option price).

5.2 Literature Review

Several different types of contract are shown to coordinate the supply chain and divide its profit; buy back, revenue sharing, quantity flexibility, sales rebate, and quantity discount contracts. In recent years, researchers have examined option contracts to help coordinate a supply chain. Table 5.1 summarizes literature about the contract policies in supply chain coordination.

Taylor (2000) incorporates a buy back contract with a sales rebate contract to coordinate the newsvendor with an effort dependent demand. Donohue (2000) studies buy back contracts in a model with multiple production opportunities and improving the demand forecasts. For the literature related to revenue sharing contracts, Mortimer (2000) provides

a detailed econometric study of the impact of revenue sharing contracts in the video rental industry. Dana and Spier (2001) study these contracts in the context of a perfectly competitive retail market. Gerchak, Cho and Ray (2001) consider a video retailer that decides how many tapes to purchase and how long to keep them.

In addition, there are a number of papers about quantity- flexibility contracts. Cachon and Lariviere (2001) and Lariviere (2002) study the interaction between quantity flexibility contracts and forecast sharing. Plambeck and Taylor (2002) study quantity flexibility contracts with more than one downstream firm and ex-post renegotiation. The sales rebate contract is studied in Taylor (2000) and Krishan, Kapuscinski and Butz (2001). In Taylor (2000) effort is chosen simultaneously with the order quantity, whereas Krishan, Kapuscinski and Butz (2001) focus on the case in which the retailer chooses an order quantity, a signal of demand is observed and then effort is exerted.

The literature about the option contracts is studied by Erkoc and Wu (2002). They propose two channel coordination contracts, and discuss how such contracts can be tailored for situations where the supplier has the option of not complying with the contract, and when the buyer's demand information is only partially updated during the supplier's capacity lead-time. Cheng, Ettl, Lin, Schwarz and Yao (2003) develop an option model to quantify and price a flexible supply contract, by which the buyer (a manufacturer), in addition to a committed order quantity, can purchase option contracts and decide whether or not to exercise them after demand is realized. However, unlike our study, they assume the retail price is constant. Martnez-de-Albenz and Simchi-Levi (2003) consider the impact of a supply option contract on the newsvendor and derive an optimal replenishment policy for a portfolio consisting of long-term and option contracts. Burnetas and Ritchken (2005) investigate the role of option contracts in a supply chain when the demand curve is sloping downward. They show that the introduction of the option contracts causes the wholesale price to increase and the volatility of the retail price to decrease.
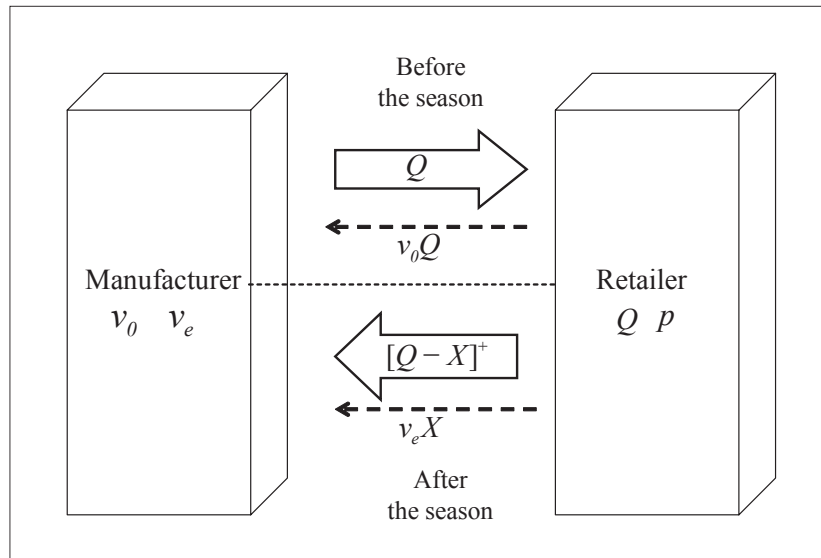
Figure 5.1: Transactions between parties.

Our study is to consider contracting arrangements in a supply chain for a newsvendor environment. Our approach, however, differs in that we do not assume the retail price and the demand distribution as given; rather, we assume demand distribution is influenced by the discount of the retail price, because, in practice, the retailer usually has a right to discount the retail price to some extent and the demand is relative to the retail price.

The rest of this study is organized as follows. In section 5.3, we describe the channel structure including the meanings of notations and the model assumptions for this study. Section 5.4 describes the types of supply chain decisions as closed form solutions (i.e., optimal option purchase and strike prices, optimal order quantity) for the manufacturer and the retailer respectively. In section 5.5, we demonstrate the results with numerical examples. Finally, we conclude in Section 5.6 with possible extensions.

Table 5.2: List of notations.

p_o :	Original selling price.
p :	Discounted selling price.
h:	Holding cost.
c:	Production cost.
X :	r.v. of Demand.
Q :	Order quantity.
x :	Value of demand.
$f(x)$:	pdf of X with discounted retail price
$f_0(x)$:	pdf of X with initial retail price
$F(x)$:	cdf of X with discounted retail price
$F_0(x)$:	cdf of X with initial retail price
v_0:	Option Purchase price (e.g.,premium).
v_e:	Option Strike price.
γ:	Proportion of the channel profit got by retailer
V_c:	Profit in the centralized system.
V_r:	Profit of retailer in the decentralized system.
V_m:	Profit of manufacturer in the decentralized system.

5.3   Channel Structure with an Option Contract

We consider a single period two-stage supply chain option contract between a manufacturer and a retailer who sells a short life-cycle product to the customers. The retailer can discount a retail price in an appropriate range. The interaction takes place between the retailer and the manufacturer in a principal agent setting where the manufacturer is the leader. Before the season the manufacturer provides the option contract consisting of the option purchase and the exercise price, then the retailer determines the retail price and the order quantity. Under the forced compliance environment, the manufacturer must build enough quantity to satisfy the final order up to the number of the order quantity.

After the demand is realized the retailer exercises the option accordingly. So in this setting there is no remainder, for the retailer has a right to return leftovers by return policy as shown in Figure 5.1. In this study it is also assumed that the unsatisfied demand becomes lost sales and there is no penalty cost for lost sales. Throughout the study, the parameters and the decision variables of the model are given in Table 5.2.

A discounted retail price could affect the expected demand and the variance of the demand. Generally, with discounting of the retail price, the expected demand and the variance will increase. We make the following assumptions concerning the model parameters:

$$c < v_0 + v_e$$
$$c < p < p_0 \tag{5.1}$$

These conditions rule out the cases where the contract cannot establish between the two parties. The first condition ensures that the manufacturers marginal profit is nonnegative. The second condition regulates the limits of the discounted retail price.

We compare two channel structures. In the Centralized System (CS), we assume that there is one decision-maker who controls the channel and hence makes decisions to optimize the total system profits. Observe that under this scenario, the option and the exercise prices play no role in determining the system profits. The only appropriate decision variables are the order quantity and the retail price decisions for the single period at the beginning of the season. The optimal solution to the CS is called as the first-best solution.

In the Decentralized System (DS), the retailer and the manufacturer play in a principal agent setting where the manufacturer is the leader and the retailer, the follower. This is reasonable whenever the manufacturer has more influence in the channel than the retailer. In this situation, for a given set of option prices announced by the manufacturer, the retailer places orders and determines the appropriate discounted retail price in the single period, which maximizes their expected profits. These orders act as implicit demand functions for the manufacturer. The CS is used as a benchmark case to investigate whether there exists an appropriate decentralization mechanism (through prices and/or quantities) such that the DS will achieve the first-best solution.

To establish a benchmark for the performance comparison, we analyze the centralized system and the details are in Appendix D.1. In CS, the expected profit function of the system and optimal order quantity are expressed as follows:

$$
\begin{aligned}
V_C(Q) &= p \cdot E[[X \wedge Q]] - h \cdot E[[Q - X]^+] - cQ \\
&= (p - c)Q - (p + h) \cdot \int_0^Q F_0(x)dx \\
&\text{where } [x]^+ = \max\{x, 0\}, [x \wedge y] = \min\{x, y\} \\
Q^* &= F_0^{-1}\left(\frac{p - c}{p + h}\right)
\end{aligned}
\tag{5.2}
$$

This reduces the problem to an optimization problem over the single variable p. Using the single variable optimal theory, we can find the optimal retail price p^* or demonstrate that $V_C(Q^*, p^*)$ is uni-modal or concave. The value for p^* can be found by using a one-dimensional search algorithm within a closed interval $[c, p_0]$ in this research.

To analyze the DS, first we need to develop the expressions for the expected profits of each system. In our setting, the retailer has a right to return the leftovers to the manufacturer without penalty. For clarity of exposition, we derive the individual profit function of the manufacturer and the retailer respectively in the DS as follows:

$$
\begin{aligned}
V_m(v_0, v_e) &= (v_0 - c)Q + v_e \cdot E[X \wedge Q]] - h \cdot E[[Q - X]^+] \\
&= (v_0 + v_e - c)Q - (v_e + h) \cdot \int_0^Q F(x)dx \\
V_r(Q, p) &= (p - v_e) \cdot E[[X \wedge Q]] - v_0 Q \\
&= (p - v_e - v_0)Q - (p - v_e) \cdot \int_0^Q F(x)dx
\end{aligned}
\tag{5.3}
$$

5.4 Decisions for a Supply Chain Coordination

We want to provide closed form solutions for the option prices, the order quantity, and the discounted retail price. We consider that demand distribution is influenced by the retail price. It demonstrates that the options redistribute the risk between two parties in shifting part of the retailer's risk due to demand uncertainty to the manufacturer. The manufacturer is compensated by the additional revenue obtained from the option contract. Any share of this profit improvement can be represented by an option contract through a suitable choice of the contract parameters. Under certain conditions, this profit sharing mechanism with an option contract will achieve the channel coordination by eliminating the double marginalization problem in the supply chain.

5.4.1 The Manufacturer's Optimal Option Pricing Decisions

In this study we provide an option contract to coordinate the decentralized supply chain with stochastic demand. By setting proper contract parameters, channel coordination is achieved. Moreover the manufacturer who shares the risk of demand uncertainty can be compensated by establishing favorable contract parameters in order to take a larger portion of the channel profit. The contractual arrangements we designed between the manufacturer and the retailer allow the decentralized supply chain to perform as well as a centralized one. In order to eliminate the double marginalization problem and to induce the perfect coordination, the relationship between the profit functions is established as follows:

$$
\begin{aligned}
V_r &= \gamma V_C \\
V_m &= (1 - \gamma)V_C
\end{aligned}
$$

(5.4)

where γ is a retailer's portion of the channel profit. Through derivations of Eq.(5.4) in Appendix D.3, the manufacturer should provide the explicit contract parameters as follows:

$$
\begin{aligned}
v_0 &= \gamma(h + c) \\
v_e &= (1 - \gamma)p^* - \gamma h
\end{aligned}
$$

$$(5.5)$$

The Eq.(5.5) represents the optimal option purchase and the strike price for the manufacturer. The result shows that the manufacturer's optimal option purchase price is not dependent on retail price while the optimal strike price is dictated by the optimal discounted retail price.

Furthermore, the result of Eq.(5.5) can support a division of the channel profit between the manufacturer and the retailer by varying the contract parameters (i.e., the option purchase or the strike price).

$$
\begin{aligned}
\gamma &= \frac{p^* - v_e}{p^* + h} \\
\gamma &= \frac{v_0}{h + c}
\end{aligned}
$$

$$(5.6)$$

5.4.2 The Retailer's Optimal Order Quantity and Discount Retail price

We attain the retailer's optimal order quantity and the discount retail price by solving the system optimal problem. Sometimes the retailer insists on no discounting in his retail price so as to keep the initial retail price p_0 to get maximum margin from each sale. With a contractual arrangement for coordination, the retailer could agree to a discount retail price to increase expected demand for the product to improve the profitability of the overall

supply chain. In that case, we assume that the demand (X) is a random variable, which depends on the discount retail price p. The expected value of X is a concave decreasing function in p where $c < p < p_0$.

We can express the expected demand (μ) with discount as $\mu_0 + \mu_0 \omega (p_0 - p)^\alpha$ where ω and α are empirically determined constants which indicate the effectiveness of the discount. A condition of $\omega = 0$ indicates that the expected demand is independent of the retail price. For $\omega > 0$, the larger the value of α, the more effective is the discounting. Three cases can be identified for the variance of X:

1. A retail price discount increases the mean demand but does not change the demand variance. Thus $\sigma^2 = \sigma_0^2$: This case will be referred to as the fixed variance case (FVC).

2. A retail price discount increases both the mean and the variance of demand in a proportional fashion. Thus, the coefficient of variation is a constant given by $CV = CV_0$: This case will be referred to as the fixed coefficient of variation case (FCVC). The derivation is in Appendix D.4.1.

3. A retail price discount increases the variance of demand at a faster rate than it increases the mean demand. Thus, CV is an increasing function of the $(p_0 - p)$: This case will be referred to as the increasing coefficient of variation case (ICVC). The ICVC reflects the stronger effects of other factors on the demand at a big discount. For example, the effects of unanticipated economic conditions or competitor's actions may lead to very large deviations from the expected demand at a big discount. The derivation is in Appendix D.4.2.

Fixed Variance Case (FVC)

We can express the expected demand and variance with discounted retail price as follows:

$$
\begin{aligned}
\mu &= \mu_0 + \mu_0\omega(p_0 - p)^\alpha \\
\sigma^2 &= \sigma_0^2 \\
&\text{where } \omega, \alpha > 0
\end{aligned}
\tag{5.7}
$$

The profit function and the optimal order quantity of a centralized system can be expressed as follows:

$$
\begin{aligned}
V_C(Q) &= (p - c)Q - (p + h) \cdot \int_0^Q F(x)dx \\
&= (p - c)Q - (p + h) \cdot \int_0^Q F_0(x - \mu_0\omega(p_0 - p)^\alpha)dx \\
Q^* &= F_0^{-1}(\frac{p - c}{p + h}) + \mu_0\omega(p_0 - p)^\alpha
\end{aligned}
\tag{5.8}
$$

For example, we apply a uniform distribution of demand on a centralized system with FVC and get an optimal order quantity solution as a consistent form with Eq.(5.8) above.

$$
\begin{aligned}
V_C(Q) &= (p - c)Q - (p + h) \cdot \int_0^Q \frac{x - \mu_0\omega(p_0 - p)^\alpha - a}{b - a}dx \\
\frac{\partial V_C(Q)}{\partial Q} &= (p - c) - \frac{p + h}{b - a}(Q - \mu_0\omega(p_0 - p)^\alpha - a) = 0 \\
Q^* &= \frac{p - c}{p + h}(b - a) + a + \mu_0\omega(p_0 - p)^\alpha
\end{aligned}
$$

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$$\begin{aligned} &= F_0^{-1}(\frac{p-c}{p+h}) + \mu_0\omega(p_0-p)^\alpha \\ &= Q_0^* + \mu_0\omega(p_0-p)^\alpha \end{aligned}$$

$$(5.9)$$

5.5 An Illustrative Example

In this section, the numerical example shows a mechanism of supply chain coordination with an option contract. The following example data is used for the analysis.

$$c = 55, h = 25, p_0 = 95, \gamma = 0.125, \mu_0 = 5,000, \sigma_0 = 1,000, \omega = 1.5, \alpha = 0 \qquad (5.10)$$

The illustration will consider a normal distribution of the demand. Based on the results shown in the following figures, we observe the following insights.

First, we consider the case that each party agrees to a total option price (i.e., $v_0 + v_e = 90$) and the retailer could decide his order quantity without discounting his retail price.

Observation 5.1 *According to Figure 5.2, optimal order quantity to maximize the supply chain profit is independent of the option prices between parties.*

Figure 5.2 shows us a consistency between Eq.(5.8), of the optimal order quantity and the result of this numerical analysis. To maximize the overall supply chain profit, the retailer should decide his order quantity based on Eq.(5.8). In these arbitrary combinations of option purchase price and strike price, the retailer's portion of the channel profit (γ) fluctuates between 1.2% to 13.9% even if the retailer does order the optimal order quantity for coordination. Therefore, the next question is what combination of option purchase price and strike price can lead to guarantee the negotiated retailer's portion of the channel profit (γ).
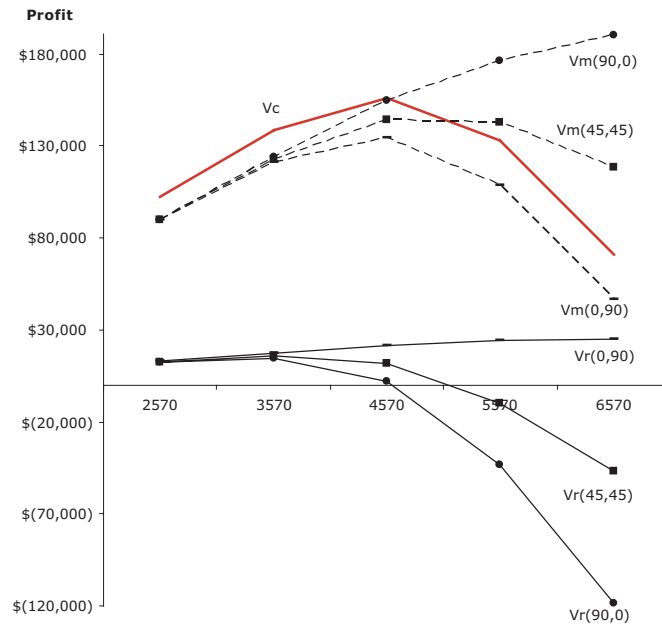
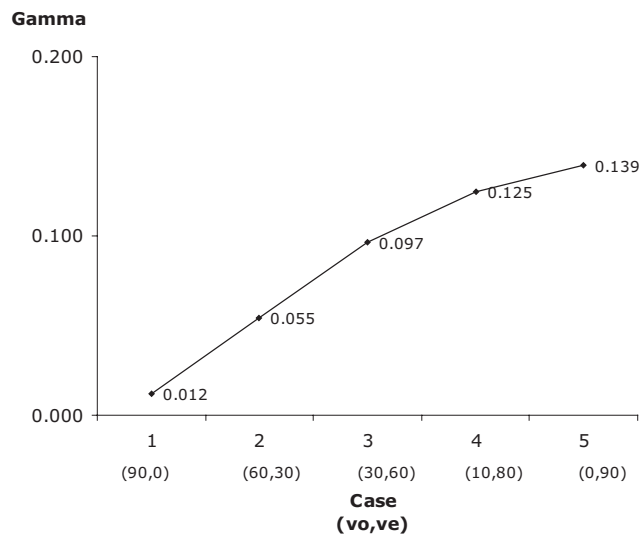Figure 5.2: Profits in the arbitrary option prices.
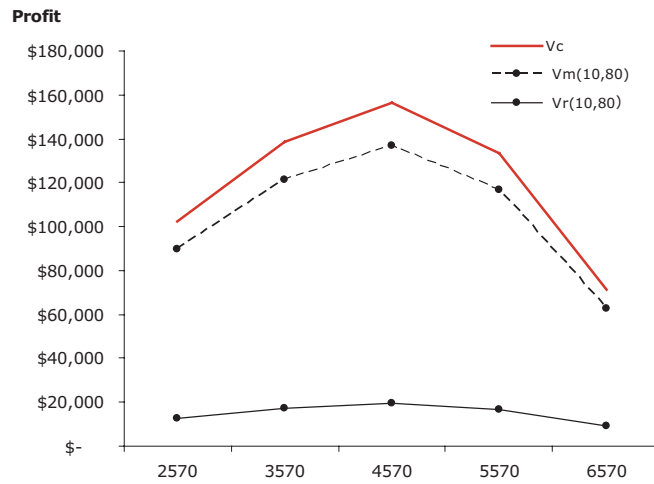

Figure 5.3: γ in the arbitrary option prices.

Figure 5.4: Profits in the optimal option price.
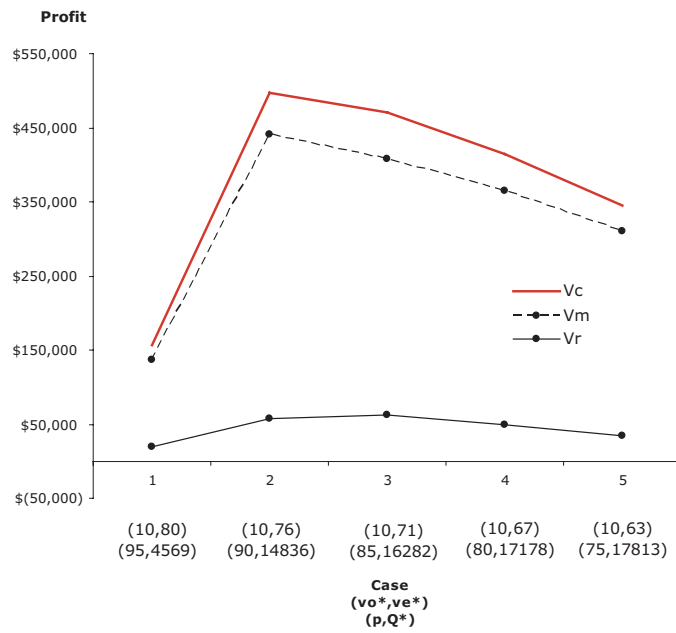
Figure 5.5: Profits in the discount retail prices in FVC.

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Observation 5.2 *According to Figure 5.3 , optimal combination of the option purchase price and the strike price (i.e., $v_0 = 10, v_e = 80$) from Eq.(5.5) guarantees the negotiated retailer's portion of the channel profit (0.125) when the retailer does order the optimal order quantity for coordination.*

The total option price (i.e., $v_0 + v_e = 90$) regulates a division of the channel profit between the manufacturer and the retailer. In this case, we get 12.5% of the retailer's portion of the channel profit (γ) based on the Eq.(5.6). On the other hand, when we know the retailer's portion of the channel profit (γ), we can get an optimal combination of the option purchase price and the strike price (i.e., $v_0 = 10, v_e = 80$) with Eq.(5.5) to coordinate the supply chain.

Observation 5.3 *According to Figure 5.4, with the optimal combination of the option purchase price and the strike price (i.e., $v_0 = 10, v_e = 80$), the negotiated retailer's portion of the channel profit (γ) is protected even if the retailer does not order the optimal order quantity for coordination.*

Second, we consider the discount retail price with FVC (Fixed Variance Case). The retailer decreases a retail price from the initial retail price p_0 to a discount price p . A retail price discount increases the mean demand but does not change the demand variance in FVC. A discount retail price affects the optimal order quantity and the profits of each party.

Observation 5.4 *According to Figure 5.5, we can get an optimal state of decision variables to maximize the profit of the centralized supply chain. In this example the optimal price is decided at the point of $5 discount from the initial retail price of $95 with optimal order quantity (14,836) and $v_0 = 10, v_e = 76$.*

It is rather difficult to get a closed form solution for the optimal discount retail price because it is complicated to derive analytically. With simulations and other tools, we can

attain the solutions. We show one solution with a numerical example. The result could be varied with changes of parameters (i.e., ω, α). We can verify our derivations of Appendix D.4 for the cases of FCVC and ICVC.

5.6 Conclusion

We develop an option contract to hedge the risk of the demand uncertainty. This option contract leads to an improved product availability by hedging the risk in the supply chain and to maximize the overall supply chain profits at the end of the season through coordination of the option prices and the order quantity. We consider the discount of retail price. In addition, we examine that the discounted retail price affects the expected product demand and the variance in several cases (i.e., FVC, FCVC, ICVC).

In this study we investigate the role of the option contract in a two-stage supply chain system. We use a single period model with and without discounting the retail price and the stochastic demand. We derive a closed form solution for the option purchase and strike price for the manufacturer and the optimal order quantity for the retailer. Furthermore, the analysis shows that the manufacturer's optimal option purchase price is independent of the retail price while the optimal strike price is dictated by the optimal discounted retail price.

Several extensions of our research could be investigated. Current models are too dependent on single shot contracting. Most supply chain interactions occur over long periods of time with many opportunities to renegotiate or to interact with the spot markets. More research is needed on how multiple suppliers compete for the affection of the multiple retailers, i.e., additional emphasis is needed on many-to-one or many-to-many supply chain structures.

Chapter 6

Conclusion

This research developed several analytical models for typical supply chain situations to help inventory decision-makers who need mathematical models to grasp the big picture of supply chain inventory problems before making executive decisions. Additionally, we derived closed form solutions for each model and found several managerial insights from our models through sensitivity analysis of numerical examples.

First, we developed a decision model considering customer impatience with stochastic demand and time-sensitive shortages. While the majority of the research literature considered time-dependent backlogging within the context of continuous review models with deterministic demand, this research studied time-dependent partial backlogging in the single period inventory problem with stochastic demand. We used concepts from utility theory to formally classify customers in terms of their willingness to wait for the supplier to replenish shortages. We conducted sensitivity analyses to determine the most and least opportune conditions for distinguishing between customer risk-behaviors. Our results suggested that the expected value of risk profile information (EVRPI) is most significant when the difference between the unit lost sales cost c_{LS} and the unit backorder cost c_B is large. Our results also indicated that EVRPI increases as the lost sales threshold M increases.

Second, we established an additional model to optimize backorder lead-time (response time) in a two-stage system with time-dependent partial backlogging and stochastic demand. We considered backorder cost as a function of response time while other literature regarded the backorder cost as a constant. A representative expected total cost function was derived and the closed form optimal solution was determined for a demand distribution. Our result showed that significant cost savings can be achieved by implementing the derived optimal solution as opposed to the prior model solution.

Third, we developed an inventory decision model applying an option-pricing framework (e.g., straddle) for determining order quantity under the situation of partial backlogging and uncertain demand. We compared the results between the traditional and option approaches with a numerical example. The result showed the advantages (i.e., total cost reduction) of an inventory decision with the real option framework under a highly uncertain demand situation. Specifically, the option approach provided decision-makers with positive optimal order quantity decisions even under the situation of high volatility of demand (σ_r) and low fill rate while the traditional approach gave them a negative optimal order quantity decision. We have shown that an option- pricing approach looks promising for determining inventory decisions under a highly uncertain demand situation as well.

Finally, we implemented an option contract to hedge the risk of the demand uncertainty. Our result showed that the option contract could lead to improvement of product availability by hedging the risk in the supply chain and maximizing overall supply chain profits at the end of the season through coordination of option prices and order quantity. We considered the discount of retail price. In addition, we examined the case where the discounted retail price affects the expected product demand and the variance in several cases (i.e., FVC, FCVC, ICVC). We derived closed form solutions of the appropriate option prices for the manufacturer as an incentive for the retailer to establish optimal pricing and order quantity decisions for coordinating the channel. Furthermore, the result showed that the manufacturer's optimal option purchase price is independent of retail price while the optimal strike price is dictated by the optimal discounted retail price.

This research gave decision-makers insights into how to implement the situation of demand uncertainty and shortage into a mathematical model in a two-stage supply chain and showed them what differences these proposed analytical models make as opposed to the traditional models. Even though each analytical model is simple but each provided an effective overall view of the supply chain system by abstracting the features of a supply chain system as a set of parameterized functions. We implemented time-sensitive shortages

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into an inventory model under emergency replenishment. In an effort to characterize diverse customer responses to shortages, we explored several types of decreasing backorder rate functions. In dealing with demand uncertainty, we adopted an option-pricing model. This showed that the tool of financial economics can be used to help make inventory stocking decisions. This research also developed an option contract in a newsvendor problem with price dependent stochastic demand and showed that the option contract could help coordination of a supply chain through improving the product availability and the overall supply chain profits.

This study can be extended in several ways. A useful extension would be to apply this approach to a multi-period problem. A multi-period problem is more likely to be applicable to real problems. Current models are too dependent on single shot contracting. Most supply chain interactions occur over long periods of time with many opportunities to renegotiate or to interact with the spot markets. More research is needed on how multiple suppliers compete for the affection of the multiple retailers (e.g., many-to-one or many-to-many structure).

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Appendices

<center>Appendix A</center>

<center>Modeling Customer Impatience in an Inventory Problem with Stochastic Demand and
Time-Sensitive Shortages</center>

A.1 Proof of Theorem 2.1

$TC(Q)$ given by Eq. (2.1) can be written as

$$
\begin{aligned}
TC(Q) &= c_O Q + c_H \int_0^Q (Q-x)f(x)\mathrm{d}x + c_{LS} \int_Q^\infty (x-Q)f(x)\mathrm{d}x \\
&\quad + (c_B - c_{LS}) \int_Q^{Q+M} (x-Q)\beta(x-Q)f(x)\mathrm{d}x.
\end{aligned} \tag{A.1}
$$

The first and second order derivatives of $TC(Q)$ are then

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} &= c_O + c_H \int_0^Q f(x)\mathrm{d}x - c_{LS} \int_Q^\infty f(x)\mathrm{d}x \\
&\quad + (c_B - c_{LS}) \int_Q^{Q+M} \left[-\beta(x-Q) + (x-Q)\beta'(x-Q) \right] f(x)\mathrm{d}x
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 TC(Q)}{\mathrm{d}Q^2} &= (c_H + c_{LS})f(Q) + (c_{LS} - c_B)f(Q) \\
&\quad + (c_{LS} - c_B) \int_Q^{Q+M} \left[2\beta'(x-Q) - (x-Q)\beta''(x-Q) \right] f(x)\mathrm{d}x.
\end{aligned} \tag{A.3}
$$

The first term in Eq.(A.3) is obviously nonnegative. Since $c_{LS} - c_B > 0$ based on Assumption 2.1, it follows that the second term is also nonnegative. Thus the third term is nonnegative if and only if the integrand is nonnegative. That is, the optimality condition holds if and only if

$$
2\beta'(x-Q) - (x-Q)\beta''(x-Q) \geq 0 \quad \text{for all} \ \ x \in [Q, Q+M). \tag{A.4}
$$

A.2 Proof of Theorem 2.3

For illustration, we will show that a decision-maker is risk-seeking if and only if his back order rate function is convex. The proofs of parts 1 and 2 of Theorem 2.3 are similar.

($\Rightarrow$) By definitions 2.3 and 2.4, we have $RP_W(L) > 0$ and $E_W[\mathcal{L}] - CE_W(\mathcal{L}) > 0$. Now $\beta(CE_W(\mathcal{L}))$ is the convex hull of $\{t_1, \ldots, t_n\}$, which can be expressed as $\beta(CE_W(\mathcal{L})) = $

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$p_1\beta(t_1) + \ldots + p_n\beta(t_n)$, where $p_1 + \cdots + p_n = 1$. Since $\beta(y)$ is a decreasing function by Property 2.1 and $E_W[\mathcal{L}] - CE_W(\mathcal{L}) > 0$, it follows that $\beta(E_W[\mathcal{L}]) - \beta(CE_W(\mathcal{L})) < 0$, or equivalently,

$$\beta(p_1t_1 + \cdots + p_nt_n) < p_1\beta(t_1) + \cdots + p_n\beta(t_n). \tag{A.5}$$

Since Eq. (A.5) holds, it follows that $\beta(y)$ is a convex function and $\beta''(y) > 0$ (see, for example, Jeter 1994 page 217).

$(\Leftarrow)$

Since $\beta(y)$ is a convex function, it follows that Eq. (A.5) holds by Proposition 2.1 shown in Jeter 1994 (page 217). Since $CE_W(\mathcal{L})$ is the convex hull of $\{t_1, \ldots, t_n\}$ and $p_1t_1 + \cdots + p_nt_n = E_W[\mathcal{L}]$, Eq. (A.5) becomes $\beta(E_W[\mathcal{L}]) - \beta(CE_W(\mathcal{L})) < 0$. Based on the last inequality, we know that $E_W[\mathcal{L}] - CE_W(\mathcal{L}) > 0$ by Property 2.1. Therefore by definitions 2.3 and 2.4, the latter inequality implies risk seeking.

A.3 Proof of Proposition 2.2

Since

$$\frac{\mathrm{d}\beta(x-Q)}{\mathrm{d}(x-Q)} = \begin{cases} -ae^{-a(x-Q)} < 0, & x \in [Q, Q+M) \\ 0, & x \in [Q+M, \infty) \end{cases}$$

Property 2.1 holds. Also,

$$\frac{\mathrm{d}^2\beta(x-Q)}{\mathrm{d}(x-Q)^2} = a^2e^{-a(x-Q)} > 0$$

shows that the third part of Theorem 2.3 holds. Now

$$\lim_{x-Q\to 0^+} \max\left\{e^{-a(x-Q)}, 0\right\} = 1$$

shows that Property 2.2 holds, and

$$\lim_{x-Q\to\infty} \max\left\{e^{-a(x-Q)}, 0\right\} = 0$$

shows that Property 2.3 holds.

A.4 Proof of Proposition 2.3

The derivative is computed as

$$\frac{\mathrm{d}\beta(x-Q)}{\mathrm{d}(x-Q)} = \begin{cases} -\frac{\pi}{2M}\sin\left(\frac{x-Q}{2M}\pi\right), & x \in [Q, Q+M) \\ 0, & x \in [Q+M, \infty) \end{cases}$$

If we let $z = (\pi/2M)(x-Q)$, then $\mathrm{d}\beta(x-Q)/\mathrm{d}(x-Q)$ becomes $-z(x-Q)\sin z$, where $z \in (0, \pi/2)$. Since $-z(x-Q) < 0$ and $\sin z > 0$ in this interval, we have $\mathrm{d}\beta(x-Q)/\mathrm{d}(x-Q) < 0$ which shows that Property 2.1 holds.Also,

$$\frac{\mathrm{d}^2\beta(x-Q)}{\mathrm{d}(x-Q)^2} = -\frac{\pi^2}{4M^2}cos(\frac{x-Q}{2M}\pi) < 0$$

shows that the first part of Theorem 2.3 holds. Now

$$\lim_{x-Q\to 0^+} \max\left\{\cos\left(\frac{x-Q}{2M}\pi\right), 0\right\} = 1$$

shows that Property 2.2 holds, and

$$\lim_{x-Q\to\infty} \max\left\{\cos\left(\frac{x-Q}{2M}\pi\right), 0\right\} = \lim_{x-Q\to M}\cos\left(\frac{x-Q}{2M}\pi\right) = 0$$

shows that Property 2.3 holds.

Appendix B

Optimizing Backorder Lead-Time and Order Quantity in a Supply Chain
with Partial backlogging and Stochastic Demand

B.1 Derivations of Optimality for the Proposed Model

B.1.1 Optimal Order Quantity

We can get a feasible point, a unique minimizer, at which the first-order conditions hold $\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}Q} = 0$.

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} &= c_O + c_H \int_0^Q f(x)\mathrm{d}x - c_{LS} \int_Q^\infty f(x)\mathrm{d}x + (c_{LS} - c_B(t)) \int_Q^\infty \beta(t)f(x)\mathrm{d}x = 0 \\
&= c_O + c_H F(Q) - c_{LS}(1 - F(Q)) + (c_{LS} - c_B(t))\beta(t)(1 - F(Q)) = 0
\end{aligned}
$$

$$
\begin{aligned}
F(Q^*) &= \frac{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - c_O}{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} + c_H} \\
Q^* &= F^{-1}\left[\frac{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - c_O}{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} + c_H} \right]
\end{aligned}
\qquad (\text{B.1})
$$

B.1.2 Sufficient Condition for Convexity and Optimal Response Time

The first and second order derivatives of $TC(Q,t)$ are then

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}t} &= \{c_B'(t)\beta(t) + c_B(t)\beta'(t) - c_{LS}\beta'(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x \\
\frac{\mathrm{d}^2 TC(Q,t)}{\mathrm{d}t^2} &= \{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) + c_B(t)\beta''(t) - c_{LS}\beta''(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x \\
&= \{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) + (c_B(t) - c_{LS})\beta''(t)\} \int_Q^\infty (x - Q)f(x)\mathrm{d}x
\end{aligned}
$$

$$(\text{B.2})$$

Clearly the second part of Eq.(B.2) is non-negative if the condition of Eq.(B.3) is satisfied.

$$
c_B''(t)\beta(t) + 2c_B'(t)\beta'(t) \quad + \quad (c_B(t) - c_{LS})\beta''(t) \geq 0
$$

$$c_{LS} - c_B(t) \leq \frac{c_B''(t)\beta(t) + 2c_B'(t)\beta'(t)}{\beta''(t)}$$

$$c_{LS} - c_B(t) \leq \frac{b^2 N e^{-bt} e^{-at} + 2b N e^{-bt} a e^{-at}}{a^2 e^{-at}}$$

$$where \quad c_B(t) = N e^{-bt}, \beta(t) = e^{-at}$$

$$c_{LS} - c_B(t) \leq \frac{2ab + b^2}{a^2} N e^{-bt}$$

$$c_{LS} - c_B(t) \leq \frac{2ab + b^2}{a^2} c_B(t)$$

$$c_{LS} \leq \left(\frac{a+b}{a}\right)^2 c_B(t)$$

$$(B.3)$$

Thus, $TC(Q,t)$ is convex in t for any fixed Q where the condition of Eq.(B.3) is satisfied. The optimal value t^* corresponding to a fixed value Q is obtained by setting the first derivative part of Eq.(B.2) equal to zero and solving for t, which yields Eq.(B.4) The value t^* is unique because Eq.(B.2) shows TC(Q, t) is strictly convex in t for any fixed Q.

$$c_B'(t)\beta(t) = -c_B(t)\beta'(t) + c_{LS}\beta'(t)$$

$$\beta(t) = \frac{\{c_{LS} - c_B(t)\}\beta'(t)}{c_B'(t)}$$

$$e^{-at} = \frac{-\{c_{LS} - N e^{-bt}\} a e^{-at}}{-b N e^{-bt}}$$

$$b N e^{-bt} = a\{c_{LS} - N e^{-bt}\}$$

$$(a+b)N e^{-bt} = a c_{LS}$$

$$e^{-bt} = \frac{a c_{LS}}{(a+b)N}$$

$$-bt = ln\frac{a c_{LS}}{(a+b)N}$$

$$t^* = ln\left(\frac{(a+b)N}{a \cdot c_{LS}}\right) b^{-1} \qquad (B.4)$$

B.1.3   Relationship between Optimal Backorder cost and Lost sales cost

$$c_B(t^*) = N e^{-bt^*}$$

$$= N e^{-b \cdot b^{-1} ln\left(\frac{(a+b)N}{a \cdot c_{LS}}\right)}$$

$$
\begin{aligned}
&= Ne^{ln\left(\frac{a \cdot c_{LS}}{(a+b)N}\right)} \\
&= N\frac{a \cdot c_{LS}}{(a+b)N} \\
&= \frac{a \cdot c_{LS}}{(a+b)}
\end{aligned}
$$

(B.5)

B.2 Derivations of Optimality for the Proposed Model with Backorder Setup Cost

B.2.1 Optimal Order Quantity

We can get a feasible point, a unique minimizer, at which the first-order conditions hold $\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}Q} = 0$.

$$
\frac{\mathrm{d}TC(Q)}{\mathrm{d}Q} = c_O + c_H F(Q) - \left[c_B(t)\beta(t) + c_{LS}(1 - \beta(t)) - \alpha\beta^2(t)\right](1 - F(Q)) = 0
$$

$$
\begin{aligned}
F(Q^*) &= \frac{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - \alpha\beta^2(t) - c_O}{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - \alpha\beta^2(t) + c_H} \\
Q^* &= F^{-1}\left[\frac{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - \alpha\beta^2(t) - c_O}{\beta(t)c_B(t) + (1 - \beta(t))c_{LS} - \alpha\beta^2(t) + c_H}\right]
\end{aligned}
$$

(B.6)

B.2.2 Sufficient Condition for Convexity and Optimal Response Time

The first and second order derivatives of $TC(Q,t)$ are then

$$
\begin{aligned}
\frac{\mathrm{d}TC(Q,t)}{\mathrm{d}t} &= \{c'_B(t)\beta(t) + c_B(t)\beta'(t) - c_{LS}\beta'(t) + \alpha\beta'(t)\}\int_Q^\infty (x - Q)f(x)\mathrm{d}x \\
\frac{\mathrm{d}^2TC(Q,t)}{\mathrm{d}t^2} &= \{c''_B(t)\beta(t) + 2c'_B(t)\beta'(t) + (c_B(t) - c_{LS} + \alpha)\beta''(t)\}\int_Q^\infty (x - Q)f(x)\mathrm{d}x
\end{aligned}
$$

(B.7)

Clearly the second part of Eq.(B.7) is non-negative if the condition of Eq.(B.8) is satisfied.

$$
\begin{aligned}
c''_B(t)\beta(t) + 2c'_B(t)\beta'(t) \quad + \quad & (c_B(t) - c_{LS} + \alpha)\beta''(t) \geq 0 \\
c_{LS} - c_B(t) \quad \leq \quad & \frac{c''_B(t)\beta(t) + 2c'_B(t)\beta'(t)}{\beta''(t)} + \alpha
\end{aligned}
$$

$$c_{LS} - c_B(t) \leq \frac{2ab + b^2}{a^2} Ne^{-bt} + \alpha$$

$$c_{LS} \leq \left(\frac{a+b}{a}\right)^2 c_B(t) + \alpha$$

(B.8)

Thus, $TC(Q, t)$ is convex in t for any fixed Q where the condition of Eq.(B.8) is satisfied, and the optimal value t^* corresponding to a fixed value Q is obtained by setting the first part of Eq.(B.7) equal to zero and solving for t, which yields Eq.(B.9) The value t^* is unique because Eq.(B.7) shows $TC(Q, t)$ is strictly convex in t for any fixed Q.

$$c_B'(t)\beta(t) = -c_B(t)\beta'(t) + c_{LS}\beta'(t) - \alpha\beta'(t)$$

$$\beta(t) = \frac{\{c_{LS} - c_B(t) - \alpha\}\beta'(t)}{c_B'(t)}$$

$$t^* = ln\left(\frac{(a+b)N}{a \cdot (c_{LS} - \alpha)}\right) b^{-1}$$

(B.9)

B.2.3 Relationship between Optimal Backorder cost and Lost sales cost

$$c_B(t^*) = Ne^{-b \cdot b^{-1} ln\left(\frac{(a+b)N}{a \cdot (c_{LS} - \alpha)}\right)}$$

$$= Ne^{ln\left(\frac{a \cdot (c_{LS} - \alpha)}{(a+b)N}\right)}$$

$$= N\frac{a \cdot (c_{LS} - \alpha)}{(a+b)N}$$

$$= \frac{a \cdot (c_{LS} - \alpha)}{(a+b)}$$

(B.10)

Appendix C
The Effects of an Option Approach to Stochastic Inventory Decisions

C.1 Derivations of Optimality for the Partial Back logging Newsvendor Approach

We need to check the total cost function is convex function or not. The convexity of the total cost function is insured since second derivation is always positive as all terms are non negatives.

$$\frac{\mathrm{d}E[TC(Q)]}{\mathrm{d}Q} = c_O + c_H \int_0^Q f(x)\mathrm{d}x - c_{LS} \int_Q^\infty f(x)\mathrm{d}x + (c_{LS} - c_B) \int_Q^\infty \beta f(x)\mathrm{d}x$$

$$\frac{\mathrm{d}^2 E[TC(Q)]}{\mathrm{d}Q^2} = \{c_H + \beta c_B + (1-\beta)c_{LS}\}f(Q) \tag{C.1}$$

We have a convex objective function and we can get a feasible point, unique minimizer, at which the first-order conditions hold $\frac{\mathrm{d}E[TC(Q)]}{\mathrm{d}Q} = 0$.

$$\frac{\mathrm{d}E[TC(Q)]}{\mathrm{d}Q} = c_O + c_H \int_0^Q f(x)\mathrm{d}x - c_{LS} \int_Q^\infty f(x)\mathrm{d}x + (c_{LS} - c_B) \int_Q^\infty \beta f(x)\mathrm{d}x = 0$$

$$= c_O + c_H F(Q) - c_{LS}(1 - F(Q)) + (c_{LS} - c_B)\beta(1 - F(Q)) = 0$$

$$F(Q) = \frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}$$

$$Q^* = F^{-1}[\frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}] \tag{C.2}$$

$\frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}$ is an optimal over stocking probability and then optimal stock out probability is equal to $\{1 - \frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}\}$ and that we can represent stock out probability is following.

$$P_s^* = \frac{c_O + c_H}{\beta c_B + (1-\beta)c_{LS} + c_H} \tag{C.3}$$

C.2 Derivation of Total Cost Function with Real Option Framework

Eq.(4.5) can be rearranged as follows:

$$TC(Q) = \begin{cases} \{c_B\beta + (1-\beta)c_{LS}\}D - \{c_B\beta + (1-\beta)c_{LS} - c_O\}Q & \text{if } D \geq Q \\ (c_O + c_H)Q - c_H D & \text{if } D < Q \end{cases} \tag{C.4}$$

Then, Based on Eq.(C.4), the present total cost of cash flows can be expressed as Eq.(4.6).

$$
\begin{aligned}
E[TC(Q)] \;=\; & e^{-r_f T} \cdot E[\{c_B\beta + (1-\beta)c_{LS}\}D \\
& - \{c_B\beta + (1-\beta)c_{LS} - c_O\}Q|D \geq Q] \cdot P(D \geq Q) \\
& + e^{-r_f T} \cdot E[(c_O + c_H)Q - c_H D|D < Q] \cdot P(D < Q)
\end{aligned}
\tag{C.5}
$$

Eq.(4.8) can be derived as following

$$
\begin{aligned}
E[TC_a(Q)] \;=\; & e^{-r_f T} \cdot E[\{c_B\beta + (1-\beta)C_{LS}\}D|D \geq Q] \cdot P(D \geq Q)] \\
& - e^{-r_f T} \cdot E[\{c_B\beta + (1-\beta)C_{LS} - c_O\}Q|D \geq Q] \cdot P(D \geq Q) \\
\;=\; & e^{-r_f T} \cdot D_0 \cdot \{c_B\beta + (1-\beta)c_{LS}\} \int_{ln(\frac{Q}{D_0})}^{\infty} rf(r)\mathrm{d}r \\
& - e^{-r_f T} \cdot D_0 \cdot \{c_B\beta + (1-\beta)c_{LS} - c_O\}\frac{Q}{D_0} \int_{ln(\frac{Q}{D_0})}^{\infty} f(r)\mathrm{d}r \\
\;=\; & e^{-r_f T} \cdot D_0 \cdot \{c_B\beta + (1-\beta)c_{LS}\}e^{r_f T}N(d_1) \\
& - e^{-r_f T} \cdot Q\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2) \\
\;=\; & D_0\{c_B\beta + (1-\beta)c_{LS}\}N(d_1) - e^{-r_f T} \cdot Q\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2) \\
& \text{where } \; d_1 = \frac{ln(\frac{D_0}{Q}) + (r_f + \frac{\sigma_r^2}{2})T}{\sigma_r \sqrt{T}}, \quad d_2 = \frac{ln(\frac{D_0}{Q}) + (r_f - \frac{\sigma_r^2}{2})T}{\sigma_r \sqrt{T}}
\end{aligned}
\tag{C.6}
$$

Eq.(4.9) can be derived as following

$$
\begin{aligned}
E[TC_b(Q)] \;=\; & e^{-r_f T} \cdot [E[(c_O + c_H)Q|D < Q]P(D < Q)] - E[\{c_H D|D < Q] \cdot P(D < Q)] \\
\;=\; & e^{-r_f T} \cdot D_0(c_O + c_H)E[\frac{Q}{D_0}|ln(\frac{D}{D_0}) < ln(\frac{Q}{D_0})]P(ln(\frac{D}{D_0}) < ln(\frac{Q}{D_0})) \\
& - e^{-r_f T} \cdot D_0 c_H E[ln(\frac{D}{D_0})|ln(\frac{D}{D_0}) < ln(\frac{Q}{D_0})]P(ln(\frac{D}{D_0}) < ln(\frac{Q}{D_0})) \\
\;=\; & e^{-r_f T} \cdot Q(c_O + c_H) \int_{0}^{ln(\frac{Q}{D_0})} f(r)\mathrm{d}r - e^{-r_f T} \cdot D_0 c_H \int_{0}^{ln(\frac{Q}{D_0})} rf(r)\mathrm{d}r \\
\;=\; & e^{-r_f T} \cdot Q(c_O + c_H)N(-d_2) - e^{-r_f T} \cdot D_0 \cdot c_H e^{r_f T}N(-d_1) \\
\;=\; & e^{-r_f T} \cdot Q(c_O + c_H)N(-d_2) - D_0 \cdot c_H N(-d_1) \\
& \text{where } \; d_1 = \frac{ln(\frac{D_0}{Q}) + (r_f + \frac{\sigma_r^2}{2})T}{\sigma_r \sqrt{T}}, \quad d_2 = \frac{ln(\frac{D_0}{Q}) + (r_f - \frac{\sigma_r^2}{2})T}{\sigma_r \sqrt{T}}
\end{aligned}
\tag{C.7}
$$

C.3 Derivation of Optimal Fill rate and Order Quantity

Eq.(4.14) can be derived as following.

$$
\begin{aligned}
\frac{d[E[TC(Q)]]}{dQ} &= e^{-r_f T}(c_O + c_H)N(-d_2) - e^{-r_f T}\{c_B\beta + (1-\beta)c_{LS} - c_O\}N(d_2) = 0 \\
&\quad e^{-r_f T}[(c_O + c_H) - \{c_H + c_B\beta + (1-\beta)c_{LS}\}N(d_2)] = 0 \\
&\quad \{c_H + c_B\beta + (1-\beta)c_{LS}\}N(d_2) = c_O + c_H \\
&\quad N(d_2) = \frac{c_O + c_H}{\{c_B\beta + (1-\beta)c_{LS} + c_H\}} \\
&\quad N(-d_2) = \frac{\beta c_B + (1-\beta)c_{LS} - c_O}{\beta c_B + (1-\beta)c_{LS} + c_H}
\end{aligned}
$$

$$(C.8)$$

Eq.(4.15) is derived as following.

$$
\begin{aligned}
\frac{d[E[TC(Q)]]^2}{d^2Q} &= \frac{d}{dQ}[-e^{-r_f T}[(c_O + c_H) - \{c_H + c_B\beta + (1-\beta)c_{LS}\}N(d_2)]] \\
&= [-e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}\frac{d[N(d_2)]}{dQ} \\
&= [-e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}\frac{d}{dQ}[N\{\frac{ln(\frac{D_0}{Q}) + (r_f - \frac{\sigma_r^2}{2})T}{\sigma_r\sqrt{T}}\} \\
&= [-e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}f(d_2)\frac{d}{dQ}[\frac{ln(\frac{D_0}{Q})}{\sigma_r\sqrt{T}}] \\
&= [-e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}f(d_2)\frac{d}{dQ}[\frac{-lnQ}{\sigma_r\sqrt{T}}] \\
&= [-e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}f(d_2)\{-(Q\sigma_r\sqrt{T})^{-1}\} \\
&= [e^{-r_f T}\{c_H + c_B\beta + (1-\beta)c_{LS}\}(Q\sigma_r\sqrt{T})^{-1}f(d_2)
\end{aligned}
$$

$$(C.9)$$

Eq.(4.16) is derived as following.

$$
\begin{aligned}
d_2 &= N^{-1}[\frac{c_O + c_H}{\{c_B\beta + (1-\beta)c_{LS} + c_H\}}] \\
\frac{ln(\frac{D_0}{Q}) + (r_f - \frac{\sigma_r^2}{2})T}{\sigma_r\sqrt{T}} &= N^{-1}[\frac{c_O + c_H}{c_B\beta + (1-\beta)c_{LS} + c_H}] \\
ln(\frac{D_0}{Q}) &= -(r_f - \frac{\sigma_r^2}{2})T + \sigma_r\sqrt{T}N^{-1}[\frac{c_O + c_H}{c_B\beta + (1-\beta)c_{LS} + c_H}
\end{aligned}
$$

$$\frac{D_0}{Q} = Exp[-(r_f - \frac{\sigma_r^2}{2})T + \sigma_r\sqrt{T}N^{-1}[\frac{c_O + c_H}{c_B\beta + (1-\beta)c_{LS} + c_H})]]$$

$$Q^* = \frac{D_0 e^{(r_f - \frac{\sigma_r^2}{2})T}}{e^{\sigma_r\sqrt{T}N^{-1}[\frac{c_O + c_H}{c_B\beta + (1-\beta)c_{LS} + c_H}]}} \tag{C.10}$$

Coordinating a Two-stage Supply Chain Based on Option Contract

D.1 Derivations of Profit Function for CS

$$
\begin{aligned}
V_C(Q) &= p \cdot E[[X \wedge Q]] - h \cdot E[[Q - X]^+] - cQ \\
&= p \cdot E[Q - [Q - X]^+] - h \cdot E[[Q - X]^+] - cQ \\
&= pQ - (p + h) \cdot E[[Q - X]^+] - cQ \\
&= (p - c)Q - (p + h) \cdot E[[Q - X]^+] \\
&= (p - c)Q - (p + h) \cdot \int_0^Q F_0(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \ \ E[[Q - X]^+] &= E[\max[Q - X, 0]] \\
&= E[-[X - Q \wedge 0]] \\
&= E[-[X \wedge Q] + Q] \\
&= E[-[X \wedge Q]] + Q \\
&= -E[X \wedge Q] + Q \\
&= -E[Q - [Q - X]^+] + Q \\
&= -Q + E[[Q - X]^+] + Q \\
&= -Q + \int_0^Q (Q - x)f_0(x)dx + Q \\
&= -\left[Q - \int_0^Q F_0(x)dx \right] + Q \\
&= -\left[\int_0^Q 1dx - \int_0^Q F_0(x)dx \right] + Q \\
&= -\left[\int_0^Q 1 - F_0(x)dx \right] + Q \\
&= \int_0^Q F_0(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_C(Q)}{\partial Q} &= (p - c) - (p + h)F_0(Q) = 0 \\
Q_0^* &= F_0^{-1}\left(\frac{p - c}{p + h} \right)
\end{aligned}
$$

$$\text{(D.1)}$$

D.2   Derivation of Profit Functions for DS

$$
\begin{aligned}
V_r(Q, p) &= (p - v_e) \cdot E[[X \wedge Q]] - v_0 Q \\
&= (p - v_e) \cdot E[Q - [Q - X]^+] - v_0 Q \\
&= (p - v_e - v_0)Q - (p - v_e) \cdot E[[Q - X]^+] \\
&= (p - v_e - v_0)Q - (p - v_e) \cdot \int_0^Q (Q - x)f(x)dx \\
&= (p - v_e - v_0)Q - (p - v_e) \cdot \int_0^Q F(x)dx
\end{aligned}
$$

(D.2)

$$
\begin{aligned}
V_m(v_0, v_e) &= (v_0 - c)Q + v_e \cdot E[X \wedge Q]] - h \cdot E[[Q - X]^+] \\
&= (v_0 - c)Q + v_e \cdot E[Q - [Q - X]^+] - h \cdot E[[Q - X]^+] \\
&= (v_0 - c)Q + v_e Q - v_e \cdot E[[Q - X]^+] - h \cdot E[[Q - X]^+] \\
&= (v_0 + v_e - c)Q - (v_e + h) \cdot E[[Q - X]^+] \\
&= (v_0 + v_e - c)Q - (v_e + h) \cdot \int_0^Q (Q - x)f(x)dx \\
&= (v_0 + v_e - c)Q - (v_e + h) \cdot \int_0^Q F(x)dx
\end{aligned}
$$

(D.3)

D.3   Derivation of Optimal Option Prices

$$
V_r = \gamma V_C
$$

$$
(p - v_e - v_0)Q - (p - v_e) \cdot \int_0^Q F(x)dx = \gamma \left[(p - c)Q - (p + h) \cdot \int_0^Q F_0(x)dx \right]
$$

$$
\{p - v_e - v_0 - \gamma(p - c)\}Q + \{-(p - v_e) + \gamma(p + h)\} \cdot \int_0^Q F_0(x)dx = 0
$$

$$
\begin{aligned}
p - v_e - v_0 - \gamma(p - c) &= 0 \\
(p - v_e) - \gamma(p + h) &= 0
\end{aligned}
$$

$$\begin{aligned}
v_e &= p - \gamma(p+h) \\
v_0 &= (p - v_e) - \gamma(p - c) \\
&= p - \{p - \gamma(p+h)\} - \gamma(p - c) \\
&= \gamma(h + c)
\end{aligned}$$

$$\text{(D.4)}$$

$$V_m = (1 - \gamma)V_C$$

$$(v_0 + v_e - c)Q - (v_e + h) \cdot \int_0^Q F(x)dx = (1 - \gamma)\left[(p - c)Q - (p + h) \cdot \int_0^Q F_0(x)dx\right]$$

$$(v_0 + v_e - p + \gamma p - \gamma c)Q - (v_e - p + \gamma p + \gamma h) \cdot \int_0^Q F(x)dx = 0$$

$$\begin{aligned}
v_0 + v_e - p + \gamma p - \gamma c &= 0 \\
v_e - p + \gamma p + \gamma h &= 0 \\
v_e &= p - \gamma(p+h) \\
v_0 &= (p - v_e) - \gamma(p - c) \\
&= p - \{p - \gamma(p+h)\} - \gamma(p - c) \\
&= \gamma(h + c)
\end{aligned}$$

$$\text{(D.5)}$$

D.4 Derivation of Optimal Order Quantity with Discounting

D.4.1 Fixed Coefficient of Variance Case (FCVC)

$$CV(X) = CV(X_0) = \frac{\sigma_0}{\mu_0} = \frac{\sigma}{\mu}$$
$$\mu = \mu_0 + \mu_0 \omega (p_0 - p)^\alpha$$
$$\sigma = \sigma_0 + \sigma_0 \omega (p_0 - p)^\alpha$$

Uniform Distribution of Demand Case:

$$V_C(Q) = (p - c)Q - (p + h) \cdot \int_0^Q \frac{x - a}{b - a} dx$$

$$
\begin{aligned}
&= (p-c)Q - \frac{(p+h)}{b-a} \int_0^Q (x-a)dx \\
\frac{\partial V_C(Q)}{\partial Q} &= (p-c) - \frac{p+h}{b-a}(Q-a) = 0 \\
Q^* &= \frac{p-c}{p+h}(b-a) + a \\
&= \frac{p-c}{p+h}(b_0 - a_0)(1 + \omega(p_0 - p)^\alpha) + a_0(1 + \omega(p_0 - p)^\alpha) \\
&= \left(\frac{p-c}{p+h}(b_0 - a_0) + a_0 \right)(1 + \omega(p_0 - p)^\alpha) \\
&= F_0^{-1}(\frac{p-c}{p+h})(1 + \omega(p_0 - p)^\alpha) \\
&= Q_0^*(1 + \omega(p_0 - p)^\alpha)
\end{aligned}
$$

$$(\text{D.6})$$

Exponential Distribution of Demand Case:

$$
\begin{aligned}
V_C(Q) &= (p-c)Q - (p+h) \cdot \int_0^Q 1 - e^{-\frac{1}{\mu}x}dx \\
\frac{\partial V_C(Q)}{\partial Q} &= (p-c) - (p+h)(1 - e^{-\frac{1}{\mu}Q}) = 0 \\
e^{-\frac{1}{\mu}Q} &= 1 - \frac{p-c}{p+h} \\
-\frac{1}{\mu}Q &= \ln\left(1 - \frac{p-c}{p+h}\right) \\
Q^* &= -\mu \ln\left(\frac{h+c}{p+h}\right) \\
&= -\mu_0(1 + \omega(p_0 - p)^\alpha)\ln\left(\frac{h+c}{p+h}\right) \\
&= -\mu_0 \ln\left(\frac{h+c}{p+h}\right)(1 + \omega(p_0 - p)^\alpha) \\
&= F_0^{-1}\left(\frac{p-c}{p+h}\right)(1 + \omega(p_0 - p)^\alpha) \\
&= Q_0^* \cdot (1 + \omega(p_0 - p)^\alpha)
\end{aligned}
$$

$$(\text{D.7})$$

D.4.2 Increasing Coefficient of Variance Case (ICVC)

$$CV(X) = (1 + \delta p^\theta) CV(X_0)$$

$$\mu = \mu_0 (1 + \omega(p_0 - p)^\alpha)$$
$$\frac{\sigma}{\mu} = (1 + \delta(p_0 - p)^\theta) \frac{\sigma_0}{\mu_0}$$

Uniform Distribution of Demand Case:

$$\frac{b + a}{2} = \frac{b_0 + a_0}{2}(1 + \omega(p_0 - p)^\alpha)$$
$$\frac{b - a}{b + a} = (1 + \delta(p_0 - p)^\theta)\frac{b_0 - a_0}{b_0 + a_0}$$

$$a = (1 + \omega(p_0 - p)^\alpha)\left\{a_0 - \frac{\delta(p_0 - p)^\theta}{2}(b_0 - a_0)\right\}$$
$$b = (1 + \omega(p_0 - p)^\alpha)\left\{b_0 + \frac{\delta(p_0 - p)^\theta}{2}(b_0 - a_0)\right\}$$

$$
\begin{aligned}
V_C(Q) &= (p - c)Q - (p + h) \cdot \int_0^Q \frac{x - a}{b - a}dx \\
&= (p - c)Q - \frac{(p + h)}{b - a}\int_0^Q (x - a)dx \\
\frac{\partial V_C(Q)}{\partial Q} &= (p - c) - \frac{p + h}{b - a}(Q - a) = 0 \\
Q^* &= \frac{p - c}{p + h}(b - a) + a \\
&= \frac{p - c}{p + h}(1 + \omega(p_0 - p)^\alpha)\left\{(b_0 - a_0) + \delta(p_0 - p)^\theta(b_0 - a_0)\right\} \\
&\quad + (1 + \omega(p_0 - p)^\alpha)\left\{a_0 - \frac{\delta(p_0 - p)^\theta}{2}(b_0 - a_0)\right\} \\
&= (1 + \omega(p_0 - p)^\alpha)\left[Q_0^* + \delta(p_0 - p)^\theta(b_0 - a_0)\left\{\frac{p - c}{p + h} - \frac{1}{2}\right\}\right]
\end{aligned}
$$

$$\text{(D.8)}$$

Exponential Distribution of Demand Case:

Exponential distribution is not applicable as its CV is always equal to one, which is valid only for the FCVC.