On the Conjugacy Theorems of Cartan and Borel Subalgebras

by

Mary Clair Thompson

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Approved by:

Tin-Yau Tam, Chair, Professor of Mathematics
Randall R. Holmes, Professor of Mathematics
Huajun Huang, Assistant Professor of Mathematics

Abstract

We study the conjugacy theorems of Cartan subalgebras and Borel subalgebras of general Lie algebras. We present a history of the problem, along with two proofs of the theorems which stay completely within the realm of Lie algebras. The first is a reworking by Humphreys of an earlier proof, relying upon the ideas of Borel subalgebras and using double induction. The second proof is a newer proof presented by Michael which substantially simplifies the theory.

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Chapter 1

Introduction

A Cartan subalgebra of a Lie algebra L over the field $\mathbb{F}$, often abbreviated CSA, is a subalgebra H of L that is (1) nilpotent and (2) self-normalizing. For example, any nilpotent Lie algebra is its own Cartan subalgebra.

Any nilpotent subalgebra is also solvable, thus contained in a maximal solvable subalgebra of L. Such subalgebras are of sufficient importance to merit their own name, Borel subalgebras, and are often denoted BSAs.

In a Lie algebra over an algebraically closed field of characteristic 0, it is a truly remarkable fact that any pair of CSAs are conjugate, in the sense that the subgroup $\mathcal{E}(L)$ of $\operatorname{Int} L$ acts transitively on the set of CSAs. Indeed the BSAs are also conjugate under the same set of automorphisms. This paper will examine two different elementary proofs of the conjugacy theorems.

CSAs over $\mathbb{C}$ were first introduced by Élie Cartan in his 1894 doctoral dissertation [8] in order to better study complex semisimple Lie algebras. His work was a major contribution to Lie algebras in that Cartan completed the classification of the complex semisimple algebras which Wilhelm Killing had begun. CSAs play an important role in the structure theory of semisimple Lie algebras. Due to their importance and to Cartan's contribution to the theory, Chevalley [9] proposed naming them Cartan subalgebras.

If the field is algebraically closed and of characteristic 0 and the algebra is finite dimensional, Chevalley [9] proved that two CSAs of L are conjugate via $\operatorname{Int} L$. Also see the comments of Borel [4, p.148] on the history of the development. The special case where L is semisimple had already been proved previously by Weyl [23] (also see Hunt [12] for a metric proof) using analytic methods ($\mathbb{F} = \mathbb{C}$) and by Weil [22] by topological methods (also see

Hopf and Samelson [10]). However, Chevalley assumed that more generally the field is algebraically closed and has characteristic zero so that he did not have a ready-made analogue of the adjoint group $\operatorname{Int} L$ to perform the conjugacy [4, p.148]. Chevalley's proof uses the methods of algebraic geometry. Of particular importance in the proof is the use of Plücker coordinates.

Winter [25], based on the techniques developed by Mostow, gave an elementary algebraic (non-geometric) proof of the conjugacy theorem. The proof presented in Humphreys' book [11] follows the approach of Winter [25].

Michael [16] gave a new elementary proof for the conjugacy theorem of CSAs of a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. The approach fits into the theme of the presentation in Bourbaki [7].

Humphreys begins his proof of the conjugacy of CSAs by proving the theorem directly for solvable Lie algebras. However, his focus soon shifts to BSAs, and after establishing several properties of BSAs and providing connections between the mechanics of BSAs and CSAs, he proceeds to prove that BSAs of a semisimple Lie algebra are conjugate. The proof is highly technical and employs double induction.

Humphreys then shows that the BSAs of a general Lie algebra L are in 1-1 correspondence with the BSAs of the semisimple Lie algebra L/R. As the conjugacy of BSAs of semisimple algebras has already been established, the conjugacy in a general algebra follows readily. Humphreys finally returns his attention to CSAs; since any CSA is contained in a BSA, the conjugacy of CSA is almost immediate.

The proof contributed by Michael begins, as with that of Humphreys, with a proof of the conjugacy of CSAs of solvable Lie algebras; this particular proof is a reworking of a proof given in Bourbaki [5].

Similar to Humphreys approach, Michael relies heavily upon BSAs. However, he uses properties of BSAs that are readily established and involve relatively simple facts from linear algebra. In particular, he uses dimension arguments and orthogonal complements to simplify the proofs.

Michael extends the vector space properties of BSAs in semisimple Lie algebras in such a way that the conjugacy of CSAs of semisimple Lie algebras is relatively easily proven. He then uses a "connecting lemma" to pull the argument back to the general case, thus establishing the conjugacy of CSAs of any Lie algebra.

Finally, in order to establish the conjugacy of BSAs, Michael proves that any BSA must contain a CSA. The proof of the conjugacy of BSAs then follows by a standard argument.

Chapter 2

Definitions and Preliminaries

All Lie algebras L are finite dimensional, and the field $\mathbb{F}$ is algebraically closed with characteristic 0.

Definition 2.1. ([11, p.1]) Let L be a vector space over a field $\mathbb{F}$. Then L is a Lie algebra if it is endowed with a bilinear operation $[\cdot, \cdot]$ such that for all x, y, and $z \in L$,

1. $[x, x] = 0$.
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (the Jacobi Identity).

A *subalgebra* of C of L is a vector subspace of L that is closed under $[\cdot, \cdot]$.

Example 2.2. The following are Lie algebras:

1. The general linear algebra $\mathfrak{gl}(V)$ (V a vector space over $\mathbb{F}$) is the set $\mathrm{End}\,(V)$ endowed with the operation $[x, y] = xy - yx$, $x, y \in \mathfrak{gl}(V)$. Here $\mathrm{End}\,(V)$ denotes the set of all endomorphisms of V.
2. The algebra of $n \times n$ matrices $\mathfrak{gl}(n, \mathbb{F})$ over $\mathbb{F}$, with $[\cdot, \cdot]$ given by $[A, B] = AB - BA$, $A, B \in \mathfrak{gl}(n, \mathbb{F})$.
3. L in which $[\cdot, \cdot]$ is trivially defined, i.e. $[x, y] = 0$ for every pair of elements x and y of L; in such a case we call L *abelian*.

Definition 2.3. ([11, p.7]) The normalizer of a subalgebra C of L is

$$N_L(C) := \{x \in L \mid [x, C] \subset C\}.$$

We call C *self-normalizing* if $N_L(C) = C$. The *centralizer* of C is

$$Z_L(C) := \{x \in L \mid [x, C] = 0\}.$$

We write $Z(L) = Z_L(L)$ and call it the *center* of L. So L is abelian if and only if $Z(L) = L$.

Definition 2.4. ([11, p.6]) An ideal I of L is a subalgebra such that $[x, y] \in I$ for all $x \in L$ and $y \in I$.

We note that all ideals are two-sided, for bilinearity combined with condition (1) of Definition 2.1 imply that $[x, y] = -[y, x]$ for any $x, y \in L$. If L has no ideals except itself and 0, then we call L *simple*.

Given an ideal I of L, the quotient space L/I is endowed with a bracket: $[x+I, y+I] := [x, y] + I$, $x, y \in L$. The operation is unambiguous so that L/I is a Lie algebra, called the *quotient algebra*.

In general, the vector space sum $I + J = \{x + y \mid x \in I, y \in J\}$ of two subalgebras I and J of L need not be a subalgebra; however, if I is an ideal and J is a subalgebra, then $I + J$ is indeed a subalgebra. If both I and J are ideals, then $I + J$ is an ideal as well.

Definition 2.5. ([11, p.7]) A vector space homomorphism $\phi : L \to L'$ is a Lie algebra homomorphism if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all x, $y \in L$. If $\phi : L \to L$ is an isomorphism, we call ϕ an automorphism. The group of all automorphisms of L is denoted by $\operatorname{Aut} L$.

The image $\operatorname{Im}\phi$ is a subalgebra of L' and the kernel $\operatorname{Ker}\phi$ is an ideal of L. The inverse image $\phi^{-1}(C')$ of a subalgebra C' of L' is a subalgebra of L.

Theorem 2.6. [11, p.7] (First Isomorphism Theorem) If $\phi : L \to L'$ is a Lie algebra homomomorphism, $\mathrm{Ker}\phi = K$, then $L/K \cong \mathrm{Im}\,\phi$, and the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[dr, "\phi"'] & L/K \arrow[d, "\psi"] \\ & L' \end{tikzcd}$$

Proof. Let π be the canonical projection of L onto L/K, and define $\psi : L/K \to L'$ by $\psi(x+K) = \phi(x)$. Then $\psi(\pi(x)) = \psi(x+K) = \phi(x)$, and the diagram commutes.

Given $a = \phi(x) \in \mathrm{Im}\,\phi$, we have $\psi(x+K) = a$, and ψ is onto. If $\psi(x+K) = \psi(y+K)$ we have $0 = \psi((x-y)+K) = \phi(x-y)$, and $x-y \in \mathrm{Ker}\phi$. So $\pi(x-y) \in K$, i.e. $x+K = y+K$, and ψ is one to one, thus an isomorphism. □

Given a Lie algebra L, denote by $\mathrm{ad}\,x \in \mathrm{End}\,L$ the endomorphism defined by

$$\mathrm{ad}\,x(y) = [x, y], \qquad y \in L.$$

If the subspace C of L is $\mathrm{ad}\,x$ stable, we denote the restriction of $\mathrm{ad}\,x$ to C by $\mathrm{ad}_C x : C \to C$.

Definition 2.7. An endomorphism t is nilpotent if $t^k = 0$ for some $k > 0$. An endomorphism is semisimple if the roots of its minimal polynomial over $\mathbb{F}$ are distinct.

An element $x \in L$ is called *ad-nilpotent* if the endomorphism $\mathrm{ad}\,x$ is a nilpotent one. Similarly, we call x *ad-semisimple* if $\mathrm{ad}\,x$ is a semisimple endomorphism.

Definition 2.8. ([11, p.8]) The adjoint map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a representation of L, i.e., a homomorphism of L with a Lie algebra of endomorphisms. Clearly the kernel of ad, denoted $\mathrm{Ker}\,\mathrm{ad}$, is the center $Z(L)$ of L.

As we shall see, the adjoint representation has many useful properties which make it of fundamental importance in the study of Lie algebras.

We now introduce two important classes of Lie algebras, namely, nilpotent and solvable algebras.

Definition 2.9. ([11, p.11]) The Lie algebra L is nilpotent if the descending central (or lower central) series defined by $L^0 := L$, $L^1 := [L, L], \dots, L^i := [L, L^{i-1}]$ terminates.

Example 2.10. The algebra of all strictly upper triangular matrices $\mathfrak{n}(n, \mathbb{F})$ over $\mathbb{F}$ is nilpotent.

It is not hard to see that $\operatorname{ad} x \in \mathfrak{gl}(L)$ is nilpotent for all $x \in L$ (i.e. x is ad-nilpotent) if L is nilpotent. The converse is true and is known as Engel's theorem ([11, p.12-13] or [5, p.39-40]).

Theorem 2.11. (Engel)

1. Let $L \subset \mathfrak{gl}(V)$ be a subalgebra of $\mathfrak{gl}(V)$, V finite dimensional. If L consists of nilpotent endomorphsims and $V \neq 0$, then there exists nonzero $v \in V$ such that $\ell v = 0$ for all $\ell \in L$.

2. A Lie algebra L is nilpotent if and only if $\operatorname{ad} x \in \operatorname{End} L$ is a nilpotent endomorphism for all $x \in L$.

Definition 2.12. ([11, p.10]) The Lie algebra L is solvable if the derived series defined by $L^{(0)} := L$, $L^{(1)} := [L, L], \dots, L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$ terminates, i.e., if there is $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Every nilpotent algebra is solvable: clearly $L^{(i)} \subset L^i$ for all i, so if the descending central series terminates, the derived series does as well.

The following is Lie's theorem ([11, p.15-16] or [5, p.46]) on the characterization of solvable algebras.

Theorem 2.13. (Lie) The Lie algebra L is solvable if and only if the derived algebra $[L, L]$ is nilpotent.

An alternate characterization of solvable Lie algebras is given by Cartan's Criterion([11, p.20] or [5, p.48]):

Theorem 2.14. (Cartan's Criterion) The Lie algebra L is solvable if and only if $\mathrm{tr}\,(\mathrm{ad}\,x\,\mathrm{ad}\,y) = 0$ for all $x \in [L, L]$, $y \in L$.

Every Lie algebra L has a unique maximal solvable ideal R [11, p.11]: for if I is any other solvable ideal of L, then $R + I$ is a solvable ideal as well. By the maximality of R, $R + I = R$, or $I \subset R$.

Definition 2.15. ([11, p.11]) The radical of L, denoted by $R = \mathrm{Rad}\,L$, is the unique maximal solvable ideal. If $R = 0$, then L is called semisimple.

Lemma 2.16. ([11, p.11]) Let L be a Lie algebra with radical R. Then the Lie algebra L/R is semisimple, i.e., the radical of L/R is 0.

Proof. Let R' be the radical of L/R and $\pi : L \to L/R$ be the canonical projection. Since π is a homomorphism, $\pi^{-1}(R')$ is an ideal of L. As R' is solvable, we know that the derived series of R' terminates, that is $(R')^{(n)} = R$ for some n; then $\pi((\pi^{-1}(R'))^{(n)}) \subset (R')^{(n)}$ means that $(\pi^{-1}(R'))^{(n)} \subset R$, that is $\pi^{-1}(R')$ is a solvable ideal, and by maximality is contained in R. □

Indeed the radical R of L is the smallest ideal of L such that L/R is semisimple.

Given $x \in L$, $\mathrm{ad}\,x$ is a vector space endomorphism of L. According to Jordan-Chevalley decomposition [11, p.17-18], L is the direct sum of generalized eigenspaces

$$L_a(\mathrm{ad}\,x) := \mathrm{Ker}\,(\mathrm{ad}\,x - a \cdot 1)^m$$

where m is the multiplicity of the eigenvalue a of $\mathrm{ad}\,x$. Each $L_a(\mathrm{ad}\,x)$ is invariant under $\mathrm{ad}\,x$ and the restriction of $\mathrm{ad}\,x$ to $L_a(\mathrm{ad}\,x)$ is the sum of a scalar multiple (namely a) of the identity and a nilpotent endomorphism. For each nonzero $x \in L$, 0 is an eigenvalue of $\mathrm{ad}\,x$ since $\mathrm{ad}\,x(x) = [x, x] = 0$, so that we have $L_0(\mathrm{ad}\,x) \neq 0$. We set $L_a(\mathrm{ad}\,x) = 0$ if a is not an eigenvalue of $\mathrm{ad}\,x$. Thus we have the following

Lemma 2.17. ([11, p.78]) Given $x \in L$, the Lie algebra L may be decomposed as

$$L = \coprod_{a \in \mathbb{F}} L_a(\operatorname{ad} x) = L_0(\operatorname{ad} x) \oplus L_*(\operatorname{ad} x)$$

where $L_*(\operatorname{ad} x)$ denotes the sum of those $L_a(\operatorname{ad} x)$ such that $a \neq 0$. In addition, any subalgebra K of L that is stable under $\operatorname{ad} x$ can be written as $K = K_0(\operatorname{ad} x) \oplus K_*(\operatorname{ad} x)$, where $K_i(\operatorname{ad} x) = K \cap L_i(\operatorname{ad} x)$.

Lemma 2.18. [11, p.78] If $a, b \in \mathbb{F}$, then $[L_a(\operatorname{ad} x), L_b(\operatorname{ad} y)] \subset L_{a+b}(\operatorname{ad} x)$. In particular $L_0(\operatorname{ad} x)$ is a subalgebra of L. When $a \neq 0$, each element of $L_a(\operatorname{ad} x)$ is ad-nilpotent.

Proof. Binomial expansion [11, p.79] yields

$$\begin{aligned}
(\operatorname{ad} x - (a+b))^m [y, z] &= (\operatorname{ad} x - a - b)^m [y, z] \\
&= \sum_{i=0}^{m} \binom{m}{i} [(\operatorname{ad} x - a)^i y, (\operatorname{ad} x - b)^{m-i} z].
\end{aligned}$$

For sufficiently large m, all terms on the right side are 0 for $y \in L_a(\operatorname{ad} x)$ and $z \in L_b(\operatorname{ad} x)$. So $[L_a(\operatorname{ad} x), L_b(\operatorname{ad} y)] \subset L_{a+b}(\operatorname{ad} x)$. Then $L_0(\operatorname{ad} x)$ is a subalgebra of L.

Each $z \in L$ can be written $z = z_0 + z_{a_1} + \ldots + z_{a_n}$ with $z_{a_i} \in L_{a_i}(\operatorname{ad} x)$ by Lemma 2.17. When $a \neq 0$ and $y \in L_a(\operatorname{ad} x)$, we have $(\operatorname{ad} y)^{r_i}(z_{a_i}) \in L_{r_i a + a_i}(\operatorname{ad} x) = 0$ for sufficiently large r_i, since there are finitely many eigenvalues for $\operatorname{ad} x$ (recall that L is finite dimensional). We need merely choose $r = \sum r_i$ to force $(\operatorname{ad} y)^r(z) = 0$. Thus $\operatorname{ad} y$ is a nilpotent endomorphism, i.e., elements of $L_a(\operatorname{ad} x)$ are ad-nilpotent. □

We note that $y \in L_0(\operatorname{ad} x)$ need not be ad-nilpotent.

Definition 2.19. [11, p.82] We call $x \in L$ strongly ad-nilpotent if $x \in L_a(\operatorname{ad} y)$ for some $a \neq 0$. The set of all strongly ad-nilpotent elements of L will be denoted $\mathcal{N}(L)$.

By Lemma 2.18 strongly ad-nilpotent elements are ad-nilpotent.

Definition 2.20. [7, p.1] Given a nilpotent subalgebra H of L, $\alpha \in H^*$, and $h \in H$, we set

$$L_{\alpha,h} := L_{\alpha(h)}(\operatorname{ad} h) = \operatorname{Ker}(\operatorname{ad} h - \alpha(h) \cdot 1)^m$$

where m is the algebraic multiplicity of the eigenvalue $\alpha(h)$, and set

$$L_\alpha(H) := \cap_{h \in H} L_{\alpha,h}.$$

When H is understood, we will often denote $L_\alpha(H)$ by L_α.

Theorem 2.21. ([7, p.8]) If H is a nilpotent subalgebra of L, then L may be decomposed as

$$L = \coprod_{\alpha \in H^*} L_\alpha(H).$$

The decomposition is called the root space decomposition.

Lemma 2.22. [11, p.14] A nilpotent Lie algebra L contains no proper self-normalizing subalgebras.

Proof. Indeed, a Lie algebra L containing a proper self-normalizing subalgebra K must also contain a (nonzero) $x_1 \in L \backslash K$, and (as K is self-normalizing) a corresponding $k_1 \in K$ such that $x_2 = [k_1, x_1] \notin K$. Then there is a $k_2 \in K$ with $[k_2, x_2] = x_3 \notin K$; thus we have a nonzero sequence $x_1, \ldots, x_n, \ldots$ with $x_i \in L^{i-1}$. Then the descending central series of L is non-terminating, i.e. L is not nilpotent. □

Lemma 2.23. Let L and L' be isomorphic as Lie algebras via ϕ. Then the subalgebra C of L is self-normalizing if and only if $\phi(C)$ is self-normalizing.

Proof: It suffices to show one implication. By virtue of the isomorphism ϕ, elements of L and L' are in 1 to 1 correspondence; we are thus justified in denoting $\phi^{-1}(x') = x$ for any $x' \in L'$. In addition, preimages of subalgebras of L' are isomorphic subalgebras in L; thus for any subalgebra C' of L, we have $C' \cong C \subset L$, where $\phi^{-1}(C') = C$.

Suppose that the subalgebra C of L is self-normalizing, and let $x'(=\phi(x)) \in N_{L'}(C')$, that is $[x', C'] \subset C'$. So $\phi([x,C]) = [\phi(x), \phi(C)] \subset \phi(C)$. But ϕ is an isomorphism, so $[x, C] \subset C$. As C is self-normalizing, we have $x \in C$ so that $\phi(x) \in \phi(C)$ and $\phi(C)$ is self-normalizing.

Lemma 2.24. Let $\phi \in \operatorname{End} V$, where V is a vector space. If $K \subset V$ is an invariant subspace under ϕ and contains the eigenspace corresponding to the eigenvalue 0, then the induced endomorphism $\phi' : V/K \to V/K$ defined by $\phi'(v+K) = \phi(v)+K$ has no nonzero eigenvalues.

Proof. Choose a basis $B_K = \{k_1, \ldots, k_t\}$ of K and extend B_K to a basis $B_V = \{k_1, \ldots, k_t, v_1, \ldots, v_{n-t}\}$ of V. With respect B_V, the matrix (denoted by M_ϕ) of ϕ is in block form

$$M_\phi = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Clearly A is the matrix of $\phi|_K : K \to K$ with respect to B_K. The eigenvalues of ϕ are those of A and C so that C has only non-zero eigenvalues . The map ϕ' is a well-defined endomorphism on the quotient (vector) space V/K. With respect to the basis $\{v_1 + K, \ldots, v_{n-t} + K\}$ of V/K, the matrix of ϕ' is C: for each $i = 1, \ldots, n-t$,

$$\begin{aligned} \phi'(v_i + K) &= \phi(v_i) + K \\ &= (b_{1i}k_1 + \cdots + b_{ti}k_t + c_{1i}v_1 + \ldots + c_{n-t,i}v_{n-t}) + K \\ &= (c_{1i}v_1 + \ldots + c_{n-t,i}v_{n-t}) + K \\ &= c_{1i}(v_1 + K) + \ldots + c_{n-t,i}(v_{n-t} + K). \end{aligned}$$

□

Lemma 2.25. Let $\mathcal{F} \subset \operatorname{End} \mathcal{V}$ be a commuting diagonalizable family, where V is a vector space over $\mathbb{F}$. Suppose $W \subset V$ is a subspace stable under $\mathcal{F}$. Then $\mathcal{F}_W := \{T|_W \in \operatorname{End} W : T \in \mathcal{F}\} \subset \operatorname{End} \mathcal{W}$ is a commuting diagonalizable family.

Proof. If $S, T \in \mathcal{F}$, then $S|_W \circ T|_W = (S \circ T)|_W = (T \circ S)|_W = T|_W \circ S|_W$. Hence $\mathcal{F}_W$ is a commuting family. It suffices to show that each $T|_W \in \operatorname{End} W$ is diagonalizable, i.e., each $T|_W$ has k linearly independent eigenvectors ($k := \dim W$), or the geometric multiplicity m of each eigenvalue λ of $T|_W$ is 1. Now $w \in \operatorname{Ker}(T|_W - \lambda \cdot 1_W)^m$ implies that $w \in W$ and $0 = (T|_W - \lambda \cdot 1_w)^m w = (T - \lambda \cdot 1)^m w$. But T is diagonalizable so that the geometric multiplicity of the eigenvalue λ of T is 1, i.e., $(T - \lambda \cdot 1)w = 0$, i.e., $m = 1$. □

Recall from Definition 2.19 that $\mathcal{N}(L)$ denotes the set of strongly ad-nilpotent elements of L.

Lemma 2.26. ([11, p.82]) For any epimorphism $\phi : L \to L'$,

$$\phi(L_a(\operatorname{ad} y)) = L'_a(\operatorname{ad}(\phi(y))), \quad y \in L. \tag{2.1}$$

So $\phi(\mathcal{N}(L)) = \mathcal{N}(L')$.

Proof. Given a Lie algebra homomorphism $\phi : L \to L'$, that $\phi(L_a(\operatorname{ad} y)) \subset L'_a(\operatorname{ad}(\phi(y)))$ is readily seen: for $x \in L_a(\operatorname{ad} y)$ means that $(\operatorname{ad} y - a \cdot 1)^k(x) = 0$ for some $k \in \mathbb{N}$. So

$$((\operatorname{ad} \phi(y)) - a \cdot 1)^k(\phi(x)) = \phi((\operatorname{ad} y - a \cdot 1)^k(x)) = 0,$$

i.e., $\phi(x) \in L'_a(\operatorname{ad} \phi(y))$.

On the other hand, any $x' \in L'_a(\operatorname{ad} \phi(y))$ has a preimage x in L by the surjectivity of ϕ. We employ Jordan-Chevalley decomposition [11, p.17] to write

$$L = L_{a_1}(\operatorname{ad} y) \oplus \ldots \oplus L_{a_n}(\operatorname{ad} y),$$

where $a_1, \ldots, a_n$ are the eigenvalues of $\operatorname{ad} y$. Then we may decompose x accordingly:

$$x = x_{a_1} + \ldots + x_{a_n}, \quad x_t \in L_{a_t}(\operatorname{ad} y).$$

By the above, $\phi(x_t) \in L'_{a_t}(\operatorname{ad}\phi(y))$. In addition, the sum $L'_{a_1}(\operatorname{ad}\phi(y)) \oplus \ldots \oplus L'_{a_n}(\operatorname{ad}\phi(y))$ is direct. Suppose $a = a_i$. Then

$$\begin{aligned} L'_{a_i}(\operatorname{ad}\phi(y)) \ni x' - \phi(x_{a_i}) &= \phi(x) - \phi(x_{a_i}) \\ &= \phi(x_{a_1}) + \cdots + \phi(x_{a_{i-1}}) + \phi(x_{a_{i+1}}) + \cdots + \phi(xa_n) \\ &\in L'_{a_1}(\operatorname{ad}\phi(y)) \oplus \cdots \oplus L'_{a_i}\widehat{(\operatorname{ad}\phi}(y)) \oplus \cdots \oplus L'_{a_n}(\operatorname{ad}\phi(y)) \end{aligned}$$

where the hat indicates the term $L'_{a_i}(\operatorname{ad}\phi(y))$ is deleted. But $x - \phi(x_{a_i}) \in L'_{a_i}(\operatorname{ad}\phi(y))$. As the sum is direct, we have $x' - \phi(x_{a_i}) = 0$, i.e. $x' = \phi(x_{a_i})$ and we have found $x_{a_i} \in L_{a_i}(\operatorname{ad} y)$ such that $\phi(x_{a_i}) = x'$. So $\phi(L_a(\operatorname{ad} y)) = L'_a(\operatorname{ad}(\phi(y)))$.

Recall that $\mathcal{N}(L)$ is defined as the set of all $x \in L$ such that $x \in L_a(\operatorname{ad} y)$, where a is a nonzero eigenvalue of the endomorphism $\operatorname{ad} y$. From the above, it is clear that $\phi(\mathcal{N}(L)) = \mathcal{N}(L')$. $\square$

Chapter 3

Properties of Cartan Subalgebras

We introduce Cartan subalgebras and discuss properties that will be useful in both of the conjugacy proofs.

Definition 3.1. [11, p.80] Let L be a Lie algebra over $\mathbb{F}$. A Cartan subalgebra H (abbreviated CSA) of L is a self-normalizing nilpotent subalgebra of L.

Unfortunately, this definition does not imply the existance of CSAs of a given Lie algebra.

Both conjugacy proofs will employ the following definition:

Definition 3.2. ([11, p.83]) A Borel subalgebra (abbreviated BSA) B of L is a maximal solvable subalgebra of L.

Note that BSAs are subalgebras, while $\operatorname{Rad} L$ is required to be an maximal solvable ideal.

A helpful propety of Borel subalgebras is the following:

Lemma 3.3. ([11, p.83]) Every BSA B of L is self-normalizing.

Proof. If $x \in L$ normalizes B, we may create $B + \mathbb{F}x$, which is certainly a subalgebra of L since

$$[B + \mathbb{F}x, B + \mathbb{F}x] \subset [B, B] + [B, \mathbb{F}x] + [\mathbb{F}x, \mathbb{F}x].$$

The last term is zero, and the term $[B, \mathbb{F}x]$ is inside of B since x is a normalizer. Clearly $[B + \mathbb{F}x, B + \mathbb{F}x] \subset B$, so $B + \mathbb{F}x$ is solvable. Now B is Borel (maximal solvable) so that x must be an element of B. $\square$

While the proof of the following lemma is beyond the scope of this paper, we shall use the result to establish several properties of CSAs:

Lemma 3.4. [7, p.8] If H is a nilpotent subalgebra of L, then there is an $x \in L$ such that $L_0(\operatorname{ad} x) = L_0(H)$.

We note that $H \subset L_0(H)$ as H is nilpotent.

Lemma 3.5. [7, p.14] Let H be a nilpotent subalgebra of L. Then H is a CSA of L if and only if $L_0(H) = H$.

Proof. We first note that any nilpotent subalgebra H is contained in $L_0(H)$. If $L_0(H) = H$, H is self-normalizing (see [7, p.10]), thus a CSA. On the other hand, if $H \subsetneq L_0(H)$, considering the nilpotent subalgebra $\operatorname{ad} H$ of $\mathfrak{gl}(L_0(H)/H)$, we may apply Engel's theorem to the (nontrivial) quotient algebra $L_0(H)/H$ to find an $x \in L_0(H) \backslash H$ such that $[x, H] \subset H$, that is $x \in \mathcal{N}_L(H)$, and H is not self-normalizing, i.e. not a CSA.

□

Theorem 3.6. [7, p.14] Every CSA H of L may be written in the form $H = L_0(\operatorname{ad} x)$ for some $x \in L$.

Proof. Lemmas 3.4 and 3.5 allow us to find an $x \in L$ such that $L_0(\operatorname{ad} x) = L_0(H) = H$.

□

Definition 3.7. [16, p.156] The rank of L, $\operatorname{rank} L$, is defined as $\min\{\dim L_0(\operatorname{ad} x) \mid x \in L\}$. Any x with $\operatorname{rank} L = \dim L_0(\operatorname{ad} x)$ is called regular.

Lemma 3.8. [7, p.17] Let H be a subalgebra of L. Then every regular element of L contained in H is also regular in H.

Theorem 3.9. [7, p.18] Let x be a regular element of L. Then $L_0(\operatorname{ad} x)$ is a CSA of L.

Proof. If x is regular in L, set $H := L_0(\operatorname{ad} x)$. By definition, $H_0(\operatorname{ad} x) = H$. By the previous lemma, x is a regular element of H, so $\operatorname{rank} H = \dim H$. Then for all $h \in H$,

$\dim H_0(\operatorname{ad} h) = \dim H$, so $\operatorname{ad}_H h$ is a nilpotent endomorphism of H. By Lie's Theorem, then, H is nilpotent; so we have $H \subset L_0(H) \subset L_0(\operatorname{ad} x) = H$, and by 3.5, H is a CSA of L.

□

As a consequence of the previous theorem, we now know that CSAs of a finite dimensional Lie algebra over an algebraically closed, characteristic 0 field $\mathbb{F}$ always exist.

Lemma 3.10. [7, p.13] A CSA H of L is a maximal nilpotent subalgebra.

Proof. If H is a nilpotent subalgebra of L containing a CSA H', we note that H' is self-normalizing not only in L, but also in H; by Lemma 2.22, $H = H'$.

□

Compare the following with [11, p.79].

Lemma 3.11. If $K \subset L$ is a subalgebra of L containing a CSA H of L, then K is self-normalizing.

Proof. By Theorem 3.6, we may write $H = L_0(\operatorname{ad} x) \subset K$ for some $x \in L$, and make the following observations:

(1) $[x, x] = 0$ so $x \in H \subset K$; thus $[N_L(K), x] \subset K$.

(2) In particular $\operatorname{ad} x(K) \subset K$ since $K \subset N_L(K)$, i.e., K is an invariant subspace of $N_L(K)$ under the endomorphism $\operatorname{ad} x$. In addition, K contains the eigenspace of $\operatorname{ad} x$ corresponding to the eigenvalue 0 since $L_0(\operatorname{ad} x) \subset K$.

By Lemma 2.24, the endomorphism $\operatorname{ad} x$ acts on $N_L(K)/K$ with no zero eigenvalues. By (1), every coset of $N_L(K)/K$ is mapped by $\operatorname{ad} x$ into K, i.e.,

$$\operatorname{ad} x(m + K) = \operatorname{ad} x(m) + K = K$$

where $m \in N_L(K)$. In other words, $\operatorname{ad} x$ acts trivially on $N_L(K)/K$. On the other hand, by (2), $\operatorname{ad} x$ has no zero eigenvalues on $N_L(K)/K$. So K is the only possible coset of $N_L(K)/K$ and we conclude that $N_L(K) = K$. □

Lemma 3.12. ([11, p.81]) If $\phi : L \to L'$ is an epimorphism of Lie algebras, then $\phi(H)$ is a CSA of L' for every CSA H of L. If H' is a CSA of L', there is a CSA H of L such that $\phi(H) = H'$.

Proof. Obviously $\phi(H)$ is nilpotent so we need only show that it is self-normalizing. Let $A = \mathrm{Ker}\phi$. Then by the First Isomorphism Theorem (Theorem 2.6), $L/A \cong \mathrm{Im}\,\phi = L'$ and we have the induced (Lie algebra) isomorphism $\psi : L/A \to L'$, defined by $\psi(x + A) = \phi(x)$. The function ψ allows us to isomorphically identify the subalgebra H/A of L/A with $\phi(H)$, for by the above $\psi(H/A) = \phi(H)$. By Lemma 2.23 $\phi(H)$ self-normalizing is equivalent to H/A being self-normalizing. This will be simpler to prove, so we focus our attention upon L/A, in particular the subalgebra H/A of L/A. In L itself, H is a subalgebra and A is an ideal, so $H + A \subset L$ is a subalgebra of L. This subalgebra contains the CSA H. By Lemma 3.11, $H + A$ is self-normalizing as a subalgebra of L.

Now suppose that $H/A \subset L/A$ is normalized by the coset $x + A$, i.e., $[x, H]/A = [x + A, H/A] \subset H/A$. We must have $[x, H] \subset H + A$ so that

$$[x, H + A] = [x, H] + [x, A] \subset H + A \subset L$$

(note that $[x, A] \subset A$ since A is an ideal). So x normalizes $H + A$. By the above, $x \in H + A$ so that that $x + A \in H/A$. So H/A is self-normalizing as a subalgebra of L/A, and $\phi(H)$ is, as well; thus $\phi(H)$ is a CSA of L'.

Finally, if H' is a CSA of L', Set $K := \phi^{-1}(H')$. We may choose a CSA H of K; then $\phi(H)$ is a CSA of H' by the above, that is $\phi(H)$ is a self-normalizing, nilpotent subalgebra of H'. But we may view H' as a Lie algebra in its own right, and note that H' is nilpotent. As CSAs are maximal nilpotent subalgebras (Lemma 3.10), $\phi(H) = H'$. We must show that H is a CSA of L. If $x \in L$ normalizes H, then $\phi(x)$ normalizes $\phi(H) = H'$ so that $\phi(x) \in \phi(H)$, i.e. $x \in H + \mathrm{Ker}\phi$. But $\mathrm{Ker}\phi \subset K$ so that $x \in H + K \subset K$. Now $x \in N_K(H) = H$ since H is a CSA of K. See [7, Corollary 2, p.18]. □

Chapter 4

Humphreys' Proof

We first prove the conjugacy of CSAs following Humphreys' approach. Indeed as Humphreys [11, p.88] points out the approach is from Winter [25, Section 3.8, p.92-99] and was inspired by Mostow [23, p.vii]. Since some of Humphreys' arguments are very brief, elaboration is needed.

Definition 4.1. ([11, p.9,82]) $\operatorname{Int} L$ is the subgroup of $\operatorname{Aut} L$ generated by all

$$\exp(\operatorname{ad} x) := 1 + \operatorname{ad} x + (\operatorname{ad} x)^2/2! + (\operatorname{ad} x)^3/3! + \cdots$$

where $x \in L$ is ad-nilpotent. Note that the sum has a finite number of terms, for $(\operatorname{ad} x)^n = 0$ for some n. We define $\mathcal{E}(L)$ as the subgroup of $\operatorname{Int} L$ generated by all $\exp(\operatorname{ad} x)$ such that $x \in \mathcal{N}(L)$ (see Definition 2.19).

Remark 4.2. ([11, p.82]) Given a subalgebra K of L, we note that $\mathcal{N}(K) \subset \mathcal{N}(L)$, and define $\mathcal{E}(L;K)$ as the subgroup of $\mathcal{E}(L)$ generated by $\exp \operatorname{ad} x$, where $x \in \mathcal{N}(K)$. Thus $\mathcal{E}(K)$ is precisely the restriction of the automorphisms of $\mathcal{E}(L;K)$ to K. In particular, given $\tau' \in \mathcal{E}(K)$, we may extend τ' to $\tau \in \mathcal{E}(L)$, where $\tau|_K = \tau'$.

It turns out that if L is semisimple, $\mathcal{E}(L) = \operatorname{Int} L$.

Lemma 4.3. ([11, p.82]) Let $\phi : L \to L'$ be an epimorphism of Lie algebras. For any $\sigma' \in \mathcal{E}(L')$, there exists $\sigma \in \mathcal{E}(L)$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\sigma"'] & L' \arrow[d, "\sigma'"] \\ L \arrow[r, "\phi"] & L' \end{tikzcd}$$

i.e., $\sigma' \circ \phi = \phi \circ \sigma$.

Proof. As $\mathcal{E}(L')$ is generated by $\sigma' = \exp \operatorname{ad} x'$, where $x' \in \mathcal{N}(L')$, it suffices to show the theorem true for such σ'. By Lemma 2.26, we may choose $x \in \mathcal{N}(L)$ such that $\phi(x) = x'$, and we set $\sigma := \exp \operatorname{ad} x \in \mathcal{E}(L)$. For any $z \in L$,

$$\begin{aligned}(\phi \circ \sigma)(z) &= \phi(\exp \operatorname{ad} x(z)) = \phi(z + [x, z] + (1/2)[x, [x, z]] + \cdots) \\ &= \phi(z) + [\phi(x), \phi(z)] + (1/2)[\phi(x), [\phi(x), \phi(z)]] + \cdots \\ &= \phi(z) + [x', \phi(z)] + (1/2)[x', [x', \phi(z)]] + \cdots \\ &= \exp \operatorname{ad} x'(\phi(z)) \\ &= (\sigma' \circ \phi)(z).\end{aligned}$$

□

One of our main goals is to show that CSAs of a Lie algebra L are conjugate via $\mathcal{E}(L)$; we first handle the special case when L is solvable.

Theorem 4.4. ([11, p.82]) CSAs of a solvable L are conjugate via $\mathcal{E}(L)$.

Proof: We proceed by induction. The theorem is obvious when $\dim L = 1$.

If L is nilpotent, there is nothing to prove, for L itself will be its only self-normalizing subalgebra by Lemma 2.22.

Thus we may assume that L is solvable but not nilpotent. As the last term of the derived series of L must be abelian, we are guaranteed the existence of non-zero abelian ideals of L; choose one, A, of minimum dimension. Set $L' := L/A$, the homomorphic image of L under the canonical projection $\phi : L \to L/A$. Then for CSAs H_1, H_2 of L, $\phi(H_1) = H_1'$ and $\phi(H_2) = H_2'$ are themselves CSAs of L' by Lemma 3.12. By the induction hypothesis, since $\dim L/A = \dim L - \dim A < \dim L$, these are conjugate via $\sigma' \in \mathcal{E}_{L'}$. By Lemma 4.3, $\sigma \in \mathcal{E}(L)$ exists such that $\sigma'\phi = \phi\,\sigma$.

Setting $K_i := \phi^{-1}(H_i')$, $i = 1, 2$, we note that the K_i are subalgebras of L. We have $\phi\,\sigma(K_1) = \sigma'\phi(K_1) = \sigma'(H_1') = H_2'$. But $\phi^{-1}(H_2') = K_2$, so $\sigma(K_1) \subset K_2$. Similarly, $\sigma(K_2) \subset K_1$. As σ is an automorphism, we must have $\sigma(K_1) = K_2$.

$$\begin{array}{rcl} L & \xrightarrow{\ \phi\ } & L' \\ \bigcup & & \\ H_1 \subset K_1 & \longrightarrow & H_1' \\ \Big\downarrow \sigma & & \Big\downarrow \sigma' \\ H_2 \subset K_2 & \longrightarrow & H_2' \end{array}$$

We consider two cases.

Case 1. $K_2 \subsetneq L$. Then again by the induction hypothesis, we have $\tau' \in \mathcal{E}(K_2)$ with $\tau'\sigma(H_1) = H_2$, since by Lemma 3.12 $\sigma(H_1)$ as well as H_2 are CSAs of K_2. Extend τ' to $\tau \in \mathcal{E}(L)$ to complete the proof (see remark 4.2).

Case 2. $K_2 = L$. As before, we have $\sigma(K_1) = K_2 = L$, so $K_1 = K_2 = L$. Now $L/A = \phi(L) = \phi(K_1) = H_1' \subset L/A$; so $L/A = H_1' = \phi(H_1)$. For any $y \in L$, $\phi(y) = y + A \in H_1/A$, which means $y \in H_1 + A \subset L$. A similar argument applies for H_2, so we have

$$L = H_1 + A = H_2 + A.$$

Now by Lemma 3.6, we write

$$H_2 = L_0(\operatorname{ad} x)$$

for some $x \in L$. Since A is an ideal, it is stable under $\operatorname{ad} x$ (i.e., $\operatorname{ad}_A x : A \to A$) and Lemma 2.17 allows us to write

$$A = A_0(\operatorname{ad} x) \oplus A_*(\operatorname{ad} x).$$

We will show that $A_0(\operatorname{ad} x)$ is an ideal of L: on one hand $[H_2, A_0(\operatorname{ad} x)] = [L_0(\operatorname{ad} x), A_0(\operatorname{ad} x)] \subset L_0(\operatorname{ad} x)$ since $A_0(\operatorname{ad} x) \subset L_0(\operatorname{ad} x)$; on the other hand $[H_2, A_0(\operatorname{ad} x)] \subset [H_2, A] \subset A$ since A is an ideal ($[A_0, A] = 0$ since A is abelian). Thus $[H_2, A_0(\operatorname{ad} x)] \subset L_0(\operatorname{ad} x) \cap A = A_0(\operatorname{ad} x)$.

As an ideal of L, $A_0(\operatorname{ad} x)$ must be trivial, otherwise by the minimality of A we have $A = A_0(\operatorname{ad} x)$. But this is impossible, for it would force $A \subset L_0(\operatorname{ad} x) = H_2$, that is $L = H_2 + A = H_2$, a nilpotent algebra. Note that the argument is symmetric, thus applies to H_1 as well.

Since $A_0(\operatorname{ad} x) = 0$, we have $A = A_*(\operatorname{ad} x) \subset L_*(\operatorname{ad} x)$, and since $H_2 = L_0(\operatorname{ad} x)$, the sum $L = H_2 + A$ is direct and we have

$$A = A_*(\operatorname{ad} x) = L_*(\operatorname{ad} x).$$

Since $L = H_1 + A$, we may write $x = y + z$, with $y \in H_1$ and $z \in L_*(\operatorname{ad} x)$. The endomorphism $\operatorname{ad} x$ is invertible on $A = L_*(\operatorname{ad} x)$, for $L_*(\operatorname{ad} x)$ clearly has no zero eigenvalues under $\operatorname{ad} x$. Thus a $z' \in L_*(\operatorname{ad} x)$ exists such that $z = [x, z']$. As A is abelian, $(\operatorname{ad}_A z')^2 = 0$; in addition, A is an ideal and $H_1 \cap A = 0$, so $\operatorname{ad} z'(H_1) = 0$. Thus $(\operatorname{ad} z')^2 = 0$. Thus $\exp \operatorname{ad} z' = 1 + \operatorname{ad} z'$ and in particular,

$$\exp \operatorname{ad} z'(x) = x + [z', x] = x - [x, z'] = x - z = y.$$

By Lemma 2.26 with $\phi := \exp \operatorname{ad} z' \in \operatorname{Aut} L$, we have

$$\exp \operatorname{ad} z'(L_0(\operatorname{ad} x)) = L_0(\operatorname{ad} (\exp \operatorname{ad} z'(x))) = L_0(\operatorname{ad} y).$$

By Lemma 3.12 $H := L_0(\operatorname{ad} y)$, as the isomorphic image of the CSA $H_2 = L_0(\operatorname{ad} x)$ of L, is also a CSA. Now $y \in H_1$, a nilpotent subalgebra, implies that $\operatorname{ad} y$ acts as a nilpotent endomorphism on H_1 (by Engel's theorem). Then by $H_1 \subset L_0(\operatorname{ad} y) = H$, we have $H = H_1$, for both are maximum nilpotent subalgebras (Lemma 3.10).

Now $\exp \operatorname{ad} z' \in \operatorname{Aut} L$ sends H_2 to H_1; it remains to show that $\exp \operatorname{ad} z' \in \mathcal{E}(L)$. By Lemma 2.17 we write $z' \in L_*(\operatorname{ad} x)$ as a sum of strongly ad-nilpotent elements, say

$z' = \sum_a z'_a$, with $z'_a \in L_a(\operatorname{ad} x)$, a a non-zero eigenvalue for $\operatorname{ad} x$. Then

$$\exp \operatorname{ad} z' = 1 + \operatorname{ad} z' = 1 + \operatorname{ad} (\sum_a z'_a) = 1 + \sum_a \operatorname{ad} z'_a.$$

But each z'_a is an element of the abelian ideal A, so $(\operatorname{ad} z'_a)(\operatorname{ad} z'_b) = 0 \in \operatorname{End} L$. Thus

$$1 + \sum_a \operatorname{ad} z'_a = \prod_a (1 + \operatorname{ad} z'_a) = \prod_a \exp \operatorname{ad} z'_a \in \mathcal{E}(L)$$

since $(\operatorname{ad} z'_a)^2 = 0$ (indeed $(\operatorname{ad} z')^2 = 0$). Thus H_1 and H_2 are conjugate via an element of $\mathcal{E}(L)$.

□

In order to prove the general case, we will employ several useful properties of Borel subalgebras.

Lemma 4.5. ([11, p.83]) The BSAs of L are in natural 1-1 correspondence (with respect to the canonical projection) with the BSAs of the semisimple L/R, where R denotes the radical of L.

Proof. The radical R of L is a maximal solvable ideal, thus $B + R$ is a solvable subalgebra of L for any BSA B of L since B is solvable. Thus we have $R \subset B$ by the maximality of B.

Now B/R is solvable in L/R. Any subalgebra $K' \subset L/R$ properly containing B/R is not solvable, otherwise the subalgebra $K := \pi^{-1}(K')$ of L containing B would be solvable, forcing $B \subsetneq K$, a contradiction. Thus B/R is indeed a BSA of L/R.

Conversely if $B' \subset L/R$ is a BSA of L/R, then the subalgebra $B := \pi^{-1}(B')$ of L is solvable . Any subalgebra $\hat{B}$ properly containing B is not solvable, otherwise the solvable $\hat{B}/R$ properly contains the BSA $B' = B/R$ of L/R. □

When L is semisimple, the abstract Jordan decomposition [11, p.24] asserts that any element x of L may be decomposed as $x = x_s + x_n$, where the ad-semisimple part x_s of x

and the ad-nilpotent part x_n of x are also elements of L. The same is true for any BSA of a semisimple L:

Lemma 4.6. [11, p.85] Any Borel subalgebra B of a semisimple L contains the semisimple and nilpotent parts of all of its elements.

Proof. Jordan-Chevalley decomposition implies that if the endomorphism σ maps a subspace B of L into $A \subset B$, then both the nilpotent and semisimple parts of σ map B into A [11, p.17]. For each $x \in B$, view $\operatorname{ad} x \in \operatorname{End} L$. Then by Lemma 3.3, $x \in B$ if and only if $\operatorname{ad} x$ maps B into itself. Thus $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ map B into itself, i.e. x_s and x_n are normalizers of B, thus elements of B. We conclude that any Borel subalgebra contains the semisimple and nilpotent parts of all of its elements. □

A subalgebra T of L is said to be toral if T consists of ad-semisimple elements. Toral subalgebras certainly exist in a semisimple Lie algebra. Any Lie algebra consisting entirely of ad-nilpotent elements is a nilpotent algebra by Engel's Theorem (Theorem 2.11). A semisimple Lie algebra L has no solvable ideals and cannot be nilpotent; we conclude that L has an ad-semisimple element, thus a nonzero toral subalgebra T.

Lemma 4.7. ([11, p.80]) The CSAs of a semisimple L are precisely the maximal toral subalgebras of L. In particular, CSAs of semisimple Lie algebras are abelian.

If H is a CSA of the semisimple L, then $\operatorname{ad} H$ is an abelian subalgebra of semisimple endomorphisms in $\operatorname{End} L$ and thus $\operatorname{ad} H$ is simultaneously diagonalizable according to linear algebra. So given $h \in H$, we have $L_{\alpha,h} = L_{\alpha(h)}(\operatorname{ad} h) = \operatorname{Ker}(\operatorname{ad} h - \alpha(h) \cdot 1)$, i.e., $m = 1$, and thus $L_\alpha(H)$ takes the special form

$$L_\alpha(H) = \{x \in L \mid [h, x] = \alpha(h)x \text{ for all } h \in H\},$$

i.e., each nonzero $x \in L_\alpha(H)$ is a common eigenvector for all endomorphisms $\operatorname{ad} h$, $h \in H$. Note that $L_0(H)$ is simply the centralizer of H. The α such that $L_\alpha(H) \neq 0$ are called roots of L with respect to H, a subcollection of H^* which we will denote by $\Phi(H)$. We will frequently make use of the root space decomposition of a semisimple L.

Lemma 4.8. ([11, p.35]) Given a CSA H of the semisimple L, we have the root space decomposition

$$L = H \oplus \coprod_{\alpha \in \ \Phi(H)} L_\alpha(H).$$

We may fix a base Δ of $\Phi(H)$ and note that

$$B(\Delta) := H + \coprod_{\alpha \succ 0} L_\alpha(H)$$

is a Borel subalgebra of L, which we call a **standard Borel** with respect to H. In addition,

$$N(\Delta) := [B(\Delta), B(\Delta)] = \coprod_{\alpha \succ 0} L_\alpha(H)$$

is nilpotent [11, p.84].

The following result on semisimple algebras is similar to Lemma 2.18:

Lemma 4.9. [11, p.36] Let H be a CSA of the semisimple Lie algebra L, and let α, $\beta \in H^*$. Then $[L_\alpha(H), L_\beta(H)] \subset L_{\alpha+\beta}(H)$.

Lemma 4.10. ([11, p.84]) Let H be a CSA of the semisimple L. All standard Borel subalgebras of L relative to H are conjugate via $\mathcal{E}(L)$.

The second main goal in this chapter is the following conjugacy theorem for BSAs.

Theorem 4.11. ([11, p.84]) The BSAs of a semisimple Lie algebra L are conjugate via $\mathcal{E}(L)$.

Proof. We will employ induction on $\dim L$. The proof is trivial if $\dim L = 1$, for L is itself its only BSA.

Let L be semisimple and H a CSA, and fix a base Δ and thus a standard BSA B with respect to H. We will show that any other BSA B' of L is conjugate to B via an element of

$\mathcal{E}(L)$. If $B \cap B' = B$, then $B = B'$ by the maximality of B, and there is nothing to prove. Thus a downward induction on the $\dim(B \cap B')$ is appropriate: we may assume that a BSA whose intersection with B has dimension greater than $\dim(B \cap B')$ is already a conjugate of B.

We divide the proof into cases, and the first case into two subcases.

Case 1: $B \cap B' \neq 0$.

Let $N' \subset B \cap B'$ be the set of ad-nilpotent elements of $B \cap B'$, i.e., $x \in B \cap B'$ and $\operatorname{ad} x \in \operatorname{End} L$ is nilpotent.

Subcase (i): Suppose $N' \neq 0$. By Lemma 4.8, B may be decomposed as

$$B = H + \coprod_{\alpha \succ 0} L_\alpha(H), \quad N := [B, B] = \coprod_{\alpha \succ 0} L_\alpha(H)$$

where H is the fixed CSA of L. So by Lie's theorem, the derived algebra $N = [B, B] = \coprod_{\alpha \succ 0} L_\alpha$ is nilpotent ($N' = N \cap B'$). Now as $[B \cap B', B \cap B']$ is properly contained in both $[B, B]$ and $[B', B']$, we see that $[B \cap B', B \cap B'] \subset B \cap B'$ is nilpotent. Thus, by Engel's theorem again, $[B \cap B', B \cap B'] \subset N'$, so N' is an ideal of $B \cap B'$. As L is semisimple, it is allowed no proper solvable (thus no nilpotent) ideals, so N' is not an ideal of L and

$$K := N_L(N') \subsetneq L.$$

We are going to show that $B \cap B'$ is properly contained in both $B \cap K$ and $B' \cap K$. Clearly from the above $B \cap B' \subset B \cap K$ and $B \cap B' \subset B' \cap K$. For each $x \in N' \subset B$, $\operatorname{ad} x$ is a nilpotent endomorphism of B (as well as of B') since B (respectively B') is stable under $\operatorname{ad} x$. But $B \cap B'$ is also stable under $\operatorname{ad} x$, so the induced action of $\operatorname{ad} x$ on the vector space $B/(B \cap B')$ is nilpotent (see the proof of Lemma 2.24). This is true for all $x \in N'$, so by Engel's Theorem, there is a non-zero $y + B \cap B' \in B/(B \cap B')$ killed by all elements of N'. We have found a $y \in B \setminus B \cap B'$ such that $[N', y] \subset B \cap B'$. In addition, $[N', y] \subset [B, B]$ (since $y \in B$ and $N' \subset B$), which is nilpotent. So every $[x, y] \in [N', y]$ is nilpotent, forcing

$[x, y] \in N'$ which is the set of nilpotent elements of $B \cap B'$. Thus $y \in N_B(N') = B \cap K$, but $y \notin B \cap B'$, i.e., $B \cap B' \subsetneq B \cap K$. The argument is symmetric for B and B' so

$$B \cap B' \subsetneq B \cap K, \quad B \cap B' \subsetneq B' \cap K.$$

As BSAs, B and B' are solvable in L, so $B \cap K$ and $B' \cap K$ are solvable subalgebras of K. We choose BSAs of K, C and C', containing $B \cap K$, $B' \cap K$ respectively (see the figure).

$$\begin{tikzcd}
& L \arrow[d, dash] & \\
& K \arrow[dl, dash] \arrow[dr, dash] & \\
C \arrow[d, dash] & & C' \arrow[d, dash] \\
B \cap K \arrow[dr, dash] & & B' \cap K \arrow[dl, dash] \\
& B \cap B' \arrow[d, dash] & \\
& N' &
\end{tikzcd}$$

Since $K \subsetneq L$, by the first induction hypothesis (on $\dim L$), C and C' are conjugate via $\sigma' \in \mathcal{E}(K)$. So there is $\sigma \in \mathcal{E}(L; K) \subset \mathcal{E}(L)$ such that $\sigma(C') = C$ [remark 4.2]. Now $B \cap B'$ is a proper subalgebra of both $B \cap K \subset C$ and $B' \cap K \subset C'$. Note that C may not be a BSA of L, but as it is solvable, it is contained in a BSA M of L. We have

$$B \cap B' \subsetneq B \cap K \subset B \cap C \subset B \cap M,$$

so $\dim(B \cap M) > \dim(B \cap B')$. By the second induction hypothesis (downward, on $\dim(B \cap B')$), there is a $\tau \in \mathcal{E}(L)$ with $\tau(M) = B$. We have $\tau(C) \subset B$, i.e.,

$$\tau\sigma(C') \subset B.$$

Since $B \cap B' \subsetneq B' \cap K$, we have $\tau\sigma(B \cap B') \subsetneq \tau\sigma(B' \cap K)$. Clearly $B' \cap K = B' \cap C'$, so

$$\tau\sigma(B \cap B') \subsetneq \tau\sigma(B' \cap K) = \tau\sigma(B' \cap C') \subset \tau\sigma(B') \cap \tau\sigma(C') \subset \tau\sigma(B') \cap B.$$

Then $\dim(B \cap \tau\sigma(B')) > \dim \tau\sigma(B \cap B') = \dim(B \cap B')$. Clearly B and $\tau\sigma(B')$ are BSAs of L. Then we use the second induction hypothesis (downward, on $\dim(B \cap B')$) to see that B is conjugate to $\tau\sigma(B')$ via an element of $\mathcal{E}(L)$.

Subcase (ii): Suppose $N' = 0$. Hence $B \cap B'$ contains no nonzero nilpotent elements. By Lemma 4.6, any BSA contains both the semisimple and nilpotent parts of any of its elements, so all elements of $B \cap B'$ are semisimple, i.e.,

$$T := B \cap B' \neq 0$$

is a toral subalgebra of the semisimple L. Recall that $B = H + N$ is a standard Borel and the subalgebra $N = [B, B]$ of L consists entirely of nilpotent elements. Since 0 is the only element which is both nilpotent and semisimple, $T \cap N = 0$. For any $x \in B$, $[x, T] \subset T$ means that $[x, T] \subset T \cap N = 0$. Thus

$$N_B(T) = C_B(T).$$

As T is toral, $T \subset C_B(T)$, and we may choose a maximal toral subalgebra C of $C_B(T)$ containing T. By definition, C is nilpotent and self-normalizing (in $C_B(T)$). So we have

$$T \subset C \subset C_B(T) = N_B(T) \subset N_B(C).$$

If $n \in N_B(C) \subset B$, then for each $t \in T \subset C$, $(\operatorname{ad} t)^k n = 0$ for some $k \in \mathbb{N}$ since C is nilpotent. However, $\operatorname{ad} t \in \operatorname{End} L$ is a semisimple endomorphism since T is toral. So $L_0(\operatorname{ad} t) = \operatorname{Ker} \operatorname{ad} t = \operatorname{Ker}(\operatorname{ad} t)^k$. Then we may choose $k = 1$, i.e., $n \in N_B(C) \subset C_B(T)$. But C is its own normalizer in $C_B(T)$. Then

$$C = N_{C_B(T)}(C) = N_B(C).$$

As a nilpotent self-normalizing subalgebra of B, C is a CSA of the solvable algebra B. Clearly H is a CSA of both B and L. By Theorem 4.4, C is conjugate, via an element of $\mathcal{E}(B)$ (hence via $\mathcal{E}(L)$), to H.

Thus without loss of generality we may assume that $T \subset H$.

Now we consider two subcases.

(A) $T = H$. Now $H = B \cap B' \subsetneq B'$ (otherwise $B = B'$ and there is nothing to prove). Recall that $B = H + \coprod_{\alpha \succ 0} L_\alpha$. Notice that $[H, B'] = [B \cap B', B'] \subset B'$, i.e., B' is stable under $\operatorname{ad}_L H$. Since $\operatorname{ad}_L H$ is simultaneously diagonalizable (as $\operatorname{ad} H \subset \operatorname{End} L$ is a commuting family of semisimple endomorphisms), $\operatorname{ad}_{B'} H \subset \operatorname{End} B'$ is also simultaneously diagonalizable by Lemma 2.25. This allows us to decompose B' into root spaces under H: $B' = H + \coprod_\beta B'_\beta$ where

$$B'_\beta = \{x \in B' \mid [h, x] = \beta(h)x \text{ for all } h \in H\}.$$

Clearly each $\beta \in \Phi$ and $B'_\beta = L_\beta$ since $\dim L_\beta = 1$ [11, p.39]. Since $H = B \cap B' \subsetneq B'$, we must have $B' = H + \coprod_{\alpha \prec 0} L_\alpha$. So there is at least one $\alpha \prec 0$ relative to Δ with $0 \neq L_\alpha \subset B'$. The reflection τ_α [11, p.42] sends α into $-\alpha$, H is preserved, and $\Delta' = \tau_\alpha(\Delta)$ is a base. Then B' is conjugate to the standard Borel $B'' := B(\Delta')$ whose intersection with B includes $H + L_{-\alpha}$. So the second induction hypothesis (downward, on $\dim B \cap B'$) applies and B'' is conjugate to B.

Remark: Instead of using τ_α, one can use the longest element $\omega_0 \in W$ [17, p.88] of the Weyl group W to get the conclusion immediately, i.e., B and B' are conjugate via ω_0, since $\omega_0(\Phi^+) = -\Phi^+$ where Φ^+ denotes the positive roots.

(B) $T \subsetneq H$. Then we have two subcases:

(B1) $B' \subset C_L(T)$. The semisimple L has zero center and we are assuming $T = B \cap B' \neq 0$. So $C_L(T) \subsetneq L$ as T itself is abelian, and we have $\dim C_L(T) < \dim L$. We are thus justified in using the first induction hypothesis on $C_L(T)$. Now $H = C_L(H)$ [11, p.36] and $T \subset H$ so that $H = C_L(H) \subset C_L(T)$. Since H is abelian and thus solvable, there is a BSA B'' of $C_L(T)$ containing H. By the first induction hypothesis, there exists $\sigma \in \mathcal{E}(L; C_L(T)) \subset \mathcal{E}(L)$ with $\sigma(B') = B''$, since B' and B'' are BSAs of $C_L(T)$. By virtue of the fact that B'' is the isomorphic image of the BSA B' of L, B'' is a BSA of L as well. Moreover B'' contains H, so

$$\dim(B \cap B'') \geq \dim H > \dim T = \dim(B \cap B').$$

By the second induction hypothesis (downward, on $\dim(B \cap B')$), B and B'' are conjugate via $\mathcal{E}(L)$, thus B and B' are, as well.

(B2) $B' \not\subset C_L(T)$. As in (A), since we are assuming $T \subsetneq H$, $\mathrm{ad}_{B'} T$ is simultaneously diagonalizable. Since $B' \not\subset C_L(T)$, there is $t' \in T$ and a common eigenvector $x \in B' \backslash C_L(T)$ of the endomorphisms of $\mathrm{ad}_{B'} T$ such that $[t', x] = c'x$, where $c' \in \mathbb{F}$ is nonzero. Setting $t := t'/c'$, we have $[t, x] = x$. Let $\Phi_t := \{\alpha \in \Phi : \alpha(t) > 0 \text{ and rational}\} \subset \Phi$. Given α, $\beta \in \Phi_t$, $\alpha + \beta \in \Phi_t$. Then

$$S := H + \coprod_{\alpha \in \Phi_t} L_\alpha$$

is subalgebra of L. Notice that $x \in S$ since x is of the form

$$x = x_h + \sum_{\alpha \in \Phi} x_\alpha, \quad x_\alpha \in L_\alpha, x_h \in H$$

and

$$x_h + \sum_{\alpha \in \Phi} x_\alpha = x = [t, x] = \sum_{\alpha \in \Phi} [t, x_\alpha] = \sum_{\alpha \in \Phi} \alpha(t) x_\alpha$$

so that $x_h = 0$ and $x_\alpha = 0$ if $\alpha \notin \Phi_t$. By Lemma 4.9 S is solvable and so lies in a BSA B''. Since $T \subset B''$ and $x \in B''$

$$B \cap B' = T \subsetneq T + \mathbb{F}x \subset B'' \cap B'.$$

we see that $\dim(B'' \cap B') > \dim(B \cap B')$; by the second induction hypothesis, B'' and B' are conjugate. In addition, $\dim(B'' \cap B) > \dim(B \cap B')$ since $H \not\subset B \cap B'$; thus B'' and B are conjugate, so B' and B are as well.

Case 2: $B \cap B' = 0$. As a standard BSA of L, $B = H + \coprod_{\alpha \succ 0} L_\alpha(H)$. But L itself is semisimple, thus allowing us to decompose $L = H \oplus \coprod_{\alpha \in \ \Phi(H)} L_\alpha(H)$. We know that α is a root if and only if $-\alpha$ is a root [11, p.37]; so $\dim B > (\dim L)/2$. However, since $B \cap B' = 0$, we employ a standard argument regarding dimension in vector spaces to see that

$$\dim B + \dim B' = \dim(B + B') \leq \dim L.$$

Thus B' must be a "small" Borel, i.e., $\dim B' < (\dim L)/2$. Choose a maximal toral subalgebra T of B'. If $T = 0$, then B' consists solely of ad-nilpotent elements. By Engel's theorem B' is nilpotent, and as it is Borel, it is also self-normalizing by Lemma 3.3. In other words, B' is a CSA of L; but this is impossible, for on one hand each CSA of the semisimple L is toral by Lemma 4.7, while on the other hand $T = 0$. So we must have $T \neq 0$. Once again applying Lemma 4.7, choose a CSA H_0 of L containing T, and B'' a standard Borel with respect to H_0. Then $B' \cap B'' \neq 0$, and by Case 1 of the proof, B' and B'' are conjugate so that $\dim B'' = \dim B' < (\dim L)/2$. However B'' is standard so that $\dim B'' > (\dim L)/2$, a contradiction. □

Theorem 4.12. ([11, p.84]) The BSAs of a Lie algebra L are conjugate under $\mathcal{E}(L)$.

Proof. We have previously established the theorem for semisimple Lie algebras; if L is not semisimple, then we construct the semisimple Lie algebra L/R $(\neq L)$. Now the BSAs of L are in 1-1 correspondence with the BSAs of L/R by Lemma 4.5. Given BSAs B_1 and B_2 of L, we identify them with the BSAs B_1' and B_2' of L/R. Now B_1' and B_2' are conjugate via an element σ' of $\mathcal{E}(L/R)$, and by Lemma 4.3, there is a $\sigma \in \mathcal{E}(L)$ such that $\phi \circ \sigma = \sigma' \circ \phi$, where ϕ is the canonical projection. So

$$\phi\sigma(B_1) = \sigma'\phi(B_1) = \sigma'(B_1') = B_2',$$

thus $\sigma(B_1) = B_2$. □

Theorem 4.13. ([11, p.84]) The CSAs of L are conjugate under $\mathcal{E}(L)$.

Proof. Any nilpotent subalgebra is solvable, thus contained in a maximal solvable subalgebra. So given a pair H, H' of CSAs of L, there is a pair B, B' of BSAs containing H, H' respectively. By Theorem 4.12, there is $\sigma \in \mathcal{E}(L)$ such that $\sigma(B) = B'$. We have two CSAs of the solvable B', namely $\sigma(H)$ and H', which are conjugate via $\tau' \in \mathcal{E}(B')$ thanks to Theorem 4.4. We have $\tau'\sigma(H) = H'$; once again, we extend τ' to $\tau \in \mathcal{E}(L)$. Then we have an endomorphism $\tau\sigma \in \mathcal{E}(L)$ such that $\tau\sigma(H) = H'$. □

Corollary 4.14. Every BSA of a Lie algebra L contains a CSA of L.

Proof. We have previously established the existence of CSAs in any Lie algebra (see remark following Theorem 3.9). Given a CSA H of L, H is contained in some BSA B of L. For any BSA B' of L, B and B' are conjugate via $\phi \in \mathcal{E}(L)$, thus B' contains the CSA $\phi(H)$. □

Chapter 5

Michael's Proof

Bourbaki [7, p.22-23] provides a proof for the conjugacy of CSAs using algebraic geometry; the approach is very different from that of Humphreys.

Michael's approach does not use algebraic geometry and is along the presentation of Bourbaki [5, 7]. Similar to Humphreys, he begins by proving the conjugacy of CSAs of solvable Lie algebras. The proof is similar to that in [7, p.25] and will be omitted here.

His next task is to provide a few lemmas to ease the way to a full proof. We will utilize Lemma 3.12, as well as various technical lemmas. We will also need Definition 2.20 in the following discussion:

$$L_{\alpha,h} := L_{\alpha(h)}(\operatorname{ad} h) = \operatorname{Ker}(\operatorname{ad} h - \alpha(h)\cdot 1)^m,$$

where m is the algebraic multiplicity of the eigenvalue $\alpha(h)$, and

$$L_\alpha(H) := \cap_{h\in H} L_{\alpha,h}$$

where H is a nilpotent subalgebra of L. Clearly $L_\alpha(H) \subset \mathcal{N}(L)$, indeed each $L_{\alpha,h} \subset \mathcal{N}(L)$ (Definition 2.19).

Definition 5.1. ([7, p.22], [16, p.156]) Given a Lie algebra L and CSA H, we denote by $\mathcal{E}_L(H)$ the subgroup of $\operatorname{Int} L$ generated by $\exp \operatorname{ad} x$, where $x \in L_\alpha(H)$ for some $\alpha \neq 0$.

Michael's proof relies upon several results which can be viewed as analogues of Lemma 2.26 and Lemma 4.3, which we will state in the following lemmas.

Recall Lemma 2.26: an epimorphism $\phi : L \to L'$ has the property that $\phi(L_a(\operatorname{ad} y)) = L_a(\operatorname{ad} \phi(y))$. The following lemma for $L_\alpha(H)$ corresponds to Lemma 2.26:

Lemma 5.2. Let $\phi : L \to L'$ be an epimorphism, and H a nilpotent subalgebra of L. Then $\phi(L_\alpha(H)) = L_{\alpha'}(\phi(H))$, where $\alpha' \in (\phi(H))^*$ is given by $\alpha'(\phi(h)) = \alpha(h)$.

Proof. Define α' as above; it is clear that $\phi(L_{\alpha(h)}(\operatorname{ad} h)) = L'_{\alpha'(\phi(h))}(\operatorname{ad} \phi(h))$ for all $h \in H$, i.e. $\phi(L_\alpha(H)) = L_{\alpha'}(\phi(H))$.

□

As analogue to Lemma 4.3, we have

Lemma 5.3. Given an epimorphism $\phi : L \to L'$ and $\sigma' \in \mathcal{E}_{L'}(H')$, H' a CSA of L', there is a CSA H of L with $\phi(H) = H'$ and $\sigma \in \mathcal{E}_L(H)$ such that $\sigma' \circ \phi = \phi \circ \sigma$.

Proof. Lemma 3.12 allows us to choose a CSA H of L such that $\phi(H) = H'$. Let $\sigma := \exp \operatorname{ad} x'$ be a generator of $\mathcal{E}_{L'}(H')$, i.e. $x' \in L'_{\alpha'}(H')$; by Lemma 5.2 there is $x \in L_\alpha(H)$ such that $\phi(x) = x'$. Then $\sigma := \exp \operatorname{ad} x \in \mathcal{E}_L(H)$ is the promised automorphism. □

Similar to Remark 4.2, we note the following:

Lemma 5.4. Given a Lie algebra L and subalgebra B containing a CSA H of L, any $\sigma \in \mathcal{E}_B(H)$ may be extended to a $\sigma' \in \mathcal{E}_L(H)$ such that $\sigma'|_B = \sigma$.

Proof. A generator $\exp \operatorname{ad}_B x$ of $\mathcal{E}_B(H)$ may be viewed as a generator of $\mathcal{E}_L(H)$: for $x \in B_\alpha(H)$ means that $x \in L_\alpha(H)$, as well. Thus $\exp \operatorname{ad}_L x \in \mathcal{E}_L(H)$. □

Lemma 5.5. Given a Lie algebra L, CSA H, and $u \in \mathcal{E}_L(H)$, we have $\mathcal{E}_L(u(H)) = u(\mathcal{E}_L(H))u^{-1}$.

Proof. By Lemma 3.12, $u(H)$ is a CSA, thus $\mathcal{E}_L(u(H))$ is defined.

Given a generator $\exp \operatorname{ad} x$ of $\mathcal{E}_L(u(H))$, we note that

1. $(\operatorname{ad} u(h) - \alpha(u(h))I)^k(x) = 0$ for all $h \in H$
2. $\operatorname{ad} u(h) = u \circ (\operatorname{ad} h) \circ u^{-1}$

Define $\alpha' \in H^*$ by $\alpha'(h) := \alpha(u(h))$. Then for all $h \in H$,

$$0 = (\operatorname{ad} u(h) - \alpha(u(h))I)^k(x) = u \circ (\operatorname{ad} h - \alpha'(h)I)^k \circ u^{-1}(x).$$

So $u^{-1}(x) \in L_{\alpha'}(H)$, and $\exp \operatorname{ad}(u^{-1}(x)) \in \mathcal{E}_L(H)$. Then we readily see that

$$\exp \operatorname{ad} x = u(\exp \operatorname{ad}(u^{-1}(x)))u^{-1} \in u(\mathcal{E}_L(H))u^{-1}.$$

Thus $\mathcal{E}_L(u(H)) \subset u(\mathcal{E}_L(H))u^{-1}$.

On the other hand, given $y \in L_\beta(H)$, we wish to show that $u(\exp \operatorname{ad} y)u^{-1} \in \mathcal{E}_L(u(H))$. Clearly $u(\exp \operatorname{ad} y)u^{-1} = \exp \operatorname{ad} u(y)$. Defining $\beta' \in (u(H))^*$ by $\beta'(u(h)) = \beta(h)$, we see by Lemma 5.3 that $u(y) \in L_{\beta'}(u(H))$. The lemma follows. □

The following is a refinement of [7, p.25-26] and the proof is almost identical.

Theorem 5.6. If L is solvable and H_1, H_2 are CSAs of L, there exist $u_i \in \mathcal{E}_L(H_i)$, $i = 1, 2$, such that $u_1(H_1) = u_2(H_2)$.

Corollary 5.7. ([16, p.158]) If L is solvable and H_1, H_2 are CSAs of L, $\mathcal{E}_L(H_1) = \mathcal{E}_L(H_2)$.

Proof. Using Lemma 5.5 and Theorem 5.6, we have

$$\mathcal{E}_L(H_1) = u_1\mathcal{E}_L(H_1)u_1^{-1} = \mathcal{E}_L(u_1(H_1)) = \mathcal{E}_L(u_2(H_2)) = u_2\mathcal{E}_L(H_1)u_2^{-1} = \mathcal{E}_L(H_2).$$

□

So if L is solvable, the choice of H is inconsequential and we may denote $\mathcal{E}_L(H) = \mathcal{E}_L$.

Corollary 5.8. ([16, p.158]) Assume that L is solvable.

1. $\mathcal{E}_L$ acts transitively on the set of CSAs of L.

2. Any CSA of L has dimension $\operatorname{rank} L$.

3. $L_0(\operatorname{ad} x)$ is a CSA of L if and only if x is regular.

Proof. (1) is clear from Lemma 3.12 and Corollary 5.7. For (2), we note that $\mathcal{E}_L$ is a group of automorphisms; so by (1), all CSAs have the same dimension. By Lemma 3.9, this must be the rank of L. Lemma 3.6 combined with Lemma 3.9 immediately give us (3). ∎

Definition 5.9. ([11, p.21]) Given a Lie algebra L, the Killing form κ is a symmetric bilinear form on L given by

$$\kappa(x, y) = \mathrm{tr}\,(\mathrm{ad}\,x)(\mathrm{ad}\,y).$$

Elements x and y of L are orthogonal via κ if $\kappa(x, y) = 0$. Two subspaces H_1 and H_2 are orthogonal, $H_1 \perp H_2$, if their elements are mutually orthogonal via κ.

Remark 5.10. We will have occasion to employ a useful identity regarding the Killing form. Note that, since $[\mathrm{ad}\,x, \mathrm{ad}\,y] \in \mathfrak{gl}(L)$, we know the action of $[\cdot, \cdot]$:

$$\mathrm{ad}\,[x, y] = [\mathrm{ad}\,x, \mathrm{ad}\,y] = \mathrm{ad}\,x\ \mathrm{ad}\,y - \mathrm{ad}\,y\ \mathrm{ad}\,x.$$

Then

$$\kappa([x, y], z) = \mathrm{tr}\,([\mathrm{ad}\,x, \mathrm{ad}\,y]\mathrm{ad}\,z) = \mathrm{tr}\,(\mathrm{ad}\,x\ \mathrm{ad}\,y\ \mathrm{ad}\,z) - \mathrm{tr}\,(\mathrm{ad}\,y\ \mathrm{ad}\,x\ \mathrm{ad}\,z).$$

But we may commute matrices without changing the trace, so the above is equal to

$$\mathrm{tr}\,(\mathrm{ad}\,x\ \mathrm{ad}\,y\ \mathrm{ad}\,z) - \mathrm{tr}\,(\mathrm{ad}\,x\ \mathrm{ad}\,z\ \mathrm{ad}\,y) = \mathrm{tr}\,(\mathrm{ad}\,x[\mathrm{ad}\,y, \mathrm{ad}\,z]) = \kappa(x, [y, z]).$$

Thus $\kappa([x, y], z) = \kappa(x, [y, z])$.

Theorem 5.11. ([11, p.22]) Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate, i.e., the radical S of the Killing form, defined by

$$S := \{x \in L \mid \kappa(x, y) = 0 \ \text{ for all } \ y \in L\}$$

is 0.

In general, if V is a finite dimensional vector space over $\mathbb{F}$ with nondegenerate symmetric form [15, Chapter XV] κ and if $W \subset V$ is a subspace, we denote by

$$W^\perp := \{v \in V | \kappa(v, w) = 0 \text{ for all } w \in W\}$$

the orthogonal complement of W with respect κ. See [15, Chapter XV], [24] for the general theory of bilinear form. An orthogonal basis [15, p.575] always exists for V with $\dim V \geq 1$ (but not necessarily an orthonormal basis, for example, κ on $\mathbb{F}^2$ defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Moreover κ induces an isomorphism between V^* and V: $f \mapsto (x, \cdot)$, $f \in V^*$ and $x \in V$. The restriction of κ on a subspace W is nondegenerate if and only if $W \cap W^\perp = 0$.

Lemma 5.12. Let V be a finite dimensional vector space with nondegenerate symmetric bilinear form κ. Let W, V_1, V_2 be subspaces of V. Then

1. $\dim W + \dim W^\perp = \dim V$ (but $W + W^\perp = V$ is not necessarily true).

2. $(W^\perp)^\perp = W$.

3. $V_1 \subset V_2$ if and only if $V_2^\perp \subset V_1^\perp$; $V_1 \subsetneq V_2$ if and only if $V_2^\perp \subsetneq V_1^\perp$.

4. $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$.

5. $V_1^\perp + V_2^\perp = (V_1 \cap V_2)^\perp$.

Proof. 1. Denote by $f : V \to W^*$ the map defined by $f(v)(w) = \kappa(v, w)$, $w \in W$, $v \in V$. Clearly $\mathrm{Ker} f = W^\perp$ and $\mathrm{Im}\, f = W^*$ so that $\dim \mathrm{Im}\, f = \dim W^* = \dim W$. Then apply the dimension theorem

$$\dim \mathrm{Ker} f + \dim \mathrm{Im}\, f = \dim V.$$

2. Clearly $W \subset (W^\perp)^\perp$. Then from (1) $\dim W + \dim W^\perp = \dim V = \dim W^\perp + \dim (W^\perp)^\perp$ so that $\dim W = (W^\perp)^\perp$. Hence $(W^\perp)^\perp = W$.

3. The first part is clear and the second part is from (1) and (2) since $\dim V_1^\perp = \dim V - \dim V_1 > \dim V - \dim V_2 = \dim V_2^\perp$.

4. From (3) $(V_1 + V_2)^\perp \subset V_i^\perp$, $i = 1, 2$, since $V_i \subset V_1 + V_2$. Thus $(V_1 + V_2)^\perp \subset V_1^\perp \cap V_2^\perp$. For any $w \in V_1^\perp \cap V_2^\perp$ and for all $v = v_1 + v_2 \in V_1 + V_2$, where $v_i \in V_i$, $i = 1, 2$, we have $\kappa(w, v_1 + v_2) = \kappa(w, v_1) + \kappa(w, v_2) = 0$. We may choose either of v_1, v_2 to be 0, so $w \in (V_1 + V_2)^\perp$.

5. By (2) each subspace W of V is the orthogonal complement of some subspace, namely, $W^\perp$. So it suffices to show $V_1 + V_2 = (V_1^\perp \cap V_2^\perp)^\perp$ and again by (2), it is simply (4).

□

We know focus our attention on semisimple Lie algebras.

Lemma 5.13. ([16, p.159]) Let L be a semisimple Lie algebra, H a CSA of L, and B a solvable subalgebra of L containing H. Then

1. $B = H \oplus [B, B]$

2. The set $[B, B]$ coincides with the set of ad-nilpotent elements in L contained in B.

Proof. By virtue of the fact that H is a CSA of L, H is in turn a CSA of B. Since $[H, B] \subset B$, $\mathrm{ad}_B H \subset \mathrm{End}\, B$ is a simultaneously diagonalizable family by Theorem 2.25. Thus B has root space decomposition $B = H \oplus \sum_{\alpha \in \Phi(H)} B_\alpha(H)$. By Lemma 3.6, we write $H = L_0(\mathrm{ad}\, s)$ for some $s \in H$. Note that the restriction of $\mathrm{ad}\, s$ to $B^+(H) := \sum_{\alpha \neq 0} B_\alpha(H)$ is bijective, for $B^+(H)$ contains no eigenvectors of $\mathrm{ad}\, s$ with corresponding eigenvalue 0. We conclude that any $x \in B^+(H)$ can be written $x = [s, m]$ for some $m \in B^+(H)$; so we have $B^+(H) \subset [B, B]$. Now B is solvable, thus $[B, B]$ is nilpotent; in addition, H is semisimple. Thus any $y \in H \cap [B, B]$ is both ad-nilpotent and ad-semisimple, that is $y = 0$. We have $B = H \oplus \sum_{\alpha \neq 0} B^\alpha(H) \subset H \oplus [B, B] \subset B$, and (1) follows.

The second statement follows from

$$[B,B] = [H \oplus \sum_{\alpha\in\Phi(H)} B_\alpha(H), H \oplus \sum_{\alpha\in\Phi(H)} B_\alpha(H)] = \sum_{\alpha\in\Phi(H)} B_\alpha(H).$$

On one hand, we know that the only endomorphism of L that is simultaneously nilpotent and semisimple is 0. Since elements of H are ad-semisimple, the ad-nilpotent elements of L contained in B must be in $[B,B]$. On the other hand, by Lemma 4.9 and the fact that $B_\alpha(H) \subset L_\alpha(H)$, $\operatorname{ad}_L x$ is a nilpotent endomorphism for any $x \in [B,B]$. □

We may now prove

Lemma 5.14. ([16, p.159]) A subalgebra B of a semisimple Lie algebra L is a BSA if and only if $[B,B] = B^\perp$.

Proof. Since L is semisimple, the subalgebra $H \subset L$ is solvable if and only if $\operatorname{ad} H$ is solvable. For the forward implication, by Cartan's criterion [11, p.20] B is solvable if and only if $\kappa(B,[B,B]) = 0$. Thus $[B,B] \subset B^\perp$. Suppose that the inclusion is proper; by Lemma 5.12, $B \subsetneq [B,B]^\perp$. We set $P := [B,B]^\perp/B$, a (nontrivial) vector space quotient. The the matrices of $\operatorname{ad} B$ form a solvable subalgebra of $\mathfrak{gl}(P)$. By Lie's Theorem, the matrices of $\operatorname{ad} B$ are upper triangular (with respect to the proper basis of P); thus there is an $x \in [B,B]^\perp \setminus B$ such that $[B,x] \subset B + \mathbb{F}x$. Set $B_1 := B + \mathbb{F}x$. Then B_1 is a subalgebra of L.

By Lemma 5.12, $B_1^\perp = (B+\mathbb{F}x)^\perp = B^\perp \cap \mathbb{F}x^\perp$. Now $[B_1,B_1] \subset [B,B] + [B,\mathbb{F}x]$; by Cartan's Criterion, since B is solvable, $[B,B] \subset B^\perp$. But x was chosen from $[B,B]^\perp$, so we also have $[B,B] \subset (\mathbb{F}x)^\perp$; thus $[B,B] \subset B^\perp \cap \mathbb{F}x^\perp = B_1^\perp$. For any $a,b \in B$, $\kappa([a,x],b) = -\kappa(x,[a,b]) = 0$, since $x \in [B,B]^\perp$. Thus $[B,\mathbb{F}x] \subset B^\perp$. Finally, given $a \in B$, we note that $\kappa([a,x],x) = \kappa(a,[x,x]) = 0$, allowing us to write $[B,\mathbb{F}x] \subset (\mathbb{F}x)^\perp$. So $[B,\mathbb{F}x] \subset B_1^\perp$ as well; we conclude that $[B_1,B_1] \subset B_1^\perp$. Once again using Cartan's Criterion, B_1 is solvable; but $B \subsetneq B_1$ was chosen as Borel, a contradiction. We are forced to have $[B,B] = B^\perp$.

We now show the reverse implication. Suppose $[B, B] = B^\perp$, and choose a solvable subalgebra B_1 of L containing B. Then

$$[B_1, B_1] \subseteq B_1^\perp \subseteq B^\perp = [B, B] \subseteq [B_1, B_1],$$

and all inclusions are forced to be equalities. So $B_1^\perp = B^\perp$ implies $B_1 = B$ by Lemma 5.12, and B is a BSA. □

Theorem 5.15. ([16, p.160]) Let L be semisimple with CSAs H_1 and H_2, and let B_1, B_2 be BSAs containing H_1 and H_2, respectively. Then

1. $B_1 \cap B_2$ contains a CSA of L.

2. There exist $u_i \in \mathcal{E}_L(H_i)$ $(i = 1, 2)$ such that $u_1(H_1) = u_2(H_2)$.

Proof. 1. Define $N_i := [B_i, B_i]$; then by Lemma 5.13 and Lemma 5.14, we have $B_i = H_i \oplus N_i$ and $B_i = N_i^\perp$. Since N_i is the set of all ad-nilpotent elements of L contained in B_i, we know that $B_1 \cap N_2 = N_1 \cap N_2 = B_2 \cap N_1$ is the set of all ad-nilpotent elements of $B_1 \cap B_2$. By Lemma 5.12

$$B_1 = N_1^\perp \subset (N_1 \cap N_2)^\perp = (B_1 \cap N_2)^\perp = B_1^\perp + N_2^\perp = N_1 + B_2.$$

We have shown that $B_1 \subset N_1 + B_2$, thus

$$B_1 = N_1 + (B_1 \cap B_2).$$

Let $r_i = \dim H_i$. By symmetry, we may assume $r_1 \leq r_2$. By Corollary 5.8, $r_i = \operatorname{rank} B_i$. By Lemma 3.6 the CSA H_1 may be written as $H_1 = L_0(\operatorname{ad} z)$ for some $z \in H_1$. Since $B_1 = N_1 + (B_1 \cap B_2) = H_1 + N_1$, we may choose $n \in N_1$ such that

$$w := z + n \in B_1 \cap B_2.$$

By Lie's theorem, there is a basis of L so that $\operatorname{ad} w$, $\operatorname{ad} z$ and $\operatorname{ad} n$, in matrix form, are upper triangular. In addition, because $\operatorname{ad} n$ is nilpotent, it is also strictly upper triangularizable; thus the diagonal entries of $\operatorname{ad} z$ are precisely those of $\operatorname{ad} w$, i.e. the pair of endomorphisms share eigenvalues. Now $\dim L_0(\operatorname{ad} z)$ and $\dim L_0(\operatorname{ad} w)$ are the algebraic multiplicity of 0 as an eigenvalue of $\operatorname{ad} z$ and $\operatorname{ad} w$, respectively; thus $\dim L_0(\operatorname{ad} z) = \dim L_0(\operatorname{ad} w)$. So

$$r_1 = \dim H_1 = \dim L_0(\operatorname{ad} z) = \dim L_0(\operatorname{ad} w) \geq \dim(B_i)_0(\operatorname{ad} w) \geq \operatorname{rank} B_i = r_i \geq r_1.$$

Thus each step above is an equality, and we have $\operatorname{rank} B_1 = r_1 = r_2 = \operatorname{rank} B_2$. So on one hand, every CSA of L has dimension $\operatorname{rank} L = \dim L_0(\operatorname{ad} z)$, while on the other hand, by Lemma 3.9, if $H = L_0(\operatorname{ad} x)$ has dimension $\operatorname{rank} L$, then H is a CSA. So a subalgebra is a CSA if and only if $H = L_0(\operatorname{ad} x)$ for some regular x. In particular, $L_0(\operatorname{ad} w) \subset B_1 \cap B_2$ is a CSA, and (1) follows.

Considering $H = L_0(\operatorname{ad} w)$ and H_i as CSAs of the (solvable) B_i, we may choose $u_i \in \mathcal{E}_{B_i}(H_i)$ such that $u_1(H_1) = H = u_2(H_2)$ by Theorem 5.6. We employ Lemma 5.4 to extend each u_i to an element $u_i' \in \mathcal{E}_L(H_i)$, and (2) follows. □

The following theorem will fill in the gap between the solvable and semisimple cases.

Theorem 5.16. ([16, p.158]) Let L be a Lie algebra and $\phi : L \to L/R$ the canonical homomorphism of L, where $R := \operatorname{Rad} L$. Then the following statements for CSAs H_1, H_2 of L are equivalent:

1. There exist $u_i \in \mathcal{E}_L(H_i)$ such that $u_1(H_1) = u_2(H_2)$.

2. There exist $v_i \in \mathcal{E}_{L/R}(\phi(H_i))$ such that $v_1(\phi(H_1)) = v_2(\phi(H_2))$.

Proof. That (1) implies (2) is straightforward: given $u_i \in \mathcal{E}_L(H_i)$, $u_i = \prod \exp \operatorname{ad} x$, define $v_i \in \operatorname{End}(L/R)$ by $v_i = \prod \exp \operatorname{ad}(x + R)$; then $v_i(y + R) = u_i(y) + R$, so $v_1(\phi(H_1)) = v_2(\phi(H_2))$. All that remains to show is that the v_i are indeed elements of $\mathcal{E}_{L/R}(\phi(H_i))$. Now $u_i = \prod \exp(\operatorname{ad} x)$ where $x \in L_\alpha(H_i)$, and by Lemma 5.2, $\phi(x) \in \phi(L_\alpha(H_i))$ means that

$x + R \in L/R_{\alpha'}(\phi(H_i))$. Thus $v_i \in \mathcal{E}_{L/R}(\phi(H_i))$. By construction, we have $v_i \in \mathcal{E}_{L/R}(H_i)$ satisfying $v_1(\phi(H_1)) = v_2(\phi(H_2))$.

The reverse implication requires more work: by Lemma 5.3, given $v_1(\phi(H_1)) = v_2(\phi(H_2))$, we have $u_i \in \mathcal{E}_L(H_i)$ with $\phi(u_1(H_1)) = \phi(u_2(H_2))$, that is $u_1(H_1)/R = u_2(H_2)/R$. Consider the set $u_1(H_1) + R = u_2(H_2) + R$ as a subalgebra of L, called T; viewed as a Lie algebra, we see that each $u_i(H_i)$ is a CSA of T. Thus the $u_i(H_i)$ are solvable subalgebras of T; R, the radical of L, is also solvable, so T itself is solvable. We already know that CSAs of solvable Lie algebras are conjugate (5.6), so we may choose $u_i' \in \mathcal{E}_T(u_i(H_i))$ such that $u_1'u_1(H_1) = u_2'u_2(H_2)$. By Lemma 5.4 the $\mathfrak{u}_i'$ may be extended to u_i'' in $\mathcal{E}_L(u_i(H_i))$.

Then $u_i'u_i \in \mathcal{E}_T(u_i(H_i))u_i$ can be written as $u_i''u_i \in \mathcal{E}_L(u_i(H_i))u_i$. By Lemma 5.5 $\mathcal{E}_L(u_i(H_i))u_i = u_i\mathcal{E}_L(H_i) = \mathcal{E}_L(H_i)$, and we conclude that $u_i''u_i \in \mathcal{E}_L(H_i)$. □

Theorem 5.17. ([16, p.160]) Given CSAs H_1 and H_2 of the Lie algebra L, there exist $u_i \in \mathcal{E}_L(H_i)$ such that $u_1(H_1) = u_2(H_2)$.

Proof. L/R is semisimple when $R = \operatorname{Rad} L$. By Theorem 5.15, with $\phi : L \to L/R$ the canonical projection, $\phi(H_1)$ and $\phi(H_2)$ are conjugate, and by Theorem 5.16 H_1 and H_2 are as well. □

Theorem 5.18. ([16, p.161]) For any Lie algebra L,

1. $\mathcal{E}_L(C)$ does not depend on the choice of the CSA C, and may be denoted $\mathcal{E}_L$.
2. $\mathcal{E}_L$ acts transitively on the set of CSAs of L.
3. Any CSA has dimension $\operatorname{rank} L$.
4. The element $x \in L$ is regular if and only if $L_0(\operatorname{ad} x)$ is a CSA, and any CSA of L may be written in this form.

Proof. The proof of (1) is identical to the proof of Corollary 5.7; (2), (3), and (4) are similar to Corollary 5.8. □

In order to establish the conjugacy of BSAs, we shall need several preliminary results.

Lemma 5.19. ([16, p.161]) Any BSA of a semisimple Lie algebra L contains a CSA of L.

Proof. Given a BSA B of L, choose a CSA H of B. We will show that H is actually a CSA of L as well; to do so we simply need $N_L(H) = H$. Since H is nilpotent, by Theorem 2.21

$$L = \coprod_{\alpha \in H^*} L_\alpha(H).$$

Given $h \in H$, we use abstract Jordan decomposition ([11, p.24]) to decompose h (in L) as $h = h_s + h_n$, where h_s and h_n are the ad-semisimple and ad-nilpotent parts, respectively, of h. We need both of the pieces h_s and h_n to be elements of H. Now $\operatorname{ad} h_n$ may be put into strictly upper triangular form and $\operatorname{ad} h_s$ may be diagonalized simultaneously, so the action of $\operatorname{ad} h_s$ on an element $x \in L_\alpha(H)$ is simply multiplication by $\alpha(h)$. Of particular importance is the fact that $[h_s, L_0(H)] = 0$, so $h_s \in Z(L_0(H))$. Now $\operatorname{ad} h_s$ is a polynomial in $\operatorname{ad} h$ ([11, p.17]), so $\operatorname{ad} h_s(H) \subset H$, thus $h_s \in N_L(H)$. Also, $h \in B$ implies that $\operatorname{ad} h(B) \subset B$, so $h_s \in N_L(B)$, indeed $h_s \in N_L(B) \cap N_L(H)$. By Lemma 3.3, $B = N_L(B)$, so $h_s \in B \cap N_L(H) = N_B(H) = H$; we have $h_s \in H$, so $h_n \in H$ as well. Thus H contains the semisimple and nilpotent parts of all of its elements.

Now $L_0(H)$ is reductive by [7, p.10], so we may use [5, p.56] to write

$$L_0(H) = Z(L_0(H)) \oplus [L_0(H), L_0(H)].$$

As $H \subset L_0(H)$, $[Z(L_0(H)), H] = 0$, so $Z(L_0(H)) \subset N_L(H)$. Given $x \in N_L(H)$, $\operatorname{ad} h(x) \in H$ implies that $(\operatorname{ad} h)^n(\operatorname{ad} h(x)) = 0$ for sufficiently large n, since H is nilpotent. Thus we know that $N_L(H) \subset L_0(H)$. We have $Z(L_0(H)) \subset N_L(H) \subset L_0(H)$, so utilizing the decomposition of $L_0(H)$, we write

$$N_L(H) = Z(L_0(H)) \oplus (N_L(H) \cap [L_0(H), L_0(H)]).$$

Thus we need to show

1. $Z(L_0(H)) \subset H$

2. $N_L(H) \cap [L_0(H), L_0(H)] \subset H$

The decomposition of L yields a decomposition of B:

$$B = B \cap L = B \cap \coprod_{\alpha \in H^*} L_\alpha(H) = \coprod_{\alpha \in H^*} B_\alpha(H),$$

where $B_\alpha := B \cap L_\alpha$. Set $B^+(H) := \coprod_{\alpha \neq 0} B_\alpha(H)$. As H is a CSA of B, we know that $B_0(H) = H$ by Lemma 3.5. So the decomposition of B is given by

$$B = H \oplus B^+(H).$$

Since we have finitely many nonzero $\alpha \in H^*$ in the above decomposition such that $B_\alpha(H) \neq 0$, we can find a nonzero $\beta \in H^*$ such that β is not contained in the union of the orthogonal complements of α's. Transporting back to H, that means that there exists $h \in H$ such that $\alpha(h) \neq 0$ for any nonzero $\alpha \in H^*$ such that $B_\alpha(H) \neq 0$. Thus $\operatorname{ad} h$ acts bijectively on $B^+(H)$, so $B^+(H) \subset [B, B]$. Then

$$[B, B] = B \cap [B, B] = (H \cap [B, B]) \oplus B^+(H).$$

By [11, p.36], $\kappa(L_\alpha(H), L_\beta(H)) \neq 0$ when $\alpha + \beta \neq 0$, and in particular since $Z(L_0(H)) \subset L_0(H)$, we have $\kappa(Z(L_0(H)), B^+(H)) = 0$. This forces $B^+(H) \subset (Z(L_0(H)))^\perp$. In addition, $[Z(L_0(H)), H \cap [B, B]] = 0$ since $H \cap [B, B] \subset H \subset L_0(H)$.

As B is solvable, $[B, B]$ is nilpotent. Then for each $h \in H \cap [B, B]$, we know that $\operatorname{ad}_B h$ is a nilpotent endomorphism. Given $x \in Z(L_0(H))$ and $h \in H \cap [B, B]$, $\operatorname{ad} x \ \operatorname{ad} y$ is also a nilpotent endomorphism of B. We conclude that $\operatorname{tr}(\operatorname{ad} x \ \operatorname{ad} y) = 0$. Thus $H \cap [B, B] \subset (Z(L_0(H)))^\perp$. We have previously shown that $B^+(H) \subset (Z(L_0(H)))^\perp$; as $[B, B] = (H \cap$

$[B, B]) \oplus B^+(H)$, we have shown

$$[B, B] \subset Z(L_0(H))^\perp.$$

As L is semisimple, we know by Lemma 5.14 that $[B, B] = B^\perp$. Then $B^\perp = [B, B] \subset Z(L_0(H))^\perp$ implies

$$Z(L_0(H)) \subset B.$$

Then $Z(L_0(H)) \subset B \cap N_L(H) = H$, i.e. we have shown (1): $Z(L_0(H)) \subset H$.

We still need to show (2), that

$$N_L(H) \cap [L_0(H), L_0(H)] \subset H.$$

Suppose that $z \in N_L(H) \cap [L_0(H), L_0(H)]$. Then

$$[H + \mathbb{F}z, H + \mathbb{F}z] \subset [H, H] + [H, \mathbb{F}z] \subset H.$$

Thus $H + \mathbb{F}z$ must be a solvable subalgebra of L, and by Cartan's Criterion,

$$\kappa(H + \mathbb{F}z, [H + \mathbb{F}z, H + \mathbb{F}z]) = 0.$$

In addition, since L is semisimple and $H + \mathbb{F}z$ is solvable, $[H + \mathbb{F}z, H + \mathbb{F}z]$ is precisely the set of ad-nilpotent elements of L contained in $H + \mathbb{F}z$ by Lemma 5.13. Given $h \in H$, $\mathrm{ad}_L h_n$ is nilpotent and $h_n \in H$; so $h_n \in [H + \mathbb{F}z, H + \mathbb{F}z]$. Thus $\kappa(z, h_n) = 0$. Recall that $[h_s, L_0(H)] = 0$; so $h_s \in Z(L_0(H))$, and

$$\kappa(z, h_s) \in \kappa([L_0(H), L_0(H)], Z(L_0(H))) = \kappa(L_0(H), [L_0(H), Z(L_0(H))]),$$

where the last equality follows by Remark 5.10. But $[L_0(H), Z(L_0(H))] = 0$, so

$$\kappa(L_0(H), [L_0(H), Z(L_0(H))]) = 0.$$

We have $\kappa(z, h_s) = 0$, giving us $\kappa(z, h) = 0$ for any $h \in H$. Now z was an arbitrary element of $N_L(H) \cap [L_0(H), L_0(H)]$, so

$$N_L(H) \cap [L_0(H), L_0(H)] \subset H^{\perp}.$$

Clearly $N_L(H) \cap [L_0(H), L_0(H)] \subset L_0(H)$, and since $B^+(H) \subset \coprod_{\alpha \in H^*} L_\alpha(H)$, we know that $L_0(H) \subset (B^+(H))^{\perp}$. Then $B = H \oplus B^+(H)$ implies that

$$N_L(H) \cap [L_0(H), L_0(H)] \subset H^{\perp} \cap (B^+(H))^{\perp} = B^{\perp}.$$

Recall that $B^{\perp} = [B, B]$. Then

$$N_L(H) \cap [L_0(H), L_0(H)] \subset B^{\perp} \cap N_L(H) \subset B \cap N_L(H) = H,$$

and we have shown (2).

To conclude, we have shown that

$$N_L(H) = Z(L_0(H)) \oplus (N_L(H) \cap [L_0(H), L_0(H)])$$

and as each piece of the summand is a subset of H, it follows that $N_L(H) = H$, i.e. H is a CSA of L.

□

Lemma 5.20. The intersection of two BSAs of a semisimple L contains a CSA.

Proof. Follow immediately by Theorem 5.15 and Lemma 5.19. □

Lemma 5.21. ([16, p.163]) The BSAs of a semisimple Lie algebra L are conjugate under $\mathcal{E}_L$.

Proof. Given a pair B_1, B_2 of BSAs, by Lemma 5.20 their intersection contains a CSA H. The B_i are stable under H, indeed the matrices of $\operatorname{ad}_{B_i} H$ are simultaneously diagonalizable. So we may decompose B_i as

$$B_i = H + \coprod_{\alpha \in \Phi_i} B_{i,\alpha},$$

where $\Phi_i \subset \Phi$. But if $\alpha \in \Phi_i$ then $B_{i,\alpha} = L_\alpha$. We may then use a standard argument involving the Weyl group to permute B_1 onto B_2. (See [11, p.75]) □

Theorem 5.22. ([16, p.161]) Each BSA B of a Lie algebra L contains a CSA of L.

Proof. Consider the image of B under the canonical homomorphism $\phi : L \to L/R$, where R is the radical of L. By Lemma 4.5 $\phi(B) = B'$ is a BSA of the semisimple L/R and by Lemma 5.19 B' contains a CSA H' of L/R. We may find a CSA H of L such that $\phi(H) = H'$ by Lemma 3.12. H is contained in a BSA $\tilde{B}$, and the BSA $\phi(\tilde{B}) = \tilde{B}'$ of L/R may be permuted via an element σ' of $\mathcal{E}_{L/R}$ onto B' by Lemma 5.21. By Lemma 5.3 there is a $\sigma \in \mathcal{E}_L$ with $\sigma'\phi = \phi\sigma$, i.e.

$$\sigma'\phi(\tilde{B}) = \sigma'(\tilde{B}') = B' = \phi\sigma(\tilde{B}).$$

But the BSAs of L are in 1-1 correspondence with the BSAs of L/R by Lemma 4.5, so $\sigma(\tilde{B}) = B$, and in particular $\sigma(H)$ is a CSA of L contained in B. □

Theorem 5.23. ([16, p.163]) The following statements hold in any Lie algebra L:

1. The intersection of two BSAs contains a CSA.

2. The BSAs of L are conjugate under $\mathcal{E}_L$.

Proof. Both statements have already been established for semisimple L; using a process similar to that in the proof of Theorem 5.22, the theorem follows. □

Chapter 6

Some remarks

Cartan subalgebras exist for finite dimensional Lie algebras whenever the base field is infinite. Indeed Barnes [1] (see also [2]) shows that if L is a Lie algebra of dimension n over a field $\mathbb{F}$ of at least $n-1$ elements, then there exists a Cartan subalgebra of L. Barnes also showed that every finite-dimensional solvable Lie algebra has a Cartan subalgebra over any field.

When $\mathbb{F}$ has no more than $\dim_{\mathbb{F}} L$ elements, the existence of Cartan subalgebras of finite dimensional Lie algebras is still an open problem [11, p.80] [19, p.509].

In [3] Billig and Pianzola give an example of a Lie algebra L of countable dimension which has no Cartan subalgebras (the definition of CSA given in [3] reduces to the classical case if L is finite dimensional).

Real case: In general, Cartan subalgebras in a real Lie algebra g are not necessarily conjugate. Example: In $\mathfrak{sl}_2(\mathbb{R})$ there are two essentially different Cartan subalgebras

$$H_1 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}, \qquad H_2 = \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

Notice that in $\mathfrak{sl}_2(\mathbb{C})$ these subalgebras are conjugate.

Kostant [14] and Sugiura [21] gave the theory of conjugacy classes of Cartan subalgebras for a real simple (and hence semisimple) noncompact Lie algebra L.

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