The Inverse Domination Number Problem, DI-Pathological Graphs, and Fractional Analogues

by

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Abstract

The conjecture that $\alpha(G) \geq \gamma'(G)$ is unproven where $\alpha(G)$ is the vertex independence number and $\gamma'(G)$ is the inverse domination number of a simple graph G. We have found the conjecture to be true for all graphs with domination number less than 5 and for many other infinite classes of graphs. We examine related questions involving DI-pathological graphs which are graphs such that every maximal independent set intersects with every minimum dominating set. Finally, we use two central results in linear programming to characterize minimum fractional total dominating functions as well as maximum fractional open neighborhood packings for certain graphs.

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Chapter 1

Introduction

1.1 Basic Definitions

Throughout this dissertation, the graph $G = (V, E)$ will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

If $v \in V$, then the *open neighborhood of* v, denoted $N_G(v)$, is $\{u \in V : uv \in E\}$ and the *closed neighborhood of* v, denoted $N_G[v]$, is $\{v\} \cup N_G(v)$. If G is the only graph in the discussion, N will replace N_G. If $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = S \cup N(S)$.

Let G and H be two graphs with $V(G) \cap V(H) = \emptyset$. Then $G + H$ is the graph such that $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. $G \vee H$ is the graph such that $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{vu : v \in V(G) \text{ and } u \in V(H)\}$.

A set $S \subseteq V$ is *dominating* (in G) if $V = N[S]$. The *domination number* of G is $\gamma(G)$= $\min[|S| : S \subseteq V$ is dominating]. A *minimum dominating set* in G is a dominating set $S \subseteq V$ such that $|S| = \gamma(G)$. A *minimal dominating set* is a dominating set no proper subset of which is a dominating set. A set of vertices which is dominating and disjoint from a minimum dominating set is an *inverse dominating set*. When G has no isolated vertices, the *inverse domination number* of G is $\gamma'(G) = \min[|B|$: $B \subseteq V \setminus S$ for some minimum dominating set $S \subseteq V$, and B is dominating in G]. $\gamma'(G)$ is well defined when G has no isolated vertices by Theorem 1.1, below. Clearly $\gamma(G) \leq \gamma'(G)$.

To see an example of an inverse dominating set, consider the graph H in Figure 1.1 where $\gamma(H) = 1$, and $\{y\}$ is the unique minimum dominating set. Thus any inverse dominating set cannot include y. Therefore $\{x, z\}$ is a minimum inverse dominating set and $\gamma'(H) = 2$.

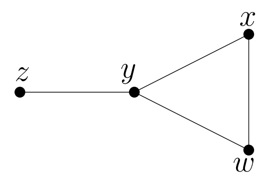

Figure 1.1: H

Theorem 1.1 ([9], [10]) *If G has no isolated vertices and S is a minimal dominating set in G, then $V \setminus S$ is dominating in G.*

Proof Suppose otherwise. Then there exists $v \in V(G)$ such that $N[v] \cap (V \setminus S) = \emptyset$. Therefore $N[v] \subseteq S$. But that is a contradiction to the minimality of S since $S \setminus v$ would then also be a dominating set in G.

■

For a graph G, the *closed neighborhood packing number*, $\pi(G)$, is the maximum number of disjoint closed neighborhoods in G. The *open neighborhood packing number*, $\pi^0(G)$, is the maximum number of disjoint open neighborhoods in G. As we will see in Chapter 4, $\pi(G)$ and $\pi^0(G)$ are very closely related to $\gamma(G)$ and $\gamma_t(G)$ respectively.

A set $I \subseteq V$ is *independent* if no two vertices of I are adjacent in G. A *maximal independent set* is one not properly contained in any other independent set. Clearly a maximal independent set in G is dominating in G. The (upper) *independence number* $\alpha(G)$ and the *lower* independence number $\imath(G)$ are defined by $\alpha(G) = \max[|I|\colon\ I \subseteq V$ is independent] and $\imath(G) = \min[|I|\colon\ I \subseteq V$ is maximal independent]. By remarks above, clearly $\gamma(G) \leq \imath(G) \leq \alpha(G)$.

Many of the above definitions can be adapted to give analogues pertinent to the study of fractional graph theory. A *fractional dominating function* on G is a function $f : V(G) \to [0, 1]$ such that for every $v \in V(G)$, $\sum_{u \in N[v]} f(u) \geq 1$. The *fractional domination number* of G is then $\gamma_f(G) = min[\sum_{v \in V(G)} f(v) : f$ is a fractional dominating function on $G]$. Since the characteristic function of a dominating set is a fractional dominating function, $\gamma_f(G) \leq \gamma(G)$.

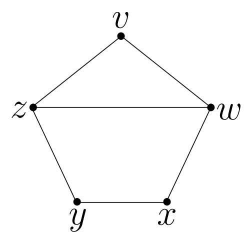

Figure 1.2: G

To see an example of a fractional dominating function., consider the graph G in Figure 1.2. Here one of the minimum dominating sets is $D := \{v, x\}$, so the characteristic function on D, $f_0 : V(G) \to [0,1]$ such that $f_0(v) = f_0(x) = 1$ and $f_0(w) = f_0(y) = f_0(z) = 0$, is a fractional dominating function. However $f_1 : V(G) \to [0,1]$ such that $f_1(v) = 0$, $f_1(z) = f_1(w) = \frac{1}{2}$, and $f_1(y) = f_1(x) = \frac{1}{4}$ is also a fractional dominating function, and it can be shown that it is minimum. Proving that f_1 is minimum will come from arguments made later in Chapter 4. Therefore $\gamma(G) = 2$ while $\gamma_f(G) = \sum_{u \in V(G)} f_1(u) = \frac{3}{2}$.

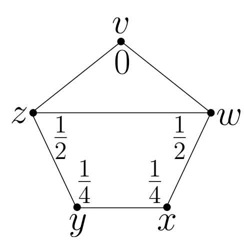

Figure 1.3: G with the weightings assigned by f_1

An *inverse fractional dominating function* of G is a fractional dominating function g of G that satisfies $g(v) \leq 1 - f(v)$ for all $v \in V(G)$, for some minimum fractional dominating function f of G. From [8], if G has no isolated vertices and f is a minimum fractional dominating function of G, then $1 - f$ is a dominating function on G. Therefore such a function g exists on graph G with no isolated vertices. The *inverse fractional domination number* of G is $(\gamma_f)'(G) = min_g[\sum_{v \in V(G)} g(v)]$ where the minimum is taken over all inverse fractional dominating functions. Since every inverse fractional dominating function of G is a fractional dominating function of G, $\gamma_f(G) \leq (\gamma_f)'(G)$.

To see an example of an inverse fractional dominating function, again consider the graph G in Figure 1.2. As mentioned above, f_1 is a minimum fractional dominating function. Therefore an inverse fractional dominating function, $f_2 : V(G) \to [0,1]$, could be any fractional dominating function such that $f_2(z), f_2(w) \le \frac{1}{2}$, and $f_2(y), f_2(x) \le \frac{3}{4}$. Therefore $f_2(z) = f_2(w) = f_2(x) = \frac{1}{2}$ and $f_2(y) = f_2(v) = 0$ is such an inverse fractional dominating function, and in fact since $\sum_{u \in V(G)} f_2(u) = \frac{3}{2}$ and $\gamma_f(G) \le (\gamma_f)'(G)$, $(\gamma_f)'(G) = \frac{3}{2}$. It can also be seen that $(\gamma_f)'(G) = \frac{3}{2}$ due to the fact that $f_1 \le 1 - f_1$. This demonstrates the obvious fact that if there exists a minimum fractional dominating function g on a graph G such that $g(v) \le \frac{1}{2}\ \forall v \in V(G)$, then $(\gamma_f)'(G) = \gamma_f(G)$.

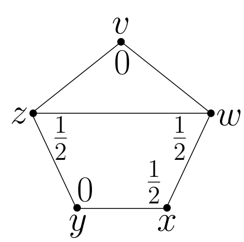

Figure 1.4: G with the weightings assigned by f_2

For another fractional analogue of the inverse dominating number, first let D be a minimum dominating set of a graph G. A *fractional inverse dominating function* g of G (with respect to D) is a fractional dominating function of G such that $g(v) = 0$ for all $v \in D$. In other words, $g \le$ the characteristic function of $V(G) \setminus D$. The *fractional inverse domination number* of G is $(\gamma')_f(G) = min_g[\sum_{v \in V(G)} g(v)]$ where the minimum is taken over all fractional inverse dominating functions g (with respect to minimum dominating sets D). Because the characteristic function of a dominating set of vertices that is in the complement of a minimum dominating set is also a fractional inverse dominating function, $(\gamma')_f(G) \le \gamma'(G)$.

To see an example of a fractional inverse dominating function, again consider graph G in Figure 1.2. As before, $D = \{v, x\}$ is a minimum dominating set for G, and $f_0 : V(G) \to [0,1]$ such that $f_0(v) = f_0(x) = 1$ and $f_0(w) = f_0(y) = f_0(z) = 0$ is the characteristic function on D. Therefore a fractional inverse dominating function, $f_3 : V(G) \to [0,1]$, could be any fractional dominating function satisfying $f_3(v) = f_3(x) = 0$. Thus f_3 such that $f_3(w) = f_3(y) = f_3(z) = \frac{1}{2}$, and $f_3(v) = f_3(x) = 0$ is a fractional inverse dominating function. Since $\sum_{u \in V(G)} f_3(u) = \frac{3}{2}$ and $\gamma_f(G) \leq (\gamma')_f(G)$, $(\gamma')_f(G) = \frac{3}{2}$.

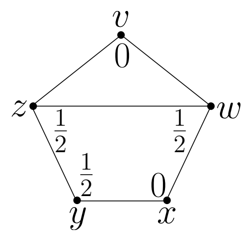

Figure 1.5: G with the weightings assigned by f_3

A function $\phi : V(G) \to [0,1]$ is *fractional independent* if for any pair v, w of adjacent vertices of G, $\phi(v) + \phi(w) \leq 1$; i.e., ϕ is a fractional *packing* of the "hypergraph" G. The *fractional independence number* is $\alpha_f(G) = max_{\phi}[\sum_{u \in V(G)} \phi(u)]$ where the maximum is taken over all fractional independent functions on G.

A *fractional clique-independent function* of G is a function $\hat{\phi} : V(G) \to [0,1]$ such that for all cliques K of G, $\sum_{v \in V(K)} \hat{\phi}(v) \leq 1$. The *fractional clique-independence number* of G is $\hat{\alpha_f}(G) = max_{\hat{\phi}}[\sum_{v \in V(G)} \hat{\phi}(v)]$ where the maximum is taken over all fractional clique-independent functions. Clearly $\alpha(G) \leq \hat{\alpha_f}(G) \leq \alpha_f(G)$.

To see examples of fractional independent and fractional clique-independent functions, once again consider graph G from Figure 1.2. Here $\alpha(G) = 2$. The function $g_1 : V(G) \to [0,1]$ defined such that $g_1(u) = \frac{1}{2} \forall u \in V(G)$ is certainly fractional independent, and it can be shown that g_1 is in fact maximum. Thus $\alpha_f(G) = \sum_{u \in V(G)} g_1(u) = \frac{5}{2}$. However, g_1 is not fractional clique-independent since $g_1(v) + g_1(w) + g_1(z) > 1$ and $\{v, w, z\}$ form a clique of size three in G. The function $g_2 : V(G) \to [0,1]$ defined such that $g_2(x) = g_2(y) = \frac{1}{2}$, and $g_2(v) = g_2(w) = g_2(z) = \frac{1}{3}$. g_2 is certainly clique-independent, and it can be shown that it is maximum. Thus $\hat{\alpha_f}(G) = 2$.

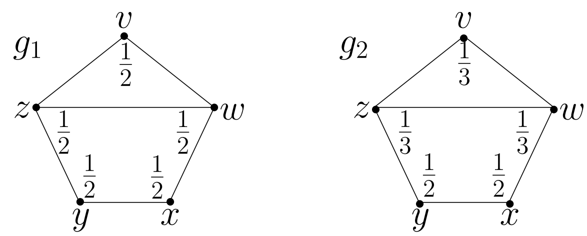

Figure 1.6: G with the weightings assigned by g_1 and g_2

A *total dominating set* of G is a set of vertices, $S \subseteq V(G)$, such that for every $v \in V(G)$, there exists a $u \in S$ with $v \in N(u)$. Notice that if G has an isolated vertex, no total dominating set exists. If G has no isolates, then the *total domination number of* G, denoted $\gamma_t(G)$, is the smallest size of total dominating set in G. Since every total dominating set is certainly dominating, it is clear that $\gamma(G) \leq \gamma_t(G)$.

A *fractional total dominating function* on G is a function $f : V(G) \to [0, 1]$ such that for every $v \in V(G)$, $\sum_{u \in N(v)} f(u) \geq 1$. As in the non-fractional case, notice that if G has an isolated vertex, then there is no fractional total dominating function on G. If G has no isolates, then the *fractional total domination number* of G is $(\gamma_t)_f(G) = min[\sum_{v \in V(G)} f(v) : f$ is a fractional total dominating function on $G]$. Since any fractional total dominating function on G is also a fractional dominating function, $\gamma_f(G) \leq (\gamma_t)_f(G)$.

To see an example of a fractional total dominating function consider the cycle on five vertices, C_5. Clearly $\gamma(G) = 2$, and $\gamma_t(G) = 3$. The function $h_1 : V(G) \to [0, 1]$ that assigns weights of $\frac{1}{3}$ to every vertex is clearly fractional dominating, but it is not total fractional dominating. In fact, it can be shown that h_1 is minimum and $\gamma_f(C_5) = \frac{5}{3}$. An example of a fractional total dominating function is $h_2 : V(G) \to [0, 1]$ defined by $h_2(u) = \frac{1}{2}$ for all $u \in C_5$. It can be shown that h_2 is minimum and $(\gamma_t)_f(C_5) = \frac{5}{2}$.

A *fractional closed neighborhood packing* of G is a function $g : V(G) \to [0, 1]$ such that for every $v \in V(G)$, $\sum_{u \in N[v]} g(u) \leq 1$. The *fractional closed neighborhood packing number* for G is then $\pi_f(G) = max[\sum_{v \in V(G)} g(v) : g$ is a fractional closed neighborhood packing of $G]$. A *fractional open neighborhood packing* of G is a function $\hat{g} : V(G) \to [0, 1]$ such that for

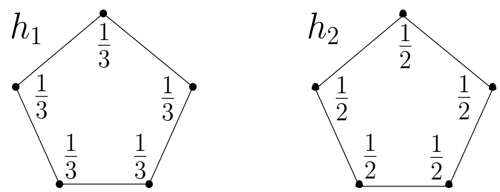

Figure 1.7: C_5 with the weightings assigned by h_1 and h_2

every $v \in V(G)$, $\sum_{u \in N(v)} \hat{g}(u) \leq 1$. The *fractional open neighborhood packing number* for G is then $\pi_f^0(G) = max[\sum_{v \in V(G)} \hat{g}(v) : \hat{g}$ is a fractional open neighborhood packing of $G]$. As will be explained in Chapter 4, it is true that for a simple graph G with no isolated vertices, $\gamma_f(G) = \pi_f(G)$ and $(\gamma_t)_f(G) = \pi_f^0(G)$.

Now suppose G is a graph with no isolated vertices. The following inequalities follow directly from the above definitions.

$$\begin{array}{lccccccccc}
i) & & & \gamma(G) & \leq & \imath(G) & \leq & \alpha(G) & \leq & \hat{\alpha_f}(G) \leq \alpha_f(G) \\
ii) & \gamma_f(G) & \leq & \gamma(G) & \leq & \gamma'(G) & & & & \\
iii) & \gamma_f(G) & \leq & (\gamma_f)'(G) & & & & & & \\
iv) & \gamma_f(G) & \leq & (\gamma')_f(G) & \leq & \gamma'(G) & & & & \\
v) & \gamma_f(G) & \leq & (\gamma_t)_f(G) & \leq & \gamma_t(G) & & & &
\end{array}$$

In [8] the following two additional inequalities were also proven.

$$\begin{array}{lccc}
vi) & (\gamma_f)'(G) & \leq & \alpha_f(G) \\
vii) & (\gamma')_f(G) & \leq & \alpha_f(G)
\end{array}$$

These next five inequalities are expected to be true but have not been proven. Notice that if inequality *viii* were proven true, then this would immediately imply that inequality *ix* would be true as well due to inequality *iv*. Inequality x is the original Kulli and Sigarkanti conjecture discussed in the next section. Inequalities *xi* and *xii*, if true, are stronger, respectively, than *vi* and *vii*.

viii)	$(\gamma_f)'(G)$	$\leq$	$(\gamma')_f(G)$	
ix)	$(\gamma_f)'(G)$	$\leq$	$\gamma'(G)$	
x)			$\gamma'(G)$	$\leq \alpha(G)$
xi)	$(\gamma_f)'(G)$	$\leq$	$\hat{\alpha_f}(G)$	
xii)	$(\gamma')_f(G)$	$\leq$	$\hat{\alpha_f}(G)$	

1.2 History

The study of domination is nothing new in Graph Theory. Since the 1950's, with the growth of computer science, the study of dominating sets in graphs has expanded rapidly. There have been over one thousand papers written on domination in graphs, and most of those papers have been written in the last 35 years.

One little known paper written in 1991 by Kulli and Sigarkanti [9] introduced the inverse domination number, $\gamma'(G)$. It was shown that $\gamma'(G)$ is well defined when G has no isolated vertices, and it was asserted that when G has no isolated vertices then $\gamma'(G) \leq \alpha(G)$. The attempted proof of this assertion was invalid; this was noticed, in due course, by Gayla Domke, Jean Dunbar, Teresa Haynes, Steve Hedetniemi, and Lisa Markus, who transmitted the open question to those interested in domination. (The first and third authors of [6] heard about the problem from Haynes, Hedetniemi, and Markus in 2000 or 2001, at a conference at Clemson University. Hedetniemi offered a prize for a resolution of the problem: a copy of [2]. The question appeared as a conjecture in [1].)

1.3 Outline

Much of Chapter 2 was inspired by the observation that if a graph G has a minimum dominating set D and a maximally independent set I which are disjoint, then $\gamma'(G) \leq \alpha(G)$. Therefore a graph G such that $\alpha(G) < \gamma'(G)$, if such a graph exists, must be found among those with no isolated vertices and the additional property that every minimum dominating set of G must have nonempty intersection with every maximally independent set of G. This class of graphs, the DI-pathological graphs, is examined in Chapter 3. In Chapter 4, two important results of linear programming are used to examine many fractional analogues to the Kulli and Sigarkanti conjecture continuing the work previously started in [8]. In Chapter 5, the tools explained in Chapter 4 are applied to minimum fractional total dominating functions, and these functions are totally characterized for certain graphs.

Chapter 2

The Inverse Domination Number

In this chapter, we consider the inverse domination number problem originally stated in [9] which conjectures that $\gamma'(G) \leq \alpha(G)$ for every simple graph with no isolated vertices. Observe that if $G = K_{1,t}$ then $\imath(G) = 1 = \gamma(G)$ and $\alpha(G) = t = \gamma'(G)$; a sign that this problem may be peskier than might have been evident at first glance.

We will say that G has Property DI if there exists a minimum dominating set $D \subseteq V$ and a maximal independent (and therefore dominating) set $I \subseteq V \setminus D$. If such a D and I exist, then straight from the definitions we have $\gamma'(G) \leq |I| \leq \alpha(G)$.

Lemma 2.1 *If G has property DI then G has no isolated vertices.*

Proof Suppose G did have an isolated vertex, v. Clearly, v must be in every dominating set. Therefore there cannot be two disjoint dominating sets. ∎

Therefore, a graph G such that $\alpha(G) < \gamma'(G)$, if there are any, will be found among graphs with no isolated vertices <u>not</u> having Property DI. Such graphs not having Property DI will be called *DI-Pathological.* We may as well look among connected graphs, because G has Property DI if and only if each component of G does, and if $\alpha(G) < \gamma'(G)$, then $\alpha(H) < \gamma'(H)$ for some component H of G.

2.1 Trees

Lemma 2.2 *If $D \subseteq V$ and I is maximal among the independent subsets of $V \setminus D$, then I is dominating in G if and only if $D \subseteq N(I)$.*

Proof Because I is maximal among the independent subsets of $V \setminus D$, $V \setminus D \subseteq N[I]$. Therefore $V = N[I]$ if and only if each vertex of D is adjacent to some vertex of I. ■

Corollary 2.1 *If G has a minimum dominating set D such that there is an independent set $I \subseteq V \setminus D$ with $D \subseteq N(I)$, then G has Property DI.*

Proof Let $\tilde{I}$ be an independent set such that $I \subseteq \tilde{I} \subseteq V \setminus D$, maximal among all independent subsets of $V \setminus D$. Then $D \subseteq N(I) \subseteq N(\tilde{I})$ so $\tilde{I}$ is a dominating independent set in the complement of D. ■

Theorem 2.1 *Every tree of order >1 has Property DI.*

Proof Clearly K_2 has Property DI. Let T be a tree of order $n \geq 3$, and let D be a minimum dominating set in T containing no leaf of T. Let u be a leaf of T. Order $V(T)$ by the rule: $v \leq w$ if and only if the path in T from u to w contains v.

Because T is a tree, every $v \in V(T) \setminus \{u\}$ has a unique immediate predecessor, $i.p.(v)$, in this ordering. Since D contains no leafs of T, every vertex in D has at least one immediate successor, in this ordering.

Now we describe an algorithm that will result in a minimum dominating set $\tilde{D}$ in T and an independent set $I \subseteq V(T) \setminus \tilde{D}$ such that $\tilde{D} \subseteq N_T(I)$. By Corollary 2.1, this will establish that T has Property DI.

We start with $\tilde{D} = D$ and $I = \{u\}$, and move out from u, processing vertices as we go, putting some in I, removing from or adding to $\tilde{D}$, and leaving some as they were. Obviously certain purposes must be served: $\tilde{D}$ must continue to be a dominating set in T, $|\tilde{D}|$ must not change, $I \subseteq V(T) \setminus \tilde{D}$ must be independent; for this it suffices to verify that each vertex in $V(T) \setminus \tilde{D}$ added to I is not adjacent to any vertex already in I. Finally, arrangements must be made so that, at the end, I dominates $\tilde{D}$.

From the many ways of organizing the algorithm, choose one satisfying: when the algorithm begins the processing of w, all of the predecessors of w have been processed, and none of the successors. The satisfying of the requirements in the paragraph above can be checked as we go along; in the end, I will dominate $\tilde{D}$ if, after each processing episode, the current I dominates $\tilde{D} \cap [\{w\} \cup \{\text{predecessors of } w\}]$.

So, suppose w is an unprocessed vertex of T, $w \neq u$, and all the predecessors of w and none of the successors of w have been processed. Let $v = i.p.(w)$. There are 6 cases to consider.

1. If $w \in \tilde{D}$ and $v \in I$, do nothing and move on. [Note that if w is the support vertex adjacent to u, then $w \in D = \tilde{D}$ and $u \in I$, so we are in case 1.]

2. If $w \in \tilde{D}$ and $v \notin I \cup \tilde{D}$, then:

   (a) If w has an immediate successor $x \in V(T) \setminus \tilde{D}$, put x in I, leave w in $\tilde{D}$, and move on. [Both w and x have now been processed.]

   (b) If all immediate successors of w are in $\tilde{D}$ then replace $\tilde{D}$ with $(\tilde{D} \setminus \{w\}) \cup \{v\}$, put w in I, and move on. [No immediate successor of w has been processed, in this case; when each gets its turn, we will be in case 1.]

   Remark: if $w \in \tilde{D}$ then, since w has not been processed earlier, $w \in D$ and so, as noted above, w does have successors.

3. If $w, v \in \tilde{D}$ then, because $\tilde{D}$ is a minimum dominating set, w must have an immediate successor x which is not in $\tilde{D}$. Leave w in $\tilde{D}$, put x in I, and move on. [Both w and x have been processed.]

4. If $w \notin \tilde{D}$ and $v \in \tilde{D}$, do nothing and move on.

5. If $w \notin \tilde{D}$ and $v \in I$, do nothing and move on.

6. If $w \notin \tilde{D}$ and $v \notin \tilde{D} \cup I$, put w in I.

It is straightforward to see, in each case, that each new member of I is adjacent to none of the previously inducted members of I, so I remains independent. Further, at each stage, after the processing episode I dominates all members of $\tilde{D}$ that have been processed at that point. Finally, in case 2(b), the only case in which $\tilde{D}$ is modified, by exchanging w for v, it is clear that the new $\tilde{D}$ is still dominating, and with the same number of vertices as the old $\tilde{D}$. ■

Corollary 2.2 *If F is a forest with no isolated vertices, then $\gamma'(F) \leq \alpha(F)$.*

After proving Theorem 2.1 we discovered that it answers a problem posed in [3], and that the problem is also solved in [4], with a similar but more economical proof. We have given our clunkier proof because it is algorithmic, and because we think it better illuminates the following theorem.

Theorem 2.2 *If T is a tree of order ≥ 2, and D is a minimum dominating set in T containing at most one leaf, then there is an independent dominating set $I \subseteq V(T) \setminus D$.*

We proposed Theorem 2.2 originally as a conjecture after proving Theorem 2.1. After transmitting this problem to Michael Henning, he and two collaborators found our conjecture to be true. The proof of this will appear in [5].

It might be useful to know for which minimum dominating sets $D \subseteq V(T)$, T as above, there is an independent dominating set $I \subseteq V(T) \setminus D$. Theorem 2.2 asserts that any minimum dominating set for a tree containing at most one leaf is such a set.

In the smallest example in which D contains 2 or more leafs of T and there is no independent dominating set $I \subseteq V(T) \setminus D$, T is P_4, the path on 4 vertices, and D consists of the two end vertices of the path.

2.2 $\gamma(G) \leq 2$

Theorem 2.3 *If $\gamma(G) \leq 2$ and G has no isolated vertices then G has Property DI unless $G = K_{m,n}$ for some $m, n > 2$.*

Proof If $\gamma(G) = 1$, let $D = \{v\}$ be any minimum dominating set. By Lemma 2.2 any independent set I maximal among independent subsets of $V \setminus D$ is a dominating set, so G has Property DI.

Suppose that $\gamma(G) = 2$. Suppose that G does not have Property DI. Then for any minimum dominating set $D = \{x, y\}$ in G, $N(x) \cap N(y) = \emptyset$, for if $z \in N(x) \cap N(y) \subseteq V \setminus D$ then $\{z\}$ is an independent subset of $V \setminus D$ such that $x, y \in N(\{z\})$, so G would have Property DI after all, by Corollary 2.1.

Let $N(x, D) = N(x) \setminus D$, $N(y, D) = N(y) \setminus D$, which partition $V \setminus D$, by the observation just above. Both sets are nonempty because $\gamma(G) = 2$ and G has no isolated vertices. By the same argument as above, appealing to Corollary 2.1, there cannot exist $u \in N(x, D)$ and $v \in N(y, D)$ which are not adjacent in G; that is, every vertex in $N(x, D)$ is adjacent to every vertex in $N(y, D)$. Therefore, if $u \in N(x, D)$ and $v \in N(y, D)$ then $D' = \{u, v\}$ is a minimum dominating set in G. Applying what has been shown about D to D', we conclude that $N(u, D')$ and $N(v, D')$ are disjoint and that $x \in N(u, D')$ and $y \in N(v, D')$ are adjacent. We also conclude that $N(x, D)$ and $N(y, D)$ are each independent because if, for instance, some $u, w \in N(x, D)$, $u \neq w$, are adjacent, then, taking any $v \in N(y, D)$, we would have that $w \in N(u, D') \cap N(v, D')$, $D' = \{u, v\}$.

Thus G is a complete bipartite graph with bipartition $N(x, D) \cup \{y\}$, $N(y, D) \cup \{x\}$. Say $G \cong K_{m,n}$, $m \leq n$. Then $2 \leq m$ because $\gamma(G) = 2$. If $m = 2$ then G does have Property DI; just take D to consist of the 2 vertices on one side of the bipartition and I to be the other side of the bipartition. Therefore, $m, n > 2$. ∎

Corollary 2.3 follows from the fact that Property DI implies $\gamma' \leq \alpha$, and that if $G = K_{m,n}$, $m, n \geq 2$, then $\gamma'(G) = \gamma(G) = 2$.

Corollary 2.3 *If $\gamma(G) \leq 2$ and G has no isolated vertices then $\gamma'(G) \leq \alpha(G)$.*

Corollary 2.4 *Suppose that $\gamma(G) = 2$ and G has no isolated vertices. Then the following are equivalent:*

(a) $G = K_{m,n}$ for some $m,n > 2$.

(b) G does not have Property DI.

(c) G does not have Property DI and for each $e \in E(G)$, $G - e$ does have Property DI.

(d) G does not have Property DI and for each $e \in E(\bar{G})$, $G \cup e$ does have Property DI.

Proof (a) and (b) are equivalent by Theorem 2.3, and the observation that $K_{m,n}$, $m, n > 2$, does not have Property DI. Clearly (c) or (d) implies (b). If $m, n > 2$ then adding or removing an edge to $K_{m,n}$ results in a connected graph with domination number 2 which is not $K_{a,b}$ for any a, b. By Theorem 2.3, the modified graph must have Property DI. Thus (a) implies (c) and (d). ∎

2.3 $3 \leq \gamma(G) \leq 4$

Lemma 2.3 *Suppose that G has no isolated vertices, and that $\gamma'(G) = \alpha(G) + c$ for some $c \geq 1$. Suppose that D is a minimum dominating set in G and $I \subseteq S = V \setminus D$ is an independent set of vertices, maximal among independent subsets of S. Let $D_0 = D \setminus N(I)$, $a = a(D, I) = \alpha(< D_0 >)$, and $b = b(D, I) = min[|S_0|; S_0 \subseteq S$ and $D_0 \subseteq N(S_0)]$. Then $a + c \leq b \leq |D_0|$.*

Proof Note that because $\gamma'(G) > \alpha(G)$, G does not have Property DI and therefore, by Corollary 2.1, D_0 is necessarily nonempty. Also, because G has no isolated vertices, every vertex in D must have a neighbor outside of D, i.e., in S. Therefore b is well defined, and $b \leq |D_0|$.

Let $S_0 \subseteq S$ satisfy $|S_0| = b$ and $D_0 \subseteq N(S_0)$. Then $I \cap S_0 = \emptyset$, and $I \cup S_0$ dominates G. Therefore $\alpha(G) + c = \gamma'(G) \leq |I \cup S_0| = |I| + |S_0| = |I| + b \leq \alpha(G) - a + b$, using the obvious inequality $|I|+a \leq \alpha(G)$. Rearranging, $\alpha(G)+c \leq \alpha(G)-a+b$, we have $a+c \leq b$. ∎

Corollary 2.5 *In the circumstances of Lemma 2.3,* $|D_0| \geq a + c \geq 1 + c \geq 2$ *and* $< D_0 >$ *must have at least* $c \geq 1$ *edges.*

Proof The first assertion follows directly from the conclusion of Lemma 2.3. The second assertion arises from the fact that adding an edge to a graph can decrease its independence number by at most one, so, since $|D_0| - c \geq a = \alpha(< D_0 >)$, $< D_0 >$ must be obtained from the empty graph on $|D_0|$ vertices by inserting at least c edges. ∎

Theorem 2.4 *If* G *has no isolated vertices and* $\gamma(G) \leq 4$ *then* $\gamma'(G) \leq \alpha(G)$.

Proof Suppose, on the contrary, that $c = \gamma'(G) - \alpha(G) \geq 1$. By Corollary 2.3, $\gamma(G) = 3$ or 4. First suppose that $\gamma(G) = 3$ and that $D = \{x, y, z\}$ is a minimum dominating set in G. Let $S = V \setminus D$, and let $N_x = N(x) \cap S$, $N_y = N(y) \cap S$, and $N_z = N(z) \cap S$. By Corollary 2.5, every set I maximal among independent subsets of S must be contained in only one of N_x, N_y, N_z – otherwise, the corresponding D_0 would have at most one element. From this observation it follows that N_x, N_y, N_z are disjoint, and that any two vertices not in the same set, among these, must be adjacent. Therefore, taking one vertex from each of N_x, N_y, N_z, we obtain a dominating set of size 3 in $S = V \setminus D$, whence $\gamma'(G) \leq 3 = \gamma(G) \leq \alpha(G)$, contrary to supposition.

Now suppose that $\gamma(G) = 4$ and $D = \{x_1, x_2, x_3, x_4\}$ is a minimum dominating set in G. As above, let $S = V \setminus D$ and $N_i = N(x_i) \setminus D = N(x_i) \cap S$, $i = 1, 2, 3, 4$. The N_i are nonempty because G has no isolated vertices. By Corollary 2.5, each maximal independent subset of S can dominate at most 2 elements of D, and so the same is true of any independent subset of S.

We show that there must exist i, j, $1 \leq i < j \leq 4$, and $u \in N_i$, $v \in N_j$, such that u, v are not adjacent. If not, then for every such i, j, every edge uv, $u \in N_i$, $v \in N_j$, $u \neq v$, is an edge of G. Choosing a representative from each N_i, we obtain a dominating set with no more than 4 elements in $V \setminus D$, whence $\gamma'(G) \le 4 = \gamma(G) \le \alpha(G)$, contrary to supposition.

So, without loss of generality, suppose that $u \in N_3, v \in N_4$, $u \neq v$, and u and v are not adjacent. Let I be any maximal independent subset of S containing u and v. Then $I \subseteq (N_3 \cup N_4) \setminus (N_1 \cup N_2)$, and the D_0 of Lemma 2.3 is $\{x_1, x_2\}$. Then $u, v \notin N_1 \cup N_2$ and u, v must dominate $N_1 \cup N_2$. Further, with a, b as in Lemma 2.3, by Lemma 2.3 we have that $2 \le a + c \le b \le |D_0| = 2$. We conclude that $a = c = 1$ and $b = 2$. Therefore $N_1 \cap N_2 = \emptyset$ (because $b = 2$), and $x_1x_2 \in E(G)$ (because $a = 1$).

Now, if $y \in N_1$ and $z \in N_2$ were not adjacent, then by the reasoning applied to u and v, y, z must dominate $N_3 \cup N_4$, and so u, v, y, z would be a dominating set in G in $S = V \setminus D$, whence $\gamma' = \gamma \le \alpha$, again contrary to supposition. So every edge yz, $y \in N_1$, $z \in N_2$, is in G. Choose any $y \in N_1$. Then $D' = \{x_1, y, x_3, x_4\}$ is a minimum dominating set, because $x_1x_2 \in E(G)$ and y dominates N_2. But then u, v, x_2 is an independent set in $S' = V \setminus D'$ which dominates 3 elements of D', namely x_1, x_3, and x_4, which is impossible by Corollary 2.5. ∎

Chapter 3

DI-Pathological Graphs

3.1 Minimal DI-Pathological Graphs with Domination Number Three

A graph G is said to be *DI-pathological* if every minimum dominating set in G intersects every maximally independent set of G. In other words, G is DI-pathological if and only if G does not have property DI defined in Chapter two.

Therefore the result in Theorem 2.3 could be restated as follows: *for graphs G with no isolated vertices and with $\gamma(G) = 2$, G is DI-pathological if and only if $G \cong K_{m,n}$ for some $m, n \geq 3$.*

In this section, we find the DI-pathological graphs with domination number three with the least number of vertices and edges.

First of all, it is a trivial fact that if a graph has an isolated vertex, then certainly every minimum dominating set must intersect every maximally independent set at that vertex. Therefore, technically, the graph with the least number of vertices or edges and that has domination number three and is DI-pathological is the complement of K_3. Our interest, however, is in DI-pathological graphs with no isolated vertices.

Now, if we restrict our graphs to those with no isolated vertices, then at least one component of the graph must be DI-pathological if the whole graph is DI-pathological. As noted before, if $\gamma(G) = 1$, then G is not DI-pathological. Therefore, among those disconnected graphs of domination number three with no isolated vertices that are DI-pathological, every one must have some component H such that $\gamma(H) = 2$ and H is DI-pathological. By the statement above, $H \cong K_{m,n}$ for $m, n \geq 3$. Therefore, the smallest disconnected DI-pathological graph with domination number three and no isolated vertices is $K_{3,3} + K_2$. And every such graph is of the form $K_{m,n} + (K_1 \vee H)$, $m, n \geq 3$, with $\vee$ denoting the join operation and H nonvoid.

Now we will turn our attention to connected DI-pathological graphs of domination number three.

Definition 3.1

Let D be a minimum dominating set of a graph G. Let $x \in D$. Then, the *D-private neighborhood of x*, written $P_D(x)$, is $\{v \in V(G) \setminus D : v \in N(x) \text{ and } v \notin N(D \setminus \{x\})\}$. In other words, $P_D(x)$ is the collection of vertices outside of D which are dominated only by one vertex, x, of D. Vertices of $P_D(x)$ are called *D-private neighbors of x.*

Lemma 3.1 *If D is a minimum dominating set in G, and $J \subseteq V(G) \setminus D$ is an independent set such that $D \subseteq N(J)$, then G is not DI-pathological.*

Proof Suppose D is a minimum dominating set in G, and $J \subseteq V(G) \setminus D$ is an independent set such that $D \subseteq N(J)$. Then let I be a maximally independent subset of $V(G) \setminus D$ such that $J \subseteq I$. Because I is a maximally independent subset of $V(G) \setminus D$, $(V(G) \setminus D) \subseteq N(I)$, and since $J \subseteq I$, $D \subseteq N(I)$. Therefore I is maximally independent in G, and $D \cap I = \emptyset$, so G is not DI-pathological. ■

Remark: Lemma 3.1 is a restatement of Corollary 2.1.

Theorem 3.1 *Let G be a connected DI-pathological graph with $\gamma(G) = 3$, and let D be a minimum dominating set in G. Then for all $x \in D$, $|P_D(x)| \geq 2$.*

Proof Let $D = \{x_1, x_2, x_3\}$ be a minimum dominating set for G, and let $S = V(G) \setminus D$. First, assume $|P_D(x_1)| = 0$.

Then $N(x_1) \cap S \subseteq [N(x_2) \cup N(x_3)] \cap S$, and since D is minimum, $x_1x_2, x_1x_3 \notin E(G)$. Pick $v \in N(x_1)$. Such a v exists because G is connected. Without loss of generality, since $|P_D(x_1)| = 0$, say $v \in N(x_2)$. Then $D_1 := \{v, x_2, x_3\}$ is another minimum dominating set.

Now, consider the subgraph $\hat{G} := < V(G) \setminus \{x_1\} >$. Certainly $\{x_2, x_3\}$ is a minimum dominating set of $\hat{G}$.

Claim: $\hat{G}$ has no minimum dominating set and maximal independent set which are disjoint.

Suppose $\hat{D}$ and $\hat{I}$ were such a pair of a minimum dominating set and a maximally independent set for $\hat{G}$. Then $v \notin \hat{D}$ since if it were, then $\hat{D}$ would dominate G implying that $\gamma(G) < 3$. Also, no vertex in $N_G(x_1)$ is in $\hat{I}$: otherwise, $\hat{I}$ would be an independent dominating set in G disjoint from $\hat{D} \cup \{x_1\}$, a minimum dominating set of G. This cannot be, since G is DI-pathological. Therefore $\hat{D} \cup \{v\}$ is a minimum dominating set for G, and $\hat{I} \cup \{x_1\}$ is an independent dominating set in G. This again contradicts the fact that G is DI-pathological. Hence no such $\hat{D}$ and $\hat{I}$ exist and the claim is true.

Thus $\hat{G}$ is DI-pathological with no isolated vertices and $\gamma(\hat{G}) = 2$, and hence $\hat{G}$ is a complete bipartite graph with parts of size at least 3, by Theorem 2.3. Therefore $\{x_2, x_3\}$ are in different parts of $\hat{G}$. Now, since $\hat{G}$ is a complete bipartite graph and $v \in N_G(x_1) \cap N_G(x_2)$, $\{x_2, v\}$ is a minimum dominating set for G which is a contradiction since $\gamma(G) = 3$. Thus $|P_D(x_1)| \neq 0$.

Now suppose $|P_D(x_1)| = 1$.

Say $P_D(x_1) = \{v\}$. Let $\hat{D} = \{v, x_2, x_3\}$. Note that $\hat{D}$ is also a minimum dominating set for G. Now $x_1x_2, x_1x_3 \notin E(G)$ since otherwise $\hat{D}$ would be a minimum dominating set such that $|P_{\hat{D}}(v)| = 0$, and, as shown above, $|P_{\hat{D}}(v)| > 0$.

Now, suppose there exists $u \in N(x_2) \cap N(x_3) \cap S$. Then $\{u\}$ must dominate $N(x_1)$ since otherwise G would not be DI-pathological; if $w \in N(x_1) \setminus N(u)$ then $J = \{u, w\}$ is an independent set in $V(G) \setminus D$ such that $D \subseteq N(J)$, whence Lemma 3.1 delivers the contradictory conclusion. In particular, $uv \in E(G)$. But then $\{u\}$ dominates $\hat{D}$, a minimum dominating set of G, giving rise to a contradiction again, by Lemma 3.1, since G is DI-pathological. Therefore $N(x_2) \cap S \cap N(x_3) = \emptyset$.

Since $|P_D(x_i)| \geq 1$, $i = 2, 3$, both $N(x_2) \cap S$ and $N(x_3) \cap S$ must be nonempty, so let $p \in N(x_2) \cap S$ and let $q \in N(x_3) \cap S$.

Suppose p and q are non-adjacent. Then $pv, qv \notin E(G)$ since otherwise $\{p, q\}$ would be an independent set disjoint from $\hat{D}$ dominating $\hat{D}$. Therefore $\{p, q, v\} \subseteq V(G) \setminus D$ is independent and dominates D, giving rise to a contradiction, and hence no such pair of nonadjacent vertices p and q exist.

Therefore, if $p \in N(x_2) \cap S$, $\{p\}$ dominates $N(x_3) \cap S$, and similarly if $q \in N(x_3) \cap S$, $\{q\}$ dominates $N(x_2) \cap S$. Thus $\{p, q, v\}$ is a minimum dominating set. This implies that $x_2x_3 \in E(G)$ since if not then $\{x_1, x_2, x_3\}$ would be an independent set dominating $\{p, q, v\}$. Since G is connected, there exists some edge connecting $\{x_1, v\}$ to $N[\{x_2, x_3\}]$.

Suppose $x_1p \in E(G)$ where without loss of generality p is some vertex of $N(x_2) \cap S$. Then $\{x_2, p, v\}$ is a minimum dominating set. x_1 dominates $\{p, v\}$, and therefore x_1 cannot be nonadjacent to any vertex of $N(x_2) \cap S$. Therefore x_1 dominates $N(x_2) \cap S$. This implies $\{x_1, x_3\}$ is a dominating set for G of size two which contradicts $\gamma(G) = 3$.

The same contradiction is reached if we say $vp \in E(G)$ or if x_1q or $vq \in E(G)$ for some $q \in N(x_3) \cap S$. Thus since G is connected, and one of these edges must exist, this is a contradiction. Therefore $|P_D(x_1)| \neq 1$, and hence $|P_D(x)| \geq 2$ for all $x \in D$. ■

Corollary 3.1 *If G is a connected DI-pathological graph with $\gamma(G) = 3$, then $|V(G)| \geq 9$.*

Proof For all x, y in a minimum dominating set D, $|P_D(x)| \geq 2$, and $P_D(x) \cap P_D(y) = \emptyset$. Since there are three vertices in D, $|V(G)| \geq 9$. ■

Corollary 3.2 *The graph G in Figure 3.1 is the unique connected, DI-pathological graph with $\gamma(G) = 3$, with the least number of vertices and edges.*

Proof First, to see that G is indeed DI-pathological, note that it has a unique minimum dominating set of $\{x_1, x_5, x_9\}$. Now if there were some maximally independent set I disjoint from $\{x_1, x_5, x_9\}$, then it must contain either x_4 or x_6. Without loss of generality, say that

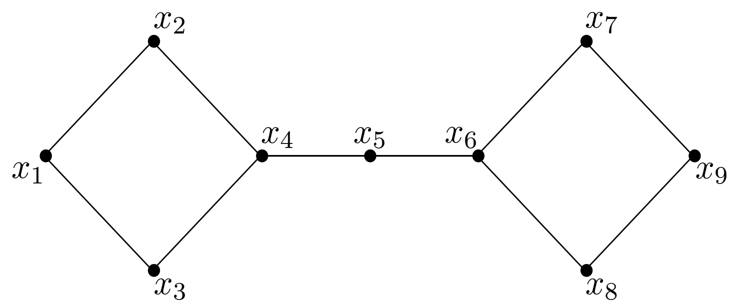

Figure 3.1: G

I contains x_4. Then I cannot be independent and dominating without having x_1 as one of its vertices. Therefore G is in fact DI-pathological.

Note also that G is connected, and $\gamma(G) = 3$. By Corollary 3.1, G has the minimum number of vertices that any such graph could have. Now suppose that H is connected, DI-pathological on 9 vertices, $\gamma(H) = 3$, and $E(H) \leq 10$. We aim to show that $H \cong G$. This will show not only that G is the unique connected DI-pathological graph with domination number 3 with the least number of edges among those with the least number of vertices, 9, but also that G is the unique such graph with the least number of vertices among those with the least number of edges, which is 10. To see this, suppose it has been shown that $H \cong G$. It then follows that the minimum number of edges in a connected DI-pathological graph with domination number 3 is 10, because, by showing $H \cong G$ it is shown that among such graphs with 9 vertices the minimum number of edges is 10, and any graph with 10 or more vertices and 9 or fewer edges is either disconnected or a tree; by Theorem 2.1, every tree with domination number 3 is not DI-pathological.

Let $D = \{x_1, x_2, x_3\}$ be a minimum dominating subset of $V(H)$. By Theorem 3.1, and the fact that $|V(G)| = 9$, the other six vertices in H are $u_1, u_2 \in N(x_1) \setminus (N(x_2) \cup N(x_3))$, $v_1, v_2 \in N(x_2) \setminus (N(x_1) \cup N(x_3))$, and $w_1, w_2 \in N(x_3) \setminus (N(x_1) \cup N(x_2))$. So H looks like Figure 3.2 with at most four edges yet to be added.

To see that $\{u_1, u_2\}$, $\{v_1, v_2\}$, and $\{w_1, w_2\}$ are independent suppose that, say, $u_1u_2 \in E(H)$. Then $D' := \{u_1, x_2, x_3\}$ is a minimum dominating set in H. If some edge v_iw_j is *not*

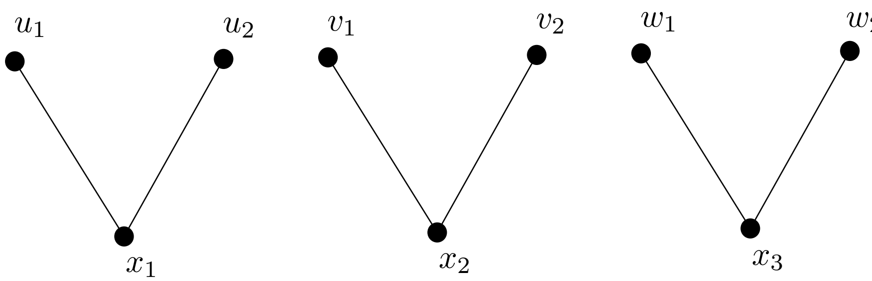

Figure 3.2: H minus 4 edges

in H, then $\{x_1, v_i, w_j\}$ is an independent set of vertices dominating D' contradicting that H is DI-pathological. Therefore all four edges v_iw_j are in H. But, with u_1u_2 as an edge, that implies that $|E(H)| \geq 11$ which is a contradiction.

Let $K = H - D$. Then K is tripartite with parts $\{u_1, u_2\}$, $\{v_1, v_2\}$, and $\{w_1, w_2\}$ and at most four edges. First of all, K contains no isolated vertices. To see this, suppose that, say, u_1 is isolated in K. Then because H is DI-pathological, every edge v_iw_j is in H. (Otherwise, there would be an independent set in $V(H) \setminus D$ which dominates D.) But then there would be ten edges in H, and no edge from $\{x_1, u_1, u_2\}$ to the rest of H, contradicting that H is connected.

Now if $|E(K)| < 4$, then since K has no isolated vertices, there must be three edges forming a perfect matching in K. If this were the case, however, there would be an independent set $I \subseteq V(H) \setminus D$ such that I would dominate D in H contradicting that H is DI-pathological. Therefore $|E(K)| = 4$.

There exists a vertex of K, say u_1, such that $deg(u_1) = 1$. And, without loss of generality say that $u_1v_1 \in E(K)$. If $w_iv_2 \notin E(K)$, $i \in \{1, 2\}$, then $\{w_i, v_2, u_1\} \subseteq V(H) \setminus D$ would be an independent set of vertices of $H - D$ dominating D. Therefore $w_1v_2, w_2v_2 \in E(K)$. Since $|E(K)| = 4$, and u_2 is not isolated, $w_1v_1, w_2v_1 \notin E(K)$. If $u_2v_1 \notin E(K)$, then $\{u_2, v_1, w_1\} \subseteq V(G) \setminus D$ would be an independent set of H dominating D. Therefore $u_2v_1 \in E(K)$; K must be isomorphic to the graph in Figure 3.3 and $H \cong G$.

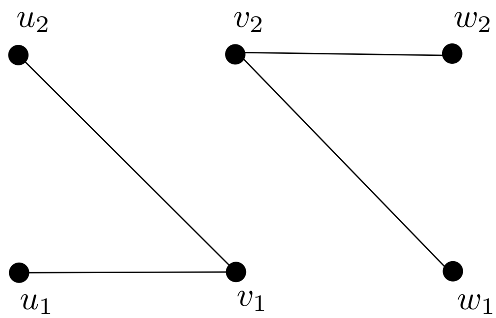

Figure 3.3: K

■

Definition 3.2

Let B_n for $n \geq 3$ be the graph of two 4-cycles joined by a path of length $3n - 7$.

Notice that B_3 is the graph G in Figure 3.1. Also following the same basic argument in Corollary 3.2, B_n has a unique minimum dominating set of size n, and B_n is clearly DI-pathological for all $n \geq 3$. These observations lead to the following conjecture.

Conjecture The unique connected, DI-pathological graph G with the fewest number of edges and the fewest number of vertices such that $\gamma(G) = n$ is B_n for $n \geq 3$.

3.2 DI-Pathological Graphs

Definition 3.3

Let H be a simple graph. An *exploded* H is a graph G such that for all $v \in V(H)$, v is replaced by an independent set A_v of size at least one, and every vertex of A_v is adjacent to every vertex of A_u if and only if v is adjacent to u in H. G can also be called an *explosion of* H.

Definition 3.4 Π_n

Π_n is the class of graphs such that $G \in \Pi_n$ if and only if G is an explosion of a path on n vertices.

Definition 3.5 χ_n

χ_n is the class of graphs such that $G \in \chi_n$ if and only if G is an explosion of a cycle on n vertices.

Hereafter, if $G \in \Pi_n \cup \chi_n$, the independent sets replacing vertices in the path or cycle of which G is an explosion will be listed $A_1, \ldots, A_n$ corresponding to vertices around the cycle or along the path, starting at an end. In the case of cycles, the indices i on the A_i are adjusted mod n.

Lemma 3.2 *Suppose that H is a graph with no isolated vertices and G is an explosion of H, with independent sets A_v replacing the vertices $v \in V(H)$. If I is a maximally independent set of vertices of G, $v \in V(H)$, and $I \cap A_v \neq \emptyset$, then $A_v \subseteq I$.*

Proof Since I is maximally independent, I is dominating in G. Suppose that $a \in A_v \cap I$ and $b \in A_v \setminus I$. Because I is independent, and $a \in I$, I can contain no vertices of any A_u, u a neighbor of v in H. But then $b \notin N_G[I]$, contradicting the fact that I is dominating in G. ■

Corollary 3.3 *With H, G, and the A_v as in the previous lemma, $I \subseteq V(G)$ is maximally independent in G if and only if for some maximally independent $J \subseteq V(H)$, $I = \cup_{v \in J} A_v$.*

Lemma 3.3 *With H, G, and the A_v, $v \in V(H)$ as in the previous lemma, if D is a minimal dominating set in G then for all $v \in V(H)$, either $A_v \subseteq D$ or $|A_v \cap D| \leq 1$.*

Proof If $|D \cap A_v| \geq 2$, then clearly, deleting one vertex of $D \cap A_v$ from D would result in a dominating set which is a proper subset of D unless the deleted vertex and thus every vertex of $D \cap A_v$ dominates only itself. Therefore, $|D \cap A_v| \geq 2$ implies $A_v \subseteq D$. ■

Lemma 3.4 *With H, G, and the A_v, $v \in V(H)$, as previously, if $v \in V(H)$, $|A_v| \geq 3$, and D is a minimum dominating set for G, then $|D \cap A_v| \leq 1$.*

Proof By Lemma 3.3, if $|D \cap A_v| > 1$ then $A_v \subseteq D$. If $A_v \subseteq D$, then exchanging two vertices of A_v for one vertex of some A_u, $u \in N_H(v)$, gives a smaller dominating set in G than D. ■

Lemma 3.5 *Let H, G, and the A_v, $v \in V(H)$, be as previously. Suppose $v \in V(H)$ and $|A_v| \geq 3$. If D is a minimum dominating set for G, and $D \cap A_v \neq \emptyset$, then there exists $u \in N_H(v)$ such that $A_u \cap D \neq \emptyset$.*

Proof Suppose $d \in D \cap A_v$. By Lemma 3.4, $(A_v \setminus \{d\}) \cap D = \emptyset$. Therefore, since D is dominating, there exists some $u \in N_H(v)$ such that $A_u \cap D \neq \emptyset$. ■

Recall that a set $D \subseteq V(G)$ is *totally dominating* if and only if $V(G) = N(D)$, and that the *total domination number* is defined to be $\gamma_t(G) = \min[|D|\colon D$ is a totally dominating set of $V(G)]$.

Corollary 3.4 *If H is a graph with no isolated vertices, and G is an explosion of H with $|A_v| \geq 2$ for all $v \in V(H)$, then $\gamma(G) = \gamma_t(H)$. If S is a minimum total dominating set in H and $D \subseteq V(G)$ is formed by taking one representative from A_v for each $v \in S$, then D is a minimum dominating set in G. If $|A_v| \geq 3$ for all $v \in V(H)$, then every minimum dominating set D of G is obtained as described above, by choosing one representative from A_v for each v in some minimum total dominating set in H.*

Proof Suppose S is a minimum total dominating set in H and D is formed as described. Label the vertex of D that is a representative of A_v, $\hat{v}$. First, to see that D is a dominating set in G, note that for each $v \in S$, $N_G(\hat{v}) = \cup\{A_u : u \in N_H(v)\}$. Since S is total dominating in H, $\cup\{N(v) : v \in S\} = V(H)$, and hence D is indeed dominating in G. Therefore, since a total dominating set in H gives rise to a dominating set in G of the same size, $\gamma(G) \le \gamma_t(H)$.

Suppose D is a minimum dominating set in G. Let $\tilde{S} \subseteq V(H)$ be the set $\{v \in V(H) : D \cap A_v \neq \emptyset\}$. By Lemmas 3.3 and 3.4, any A_v such that $D \cap A_v \neq \emptyset$ must be such that $|D \cap A_v| \in \{1, 2\}$. Therefore, if $|A_v| \ge 3$ for all $v \in V(H)$, D consists of one representative of each A_v, $v \in \tilde{S}$.

If $v \in \tilde{S}$, and $|A_v| = |D \cap A_v| = 2$, then another minimum dominating set in G may be obtained by replacing one vertex of D in A_v by a vertex in A_u for some u adjacent to v in H. (Because D is minimum dominating and $A_v \subseteq D$, $|A_v| = 2$, no such A_u can contain a vertex in D.) Continuing in this way we obtain another minimum dominating set for G at most one vertex from each A_w, $w \in V(H)$. Let us continue to call this set D, and let $\tilde{S}$ be defined as before. Note that if $|A_w| \ge 3$ for all $w \in V(H)$, then the new D is the D we started with.

It remains only to show that $\tilde{S}$ is a total dominating set in H, for then we have that $\gamma_t(H) \le |\tilde{S}| = |D| = \gamma(G)$, so $\gamma_t(G) = \gamma(G)$ and every D formed in G from a minimum total dominating set in H as described is minimum dominating in G. Also, by previous remarks, if $|A_v| \ge 3$ for all $v \in V(G)$, every minimum dominating set in G must be obtainable in this way.

But it is quite clear that $\tilde{S}$ is a total dominating set in H, for if $w \in V(H) \setminus N_H(\tilde{S})$ then no vertex in A_w is in $N_G(D)$; since $V(G) = N_G[D]$, it would follow that $A_w \subseteq D$, but $|A_w| \ge 2$ and arrangements have been made so that $|A_w \cap D| \le 1$. ■

Lemma 3.6 *Let H, G, and the A_v, $v \in V(H)$, be as previously, and suppose that H is a path or cycle of order n. Suppose that $D \subseteq V(G)$, and for some $i \in \{1, \ldots, n\}$, $D \cap A_j \neq \emptyset$ for all $j \in \{i, i+1, i+2\}$. Then every maximal independent set of vertices in G intersects D.*

Proof If I were a maximally independent set of vertices of G disjoint from D, then by Lemma 3.2 $I \cap A_j = \emptyset$, $j = i, i+1, i+2$. But then $I \cup A_{i+1}$ is an independent set of vertices properly containing I. ■

Lemma 3.7 *Let H, G, and the A_v, $v \in V(H)$, be as previously, H a path or cycle of order n, $n \geq 6$, and $|A_v| \geq 3$ for all $v \in V(H)$. If D is a dominating set in G such that there exist six consecutive integers $\{i, i+1, i+2, i+3, i+4, i+5\}$, $1 \leq i \leq n-5$, such that $A_j \cap D \neq \emptyset$, $j \in \{i, i+1, i+4, i+5\}$ and $A_j \cap D = \emptyset$, $j \in \{i+2, i+3\}$, then there is no maximally independent (and hence dominating) subset of $V(G)$ disjoint from D. If H is a cycle, the same holds without the requirement $1 \leq i \leq n-5$, reading indices mod n.*

Proof By Lemma 3.2 if there existed a dominating independent set I of G disjoint from D, then $I \cap A_{i+2} \neq \emptyset$ in order that I dominate A_{i+1}. But, by the same argument, $I \cap A_{i+3} \neq \emptyset$ in order that I dominate A_{i+4}. This is a contradiction because I is independent. ■

The following lemma is only true for graphs $G \in \Pi_n$.

Lemma 3.8 *If $G \in \Pi_n$, $n \geq 2$, and D is a minimum dominating set in G such that $D \cap A_1 \neq \emptyset$ and $|A_1| \geq 3$ or $D \cap A_n \neq \emptyset$ and $|A_n| \geq 3$, then no maximally independent subset of $V(G)$ is disjoint from D.*

Proof If, say, $|A_1| \geq 3$ and $D \cap A_1 \neq \emptyset$, then by Lemma 3.5 $D \cap A_2 \neq \emptyset$. By Lemma 3.2, if I is a maximal independent set of vertices of G disjoint from D then $I \cap A_i = \emptyset$, $i = 1, 2$. But then $I \cup A_1$ is an independent set in G properly containing I. ■

Theorem 3.2 *If $G \in \Pi_n$, an explosion of $H \cong P_n$, $n \geq 2$, $n \neq 4, 7, 10$, and $|A_v| \geq 3$ for all $v \in V(H)$, then G is DI-pathological.*

Proof If $n = 2$, then $G \cong K_{m,n}$ for some $m, n \geq 3$. By Theorem 2.2, G is DI-pathological. Since $|A_v| \geq 3$ for all $v \in V(H)$, by Corollary 3.4 D is formed by choosing one vertex from each A_v, $v \in S$, where S is a minimum total dominating set in $H \cong P_n$. If $n = 3$, then any minimum dominating set D for G must have the property that either $D \cap A_1 \neq \emptyset$ or $D \cap A_3 \neq \emptyset$. In either case, by Lemma 3.8, G is DI-pathological.

Now, if $n \geq 5$, then we must consider three cases.

(i) $n \equiv 0 (mod\ 4)$

Note that since $n \neq 4$, $n \geq 8$. Any minimum total dominating set S of P_n has the property that there exist six consecutive vertices of P_n, $\{i, i+1, i+2, i+3, i+4, i+5\}$ such that $\{i, i+1, i+4, i+5\} \in S$. The conclusion that G is DI-pathological now follows from Corollary 3.4 and Lemma 3.7.

(ii) $n \equiv 1 (mod\ 4)$

Any minimum total dominating set S of P_n has the property that there are three consecutive vertices, $\{i, i+1, i+2\} \in S$. Thus in any minimum dominating set D of G, there must exist a set of three consecutive integers $\{i, i+1, i+2\}$ with the property that $D \cap A_i, D \cap A_{i+1}, D \cap A_{i+2} \neq \emptyset$. By Corollary 3.4 and Lemma 3.6, no maximally independent set of $V(G)$ could be disjoint from D.

(iii) $n \equiv 2, 3 (mod\ 4)$

Here, any minimum total dominating set S of P_n falls into at least one of the following two categories.

(a) There exist three consecutive integers $\{i, i+1, i+2\} \in S$. In this case, by Corollary 3.4 and Lemma 3.6 there is no maximally independent set disjoint from any minimum dominating set D in G, derived from such an S.

(b) As long as $n \neq 7, 10$, either $\{1\} \subseteq S$, $\{n\} \subseteq S$, or there exist six consecutive vertices of P_n $\{i, i+1, i+2, i+3, i+4, i+5\}$ such that $\{i, i+1, i+4, i+5\} \subseteq S$.

By Corollary 3.4 and Lemmas 3.8 and 3.7, no maximally independent subset of $V(G)$ is disjoint from any minimum dominating set D derived from such an S.

In every case there is are no minimum dominating sets which are disjoint from any maximally independent sets. Thus G is DI-pathological. ■

Theorem 3.3 *If $G \in \chi_n$, $n \geq 4$, $n \neq 6$, and $|A_v| \geq 3$ for all $v \in V(H)$ where $H \cong C_n$, then G is DI-pathological.*

Proof Let $G \in \chi_n$ as above. As in the proof of the previous theorem, by Corollary 3.4 any minimum dominating set D of G will be formed by choosing one representative from each of the sets A_v $v \in S$, a minimum total dominating set of H. We must consider the following three cases.

(i) $n \equiv 0(mod\ 4)$

When $n = 4$, $G \equiv K_{r,s}$, $r, s \geq 6$, so assume $n \geq 8$. Any minimum total dominating set S of C_n is such that there exists six consecutive vertices of C_n $\{i, i{+}1, i{+}2, i{+}3, i{+}4, i{+}5\}$ such that $\{i, i+1, i+4, i+5\} \subseteq S$. Thus any minimum dominating set D of G will be such that A_i, A_{i+1}, A_{i+4}, A_{i+5} $\cap D \neq \emptyset$ and A_{i+2}, A_{i+3} $\cap D = \emptyset$. By Lemma 3.7, no maximally independent set exists which is disjoint from D.

(ii) $n \equiv 1(mod\ 4)$

Any minimum total dominating set S of C_n is such that there are three consecutive vertices of C_n, $\{i, i+1, i+2\} \subseteq S$. By Lemma 3.6, there is no maximally independent set disjoint from any minimum dominating set D in G.

(iii) $n \equiv 2, 3(mod\ 4)$

Note that since $n \neq 6$, $n \geq 7$. Therefore, a minimum total dominating set S in $H \cong C_n$ falls into at least one of the following two categories.

(a) There exist three consecutive vertices of C_n, $\{i, i+1, i+2\} \subseteq S$. In this case, by Lemma 3.6 there is no maximally independent set disjoint from any minimum dominating set D derived from such an S.

(b) There exist six consecutive vertices of C_n, $\{i, i+1, i+2, i+3, i+4, i+5\}$, such that $\{i, i+1, i+4, i+5\} \subseteq S$. By Lemma 3.7, no maximally independent subset of $V(G)$ exists that is disjoint from any minimum dominating set D derived from such an S.

In every case, there are no minimum dominating sets which are disjoint from any maximally independent sets. Therefore G is DI-pathological. ■

Since it has been shown that many exploded paths and cycles are DI-pathological, it is natural to examine if other graphs can be exploded giving rise to other DI-pathological graphs. It turns out that many graphs may fall into this category. In fact, it is an easy exercise to see that even any exploded Petersen graph with $|A_v| \geq 3$ for all vertices v is also DI-pathological.

Another natural question would be if all DI-pathological graphs could be characterized as explosions of simple graphs. It will be shown that this is not the case. In fact the theorem below shows that DI-pathological graphs of a certain size can take various forms.

Definition 3.6

A graph G is said to have an *exploded tail of length* n if there exists $S \subseteq V(G)$ such that $< S > \cong H \in \Pi_n$ for some n, and the only edges connecting S to $V(G) \setminus S$ are incident to vertices of $A_1 \subseteq S$.

Definition 3.7

A graph G is said to have an *exploded ear of length* n if there exists $S \subseteq V(G)$ such that

$< S > \cong H \in \Pi_n$ for some n, and the only edges connecting S to $V(G) \setminus S$ are incident to vertices in either A_1 or $A_n \subseteq S$; and there is at least one edge with an end in A_i and the other end in $V(G) \setminus S$, for each $i \in \{1, n\}$.

Theorem 3.4 *If G is a simple graph with an exploded tail of order $n_1 \geq 12$ or an exploded ear of order $n_2 \geq 14$, and with $|A_i| \geq 3$, $i = 1, 2, \ldots, n_j$, $j = 1, 2$, then G is DI-pathological.*

Proof Let G be such a graph, and let D be a minimum dominating set for G. Clearly, whether G has an exploded tail of order $n_1 \geq 12$ or an exploded ear of order $n_2 \geq 14$, there must exist six consecutive integers $\{i, i+1, i+2, i+3, i+4, i+5\}$ such that $A_j \cap D \neq \emptyset$, $j \in \{i, i+1, i+4, i+5\}$, and $A_{i+2} \cap D = A_{i+3} \cap D = \emptyset$. By the same logic as in Lemma 3.7, there is no maximally independent set for G that is disjoint from D. ■

At first glance, it may seem that in the previous theorem the requirement that $n_1 \geq 12$ and $n_2 \geq 14$ is a little bit of an overkill. However, we have found that these are optimal bounds for n. In other words, there are non-DI-pathological graphs such that they an exploded tail of order 11, and there are also non-DI-pathological graphs such that they have an exploded ear of order 13. Two such graphs are given in Figures 3.4 and 3.5.

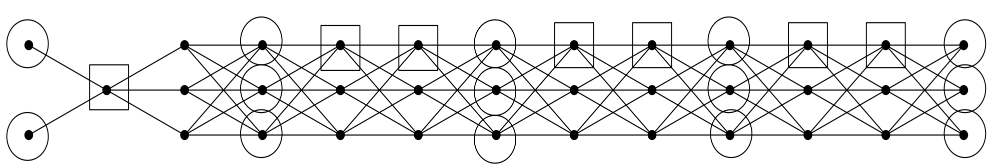

Figure 3.4: A non-DI-pathological graph G with an exploded tail of order 11

It can clearly be seen that the graph G in Figure 3.4 has an exploded tail of order 11, and it is not difficult to verify that $\gamma(G) = 7$. Thus the vertices that are boxed form a minimum dominating set that is, in fact, disjoint from the maximally independent set of vertices that are circled. The graph H in Figure 3.5 has an exploded ear of order 13, and

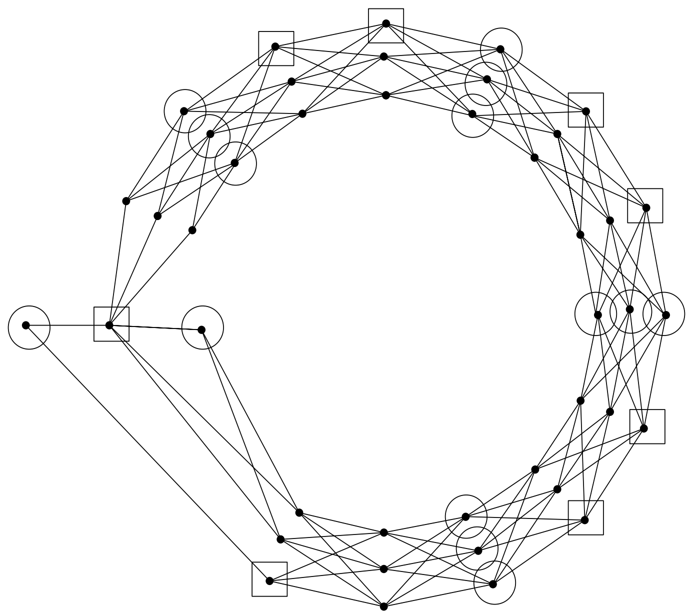

Figure 3.5: A non-DI-pathological graph H with an exploded ear of order 13

$\gamma(H) = 8$. As before, it is clear to see that the boxed vertices form a minimum dominating set disjoint from the maximally independent set of vertices that are circled.

The previous theorem sheds some light on how difficult it would be to classify all DI-pathological graphs. The "non-tail" or "non-ear" part of the graph could be almost anything.

One also notices that every DI-pathological graph mentioned so far in this paper has at least one set of clones, where two vertices x and y are called *clones* in G if $N(x) = N(y)$. So it may seem that the existence of clones might be a necessity for DI-pathological graphs. Even this characteristic is not shared by all DI-pathological graphs. For example, the graph of two cycles on 7 vertices joined by a path of length 2 is DI-pathological but has no clones. In fact if G is the graph composed of two cycles, C_m and C_n joined by a path of length k where $m, n > 4$, $m, n \equiv 1(mod\ 3)$, and $k \equiv 2(mod\ 3)$, then G is DI-pathological and has no clones.

Chapter 4

The Principle of Strong Duality and the Principle of Complementary Slackness

In Chapter 1 were introduced many fractional analogues to such parameters as the domination number, the total domination number, the inverse domination number, the closed neighborhood packing number, the open neighborhood packing number, and the independence number. Since the problems of finding these parameters can be viewed as integer programs and the problems of finding their fractional analogues can be viewed as linear programs, an extremely helpful tool is the Principle of Strong Duality. This is the central result in the theory of linear programming, and a thorough examination of application of the Principle of Strong Duality and its application to fractional graph theory can be found in [11]. Here we purloin from [11] the basic definitions and results in linear programming pertinent to our aims.

A *linear program* (LP) is an optimization problem that can be expressed in the form "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x}\leq \mathbf{b}$", where $\mathbf{b}$ is an m-vector, $\mathbf{c}$ is an n-vector, A is an m-by-n matrix, and $\mathbf{x}$ varies over all the n-vectors with nonnegative entries. (Inequalities are coordinate-wise.) The problem "minimize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x}\geq\mathbf{b}$" is also a linear program; again, we assume that $\mathbf{x}\geq 0$. It is easy to see that problems with equality constraints or with unconstrained variables can be put into the above form, so these variations may be considered. For our purposes, LPs always take the standard forms introduced here.

An *integer program* (IP) is an optimization problem of the same form as a linear program except that the vector $\mathbf{x}$ is subject to the additional constraint that all its entries must be integers.

In an LP or an IP, the expression $\mathbf{c}^t\mathbf{x}$ is called the *objective function*, a vector $\mathbf{x}$ satisfying the constraints $A\mathbf{x}\leq \mathbf{b}$, $\mathbf{x}\geq 0$ is called a *feasible solution*, and the optimum of the objective function over all feasible solutions is called the *value* of the program. It is natural to assign the value $-\infty$ to a maximization program with no feasible solutions and the value $+\infty$ if the objective function is unbounded on feasible solutions. The linear program obtained from an integer program P by dropping the constraint that the entries of $\mathbf{x}$ be integers is called the *linear relaxation* of P.

If P is the (linear or integer) program "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x}\leq \mathbf{b}$, $\mathbf{x}\geq 0$", then the program "minimize $\mathbf{b}^t\mathbf{y}$ subject to $A^t\mathbf{y}\geq\mathbf{c}$, $\mathbf{y}\geq 0$" is called the *dual* of P. If $\mathbf{x}$ is a feasible solution for P and $\mathbf{y}$ is a feasible solution for the dual of P, then because $\mathbf{x}$, $\mathbf{y} \geq 0$, we have the *weak duality inequality.*

$$\mathbf{c}^t\mathbf{x} = \mathbf{x}^t\mathbf{c} \leq \mathbf{x}^t A^t\mathbf{y} = (A\mathbf{x})^t\mathbf{y} \leq \mathbf{b}^t\mathbf{y}$$

This implies that the value of P is less than or equal to the value of the dual of P. In fact, if P is a linear program, more is true.

Theorem 4.1 ***The Principle of Strong Duality***

A linear program and its dual have the same value.

4.1 The Fractional Domination Number and the Fractional Closed Neighborhood Packing Number

For a graph G, the domination number, $\gamma(G)$, and the closed neighborhood packing number, $\pi(G)$, were defined in Chapter 1. The problem of finding $\gamma(G)$ and $\pi(G)$ are dual integer programs. Consequently, the problem of finding their fractional analogues, $\gamma_f(G)$ and $\pi_f(G)$ are dual linear programs. By the Principle of Strong Duality, therefore, $\gamma_f(G) = \pi_f(G)$.

When attempting to find $\gamma_f(G)$ or $\pi_f(G)$ for a graph G, one need only find two functions $g, h : V(G) \to [0,1]$ such that g is fractional dominating, h is a fractional closed neighborhood packing function, and $\sum_{v\in V(G)} g(v) = \sum_{v\in V(G)} h(v)$. This then immediately shows that g is minimum, h is maximum, and $\sum_{v\in V(G)} g(v) = \sum_{v\in V(G)} h(v) = \gamma_f(G) = \pi_f(G)$.

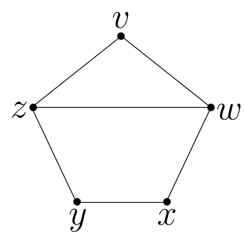

Figure 4.1: G

It then becomes a simple exercise to find $\gamma_f(G)$ for a graph G such as Figure 4.1, which is a copy of the C_5 with a chord, that was represented in Figure 1.2 in Chapter 1. Recall the function $f_1 : V(G) \to [0,1]$ from Chapter 1 such that $f_1(v) = 0$, $f_1(z) = f_1(w) = \frac{1}{2}$, and $f_1(y) = f_1(x) = \frac{1}{4}$ which is clearly fractional dominating. In Chapter 1, the claim was made that f_1 was minimum and therefore that $\gamma_f(G) = \sum_{u \in V(G)} f_1(u) = \frac{3}{2}$. To now verify this, consider the fractional closed neighborhood packing of G, $h : V(G) \to [0,1]$, defined by $h(v) = h(z) = h(x) = \frac{1}{2}$ and $h(w) = h(y) = 0$. Since $\sum_{u \in V(G)} h(u) = \frac{3}{2}$ as well, $\gamma_f(G)$ does indeed equal $\frac{3}{2}$.

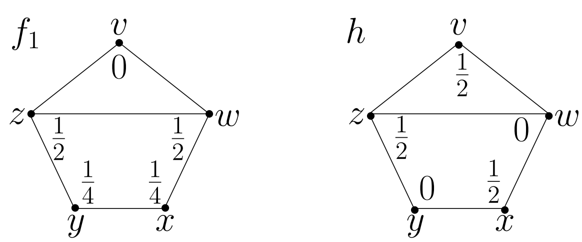

Figure 4.2: f_1, a fractional dominating functions and h, a fractional closed neighborhood packing

4.2 The Fractional Total Domination Number and the Fractional Open Neighborhood Packing Number

For a graph G with no isolated vertices, the problem of finding the total domination number, $\gamma_t(G)$, also has an integer dual, and it is the problem of finding the open neighborhood packing number, $\pi^0(G)$. And, just as in the previous section, the problem of finding their fractional analogues $(\gamma_t)_f(G)$ and $\pi^0_f(G)$ are dual linear programs. Therefore the problem of finding $(\gamma_t)_f(G)$ for a graph G with no isolates simplifies into finding functions $g, h : V(G) \to [0,1]$ such that g is a fractional total dominating function, h is a fractional open neighborhood packing, and $\sum_{v \in V(G)} g(v) = \sum_{v \in V(G)} h(v)$. This sum is then the fractional total domination number and the fractional open neighborhood packing number.

To see an example, consider the cycle on 5 vertices, C_5. In Chapter 1, the claim was made that $g : V(C_5) \to [0,1]$ defined by $g(u) = \frac{1}{2}$ for all $u \in C_5$ is a minimum fractional total dominating function. This becomes obvious when one notices that g is also a fractional open neighborhood packing of C_5. In fact, it will be seen in Chapter 5 that g is the *only* fractional total dominating function and the *only* fractional open neighborhood packing for C_5. Thus clearly $(\gamma_t)_f(C_5) = \pi^0_f(C_5) = \frac{5}{2}$.

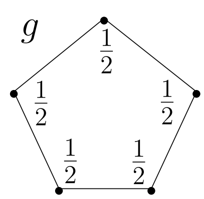

Figure 4.3: g, a fractional total dominating function *and* a fractional open neighborhood packing for C_5

The example in Figure 4.3 helps to illustrate the following theorem.

Theorem 4.2 *For $n \geq 3$, $(\gamma_t)_f(C_n) = \frac{n}{2}$.*

Proof The function $g : V(C_n) \to [0,1]$ defined by $g(u) = \frac{1}{2}$ for all $u \in C_n$ is both a minimum fractional total dominating function and a fractional open neighborhood packing of

C_n. Thus, by the principle of strong duality, since $\sum_{u \in V(C_n)} g(u) = \frac{n}{2}$, $(\gamma_t)_f(C_n) = \frac{n}{2}$. ■

This next theorem makes use of the principle of strong duality to find $(\gamma_t)_f(P_n)$ for $n \geq 2$.

Theorem 4.3 *For $n \geq 2$*

$$(\gamma_t)_f(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (mod\ 4) \\ \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 (mod\ 4) \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 (mod\ 4) \\ \lceil \frac{n}{2} \rceil & \text{if } n \equiv 3 (mod\ 4). \end{cases}$$

Proof For simplicity, label the vertices of P_n sequentially along the path starting at one end as follows: $\{x_1, x_2, \ldots, x_n\}$. This proof now divides into four cases.

Case 1: $n \equiv 0(mod\ 4)$

Let $g : V(P_n) \to [0, 1]$ such that $g(x_i) = \begin{cases} 1 & \text{if i} \equiv 2, 3(mod\ 4) \\ 0 & \text{if i} \equiv 0, 1(mod\ 4). \end{cases}$

g is both a fractional total dominating function and a fractional open neighborhood packing of P_n. Thus, by the principle of strong duality, since $\sum_{u \in V(P_n)} g(u) = \frac{n}{2}$, $(\gamma_t)_f(P_n) = \frac{n}{2}$.

Case 2: $n \equiv 1(mod\ 4)$

Let $g : V(P_n) \to [0, 1]$ such that $g(x_i) = \begin{cases} 1 & \text{if i} \equiv 2, 3(mod\ 4)\text{, or if } i = n-1 \\ 0 & \text{if i} \equiv 0, 1(mod\ 4)\text{, } i \neq n-1. \end{cases}$

Let $h : V(P_n) \to [0, 1]$ such that $h(x_i) = \begin{cases} 1 & \text{if i} \equiv 1, 2(mod\ 4) \\ 0 & \text{if i} \equiv 0, 3(mod\ 4). \end{cases}$

g is a fractional total dominating function of P_n, and h is a fractional open neighborhood packing of P_n. Thus, by the principle of strong duality, since $\sum_{u \in V(P_n)} g(u) = \lceil \frac{n}{2} \rceil = \sum_{u \in V(P_n)} h(u)$, $(\gamma_t)_f(P_n) = \lceil \frac{n}{2} \rceil$.

Case 3: $n \equiv 2(mod\ 4)$

Let $g : V(P_n) \to [0, 1]$ such that $g(x_i) = \begin{cases} 1 & \text{if i} \equiv 1, 2(mod\ 4) \\ 0 & \text{if i} \equiv 0, 3(mod\ 4). \end{cases}$

g is both a fractional total dominating function and a fractional open neighborhood packing of P_n. Thus, by the principle of strong duality, since $\sum_{u\in V(P_n)} g(u) = \frac{n}{2}+1$, $(\gamma_t)_f(P_n) = \frac{n}{2}+1$.

Case 4: $n \equiv 3(mod\ 4)$

Let $g : V(P_n) \to [0,1]$ such that $g(x_i) = \begin{cases} 1 & \text{if i} \equiv 1,2(mod\ 4) \\ 0 & \text{if i} \equiv 0,3(mod\ 4). \end{cases}$

g is both a fractional total dominating function and a fractional open neighborhood packing of P_n. Thus, by the principle of strong duality, since $\sum_{u\in V(P_n)} g(u) = \lceil\frac{n}{2}\rceil$, $(\gamma_t)_f(P_n) = \lceil\frac{n}{2}\rceil$. ∎

4.3 The Fractional Independence Number, the Fractional Clique-Independence Number, and their Dual Linear Programs

The problem of finding the independence number, $\alpha(G)$, of a graph G also has an integer dual, and it is the problem of finding the edge covering number, $c(G)$. An *edge covering* of the graph G is a set of edges $E \subseteq E(G)$ such every vertex of G is incident to at least one edge of E; $c(G)$ is then the least number of edges in an edge covering. Note that $c(G)$ is defined only if G has no isolated vertices. The fractional analogue to this is $c_f(G)$, the fractional edge covering number. A *fractional edge covering* is a function $\psi : E(G) \to [0,1]$ such that for each $v \in v(G)$, the sum of the weightings of the edges incident with v is ≥ 1. Thus $c_f(G) = min\{\sum_{e\in E(G)} \psi(e) : \psi$ is a fractional edge covering on $G\}$. The problem of finding $\alpha_f(G)$ and the problem of finding $c_f(G)$ are dual linear programs, and hence $\alpha_f(G) = c_f(G)$.

Thus the problem of finding $\alpha_f(G)\Big(= c_f(G)\Big)$ simplifies into finding a fractional independent function ϕ and a fractional edge covering ψ such that $\sum_{v\in V(G)} \phi(v) = \sum_{e\in E(G)} \psi(e)$. To see an example, consider once again the graph G in Figure 4.1. In Chapter 1, the claim was made that the function $\phi : V(G) \to [0,1]$ such that $\phi(u) = \frac{1}{2}$ for all $u \in V(G)$ is a maximum fractional independent function. In order to see this, consider the function $\psi : E(G) \to [0,1]$ such that $\psi(e)) = \begin{cases} 0 & \text{if } e \text{ is } zw \\ \frac{1}{2} & \text{otherwise} \end{cases}$.

ψ is clearly a fractional edge covering with $\sum_{e\in E(G)} \psi(e) = \sum_{v\in V(G)} \phi(v) = \frac{5}{2}$. Therefore $\alpha_f(G)$ is indeed $\frac{5}{2}$.

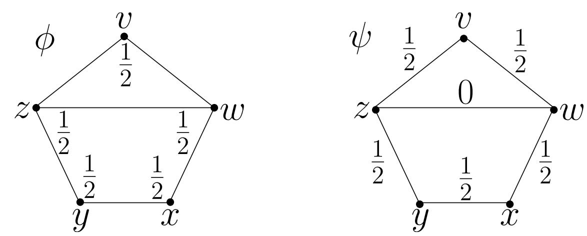

Figure 4.4: ϕ a fractional independent function and ψ a fractional edge covering for G

The problem of finding the fractional clique-independence number, $\hat{\alpha_f}$, has the dual linear program of the problem of finding the *fractional clique covering number*, $\hat{c_f}$. Before we can define $\hat{c_f}$, we must first define a fractional clique covering on G. Let $\mathcal{K}$ be the set of all cliques of G. A fractional clique covering on G is then a function $\hat{\psi} : \mathcal{K} \to [0, 1]$ such that $\sum_{K:v\in K} \hat{\psi}(K) \geq 1$ for all $v \in V(G)$. $\hat{c_f}$ is defined to be the $min\{\sum_{K\in\mathcal{K}} \hat{\psi}(K) : \hat{\psi}$ is a fractional clique covering on $G\}$.

In Chapter 1, the claim was made that the function $\hat{\phi} : V(G) \to [0, 1]$ such that $\hat{\phi}(x) = \hat{\phi}(y) = \frac{1}{2}$ and $\hat{\phi}(z) = \hat{\phi}(v) = \hat{\phi}(w) = \frac{1}{3}$ was a maximum clique-independent function on the graph G in Figure 4.1. In order to show that $\hat{\phi}$ is maximum, we need only find a fractional clique covering of G, $\hat{\psi}$, such that the sum of the weights on the cliques is equal to 2. Thus consider the following function. Let $\hat{\psi} = \begin{cases} 1 & \text{if K is } \{v, w, z\} \text{ or } \{y, x\} \\ 0 & \text{otherwise} \end{cases}$. $\hat{\psi}$ is certainly a fractional clique covering since every vertex of G is in a clique that has weight equal to 1. And, since the sum of all the weights on the cliques of G is 2, $\hat{\alpha_f} = 2$.

4.4 The Principle of Complementary Slackness

The Principle of Complementary Slackness is an extremely important corollary to the proof of the Principle of Strong Duality, and similarly is a powerful tool in fractional graph theory. Again, a thorough explanation of this topic can be found in [11].

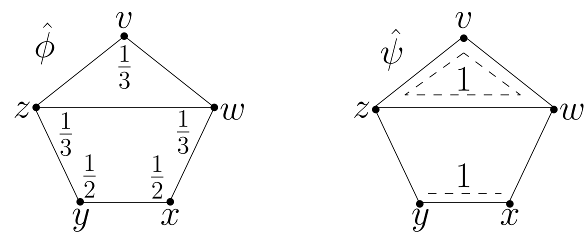

Figure 4.5: $\hat{\phi}$ a fractional clique-independent function and $\hat{\psi}$ a fractional clique covering for G

Theorem 4.4 ***The Principle of Complementary Slackness***

Let $\boldsymbol{x}^*$ *be any optimal solution to the bounded, feasible linear program, maximize* $\boldsymbol{c}^t\boldsymbol{x}$ *subject to* $A\boldsymbol{x}\leq b$, $\boldsymbol{x}\geq 0$, *and let* $\mathbf{y}^*$ *be any optimal solution to the dual, minimize* $\boldsymbol{b}^t\boldsymbol{y}$ *subject to* $A^t\boldsymbol{y}\geq \boldsymbol{c}$, $\boldsymbol{y} \geq 0$. *Then*

$$\boldsymbol{x}^* \cdot (A^t\boldsymbol{y}^* - \boldsymbol{c}) = \boldsymbol{y}^* \cdot (A\boldsymbol{x}^* - \boldsymbol{b}) = 0.$$

It is useful to rewrite this theorem in the contexts in which is will be applied in this thesis. In these restatements, it is important to notice that since $\mathbf{x}^*$ and $(A^t\mathbf{y}^*$ - $\mathbf{c})$ are both nonnegative, if some coordinate of $\mathbf{x}^*$ or $(A^t\mathbf{y}^*$ - $\mathbf{c})$ is nonzero, then the corresponding coordinate of the other must be zero $\Big($similarly for $\mathbf{y}^*$ and (A$\mathbf{x}^*$ - $\mathbf{b}$)$\Big)$. It is also important to note that in all of the cases discussed in this paper, the constraint vector for these linear programs, either $\mathbf{b}$ or $\mathbf{c}$, is the vector where 1 is the entry in every component. Thus we have the following corollaries to the Principle of Complementary Slackness.

Corollary 4.1 ***The Principle of Complementary Slackness applied to fractional dominating functions and fractional closed neighborhood packings***.

Let G *be a graph with* $v \in V(G)$. *If* $g(v) > 0$ *for some minimum fractional dominating function on* G, *then* $h(N[v]) = 1$ *for every maximum fractional closed neighborhood packing* h *of* G; *and, if* $h(v) > 0$ *for some maximum fractional closed neighborhood packing of* G, *then* $g(N[v]) = 1$ *for every minimum fractional dominating function on* G.

Corollary 4.1 implies the following two facts:

(i) If there exists a maximum fractional closed neighborhood packing h such that $h(N[v]) < 1$, then $g(v) = 0$ for every minimum fractional dominating function g of G.

(ii) If there exists a minimum fractional dominating function g such that $g(N[v]) > 1$, then $h(v) = 0$ for every maximum fractional closed neighborhood packing h of G.

Corollary 4.2 is almost identical to that of 4.1 and has very similar implications.

Corollary 4.2 ***The Principle of Complementary Slackness applied to fractional total dominating functions and fractional open neighborhood packings.*** *Let G be a graph with $v \in V(G)$. If $g(v) > 0$ for some minimum fractional total dominating function on G, then $h(N(v)) = 1$ for every maximum fractional open neighborhood packing h of G; and, if $h(v) > 0$ for some maximum fractional open neighborhood packing of G, then $g(N(v)) = 1$ for every minimum fractional total dominating function on G.*

As in Corollary 4.1, Corollary 4.2 implies the following two facts:

(i) If there exists a maximum fractional open neighborhood packing h such that $h(N(v)) < 1$, then $g(v) = 0$ for every minimum fractional total dominating function g of G.

(ii) If there exists a minimum fractional total dominating function g such that $g(N(v)) > 1$, then $h(v) = 0$ for every maximum fractional open neighborhood packing h of G.

Corollary 4.2 is very useful in the characterization of minimum fractional total dominating functions and maximum fractional open neighborhood packings which is much of the aim of Chapter 5.

It is worth mentioning that the Principle of Complementary Slackness is applicable to maximum fractional independent functions and maximum fractional clique-independent functions along with their dual linear programs. This topic, however, is not explored in depth in this thesis.

Chapter 5

Total Domination Null and Open Neighborhood Packing Null Vertices

5.1 Fractional Total Domination

As explained in Chapter 1, a function $g : V(G) \to [0,1]$ is total dominating on G if $\sum_{v \in N(u)} g(v) \geq 1$ for all $u \in V(G)$. The fractional total domination number is defined by $(\gamma_t)_f(G) = \min\{\sum_{v \in V(G)} g(v)$: g is a fractional total dominating function on $G\}$. As mentioned in Chapter 4, the problem of finding $\pi_f^0(G)$ is a dual linear program to that of finding $(\gamma_t)_f(G)$.

In [7] the following definitions of domination null and packing null vertices were given as follows: A vertex $v \in V(G)$ is *domination null* if and only if $g(v) = 0$ for every minimum fractional dominating function g on G. A vertex $v \in V(G)$ is *packing null* if and only if $h(v) = 0$ for every maximum fractional packing h of G. Continuing the work started in this paper, I define two analogous terms corresponding to fractional total dominating functions and fractional open neighborhood packings.

A vertex $v \in V(G)$ is *total domination null* if and only if $g(v) = 0$ for every minimum fractional total dominating function g on G. A vertex $v \in V(G)$ is *open neighborhood packing null* if and only if $h(v) = 0$ for every maximum fractional open neighborhood packing h of G.

For a simple graph G, let $\mathcal{G}_G = \{g : V(G) \to [0,1] \mid g$ is a minimum fractional total dominating function of $G\}$, and let $\mathcal{H}_G = \{h : V(G) \to [0,1] \mid h$ is a maximum fractional open neighborhood packing of $G\}$.

The following lemma is very helpful in the pursuit of characterization of minimum fractional total dominating functions and maximum fractional open neighborhood packings for certain graphs.

Lemma 5.1 *If there exists a graph G and a function $f : V(G) \to [0,1]$ such that $f(v) > 0$ for all $v \in V(G)$ and $f \in \mathcal{G}_G \cap \mathcal{H}_G$, then $\mathcal{G}_G = \mathcal{H}_G$.*

Proof By the principle of complementary slackness, the following two statements are true.

(i) If $h \in \mathcal{H}_G$ then $h(N(v)) = 1$ for all $v \in V(G)$ since $f(v) > 0$ for all $v \in V(G)$ and $f \in \mathcal{G}_G$. Thus $h \in \mathcal{G}_G$.

(ii) If $g \in \mathcal{G}_G$ then $g(N(v)) = 1$ for all $v \in V(G)$ since $f(v) > 0$ for all $v \in V(G)$ and $f \in \mathcal{H}_G$. Thus $g \in \mathcal{H}_G$.

■

Corollary 5.1 *If G is regular of degree $k \geq 1$, then $\mathcal{G}_G = \mathcal{H}_G$.*

Proof Let $f : V(G) \to [0,1]$ be such that $f(v) = \frac{1}{k}$ for all $v \in V(G)$. Clearly $f \in \mathcal{G}_G \cap \mathcal{H}_G$, and $f(v) > 0$ for all $v \in V(G)$. ■

Corollary 5.2 *$\mathcal{G}_{K_{r_1,r_2,\ldots,r_t}} = \mathcal{H}_{K_{r_1,r_2,\ldots,r_t}}$ where $K_{r_1,r_2,\ldots,r_t}$ is the complete t-partite graph with parts of sizes $r_1, r_2, \ldots, r_t$ and $t \geq 2$.*

Proof Let $f : V(G) \to [0,1]$ be such that if v is in part i, then $f(v) = (\frac{1}{t-1})(\frac{1}{r_i})$. As in the previous corollary, $f \in \mathcal{G}_G \cap \mathcal{H}_G$, and $f(v) > 0$ for all $v \in V(G)$. ■

A similar lemma to that of Lemma 5.1 holds for the sets $\hat{\mathcal{G}}_G := \{g : V(G) \to [0,1] \mid g$ is a minimum fractional dominating function for $G\}$ and $\hat{\mathcal{H}}_G := \{h : V(G) \to [0,1] \mid h$ is a maximum closed neighborhood packing on $G\}$.

Lemma 5.2 *If there exists a graph G and a function $f : V(G) \to [0,1]$ such that $f(v) > 0$ for all $v \in V(G)$ and $f \in \hat{\mathcal{G}}_G \cap \hat{\mathcal{H}}_G$, then $\hat{\mathcal{G}}_G = \hat{\mathcal{H}}_G$.*

Proof The proof is identical to that of Lemma 5.1, and again comes directly from the principle of complementary slackness. ■

Similarly, the following corollary comes directly from Lemma 5.2.

Corollary 5.3 *If G is regular of degree k, then $\hat{\mathcal{G}}_G = \hat{\mathcal{H}}_G$. Also, $\hat{\mathcal{G}}_{K_{r_1,r_2,\ldots,r_t}} = \hat{\mathcal{H}}_{K_{r_1,r_2,\ldots,r_t}}$ where $K_{r_1,r_2,\ldots,r_t}$ is the complete t-partite graph with parts of sizes $r_1, r_2, \ldots, r_t$ and $t \geq 2$.*

Proof It is not hard to verify that there exist functions that meet the criteria of Lemma 5.2 for both nontrivial graphs of regular degree and complete t-partite graphs. Also, see [7]. ■

5.2 Cycles

When trying to characterize all functions $g \in \mathcal{G}_{C_n}$ and $h \in \mathcal{H}_{C_n}$, Lemma 5.1 is extremely helpful. Since C_n is regular of degree 2, $\mathcal{G}_{C_n} = \mathcal{H}_{C_n}$. For simplicity, from now on the vertices of C_n will be labeled $\{x_1, x_2, \ldots, x_n\}$ sequentially around the cycle. i.e. $x_n x_1 \in E(C_n)$, and $x_i x_{i+1} \in E(C_n)$ for all $1 \leq i \leq n-1$. Recall that by Theorem 4.2, $(\gamma_t)_f(C_n)\Big(= \pi_f^0(C_n)\Big) = \frac{n}{2}$.

The following theorem totally answers the characterization problem for cycles.

Theorem 5.1 *If $g \in \mathcal{G}_{C_n}$ and $n \not\equiv 0(mod\ 4)$, then $g(x_i) = \frac{1}{2}$ for all $1 \leq i \leq n$. If $g \in \mathcal{G}_{C_n}$ and $n \equiv 0(mod\ 4)$, then*

$$g(x_i) = \begin{cases} s & \text{if } i \equiv 0(mod\ 4) \\ 1-s & \text{if } i \equiv 2(mod\ 4) \\ t & \text{if } i \equiv 1(mod\ 4) \\ 1-t & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

where $s, t \in [0,1]$.

Proof First, I claim that $g \in \mathcal{G}_{C_n}\Big(= \mathcal{H}_{C_n}\Big)$ if and only if $g(x_i)+g(x_{i+2}) = 1$ for all $1 \leq i \leq n$ where $i+2$ is treated as $i+2(mod\ n)$. To see this, first suppose that $g(x_i) + g(x_{i+2}) = 1$

for all i. Then g is clearly total dominating on G, and $\sum_{i=1}^{n} g(x_i) = \frac{n}{2}$. Therefore $g \in \mathcal{G}_{C_n}$. Secondly, let g be any function in $g \in \mathcal{G}_{C_n}$. Therefore $2\sum_{i=1}^{n} g(x_i) = \sum_{i=1}^{n} g(x_i) + g(x_{i+2}) \geq n = 2 \cdot \frac{n}{2} = 2\sum_{i=1}^{n} g(x_i)$ where the inequality comes from the fact that $g(x_i) + g(x_{i+2}) \geq 1$ for all $1 \leq i \leq n$ since g is total dominating on G. Thus $g(x_i) + g(x_{i+2}) = 1$ for all $1 \leq i \leq n$, and thus the claim is true.

Now if $n \not\equiv 0(mod\ 4)$, the only way that every pair $g(x_i)$ and $g(x_{i+2})$ can sum to 1 is if $g(x_i) = \frac{1}{2}$ for all $1 \leq i \leq n$. If $n \equiv 0(mod\ 4)$, however, this is possible only if

$$g(x_i) = \begin{cases} s & \text{if i} \equiv 0(mod\ 4) \\ 1-s & \text{if i} \equiv 2(mod\ 4) \\ t & \text{if i} \equiv 1(mod\ 4) \\ 1-t & \text{if i} \equiv 3(mod\ 4) \end{cases}$$

where $s, t \in [0, 1]$. ■

Corollary 5.4 *C_n has no total domination null vertices and no open neighborhood packing null vertices for $n \geq 3$.*

5.3 Paths

Let $n \geq 2$. Characterizing $\mathcal{G}_{P_n}$ and $\mathcal{H}_{P_n}$ seems to be a much messier problem that it is for cycles. However, the principle of complementary slackness again serves as a valuable tool in this section. For simplicity, from now on, we will refer to the principle of complementary slackness as PCS and label the vertices of P_n sequentially $\{x_1, x_2, \ldots, x_n\}$ along the path.

Lemma 5.3 *If v is a stem of G (i.e. v is adjacent to vertex u such that $deg(u) = 1$) and g is a fractional total dominating function on G, then $g(v) = 1$.*

Proof If $g(v) < 1$, then g would fail to dominate the open neighborhood of u which would be a contradiction to g being a total dominating function on G. ■

Lemma 5.4 *Suppose x_i is a total domination null vertex in P_n. If $i+2 \leq n$ then $g(x_{i+2}) = 1$ for any minimum fractional dominating function g on G, and if $1 \leq i-2$ then the same holds for x_{i-2}.*

Proof Suppose x_i is as above. Then $N(x_{i+1})$ is $\{x_i, x_{i+2}\}$. Thus in order for x_{i+1} to be dominated by g, $g(x_{i+2}) = 1$ since $g(x_i) = 0$. Likewise, $g(x_{i-2}) = 1$ as well. ■

Theorem 5.2 *For $n \geq 2$ the following is the chart of the total domination null and the open neighborhood packing null vertices for P_n.*

	i such that x_i is total domination null	*i such that x_i is open neighborhood packing null*	$(\gamma_t)_f(P_n)$
$n \equiv 0(mod\ 4)$	$i \equiv 0, 1(mod\ 4)$	*None*	$\frac{n}{2}$
$n \equiv 1(mod\ 4)$	$i \equiv 1(mod\ 4)$	$i \equiv 3(mod\ 4)$	$\lceil \frac{n}{2} \rceil$
$n \equiv 2(mod\ 4)$	*None*	$i \equiv 0, 3(mod\ 4)$	$\frac{n}{2} + 1$
$n \equiv 3(mod\ 4)$	$i \equiv 0(mod\ 4)$	$i \equiv 0(mod\ 4)$	$\lceil \frac{n}{2} \rceil$

Proof First of all, the right most column was shown in Chapter 4. It follows from Lemma 5.3 that $\mathcal{G}_{P_2} = \mathcal{H}_{P_2} = \{g : V(P_2) \rightarrow [0,1]$ such that $g(x_1) = g(x_2) = 1\}$, and that $\mathcal{G}_{P_3} = \mathcal{H}_{P_3} = \{g : V(P_3) \rightarrow [0,1]$ such that $g(x_1) = t$, $g(x_2) = 1$, and $g(x_3) = 1-t$ for some $t \in [0,1]\}$.

The rest of the proof will be divided into 4 parts. Let $n \geq 4$.

<u>$n \equiv 0(mod\ 4)$</u>

If $n \equiv 0(mod\ 4)$, then the constant function $h(x_i) = \frac{1}{2}$ for $1 \leq i \leq n$ is a maximum fractional open neighborhood packing of P_n. Therefore there are no open neighborhood packing null vertices. h also implies that for any minimum fractional total dominating function g, $g(N(x_i)) = 1$ for all $1 \leq i \leq n$; this is an application of the PCS. Clearly, $g(x_1) = g(x_n) = 0$ by the fact that $h(N(x_i)) = \frac{1}{2} < 1$, $i = 1, n$. Therefore, again by the PCS conclusions, or using Lemma 5.4, $g(x_2) = 1 = g(x_3)$, and therefore $g(x_4) = 0$. Thus since the weights on x_1, x_2, x_3, x_4 are $0, 1, 1, 0$ respectively, the rest follows immediately from the PCS conclusion that $g(N(x_i)) = 1$ for all i, and the only possible member of $\mathcal{G}_{P_n}$ is the function illustrated in Figure 5.1.

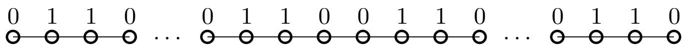

Figure 5.1: $n \equiv 0(mod\ 4)$

<u>$n \equiv 1(mod\ 4)$</u>

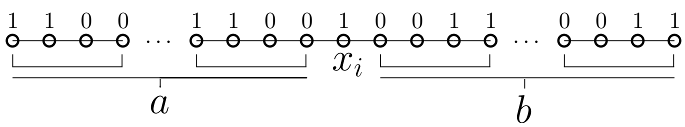

Figure 5.2: $i \equiv 1(mod\ 4)$, $a, b \geq 0$, $a + b = \frac{n-1}{4}$

In Figures 5.2 through 5.8, a is the number of 4 element sets of vertices that have the same weightings as $\{x_1, x_2, x_3, x_4\}$, counting from the left, and b is the number of 4 element sets of vertices that have the same weightings as $\{x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$, counting from the right.

The function $h \in \mathcal{H}_{P_n}$ illustrated in Figure 5.2 is such that $\sum_{u \in N(x_i)} h(u) = 0 < 1$ so therefore by PCS, $g(x_i) = 0$ for all $g \in \mathcal{G}_{P_n}$. Thus x_i is total domination null for $i \equiv 1(mod$ 4).

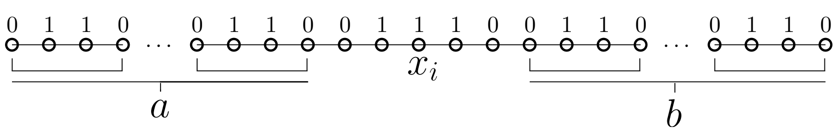

Figure 5.3: $i \equiv 3(mod\ 4)$, $a, b \geq 0$, $a + b = \frac{n-5}{4}$

The function $g \in \mathcal{G}_{P_n}$ illustrated in Figure 5.3 is such that $\sum_{u \in N(x_i)} g(u) = 2 > 1$ so therefore by PCS, $h(x_i) = 0$ for all $h \in \mathcal{H}_{P_n}$. Thus x_i is open neighborhood packing null for $i \equiv 3(mod\ 4)$.

To see that these are these are the only sets of total domination or open neighborhood packing null vertices consider the following functions $g_0, g_1 \in \mathcal{G}_{P_n}$ and $h_0 \in \mathcal{H}_{P_n}$.

$$g_0(x_i) = \begin{cases} 0 & \text{if i} \equiv 1, 2(mod\ 4),\ i \neq 2 \\ 1 & \text{if i} \equiv 0, 3(mod\ 4),\ i = 2, \end{cases} \qquad g_1(x_i) = \begin{cases} 0 & \text{if i} \equiv 0, 1(mod\ 4),\ i \neq n-1 \\ 1 & \text{if i} \equiv 2, 3(mod\ 4),\ i = n-1, \end{cases}$$

$$h_0(x_i) = \begin{cases} 0 & \text{if i} \equiv 3(mod\ 4) \\ 1 & \text{if i} \equiv 1(mod\ 4) \\ \frac{1}{2} & \text{if i} \equiv 0, 2(mod\ 4) \end{cases}$$

$\underline{n \equiv 2(mod\ 4)}$

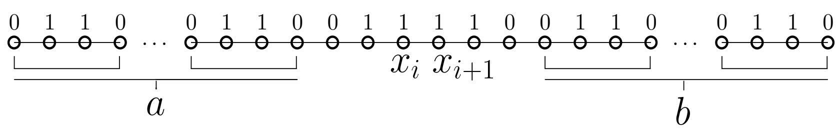

Figure 5.4: $i \equiv 3(mod\ 4)$, $a, b \geq 0$, $a + b = \frac{n-6}{4}$

The function $g \in \mathcal{G}_{P_n}$ illustrated in Figure 5.4 is such that $\sum_{u \in N(x_i)} g(u) = \sum_{u \in N(x_{i+1})} g(u) = 2 > 1$ so therefore by PCS, $h(x_i) = h(x_{i+1}) = 0$ for all $h \in \mathcal{H}_{P_n}$. Thus x_i is open neighborhood packing null for $i \equiv 0, 3(mod\ 4)$.

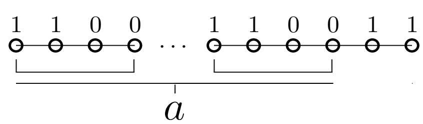

Figure 5.5: $a = \frac{n-2}{4}$

The function $h \in \mathcal{H}_{P_n}$ illustrated in Figure 5.5 is such that $h(x_i) = 1 > 0$ for all $i \equiv 1, 2(mod\ 4)$. Thus the only open packing null vertices are x_i such that $i \equiv 0, 3(mod\ 4)$.

The function represented in Figure 5.4 shows that x_i is not total domination null for all i except possibly when $i = 1$ or when $i = n$. But, note that the function $h \in \mathcal{H}_{P_n}$ illustrated in Figure 5.5 is also such that $h \in \mathcal{G}_{P_n}$. Here, $h(x_1) \neq 0$ and $h(x_n) \neq 0$. Thus, it has been shown that there are no total domination null vertices for P_n.

$n \equiv 3(mod\ 4)$

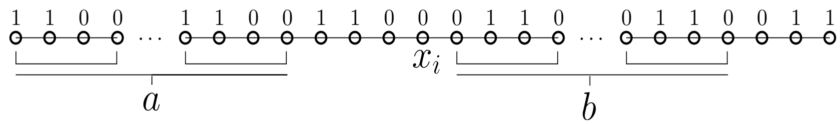

Figure 5.6: $i \equiv 0(mod\ 4)$, $a, b \geq 0$, $a + b = \frac{n-7}{4}$

The function $h \in \mathcal{H}_{P_n}$ illustrated in Figure 5.6 is such that $\sum_{u \in N(x_i)} h(u) = 0 < 1$ so therefore by PCS, $g(x_i) = 0$ for all $g \in \mathcal{G}_{P_n}$. Thus x_i is total domination null for $i \equiv 0(mod\ 4)$.

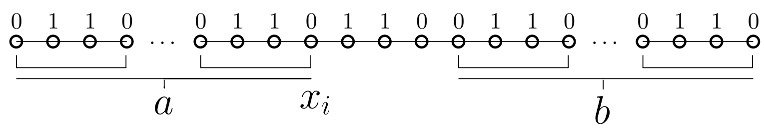

Figure 5.7: $i \equiv 0(mod\ 4)$, $a, b \geq 0$, $a + b = \frac{n-3}{4}$

The function $g \in \mathcal{G}_{P_n}$ illustrated in Figure 5.7 is such that $\sum_{u \in N(x_i)} g(u) = 2 > 1$ so therefore by PCS, $h(x_i) = 0$ for all $h \in \mathcal{H}_{P_n}$. Thus x_i is open neighborhood packing null for $i \equiv 0(mod\ 4)$.

The function illustrated in Figure 5.7 shows that x_i is not total domination null except when $i \equiv 0(mod\ 4)$ and possibly when $i = n$. The function illustrated in Figure 5.8 is in $\mathcal{G}_{P_n}$, and $g(x_n) \neq 0$. Thus the only total domination null vertices are those x_i such that $i \equiv 0(mod\ 4)$.

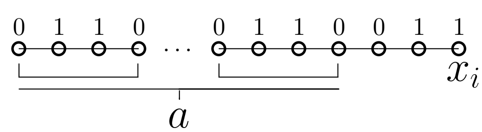

Figure 5.8: $i = n$ and $a = \frac{n-3}{4}$

The function illustrated in Figure 5.6 shows that x_i is not open neighborhood packing null except when $i \equiv 0(mod\ 4)$ and possibly when $i \in \{3, n\}$. The function illustrated in

Figure 5.8 is also in $\mathcal{H}_{P_n}$, and hence x_3 and x_n are not open neighborhood packing null. Thus the only open neighborhood packing null vertices are those x_i such that $i \equiv 0(mod\ 4)$.

■

Now that it is clear which vertices of P_n are total domination null and which are open neighborhood packing null, we can more easily characterize all functions in the sets $\mathcal{G}_{P_n}$ and $\mathcal{H}_{P_n}$.

Theorem 5.3 *If $n \equiv 0(mod\ 4)$, then there is only one function in the set $\mathcal{G}_{P_n}$, and it is*

$$g(x_i) = \begin{cases} 0 & \text{if } i \equiv 0, 1(mod\ 4) \\ 1 & \text{if } i \equiv 2, 3(mod\ 4) \end{cases}$$

Proof Let $g \in \mathcal{G}_{P_n}$. Theorem 5.2 states that $g(x_i) = 0$ for all $i \equiv 0, 1(mod\ 4)$. Then, by Lemma 5.4, it must be that $g(x_i) = 1$ for all $i \equiv 2, 3(mod\ 4)$. ■

Theorem 5.4 *If $n \equiv 1(mod\ 4)$, then $\mathcal{G}_{P_n}$ is the set of functions $g : \{x_1, x_2, \ldots, x_n\} \to [0, 1]$ such that*

(a) $g(x_i) = \begin{cases} 0 & \text{if } i \equiv 1(mod\ 4) \\ 1 & \text{if } i \equiv 3(mod\ 4) \end{cases}$

(b) $g(x_2) = g(x_{n-1}) = 1$

(c) $\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} g(x_{2k}) = \lceil \frac{n}{4} \rceil$

(d) $g(x_i) + g(x_{i+2}) \geq 1$ *for* $i \equiv 0(mod\ 2)$.

Proof Suppose $g \in \mathcal{G}_{P_n}$. Part (a) is again a direct result of Theorem 5.2 and Lemma 5.4. Part (b) is direct result of Lemma 5.3. Part (c) is true because $(\gamma_t)_f = \lceil \frac{n}{2} \rceil$ and, by part (a), $\sum_{i \equiv 3(mod 4)} g(x_i) = \lfloor \frac{n}{4} \rfloor$ and $\sum_{i \equiv 1(mod 4)} g(x_i) = 0$. Part (d) is obvious since g is a total dominating function on P_n.

It is straightforward to see that if $g : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ satisfies (a) through (d), then $g \in \mathcal{G}_{P_n}$. ■

Theorem 5.5 *If $n \equiv 2(mod\ 4)$, then $\mathcal{G}_{P_n}$ is the set of functions $g : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ such that*

(a) $\sum_{k=1}^{n} g(x_k) = \frac{n}{2} + 1$

(b) $g(x_2) = g(x_{n-1}) = 1$

(c) $g(x_i) + g(x_{i+2}) \geq 1$ *for* $1 \leq i \leq n-2$.

Proof Suppose $g \in \mathcal{G}_{P_n}$. Part (a) follows from Theorem 5.2 and the definition of $(\gamma_t)_f(P_n)$. Part (b) is direct result of Lemma 5.3. Part (c) is obvious since g is a total dominating function on P_n.

It is straightforward to see that any $g : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ satisfying (a), (b), and (c) is a minimum fractional total dominating function on G. ■

Theorem 5.6 *If $n \equiv 3(mod\ 4)$, then $\mathcal{G}_{P_n}$ is the set of functions $g : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ such that*

(a) $g(x_i) = \begin{cases} 0 & \text{if } i \equiv 0(mod\ 4) \\ 1 & \text{if } i \equiv 2(mod\ 4) \end{cases}$

(b) $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(x_{2k+1}) = \lceil \frac{n}{4} \rceil$

(c) $g(x_i) + g(x_{i+2}) \geq 1$ *for* $i \equiv 0(mod\ 2)$.

Proof Suppose $g \in \mathcal{G}_{P_n}$. Part (a) is again the direct result of Theorem 5.2 and Lemmas 5.4 and 5.3. Part (b) is true because $(\gamma_t)_f = \lceil \frac{n}{2} \rceil$ and, by part (a), $\sum_{i \equiv 2(mod4)} g(x_i) = \lceil \frac{n}{4} \rceil$ and $\sum_{i \equiv 0(mod4)} g(x_i) = 0$. Part (c) is obvious since g is a total dominating function on P_n.

The sufficiency of (a), (b), and (c) for $g : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ to be a minimum fractional total dominating function on G is straightforward. ■

Similarly, because of Theorem 5.2, it is not difficult to characterize all of the functions in the set $\mathcal{H}_{P_n}$.

Theorem 5.7 *If $n \equiv 0(mod\ 4)$, then $\mathcal{H}_{P_n}$ is the set of functions $h : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ such that*

(a) $h(x_i) + h(x_{i+2}) = 1$ *for* $i \equiv 1, 2(mod\ 4)$

(b) $h(x_i) + h(x_{i+2}) \leq 1$ *for* $i \equiv 0, 3(mod\ 2)$.

Proof Let $h \in \mathcal{H}_{P_n}$. Since there are $\frac{n}{2}$ disjoint pairs $\{x_i, x_{i+2}\}$, $i \equiv 1, 2(mod\ 4)$, h is a fractional open neighborhood packing of P_n, and $(\gamma_t)_f(P_n) = \pi_f^0(P_n) = \frac{n}{2}$, each pair $\{x_i, x_{i+2}\}$ must be such that $h(x_i) + h(x_{i+2}) = 1$, proving (a). Part (b) is a result of $h \in \mathcal{H}_{P_n}$.

On the other hand, it is straightforward to see that if $h : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ satisfies (a) and (b) then $h \in \mathcal{H}_{P_n}$. ■

Theorem 5.8 *If $n \equiv 1(mod\ 4)$, then $\mathcal{H}_{P_n}$ is the set of functions $h : \{x_1, x_2, \ldots, x_n\} \to [0,1]$ such that*

(a) $h(x_i) = \begin{cases} 0 & \textit{if } i \equiv 3(mod\ 4) \\ 1 & \textit{if } i \equiv 1(mod\ 4) \end{cases}$

(b) $h(x_i) + h(x_{i+2}) = 1$ *for* $i \equiv 2(mod\ 4)$

(c) $h(x_i) + h(x_{i+2}) \leq 1$ *for* $i \equiv 0(mod\ 4)$

Proof Let $h \in \mathcal{H}_{P_n}$. To show (a), first it is clear by Theorem 5.2 that $h(x_i) = 0$ for $i \equiv 3(mod\ 4)$. Also, since there are $\frac{n-1}{4}$ disjoint sets of the form $\{x_i, x_{i+2}\}$ such that $i \equiv 2(mod\ 4)$, and

$h \in \mathcal{H}_{P_n}$, $\sum_{i\equiv 0(mod 2)} h(x_i) \leq \frac{n-1}{4}$. Therefore because $\pi_f^0(P_n) = \sum_{i=1}^{n} h(x_i) = \lceil \frac{n}{2} \rceil$, $h(x_i) = 1$ for $i \equiv 1(mod\ 4)$. (b) then follows immediately because $\pi_f^0(P_n) = \lceil \frac{n}{2} \rceil$. Finally (c) is a result of the fact that $h \in \mathcal{H}_{P_n}$.

On the other hand, it is straightforward to see that if $h : \{x_1, x_2, \ldots, x_n\} \to [0, 1]$ satisfies (a), (b), and (c) then $h \in \mathcal{H}_{P_n}$. ■

Theorem 5.9 *If $n \equiv 2(mod\ 4)$, then $\mathcal{H}_{P_n}$ has only one member, and it is exactly $h : \{x_1, x_2, \ldots, x_n\} \to [0, 1]$ such that* $h(x_i) = \begin{cases} 0 & \text{if } i \equiv 0, 3(mod\ 4) \\ 1 & \text{if } i \equiv 1, 2(mod\ 4) \end{cases}$

Proof Let $h \in \mathcal{H}_{P_n}$. By Theorem 5.2, it is clear that $h(x_i) = 0$ for $i \equiv 0, 3(mod\ 4)$. Since $\pi_f^0(P_n) = \lceil \frac{n}{2} \rceil + 1$, it is forced that $h(x_i) = 1$ for $i \equiv 1, 2(mod\ 4)$. ■

Theorem 5.10 *If $n \equiv 3(mod\ 4)$, then $\mathcal{H}_{P_n}$ is the set of functions $h : \{x_1, x_2, \ldots, x_n\} \to [0, 1]$ such that*

(a) $h(x_i) = \begin{cases} 0 & \text{if } i \equiv 0(mod\ 4) \\ 1 & \text{if } i \equiv 2(mod\ 4) \end{cases}$

(b) $h(x_i) + h(x_{i+2}) = 1$ *for* $i \equiv 1(mod\ 4)$

(c) $h(x_i) + h(x_{i+2}) \leq 1$ *for* $i \equiv 3(mod\ 4)$, $i < n$.

Proof Let $h \in \mathcal{H}_{P_n}$. Again, it is clear by Theorem 5.2 that $h(x_i) = 0$ for $i \equiv 0(mod\ 4)$. There are $\frac{n+1}{4}$ disjoint sets of the form $\{x_i, x_{i+2}, x_{i+3}\}$ such that $i \equiv 1(mod\ 4)$. Certainly $h(x_i) + h(x_{i+1}) + h(x_{i+2}) \leq 2$ for each such set. Therefore, since $\pi_f^0(P_n) = \lceil \frac{n}{2} \rceil$, it must be that $h(x_i) + h(x_{i+2}) = 1$ and $h(x_{i+1}) = 1$ for $i \equiv 1(mod\ 4)$ proving (a) and (b). (c) is again a result of the fact that $h \in \mathcal{H}_{P_n}$.

On the other hand, it is straightforward to see that if $h : \{x_1, x_2, \ldots, x_n\} \to [0, 1]$ satisfies (a), (b), and (c) then $h \in \mathcal{H}_{P_n}$. ■

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