Fixed Points and Periodic Points of Orientation Reversing Planar Homeomorphisms

by

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Abstract

Topological dynamics on surfaces is studied. The primary objects of study are orientation reversing homeomorphisms of the plane, but most of the results apply also to the orientation reversing homeomorphisms of the 2-sphere. The starting point for the present dissertation was the following problem of Krystyna Kuperberg from 1989. Suppose h is an orientation reversing homeomorphism of the plane, and there are at least n bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. Must there be at least $n+1$ fixed points of h in X? This question is answered in the affirmative. Several other results concerning this isotopy class of homeomorphisms are proved. A separate topic of the present dissertation is constituted by periodic point theorems for plane separating circle-like continua. One of them is the following theorem. Let $f : \mathcal{C} \to \mathcal{C}$ be a self-map of the pseudo-circle $\mathcal{C}$. Suppose that $\mathcal{C}$ is embedded into an annulus $\mathbb{A}$, so that it separates the two components of the boundary of $\mathbb{A}$. Let $F : \mathbb{A} \to \mathbb{A}$ be an extension of f to $\mathbb{A}$ (i.e. $F|\mathcal{C} = f$). If F is of degree d then f has at least $|d-1|$ fixed points. This result generalizes to all plane separating circle-like continua. In addition, several other aspects of topological dynamics on planar continua are studied relating to Sarkovskii's theorem.

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Chapter 1

Introduction

The purpose of the present chapter is to outline the main results of this thesis (Theorem 1.1 - Theorem 1.7) in the context of the results already known from the literature. These main results are independently restated in the subsequent sections, where the proofs of the results are contained. In order to facilitate reading of the present work, each chapter is aimed to be as self-contained as possible so that the reader can choose which theorems to read without getting lost in chapters unrelated to their interest.

1.1 Brouwer homeomorphisms

Two isotopy classes partition the family of plane homeomorphisms. The orientation preserving homeomorphisms are in the isotopy class of the identity map $\mathrm{id}(x, y) = (x, y)$. The reflection map $r(x, y) = (-x, y)$ determines the isotopy class of orientation reversing homeomorphisms. The study of topological dynamics of planar homeomorphisms has a long history. In the beginning of the twentieth century L. E. J. Brouwer studied orientation preserving planar homeomorphisms without fixed points (in the literature referred to as Brouwer homeomorphisms), and in 1912 showed [20] that any such homeomorphism must be a translation. By a translation Brouwer meant a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the orbit $\{T^n(x) : n \in \mathbb{N}\}$ of any point x is unbounded; that is, it is not contained in any closed disk. Brouwer's work, followed by the work of B. de Kerékjártó [39], W. Scherrer [58], H. Terasaka [61], and E. Sperner [60] (see also [29],[59]), resulted in the following characterization of Brouwer homeomorphisms. Suppose H preserves the orientation and is fixed point free. Then,

(OP1) H is periodic point free; i.e. $H^p(x) \neq x$ for any point x and any integer p, (OP2) Any point in the plane is a wandering point; i.e. for any point $x \in \mathbb{R}^2$ there is a small disk neighborhood D_x of x such that $\{H^n(D) : n \in \mathbb{Z}\}$ are pairwise disjoint,

(OP3) For any point $x = (x_1, x_2)$ there is an open invariant set U (called a translation domain) that contains x, where $H|U$ is conjugate to the translation $\sigma : (x_1, x_2) \mapsto (x_1, x_2 + 1)$; i.e. there is an embedding $\phi_x : \mathbb{R}^2 \to U$ such that $H|U \circ \phi_x = \phi_x \circ \sigma$.

1.2 Fixed points in nonseparating plane continua and the bounded orbit conjecture

A continuum is a connected and compact set that contains at least 2 points. Suppose h is an orientation preserving homeomorphism h with an invariant (i.e. $h(X) = X$) nonseparating plane continuum X. It follows from Brouwer theory that such a homeomorphism has a fixed point in the plane. Must it have a fixed point in the invariant continuum? In 1951, in Annals of Mathematics [24], M. L. Cartwright and J. E. Littlewood answered this question in affirmative. The motivation for their work emerged naturally from the study of differential equations. For example, it had been known that van der Pol's equation

$$\ddot{x} + k(x^2 - 1)\dot{x} + \omega^2 x = bk \cos 2\pi t$$

led to the invariant set whose boundary was not locally connected. This set exhibited a very complicated structure, and possibly contained indecomposable continua. Later, in 1966, it was shown by V. A. Pliss [55] that any nonseparating plane continuum is the maximal bounded closed set invariant under a transformation F, where F is a solution of certain dissipative system of differential equations (see also [2]). Because the continua invariant under plane homeomorphisms could exhibit such an arbitrarily complicated structure, it was not known at the time Cartwright-Littlewood theorem was proved, how the existing tools of algebraic topology could be applied to the study of this problem. Despite this fact, in 1954 O. H. Hamilton [32] offered an alternative short proof of Cartwright-Littlewood theorem, deriving it directly from the theorem of Brouwer. Algebraic topology proved helpful after all when, in 1977, Morton Brown [21] provided yet another short proof of the same theorem by the means of covering spaces and the theorem of Brouwer.

It may be surprising that it took more than a quarter of the century to determine if the Cartwright-Littlewood theorem was true also for orientation reversing plane homeomorphisms. This difficult open question was settled in 1978 by H. Bell [5] who showed that such a theorem was true indeed. The main difficulty which Bell had to face was the fact that the bounded orbit conjecture had not been resolved by that time. The bounded orbit conjecture assumed that any plane homeomorphism h that has an orbit of each point bounded must have a fixed point. It follows from the theorem of Brouwer that even one bounded orbit forces h to have a fixed point if h preserves the orientation. However, an easy example shows that this is not the case for orientation reversing homeomorphisms. To see that, it is enough to consider a homeomorphism h given by $h(x_1, x_2) = (-x_1, x_2 - |x_1| + 1)$ if $|x_1| < 1$, and $h(x_1, x_2) = (-x_1, x_2)$ for $|x_1| \geq 1$ (see Figure 1.2). Clearly, any point with $|x_1| > 1$ is of period 2, but no point is fixed under h. Surprisingly, the bounded orbit conjecture was disproved by S. Boyles [18], [19] who, in 1980, constructed an example of a fixed point free orientation reversing homeomorphism of the plane with every orbit bounded. This result emphasized the contrast between the two isotopy classes of plane homeomorphisms.

1.3 Fixed point index

There are many other surprising differences between dynamics of orientation preserving and orientation reversing homeomorphisms. Recall that a fixed point index of a homeomorphism h at an isolated fixed point x_o is the degree of the map $z \rightarrow \frac{z-f(u(z))}{||z-f(u(z))||}$, where $u : \mathbb{S}^1 \rightarrow C$ is an orientation preserving parametrization of a simple closed curve C around x_o, and C is contained within a neighborhood D of x_o that contains no other fixed point of h. M. Brown [23] proved that each integer may occur as the fixed point index at the origin of an orientation preserving plane homeomorphism. However, in the same paper, he

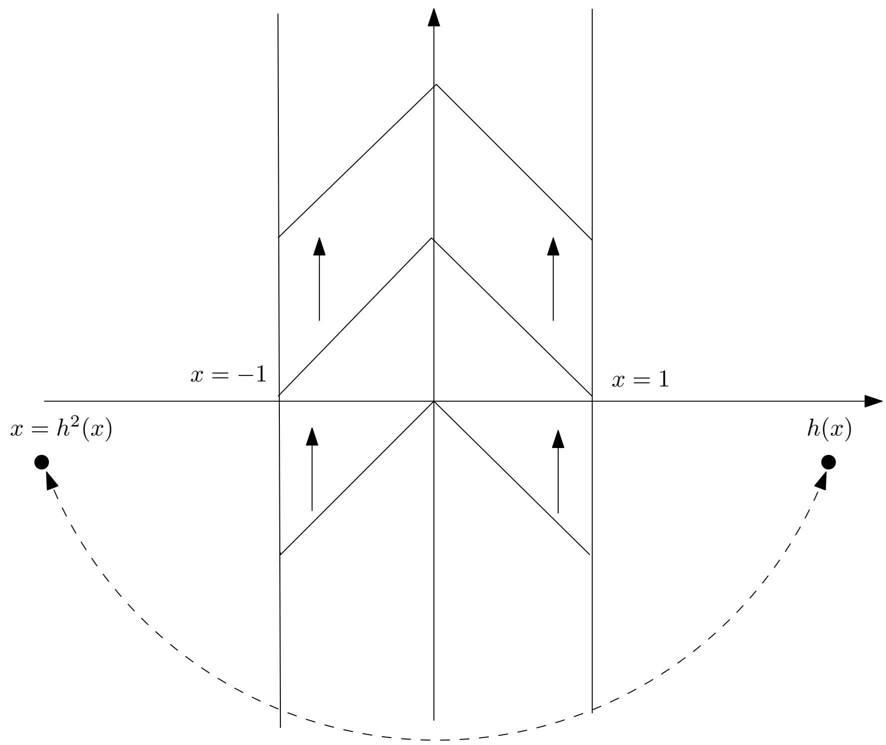

Figure 1.1: Fixed point free orientation reversing homeomorphism with two half-planes of bounded 2-periodic orbits

announced that this is not the case for orientation reversing homeomorphisms as, in such a case, only $-1, 0$ and 1 are possible fixed point indexes at an isolated fixed point. Despite his intent, Brown never actually provided the proof for the later result, but it was furnished by Marc Bonino in [13].

1.4 The structure of the fixed point set

How complicated can the fixed point set of an orientation reversing plane homeomorphism be? M. Brown and J. M. Kister [22] showed that if F is a fixed point set of a homeomorphism f of a connected topological manifold M, then either each component of $M \setminus F$ is invariant under f, or there are exactly two components of $M \setminus F$ and f interchanges them. Since, in the plane, the bounded and unbounded complementary domains cannot be interchanged, it follows that the fixed point set of any homeomorphism that reverses the orientation cannot separate the plane. A pseudo-arc, first constructed by B. Knaster in [40], and then independently by E. E. Moise [51], was characterized by R. H. Bing in [10] as the chainable hereditarily indecomposable continuum. A continuum is chainable (or arc-like) if it is the inverse limit of arcs. A continuum is indecomposable if it is not a union of two proper subcontinua, and it is hereditarily indecomposable if each of its proper subcontinua is indecomposable. Hereditarily indecomposable continua exhibit a very complicated structure. D. Bellamy and W. Lewis [6] showed that an orientation reversing homeomorphism can have the pseudo-arc as an invariant set. Although invariant sets of the homeomorphisms in this isotopy class may indeed be pathological in their structure, it turns out that any component of the fixed point set must be either a line, an arc or a point. This follows from the result of D. B. Epstein [26] who showed that any such component of the fixed point set of an orientation reversing homeomorphism of the 2-sphere $\mathbb{S}^2$ must be either a circle, an arc, or a point. Consequently, the classification for the planar fixed point sets follows by compactification of the plane with a point at infinity.

1.5 Fixed points in separating plane continua

Obviously, the Cartwright-Littlewood theorem is not true in general for plane separating continua, with the rotation of the circle as the simplest example. However, under additional assumptions, orientation preserving homeomorphisms do have fixed points in such continua. Specifically, let X be a continuum invariant under such a homeomorphism h. Suppose that X separates the plane into exactly two components. In 1992 M. Barge and R. Gillette [3] showed that if there is a fixed prime end in the prime-end compactification of either of the two domains, then there must be a fixed point in X. This result was generalized by Barge and K. Kuperberg [4] to the case when X has finitely many complementary domains: If a prime end is fixed in the prime-end compactification of each of the domains then there is a fixed point in X. The reader less familiar with the theory of prime ends is referred to Section A.3 of Appendix.

A more powerful result had been discovered by Kuperberg for orientation reversing homeomorphisms earlier in 1989 [41], when she extended the result of Bell to separating plane continua. Namely, let h be such a homeomorphism with a continuum X invariant (i.e. $h(X) = X$). Suppose there is at least one bounded component of $\mathbb{R}^2 \setminus X$ that is invariant under h. Kuperberg proved that, in this setting, there must be two fixed points of h in X. Subsequently, she also showed [42] that h must have at least $k+2$ fixed points in X, whenever there are n bounded invariant components of $\mathbb{R}^2 \setminus X$, where $n \geq 2^k$. Motivated by these results she raised the following question.

Question. [K. Kuperberg, 1989, [42]] Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation reversing homeomorphism with a continuum X invariant. Suppose there are n components of $\mathbb{R}^2 \setminus X$ that are invariant under h (including the unbounded one). Must there be n fixed points in X?

In the present dissertation, in Chapter 2, we shall answer this question in the affirmative. More precisely, we will prove the following stronger result.

Theorem 1.1. *[16] Let $M \in \{\mathbb{R}^2, \mathbb{S}^2\}$ and let $h : M \to M$ be an orientation reversing homeomorphism with a continuum X invariant, and suppose there are at least n components of $M \setminus X$ that are invariant under h. Then $Fix(X, h) = \{x \in X : h(x) = x\}$, the set of fixed points of h in X, has at least n components.*

1.6 Periodic points of orientation reversing homeomorphisms

At this point one may ask if, in the above setting, anything can be said about the existence of points of other periods in X. Before we answer this question let us mention that in 2004 M. Bonino [14] established a full analogy of the Brouwer theory for orientation reversing homeomorphisms of $\mathbb{S}^2$ (characterization for $\mathbb{R}^2$ is obtained by compactification of the plane by a point at ∞). More specifically he showed that, in some sense, one may replace fixed points with points of period 2 in the theory of Brouwer homeomorphisms. He then obtained a parallel characterization for any 2-periodic point free orientation reversing homeomorphism G. Namely,

(OR1) G has no periodic points except fixed points,

(OR2) Every point that is not fixed is a wandering point,

(OR3) For any point $x = (x_1, x_2)$ that is not in the fixed points set $\mathrm{Fix}(G)$, there is a topological embedding $\phi_x : O \to \mathbb{R}^2 \setminus \mathrm{Fix}(G)$ such that

  (a) O is either $\mathbb{R}^2$ or $\{(y_1, y_2) \in \mathbb{R}^2 : y_2 \neq 0\}$ or $\mathbb{R}^2 \setminus \{(0,0)\}$,

  (b) $x \in \phi_x(O)$

  (c) if $O = \mathbb{R}^2$ or $O = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \neq 0\}$ then

    (i) $G \circ \phi_x = \phi_x \circ \delta | O$, where $\delta(y_1, y_2) = (y_1 + 1, -y_2)$

    (ii) for every $y \in \mathbb{R}$, $\phi_x((\{y\} \times \mathbb{R}) \cap O)$ is a closed subset of $\mathbb{R}^2 \setminus \mathrm{Fix}(G)$

  (d) if $O = \mathbb{R}^2 \setminus \{(0,0)\}$ then $G \circ \phi_x = \phi_x \circ \rho | O$, where $\rho(y_1, y_2) = \frac{1}{2}(y_1, -y_2)$

1.7 Periodic points in invariant continua

Note that it follows from the above characterization that if $h : \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation reversing homeomorphism of $\mathbb{S}^2$ onto itself with an orbit $\mathcal{O}$ of period $k > 2$ then h must also have an orbit $\mathcal{O}'$ of period 2. Using Nielsen's theory Bonino strengthened his result in [15] showing that if h has a k-periodic orbit $\mathcal{O}$ with $k > 2$, then there is a 2-periodic orbit $\mathcal{O}'$ such that $\mathcal{O}$ and $\mathcal{O}'$ are linked. Two orbits $\mathcal{O}$ and $\mathcal{O}'$ are linked in the sense of Bonino if one cannot find a Jordan curve $C \subseteq \mathbb{S}^2$ separating $\mathcal{O}$ and $\mathcal{O}'$ which is freely isotopic to $h(C)$ in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$. C and $h(C)$ are freely isotopic in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$ if there is an isotopy $\{i_t : \mathbb{S}^1 \to \mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}') : 0 \leq t \leq 1\}$ from $i_0(\mathbb{S}^1) = C$ to $i_1(\mathbb{S}^1) = h(C)$; i.e. $i_t(\mathbb{S}^1)$ is a Jordan curve for any t ($\mathbb{S}^1$ denotes the unit circle). Exploiting heavily results from the second paper we will show that, under certain assumptions, one can find a 2-periodic point in an invariant continuum.

Theorem 1.2. *[16] Let $M \in \{\mathbb{R}^2, \mathbb{S}^2\}$ and let $h : M \to M$ be an orientation reversing homeomorphism with a continuum X invariant (i.e. $h(X) = X$). Suppose h has a k-periodic orbit in X with $k > 2$.*

(i) If X does not separate the plane then h has a 2-periodic orbit in X.

(ii) If X separates the plane then h has a 2-periodic orbit in X, or there is a 2-periodic component U of $M \setminus X$.

The above result seems to be related to Sarkovskii's theorem [1]. Sarkovskii's theorem [57] asserts that any map of the real line $\mathbb{R}^1$ into itself that has a periodic point of period n must also have a periodic point of period m whenever $n \triangleleft m$, where $\triangleleft$ is an ordering of the natural numbers defined as follows: $3 \triangleleft 5 \triangleleft 7 \triangleleft \ldots \triangleleft 3 \cdot 2 \triangleleft 5 \cdot 2 \triangleleft \ldots \triangleleft 3 \cdot 2^2 \triangleleft 5 \cdot 2^2 \triangleleft \ldots \triangleleft \ldots \triangleleft 2^3 \triangleleft 2^2 \triangleleft 2 \triangleleft 1$. Some generalizations of this theorem are known to be true for self-maps of certain classes of continua, and this topic is further discussed in section 1.13, as well as Chapter 6.

1.8 Measures preserved by homeomorphisms

Must an orientation reversing homeomorphism of the plane, under certain additional assumptions, preserve a measure that, in some sense, behaves similar to the Lebesgue measure? Note that this is the case in the framework of orientation preserving homeomorphisms. In response to a question of Brown, S. Baldwin and E. Slaminka [1] showed that for a given Brouwer homeomorphism h there is a measure μ that is invariant with respect to h; i.e. $\mu(h(A)) = \mu(A)$ for every μ-measurable subset A of the plane. In addition, μ exhibits the following properties:

(M1) μ is the completion of a countably additive measure on the set of all Borel subsets of $\mathbb{R}^2$,

(M2) $\mu(A)$ is finite for any bounded subset of $\mathbb{R}^2$,

[1] Note that an important related result was independently proved by Li T. Y. and J. A. Yorke [46], and led to the development of the chaos theory.

(M3) $\mu(U) > 0$ for any nonempty open subset of $\mathbb{R}^2$,

(M4) If A is a subset of $\mathbb{R}^2$ and $f(A)$ has Lebesgue measure 0 for any homeomorphism f of $\mathbb{R}^2$, then $\mu(A) = 0$,

(M5) Lebesgue measure λ is absolutely continuous with respect to μ; i.e. $\mu(A) = 0$ implies that A is Lebesgue measurable and $\lambda(A) = 0$.

In the light of aforementioned work of Bonino, it seems interesting to point out that the above result extends to fixed point and 2-periodic point free orientation reversing homeomorphisms of the plane. In Section A.4 of Appendix we recall the construction of Baldwin and Slaminka.

1.9 The pseudo-circle as an invariant continuum

In [33] M. Handel constructed an area preserving C^∞ diffeomorphism of the plane with a minimal set (i.e. invariant and closed set that contains no other set with this property) that is a pseudo-circle. The pseudo-circle is characterized as a plane separating, hereditarily indecomposable circularly chainable continuum, whose every proper subcontinuum is a pseudo-arc [10]. In [34] J. Heath showed that the pseudo-circle admits a 2-fold cover onto itself (see [30] for related results). Her example yields a construction of a k-periodic orientation preserving homeomorphism of the plane with an invariant pseudo-circle, for any $k \geq 2$. Heath's result will be used in Chapter 3 to show a family of orientation reversing homeomorphisms of $\mathbb{S}^2$ with an invariant pseudo-circle.

Theorem 1.3. *For any $k \geq 1$ there is a $2k$-periodic orientation reversing homeomorphism of $\mathbb{S}^2$ with invariant pseudo-circle.*

It seems worthwhile to mention a characterization of 2-periodic homeomorphisms of $\mathbb{S}^2$. Kerékjártó [37] and S. Eilenberg [25] showed that such a map is topologically equivalent to either the identity (every point is fixed), a rotation (there are two fixed points), a reflection (there is a simple closed curve of fixed points), or a rotation followed by a reflection (there are no fixed points, see [48],[49] for related results). It follows from the characterization that the $2k$-periodic homeomorphisms described in Theorem 1.3 have no fixed points. This is due to the fact that the pseudo-circle separates $\mathbb{S}^2$ into two 2-periodic components, and since the invariant pseudo-circle does not contain a simple closed curve, therefore no point can be a fixed point on $\mathbb{S}^2$.

1.10 Linear order on the complementary domains of a continuum

The above examples show that orientation reversing homeomorphisms admit continua of very complicated structure as their invariant sets. Can one also find some regularity in the structure of these sets? Namely, consider the following set, that is the union of three circles.

$$X = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 = 1, \text{ where } a = 0, b = -2, 0, 2\}.$$

Notice that X is a continuum invariant under the reflection $r(x,y) = (-x,y)$. All of the fixed points of r in X lie on the line $x = 0$. Namely,

$$\mathrm{Fix}(r,X) = \{(-3,0), (-1,0), (1,0), (3,0)\}.$$

Note that $\mathrm{Fix}(r,X)$ is linearly ordered by the second coordinate. The same is true about the collection of bounded complementary domains of X. That is, the components of $B = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < 1, \text{ where } a = 0, b = -2, 0, 2\}$ are linearly ordered by the second coordinate. Since any orientation reversing planar homeomorphism is isotopic to r, it seems natural to ask the following.

Question: *For a given orientation reversing homeomorphism h with a plane separating continuum X invariant, is there a natural linear order on the collection of the bounded invariant components of $\mathbb{R}^2 \setminus X$, that will resemble the one described in the above example?*

The answer does not seem apparent, for example, in the case when X is a Lakes of Wada continuum (i.e. every point in X is in the boundary of any component of $\mathbb{R}^2 \setminus X$) or X is (hereditarily) indecomposable. In Chapter 4 we shall answer this question in the affirmative. First notice that, without loss of generality, we may assume that X contains an invariant simple closed curve in each invariant component of $\mathbb{R}^2 \setminus X$, one of which, say C_0, bounds X. Let $\mathbb{A}_U$ denote an invariant annulus containing X, determined by C_0 and one more invariant simple closed curve from U, one of the bounded components of $\mathbb{R}^2 \setminus X$. We will prove the following result, motivated also by the proof of Theorem 1.1.

Theorem 1.4. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation reversing homeomorphism with a continuum X invariant. Suppose $\mathcal{U}$ is a collection of k bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. For any two nonnegative integers p, q such that $p + q = k - 1$ there is $U \in \mathcal{U}$ such that the two Nielsen classes of $h|\mathbb{A}_U$ partition $\mathcal{U} \setminus U$ into two sets, one of which has p elements, and the other one has q elements.*

One could say that the complementary domains of X that are in the same Nielsen class with respect to the annulus $\mathbb{A}_U$ as described above, are on the same side of U. Therefore the Nielsen classes determine, in some sense, two sides in $\mathbb{A}_U$: "above" U and "below" U (cf. Figure 1.10).

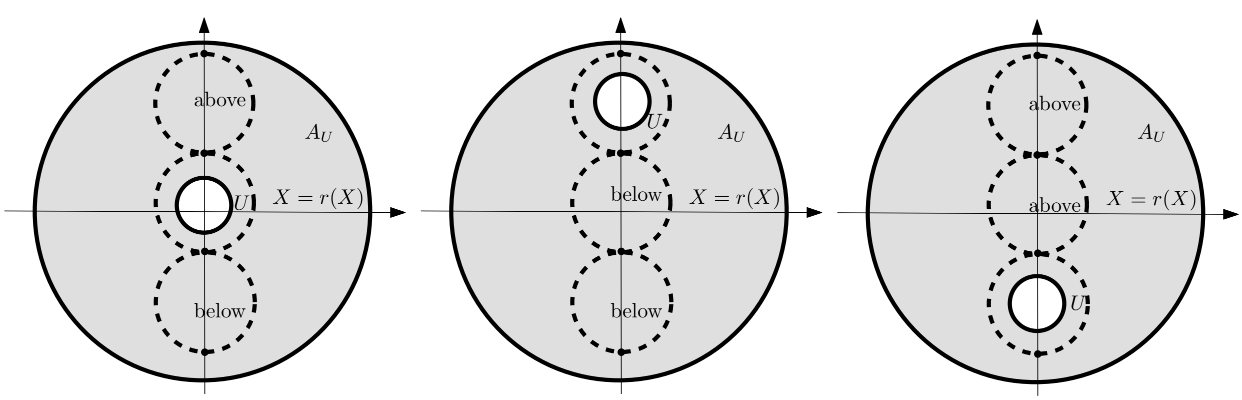

Figure 1.2: The two Nielsen classes in $\mathbb{A}_U$

1.11 Nielsen fixed point classes

The notion of a Nielsen fixed point class originates from the Nielsen fixed point theory, which emerged as a powerful method of finding fixed and periodic points of self-homeomorphisms of surfaces (i.e. 2-manifolds). Let M be a compact connected surface with or without boundary and let $\tau : \tilde{M} \to M$ be its universal covering. A Nielsen class of a continuous map $\psi : \tilde{M} \to M$ is defined as the set $\tau(\{\tilde{x} \in \tilde{M} : \tilde{\psi}(\tilde{x}) = \tilde{x}\})$ where $\tilde{\psi} : \tilde{M} \to \tilde{M}$ is a lift of ψ. Nonempty Nielsen classes define a partition of the fixed point set of ψ. Any orientation reversing self-homeomorphism h of the annulus has two nonempty Nielsen classes. By a disk with k holes we mean a compact connected 2-manifold with boundary, embeddable in $\mathbb{R}^2$, such that its boundary has $k+1$ components. A disk with 1 hole is an annulus. It can be implicitly found in the work of Krystyna Kuperberg [41] that any separating plane continuum essentially embedded into an annulus, and invariant under h, must intersect these two classes. This motivated the following theorem, in which we will provide an alternative proof of Theorem 1.1.

Theorem 1.5. *Let $h : \mathbb{D} \to \mathbb{D}$ be an orientation reversing homeomorphism of the disk with k holes. Let X be a continuum invariant under h, with $k+1$ components of $\mathbb{D} \setminus X$, each of which is invariant under h, and contains exactly one component of $\partial\mathbb{D}$. Then there are $k+1$ Nielsen classes of h, each of which intersects X; i.e. there are $k+1$ components of* $\mathrm{Fix}(h, X)$.

Some of the aforementioned results obtained by the author motivated an additional, independent area of study for the present thesis, that we describe herein.

1.12 Fixed points of self-maps of the pseudo-circle and other circle-like continua

A continuum X has the fixed point property (abbreviated f.p.p.) if for any map $f : X \to X$ there is an $x \in X$ such that $f(x) = x$. It is known that the unit interval $[0, 1]$ has the f.p.p. Recall that the pseudo-arc is a hereditarily indecomposable arc-like continuum.

It is homogeneous (R. H. Bing [8]) and homeomorphic to each of its subcontinua (Moise [51]). In [31] O.H. Hamilton proved that the pseudo-arc and all other arc-like continua posses the f.p.p.. It is known that any self-map of the unit circle $\mathbb{S}^1$ of degree d has at least $|d-1|$ fixed points. A continuum X is circle-like (or circularly chainable) if it is the inverse limit of spaces homeomorphic to $\mathbb{S}^1$. Recall that the pseudo-circle is a planar hereditarily indecomposable circle-like continuum that has the same Čech homology as $\mathbb{S}^1$, and every proper subcontinuum of which is homeomorphic to the pseudo-arc. L. Fearnley in [27], and J. T. Rogers Jr. in [56] showed that the pseudo-circle is not homogeneous (see also [43]). The pseudo-circle does not posses f.p.p. For example, by the result of Heath [34], the pseudo-circle admits a 2-fold cover onto itself and therefore it admits a rotation (induced by a deck transformation) that does not have any fixed points (see also [33], and Theorem 1.3 in the present dissertation). In [6] Bellamy and Lewis showed that the two-point compactification of the universal covering space of the pseudo-circle is the pseudo-arc. These results, from [34] and [6], were further studied and extended by K. Gammon in [30]. We shall show that many self-maps of the pseudo-circle do have a fixed point, and the number of fixed points for such maps may be estimated analogously to the self-maps of $\mathbb{S}^1$. Namely, let $\mathcal{C}$ be the pseudo-circle embedded into $\mathbb{A}$ in such a way that the winding number of each circular chain in the sequence of crooked circular chains defining $\mathcal{C}$ is one. Note that any self-map f of $\mathcal{C}$ extends to a self-map of $\mathbb{A}$. Indeed, since $\mathbb{A}$ is an Absolute Neighborhood Retract, f can be first extended to a map $\hat{f} : U \to \mathbb{A}$, where U is a closed annular neighborhood of $\mathcal{C}$. Then if $r : \mathbb{A} \to U$ is a retraction of $\mathbb{A}$ onto U the composition $F = \hat{f} \circ r$ is the desired self-map of $\mathbb{A}$ such that $F|\mathcal{C} = f$ (cf. [43]). In Chapter 5 we shall prove the following result.

Theorem 1.6. *Let $f : \mathcal{C} \to \mathcal{C}$ be a self-map of the pseudo-circle $\mathcal{C}$. Suppose that $F : \mathbb{A} \to \mathbb{A}$ is an extension of f to $\mathbb{A}$ (i.e. $F|\mathcal{C} = f$). If F is of degree d then f has at least $|d-1|$ fixed points.*

1.13 Sarkovskii-type theorem for hereditarily decomposable circle-like continua

In [50] Piotr Minc and William R.R. Transue showed that Sarkovskii's theorem extends to self-maps of hereditarily decomposable chainable continua. A continuum is decomposable if it is the union of two proper subcontinua. A continuum is hereditarily decomposable if each of its nondegenerate subcontinua is decomposable. In [44] Lewis constructed an n-periodic homeomorphism of the pseudo-arc with exactly one fixed point, for every positive integer $n > 1$ (see [62] for related results). These examples showed that Sarkovskii's theorem is not true in general for chainable continua. Also in [50] was given an example of a map of a chainable indecomposable continuum into itself having a point of period 3 but none of period 2.

On the other hand, although Sarkovskii's theorem is not true for all self-maps of the circle (consider a rotation by 120^o), an extension of this theorem is true for certain maps of $\mathbb{S}^1$. L. Block in [11] showed that if f is a self-map of $\mathbb{S}^1$, f has a fixed point and a periodic point with least period n ($n > 1$), then one of the following holds: f has a periodic point with least period m for every $n < m$, or f has a periodic point with least period m for every m satisfying $n \triangleleft m$ (see also [12]). Chris Bernhardt [7] studied the same problem in the case when $f : \mathbb{S}^1 \to \mathbb{S}^1$ does not have a fixed point. He showed that, in such a case, if s and t are the two smallest periods of points for f, and if s and t are coprime, then f has points of period $\alpha s + \beta t$ for all positive integers α and β (see also [63] for related results).

It seems natural to ask if these results for the circle could be, in some sense, extended to circle-like continua, as the results for interval maps were extended to chainable hereditarily decomposable continua in [50]. In Chapter 6 we shall show that under certain assumptions this is the case, proving the following theorem.

Theorem 1.7. *Let X be a hereditarily decomposable circle-like continuum embedded essentially into $\mathbb{A}$ (i.e. X separates the two components of the boundary of $\mathbb{A}$). Let also $f : X \to X$ be a self-map of X that extends to a map $F : \mathbb{A} \to \mathbb{A}$ (i.e. $F|X = f$). If the degree of F is*

-1 *and* f *has a point of prime odd period* p *in* X *then it has a point of any prime period* $q > p$ *in* X*, and of any period that is a power of* 2*, with possible exception for period* 2*.*

As any orientation reversing homeomorphism of the plane with a hereditarily decomposable invariant continuum is a natural candidate for applications of the above theorem, it is important to mention the following. In [35] W. T. Ingram showed that self-homeomorphisms of a chainable hereditarily decomposable continuum admit only periodic orbits with periods that are powers of 2. His result was proved independently again by X. D. Ye in [64] who, in addition, showed that such homeomorphisms, as well as self-homeomorphisms of Souslinian (each collection of nondegenerate disjoint subcontinua is countable) circle-like hereditarily decomposable continua, have zero topological entropy. The topological entropy of a homeomorphism, that measures the complexity of the dynamics of such a homeomorphism, can be defined as follows [54]. Let $T : Y \to Y$ be a homeomorphism of a compact Hausdorff space Y, and let $\mathcal{U}$ be an open cover of Y. Denote by $M(\mathcal{U})$ the minimum cardinality of the subcovers of $\mathcal{U}$. Let $\mathcal{V}$ be a refinement of $\mathcal{U}$; i.e. every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$. Set $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and for $-\infty < m \leq n < \infty$ define $\mathcal{U}_m^n = T^{-m}(\mathcal{U}) \vee T^{-(m+1)}(\mathcal{U}) \vee \ldots \vee T^{-n}(\mathcal{U})$. Then the topological entropy of T is given by $h_{top}(T) = \sup_{\mathcal{U}} \left\{\lim_{n\to\infty} \frac{1}{n} \log_2[M(\mathcal{U}_0^{n-1})]\right\}$.

Chapter 2

Fixed points and periodic points of orientation reversing planar homeomorphisms

2.1 Preliminaries

Given a set D, by Int D and ∂D we will denote respectively the interior and the boundary of D. Throughout this paper h is an orientation reversing homeomorphism of the plane $\mathbb{R}^2$ onto itself, and X is a continuum (i.e. connected and compact subset of the plane) invariant under h; that is $h(X) = X$. Denote by $Fix(X,h)$ the set of fixed points of h in X; i.e. $Fix(X,h) = \{x \in X : h(x) = x\}$. Components of $\mathbb{R}^2 \setminus X$ are called complementary domains of X. A point x (a complementary domain U of X) is k-periodic if $h^k(x) = x$ but $h^p(x) \neq x$ ($h^k(U) = U$ but $h^p(U) \neq U$) for any positive integer $p < k$. $\mathcal{O}$ is a k-periodic orbit if $\mathcal{O} = \{x, h^1(x), \ldots, h^{k-1}(x)\}$ for a k-periodic point x. Let us recall the methods of [41] and [42] that we will rely on in order to prove Theorem 1.1. Let U be a bounded complementary domain of $\mathbb{R}^2 \setminus X$ that is invariant under h. With modification of h outside of X one can ensure that there is an annulus $\mathbb{A}$ invariant under h such that $X \subseteq \mathbb{A}$. $\mathbb{A}$ is topologically a geometric annulus $\{(r,\theta) \in \mathbb{R}^2 : 1 \leq r \leq 2, 0 \leq \theta < 2\pi\}$, given in the polar coordinates, with two boundary components $\mathbb{A}^+ = \{(r,\theta) \in \mathbb{R}^2 : r = 2, 0 \leq \theta < 2\pi\}$ and $\mathbb{A}^- = \{(r,\theta) \in \mathbb{R}^2 : r = 1, 0 \leq \theta < 2\pi\}$. Continuum X is essentially inscribed into $\mathbb{A}$; i.e. $\mathbb{A}^- \subseteq U$. Now, one can consider the universal covering space of $\mathbb{A}$ given by $\tilde{\mathbb{A}} = \{(x,y) \in \mathbb{R}^2 : 1 \leq y \leq 2\}$, with the covering map $\tau : \tilde{\mathbb{A}} \to \mathbb{A}$ determined by $\tau(x,y) = (y, 2\pi x (\operatorname{mod} 2\pi))$.

Let $\tilde{h} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift homeomorphism of $h|\mathbb{A}$ (i.e. $\tau \circ h = \tilde{h} \circ \tau$). Note that for any $p = (r,\theta)$ in $\mathbb{A}$ its fiber is the set $\tau^{-1}(p) = \{(\frac{\theta}{2\pi} + n, r) : n \in \mathbb{Z}\}$, and p is a fixed point of h iff $\tau^{-1}(p)$ is invariant under $\tilde{h}$. Let p be a fixed point of h. The main ingredients from [41] and [42] that we will need are the following facts.

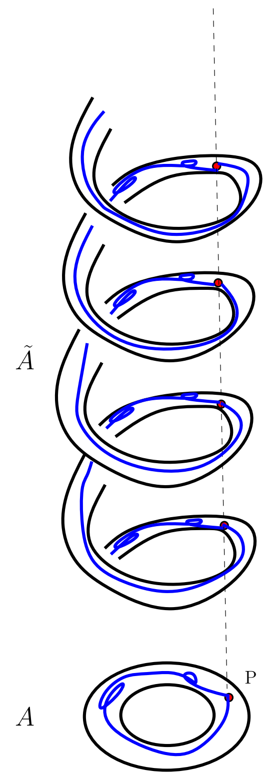

Figure 2.1: The annulus $\mathbb{A}$ and its universal covering space $\tilde{\mathbb{A}}$

1. Given a fixed point $p = (r, \theta) \in \mathbb{A}$ and a lift $\tilde{h}$ of h there is an integer $m[\tilde{h}, p]$ such that, $\tilde{h}(\frac{\theta}{2\pi} + n, r) = (\frac{\theta}{2\pi} - n + m[\tilde{h}, p], r)$ for every $(\frac{\theta}{2\pi} + n, r) \in \tau^{-1}(p)$,

2. $\tilde{h}$ has a fixed point in $\tau^{-1}(p)$ iff $m[\tilde{h}, p]$ is even,

3. if $m[\tilde{h}, p]$ is even then $\tilde{h}(x + 1, y)$ is a lift homeomorphism of h that does not have a fixed point in $\tau^{-1}(p)$.

4. $\tilde{\mathbb{A}}$ can be compactified by two points, say b_1 and b_2, so that $\tilde{X} = \tau^{-1}(X) \cup \{b_1, b_2\}$ is a continuum invariant under $\tilde{h}$, and the latter can be extended to an orientation reversing homeomorphism of the entire plane onto itself.

Let $\tilde{h}_1 : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h and $\tilde{h}_2(x, y) = \tilde{h}_1(x + 1, y)$ be another lift, fixed once and for all. For simplicity we will use the same symbols $\tilde{h}_1$ and $\tilde{h}_2$ to denote the extensions of these two lifts to the entire plane.

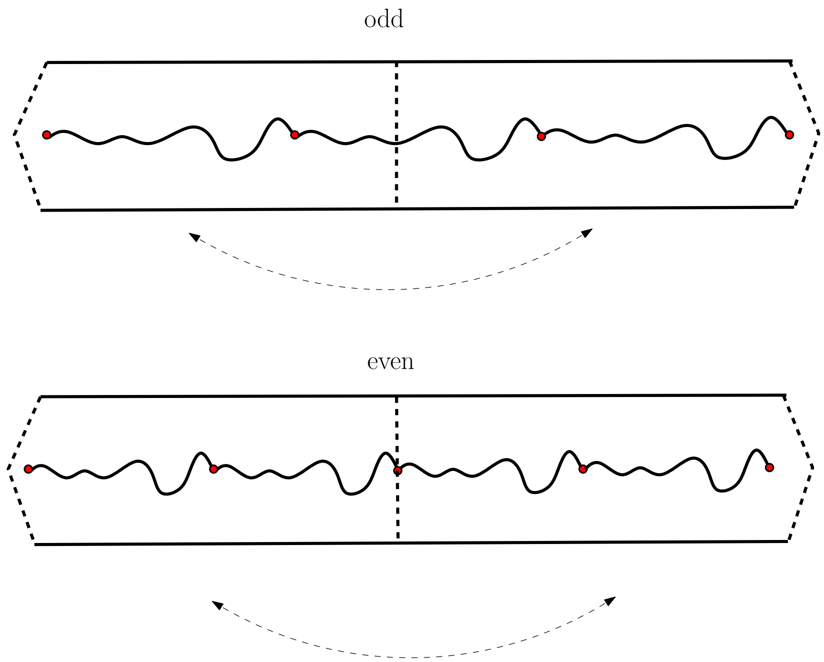

Figure 2.2: Two classes of lifts, and their action on a fiber of a fixed point

Proposition 2.1. *If Y is a subcontinuum of the set of fixed points of h then Y does not separate the plane.*

Proof. If F is a fixed point set of a homeomorphism f of a connected topological manifold M, then either each component of $M \setminus F$ is invariant under f or there are exactly two components of $M \setminus F$ and f interchanges them [22]. Since in the case of planar homeomorphisms the unbounded complementary domain of F is always invariant under h, the above implies that all components of $\mathbb{R}^2 \setminus F$ must be invariant under h. Consequently, if Y were a continuum of fixed points of h separating the plane, then Y could be essentially inscribed into the annulus $\mathbb{A}$ with $\mathbb{A}^-$ and $\mathbb{A}^+$ invariant under h, and h would induce the identity on the homology group $\mathrm{H}_1(\mathbb{A}, \mathbb{Z})$. Therefore any lift $\tilde{h}$ of h to the universal cover $\tilde{\mathbb{A}}$ would preserve the orientation on the two boundary components of $\tilde{\mathbb{A}}$, at the same time keeping them invariant. Consequently $\tilde{h}$ would be orientation preserving on $\tilde{\mathbb{A}}$, contradicting the fact that any lift of h to $\tilde{\mathbb{A}}$ must be orientation reversing. $\square$

Lemma 2.2. *Suppose p is a fixed point of h and let Y be a component of p in $Fix(X,h)$. Then*

$$m[\tilde{h}_1, p] = m[\tilde{h}_1, q](\text{mod}2)$$

for every $q \in Y$.

Proof. First, Y does not separate the plane. Suppose $m[\tilde{h}_1, p]$ is even. Let α be the fixed point of $\tilde{h}_1$ in $\tau^{-1}(p)$ and let K be a component of $\tau^{-1}(Y)$ containing α. To the contrary, suppose the above claim is false and let $q \in Y$ be such that $m[\tilde{h}_1, q]$ is odd. For every $\beta \in \tau^{-1}(q)$ we have $\beta \neq \tilde{h}_1(\beta)$. Let γ be in $K \cap \tau^{-1}(q)$. Then $\tilde{h}_1(\gamma) \neq \gamma$ and $\tilde{h}_1(\gamma) \in \tilde{h_1}(K)$. Since $\tilde{h}_1(K)$ is also a component of $\tau^{-1}(Y)$, and $\alpha \in \tilde{h}_1(K)$, therefore $K = \tilde{h}_1(K)$. Consequently, K contains two elements from the same fiber $\tau^{-1}(q) \in \tau^{-1}(Y)$. But this contradicts the following observation indicated in [21], which in turn will complete the proof.

Since $Y \subseteq \mathbb{A}$ does not separate the plane, one can choose a disk $D \subseteq \mathbb{A}$ around Y; i.e., $Y \subseteq \operatorname{Int} D$, and $\operatorname{Int} D$ being simply connected lifts to disjoint homeomorphic copies of $\operatorname{Int} D$ in $\tilde{\mathbb{A}}$. Consequently, Y lifts to disjoint homeomorphic copies in $\tilde{\mathbb{A}}$. Since K is one of them, it cannot contain two points from the same fiber $\tau^{-1}(q)$.

□

As a consequence of the above, for a given component Y of $Fix(X,h)$, one can choose any $p \in Y$ and say that $m[\tilde{h}_1, Y]$ is even (or odd) if $m[\tilde{h}_1, p]$ is of the same parity.

2.2 Proof of Theorem 1.1

Theorem 1.1: *Let $M \in \{\mathbb{R}^2, \mathbb{S}^2\}$ and let $h : M \to M$ be an orientation reversing homeomorphism with a continuum X invariant, and suppose there are at least n components of $M \setminus X$ that are invariant under h. Then $Fix(X,h)$, the set of fixed points of h in X, has at least n components.*

Proof. 1. Assume that $M = \mathbb{R}^2$. We will prove this theorem by induction. First, observe that the case when $n = 0$ is the theorem of Bell [5]. Indeed, if X is a nonseparating plane continuum then by Bell's theorem h must have a fixed point in X, and therefore there is at least one component of $Fix(X, h)$.

For the sake of induction suppose the theorem is true for $n = k - 1$. Now we will show that the theorem holds true for $n = k$.

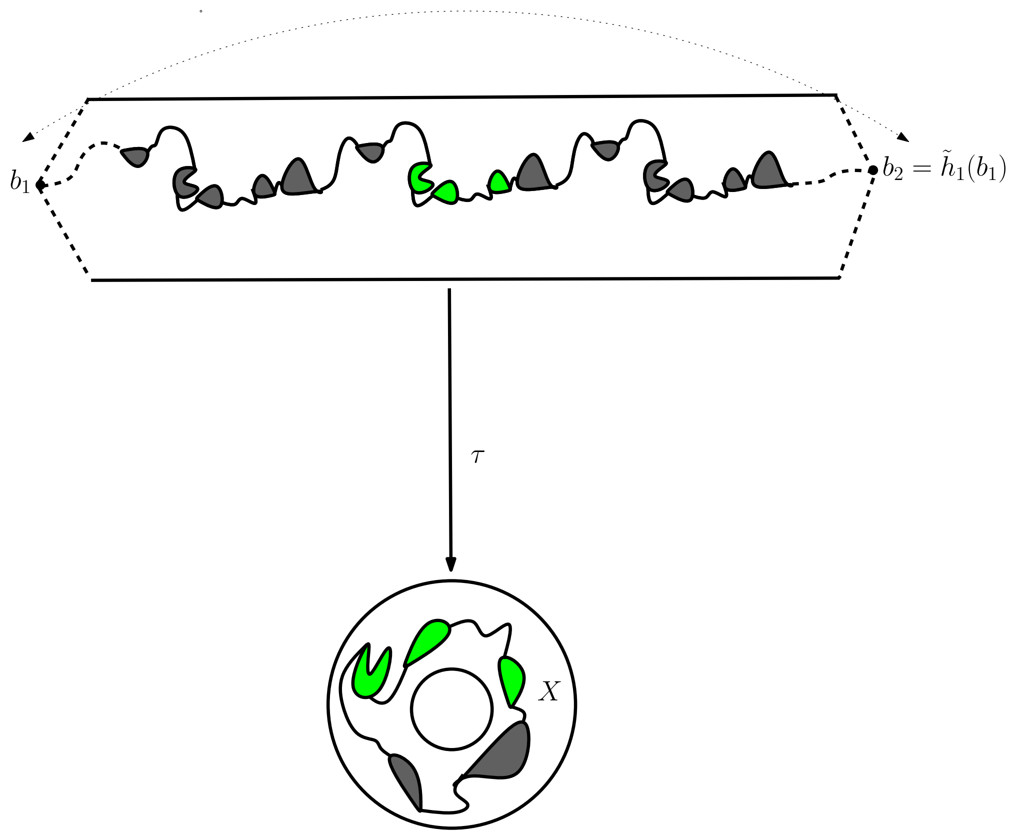

Figure 2.3: First class of lifts gives rise to p invariant complementary domains

Assume $U_1, \ldots, U_k$ are bounded complementary domains of X invariant under h, and that $\mathbb{A}^-$ is inscribed into U_k. We may assume that there is a fixed point u_i of h in each U_i. Without loss of generality assume that $u_1, \ldots, u_p$ are all fixed points of h such that there is a fixed point of $\tilde{h}_1$ in the fiber $\tau^{-1}(u_i)$, for all $i = 1, \ldots, p$. In other words, each set from $U_1, \ldots, U_p$ contains in its lift $\tau^{-1}(U_i)$ a bounded complementary domain of $\tilde{X}$ that is invariant under $\tilde{h}_1$. Equivalently, $m[\tilde{h}_1, u_i]$ is even for $i = 1, \ldots, p$ and $m[\tilde{h}_1, u_i]$ is odd for $i = p + 1, \ldots, k - 1$.

Let $q = k - 1 - p$. Note that p, q are nonnegative integers (possibly with p or q equal 0). Since $\tilde{X}$ is a continuum with p bounded complementary domains invariant under $\tilde{h}_1$, and $p \leq k-1$, by the induction hypothesis there are $p+1$ components of $Fix(\tilde{h}_1, \tilde{X})$. Let $A_1, \ldots, A_{p+1}$ be those components. For every $i = 1, \ldots, p+1$ there is a component

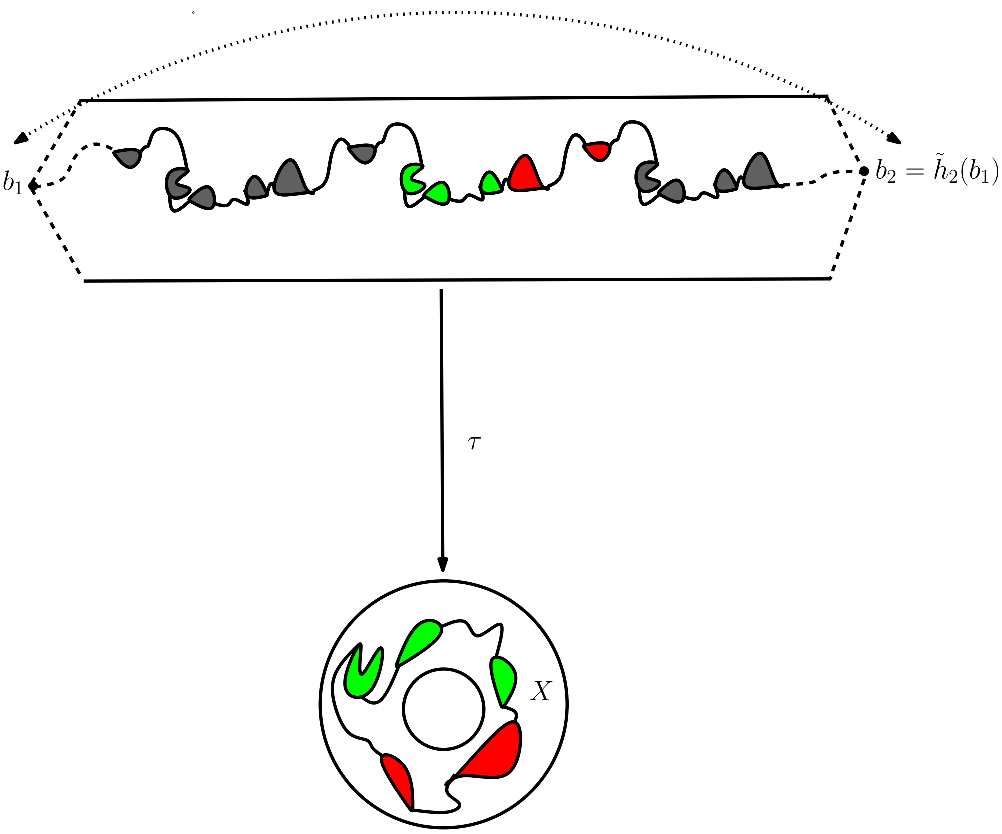

Figure 2.4: Second class of lifts gives rise to q invariant complementary domains

X_i of $Fix(h, X)$ such that $X_i = \tau(A_i)$. Note that $\tau(A_i)$ and $\tau(A_t)$ are disjoint for $i \neq t$ since any fiber of a fixed point of h contains no more that one fixed point of $\tilde{h}_1$. Therefore $\{X_i : i = 1, \ldots, p+1\}$ consists of $p+1$ distinct components of $Fix(X, h)$.

Now, $\tau^{-1}(X_i)$ is invariant under $\tilde{h}_1$, and $m[\tilde{h}_1, X_i]$ is even for every $i = 1, \ldots, p+1$. $\tau^{-1}(X_i)$ is also invariant under $\tilde{h}_2$ but contains no fixed point of $\tilde{h}_2$ since $m[\tilde{h}_2, X_i]$ is odd for every $i = 1, \ldots, p+1$. For $i = 1, \ldots, p$ no $\tau^{-1}(u_i)$ contains a fixed point of $\tilde{h}_2$ since $m[\tilde{h}_2, u_i]$ is odd. For $i = p+1, \ldots, k-1$ every $\tau^{-1}(u_i)$ contains a fixed point of $\tilde{h}_2$ since $m[\tilde{h}_2, u_i]$ is even. Therefore, there are $q = (k-1) - p$ bounded complementary domains of $\tilde{X}$ that are invariant under $\tilde{h}_2$. Again, by induction hypothesis, there must be $q+1$ components of $Fix(\tilde{h}_2, \tilde{X})$. Denote them by $C_1, \ldots, C_q$. For every $j = 1, \ldots, q+1$, $\tau(C_j)$ is a component of $Fix(h, X)$. Note that $\tau(C_j)$ and $\tau(C_t)$ are disjoint for $j \neq t$ since any fiber of a fixed point of h contains no more that one fixed point of $\tilde{h}_2$. Therefore $\{\tau(C_j) : j = 1, \ldots, q+1\}$ consists of $q+1$ distinct components of $Fix(X, h)$. Since each $\tau^{-1}(X_i)$ contains no fixed point of $\tilde{h}_2$, no $\tau(C_j)$ can coincide with any X_i. Therefore there are $p+1+q+1 = k+1$ components of $Fix(h, X)$. This completes the proof.

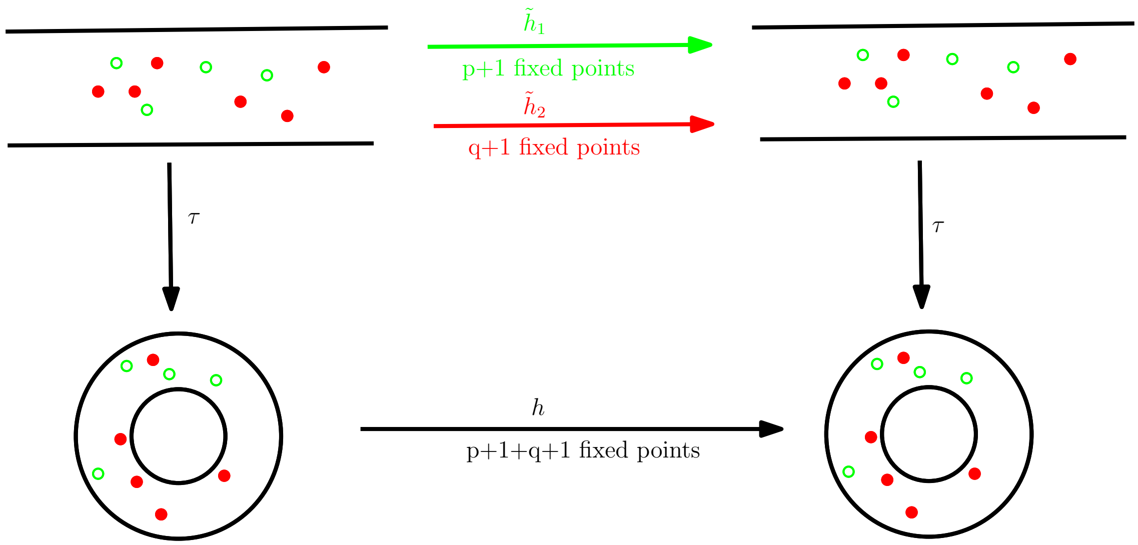

Figure 2.5: Proof of Theorem 1.1

2. Assume $M = \mathbb{S}^2$. First suppose that $\mathbb{S}^2 \setminus X$ has exactly one component U invariant under g. We can assume that there is a fixed point u of g in U. Notice that $\mathbb{S}^2 \setminus \{u\}$ is topologically the plane, and $G = g|(\mathbb{S}^2 \setminus \{u\})$, obtained by a restriction of g to $\mathbb{S}^2 \setminus \{u\}$, is an orientation reversing homeomorphism of the plane onto itself with the continuum X invariant. Now, since X has no bounded complementary domains invariant under G, by the theorem of Bell there is at least one component of $Fix(X, G) = Fix(X, g)$. Bell's theorem applies to nonseparating plane continua, but in the above case, if X separates the plane and none of the bounded complementary domains is invariant under G, then these domains can be added to X to form a nonseparating plane continuum Y with $Fix(X, G) = Fix(Y, G)$.

Second suppose that $\mathbb{S}^2 \setminus X$ has at least two components U_1 and U_2 invariant under g. Then there is an annulus $\mathbb{A}$ such that $X \subseteq \mathbb{A}$, $\mathbb{A}^- \subseteq U_1$ and $\mathbb{A}^+ \subseteq U_2$. Since U_1 and U_2 are invariant under g therefore h does not interchange $\mathbb{A}^-$ and $\mathbb{A}^+$, and one can repeat the same arguments as for $M = \mathbb{R}^2$. □

2.3 Proof of Theorem 1.2

Theorem 1.2 seems to fit well in the following context. The Cartwright-Littlewood-Bell theorem (see [24] and [5]) states that any planar homeomorphism fixes a point in an invariant nonseparating continuum. Morton Brown [21] and O.H. Hamilton [32] exhibited that, in the case of orientation preserving homeomorphisms, this theorem can be deduced directly from a theorem of Brouwer [20]. Brouwer showed that any orientation preserving homeomorphism with at least one bounded orbit must have a fixed point. Briefly, the idea behind these short proofs of the fixed point theorem was to separate the invariant continuum from the fixed point set F, and then for an open invariant component U in $\mathbb{R}^2 \setminus F$ containing X argue that U contains no fixed point, thus contradicting the theorem of Brouwer. The inspiration for a proof of Theorem 1.2 comes from these very papers, but since a set of 2-periodic points does not need to be closed (in contrast with the fixed point set), one cannot just replace the theorem of Brouwer with a theorem of Bonino from [14] and use the same arguments. Instead, we will use Bonino's result from [15] and show that no 2-periodic orbit in an invariant component of $\mathbb{R}^2 \setminus X$ can be linked to a k-periodic ($k > 2$) orbit in X.

Theorem 1.2: *Let $M \in \{\mathbb{R}^2, \mathbb{S}^2\}$ and let $h : M \to M$ be an orientation reversing homeomorphism with a continuum X invariant (i.e. $h(X) = X$). Suppose h has a k-periodic orbit in X with $k > 2$.*

(i) If X does not separate the plane then h has a 2-periodic orbit in X.

(ii) If X separates the plane then h has a 2-periodic orbit in X, or there is a 2-periodic component U of $M \setminus X$.

Proof. If $M = \mathbb{R}^2$ then compactify $\mathbb{R}^2$ by a point ∞ to obtain $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ and extend given homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ to a homeomorphism $\tilde{h} : \mathbb{S}^2 \to \mathbb{S}^2$ by setting $\tilde{h}|\mathbb{R}^2 = h$, and $\tilde{h}(\infty) = \infty$. h and $\tilde{h}$ have exactly the same k-periodic points for any $k > 1$.

By Bonino's result, there is an orbit $\mathcal{O}' \subseteq \mathbb{S}^2$ of $\tilde{h}$ of period exactly 2. We will show that any such 2-periodic orbit that lies in an invariant complementary domain of X is not linked to $\mathcal{O}$.

Suppose $\mathcal{O}' \cap X = \emptyset$ and $\mathcal{O}' \subseteq U$ for a complementary domain U of X invariant under h. Since $\mathcal{O}'$ and X are closed, there is a Jordan curve $S \subseteq \mathbb{S}^2$ separating $\mathcal{O}'$ from X. Let D be one of the two disks in $\mathbb{S}^2$ bounded by S, such that $X \subseteq \operatorname{Int} D$. Then $D \cap \mathcal{O}' = \emptyset$. Since X is invariant under $\tilde{h}$, by continuity of $\tilde{h}$, there is a disk C such that $C \subseteq \operatorname{Int} D$ and $\tilde{h}(C) \subseteq \operatorname{Int} D$. Since both C and $\tilde{h}(C)$ contain X in its interior, there is a disk $B \subseteq C \cap \tilde{h}(C)$ that contains X in its interior. Therefore C and $\tilde{h}(C)$ are freely isotopic in the annulus $D \setminus \operatorname{Int} B$, and thus freely isotopic in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$. This shows that if $\mathcal{O}' \subseteq \mathbb{S}^2 \setminus X$ is a 2-periodic orbit, then $\mathcal{O}'$ and $\mathcal{O}$ are not linked. Therefore the 2-periodic orbit $\mathcal{O}'$ linked to $\mathcal{O}$, guaranteed by the theorem of Bonino in [15], must be in X or in a 2-periodic component of $\mathbb{S}^2 \setminus X$. □

Corollary 2.3. *Let $M \in \{\mathbb{R}^2, \mathbb{S}^2\}$ and let $h : M \to M$ be an orientation reversing homeomorphism with a continuum X invariant (i.e. $h(X) = X$). Suppose there is a k-periodic component of $M \setminus X$, for $k > 2$. Then, either there is a 2-periodic orbit in X, or there is a 2-periodic component of $M \setminus X$.*

Proof. Let W be a k-periodic complementary domain of X $(k > 2)$. Without loss of generality one may assume that there is a k-periodic point w in W (w is a fixed point of h^k). Consider $Y = X \cup W \cup h(W) \cup \ldots \cup h^{k-1}(W)$. Clearly Y is a continuum invariant under h. Now apply Theorem 1.2. □

Remark: It is not apparent to the present author if one can improve Theorem 1.2 and get rid of the 2-periodic component of $\mathbb{R}^2 \setminus X$ to guarantee that, under assumptions, there

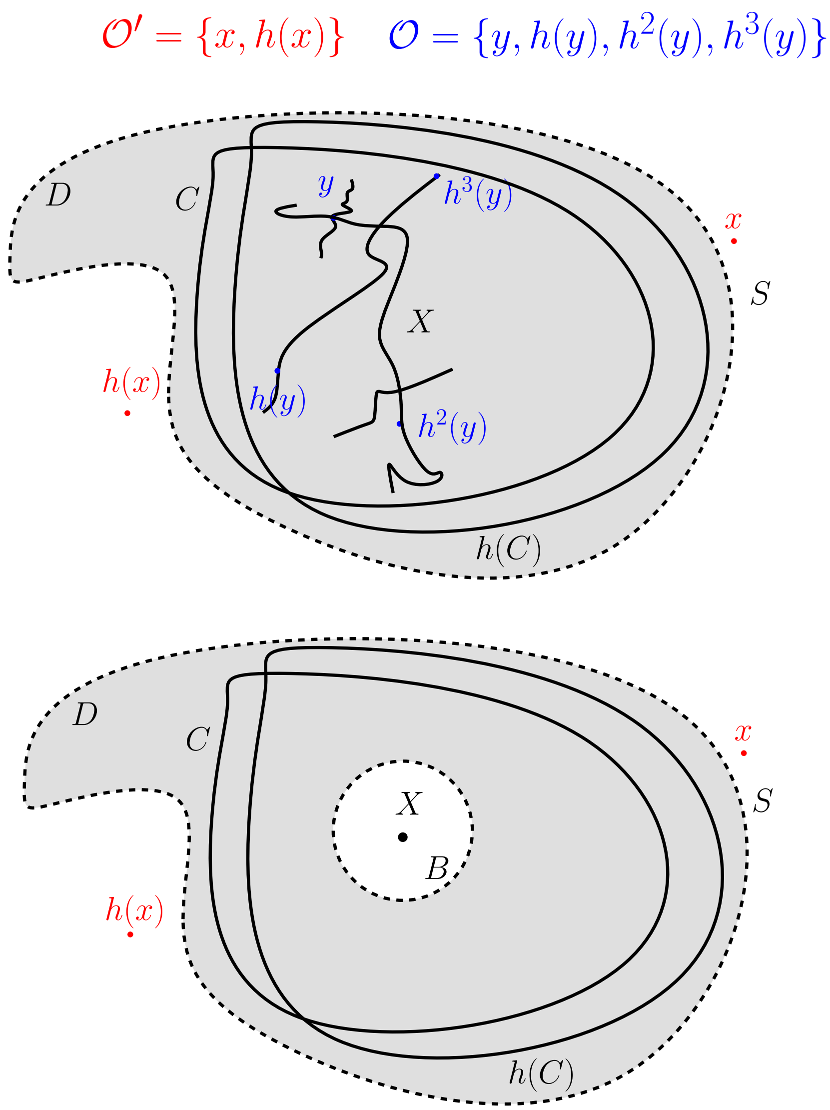

Figure 2.6: Proof of Theorem 1.2

will be a 2-periodic point in X. Nonetheless, the following example shows that one cannot do it for $\mathbb{S}^2$.

Example: Let $\mathbb{S}^2$ be given in spherical coordinates by $\mathbb{S}^2 = \{(r, \theta, \phi) : r = 1, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}$. Consider a Jordan curve $S \subseteq \mathbb{S}^2$ determined by $S = \{(r, \theta, \phi) : r = 1, \phi = \frac{\pi}{2}\}$. Let U^+, U^- be the two disks in $\mathbb{S}^2 \setminus S$ bounded by S. Fix $k > 2$ and consider an orientation reversing homeomorphism $g : \mathbb{S}^2 \to \mathbb{S}^2$ determined by

$$g(r, \theta, \phi) = (r, \theta + \frac{2\pi}{k}, \pi - \phi).$$

g interchanges U^+ and U^-, reflecting $\mathbb{S}^2$ about S and then rotating $\mathbb{S}^2$ by $\frac{2\pi}{k}$. Notice that $g^2(r,\theta,\phi) = (r, \theta + \frac{4\pi}{k}, \phi)$ and $g^k(r,\theta,\phi) = (r,\theta,\phi) = id_{\mathbb{S}^2}(r,\theta,\phi)$. Clearly, g is an orientation reversing homeomorphism of $\mathbb{S}^2$ with the continuum S invariant, and any point in S of period exactly k, but the only points of period 2 are the two poles, which are not in S.

Chapter 3

Periodic orientation reversing homeomorphisms of $\mathbb{S}^2$ with invariant pseudo-circle

Let Y be a space and $f : Y \to Y$ be a homeomorphism. Denote by $\mathrm{id}_Y : Y \to Y$ the identity map on Y. f is a k-periodic homeomorphism if $f^k = \mathrm{id}_Y$, and $f^i \neq \mathrm{id}_Y$ for any positive integer $i < k$.

In [6] Bellamy and Lewis gave an example of orientation reversing homeomorphism of the plane, with an invariant pseudo-arc. In [34] Jo Heath showed that the pseudo-circle admits a 2-fold cover onto itself. It follows from her example that there exists a 2-periodic orientation preserving homeomorphism of the plane with invariant pseudo-circle. In fact, her example yields a construction of a k-periodic orientation preserving homeomorphism of the plane with invariant pseudo-circle, for any $k \geq 2$ (see also [30]). In the present chapter we shall describe, for any $k \geq 1$, an example of a $2k$-periodic orientation reversing homeomorphism of $\mathbb{S}^2$ with an invariant pseudo-circle. The idea of the construction is to use the degree 1 embedding of the pseudo-circle into the Möbius band $\mathbb{M}$ described by Bellamy and Lewis. Then, applying arguments similar to those in [34], one can consider a $2k$-fold cover of $\mathbb{M}$ provided by an annulus $\mathbb{A}$, obtaining the desired homeomorphisms as a deck transformation.

3.1 Preliminaries in Continuum Theory

Suppose that $\mathcal{U}$ is a cover of a connected set H; i.e. $H \subseteq \bigcup \mathcal{U}$. Recall that the nerve of $\mathcal{U}$, denoted by $N(\mathcal{U})$, is an abstract simplicial complex that satisfies the following:

- the vertices of $N(\mathcal{U})$ are the elements of $\mathcal{U}$
- the simplices of $N(\mathcal{U})$ are the finite subcollections $\{U_1, \dots, U_q\}$ of $\mathcal{U}$ such that $U_1 \cap U_2 \cap \ldots \cap U_q \neq \emptyset$.

For an $\epsilon > 0$ we call a cover $\mathcal{U}$ of H an ϵ-cover if the diameter of each element of $\mathcal{U}$ is less than ϵ.

A metric continuum X is chainable, or arc-like, if one of the following conditions is satisfied.

(Ch1) For every $\epsilon > 0$ there is an ϵ-cover of X by open sets with the nerve homeomorphic to $[0, 1]$.

(Ch2) For every $\epsilon > 0$ there is an ϵ-map $f : X \to [0, 1]$; i.e. the diameter of $f^{-1}(x)$ less than ϵ for every $x \in [0, 1]$.

(Ch3) X is the inverse limit of spaces homeomorphic to $[0, 1]$.

It is well known that each of the above conditions is equivalent to any of the other two.

The $\sin(\frac{1}{x})$-curve, that is the set given by

$$\{(x, y) \in \mathbb{R}^2 : y = \sin(\tfrac{1}{x}), x \in (0, \tfrac{1}{\pi}]\} \cup (\{0\} \times [-1, 1])$$

is the simplest example of a chainable continuum that is not locally connected.

A metric continuum X is circularly chainable, or circle-like, if one of the following conditions is satisfied.

(Ci1) For every $\epsilon > 0$ there is an ϵ-cover of X by open sets with the nerve homeomorphic to the unit circle $\mathbb{S}^1$.

(Ci2) For every $\epsilon > 0$ there is an ϵ-map $f : X \to \mathbb{S}^1$; i.e. the diameter of $f^{-1}(x)$ less then ϵ for every $x \in \mathbb{S}^1$.

(Ci3) X is the inverse limit of spaces homeomorphic to $\mathbb{S}^1$.

Again, each of the above conditions is equivalent to any of the other two.

A counterpart of the $\sin(\frac{1}{x})$-curve for circle-like continua is the Warsaw circle (also known as the Austin circle). It is the space obtained from $\sin(\frac{1}{x})$-curve by adding the union of three arcs $\{\frac{1}{\pi}\} \times [-2, 0]$, $[0, \frac{1}{\pi}] \times \{-2\}$ and $\{0\} \times [-2, -1]$.

A continuum is said to be decomposable if it is the union of two of its proper subcontinua, and it is said to be indecomposable otherwise. A continuum is hereditarily indecomposable if each of its nondegenerate subcontinua is indecomposable. The easiest to construct example of an indecomposable continuum is the Janiszewski-Knaster buckethandle continuum. It is obtained by an inductive procedure depicted in Figure 3.1. It is readily seen that the buckethandle continuum is not hereditarily indecomposable. For example, it contains arcs as its subcontinua, and any such arc is decomposable.

The condition (Ch1) says that if X is chainable then for every $\epsilon > 0$ there is an ϵ-cover of X by a finite family of open sets $C = \{C(1), \ldots, C(n)\}$ such that $C(i) \cap C(j) \neq \emptyset$ if and only if $|i - j| \leq 1$. In such a case C is called an ϵ-chain, and each element $C(i)$ of the chain C is called a link. $C(i)$ is the i-th link of C. A subchain $C(i, j)$ of C is a chain of links in C whose first link is $C(i)$ and last link is $C(j)$. Suppose C, E are two chains. We say that E refines C if and only if each link in E is contained in a link of C. We say that E properly refines C if and only if each link in E is contained with its closure in a link of C. In such a case we say that E is crooked in C if whenever $E(j)$ and $E(k)$ are links of E which intersect $C(J)$ and $C(K)$, respectively, with $|J - K| \geq 3$, then $E(j, k)$ can be written as the union of three proper subchains $E(j, r), E(r, s)$ and $E(s, k)$ with $(s - r)(k - j) > 0$, and $E(r)$ is a subset of the link of $C(J, K)$ adjacent to $C(K)$, whereas $E(s)$ is a subset of the link of $C(J, K)$ adjacent to $C(J)$.

Let R be the family of open sets described by the condition $(Ci1)$. R can be considered as an ϵ-chain of open sets $R = \{R(1), \ldots, R(n)\}$ with the first and last link identified; i.e. $R(1) = R(n)$. In such a case R is called a circular ϵ-chain. Suppose R, L are two circular chains and that L refines R. In such a case we say that L is crooked in R if for every proper subchain P of R, each subchain H of L that refines P is crooked in P.

Recall that a map is 2-to-1 if the preimage of each point in the image has exactly two points. A map is reduced if no proper subcontinuum of the image has a connected preimage. A map is confluent if for each continuum X in the image, each component of the preimage of X maps onto X. It can be found in [34] that if f is a reduced, confluent, 2-to-1 map from a continuum X onto Y, then X is hereditarily indecomposable if Y is hereditarily indecomposable.

3.2 The pseudo-circle and the pseudo-arc

The pseudo-arc is characterized in [10] as a chainable hereditarily indecomposable continuum. It is homeomorphic to each of its nondegenerate subcontinua, does not separate the plane, and is homogeneous; i.e. for any two points x and y in the pseudo-arc $\mathcal{P}$ there is a self-homeomorphism $h : \mathcal{P} \to \mathcal{P}$ such that $h(x) = y$. One can define the pseudo-arc also in the terms of ϵ-chains [8]. That is, the pseudo-arc can be represented as $\mathcal{P} = \bigcap_{m\in\mathbb{N}} \bigcup D_m$, where

(P1) each D_m is a $\frac{1}{m}$-chain,

(P2) D_{m+1} properly refines D_m for every m,

(P3) the first (last) link of D_{m+1} is contained in the first (last) link of D_m,

(P4) D_{m+1} is crooked in D_m for every m.

A pseudo-circle is a hereditarily indecomposable, circle-like, non-chainable continuum embeddable in the plane. It separates the plane into two complementary domains, and each proper nondegenerate subcontinuum of it is homeomorphic to the pseudo-arc. The pseudo-circle is not homogeneous. One can define the pseudo-circle also in the terms of circular ϵ-chains [10]. That is, the pseudo-circle $\mathcal{C}$ can be represented as $\mathcal{C} = \bigcap_{m\in\mathbb{N}} \bigcup D_m$, where

(C1) each D_m is a circular $\frac{1}{m}$-chain,

(C2) D_{m+1} properly refines D_m for every m,

(C3) D_{m+1} has winding number ± 1 in D_m,

(C4) D_{m+1} is crooked in D_m for every m.

3.3 Degree 1 embedding of the pseudo-circle into the Möbius band $\mathbb{M}$

Let us also recall from [6] the main elements of the construction of an orientation reversing planar homeomorphism with invariant pseudo-arc, that inspired the results of the present chapter. The construction is outlined in the following steps.

1. There exists an essential degree one embedding of the pseudo-circle in the Möbius band. To obtain the embedding it is enough to form an appropriate sequence of circular chains in the Möbius band $\mathbb{M} = \mathbb{S}^1 \times [0,1]/\approx$, with the quotient map $q : (\theta, r) \approx (\theta+\pi, 1-r)$. To obtain the initial chain one considers the three line segments $\bar{ab}, \bar{bc}, \bar{cd}$ joining points $a = (0, \frac{1}{4}), b = (\pi, \frac{1}{3}), c = (0, \frac{1}{2})$ and $d = (\pi, \frac{3}{4}) = a$. The union of those three is a simple closed curve embedded with degree 1 into $\mathbb{M}$. This spanning curve is the centerline for a circular chain $\mathcal{D}_1$ of disks of small diameter, where each two adjacent disks intersect each other in a disk. The union of $\mathcal{D}_1$ forms another Möbius band inside $\mathbb{M}$. The subsequent circular chains $\mathcal{D}_n$ are obtain by an inductive procedure, which assures that $\mathcal{D}_n$ is crooked inside $\mathcal{D}_{n-1}$, and inscribed with degree 1 into it, so that the intersection $\bigcap_{n\in\mathbb{N}} \mathcal{D}_n$ is the pseudo-circle.

2. The universal covering of $\mathbb{M}$ is $\tilde{\mathbb{M}} = \mathbb{R} \times [0,1]$.

3. The deck transformation $\tilde{f}$ determined by the generator of the fundamental group of $\mathbb{M}$ produces an orientation reversing homeomorphism of $\tilde{\mathbb{M}}$, which is invariant on the connected lift of the pseudo-circle, and interchanges the two boundary components of $\tilde{\mathbb{M}}$.

4. The two-point compactification K of $\tilde{\mathbb{M}}$ is a disk, with the two-point compactification of the lift of the pseudo-circle homeomorphic to the pseudo-arc. The desired homeomorphism is obtained by the fact that $\tilde{f}$ extends to a homeomorphism of K with the pseudo-arc invariant.

3.4 $2k$-periodic orientation reversing homeomorphisms of $\mathbb{S}^2$ with invariant pseudo-circle

Theorem 1.3 *For any $k \geq 1$ there is a $2k$-periodic orientation reversing homeomorphism of $\mathbb{S}^2$ with invariant pseudo-circle.*

Proof. First we will describe an example of such a homeomorphism for $k = 1$. Consider the pseudo-circle $\mathcal{C}$ embedded with degree 1, as described in [6], into the Möbius band $\mathbb{M} = \mathbb{S}^1 \times [0,1]/\approx$, with the quotient map $q : (\theta, r) \approx (\theta + \pi, 1 - r)$. Then the annulus $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ provides a natural 2-fold cover of $\mathbb{M}$ with the quotient map q as the covering map.

Let $\mathcal{K} = q^{-1}(\mathcal{C})$, and notice that $\mathcal{K}$ is a continuum as $\tau^{-1}(\mathcal{C})$, the lift of $\mathcal{C}$ to the universal cover $(\tilde{\mathbb{M}}, \tau)$, is connected. Indeed, $\mathcal{K}$ must have either one or two components. If $\mathcal{K}$ had two components then $\tau^{-1}(\mathcal{C})$ would need to have two components in $\tilde{\mathbb{M}}$, as $\tau^{-1}(\mathcal{C})$ covers also $\mathcal{K}$. Note also that $\mathcal{K}$ is a separating plane continuum, as $\mathcal{K}$ separates the two boundary components of $\mathbb{A}$, by the fact that $\tau^{-1}(\mathcal{C})$ separates the two boundary components of $\tilde{\mathbb{M}}$.

Every subcontinuum of $\mathcal{C}$ is the pseudo-arc. For any such pseudo-arc there is a disk D containing it, that lifts to two disjoint homeomorphic copies of D in $\mathbb{A}$. Therefore, no proper subcontinuum $Y \subseteq \mathcal{C}$ has a connected preimage $q^{-1}(Y)$, and consequently q is a reduced map. Clearly, q is also a confluent map by the very definition of a covering map. Finally it is 2-to-1 and therefore by Lemma 1 in [34] $\mathcal{K}$ is hereditarily indecomposable as $\mathcal{C}$ is. Furthermore, $\mathcal{K}$ is circularly chainable since any circular chain covering $\mathcal{C}$ consisting of fundamental open sets lifts to a circular chain covering $\mathcal{K}$. It follows that $\mathcal{K}$ is a hereditarily indecomposable separating plane continuum, and therefore it is the pseudo-circle.

Now, define a homeomorphism $g : \mathbb{A} \to \mathbb{A}$ to be the deck transformation determined by the generator of the fundamental group of $\mathbb{M}$. g can be viewed as a rotation of $\mathbb{A}$ by 180^o followed by the reflection that interchanges the two boundary components of $\mathbb{A}$. To obtain a homeomorphism of $\mathbb{S}^2$ identify all points that lie on the same component of the boundary of $\mathbb{A}$. Since g^2 is the identity, g is 2-periodic.

To finish the proof note that, although Heath's original example was only for 2-fold cover, it is well known that the same arguments can be applied for any other n-fold cover (see [43] for an alternative proof). Therefore to obtain an example for any $2k > 2$, consider the annulus that is a $2k$-fold cover of $\mathbb{M}$, and apply the same arguments. $\square$

3.5 Further comments on antipodal maps of the pseudo-circle

One can see the 2-periodic homeomorphisms of the pseudo-circle as analogues of the antipodal map of $\mathbb{S}^1$. Any two fixed point free antipodal maps of $\mathbb{S}^1$ are isotopic, as any fixed point free homeomorphism of $\mathbb{S}^1$ must be isotopic to the identity map. Therefore it seems interesting to point out that the 2-periodic homeomorphism of the pseudo-circle $a(x)$ constructed by Heath, and homeomorphism $g(x)$ exhibited in Theorem 1.3 are, is some sense, not equivalent. This is because the pseudo-circle separates $\mathbb{S}^2$ into two complementary domains U_1 and U_2. If z is a point in the pseudo-circle accessible from U_1 then its antipode $a(z)$ has the same property. On the other hand $g(z)$, the antipode under g, is accessible from U_2, and therefore it cannot be accessible from U_1. Otherwise, since there is no fixed point for g, there would be two points accessible from both complementary domains and the pseudo-circle would be the union of two continua meeting at those two points, contradicting indecomposability of the pseudo-circle.

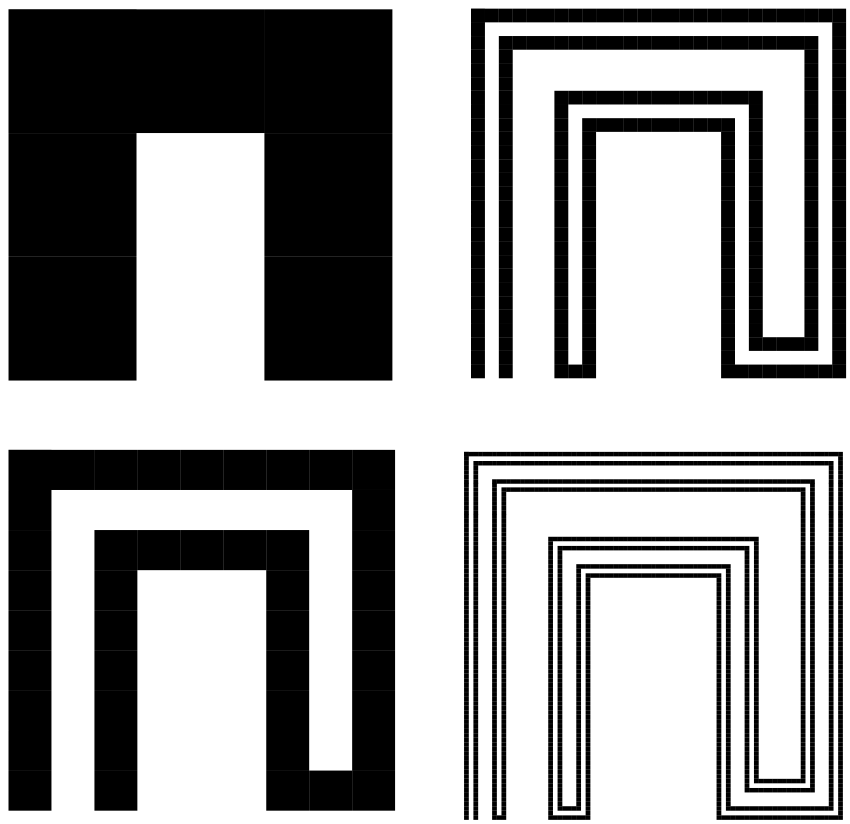

Figure 3.1: Janiszewski-Knaster (buckethandle) continuum

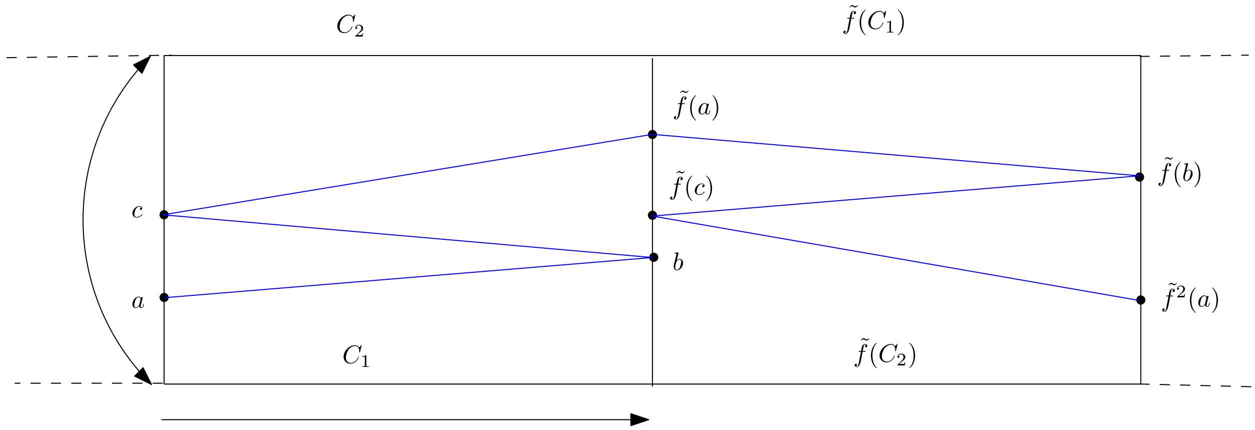

Figure 3.2: Action of the deck transformation $\tilde{f}$ on $\tilde{\mathbb{M}}$ with a part of the spanning line for the lift of the pseudo-circle

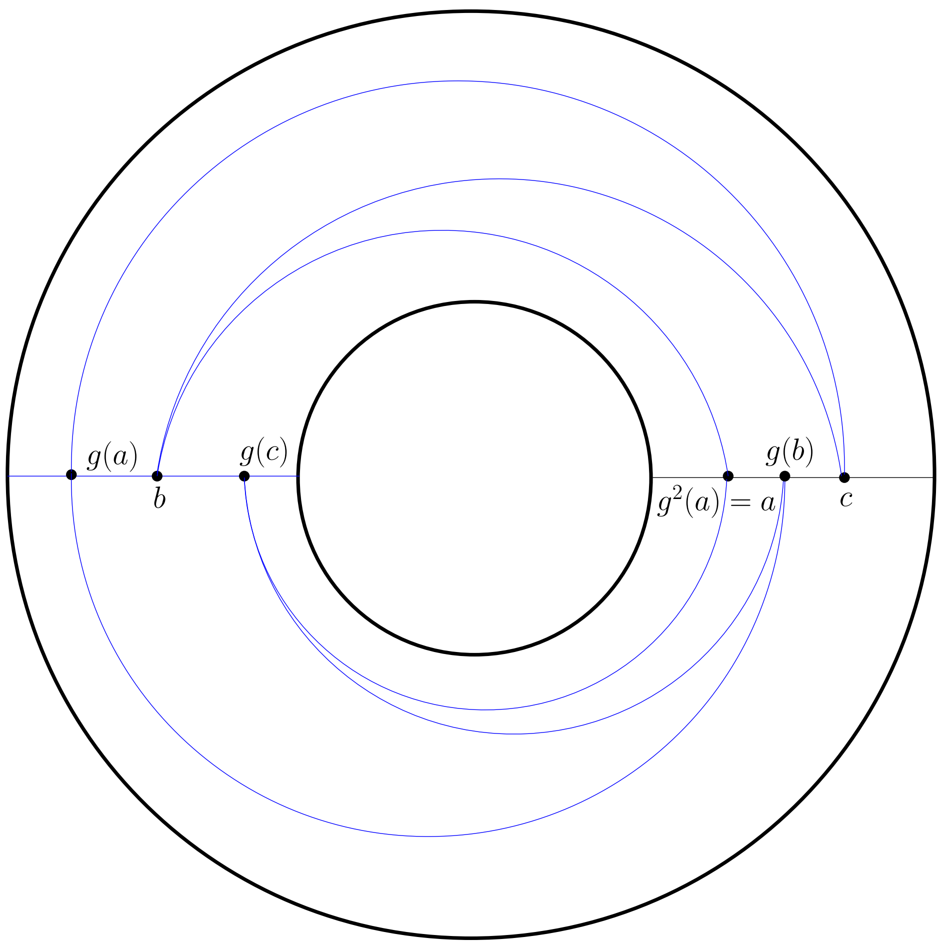

Figure 3.3: The 2-periodic homeomorphism g

Chapter 4

Linear order on the complementary domains of an invariant continuum

4.1 Preliminaries on covering spaces and Nielsen fixed point classes

We shall first recall some background from algebraic topology from [47] and [36] concerning universal covering spaces and Nielsen fixed point classes. Let $\tilde{Y} \to Y$ be the universal covering of Y. A lifting of a map $h : Y \to Y$ is a map $\tilde{h} : \tilde{Y} \to \tilde{Y}$ such that $p \circ \tilde{h} = h \circ p$. By the unique lifting property any lift of a map is uniquely determined by where it maps a single point. A covering translation (or deck transformation) is a map $\alpha : \tilde{Y} \to \tilde{Y}$ such that $p \circ \alpha = p$; i.e. a lifting of the identity map. The covering translations of $\tilde{Y}$ form a group $\mathcal{D}$ which is isomorphic to the fundamental group $\pi_1(Y)$. A Nielsen fixed point class of a map $h : Y \to Y$ is the set $\tau(\{\tilde{x} \in \tilde{Y} : \tilde{h}(\tilde{x}) = \tilde{x}\})$, where $\tilde{h} : \tilde{Y} \to \tilde{Y}$ is a lift of h. Equivalently, two fixed points x_0, x_1 of h belong to the same Nielsen fixed point class if and only if there is a path γ from x_o to x_1 such that $\gamma \cong h \circ \gamma$; i.e. γ is homotopic to $h \circ \gamma$ with the two points fixed (see p.622 of [36] for the equivalence of the two definitions). Two lifts $\tilde{h}$ and $\tilde{h}'$ determine the same Nielsen class if and only if $\tilde{h} = \alpha \circ \tilde{h}' \circ \alpha^{-1}$ for some $\alpha \in \mathcal{D}$. In such a case we call $\tilde{h}$ and $\tilde{h}'$ conjugate. Nonempty Nielsen classes define a partition of Fix(h), the fixed point set of h. Moreover, each Nielsen class is an open subset of Fix(h).

4.2 The universal covering space of the disk with k holes

Let $\mathbb{D}$ be a disk with k holes; i.e. $\mathbb{D}$ is a surface with boundary. The boundary $\partial\mathbb{D}$ of $\mathbb{D}$ consists of $k+1$ pairwise disjoint simple closed curves. Let $\tau : \tilde{\mathbb{D}} \to \mathbb{D}$ be a universal cover. $\tilde{\mathbb{D}}$ can be seen as a convex subset of the hyperbolic Poincaré disc $\mathbb{H}^2$. One can compactify $\tilde{\mathbb{D}}$ by a cantor set E_∞ of points on the boundary of $\mathbb{H}^2$. Then $\tau^{-1}(\partial\mathbb{D}) \cup E_\infty$

is the frontier of $\tilde{\mathbb{D}}$, homeomorphic to the circle. Every lift $\tilde{h} : \tilde{\mathbb{D}} \to \tilde{\mathbb{D}}$ of an orientation reversing homeomorphism $h : \mathbb{D} \to \mathbb{D}$ extends to an orientation reversing homeomorphism (also denoted by $\tilde{h}$) of $\tilde{\mathbb{D}} \cup E_\infty$ (see [15], p.427 for more).

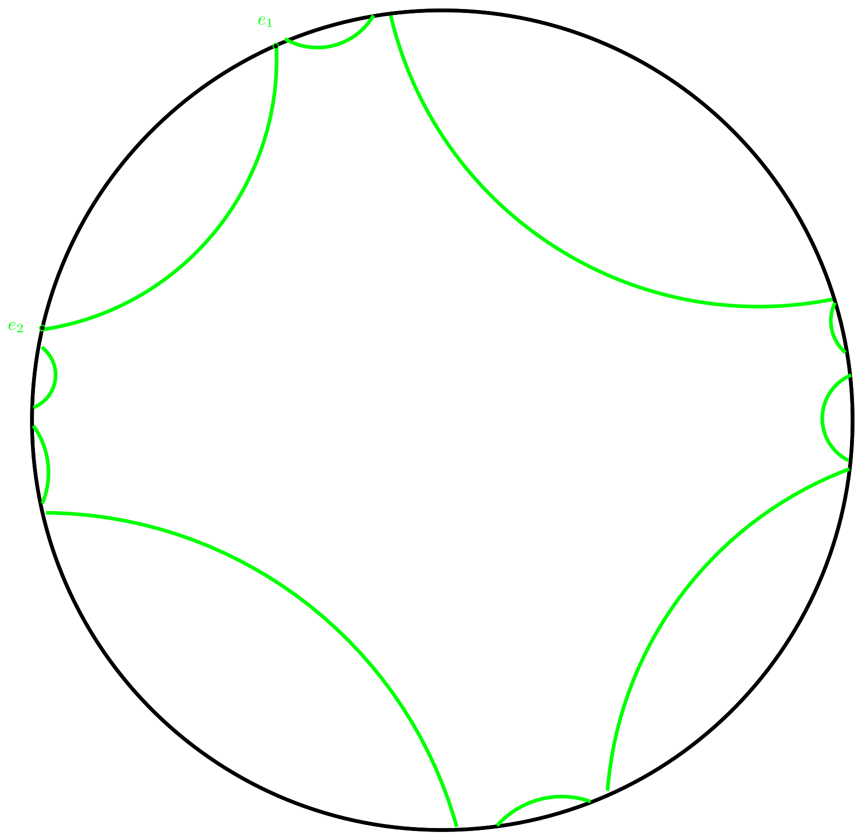

Figure 4.1: An approximation of $\tau^{-1}(\partial\mathbb{D})$ in the universal covering space of $\mathbb{D}$ with $e_1, e_2 \in E_\infty$

4.3 Nielsen classes in a disk with k holes.

Let $h : \mathbb{D} \to \mathbb{D}$ be an orientation reversing homeomorphism of $\mathbb{D}$, with $h(x_o) = x_o$. Let $\{\alpha_i : i = 1, \ldots, k\}$ be the k generators of the fundamental group at x_o, where α_i is represented by a loop around the i-th hole. Notice that $h(\alpha_i) = [\alpha_i]^{-1}$ for any i, since the holes are invariant under h, and h reverses the orientation. Denote by $\mathcal{G}(k, x_o)$ the Cayley graph of a free group on k generators, that represents the structure of $\tau^{-1}(x_o)$. Vertices of $\mathcal{G}(k, x_o)$ represent points in $\tau^{-1}(x_o)$. Edges of $\mathcal{G}(k, x_o)$ represent elements of $\tau^{-1}(\alpha_i)$. If $\tilde{h}$ is a lift of h to $\tilde{\mathbb{D}}$ then $\tilde{h}$ induces a simplicial homeomorphism of $\mathcal{G}(k, x_o)$. To simplify notation, we will denote this simplicial homeomorphism by $\tilde{h}$, that is the same symbol as the lift it is determined by.

Note that for another lift $\tilde{h}_1$ of h it may be the case that $\tilde{h}|\mathcal{G}(k, x_o) \neq \tilde{h}_1|\mathcal{G}(k, x_o)$. Moreover, it is also possible that $\tilde{h}|\mathcal{G}(k, x_o) \neq \tilde{h}|\mathcal{G}(k, x_1)$, for $x_1 \neq x_o$.

Lemma 4.1. *There are at least $k+1$ fixed point classes of h.*

Proof. Let c be any fixed point of h in $\partial\mathbb{D}$. By the unique lifting property, choose the lift $\tilde{h}_0$ of h such that $\tilde{h}_0(\tilde{c}) = \tilde{c}$. Now, define $\tilde{h}_i = \tilde{h} \circ \alpha_i$, for any $i = 1, \ldots, k$. Notice that $\tilde{h}_0, \ldots, \tilde{h}_k$ determine $k+1$ different fixed point classes of h, since no two of them are conjugate, as $\{\alpha_i : i = 1, \ldots, k\}$ is a minimal generating set for $\pi_1(X)$. $\square$

Lemma 4.2. *If $\tilde{h}$ is a lift and $\tilde{h}' = \tilde{h} \circ [\alpha_i]^2$, for some i, then $\tilde{h}$ and $\tilde{h}'$ are conjugate; i.e. $\tilde{h}$ and $\tilde{h}'$ determine the same fixed point class.*

Proof. Note that $\tilde{h} \circ \alpha_i = [\alpha_i]^{-1} \circ \tilde{h}$ since h is orientation reversing and each hole is invariant. Therefore

$$\tilde{h}' = \tilde{h} \circ [\alpha_i]^2 = \tilde{h} \circ \alpha_i \circ \alpha_i = [\alpha_i]^{-1} \circ \tilde{h} \circ \alpha_i,$$

and consequently $\tilde{h}$ and $\tilde{h}'$ are conjugate. $\square$

Lemma 4.3. *There are exactly $k+1$ fixed point classes of h.*

Proof. Let $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_k$ be the lifts described in Lemma 4.1. We shall show that these lifts determine all fixed point classes. Let $\tilde{h}$ be any lift of h. There are integers $p_1, \ldots, p_k$ such that $\tilde{h} = \tilde{h}_0 \circ [\alpha_1]^{p_1} \circ \ldots \circ [\alpha_k]^{p_k}$. Note that by Lemma 4.2

- if all p_i are even then $\tilde{h}$ is conjugate to $\tilde{h}_0$;
- if p_j is odd and p_i are even, for $i \neq j$, then $\tilde{h}$ is conjugate to $\tilde{h}_j$.

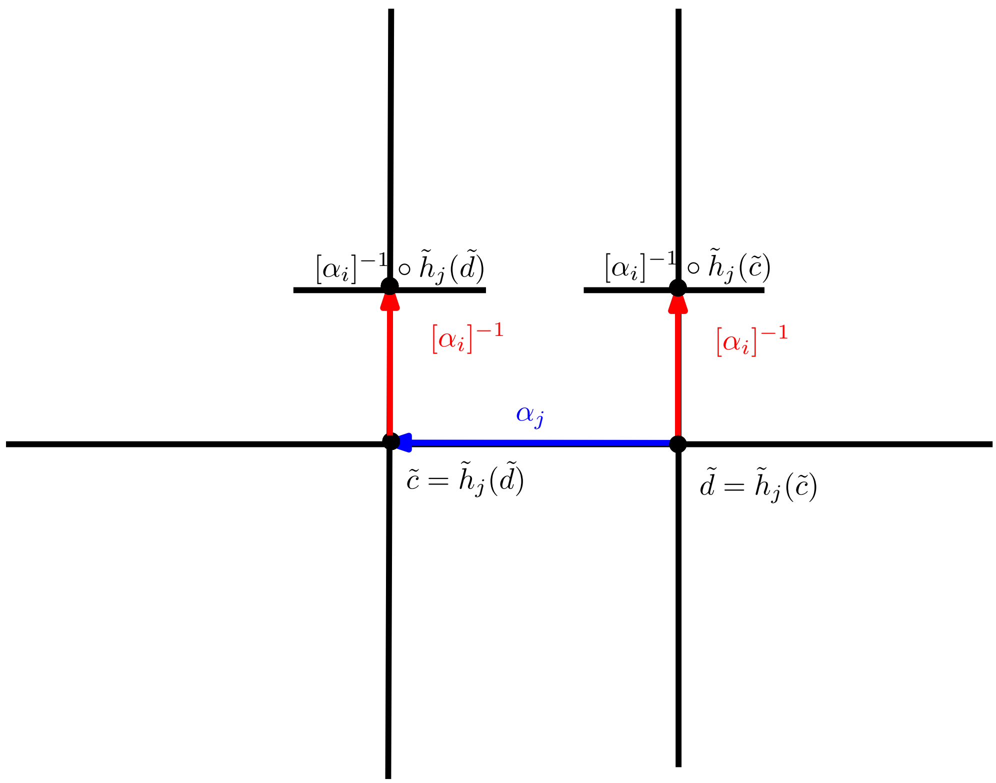

Figure 4.2: Part of the Cayley graph $\mathcal{G}(2, c) \subseteq \mathcal{G}(k, c)$ generated by α_i, α_j.

We shall show that no more than one p_i can be odd, which will complete the proof.

By what we have shown so far, it is enough to show that there does not exist a lift $\tilde{h} = \tilde{h}_0 \circ \alpha_i \circ \alpha_j$, with $i \neq j$. Suppose such a lift exists. Refer to Figure 4.3. Then $\tilde{h} = [\alpha_i]^{-1} \circ \tilde{h}_0 \circ \alpha_j = [\alpha_i]^{-1} \circ \tilde{h}_j$. Recall that $\tilde{h}_0$ fixes a vertex $\tilde{c}$ in $\mathcal{G}(k, c)$. Therefore $\tilde{h}_j$ fixes no vertex in $\mathcal{G}(k, c)$, but fixes an edge E of $\mathcal{G}(k, c)$, that belongs to a component $\tilde{K}$ of $\tau^{-1}(\alpha_j)$. Consequently $\tilde{h}(\tilde{K}) = \tilde{K}$. Notice that $\tilde{c}$ is a vertex of E, and let $\tilde{d}$ be the other vertex of E. $\tilde{c}, \tilde{d}$ are interchanged by $\tilde{h}_j$; i.e. $\tilde{h}_j(\tilde{c}) = \tilde{d}$ and $\tilde{h}_j(\tilde{d}) = \tilde{c}$. Notice that $\tilde{c}, \tilde{d}$ are in two different components of $\tau^{-1}(\alpha_i)$. Therefore $[\alpha_i]^{-1}(\tilde{c})$ and $[\alpha_i]^{-1}(\tilde{d})$ are in two different components of $\tau^{-1}(\alpha_i)$ (the deck mapping $[\alpha_i]^{-1}$ maps these components in a bijective way). But this is contradiction, since $[\alpha_i]^{-1} \circ \tilde{h}_j(\tilde{c})$ and $[\alpha_i]^{-1} \circ \tilde{h}_j(\tilde{d})$ do not share an edge. $\square$

Let $\{C_l : l = 0, \ldots, k\}$ be the components of $\partial\mathbb{D}$, where C_0 lies in the intersection of $\mathbb{D}$ with the unbounded complementary domain of $\mathbb{D}$.

Lemma 4.4. *For each Nielsen class N there is $i \neq j$ such that $|N \cap C_i| = 1$,$|N \cap C_j| = 1$, and $|N \cap C_l| = 0$ for $l \notin \{i, j\}$.*

Proof. Each C_l is a simple closed curve, and therefore there are exactly $2(k+1)$ fixed points of h in $\partial\mathbb{D}$. Since Nielsen classes partition the fixed point set of h, and each lift has exactly two fixed points in the circle $\tau^{-1}(\partial\mathbb{D}) \cup E_\infty$, each fixed point class contains exactly two fixed points from some two components C_i, C_j. We shall show that $i \neq j$.

By contradiction, suppose $i = j$, and let $c, d \in C_i$ be the two fixed points of h in C_i. Since c and d are in the same Nielsen class, there is a path γ in $\mathbb{D}$ from c to d that is homotopic to $h \circ \gamma$ with the endpoints fixed. Therefore there exists an orientation reversing homeomorphism g of $\mathbb{D}$ such that $g|\partial\mathbb{D} = h|\partial\mathbb{D}$ and $g(\gamma) = \gamma$
(g is homotopic to h, cf. proof of Lemma 4.5 and Figure 4.3). Let A and B be two arcs from c to d in C_i such that $A \cup B = C_i$. γ separates $\mathbb{D}$ into two components, one containing A, and the other containing B. Since g interchanges A and B, g also interchanges these two components. However, this contradicts the fact that each C_l is invariant, as they must be in one of the two components. □

Lemma 4.5. *Let, for every $j = 0, \ldots, 2k+1$, x_j be a fixed point of h in $\partial\mathbb{D}$, and let $x_0, x_{2k+1} \in C_0$. There is a homotopy $\{g_t : \mathbb{D} \to \mathbb{D} | t \in [0, k]\}$, and a finite sequence of arcs $\gamma^1, \ldots, \gamma^{k+1}$ such that*

- $g_0 = h$,
- $g_t|\partial\mathbb{D} = h|\partial\mathbb{D}$, *for every* $t \in [0, k]$,
- $g_k(\gamma^i) = \gamma^i$ *for any* $i = 1, \ldots, k$,
- γ^i *has endpoints in* x_{2i-2}, x_{2i-1},
- *for every* $i = 1, \ldots, k$, x_{2i-1}, x_{2i} *belong to the same component of* $\partial\mathbb{D}$.

Proof. It follows from Lemma 4.4 that there is an $x_{\sigma(1)} \in \partial\mathbb{D} \setminus C_0$ that is in the same Nielsen class as x_0. Consequently, by one of the definitions of a Nielsen class, there exists a path

$\{\gamma^1(t) : t \in [0,1]\}$ from x_0 to $x_{\sigma(1)}$, such that γ^1 and $h(\gamma^1)$ are homotopic, with the endpoints fixed. One can choose a closed disk $R^1 \subseteq \mathbb{D}$ such that $(\gamma^1 \cup h(\gamma^1)) \setminus \{x_0, x_{\sigma(1)}\}$ is contained in the interior of R^1. Therefore one can construct an orientation reversing homeomorphism $g_1 : \mathbb{D} \to \mathbb{D}$ homotopic to h, such that

- $g_1(x) = h(x)$ for any $x \notin \mathrm{Int}(R^1)$
- $g_1(\gamma^1) = \gamma^1$.

To obtain the above it is enough to use an orientation preserving homeomorphism ϕ such that $\phi(x) = x$ for $x \notin \mathrm{Int}(R^1)$, and for which $\phi(\gamma) = \gamma$, and then set $g_1(x) = \phi(h(x))$. Note that $g_1(x) = h(x)$ for any $x \in \partial\mathbb{D}$. Since any such ϕ is isotopic to the identity map on R^1, $g_1 = \phi \circ h$ will be homotopic to h.

Now, let $x_{\sigma(2)}$ be the other fixed point of h in the component $C_{\sigma(1)}$ of $\partial\mathbb{D}$ that contains $x_{\sigma(1)}$. It follows from Lemma 4.4 that there is $x_{\sigma(3)} \in \partial\mathbb{D} \setminus C_{\mu(1)}$ that is in the same Nielsen class as $x_{\sigma(2)}$. Consequently there exists a path $\{\gamma^2(t) : t \in [0,1]\}$ from $x_{\sigma(2)}$ to $x_{\sigma(3)}$, such that γ^1 and $h(\gamma^1)$ are homotopic, with the endpoints fixed. One can choose a closed disk $R^2 \subseteq \mathbb{D}$ such that $(\gamma^2 \cup g_1(\gamma^2)) \setminus \{x_{\sigma(2)}, x_{\sigma(3)}\}$ is contained in the interior of R^2. Therefore one can construct an orientation reversing homeomorphism $g_2 : \mathbb{D} \to \mathbb{D}$ such that

- $g_2(x) = g_1(x)$ for any $x \notin \mathrm{Int}(R^2)$
- $g_2(\gamma^2) = \gamma^2$.

To obtain the above it is enough to use an orientation preserving homeomorphism ψ such that $\phi(x) = x$ for $x \notin \mathrm{Int}(R^2)$, and for which $\psi(\gamma) = \gamma$, and then set $g_2(x) = \psi(g_1(x))$. Note that $g_2(x) = g_1(x)$ for any $x \in \partial\mathbb{D}$. Since any such ψ is isotopic to the identity map on R^2, $g_2 = \psi \circ g_1$ will be homotopic to g_1, and therefore to h.

It is clear that continuing the above procedure inductively one obtains a sequence of paths $\{\gamma^j : 1 \leq j \leq k\}$ invariant under an orientation reversing homeomorphism g_k that is homotopic to h in $\mathbb{D}$, and such that

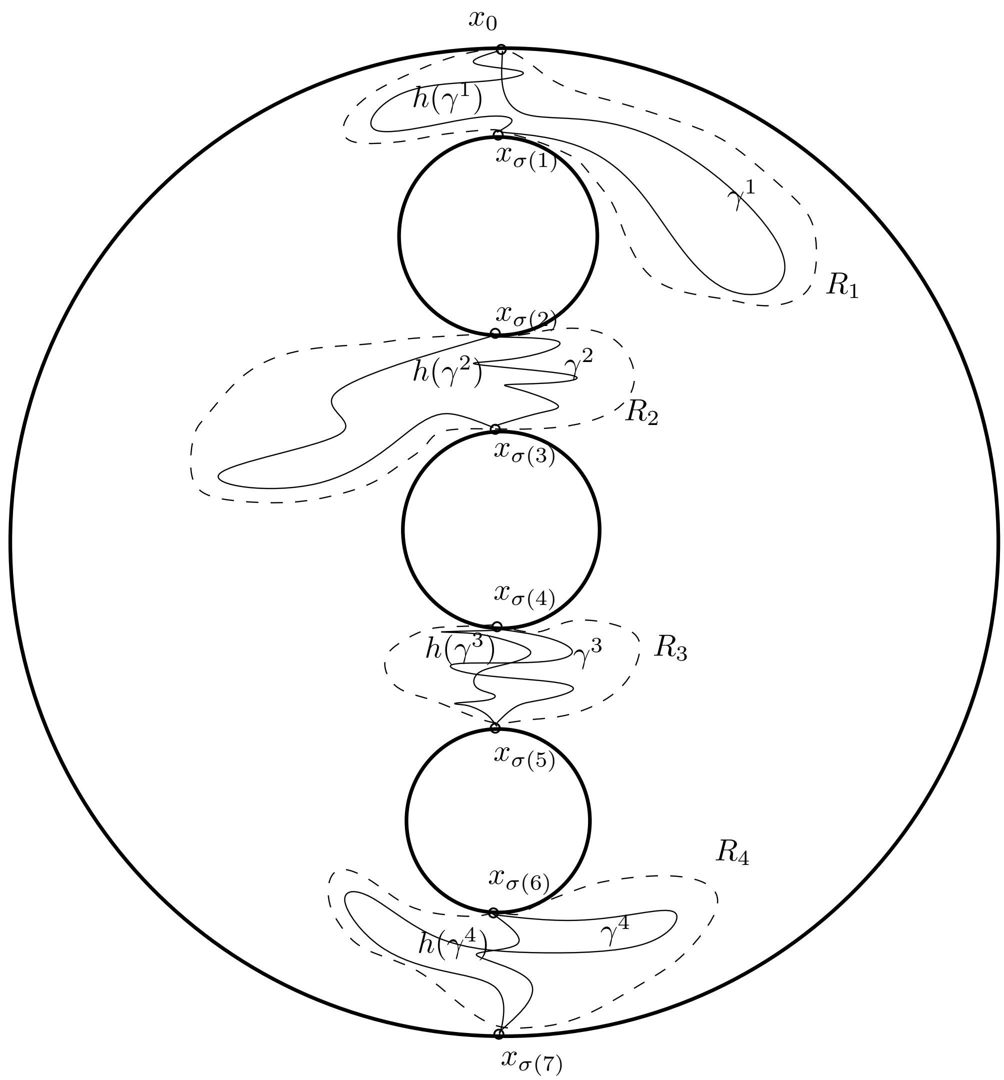

Figure 4.3: Proof of Lemma 4.5

- $\bigcup_{j=1}^{n} \gamma^j$ separates $\mathbb{D}$,
- $g_k(x) = h(x)$ for any $x \in \partial\mathbb{D}$.

$\square$

4.4 Proof of Theorem 1.4

Recall that, without loss of generality, we may assume that X contains an invariant simple closed curve in each invariant component of $\mathbb{R}^2 \setminus X$, one of which, say C_0, bounds

X. When U is a bounded component of $\mathbb{R}^2 \setminus X$, then $\mathbb{A}_U$ will denote an invariant annulus containing X, determined by C_0 and an invariant simple closed curve $C \subseteq U$.

Theorem 1.4: *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation reversing homeomorphism with a continuum X invariant. Suppose $\mathcal{U}$ is a collection of k bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. For any two nonnegative integers p, q such that $p + q = k - 1$ there is $U \in \mathcal{U}$ such that the two Nielsen classes of $h|\mathbb{A}_U$ partition $\mathcal{U} \setminus U$ into two sets, one of which has p elements, and the other one has q elements.*

Proof. There is a disk with k holes $\mathbb{D}$ containing X, such that each of the $k+1$ components of $\mathbb{D} \setminus X$ contains exactly one component of $\partial\mathbb{D}$; i.e. each of these components is invariant under h. Let $C_0, C_1, \ldots, C_k$ be these components of $\partial\mathbb{D}$, where C_0 lies in the unbounded complementary domain of X. By Lemma 4.5 there is an orientation reversing homeomorphism g homotopic to h, and a sequence of arcs $\gamma^1, \ldots, \gamma^{k+1}$ invariant under g such that

- $g_t|\partial\mathbb{D} = h|\partial\mathbb{D}$, for every $t \in [0, 1]$,
- for every $j = 0, \ldots, 2k+1$, x_j is a fixed point of g in $\partial\mathbb{D}$,
- γ^i has endpoints in x_{2i-2}, x_{2i-1},
- $x_0, x_{2k+1} \in C_0$,
- for every $i = 1, \ldots, k$, x_{2i-1}, x_{2i} belong to the same component of $\partial\mathbb{D}$.

Now, fix p and q such that $p + q = k - 1$. Consider the annulus $\mathbb{A}_{p+1}$ that is determined by C_0 and C_{p+1} as the two components of its boundary. Let $(\tilde{\mathbb{A}}, \tau)$ be its universal covering. Choose a lift $\tilde{g}$ of g that determines the Nielsen class of x_{2p+1}. Note that $x_{2p+1} \in C_{p+1} \cap \gamma^{p+1}$. Let D_i be the disk bounded by C_i. Set $S = \bigcup_{i=1}^{p+1} \gamma^i \cup \bigcup_{i=1}^{p} D_i$. Then $x_0, x_{2p+1} \in S$. Now, choose the component $\tilde{S}$ of $\tau^{-1}(S)$ that contains the fixed point of $\tilde{g}$ in $\tau^{-1}(x_{2p+1})$. Note that $\tilde{S}$ is invariant under $\tilde{h}$, as S is invariant under g. S does not separate the plane, and therefore it belongs to the Nielsen class of x_{2p+1} (cf. proof of Lemma 2.2). Additionally, $\tilde{S}$ separates $\tilde{\mathbb{A}}$ into two components, say W and $\tilde{h}(W)$, as it is a continuum (homotopy equivalent to an arc) connecting two components of the boundary of $\tilde{\mathbb{A}}$. It is easy to see that no complementary domain of X that contains C_i for $i = p+2, \ldots, k$ is in the Nielsen class of x_{2p+1}, as $\bigcup_{i=p+2}^{k} \tau^{-1}(C_i) \subseteq W \cup \tilde{h}(W)$. Since there are only two Nielsen classes for $g|\mathbb{A}_{p+1}$ it suffices to show that these classes coincide with the Nielsen classes of $h|\mathbb{A}_{p+1}$.

To see that, let $\{g_t | t \in [0,1]\}$ be a homotopy between g and h described in Lemma 4.5. There is a unique lift $\tilde{h}$ of h and a homotopy $\tilde{g}_t : \tilde{\mathbb{A}}_{p+1} \to \tilde{\mathbb{A}}_{p+1} | t \in [0,1]\}$ such that $\tilde{g}_0 = \tilde{g}$ and $\tilde{g}_1 = \tilde{h}$. It suffices to show that for a given i and $\tilde{C}_i$, a component of $\tau^{-1}(C_i)$, $\tilde{g}(\tilde{C}_i) = \tilde{h}(\tilde{C}_i)$. This will prove that C_i is in the Nielsen class of x_{2p+1} with respect to g if and only if it is in the Nielsen class of x_{2p+1} with respect to h. To finish the proof notice that $\tilde{g}_t(\tau^{-1}(C_i)) = \tau^{-1}(C_i)$ for any $i = 0, \ldots, k$ and any $t \in [0,1]$. Since each component of $\tau^{-1}(C_i)$ is a connected and isolated subset of $\tau^{-1}(C_i)$, and by the continuity of the homotopy $\{g_t | t \in [0,1]\}$ with respect to t, we must have $\tilde{g}_t(\tilde{C}_i) = \tilde{g}(\tilde{C}_i)$ for any t. Consequently $\tilde{h}(\tilde{C}_i) = \tilde{g}_1(\tilde{C}_i) = \tilde{g}(\tilde{C}_i)$, and the proof is complete. □

Theorem 1.5: *Let $h : \mathbb{D} \to \mathbb{D}$ be an orientation reversing homeomorphism of the disk with k holes. Let X be a continuum invariant under h, with $k+1$ components of $\mathbb{D} \setminus X$, each of which is invariant under h, and contains exactly one component of $\partial\mathbb{D}$. Then there are $k+1$ Nielsen classes of h, each of which intersects X; i.e. there are $k+1$ components of* $\mathrm{Fix}(h, X)$.

Proof. Let $h, \mathbb{D}$ and X be as in the theorem. Then, by Lemma 4.3 there are $k+1$ Nielsen classes of h. Let $\tilde{h}_i, i = 0, \ldots, k$ be the lifts representing each Nielsen class. Notice that $\tau^{-1}(X)$ compactified by points in E_∞ is a subcontinuum of $\mathbb{D}$, invariant under each orientation reversing homeomorphism $\tilde{h}_i$. Suppose $\{D_n : n \in \mathbb{N}\}$ is a family of disks with holes such that $\bigcap_{n \in \mathbb{N}} D_n = X$. Then $\bigcap_{n \in \mathbb{N}} \tau^{-1}(D_n) = \tau^{-1}(X)$ and each $\tau^{-1}(D_n)$ is simply connected. By Lemma 4.4 each lift $\tilde{h}$ has no fixed points in E_∞. Therefore by the theorem of Bell [5] each lift $\tilde{h}$ has a fixed point in $\tau^{-1}(X)$. This shows that the $k+1$ Nielsen fixed point classes of h intersect X. It follows from the proof of Lemma 2.2 that there are $k+1$ components of X. □

Chapter 5

A fixed point theorem for the pseudo-circle and other planar circle-like continua

The main idea in our proof of Theorem 1.6 will be to use the universal covering space $(\tilde{\mathbb{A}}, \tau)$ of the annulus $\mathbb{A}$ essentially circumscribed about the pseudo-circle. This idea, in part, originates from [43] where Krystyna Kuperberg and Kevin Gammon presented a short proof of nonhomogeneity of the pseudo-circle. The aim of using the universal cover is twofold. First, it is to make use of the known properties of lifting classes of self-maps of $\mathbb{A}$ in $\tilde{\mathbb{A}}$. This is a standard approach in Nielsen fixed point theory of compact connected polyhedra. For a class of arbitrary separating plane continua, this idea originates from [41] where Kuperberg studied fixed points of orientation reversing planar homeomorphisms in invariant separating plane continua. Second, it is to unfold any circular chain of the covering of $\mathcal{C}$ by open sets to an infinite linear chain of open sets that covers the fiber $\tau^{-1}(C)$ in $\tilde{\mathbb{A}}$. Then it uses arguments patterned on Hamilton's proof of f.p.p. for arc-like continua [31]. The reader is referred to [10],[27],[28],[30],[56], and [45] to learn more on the pseudo-arc and pseudo-circle.

5.1 Preliminaries on Nielsen classes of an annulus

Recall that for a compact connected polyhedron M with the universal covering $\tau : \tilde{M} \to M$, a Nielsen class of a map $\psi : M \to M$ is the set $\tau(\{\tilde{x} \in \tilde{M} : \tilde{\psi}(\tilde{x}) = \tilde{x}\})$, where $\tilde{\psi} : \tilde{M} \to \tilde{M}$ is a lift of ψ to $\tilde{M}$. Nonempty Nielsen classes define a partition of the fixed point set of ψ.

It is known that a self-map of $\mathbb{S}^1$ of degree d has $|d-1|$ lifting classes in its universal covering $e^{it} : \mathbb{R} \to \mathbb{S}^1$ that determine $|d-1|$ Nielsen fixed point classes. $\mathbb{A}$ exhibits the same property, and for completeness sake we will now recall how these lifting classes are defined (cf. [36]).

Let $\tau : \tilde{\mathbb{A}} \to \mathbb{A}$ be a universal cover of $\mathbb{A}$, where we can assume that $\mathbb{A} = \{(r, \theta) \in \mathbb{R}^2 : 1 \le r \le 2, 0 \le \theta < 2\pi\}$ in polar coordinates, $\tilde{\mathbb{A}} = \{(y, x) \in \mathbb{R}^2 : 1 \le y \le 2\}$, and $\tau(y, x) = (y, x(\bmod 2\pi))$. Let $F : \mathbb{A} \to \mathbb{A}$ be a map of degree d, and let $\tilde{F}_0$ be any lift of F. Set $\tilde{F}_0(r, \theta) = (\phi_0(r, \theta), \psi_0(r, \theta))$. Any other lift $\tilde{F}_k$ is determined by $\tilde{F}_k(r, \theta) = (\phi_0(r, \theta), \psi_0(r, \theta) + 2k\pi)$ for any integer k. Since the degree of F is d, $\tilde{F}_k(r, \theta + 2\pi) = (\phi_k(r, \theta), \psi_k(r, \theta) + 2d\pi)$ for any k. If (r_o, θ_o) is a fixed point of F then $\tilde{F}_0(r_o, \theta_o) = (r_o, \theta_0 + 2k\pi)$ for some k. Equivalently $\tilde{F}_k(r_o, \theta_o) = (r_o, \theta_o)$ for some k. If $\theta' = \theta_o + 2q\pi$ then $\tilde{F}_0(r_o, \theta') = (\phi_0(r_o, \theta), \psi_0(r_o, \theta) + 2q(d - 1)\pi)$. Therefore if $k = s \bmod (d-1)$ then fixed points of $\tilde{F}_k$ and fixed points of $\tilde{F}_s$ project to the same fixed points of F in $\mathbb{A}$. On the other hand if $k \ne s \bmod (d-1)$ then fixed points of $\tilde{F}_k$ and fixed points of $\tilde{F}_s$ project onto different fixed points of F in $\mathbb{A}$. This determines $|d-1|$ lifting classes of F corresponding to $|d-1|$ Nielsen fixed point classes.

We will also need one more property of liftings of F. Fix $d \ne 1$. For any k from the equality $\psi_k(r, \theta + 2\pi) - \psi_k(r, \theta) = 2d\pi$ it follows that for any θ

$$(\Delta)\ \theta + 2n\pi - \psi_k(r, \theta + 2n\pi) = [\theta - \psi_k(r, \theta)] + 2n\pi(1 - d),$$

and therefore the sign of $\theta + 2n\pi - \psi_k(r, \theta + 2n\pi)$ depends only on whether $n \to \infty$ or $n \to -\infty$.

Since $\mathbb{A}$ is given in the polar coordinates and coordinates of points in $\tilde{\mathbb{A}}$ are induced by these coordinates, it will be convenient to use the following metric $||(r, \theta) - (r', \theta')|| = |r - r'| + |\theta - \theta'|$ for both $\mathbb{A}$ and $\tilde{\mathbb{A}}$. To avoid confusion we will indicate in which of the two spaces the distance is taken by writing $|| \cdot ||_{\mathbb{A}}$ or $|| \cdot ||_{\tilde{\mathbb{A}}}$. Note that $\lim_{n\to\infty} ||x_n - x_o||_{\tilde{\mathbb{A}}} = 0$ implies $\lim_{n\to\infty} ||\tau(x_n) - \tau(x_o)||_{\mathbb{A}} = 0$.

Let $\mathcal{C}$ be the pseudo-circle embedded with degree 1 into an annulus $\mathbb{A}$, in such a way so that each crooked circular ϵ-chain defining $\mathcal{C}$ consists of closed ϵ-disks.

We shall call a cover $\mathcal{V}$ of H an infinite chain if $N(\mathcal{V})$ is connected, has infinitely (countably) many vertices, and each vertex is of degree 2. Equivalently, one can enumerate the elements of $\mathcal{V}$ by integers so that $V_i \cap V_j \ne \emptyset$ iff $|i - j| \le 1$, for any $V_i, V_j \in \mathcal{V}$. We will need the following lemma motivated by [6] (p.1147, Step 4; see also Figure 1), where it is mentioned without proof in the case of the Möbius band, instead of $\mathbb{A}$.

Lemma 5.1. *Let $\mathcal{U}^\epsilon$ be a finite ϵ-cover of $\mathcal{C}$ by closed disks, with the underlying space of $N(\mathcal{U}^\epsilon)$ homeomorphic to $\mathbb{S}^1$. Let $\tau^{-1}(U^\epsilon) = \{U_n^\epsilon : n \in \mathbb{Z}\}$ consist of disjoint homeomorphic copies of U^ϵ in $\tilde{\mathbb{A}}$. Then, for sufficiently small ϵ, $\mathcal{V}^\epsilon = \{U_n^\epsilon : n \in \mathbb{Z}, U^\epsilon \in \mathcal{U}^\epsilon\}$ is an ϵ-cover of $\tau^{-1}(\mathcal{C})$ that is an infinite chain.*

Proof. It is clear that $\mathcal{V}^\epsilon$ is a cover of $\tau^{-1}(\mathcal{C})$. Also, since for the two metrics defined above, τ is a local isometry, for sufficiently small ϵ, $\mathcal{V}^\epsilon$ is an ϵ-cover of $\tau^{-1}(\mathcal{C})$. To see that $N(\mathcal{V}^\epsilon)$ is an infinite chain, first observe that since $\bigcup \mathcal{U}^\epsilon$ is homeomorphic to $\mathbb{A}$, $\tau^{-1}(\bigcup \mathcal{U}^\epsilon) = \bigcup \mathcal{V}^\epsilon$ is homeomorphic to $\tilde{\mathbb{A}}$. Consequently $N(\mathcal{V}^\epsilon)$ is connected. Now we shall show that any vertex of $N(\mathcal{V}^\epsilon)$ is of degree 2. Let $\tilde{V}_0$ be an element of $\mathcal{V}^\epsilon$, and set $U_0 = \tau(\tilde{V}_0)$. By definition

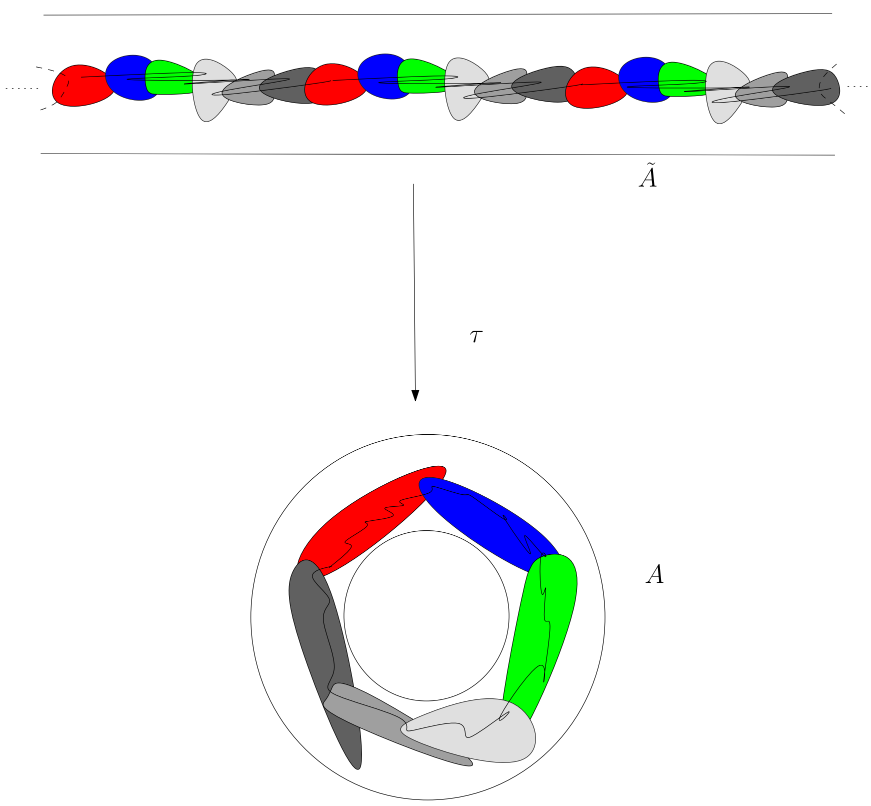

Figure 5.1: Lifting a cover $\mathcal{U}^\epsilon$ with $N(\mathcal{U}^\epsilon) \cong \mathbb{S}^1$ to a cover $\mathcal{V}^\epsilon$ with $N(\mathcal{V}^\epsilon) \cong \mathbb{R}$.

$U_0 \in \mathcal{U}^\epsilon$, and since $N(\mathcal{U}^\epsilon)$ is topologically $\mathbb{S}^1$, there are exactly two elements, say U_{-1}, U_1, of $\mathcal{U}^\epsilon$ such that $U_{-1} \cap U_0 \neq \emptyset \neq U_1 \cap U_0$. Therefore if $\tilde{W} \in \mathcal{V}^\epsilon$ then $\tilde{W} \cap \tilde{V} \neq \emptyset$ if and only if $\tau(\tilde{W}) \in \{U_{-1}, U_1\}$. For sufficiently small ϵ, it is also clear that if $\tau(\tilde{W}) = U_{-1}$ and $\tilde{W} \cap \tilde{V} \neq \emptyset$ then $\tilde{Z} \cap \tilde{V} = \emptyset$ for any $\tilde{Z} \neq \tilde{W}$ such that $\tau(\tilde{Z}) = U_{-1}$. We argue similarly if $\tau(\tilde{W}) = U_1$. □

5.2 Proof of Theorem 1.6

Theorem 1.6: *Let $f : \mathcal{C} \to \mathcal{C}$ be a self-map of the pseudo-circle $\mathcal{C}$. Suppose that $F : \mathbb{A} \to \mathbb{A}$ is an extension of f to $\mathbb{A}$ (i.e. $F|\mathcal{C} = f$). If F is of degree d then f has at least $|d-1|$ fixed points.*

Proof. Fix $d \neq 1$ and let $\tilde{F}$ be a lift of F. It is enough to show that $\tilde{F}$ has a fixed point in $\tau^{-1}(\mathcal{C})$, which will imply that there are $|d-1|$ fixed points of F in $\mathcal{C}$ (at least one for each Nielsen class).

For every m, let $\mathcal{U}^{\frac{1}{m}}$ be an $\frac{1}{m}$-cover of $\mathcal{C}$ by closed disks such that $N(\mathcal{U}^\epsilon)$ is homeomorphic to $\mathbb{S}^1$. By Lemma 5.1 $\mathcal{U}^{\frac{1}{m}}$ lifts to an $\frac{1}{m}$-cover $\mathcal{V}^{\frac{1}{m}} = \{\tau^{-1}(U^{\frac{1}{m}}) : U^{\frac{1}{m}} \in \mathcal{U}^{\frac{1}{m}}\}$ of $\tau^{-1}(\mathcal{C})$, such that $N(\mathcal{V}^{\frac{1}{m}})$ is homeomorphic to $\mathbb{R}$. Set $\mathcal{V}^{\frac{1}{m}} = \{V_i^{\frac{1}{m}} : i \in \mathbb{Z}\}$. One can enumerate elements of $\mathcal{V}^{\frac{1}{m}}$ so that $V_i^{\frac{1}{m}} \cap V_j^{\frac{1}{m}} \neq \emptyset \Longleftrightarrow |i-j| \leq 1$. Set

$$\begin{aligned} A_m = \{x \in \tau^{-1}(\mathcal{C}) : [x \in U_i^{\frac{1}{m}} \& \tilde{F}(x) \in U_j^{\frac{1}{m}}] \Rightarrow [i<j]\} \bigcup \\ \{x \in \tau^{-1}(\mathcal{C}) : x, \tilde{F}(x) \in U_i^{\frac{1}{m}} \text{ for some } i \in \mathbb{Z}\}, \\ B_m = \{x \in \tau^{-1}(\mathcal{C}) : [x \in U_i^{\frac{1}{m}} \& \tilde{F}(x) \in U_j^{\frac{1}{m}}] \Rightarrow [i>j]\} \bigcup \\ \{x \in \tau^{-1}(\mathcal{C}) : x, \tilde{F}(x) \in U_i^{\frac{1}{m}} \text{ for some } i \in \mathbb{Z}\}. \end{aligned}$$

Notice that A_m and B_m are closed and, by (Δ), $A_m \neq \emptyset$ and $B_m \neq \emptyset$. Since $A_m \cup B_m = \tau^{-1}(\mathcal{C})$ and $\tau^{-1}(\mathcal{C})$ is connected, $A_m \cap B_m \neq \emptyset$. For every m choose $x_m \in A_m \cap B_m$. We have $||\tilde{F}(x_m) - x_m||_{\tilde{\mathbb{A}}} \leq \frac{1}{m}$ and thus $\lim_{m\to\infty} ||\tilde{F}(x_m) - x_m||_{\tilde{\mathbb{A}}} = 0$. Consequently also $\lim_{m\to\infty} ||F(\tau(x_m)) - \tau(x_m)||_{\mathbb{A}} = 0$. Notice that since $\mathcal{C}$ is compact, there is $c \in \mathcal{C}$ such that $\lim_{m\to\infty} \tau(x_{\alpha(m)}) = c$, for a subsequence $\{x_{\alpha(m)}\}_{m=1}^\infty \subseteq \{x_m\}_{m=1}^\infty$. Clearly $F(c) = c$ and therefore $\tilde{F}(\tau^{-1}(c)) \subseteq \tau^{-1}(c)$. We shall show that there is $x_o \in \tau^{-1}(c)$ that is a fixed point of $\tilde{F}$. Notice that since we have already exhibited that c is a fixed point of F in $\mathcal{C}$, we may assume that there is a disk D around c of diameter less than $\frac{1}{4}$ that is invariant under F; i.e. $F(D) = D$.

To finish the proof notice that $\tau^{-1}(D) = \bigcup_{n\in\mathbb{Z}} D_n$ consist of disjoint homeomorphic copies of D. There is k such that $\tau(x_{\alpha(k)}), F(\tau(x_{\alpha(k)})) \in D$. Let D_k be the homeomorphic copy of D in the fiber $\tau^{-1}(D)$ that contains $x_{\alpha(k)}$. Note that since $||\tilde{F}(x_{\alpha(k)}) - x_{\alpha(k)}||_{\tilde{\mathbb{A}}} < \frac{1}{\alpha(k)}$ and the diameter of each D_n is less than $\frac{1}{4}$ we must have $\tilde{F}(x_{\alpha(k)}) \in D_k$. Thus $\tilde{F}(D_k) = D_k$. Now choose $x_o \in \tau^{-1}(c) \cap D_k$. Clearly $\tilde{F}(x_o) = x_o$ and the proof is complete. □

Remark 5.2. It should be clear from the proof of Theorem 1.6 that the result extends to all plane separating circle-like continua.

Corollary 5.3. *Let $f : \mathcal{C} \to \mathcal{C}$ be a self-map of the pseudo-circle $\mathcal{C}$. Suppose that $F : \mathbb{A} \to \mathbb{A}$ is an extension of f to $\mathbb{A}$ (i.e. $F|\mathcal{C} = f$). If F is of degree d then the fixed point set of f has at least $|d-1|$ components.*

Proof. It is known that each Nielsen class is an isolated and open subset of the fixed point set. Now, the fixed point set of F is partitioned into $|d-1|$ Nielsen classes. By the proof of Theorem 1.6 the fixed point set of F in $\mathcal{C}$ intersects each of them in a nonempty set. Let $N_1, \ldots, N_{|d-1|}$ be those nonempty Nielsen classes and let K be a component of the fixed point set of F in $\mathcal{C}$. K is closed and, since each N_i is a closed set, therefore $K \cap N_i$ is closed as well. Consequently, by the fact that K is connected, K cannot contain points from more then one N_i. □

5.3 Further comments on self-maps of the pseudo-circle

Theorem 1.6 shows that in coincidence properties there are some similarities between self-maps of the pseudo-circle and self-maps of $\mathbb{S}^1$. Therefore, in addition, the author would like to point out that it is possible to formulate a version of the Borsuk-Ulam theorem for self-maps of the pseudo-circle. Namely, let $a : \mathcal{C} \to \mathcal{C}$ be a fixed point free map such that

$a^2(x) = a \circ a(x) = x$ for any $x \in \mathcal{C}$. a can be viewed as the antipodal map, or the rotation by 180^o, of the pseudo-circle. Then for any self-map of the pseudo-circle $F : \mathcal{C} \to \mathcal{C}$, such that $F(\mathcal{C}) \neq \mathcal{C}$, there is a $z \in \mathcal{C}$ for which $F(z) = F(a(z))$. This is a consequence of the fact that any self-map of the pseudo-circle that is not surjective maps the pseudo-circle onto a pseudo-arc. Then one can apply the following known property of chainable continua [52] [Theorem 12.29, p.253]: for any continuum X, any chainable continuum Y, and any two maps $f, g : X \to Y$, such that either $f(X) = Y$ or $g(X) = Y$, there is a $z \in X$ such that $f(z) = g(z)$.

Chapter 6

Sarkovskii-type theorem for hereditarily decomposable circle-like continua

Recall that a point x is k-periodic (or of period k) if $h^k(x) = x$, but $h^p(x) \neq x$ for any positive integer $p < k$. Throughout this chapter X will be a hereditarily decomposable circle-like continuum embedded essentially into an annulus $\mathbb{A}$ (i.e. X separates the two components of the boundary of $\mathbb{A}$). As in earlier chapters, by $(\tilde{\mathbb{A}}, \tau)$ we will denote the universal covering of $\mathbb{A}$.

6.1 Preliminaries

Lemma 6.1. *The two-point compactification $\tilde{X}$ of $\tau^{-1}(X)$ is a chainable continuum.*

Proof. It follows from the proof of Lemma 5.1 that any circular ϵ-chain covering X lifts to an infinite ϵ-chain $\mathcal{U}$ covering $\tau^{-1}(X)$. Since the nerve of $\mathcal{U}$ is homeomorphic to $\mathbb{R}$, it is easily seen that the two-point compactification of $\tau^{-1}(X)$ admits an ϵ-chain covering it; i.e. the nerve of such a covering is homeomorphic to the arc. $\square$

Lemma 6.2. *$\tilde{X}$ is a hereditarily decomposable continuum.*

Proof. First we show that $\tilde{X}$ must be decomposable. Since X is decomposable therefore $X = K \cup L$ for two proper subcontinua K and L of X. Note that both K and L are chainable and therefore there is an arc A such that $A \cap L = \emptyset$ (so $A \cap K \neq \emptyset$) and A has its end points in two different components of the boundary of $\mathbb{A}$. Without loss of generality we may assume that A has no self-intersections and it intersects the boundary of $\mathbb{A}$ only at its end points. Let $\{K_n : n \in \mathbb{Z}\}$ be disjoint homeomorphic copies of K in $\tau^{-1}(K)$. Let also $\{L_n : n \in \mathbb{Z}\}$ be disjoint homeomorphic copies of L in $\tau^{-1}(L)$. Without loss of generality we may assume that

$$(\Omega) L_i \cap K_j \neq \emptyset \text{ if and only if } i = j \text{ or } j = i + 1.$$

Consider the set

$$\tilde{Y} = L_0 \cup \bigcup_{n=-\infty}^{n=-1} [K_n \cup L_n] \cup \bigcup_{n=1}^{n=\infty} [K_n \cup L_n].$$

Notice that there is $\tilde{A}$, a homeomorphic copy of A in $\tau^{-1}(A)$ such that $\tilde{A} \cap \tilde{Y} = \emptyset$ ($\tilde{A}$ is such that $\tilde{A} \cap K_0 \neq \emptyset$). Also, $\tilde{A}$ separates $\tilde{\mathbb{A}}$ into two components, say $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{A}}_2$. Now consider a partition of $\tilde{Y}$ into $\tilde{Y}_1 = \tilde{Y} \cap \tilde{\mathbb{A}}_1, \tilde{Y}_2 = \tilde{Y} \cap \tilde{\mathbb{A}}_2$. It is easily seen that both $\tilde{Y}_1$ and $\tilde{Y}_2$ are connected and closed, and each of them is compactified by one of the two points compactifying $\tilde{\mathbb{A}}$. Specifically,

$$\tilde{Y}_1 = L_0 \cup \bigcup_{n=-\infty}^{n=-1} [K_n \cup L_n].$$

and

$$\tilde{Y}_2 = \bigcup_{n=1}^{n=\infty} [K_n \cup L_n].$$

$\tilde{Y}_1, \tilde{Y}_2$ are closed as every K_n and L_n is an isolated and closed subset of $\tau^{-1}(K)$ and $\tau^{-1}(L)$, respectively. $\tilde{Y}_1, \tilde{Y}_2$ are connected by (Ω). Call $\tilde{X}_1, \tilde{X}_2$ compactifications of $\tilde{Y}_1$ and $\tilde{Y}_2$, respectively. Then $\tilde{X}_1 \cup (\tilde{X}_2 \cup K_0) = \tilde{X}$ is the decomposition of $\tilde{X}$ into the union of proper subcontinua.

Similar arguments apply if we choose a proper subcontinuum $\tilde{T}$ of $\tilde{X}$ such that $\tau(\tilde{T}) = X$, in order to show that $\tilde{T}$ is decomposable.

If $\tilde{T}$ is a proper subcontinuum of $\tilde{X}$ such that $T = \tau(\tilde{T}) \neq X$, then T and $\tilde{T}$ are homeomorphic (as T is chainable it does not separate the plane, and therefore lifts to disjoint homeomorphic copies in $\tilde{\mathbb{A}}$, one of which is $\tilde{T}$). Since T is decomposable, so is $\tilde{T}$. $\square$

Lemma 6.3. *Suppose $g : \mathbb{A} \to \mathbb{A}$ is a map of degree -1 and y is a point of prime odd period p. Then there is a lift $\tilde{g} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ and $\tilde{y} \in \tau^{-1}(y)$ such that $\tilde{g}^p(\tilde{y}) = \tilde{y}$.*

Proof. Recall from Chapter 2 that if x is a fixed point of g and $\tilde{g}$ is any lift of g, then for any $\tilde{x} \in \tau^{-1}(x)$ either $\tilde{g}(\tilde{x}) = \tilde{x}$ or $\tilde{g}^2(\tilde{x}) = \tilde{x}$. Also if $\tilde{g}(\tilde{x}) \neq \tilde{x}$ for every $\tilde{x} \in \tau^{-1}(x)$, and α is the deck transformation determined by the generator of the fundamental group of $\mathbb{A}$, then $\alpha\tilde{g}(\tilde{x}) = \tilde{x}$ for some $\tilde{x} \in \tau^{-1}(x)$. Moreover $\alpha^n\tilde{g}$ has a fixed point in $\tau^{-1}(x)$ for any odd n, and $\alpha^n\tilde{g}$ has no fixed point in $\tau^{-1}(x)$ for any even n. These results were applied in Chapter 2 to the situation were g is an orientation reversing homeomorphism, but they clearly depend only on the fact that g induces multiplication by -1 on the fundamental group of $\mathbb{A}$. This is because if p is a fixed point of g then the points in the fiber $\tau^{-1}(p)$ are enumerated by integers, and any lift of g induces an order-reversing bijection on $\tau^{-1}(p)$.

Let $\tilde{g}$ be any lift of g. Then $\tilde{g}^p(\tau^{-1}(y)) = \tau^{-1}(y)$. Suppose that $\tilde{g}^p(\tilde{y}) \neq \tilde{y}$ for every $\tilde{y} \in \tau^{-1}(y)$. It follows that $\alpha\tilde{g}^p$ has a fixed point in $\tau^{-1}(y)$. Consider the lift of g given by $\alpha\tilde{g}$. Then $(\alpha\tilde{g})^p = \alpha\tilde{g}^p$, as $\alpha\tilde{g} = \tilde{g}\alpha^{-1}$ and p is odd. Therefore $(\alpha\tilde{g})^p$ has a fixed point $\tilde{y}_o$ in $\tau^{-1}(y)$. □

6.2 Proof of Theorem 1.7

Theorem 1.7: *Let X be a hereditarily decomposable circle-like continuum embedded essentially into $\mathbb{A}$. Let also $f : X \to X$ be a self-map of X that extends to a map $F : \mathbb{A} \to \mathbb{A}$ (i.e. $F|X = f$). If the degree of F is -1 and f has a point of prime odd period p in X then it has a point of any prime period $q > p$ in X, and of any period that is a power of 2, with possible exception for period 2.*

Proof. Let y be a point of odd prime period p in X. By Lemma 6.3 there is a lift $\tilde{f}$ of f and $\tilde{y} \in \tau^{-1}(y)$ such that $\tilde{f}^p(\tilde{y}) = \tilde{y}$. Therefore $\tilde{f}$ has a point of odd prime period p in the two-point compactification $\tilde{X}$ of $\tau^{-1}(X)$. $\tilde{X}$ is a chainable, hereditarily decomposable continuum by Lemma 6.1 and Lemma 6.2. By the result of Minc and Transue [50] for every prime number $q > p$ there is a point $\tilde{z}_q$ of period q with respect to $\tilde{f}$. Since the two points compactifying $\tau^{-1}(X)$ are of period 2 and are interchanged, $\tau(\tilde{z}_q) = z_q$ is well defined. The period of z_q with respect to f must be a divisor of q. Since q is prime, z_q must be either of period 1 or q. If z_q were a fixed point of f then $\tilde{z}_q$ would be a fixed point or a point of period 2 of $\tilde{f}$, contradicting the fact that $\tilde{z}_q$ is a point of period q of $\tilde{f}$. Therefore z_q cannot be a fixed point of f and consequently z_q is a point of period q in X. Now we shall show that f has points of any period that is a power of 2 in X, with possible exception for period 2.

By theorem of Minc and Transue from [50] $\tilde{f}$ has points of any period that is a power of 2. If $\tilde{z}$ is a point of period 2 for $\tilde{f}$ it may be a fixed point, therefore existence of 2-periodic orbits cannot be inferred. Consequently it suffices to show that if $\tilde{z}$ is 2^k-periodic point of $\tilde{f}$ then $z = \tau(\tilde{z})$ is not 2^i-periodic point of f, for $i < k$.

Set $a = 2^k$ and $b = 2^i$. By contradiction, suppose $\tilde{z}$ is an a-periodic point of $\tilde{f}$ but z is a b-periodic point of f. Then there is a $\sigma \in \Pi_1(\mathbb{A})$, an element of the fundamental group of $\mathbb{A}$, such that $\sigma\tilde{f}^b(\tilde{z}_1) = \tilde{z}_1$ for some $\tilde{z}_1 \in \tau^{-1}(z)$, where σ also denotes the deck transformation determined by $\sigma \in \Pi_1(\mathbb{A})$.

Notice that since f^b is of degree 1 (as f is of degree -1), any lift $\tilde{\phi}$ of f^b either has every point in the fiber $\tau^{-1}(z)$ fixed or none. Moreover, there is exactly one lift of f^b that has every point in the fiber $\tau^{-1}(z)$ fixed, and every other lift does not fix any point in that fiber (the same is true for lifts of f^a). This is seen by considering all order-preserving bijections of the integers $\mathbb{Z}$ onto itself; i.e. the only such bijection that has a fixed point is the identity.

Consequently $\sigma\tilde{f}^b(\tilde{z}) = \tilde{z}$ and therefore $\sigma^{\frac{a}{b}}\tilde{f}^a(\tilde{z}) = [\sigma\tilde{f}^b]^{\frac{a}{b}}(\tilde{z}) = \tilde{z}$. It follows that $\sigma^{\frac{a}{b}}\tilde{f}^a$ is the only lift of f^a that has a fixed point in $\tau^{-1}(z)$, and therefore there is no fixed point for $\tilde{f}^a$ in $\tau^{-1}(z)$. This contradicts the fact that $\tilde{z}$ is an a-periodic point of $\tilde{f}$. This contradiction completes the proof.

$\square$

It seems interesting to ask whether Theorem 1.7 is true also for some other self-maps of circle-like hereditarily decomposable continua; i.e. those self-maps that extend to maps of the annulus with degree $\neq -1$. Also, is there a counterexample for Theorem 1.7 in the case of an indecomposable circle-like continuum?

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Appendices

Appendix A

In Section A.1 and Section A.2 of Appendix we present two independent short proofs of the Cartwright-Littlewood Theorem from [32] and [21], that motivated the proof of Theorem 1.2. In Section A.3 we recall basic facts from the theory of prime ends. As discussed in Section 1.5, this theory was used by Barge and Gillette, as well as Barge and Kuperberg in proving the fixed point theorems in [2], [4]. Section A.4 recalls the construction, due to Baldwin and Slaminka [1], of measures preserved by Brouwer homeomorphisms (see Section 1.8).

A.1 A Short Proof of the Cartwright-Littlewood Theorem by O.H. Hamilton

In this section we recall a short proof of the Cartwright-Littlewood Theorem by O.H. Hamilton from [32] that inspired Theorem 1.2.

Cartwright-Littlewood Theorem: Suppose $h : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism of the plane, with a continuum X invariant; i.e. $h(X) = X$. Then there is a point $x_o \in X$ such that $h(x_o) = x_o$.

Proof:(sketch) Let F be the fixed point set of h; i.e. $F = \{x \in \mathbb{R}^2 : h(x) = x\}$. By contradiction, suppose $F \cap X = \emptyset$. Then there is a simple closed curve C_1 separating F from X. C_1 bounds an open disk D_1 containing X. Set $h(D_1) = D_2$ and $h(C_1) = C_2$. It follows from the Brouwer fixed point theorem for a closed disk that neither of D_1 and D_2 contains the other. Therefore $C_1 \cap C_2$ contains at least two points. Therefore, by a theorem of Karékjartó [1] (see [38], p. 87) the component G of $D_1 \cap D_2$ that contains X is bounded by a simple closed curve J. Without loss we will assume that J is the unit circle.

[1]The theorem of Karékjartó may be formulated as follows: Every component of the intersection of two domains, each of which is bounded by a simple closed curve, is again bounded by a simple closed curve. For more detailed description see Proposition 3.1 in [13]

For $j = 1, 2$ the components D_{ji} of $D_j \setminus \mathrm{cl}\, G$ have each as a boundary a simple closed curve composed of L_{ij}, a subarc of J, and a subarc of C_i meeting L_{ij} at the end points. For each i and each j let L'_{ij} be a circular arc of radius $1 - \delta$ that meets L_{ij} at the end points, where $\delta > 0$ is chosen so that no two arcs L'_{ij} can meet except at the end points.

Let H_{ji} be the inner domain of of L_{ij} and L'_{ij}. There is a homeomorphism ϕ_{ji} which maps $\mathrm{cl}\, D_{ji}$ onto $\mathrm{cl}\, H_{ji}$ and leaves each point of L_{ji} fixed. Now, set $\mathrm{cl}\, D_j = \mathrm{cl}\, G \cup \bigcup_i \mathrm{cl}\, H_{ji}$ for $j = 1, 2$. Then $\phi_j(x) = id_{\mathrm{cl}\, G}(x)$ for $x \in \mathrm{cl}\, G$ and $\phi_j(x) = \phi_{ji}$ for $x \in \mathrm{cl}\, D_{ji}$ $(j = 1, 2)$ are homeomorphisms of $\mathrm{cl}\, D_j$ onto $\mathrm{cl}\, H_j$.

Now let $h' : \mathrm{cl}\, H_1 \to \mathrm{cl}\, H_2$ be defined as follows

$$h' = \phi_2 \circ h \circ \phi_1^{-1}.$$

Then $h'|X = h|X$, and h' has no fixed points in $\mathrm{cl}\, H_1$, since if $x \in \mathrm{cl}\, G$ then $h'(x) = h(x) \neq x$ and if $x \in \mathrm{cl}\, H_1 \setminus \mathrm{cl}\, G$ then $x \notin \mathrm{cl}\, H_2 = h'(\mathrm{cl}\, H_1)$.

Now, we shall extend h' to the entire plane as follows. Choose a point $x \notin \mathrm{cl}\, H_1$. Then $z = x + cv_x$, where $x \in C_1$ and v is the unit vector in the direction $0x$ (0 is the center of J), and c is a positive constant. Set $h'(z) = h'(x) + cv_{h(x)'}$. It is readily seen that h' is a homeomorphism of the plane onto itself and that $h'(z) \neq z$ for any point z in the plane. Otherwise, we would have $x = h'(x)$ for some in $\mathrm{cl}\, H_1$, as the direction from 0 to z and from 0 to $h'(z)$ would be the same and therefore $v_x = v'_x$, resulting in $h'(x) = x$.

The Cartwright-Littlewood theorem now follows from the fact that h' is a fixed point free orientation preserving homeomorphism of the plane onto itself with bounded orbits of points in X, contradicting the theorem of Brouwer from [20].

A.2 A Short Short Proof of the Cartwright-Littlewood Theorem by M. Brown

In this section we recall a very short proof of the Cartwright-Littlewood Theorem by Morton Brown from [21] that inspired Theorem 1.2.

Cartwright-Littlewood Theorem: Suppose $h : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism of the plane, with a continuum X invariant; i.e. $h(X) = X$. Then there is a point $x_o \in X$ such that $h(x_o) = x_o$.

Proof: Let F be the fixed point set of h; i.e. $F = \{x \in \mathbb{R}^2 : h(x) = x\}$. By contradiction, suppose $F \cap X = \emptyset$. Let U be a component of $\mathbb{R}^2 \setminus X$ that contains X. Notice that $h(U) = U$ and that the universal covering space $(\tilde{U}, \tau)$ of U is homeomorphic to the plane. Since X does not separate the plane therefore X is the intersection of a family of disks $\{D_n : n \in \mathbb{N}\}$ such that $D_{n+1} \subseteq D_n$. Let D_k be such that $D_k \cap F = \emptyset$. Then since D_k is simply connected subset of U therefore $\tau^{-1}(D_k)$ consists of disjoint homeomorphic copies of D_k. There is a lift $\tilde{h}$ of h, and $\tilde{D}_k$ a component of $\tau^{-1}(D_k)$, such that $\tilde{h}(\tilde{D}_k) = \tilde{D}_k$. Therefore there is $\tilde{X}$ a component of $\tau^{-1}(X)$ contained in $\tilde{D}_k$, such that $\tilde{h}(\tilde{X}) = \tilde{X}$. Note that $H = \tilde{h}|\tilde{D}_k$ is an orientation preserving self-homeomorphism of the set $\tilde{D}_k$ that is homeomorphic to the plane. H has points with bounded orbits, since any $\tilde{x} \in \tilde{X}$ exhibits this property. However, H has no fixed points in $\tilde{D}_k$, since if $\tilde{x} = H(\tilde{x})$ then $\tau(\tilde{x})$ is a fixed point of h in U, contradicting $F \cap U = \emptyset$.

A.3 Prime ends

We recall basic facts about prime ends from [53]. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of the plane and let $U \subseteq \mathbb{R}^2$ be an open, connected, simply connected region such that $h(U) = U$. For $V \subset U$ denote by $\partial_U V$ the boundary of V in U. Call V simple if V is open, connected and $\partial_U V$ is a curve of nonzero finite length with no self-intersections. Let W be another simple subset of U. We say that V divides W if $V \subseteq W$ and $\mathrm{cl}(\partial_U V) \cap \mathrm{cl}(\partial_U W) = \emptyset$. A chain is a sequence of simple sets $\{V_n : n \in \mathbb{N}\}$ such that V_n divides V_m for $n > m$. Such a chain divides W if V_n divides W for some n. It divides another chain $\{W_n : n \in \mathbb{N}\}$ if and only if it divides W_n for every n. We call two chains $\{V_n : n \in \mathbb{N}\}$ and $\{W_n : n \in \mathbb{N}\}$ equivalent if they mutually divide each other. We say that $\{V_n : n \in \mathbb{N}\}$ is a prime chain if and only if from the fact that $\{W_n : n \in \mathbb{N}\}$ divides $\{V_n : n \in \mathbb{N}\}$ it follows that these two chains are equivalent, for any chain $\{W_n : n \in \mathbb{N}\}$. A prime point is an equivalence class of prime chains. A prime point is a prime end if no chain defining it contains a trivial element. A simple set V is trivial if $\mathrm{cl}(\partial_U V)$ is a closed curve.

Let $\hat{U}$ be the set of prime points of U. It is possible to introduce a topology on $\hat{U}$, in which $\hat{U}$ becomes a closed disk, with the subset of prime ends homeomorphic to the circle. This is carried out as follows.

Let $V \subseteq U$ be an open set and let $\gamma \in \hat{U}$ be the equivalence class of $\{V_n : n \in \mathbb{N}\}$. The prime point γ divides V if $V_n \subseteq V$ for some n. Let $[V]_{div} = \{\gamma \in \hat{U} : \gamma \text{ divides } V\}$. It is easily verified that $[V \cap W]_{div} = [V]_{div} \cap [W]_{div}$ and consequently

$$\mathcal{B} = \{[V]_{div} : V \text{ is an open subset of } U\}$$

defines the basis for a topology on $\hat{U}$. With this topology $\hat{U}$ is homeomorphic with a closed disk, and the collection of prime ends is homeomorphic with $\mathbb{S}^1$. If $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of prime ends then $\lim_{n\to\infty} \gamma_n = \gamma$ if for each m there is an N such that γ_n divides V_m for every $n > N$, where $\{V_m : m \in \mathbb{N}\}$ represents the equivalence class of γ. The crucial result for the applications of the theory of prime ends to the study of dynamics of planar homeomorphisms in invariant continua is that the homeomorphism h induces a homeomorphism H on the disk of prime points $\hat{U}$. H is orientation preserving if and only if h is.

A.4 Construction of Measures preserved by Brouwer homeomorphisms

Let h be a fixed point free orientation preserving homeomorphism of the plane. We shall recall the construction of Baldwin and Slaminka from [1] which shows that there is a measure μ that is invariant with respect to h (i.e. $\mu(h(A)) = \mu(A)$) for every μ-measurable subset A of the plane, with the following additional properties:

(M1) μ is the completion of a countably additive measure on the set of all Borel subsets of $\mathbb{R}^2$, (M2) $\mu(A)$ is finite for any bounded subset A of $\mathbb{R}^2$,

(M3) $\mu(U) > 0$ for any nonempty open subset of $\mathbb{R}^2$,

(M4) If A is a subset of $\mathbb{R}^2$ and $f(A)$ has Lebesgue measure 0 for any homeomorphism f of $\mathbb{R}^2$, then $\mu(A) = 0$,

(M5) Lebesgue measure λ is absolutely continuous with respect to μ; i.e. $\mu(A) = 0$ implies that A is Lebesgue measurable and $\lambda(A) = 0$.

Recall that since h has no fixed points and no 2-periodic points, by classification of Brouwer and Bonino, any point is wandering; i.e. for every $x \in \mathbb{R}^2$ there is a translation domain V_x such that $\{h^n(V_x) : n \in \mathbb{Z}\}$ is a collection of pairwise disjoint open sets. Fix any $x \in \mathbb{R}^2$, and without loss assume V_x is of Lebesgue measure $\lambda(V_x) \leq 1$.

Lemma A.1. *[1] There is a countably additive measure μ on $\mathbb{R}^2$ satisfying $(M2)$ and $(M4)$, and the following:*

1. *$\mu(V) \leq 1$ for every translation domain V,*
2. *$\mu(B) = \lambda(B)$ for every $B \subseteq V$,*
3. *$\mu(h(A)) = h(A)$ for all μ-measurable subsets A of the plane.*

Proof. Set $\mu(A) = \sum_{-\infty}^{\infty} \lambda(f^i(A) \cap V_x)$, and say that A is μ-measurable if every term on right is well defined. $\mu(A)$ is infinite if the series fails to converge. Clearly μ is countably additive, as λ is, and $\mu(h(A)) = h(A)$ by the right hand side of the equation defining μ. Also $(M3)$ easily follows, as if $\lambda(h(A)) = 0$ then $\lambda(h^i(A)) = 0$ for any i and consequently $\mu(A) = 0$. Furthermore, since $h^i(V_x) \cap V_x = \emptyset$, $\mu(B) = \lambda(B)$ for every $B \subseteq V$.

To see that $\mu(V) \leq 1$ for every translation domain V, again observe that $h^i(V_x) \cap V_x = \emptyset$. Finally, to obtain $(M2)$, choose any bounded subset A of $\mathbb{R}^2$ and by contradiction suppose $\mu(A)$ is infinite. Then one can successively divide A into bounded subsets of smaller diameters with infinite μ-measure. This would imply that there is a point p in the plane that contains a set of infinite measure in each open neighborhood, contradicting the fact that p is wandering and has a translation domain of μ-measure 1 as its neighborhood. □

Theorem A.2. *[1] Let h be a fixed point free orientation preserving homeomorphism of the plane. Then there is a measure satisfying $(M1)-(M4)$ which is invariant with respect to h.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be an open cover of the plane with translation domains, and without loss of generality assume that each of them has Lebesgue measure not exceeding 1. Let μ_n be the measure on $\mathbb{R}^2$ determined by V_n and the application of Lemma A.1. Define measure μ by $\mu(A) = \sum_{i=1}^{\infty} 2^{-n}\mu_n(A)$. By repeating arguments from the proof of Lemma A.1, it follows that μ is bounded on bounded sets, as it is bounded on any translation domain. Clearly, properties $(M1) - (M4)$ and the invariance of measure μ follows from the definition and Lemma A.1, which completes the proof. □