The Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions

by

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Abstract

The use of technology in mathematics education has been strongly encouraged by the National Council of Teachers of Mathematics (NCTM, 2000) and the American Association of Two-year Colleges (AMATYC, 2006). Researchers have envisioned technology's potential in grand ways, including democratization of access to higher mathematics (Kaput, 1994). There are challenges to the realization of that dream. For example, innovation in technological advances often outpaces the evaluation of how those advances can be best applied (Epper & Baker, 2009).The need for improved use of technology in adult developmental mathematics education has been documented (Caverly, Collins, DeMarais, Otte, & Thomas, 2000; Epper & Baker, 2009).

At the same time, adult developmental mathematics students' need for support and help to realize their educational dreams is a vital current issue (Bryk & Treisman, 2010). This study seeks to provide insight into how the use of mathematics technology affects the internal mathematical representations possessed by adult developmental mathematics students. It is hoped that such insight may provide teachers of adult developmental mathematics students with research based understanding which will aid them in incorporating the use of technology.

Open recruitment was done on the campus of a mid-sized university in the southern United States. One subject was interviewed 7 times and then a second subject was interviewed 6 times. Each interview was video taped with three feeds to capture the subjects' interactions with both paper and technology and to record the subject's movement and facial expressions. Qualitative analysis was done with the aid of Atlas.ti software during and after data collection. Each case was considered separately, compared and contrasted and merged results were also considered. Results suggest ways in which technology can impact student thinking.

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1. Introduction

In a junior high school in the southern United States, students have access to laptop computers and regularly interact with technological representations of mathematics. These may include charts, graphs, geometric shapes, algebraic equations, or other mathematical objects. Those objects may be represented via mathematics software, internet sites, or on hand-held devices such as calculators. Many of the students at this school are accelerated beyond the normal curriculum for their grade, increasing their future educational opportunities. Across the river at a mid-sized university, over 500 adult developmental mathematics students try to make up for lost opportunities to learn. Technology laboratories are provided for them at the university, but the students may be either reluctant to use them or unaware of their potential. This situation is symptomatic of issues regarding the use of technology in mathematics education. New mathematics technology continues to be developed, and may even be made available to teachers and students, but is it being used wisely to help the students that are most in need of the help it can provide? Examining the situation of adult developmental mathematics students may be an important avenue for the examination of this question.

Statement of the Problem

The need for more attention to ways to incorporate technology into developmental education has been documented (Epper & Baker, 2009; Caverly et al., 2000). Developmental education, sometimes referred to as remedial education, refers to educational efforts which serve college students who need additional preparation in order to be successful (Payne & Lyman, 1996). Developmental mathematics “has become an insurmountable barrier for many students, ending their aspirations for higher education” (Bryk & Treisman, 2010, p. B19). Instructors may not be familiar enough with the learning barriers that adult students face to choose technology which will meet their needs. Adult learners also may not have the verbal comprehension necessary to successfully interact with software designed to supplement instruction (Li & Edmonds, 2005). The use of software designed to tutor them, provide them with extra practice, and sometimes engage them in dialog is sometimes known as computer-assisted instruction (CAI) and it is common in adult developmental mathematics programs ("CAI," 2003; Caverly et al., 2000). Other uses of technology in developmental programs include internet sites, distance learning technology, computer algebra systems, graphing calculators and spreadsheets (Epper & Baker, 2009).

The American Mathematical Association of Two-Year Colleges (AMATYC) has adopted the use of technology as one of its basic principles (American Mathematical Association of Two-Year Colleges, 2006; Epper & Baker, 2009). AMATYC’s (2006) document *Beyond Crossroads,* which seeks to provide help in implementing mathematics education standards for those teaching beginning college students, states that “Technology should be integral to the teaching and learning of mathematics” (p. 11). The description of the principle states

> Technology continues to change the face of mathematics and affect the relative importance of various concepts and topics of the discipline. Advancements in technology have changed not only *how* faculty teach, but also *what* is taught and *when* it is taught. Using some of the many types of technologies can deepen students’ learning of mathematics and prepare them for the workplace (p. 11).

Even though it is considered integral to teaching and learning and has the potential to exert a positive influence, innovation in technology has outpaced its evaluation (Epper & Baker, 2009). There are also many questions still to be resolved in developmental mathematics such as curriculum content and sequencing which affect the use of technology. Software packages may follow a broad and shallow curriculum that is inappropriate for the needs of adult developmental mathematics students. Epper and Baker (2009) have suggested several steps to improve the use of technology in adult developmental mathematics education including blending best practice with leading technological innovations, providing greater research evidence, increasing technological development for education, and overcoming resistance to change in the community college culture.

Theoretical Foundations in Representation

In order to provide the research evidence which adult developmental mathematics educators need, it is necessary to understand the use of representation in mathematics education because of the connections between technology and representation. The National Council of Teachers of Mathematics (NCTM) (2000) defined mathematical representation as both "the act of capturing a mathematical concept or relationship in some form" and "the form itself" (p. 66). They also referred to the influence of computers and calculators on representation, noting that technology has increased the number of representations available to students (NCTM, 2000). They also noted that technology allows students access to more representations, some of which students may not otherwise be able to access. New dynamic technology, which allows the movement of an object represented technologically, often affecting the movement of another connected object, transforms the possibilities for representation and may have a great impact on how mathematical objects are conceptualized and mathematical meanings are internalized

(Falcade, Laborde, & Mariotti, 2007; Moreno-Armella, Hegedus, & Kaput, 2008). Researchers have noted the connection between technology and representation and the need for further study of these connections (Hollenbeck & Fey, 2009; Stylianou, Smith, & Kaput, 2005).

Many studies over the past ten years have looked at links between the use of technology and representation (cf. Abramovich & Ehrlich, 2007; Falcade, Laborde, & Mariotti, 2007; Kaput, 1998; Yerushalmy & Shternberg, 2001). A variety of technologies in various settings have been examined, such as a professional development setting in which spreadsheets were examined as cognitive tools (Alagic & Palenz, 2006); the creation of computer technology to link middle grades classrooms for a study of multiple representations of functions (Hegedus & Kaput, 2004); the examination of the use of calculator based laboratories (CBLs, which use sensors to translate real-world information into calculator data for analysis) with students (Lapp & Cyrus, 2000) and preservice teachers (Sylianou, Smith, & Kaput, 2005). The idea of a function, central to some of these studies, is a mathematical relationship in which one set of data is matched with another set of data so that each piece of data in the input set is matched to one and only one piece of data in the output set. It is one of the most important topics in mathematics (O'Callaghan, 1998).

Purpose of the Study

This study seeks to add to the work which has been done linking technology and representation and broaden its scope to specifically address the needs of adult developmental mathematics students. It has been shown that technology has a potential impact on learners' conceptualizations and internalization of mathematical meaning (Moreno-Armella, Hegedus, & Kaput, 2008). It has also been shown that teachers of adult students need more understanding of their learning barriers (Li & Edmonds, 2005). The focus of this study is on the effect of technology on adult developmental mathematics students' understanding as evidenced by the apparent changes in their internal representations of mathematics - those mathematical forms which exist within the students' mind (Goldin, 2003). Those apparent changes and their relationship to the subject's interaction with mathematics technology will be examined. The questions specifically are these:

1. Following the introductory use of dynamic computer technology to explore mathematical concepts built upon previous knowledge, what internal representations of those concepts do developmental mathematics students possess?
2. What can be determined about the validity and usefulness of those representations?
3. How well do those representations endure over a period of time and in the company of tasks which build upon them?

Significance of the Study

The current study has the potential to assist developmental educators in meeting the needs of adult learners. It can help those educators realize the potential of technology to improve developmental mathematics instruction for all students (Epper & Baker, 2009). Bryk and Treisman (2010) recently noted the dilemma of developmental mathematics students who may work under great pressures in their personal lives. They may put all the effort they can into completing the mathematics sequences designed to help them get ahead and still fail to complete them. Describing the case of a single mother working the late shift at a supermarket and trying to go back to school, Bryk and Treisman (2010) noted that she said "I just couldn't do it anymore." They noted that for "this student and too many others, the dream stops here" (p. 19). They also noted that as many as 70 percent of students placed in developmental mathematics courses do not complete them. Technology has the potential to change such students' lives (Epper & Baker, 2009). The dream need not stop. The results of this study can provide adult developmental mathematics educators with information that will help them to better understand their students' thinking and make more informed choices as to the technology-based mathematics instruction they provide those students.

A Brief Summary of the Content to Follow

In chapter 2, the review of the literature will examine the role of technology in mathematics education, the needs and concerns of adult developmental mathematics students' mathematical representations, and ideas relating technology and representation as well as specific studies combining the two. Literature related to the ideas of constructivism will be presented in order to provide a theoretical setting for the study of students' interactions with technological representations. Chapter 3 will first discuss the theoretical basis for the methods chosen and then describe the specific procedures. The theoretical presentation will conclude with a brief look at lessons learned during the course of a pilot study. In chapter 4, following an introduction to the two subjects of the case, the progress of the teaching experiment in each of their cases will be described. This will be followed by a look at the theoretical ideas which were investigated and discovered as they emerged from the data. In chapter 5 limitations and conclusions will be presented, including a discussion of what was learned about the research questions. Implications for teachers of adult developmental mathematics students will also be noted as will suggestions for further research. It is hoped that these results may help empower adult developmental mathematics students for a viable academic future.

2. Review of the Literature

In order to understand the influence of technology use on the mathematical thinking of adult developmental mathematics students, it is important to understand issues related to technology use in mathematics education in general and issues facing adult students. Ideas surrounding the use of mathematical representations are also important because of their close connection to the use of technology. This review will begin with an examination of the benefits and challenges associated with the use of technology in mathematics education. Following this the particular needs of adult developmental mathematics students will be considered. This will include information about their general needs as well as the issues related to the use of technology which they face. The role of representation in mathematics education will be carefully examined. This examination will conclude with the presentation of an interpretive framework with which student thinking might be considered. Once this framework has been established, the connections between technology and representation will be more carefully considered and several studies relating the two will be examined. Because knowledge and the development of knowledge is to be examined, an epistemology must exist as part of that examination, and so literature related to constructivism will also be presented. This will provide a theoretical foundation for understanding not only student thinking, but the way knowledge of student thinking can be built. Conclusions and research questions will follow. Note that a glossary defining key terms is provided in Appendix A.

Technology in Mathematics Education

Educational technology continues to be reinvented at a rapid pace, and it is sometimes the case that research, access, and implementation have difficulty keeping up with the pace of invention and with each other (Fey, 1984; Fey, Hollenbeck, & Wray, 2010). In order to serve students, promote positive change, and understand the role of technology in mathematics education, both the benefits and the challenges associated with it must be considered. Table 1 is provided below as a summary of some of these ideas.

Table 1

Benefits and Concerns of the use of Technology in Mathematics Education

Benefits	Concerns
New forms of mathematical activity (Moreno-Armella et al., 2008) Opportunity for student initiative and exploration (Fey, 1984; Fey et al., 2010) The ability to visualize as mathematicians do (Cuoco & Goldenberg, 1996) Provides external reference objects which helps make their thinking explicit and clarify their ideas (Hennessy, Fung, & Scanlon, 2001) Quicker problem solving "feedback loops" (Shaffer & Kaput, 1998, p. 111) May serve the role of active listener (Connell, 1998)	Ensuring that conceptual understanding is center stage and student thinking is not replaced (Fennell & Rowan, 2001; Fey et al., 2010) Both technical and mathematical knowledge are needed to take advantage of learning possibilities (Lingefjard, 2008) Selection of appropriate tasks is important to taking full advantage of technology use (Alagic & Palenz, 2006; Gadanidis et al., 2004)

Benefits	Concerns
More direct access to mathematics structures allowing a greater impact on their minds (Moreno-Armella et al., 2008)	Time, training, and immersion in the technology are needed for those using it to learn to think with technology rather than about it (Gadanidis, 2008)
Encourages dynamic visualization (Yerushalmy et al. 1999, cited by Presmeg, 2006)	Gaps exist between the level of technology available and the practical use being made of it (Atan, Suncheleev, Shitan, & Mustafa, 2008; Hollenbeck & Fey, 2009; Oncu, Delialioglu, & Brown, 2008)
Encourages dynamic visualization (Yerushalmy et al. 1999, cited by Presmeg, 2006)	
Move more flexibly between different representations (NCTM, 2000)	

Benefits of technology to mathematics education. Digital technologies are capable of providing new forms of mathematical activity in socially rich ways (Moreno-Armella et al., 2008). They redefine the practices, content, and ways of knowing about a subject. If a certain technology is absent, different knowledge will be produced (Villarreal, 2008). When present, technology can provide an opportunity in the learning environment for student initiative and exploration (Fey, 1984; Fey et al., 2010).

Mathematicians have been known to use creative imagery and metaphor to understand and think about mathematics. Some researchers believe that through technology, students can be provided with the opportunity to "tinker with mathematical objects just as they might tinker with mechanical objects" and thus develop the ability to visualize as mathematicians do (Cuoco & Goldenberg, 1996, p. 17). Immersion in dynamic, technological mathematical environments could have a great impact on how students conceptualize mathematical objects and what they consider doing mathematics to entail (Moreno-Armella et al., 2008). Dynamic environments allow mathematical representations to be set in motion in some way. Not all technological representations of mathematics are designed to do so, as is the case with some types of computer aided instruction (CAI) software which may only involve the static input of student responses to questions (Kaput, 1992). The influence of dynamic computer environments continues to be an area of significant focus in the study of education for effective mathematical visualization (Presmeg, 2006). Researchers have conjectured that computer models can provide students the start they need to be able to engage in more advanced mental visualization of mathematical concepts (Cuoco & Goldenberg, 1996).

Technology can provide an external reference object which encourages students to make their thinking explicit and clarify their ideas (Hennessy, Fung, & Scanlon, 2001). It allows students greater opportunity to experience mathematics as an "experimental enterprise." The problem solving "feedback loops" are quicker and not dependent on symbolic manipulation (Shaffer & Kaput, 1998, p. 111). Dynamic media allow students more direct access to mathematical structures and thus allow those structures to have greater impact on the minds of students (Moreno-Armella et al., 2008). Yerushalmyet al. (1999, as cited in Presmeg, 2006) noted that the use of computer software for mathematics encourages dynamic visualization. The visual process used to classify the types of problems in their study was enhanced by the use of dynamic software which allowed the users to move flexibly between different representations.

Iterative examples developing the concept of limit and the asymptotic behavior of certain functions, and transformations are some of the mathematical topics made more accessible through the use of technology (NCTM, 2000).

Concerns regarding technology in mathematics education. Care must be taken to ensure that conceptual understanding, rich in connections to other ideas, is center stage and that students have the chance to produce their own representations of what is occurring within the technological representation (Fennell & Rowan, 2001; Goldin, 2003; Lapp & John, 2009). Sometimes materials used to represent mathematical thinking replace student's thinking rather than represent it (Fennell & Rowan, 2001). Good representations show how students are thinking. Students may be taught how to use tools such as hand-held modeling objects known as manipulatives or technological representations as the only way to solve problems and when this happens, the tool may actually interfere with learning and fail to build mathematical understanding. Forms of representations can become ends in themselves, which is not productive for students. (Fennell & Rowan, 2001). It is just as harmful to use manipulatives or technological tools in a formulaic "do as I do" way as it is to have students blindly follow algorithms. The goal is to have students use manipulatives and technological tools to think with rather than as answer machines. Van de Walle (2007) stated that "A mindless procedure with a good manipulative is still just a mindless procedure" (Van de Walle, 2007, p. 34). When the focus of teaching is on attitudes, atmosphere, and objectives, and the teaching materials are seen as a means to maximize those aspects of the lesson, then those materials are not an end in themselves (Villarreal, 2008). Rather than being used for mindless procedures, the computer may, for example, serve the role of an active listener, doing as the student tells it to do, and assisting the student in constructing their own knowledge (Connell, 1998).

Students must have both technical and mathematical knowledge to take advantage of the possibilities for learning that a technological tool provides (Lingefjard, 2008). This requires time and training if students are to learn to think with the technology. When human beings immerse themselves in using a technology, then they can learn to think with that technology, rather than about that technology (Gadanidis, 2008). Selection of problems is another important aspect of accomplishing learning goals in a technology oriented environment. This is particularly important when time must be taken to assist students in becoming proficient with the tools being used (Alagic & Palenz, 2006).

Interactive examples may be accompanied by instructions for students and teachers that have been designed to foster investigation, problem solving, and student discovery. Those examples, however, may not support the discovery envisioned by their creators (Gadanidis, Sedig, Liang, & Ning, 2004). Such tasks should be analyzed to see whether or not they support the desired pedagogy. One quality to look for is whether or not the task provides appropriate mathematical patterns related to the investigation, so that students may observe the relationships found in the objectives for the lesson. For example, if the investigation calls for students to analyze relationships between surface area and volume, are those outputs displayed together on the same graph or separately on different graphs? What would help the student more? Are different helpful representations present, such as equations, ratios, tables, graphs, and illustrations, if they would prove useful to the student? (Gadanidis et al., 2004). Even considering the benefits and attractions of interactiveness, students may be more likely to engage with an investigation involving real-life content than a purely mathematical investigation (Gadanidis et al., 2004). Those producing interactive visualizations for the classroom would benefit from using a review process including feedback from classroom teachers and suggestions for improvement.

The design of such tasks should include consideration of principles of both presentation and interaction, incorporating the maximum mathematical benefit for the student. This includes making the information intelligible, engaging the learners with the information and facilitating mental interaction with the material (Gadanidis et al., 2004).

Though technology has advanced, the use of technology has not always kept pace with its development. Access to technological tools is easier than the far greater challenge of determining how to use those tools effectively (Fey et al., 2010). There may be gaps between the level of technology available and its practical applications in mathematics education (Atan, Suncheleev, Shitan, & Mustafa, 2008; Hollenbeck & Fey, 2009). Merely using the technological tool to find solutions does not guarantee knowledge of broader principles. Care must be taken that student conjectures arising from the use of software are subject to appropriate mathematical proof (Fey et al., 2010). It should also be noted that despite the many advantages of technology, paper and pencil provides a recording media which leaves an accessible record of what happened, something which technology cannot always provide (Goldin, 2003).

Fey spoke in 1984 of the challenges of incorporating technology into mathematics education. He noted then that the traditional pattern for educational change, in which a proposal is made by professional educators, curriculum is written, and classroom implementation follows, underestimates the complexity of the actual process of change (Fey, 1984). In some instances, technology may be available to teachers, but may not be used to support student learning (Oncu, Delialioglu, & Brown, 2008). Hollenbeck and Fey (2009) illustrated how a variety of the latest technological tools, if available to teachers, could be used to enhance learning in the mathematics classroom. They concluded that appropriate use of technological instructional tools should be a high priority for mathematics education researchers (Hollenbeck & Fey, 2009). This echoes the suggestion Fey made in 1984 that, in spite of the challenges in implementation, the best use of technology in the classroom be determined without the research being limited by concerns over how to make that technology available. One of the populations in need of such work is adult learners. Following is an examination of their general needs and then a more specific look at the role of technology in their mathematics education.

Adult Developmental Mathematics Students

I will first present some general information about adult developmental mathematics education. I will then look at particular challenges faced by the students in these programs. Following this, will be an examination of the use of technology for adult developmental mathematics students.

Introduction to adult developmental mathematics education. One of the subjects of the current study instructed the tutors in the mathematics lab she frequented to "pretend" she was "in middle school." She seemed to understand and reflect the truth that many students in the U.S. begin to fall behind at the point in their education where algebra course work customarily begins (Epper & Baker, 2009). This is the mathematical content on which developmental students are assessed for placement. Developmental mathematics programs may be the key to success for many students. They seek to bridge the gap between what has been learned in high school and what is needed for success in postsecondary education (Epper & Baker, 2009). Without developmental education, many people would never have the opportunity to attend college or improve their employment possibilities (Gerlaugh, Thompson, Boylan, & Davis, 2007). Table 2 displays some of the statistics regarding adult developmental mathematics education. Title IV institutions are those which participate in certain federal student aid programs (Aud et al., 2010).

Table 2

Adult Developmental Mathematics Education Summary of Cited Statistics

Item	Percent	Source and year
All Title IV institutions offering developmental mathematics	71%	Parsad and Lewis (2003) NCES report
Public 2-year institutions offering developmental mathematics	97%	Parsad and Lewis (2003) NCES report
Public 4-year institutions offering developmental mathematics	78%	Parsad and Lewis (2003) NCES report
Freshmen surveyed at all types of institutions who were enrolled in developmental mathematics	>20%	Parsad and Lewis (2003) NCES report
Private 2–year institutions freshmen surveyed who were enrolled in developmental mathematics	33%	Parsad and Lewis (2003) NCES report
Students in 2-year colleges requiring developmental mathematics	>50%	Shwarte (2007)
Those who test into developmental mathematics that pass on the first attempt (some states)	40-50%	Trenholm (2006)
Percent of students in Nevada graduating from high school in 2006 and attending college the following fall that enrolled in remedial mathematics during their first year of college	37.6%	Fong, Huang, & Goel (2008) NCES report

More developmental courses in mathematics are generally offered than in the other two standard developmental topics, reading and writing (Parsad & Lewis, 2003). The National Center for Educational Statistics (NCES) (Parsad & Lewis, 2003) published a special report in 2003 focusing on what they call remedial education. They reported that in the year 2000, 71% of all Title IV degree-granting institutions with freshmen offered developmental courses in mathematics. When broken down further, it was noted that 97% of public 2-year institutions offered developmental courses in mathematics, as did 78% of public 4-year institutions. More than one fifth of freshmen at all institutions in the survey enrolled in developmental mathematics as freshmen. For private 2-year institutions, this figure rises to 35% (Parsad & Lewis, 2003). In a later study published by NCES, Fong, Huang, and Goel (2008) reported that of 4,653 students graduating from Nevada public high schools in 2006 and enrolling in at least one mathematic course in a Nevada public college the following school year, 37.6% enrolled in a remedial mathematics course. Schwarte (2007) stated an even higher figure, noting that more than half of the students in two-year colleges require developmental mathematics instruction to prepare them for college-level mathematics work. Trenholm (2006) noted further that in some states, only 40-50% of those that test into developmental mathematics courses pass on the first attempt. These figures show the large population being served, the need for improvement, and the implication that research into adult developmental mathematics education is important.

Lesik (2007) showed that developmental mathematics can be effective, noting that the risk of college drop-out among developmental mathematics students was significantly lower than for similar students who did not participate in developmental mathematics programs (Lesik, 2007). Her study followed 1,276 freshmen at a 4-year institution and noted that 536 eventually dropped out. She reduced the sample to those students who were similar, that is who scored "within 5 points on both sides of the cutoff score" on a placement test which determined whether or not they were assigned to developmental mathematics courses. This resulted in n = 212 (p. 597). A statistical analysis showed that the risk of a student dropping out was significantly lower for similar students who did participate in developmental mathematics than it was for those who didn't participate in developmental mathematics. After the first year, those who did participate had "an estimated risk of dropout" of 8.2% (p. 601). For those who did not participate, the risk was 27.7%.

A study by Duranczyk and Higbee (2006) showed that students and colleges both benefit from developmental mathematics programs at 4-year institutions, as well as at 2 year institutions. They surveyed 20 and interviewed 18 people who had completed developmental coursework at "a comprehensive, urban, public university in the Midwest" (p. 24). One practical aspect of being required to take developmental courses at a different location is the inconvenience of working through the registration process of two different institutions. Some respondents also indicated that the convenience of being able to easily transfer from a course in which they were enrolled, but having difficulty, to one which would bolster their chances of success at the same institution was a factor in their eventual success (Duranczyk & Higbee, 2006). Table 3 summarizes some of these ideas related to the importance and benefits of adult developmental mathematics education.

Table 3

The importance and benefits of adult developmental mathematics education

Author	Year	Major ideas
Gerlaugh, Thompson, Boylan, & Davis	2007	The importance of developmental education in opening the door to opportunity

Author	Year	Major ideas
Lesik	2007	Drop-out rates among students enrolled in developmental mathematics lower than for similar students who were not so enrolled
Duranczyk and Higbee	2006	Benefits of having developmental mathematics available at 4-year institutions shown
Galbraith and Jones	2008	Describe challenges of teaching adults

Interested parties, including public policy groups and researchers, who have noted the role of mathematics as a gatekeeper to future success, have focused attention on developmental mathematics education and the feasibility and desirability of new strategies and innovations (Epper & Baker, 2009). Regarding those who teach developmental mathematics students, Galbraith and Jones (2008) have noted, "Teaching developmental mathematics in a community college is demanding, challenging, and rewarding for those who engage in the endeavor" (p. 35). Despite the effectiveness of developmental mathematics in many cases, the numbers cited by Trenholm (2006) are unacceptably high. Alternative educational methods for this population must be considered (Trenholm, 2006). In order to assess such methods, it’s necessary to understand the needs of adult learners. A look at some of the challenges they face follows.

Challenges faced by adult developmental mathematics students. Special needs of adult developmental mathematics students include understanding and navigating the educational system, accurately assessing their own needs, managing non-cognitive factors affecting their education, and access to knowledgeable teachers and other resources. Non-cognitive factors include influences unrelated to the student’s knowledge. They include factors such as self-efficacy and motivation, which have been shown by research to be important in developmental mathematics achievement. In their study, Wadsworth, Husman, Duggan, and Pennington (2007) defined self-efficacy as a person’s belief in their own ability to be effective in managing future situations. They looked at achievement along with measures of learning strategies and self-efficacy in 89 developmental mathematics students enrolled in an online course. They measured self-efficacy by asking students to rate their confidence in their ability to complete certain types of mathematics problems relevant to the course in which they were enrolled. They found that achievement was in part affected by self-efficacy and certain learning strategies (motivation, concentration, information processing, and self-testing) (Wadsworth et al., 2007). Obiekwe (2000) in his discussion of the instrument used by Wadsworth et al. noted that concentration was thought of as a student’s ability to give attention to an academic task. He described self-testing as a student’s ability to prepare for tests and classes, and information processing as a student’s ability to process knowledge. Wadsworth et al., (2007) noted that direct instruction, a term they seem to use to indicate in-classroom instruction with a teacher present, as a supplement to computer instruction can help students improve in these areas. For example, online course instructors can meet face to face with students and discuss with those students the learning strategies they will need in order to be successful.

In addition to low self-efficacy and a need for better learning strategies, research has identified some of the other factors that affect adult students. For example, those who are placed in remedial mathematics courses are disproportionately minority and first-generation college students, placing them at risk (Epper & Baker, 2009). Some have also found that a higher percentage of developmental mathematics students have learning disabilities (Epper & Baker, 2009). First-generation college students, recent immigrants, and students of color tend to be at a disadvantage because of their family's lack of understanding of the educational system and the implications for the future of non-college high school tracking (Collins, Bollman, Eaton, Otte, & Thomas, 2000). Those whose education includes a significant time gap between the completion of their last mathematics course in high school and the beginning of their first college course may find it difficult to be successful in their college mathematics course (Collins et al., 2000). Some who struggle in college mathematics may have difficulty correctly identifying what it is that is keeping them from succeeding (Hall & Ponton, 2005)

Standardized tests, state placement tests, institutional placement tests, and the student’s history of courses taken and grades earned all may be factors in students’ college mathematics placements (Collins et al., 2000). "Students may resist or feel insulted by” those placements (Collins et al., 2000, p. 37). In addition, the placement tests they may be required to take may not only be far removed from their last mathematics course, as has been noted, but also may not match the instruction they were given. For example a student who took a calculator based curriculum in high school may not be permitted the use of a calculator on the placement exam (Collins et al., 2000).

This discussion shows that adult developmental mathematics students may need assistance outside of class to address both academic and non-academic factors. Research indicates that non-cognitive factors, such as time management and motivation, influence developmental students' success, but non-cognitive assessment is infrequently employed (Gerlaugh et al., 2007). The time demands on adult developmental mathematics students may make it difficult for them to participate in enrichment programs designed to teach skills needed for success in college (Collins et al., 2000). Many adult developmental mathematics students have multiple responsibilities outside of school and value flexibility of delivery and readily available support services (Epper & Baker, 2009).

One of the challenges developmental education faces is that faculty have come from other fields and have not had professional training in dealing with developmental students. They may also not have had training in teaching with technology (Caverly et al., 2000). Though developmental mathematics instructors may all believe that mathematical understanding is their primary goal, they do not all think of that understanding in the same way. Many believe mathematical understanding to be procedural, consisting of an ability to perform a sequence of actions. Others believe it to imply conceptual knowledge, which has been described as knowledge that is part of a network of connections (Kinney & Kinney, 2002). Developmental mathematics teaching practices currently emphasize procedural fluency over conceptual understanding (Epper & Baker, 2009). Faculty members may not have had enough professional development opportunities. They are also under pressure to quickly do the job that was not done in high school, and may feel there is not time to teach for both concept and fluency. Some argue that teaching conceptually will aid in fluency. Those who seek to make improvements in courses may, nevertheless, reduce content or increase the number of courses in which the content will be covered. Some believe there is not enough time to teach what is required without the use of technology (Epper & Baker, 2009). Table 4 summarizes some of the needs of adult learners which have been discussed.

Table 4

Needs of Adult Learners

Author	Year	Key ideas
Wadsworthet al.	2007	Adult learners achievement affected by self-efficacy and learning strategies.
Epper & Baker	2009	Adult developmental populations are disproportionately minority and first-generation college students. They also have a higher percentage of learning disabilities as well as many responsibilities outside of school. Faculty members are under pressure to quickly do a job that was not done in high school
Collinset al.	2000	Family's lack of understanding of educational system has affected many adult developmental mathematics students. A time gap between high school and college may hinder their success. "Students may resist or feel insulted by" placement tests (p. 37) Placement tests may not match the instruction that they received
Hall & Ponton	2005	They may have difficult identifying what it is that is keeping them from succeeding

Author	Year	Key ideas
Gerlaugh et al.	2007	Non-cognitive factors such as time management are important but seldom assessed
Caverly et al.	2000	Faculty may have come from other fields and lack appropriate professional development

Technology use in adult developmental mathematics education. Technology found in use in developmental mathematics classrooms includes computer assisted instruction (CAI) software, internet sites, distance learning technology, computer algebra systems, graphing calculators, and spreadsheets (Epper & Baker, 2009). Almost one third of all institutions indicated that computers are frequently used by students as instructional tools in developmental mathematics (Parsad & Lewis, 2003). Such technology can be beneficial for developmental mathematics students.

The American Mathematical Association of Two-Year Colleges (AMATYC) has adopted the use of technology as one of its basic principles (AMATYC, 2006). Technology, however, can only truly transform developmental education when it is used to foster change in student behavior, so that students take control of their own learning and persist toward the successful accomplishment of their worthy goals (Brothen, 1998).

The following discussion will start with a look at issues particular to adult students using technology. As very few studies exist which pair adult learners specifically with the use of dynamic geometry or algebra software, I will present the reader with a closer look at studies involving CAI, which is the most dominant form of computer technology used with adult learners. The technology choices adult developmental mathematics instructors must make will then be examined.

Issues in technology use for adult students. Although it has been shown that the use of technology is critical to the success of developmental mathematics education, innovation has outpaced evaluation (Epper & Baker, 2009). There are also other challenges to implementation. Those working with developmental mathematics students must carefully consider the role of race/ethnicity and prior academic performance, both of which may have bearing on the choices those students make about the use of educational resources made available to them, including computer technology (Duranczyk, Goff, & Opitz, 2006).

One factor to consider when planning for the use of technology is the relatively low rate of computer ownership of students enrolled in developmental mathematics. Computer laboratories must be available and provide the features needed (Epper & Baker, 2009). For students to take advantage of the opportunities technology provides, "it must become a seamless part of the learning environment" (Epper & Baker, 2009, p. 9). Online learning faces the problems of student discipline, cost, and faculty acceptance (Epper & Baker, 2009). Instructors' perspectives may influence their decisions as to the use of technology in their classrooms (Kinney & Kinney, 2002). Epper and Baker (2009) reported tension between procedural fluency and conceptual understanding approaches in their review of practices in developmental mathematics education. This tension had implications for the use of technology (Epper & Baker, 2009). Some have noted that both fluency and conceptual understanding are vital, but curriculum content and sequencing questions have yet to be resolved. Many developmental mathematics texts and software packages possess the "mile wide and an inch deep . . . laundry list" quality that afflicts the U.S. K-12 curriculum (Epper & Baker, 2009, p. 5). More technological applications in developmental mathematics focus on procedural fluency than on conceptual understanding due in part to the current demands of the market (Epper & Baker, 2009). Table 5 summarizes some of the issues in technology use for adult students.

Table 5

Issues in technology use for adult students

Author	Year	Concerns noted
Epper & Baker	2009	Low rate of computer ownership among students enrolled in developmental mathematics
		Online learning hampered by student discipline, cost, and faculty acceptance
		Tensions between faculty beliefs regarding procedural fluency vs. conceptual understanding affect the use of technology
		Software packages may have the same defects that textbooks have, such as a "mile wide inch deep" curriculum or a focus on procedural over conceptual understanding
Duranczyk, Goff, & Opitz	2006	Race/ethnicity and prior academic performance may have a bearing on choices students make about their use of educational resources available, including technology
Kinney & Kinney	2002	Faculty perceptions affect decisions to use technology

Computer assisted instruction. Computer assisted instruction (CAI), which provides a tutoring supplement to or in some cases replaces classroom instruction, is found by some studies to be used in more that 40% of community colleges in the U.S. and is frequently referred to in developmental mathematics education literature (Epper & Baker, 2009). Some have referred to such computer use as computer-mediated learning, defining it to be learner centered computer intervention in which the computer provides the instruction, requires student responses, provides immediate feedback, and tracks students' progress (Kinney & Kinney, 2002). This review will consider computer assisted instruction, computer-mediated learning as well as online instruction to all be under the umbrella of CAI. An examination of the issues related to this particular form of technological intervention may assist in the effective implementation of other forms of technology as well. Since CAI is the predominant form which technology use in adult developmental education currently takes, an examination of its use may also point to reasons why other forms of technology use with adult developmental mathematics students, such as the one examined in the current study, should also be considered. Table 6 summarizes some of the advantages and disadvantages to the use of CAI which will be discussed.

Table 6

Advantages and disadvantages of computer assisted instruction (CAI)

Authors	Type of work	Advantages to the use of CAI noted	Disadvantages to the use of CAI noted
Caverly et al., 2000	Conference paper summarizing issues in technology and developmental education	Allows students to move on when they are ready to do so	May provide only superficial knowledge which cannot be applied in other situations

Authors	Type of work	Advantages to the use of CAI noted	Disadvantages to the use of CAI noted
Epper & Baker, 2009	Overview summarizing the issues involved in using technology to remediate adult mathematics students	Some improved results Allows an alternative to regular class meetings Some reduction in discrimination Certain types of CAI software may emphasize meaning	
Kinney & Kinney, 2002	Surveyed 11 instructors who had taught both developmental level mathematics courses using CAI and those which did not	Students control the pace of learning, receive more instruction, receive immediate feedback with detailed explanations, move more quickly, get more practice, and remain active during instructional time	Lack of discussion Only one way of thinking presented Students fail to ask for help when needed Students can't hear conversations with others Instructors had difficulty determining how deeply students were thinking

Authors	Type of work	Advantages to the use of CAI noted	Disadvantages to the use of CAI noted
Li & Edmonds, 2005	Three basic mathematics classrooms of at-risk adult learners were compared, two of which used CAI and one which was traditionally taught. Students were tested for knowledge and attitude.	Increased confidence level Increased satisfaction level Improved ability to transfer skills to classroom settings Bridges gaps in classroom instruction Helps meet needs of diverse learners	Low literacy skills of developmental students hinder their ability to make use of CAI
Qi & Polianskaia 2007	Examined enrollment, completion, and assessment data for traditional and CAI courses		Those in CAI courses did no better than their peers in traditional courses

Authors	Type of work	Advantages to the use of CAI noted	Disadvantages to the use of CAI noted
Taylor, 2008	Surveyed freshman enrolled in intermediate algebra courses, both traditional and CAI based (online delivery) to determine both attitude and achievement	Can decrease anxiety in some cases without lowering performance	
Trenholm, 2007	Surveyed online mathematics course faculty members to determine assessment practices		Questions as to proper assessment must be addressed when online delivery is used. 78% of online developmental instructors used proctored exams.

Some studies do show improved results for those who use CAI (Epper & Baker, 2009). Research has shown that CAI increases confidence levels and satisfaction, improves student ability to transfer skills learned online to classroom settings, bridges gaps in classroom instruction, and helps meet the needs of diverse learners (Li & Edmonds, 2005). Teachers in one study felt that the uses of CAI allowed students to control the pace of their learning, to choose the path that best met their needs, to receive more instruction through multi-media, to receive immediate feed back on their work with detailed explanations, to move more quickly and get more practice, and to remain active during their instructional time, rather than passive (Kinney & Kinney, 2002). It can also be advantageous for adult learners because technology which allows developmental mathematics students to find an alternative to regular class meetings allows them to manage their many responsibilities (Epper & Baker, 2009).

Kinney and Kinney (2002) surveyed 11 instructors with experience leading two types of courses, those using CAI and traditional lecture style courses. The topics they taught ranged from elementary algebra to college algebra. Though the results showed a slight preference for CAI courses, they also found some disadvantages to the use of CAI mentioned. Those disadvantages included lack of discussion, the presentation of only one way of thinking, student failure to ask for help when they needed it, lack of opportunity for students to hear instructors' conversations with others about topics they need to understand better, and the inability of the instructor to know how deeply the students were thinking (Kinney & Kinney, 2002). Some respondents wanted students to construct more of their own knowledge and gain greater conceptual understanding. They addressed this issue by incorporating writing into the class, such as daily checkpoint questions which briefly ask student to clarify a concept or justify a task, and learning logs, which require more in depth writing than a checkpoint question (Kinney & Kinney, 2002).

Some CAI may be presented through online course delivery. Taylor (2008) conducted a study in which she surveyed 54 freshman students enrolled in intermediate algebra courses using computer software and 39 students enrolled in traditional intermediate algebra courses. She administered pre-tests and post-tests for both mathematics achievement and mathematics anxiety ratings. Results showed that the use of web-based technology can in some cases decrease anxiety in developmental mathematics students without lowering performance. Some have reported that online delivery of instruction reduces discrimination (Epper & Baker, 2009). The visual image of the student in such cases is not a factor, and students who are reluctant to speak up in class may be less so in an online environment.

Although there are benefits to online delivery, there are also disadvantages. A study by Trenholm (2007) addressed the question of how learning outcomes for online mathematics courses could be effectively assessed. It considered what percentage of such courses are proctored, what the differences might be in proctored and unproctored courses, and what faculty of online courses considered important to their assessment practices. Three survey questions were sent to about 120 online mathematics course faculty members. The total final response rate was 39%. Six courses and ten categories of assessment were considered in the analysis of the data. Results showed that half of the two-year institutions that responded used proctoring, but none of the four-year institutes used proctoring. There were significant differences in the proctoring percentage by course, with a large percentage of developmental courses using proctoring (78%), more than half of calculus course using proctoring, but only 39% of General Algebra courses and only 8% of general liberal arts courses. Data is also given about proctoring for different types of assessments. Unproctored (100% online) courses tended to rely more heavily on formative assessment, that is, assessment designed primarily to provide constructive feedback to the student so that he or she may improve. The author concluded that at "this time, in math e-learning, it appears only some form of significant proctored summative assessment instrument will ensure that educational standards and integrity are preserved" (p. 53).

Other issues that have arisen with the use of CAI involved literacy and experience. Li and Edmonds noted that "[a]t-risk students with low literacy skills are hindered by their inability to comprehend written language . . . in a CAI environment" (Li & Edmonds, 2005, p. 162). In addition, if teachers have not experienced CAI, they have no experiential basis upon which to decide about its effectiveness (Kinney & Kinney, 2002). The capabilities of the software used, the physical resources available, the method of implementation, the amount and type of teacher/student contact, and the teacher's theoretical beliefs have all affected the implementation of CAI and could reasonably be expected to affect the implementation of other types of technology as well.

Teachers who wish to make effective use of CAI may provide visually appealing web materials, and be readily available to assist students. Teacher should ensure that the format is easy to navigate. They would also better serve students by helping them connect what they are learning to their life's goals (Li & Edmonds, 2005). Instructors incorporating CAI into their courses may decide not to lecture, when the software used provides what they feel are complete initial presentations of the material. They may instead provide clarifications, assistance, feedback, and study skills training. Those who do feel the need to lecture typically use direct instruction, supplemented by whole class discussions, and opportunities for students to practice individually or collaboratively (Kinney & Kinney, 2002).

Some schools have redesigned their programs with the help of CAI so that students only take the portions of the course in which they need remediation. The course is divided into modules and the modules are combined with CAI, giving different students different software assignments. Classroom instruction is focused on conceptual understanding and study skills (Epper & Baker, 2009). Different CAI programs have different capabilities. Cognitive Tutor is a

CAI program which emphasizes meaning and fluency, multiple representations, and formative assessment (Epper & Baker, 2009). Cognitive Tutor was developed by Carnegie Learning and is described as a research-based product incorporating the opportunity for students to work with multiple representations and to view examples which are intended to help build conceptual understanding ("Carnegie Learning," 2010). Epper and Baker (2009) reported that Pellissippi State Community College in Knoxville, Tennessee redesigned their developmental mathematics program to include the use of Cognitive Tutor in combination with classroom instruction, resulting in increased success over traditional teaching methods.

Programs that use CAI are common to developmental education. They may place students at a certain level, and allow them to move on when a test indicates that they are ready to do so (Caverly et al., 2000). Such programs have some advantages, however, the behaviorist model such programs typically follow only provides students with superficial levels of knowledge. Behaviorist models may provide a stimulus and response approach without regard to conceptual understanding. Students may be able to recall information, but not be able to apply it. When technology is used as a tool in a social constructivist setting, the student has a greater chance of reaching more complex levels of understanding (Caverly et al., 2000).

Qi and Polianskaia (2007) showed that the use of CAI does not necessarily increase performance. They examined enrollment, completion, and assessment data for traditional and computer-mediated course at a community college with a population of about 4,000 students. The computer-mediation was a self-paced multi-media environment called PLATO interactive mathematics (Plato Learning, 2004). Interactive conceptual presentation, immediate feedback, skills development, and online quizzes were all part of the software. It also provided teacher tools such as tracking for student progress and time on task. After a carefulness analysis of the data which compared completion rates, pass rates, and average scores for traditional and computer-mediated courses, the researchers found that those in the computer-mediated courses did no better than their peers in traditional courses (Qi & Polianskaia, 2007). Other choices aside from CAI should be available for developmental mathematics teaching and learning.

Helping instructors make technology choices. The issues surrounding the use of CAI illustrate the importance of the manner in which technology is incorporated into developmental mathematics courses. The use of technology in and of itself does not guarantee improvement in student performance (Qi & Polianskaia, 2007). Technology as it is being used may only be fostering superficial knowledge (Caverly et al., 2000). Attention to students' learning barriers is another important consideration developmental educators face (Caverly et al., 2000). Decisions as to what technology to use and how to use it must be carefully considered in order to best meet the needs of students. Educators must have the knowledge they need to make these choices. They must be familiar with the way each particular type of technology affects student thinking and learning, since the capabilities of software is one of the factors in its implementation. Some CAI use, for example, may hamper valuable mathematical communication and instructors may find it challenging to determine how deeply students are thinking (Kinney & Kinney, 2002). On the other hand, it has been shown that the use of dynamic interactive technology can, in some students, foster new understanding of mathematical concepts with which they have previously struggled (Li & Edmonds, 2005). It has also been shown that mathematical software which allows constructive explorations can help build higher levels of understanding. Allowing students to use technology which lets them create tools for other students strengthens this knowledge even more (Kaput, 1998).

Research that provides insight into the way in which particular technologies can affect student thinking may assist instructors in deciding the method of implementation of that software. This can help them meet the needs of a diverse population of students with widely varying social and cognitive needs. Providing greater research evidence is one of the factors that can assist in this process, and help educators realize the potential of technology to improve developmental mathematics instruction for all students (Epper & Baker, 2009). Developmental educators would most likely benefit from evidence that comes from studies which bring developmental mathematics students together with various types of technology, since a greater base of evidence is needed to help them meet the challenges they face (Epper & Baker, 2009). Such studies may help broaden the range of choice those educators have for their students. In order to examine students' thinking in such a setting, an understanding of the role of representation in mathematics education is necessary.

Representation in Mathematics Education

I will first consider how representation in mathematics education has been conceptualized, beginning with a look at representational systems and the associated idea of idiosyncratic representations (those which are unique to the learner) (Smith, 2003). Following this is a look at other constructs related to representation, including visualization (the creation of a mental image to guide the representation of ideas), and symbolization (the use of symbols to organize the mind and reflect thoughts) (Moreno-Armella et al., 2008; Presmeg, 2006). A look at something vital to the incorporation of representation in mathematics education, the use of multiple representations for the same concept will lead to an examination of modeling and functions. Modeling and functions are areas of mathematics in which the way representation is used is particularly crucial. The systems of representation with which a student engages, their visualizations, the connection of multiple representations, and the use of functions to model ideas are all important considerations in the examination of technology use undertaken in the current study.

Representation can be thought of as both the language of mathematics and the process of illuminating ideas (Coulombe & Berenson, 2001). That is, it can be thought of as both a process and a product. It refers to the act of representing an idea as well as to the form used for that act (NCTM, 2000). Thought of this way, it permeates all of mathematics. Its effective use is a way of both teaching and learning mathematics (Fennell & Rowan, 2001). Recent shifts in educational practice include a "heightened awareness of representation as a cognitive and social process" as well as an increased understanding of the vital link it forms to knowledge (Monk, 2003, p. 250).

According to Cuoco and Curcio (2001), a representation is a map of correspondence between a mathematical structure and a better understood structure. This map of correspondence preserves the structure of what is being represented (Cuoco & Curcio, 2001). It "re-presents" the ideas so that a solution to a problem may be found (Smith, 2003, p. 263). Representations facilitate reasoning and support different ways of thinking. They are the tools of proof and the heart of communication (NCTM, 2000). Their effective use can provide classroom experiences which can help students "see the beauty and excitement in mathematics" (Cuoco & Curcio, 2001, p. xiii).

Managing the long term process of conceptualization is more difficult in mathematics, which encompasses a large variety of situations, procedures, and symbols (Vernaud, 1998). This challenge makes understanding representation vital to mathematics education and representations become much more than just ends in themselves. They become essential to understanding, communication, justification, connections inside and outside of mathematics, and mathematical applications (NCTM, 2000). Those that have researched the area of representation in mathematics education have used a variety of different terms in their search to clarify its use. A more detailed discussion of some of these ideas follows. For the readers' convenience, Table 7 lists some of the major contributors to the study of mathematical representation in chronological order along with the major ideas contributed by the work listed. Some of those included were contributors to influential collected works.

Table 7

Some of the major contributors to the study of representation in mathematics education

Author	Year	Major ideas
Kaput[1]	1998	Functions are rich in representational possibilities The phenomenon being represented should be at the center of the study of functions. Theoretical framework also presented, including ideas related to systems of representation, notations, inscriptions, and language.
NCTM	2000	Presented and defined representation as a process standard for teaching mathematics. Ideas include the notion that representation is both a process and a product. It refers to the act of representing an idea and to the form used for that act.

[1] Part of a two volume special edition of the *Journal of Mathematical Behavior* which focused on representation and from which several other sources used herein were taken.

Author	Year	Major ideas
Yerushalmy and Shternberg[2]	2001	Described three representational phases involved in learning algebra: graphic (a drawing of a situation), iconic (sections of graphs seen holistically and used as icons), and symbolic
Fennell & Rowan	2001	Materials used to represent mathematical ideas may replace student thinking rather then represent it. Representations should not become ends in themselves.
Pape & Tchoshanov	2001	Four implications for the use of representations as cognitive tools rather then ends in themselves: opportunity to practice, socialization, variety of techniques, and use as tools for thinking, justifying and explaining.
Goldin[3]	2003	Described systems of representation and made suggestions as to how researchers can examine students' internal representations

[2] From the 2001 yearbook of the National Council of Teachers of Mathematics, entitled *The Roles of Representation in School Mathematics* edited by Albert A. Cuoco and Frances R Curcio. Several other sources used herein were also taken from this source.

[3] This, Monk (2003) and Smith 2003 are from *A research companion to principles and standards for school mathematics* edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter.

Author	Year	Major ideas
Monk	2003	Instead of being isolated items, graphs, charts, and other representations can be tools for building understanding of mathematics.
Smith	2003	Provided examples of and discussed students' idiosyncratic representations. Discussed the importance of helping students connect those representations to standard ones. Conversations with students about how they developed their representations can help teachers infer students' internal reasoning.
Abramovich & Norton	2006	"Residual mental power" (p. 11) means that representations developed during the use of technology continue to be useful to the learner in the absence of technology
Duval	2006	Connected semiotics to mathematics Semiotic representations are tools for producing knowledge
Presmeg	2006	Summary of research on visualization in mathematics education. Ideas include the notion that students may not have sufficient training with visual representations

Author	Year	Major ideas
Falcade, Laborde, & Mariotti	2007	Technology can allow access to more representations and impact how mathematical objects are conceptualized
		Semiotic mediation can allow new, internalized meanings to be developed for mathematical tools
Moreno-Armella, Hegedus, & Kaput	2008	Mathematical notations, experiences, and the medium that relates them co-evolve. Symbolic thinking has evolved over time from static notations to dynamic technological inscriptions.
Sedig, 2008	2008	Categorized representations into two broad categories, textual (descriptive) and visual (depictive). Textual representations are semantically dense, and conveyed through rules. Visual representations are more analogical.

Representational systems within which learners operate. To better understand the role of representation, it is helpful to consider the systems of representation within which the learner operates. Those systems are generally described as either external or internal (Goldin, 2003). The words external and internal refer to the relationship of that representation to the mind of the student. If the representation exists within the mind of the student, then it is an internal representation. If the representation is found in the environment outside of the student's mind, in a textbook, on a computer screen, or on a piece of paper for example, then it is considered to be an external representation. External or internal representational systems are influenced by other constructs such as the ideas of linguistics, affective notions, and habits of mind. An examination of how some of these ideas relate to each other follows. A diagram summarizing some of the ideas that will be discussed is provided in Figure 1.

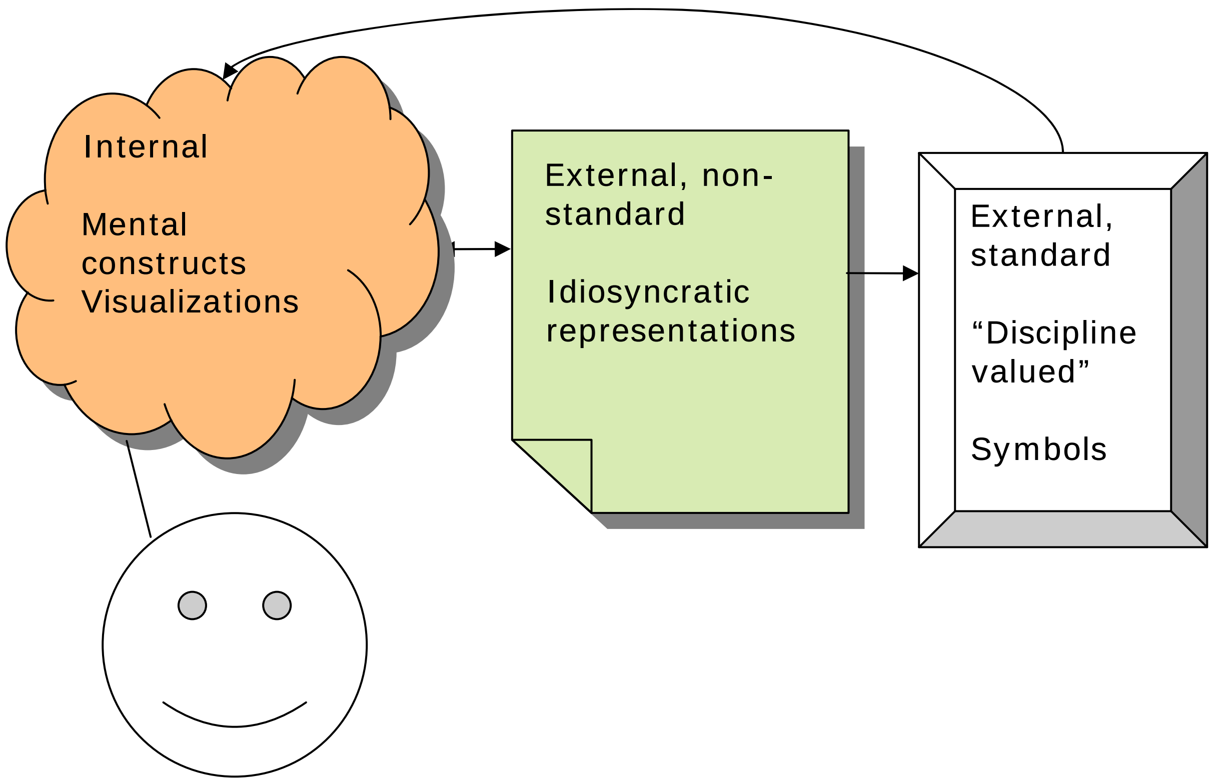

Figure 1. The interplay between internal and external representations.

In 1998, two volumes of the Journal of Mathematical Behavior addressed representation. The introduction to those two volumes, written by Goldin and Janvier (1998), described a system of representation as referring to any of the following four categories of concepts. First, it can refer to embodied representations of mathematical ideas which are external, physical situations in the environment. Second, they said that it may refer to linguistic representations in which the emphasis is on syntax and semantics. It may also refer to formal systems which use symbols, axioms, definitions, constructs, etc. Finally, it may refer to internal, individual systems which describe thinking processes and are inferred from behavior or introspection (Goldin & Janvier, 1998).

Some researchers have also referred to the affective domain as a representational system in and of itself. Goldin (2003) said:

> The affective domain refers to feelings that pertain to mathematics, to the experiencing of mathematics, or to oneself in relation to mathematics . . . affect serves a representational function in the individual and . . . as a representational system, it enhances or impedes mathematical understanding in certain ways. Local, changing states of feeling are not just experienced but utilized by problem solvers and learners to store information, to monitor, and to evoke heuristic processes (p. 280).

Frustration may signal the student to try a new strategy, signal that a problem is non-routine, and provide impetus for a more effective approach. It may signal the possibility of joy at meeting a challenge. In some students, frustration may invoke panic. Either way, it represents something about mathematics to that student, either motivational or discouraging (Goldin, 2003). Negative attitudes toward mathematics may be influenced by memories of past failures; interactions with peers, teachers, and parents; and exposure to teaching methods, certain types of mathematics, and certain learning environments (Sedig, 2008).

Representations occur externally in the physical environment or internally in the mind of the person doing mathematics (NCTM, 2000). External representational systems include conventional graphical and formal notational systems of mathematics, manipulatives, and computer-based representations (Goldin, 2003). Internal representational systems are personal to the learner and may consist of sensations, perceptions, imagined objects, or even emotional feelings. They also include visual imagery, spatial, tactile and kinesthetic representations along with students' personal conceptions and misconceptions. Each person forms their own internal representational system (Goldin, 2003).

Mathematics is more than a collection of results and conjectures; it is also a collection of habits of mind, such as those related to the learners' internal representational system (Cuoco & Goldenberg, 1996). Students should develop the ability to visualize, describe, and analyze situations mathematically (NCTM, 2000). One way that students understand a mathematical idea is through reification, the process by which something abstract becomes real to the learner and exists in his or her mind as a mental object (O'Callaghan, 1998). Students may, however, have difficulty understanding standard representations which may be second nature to the teacher (NCTM, 2000). Presmeg (2006) cited a study by Mourao (2002) which showed that students may carry prototypical visual images, such as the image of a parabola with two real roots, which are at odds with other, different but also valid visual images of the same concept. Such insufficient internal representation can lead to manipulation of external representations without attached meaning. Students with insufficient internal representations may cling to memorized rules, rather than learning with comprehension (Saul, 2001). Representations students choose themselves can help them understand and think through problems and bridge this gap of understanding (Fennell & Rowan, 2001). They must be given the opportunity to practice producing external representations and internalizing mathematical ideas (Pape & Tchoshanov, 2001). The internal and the external representational systems can then interact and represent each other's constructs in different situations (Goldin, 2003).

One of the challenges for researchers is that private representations and mental images are hard to describe (Cuoco & Curcio, 2001). Goldin (2003) noted that such research relies on observations of students' interactions with and production of external representations. He proposed task-based interviews and suggested ten principles for planning, structuring, and conducting them. Those principles include well-designed research questions and tasks, explicit interview protocols with contingency plans, encouraging free problem-solving, and maximizing interaction with the external environment through a variety of representational possibilities. Smith (2003) analyzed the student's creative process through their language and beliefs in order to see more than their external representations revealed. His conversations with them included a discussion of the development of their representations. He concluded that research into understanding how representations enable learning must proceed from inside the child (Smith, 2003). There is interplay between the external and internal representational systems; there are many mental constructs which come into play in a discussion of the internal representational system, and there are challenges involved in examining it.

Representations unique to the learner. As students work to connect the world of their own internal representational system and the concepts within it to new mathematical ideas, they may create their own unique, non-standard, idiosyncratic representations. These can help the student cross the conceptual bridge to an understanding of standard representations. The way students build representations is part of how they learn (Cuoco & Curcio, 2001). Mathematical representations allow students to organize, understand, and communicate mathematical ideas (NCTM, 2000). Monk (2003) has stated that “students . . . have surprising representational competence when their activity is within a coherent community with a sustained purpose” (p. 259). Helping students learn to choose the best representation wisely is vital. For example, teachers must be familiar with the strengths and weaknesses of different types of representations of a function, such as the quick reference aid a table provides and the global picture a graph gives the viewer (NCTM, 2000).

One way of helping students develop their representational competence and assisting them in learning to choose representations wisely is by incorporating the use of the students' idiosyncratic representations (NCTM, 2000). NCTM encourages the use of representations which are invented by and unique to the student (Smith, 2003). Students often generate nonstandard representations for successfully accomplishing tasks (Izsak & Sherin, 2003). They may also push the form of a representation beyond its original intentions (Monk, 2003). Nonstandard representations often serve better than standard ones as tools for understanding and communicating. They serve as bridges between a phenomenon and its standard mathematical representation (Monk, 2003). Such personal forms of representation, which may be very meaningful to the student, but have little resemblance to those commonly used, are an important first step in students' developing the ability to use representations wisely. They may also continue to be a tool with which students can reason and solve problems, even as their facility with standard representations develops (NCTM, 2000).

One way to help students learn to use conventional representations is to engage the class in whole group discourse about student generated representations (NCTM, 2000). Inventing representations and then reflecting on them is a way to help promote fluency in the use of graphs (Monk, 2003). Students' creation of idiosyncratic graphs, as opposed to standard ones, can be followed by group consolidation onto a common graph. This can lead to recognition of the need for standardization (Monk, 2003).

One goal in working with idiosyncratic representations is to help students make the connection between their unique representations and discipline-valued representations. This is necessary if they are to progress within the discipline of mathematics (Smith, 2003). Another is to make inferences about student understanding. Inferences can be made about students' ability to represent and understand mathematics by examining their language about their own mathematical representations together with their attitudes towards their mathematics learning (Smith, 2003). Some believe that the use of external representations of mathematical concepts to develop student understanding is directly related to students' ability to visualize with those representations (Pape & Tchoshanov, 2001). The ability to visualize would indicate that those representations are part of their internal representational system (Goldin, 2003).

Idiosyncratic representations can be a valuable tool for the student. Operating with them externally can lead to important classroom discourse and recognition of the need for standardization. They can also help the teacher make inferences about student understanding. Used externally, as they bridge to more conventional representations, they and the conventional representations they lead to can both become tools with which the student can visualize mathematics.

The role of visualization and imagery. Visualization is one of the constructs with which researchers examine the field of representation, and it is an important consideration in examining what internal representations students may possess. Visualizations and imagery may be categorized in different ways as seen in the research discussed below.

Some researchers categorize representations broadly as either textual (descriptive) or visual (depictive). Textual refers to those representations which are like language, semantically dense, and conveyed through rules. Visual representations, as opposed to textual, are more analogical in nature (Sedig, 2008). Smith (2003) cited Goldin and Kaput (1996) as stating that imagistic representations include internal imagery which may sometimes demonstrate meaning through visualization, analogy, or metaphor.

Presmeg (2006) used the term visualization to describe the creation of a visual image (mental construct) which guides the creation of mathematical inscriptions (representations). She referred to those who prefer to use visual methods as visualizers. Monk (2003) described seeing as a constructed activity, closely linked to thoughts and actions. How people see things is linked to how they think about them. Changes in the way students see things can be fostered. For example, many visual displays can be used, rather than a single visual representation, such as a traditional graph (Monk, 2003).

Presmeg (2006) listed five types of imagery used by high school learners: concrete imagery, kinesthetic imagery, dynamic imagery, memory images, and pattern imagery. Concrete imagery is a picture in the mind. Kinesthetic imagery refers to physical movement. Dynamic imagery occurs when the image itself is moved or transformed. Memory images are those which allow learners to recall formulas. Pattern imagery is "pure relationships stripped of concrete details" (Presmeg, 2006, p. 210). Different categories of imagery may overlap. Visualization is a powerful tool in algebra as well as geometry and trigonometry, however, students may not have sufficient training with visual representations. For example, dynamic imagery was shown in one study to be used effectively, but rarely, by high school students (Presmeg, 2006). Textual, visual, concrete, kinesthetic, dynamic, memory, and pattern images have all been proposed as possible ways in which students visualize mathematics. They can be considered as ideas with which to examine a student's internal representational system.

Symbolization as a construct related to representation. While visualization is directly related to a student's internal representational system, external representations are more closely related to the idea of symbolization. External representations may be idiosyncratic, but more often they employ standard mathematical symbols. The idea of symbolization is another lens through which researchers have examined representations. An examination of research shows that symbols have been discussed as representing concepts, organizing the human mind, fostering the sharing of ideas, providing a link to abstract ideas, and becoming a new internalized tool with which the student can build new knowledge.

Idiosyncratic, internal, and external representations each may provide different instances of one idea. A unifying expression for a set of multiple instances requires some sort of symbolic structure (Moreno-Armella et al., 2008). Symbols represent items from a reference field. For example, one reference field for the set of nouns is the set of material objects. Reference fields grow and transform with the shared use of symbols (Moreno-Armella et al., 2008). Symbolic structures re-design the architecture of the human mind and provide a meta-cognitive mirror in which our thought is reflected (Moreno-Armella et al., 2008).

Educators sometimes refer to such a process as symbolization. Each mathematical representation stands for some mathematical concept. In the process of symbolization, symbols and referents are sometimes experienced as separate items and sometimes experienced as the same thing (Kaput, Blanton, & Moreno, 2008). Symbolizations can be privately constructed and used by one individual, as with students' idiosyncratic representations. They can also be shared by a community, as with commonly used external representational systems, which may be the product of a long process of refinement (Kaput et al., 2008).

Symbols may sometimes be treated as objects in their own right without regard to the referent for which they stand. Whitehead (1929), as quoted by Kaput, Blanton, and Moreno (2008) stated, "Civilization advances by extending the number of important operations we can perform without thinking about them" (p. 22). Students may operate on symbols by following rules and algorithms. Using this method of operation, students may, indeed, be acting upon the marks on the paper without understanding. They may act based on the position of the symbols, not based on their meaning. On the other hand, writing which is not based on rules for manipulating symbols may be used to develop an idea or build an argument. In such instances, students may draw diagrams, for example. If so, what they write is not determined by "strict syntactically defined rules" (Kaput et al., 2008, p. 24).

Symbolization cannot be separated from conceptualization. Rather than promoting the lack of thought, it has the potential to provide a "lift-off" from concrete thinking (Kaput et al., 2008, p. 23). Algebraic symbolization historically gave rise to the understanding of new ideas such as negative and complex numbers. Through symbolization, "new mathematical worlds become possible" to students - one possible reason why algebra serves as a gateway to further mathematical development (Kaput et al., 2008, p. 23). The manner in which mathematical discourse and mathematical objects interact is a creative and continuous process, involving symbolization, which has occurred throughout history. It continues to occur in classrooms as well as in the minds of individuals (Kaput et al., 2008).

Symbols also serve as signs. Semiotics is the study of signs and their meanings. Cunningham (1992) described semiotics as "a way of thinking about the mind, and how we come to know and communicate knowledge" (p. 166). He also noted that it has an "ecumenical nature" in that it "draws from" and "informs" many other disciplines (Cunningham, 1992, pp. 166-167). He argued that "knowledge does not exist separate from the knower" and that "knowing can't be anything but personal knowledge" (Cunningham, 1992). This is not to say that reality does not exist, but that our understanding of it is constructed, and there are limitations on those constructions determined in part by the existing structure of our thoughts. Though different individuals' constructions of understanding may differ, similarities across individual constructions may reveal something about reality.

Representations can be considered signs, and their associations are produced according to a set of rules, allowing the description of a system. In this system, the semiotic representations become tools for producing new knowledge, not just for communicating a certain internal representation (Duval, 2006). Semiotic mediation refers to the use of a semiotic system or tool in social interaction so that new signs are generated to foster internalization of meanings. Under proper guidance, new meanings related to the use of a tool can be formed and developed (Falcade et al., 2007).

Internalization occurs through systems of signs and semiotic processes. A tool used under expert guidance to accomplish a task functions as a semiotic mediator as new signs are derived from the actions performed with the tool. This fosters an internalization process, producing a new internal tool. The internal tool may resemble in some respects the actual external tool and new meanings may be generated related to the use of the tool (Falcade et al., 2007). This research gives us greater insight into the role played by those symbols normally thought of as forming the body of tools for representing mathematics. It is possible to see that they can take on added meaning, producing new knowledge as well as communicating what is already known.

Connecting multiple representations. It was noted in the discussion of symbolization that seeking a unifying way to express multiple instances of the same idea may lead to the development of symbols. In a similar manner, those who encounter multiple symbols or representations for the same idea may find their understanding of that idea deepening. Students who learn to move flexibly among and choose from a variety of representations to solve problems will also be deepening their understanding of mathematics.

One way students demonstrate conceptual understanding is by their ability to move from one representation to another and coherently use different representations (Even, 1998; Hitt, 1998; O'Callaghan, 1998). As the number of representational tools a student uses expands, he or she will need to move flexibly among different representations in order to view mathematical ideas from the different perspectives those tools offer. This ability will enhance their mathematical power (NCTM, 2000). Those who are able to switch representations can use the representation most beneficial to their analysis (Even, 1998). Mathematically proficient students productively choose from a selection of possible representations in order to solve problems (NCTM, 2000). In addition, the combined use of different types of representations has the potential to cancel disadvantages each type may have. They may provide greater information when used together (Friedlander & Tabach, 2001). Monk (2003) asserted:

> The goal is not to select one or two representational forms for students to learn and use in all situations but, rather, to teach students to adapt representations to a particular context and purpose and even to use several representations at the same time (p. 260).

The use of multiple diverse representations has been shown to be important to the understanding of students at higher levels of mathematics (Santos-Trigo, 2002). Instead of being isolated items, graphs, charts, and other representations can be tools for building understanding of mathematics (Monk, 2003).

Students may resist transitioning between different representations (Friedlander & Tabach, 2001). To help encourage them, tasks can be designed which promote the use of multiple representations. Such tasks may promote frequent transitions between representations, and the use of different representations becomes a "natural need" rather than an "arbitrary requirement" (Friedlander & Tabach, 2001, p. 176). An environment where teachers and students are presenting in different representations encourages flexibility in the choice of representations. Students may be started with problems requiring the use of a specific representation and then later be asked more open-ended questions (Friedlander & Tabach, 2001). As teachers incorporate multiple representations into their classrooms, student thinking will emerge and their erroneous ideas can be addressed (Hitt, 1998). Teachers help students progress from seeing representations as ends in themselves to the act of representing by engaging them in ongoing discussions of the reasons for choosing one kind of representation over another (Monk, 2003).

There is potential vulnerability in making conceptual and representational connections (Hitt, 1998). Two representations may be mathematically equivalent but cognitively non-equivalent in that they are processed differently by different learners. Presenting ideas in ways that convey the most meaning to the most students will bring maximum benefit. This may mean using such visual cues as graphic indicators of number size, the use of proximity and color to distinguish related items, and the placement of related representations so they can all be seen at once (Gadanidis et al., 2004).

As students grow and learn about mathematics their use of representations grows from directly perceived objects and actions, to indirectly perceived items such as rational numbers, and eventually to abstract ideas such as functions (NCTM, 2000). Abstraction in mathematics is the stripping away of features not necessary for analysis. By facilitating abstraction, representational ability aids students in identifying common underlying mathematical structures which appear in different settings. It also allows them to examine essential features of problems and their mathematical relationships. Rich representations allow students to examine many aspects of this process (NCTM, 2000). It has been said that the student use of multiple representations in working with the same mathematical object is what constitutes ideal mathematics learning (Hsieh & Lin, 2008). Choosing, adapting, transitioning, communicating, abstracting, and justifying are all aspects of the use of representations for the study of mathematics. Each of these habits has the potential to emerge in the presence of multiple representations.

Uses of representations: Models and functions. Two fruitful and closely related areas for the study of how representations affect mathematics are modeling and functions. Models and functions both can be used to illustrate real-life phenomenon mathematically and both regularly involve the use of multiple representations. A look at models as a form of representations will be followed by a look at representational ideas associated with functions.

Models as representations. One form of study in the field of mathematics which employs representation as a problem solving tool is modeling. Models link different ideas together; particularly they link concepts outside of mathematics to mathematics. The process of modeling helps students to notice underlying mathematical structures in the world around them, builds the idea of isomorphism (a one to one relationship between two sets of data preserving operations within the two sets), broadens their understanding of what it means to represent something, and deepens their understanding of mathematics (Abrams, 2001).

A mathematical model is a form of representation which illustrates mathematical features of a complex phenomenon and is used to clarify situations and solve problems (NCTM, 2000). Models have the potential to provide an important service, since cognitive knowledge is closely linked with the knowledge people have of a situation being represented (Monk, 2003). Traditional word problems have usually involved the use of a specific formula or algorithm, and an easily detectable list of data to be used. This is different from authentic problem solving using modeling tools (Yerushalmy & Shternberg, 2001).

Abrams (2001) defined modeling as the process of studying questions outside of mathematics with mathematics. When used for modeling real life situations, mathematics serves as an "intellectual lens" for examining questions (Abrams, 2001, p. 269). In this sense it is not self-contained, but used as a tool in other disciplines, as well as for abstract discoveries. When mathematical modeling is ignored, some skills, including choosing appropriate representations for a situation, and recognizing common structures are neglected (Abrams, 2001). As students' understanding and use of representations develops and becomes more sophisticated they can learn to use variables, tables, equations, and graphs to model and analyze real life phenomenon (NCTM, 2000). Modeling can be supportive of emerging representations of functions (Yerushalmy & Shternberg, 2001).

Modeling involves examining two ideas with matching structures, which builds the concept of isomorphism (Abrams, 2001). It maps the real-life situation to its mathematical model. The modeling cycle consists of posing a question, selecting the representation(s), creating a model, manipulating the model, determining mathematical products, translating (interpreting mathematical results according to the setting), deriving new knowledge, and analyzing the results. Analysis leads back to the question until a sufficient model is created (Abrams, 2001). As part of the selection and creation of the representation, a real situation may first be idealized into a pseudo-concrete model before it is further abstracted into a mathematical model (Presmeg, 2006) .

If students are allowed to interpret familiar events mathematically, then they can understand representation more deeply (Coulombe & Berenson, 2001). They can also more easily access problems which can be represented in ways that are meaningful to them (Fennell & Rowan, 2001). Some students may, in the process of representing a real life situation, strip the context away while others may use the context. Stripping the context away seems in some cases to have helped the student avoid confusion (Smith, 2003). A deeper interpretation of mathematics based on familiar events and problem solving can broaden students' understanding of conventional representations beyond mere manipulation (Coulombe & Berenson, 2001). The process of modeling helps develop the mathematical lens through which a student views the world. That modeling process may take several steps through different levels of abstraction. Representations, both idiosyncratic and standard, can be part of that process along the way. Through modeling, students can improve their facility with representational tools.

Function as a context for studying representation. Functions are often used as a mathematical context for studies of representation because they are rich in representational possibilities. For example, the encouragement to regularly immerse students in mathematical experiences which involve an interplay of symbolic, numerical, and graphical forms of representation is commonly known as the rule of three (Reinford, 1998). The rule of three was meant to encourage students to take facts they see in graphs and verify them numerically and algebraically (Bridger & Bridger, 2001). Functions can be represented all three ways. Function is also considered by many to be the most important concept in all of mathematics and fundamental to the learning of mathematics (Hitt, 1998; O'Callaghan, 1998). The concept of function has been important in the study of students' obstacles with regard to representations (Goldin, 2003). The ability to interpret and translate representations can help students construct their mental images of patterns and functions and thus extend their algebraic thinking (Coulombe & Berenson, 2001).

Even (1998) referred to two ways participants in one study dealt with functions: pointwise or globally. To deal with functions in a pointwise way is to plot, read or deal with discrete points. To deal with the function in a global way is to look at its overall behavior, such as when students sketch the graph of a function and look at its maximums and minimums and other characteristics. This study suggested that those who can easily use a global analysis of changes in the graphic representation of a function have a better understanding of the relationships between graphic and symbolic representations than those that check local characteristics (Even, 1998).

When the student is able to understand the different representations of functions, those representations can serve as windows into functional relationships in particular situations (Lloyd & Wilson, 1998). Representations of functions then become valuable tools for modeling real life situations. Students having a good concept of function will be aided in solving problems (Yerushalmy & Shternberg, 2001).

Numerical representations are some of the first representations that students encounter, and provide some of their first experiences forming internal representations (Pape & Tchoshanov, 2001). At first, children may not understand that when they are counting, the last word they say represents the number of things they have counted all together. Eventually the name of the number comes to represent a set of that many objects (Pape & Tchoshanov, 2001). Later uses of numerical representations are familiar and convenient, but lack generality. This limits some of their problem solving potential (Friedlander & Tabach, 2001). It is not directly evident that when teachers use manipulatives to represent number concepts, such as base ten blocks used for learning regrouping, that students see the connection between the manipulatives and the mathematical activities they are intended to represent. Some believe that the use of such external representations of numbers to develop a student's understanding of mathematics is directly related to the student's ability to visualize with those representations (Pape & Tchoshanov, 2001). Algebraic representations have many advantages, such as conciseness, generality, and effective modeling. Sometimes, they provide the only avenue of proof. The problem comes when their exclusive use interferes with the conceptual understanding of what they represent (Friedlander & Tabach, 2001).

Verbal representations assist in understanding context, communicating results, solving problems, and working with patterns (Friedlander & Tabach, 2001). They emphasize connections to other domains of study. Verbal representations can, however, suffer from ambiguity and thus they can possibly become obstacles to communication (Friedlander & Tabach, 2001). Oral and written language as tools of communication can be considered forms of mathematical representation (Coulombe & Berenson, 2001).

A graph can be thought of as a lens through which to explore the phenomenon graphed. Learning to read graphs is referred to by some researchers as 'disciplined perception' (Monk, 2003). Graphs connect formal static definitions of function with the metaphor of motion (Falcade et al., 2007). One of the complexities in graphing as a representation is that a graph has many potential meanings. Those who are fluent graph readers can forget the difficulties others have (Monk, 2003). Inappropriate responses to visual attributes of a graph are the most frequently cited student errors. Some mathematics educators have begun to focus on the process by which a learner constructs meaning from a graphical representation in addition to focusing on the information itself (Monk, 2003).

A graph does not reach its full potential until it is used to make meaning. Monk (2003) suggested a variety of meaning making processes to aid students in making meaning from graphs, which can be transferred to the use of technological representations. Students can explore aspects of the situation graphed that were not otherwise apparent. The process of representing a context can lead to questions about it. Graphing and analyzing a well understood concept can help them understand graphing better. Important features of a graph help them construct new concepts. Understanding of the graph and the context can build at the same time. Finally, a group can build shared understanding through the common use of a graph as a window into a phenomenon (Monk, 2003).

Another approach to the study of functions in addition to the traditional Cartesian plane is to view them via mapping diagrams (Bridger & Bridger, 2001). Bridger and Bridger's (2001) use of function mapping diagrams is taken from the science of map projection, which renders an image of a globe on a flat surface. A function mapping diagram renders the x and y axes as parallel lines, with line segments connecting a point on the x-axis to its image on the y-axis. Different representations of functions have different advantages and disadvantages (Bridger & Bridger, 2001). Cartesian graphs are more useful for examining extrema, convexity, and asymptotes. Advocates of the function mapping approach feel that traditional graphs may inhibit the development of a mapping concept of functions (Bridger & Bridger, 2001). They point out that, in addition to promoting the concept of function as a mapping, they also show whether the function is an expansion or a contraction, and where and how it is one-to-one or many-to-one. They are also excellent for visualizing compositions and inverses of functions (Bridger & Bridger, 2001). Both models and functions, which are closely related, overlapping mathematical ideas, give us ample opportunity to consider the effect of representation on student understanding. In examining that understanding it will be helpful to consider an interpretive framework.

Building validity, usefulness, and endurance. One of the ideas discussed in regard to representation was the concept of a student's internal representation, which incorporates the idea of conceptualizing and symbolizing. In considering the relationship between a student's internal representational system and their success in learning mathematics, I posit that it is useful to organize key ideas into three constructs: validity, usefulness, and endurance. These constructs, summarized in Table 8, form an interpretive framework with which the effect of technology on student learning can be examined.

Table 8

Interpretive framework for ideas related to representation

Representations are		
Valid if they	Useful if they	Enduring if they
Accurately reflect the mathematics they seek to represent and are flexible enough to allow additional mathematical ideas to be built upon them. Are accompanied by sound mathematical habits of mind.	Are accessible for reasoning and sense-making, communication of mathematical ideas, and building new understanding.	Remain with the student in various situations apart from the environment in which they were initially developed. Are carried forward, built upon, and refined over a period of time.

Valid internal representations. Validity in research allows that research to be correctly interpreted (Gay, 1996). In mathematics, a representation may be considered valid if it accurately represents the mathematics it seeks to represent and is flexible enough to allow additional mathematical ideas to be built upon it. Not all internal representations are mathematically valid. Research has shown that students may hold prototypical images of mathematical concepts, but that these images may limit their thinking and force them to rely on memorization, providing very little true comprehension (Saul, 2001). One such prototypical image is the parabola with two real roots, which, when fixed in the student's mind, impedes the idea that a quadratic equation may have just one real root or none at all (Presmeg, 2006). This concern is related to the thoughts expressed by Rogers (1999) that simplified diagrams make take away from the deeper learning that other representations may provide. Prototypical images which provide no true comprehension are not accurately representing the mathematics and simplified methods which short-cut deeper understanding are too inflexible to build upon.

Part of the validity of mathematical representations is situated in the mathematical habits of mind which accompany them and which assist the student in translating them into other valid representations (Cuoco & Goldenberg, 1996; Even, 1998). The parabola with two real roots may be invalid for a student who has no accompanying habits of mind allowing the switch to one or zero real roots. It may, however, be valid for another student who does possess those habits of mind. In addition to this type of flexibility, mathematically valid representations do not merely represent each other, they also represent contextual situations or reified concepts (Kaput, 1998; O'Callaghan, 1998). Such conceptualizations indicate that the mathematics is real to the learner and accompanied by understanding (O'Callaghan, 1998).

Useful internal representations. Representations are useful when they can be selected and applied to maximize problem solving, build new knowledge, and communicate mathematical ideas to others. Students demonstrate mathematical proficiency and power when they can use representations in these ways. The representations they hold are not useful to them if they cannot communicate and solve problems with them (NCTM, 2000). Note, for example, that there is an important difference in the learning process when a teacher uses representations as part of a dynamic, active process, which facilitates sense-making as opposed to making explicit the representations with which they wish their students to solve problems. The former involves students in the act of re-presenting to themselves prior mathematical activities in ways crucial to the knowledge they are currently constructing (Cifarelli, 1998). This sense-making process requires useful representations as tools for students. They are available as what Cifarelli (1998) called “interpretive tools of understanding” (p. 241). They are general enough in the mind of the student so that their problem solving potential is not limited (Friedlander & Tabach, 2001). Using a problem based approach in teaching can give students the opportunity to construct and interpret graphs, generate data, find patterns, and interpret mathematical ideas in other ways (Coulombe & Berenson, 2001). This can allow the teacher to discover something about the usefulness of the student’s internal representations. Koedinger and Nathan (2004) referred to the process of using familiar representations to build new ones as “grounding” (p. 158). Useful representations allow students to ground their new understanding in that which they already know. They will also be available for students so that they can organize, record, and communicate mathematical ideas (Fennell & Rowan, 2001). Useful representations may be non-standard, but non-standard representations often serve well as tools for understanding or communicating (Monk, 2003). Useful internal representations provide students with mathematical power, which includes the ability to reason, communicate, discover, conjecture and connect mathematics within itself and outside itself (NCTM, 1991).

Enduring internal representations. Internal representations may be considered enduring if they stay with the student in various situations apart from the environment in which they were developed and are carried forward to later work in which they are deepened and built upon as the student progresses mathematically. Abramovich and Norton (2006) referred to representations which may develop during the use of technology, but continue to be useful to the learner in the absence of technology as providing “residual mental power” (p. 11). In their study, the use of a locus approach for solving problems with parameters, studied with the use of technological representations, resulted in the locus becoming a familiar mathematical thinking device (Abramovich & Norton, 2006). The variety of symbols used in mathematics makes long term conceptualization challenging for some students (Vernaud, 1998). A vivid and meaningful representation can be held onto, built upon, and refined over a period of time within an individual, thus aiding long-term conceptualization (Kaput, Blanton, and Moreno, 2008). Enduring representations become part of what Rogers referred to as "stored knowledge" which is part of the critical process of knowledge integration and reasoning (Rogers, 1999).

Cifarelli (1998) noted that representation has a constructive function which involves the development of mental objects which students can reflect on and transform. Enduring representations will be stored and available to the student for future work and remain part of the set of mental objects upon which they reflect in order to build new knowledge.

Building valid, useful, and enduring internal representations of mathematics within students will provide them with clarity of understanding, power in problem solving and communication, and a continually developing storehouse of mathematical knowledge. Considering these representational ideas will provide a framework with which to examine the interplay between technology and student learning. Following is a closer examination of the interplay between technology and representation.

The Connections Between Technology and Representation

The connections between technology and the use of mathematical representations are many. Technology transforms the possibilities present in mathematical representations (Moreno-Armella et al., 2008). It can allow students to gain a better understanding of the use of representations (Maximo & Ceballos, 2004). Kaput (1998) believed that technology had the potential to build grounded understanding of mathematical ideas so that different representations are doing more than just representing each other.

Among the ways technology and the use of mathematical representations influence each other is the ability of technology to link multiple representations of the same phenomenon (Hennessy et al., 2001). Technology also allows a connection between real-life phenomenon and the representations which depict them (Stylianou et al., 2005). In addition, technology can allow students to access a variety of different types of representations and develop a deeper understanding of representations (Alagic & Palenz, 2006). Technology allows students to use representations to explore problems that were previously inaccessible, such as the modeling of complex real-life phenomenon and the examination of multiple changes in mathematical parameters (NCTM, 2000). Technology also allows a direct link between a real life phenomenon and its associated representations through the use of calculator based laboratories. In this way, using computer technology provides for bidirectional interactions (the phenomenon affects the representations and this affects the phenomenon), and this exchange can become rapid, allowing student hypotheses to be quickly tested (Kaput, 1998).

Technology increases the opportunity to analyze multiple, connected representations (Yerushalmy & Shternberg, 2001). Heid and Blume (2008) noted technology's increasing sophistication and and "multirepresentational capability" (p. 58). Technology can connect mathematics with visual representations in such a way as to foster mathematical thinking and conceptual understanding (Lopez Jr, 2001). Through the promotion of multiple representations, technology imparts the advantage of flexibility to visual reasoning (Friedlander & Tabach, 2001; Presmeg, 2006). Graphing calculators can mediate students' problem solving by providing seamless switching between symbolic, graphical, and numerical representations (Hennessyet al., 2001).

Microsoft Excel's "ability to integrate multiple representations" helps students think of things in different ways. Students must also think carefully about the role of variables which is vital to their understanding of algebra (Donovan II, 2006). Interactive diagrams can allow multiple representations to be combined and changed in relationship to each other. Rogers (1999) called this process "dynalinking" (p. 423). When a computer simulation of a real-life setting was linked to an interfaced diagram showing aspects of that setting, students were able to learn more from working with the interfaced technological diagram than they had learned with static textbook diagrams. Their ability to reason with the abstract diagrams had improved (Rogers 1999). In the same way, through the use of technology graphs have become manipulable, whereas they were previously seen as static, fixed entities (Kaput, 1998).

Technology allows students access to representational ideas that otherwise might be difficult to share or visualize, such as a depiction of the movement of two variables at the same time (Falcade et al., 2007). Technology may allow students to make predictions about what a representation is telling them and then test their conjecture, allowing them to build increasing understanding of that representation (Hegedus & Kaput, 2004). Students may use technology to create different representations, leading them to make their own mathematical conjectures (Santos-Trigo, 2002).

Technology can also assist students in developing greater conceptual understanding of representational ideas such as algebraic symbolization (Abramovich & Ehrlich, 2007). For example, they are often are able to solve equations with no conceptual understanding of what they are doing, but that lack of understanding is a much greater handicap when solving inequalities. Technology can allow them to examine a series of comparisons which can give them insight into the effects of different choices of algebraic manipulation on the solution set of inequalities. This can give them a conceptual underpinning to what otherwise might be an ungrounded set of rules and procedures (Abramovich & Ehrlich, 2007).

In considering these connections and the use of technology in the classroom to enhance the use of mathematical representations, teachers may wish to have certain things in mind. For example, having students produce their own representations of what is occurring within the technological representation may be important in making explicit the implicit knowledge that teachers may take for granted (Hennessy et al., 2001). Knowledge called for with the use of technological representations includes an understanding of the method of input and mathematical interpretation of output associated with the formats the technology uses. This may include notation which varies from the notation seen in textbooks. The use of such technology in the classroom requires the teacher to carefully consider the role of representation. (NCTM, 2000). Following is an examination of several studies which examine the use of both technology and mathematical representations. Table 9 provides a summary of the some of the major studies discussed.

Table 9

A selection of studies in technology and representation

Aspect of technology	Researchers	What was done	What they found
Using relations to look at algebra from a geometric perspspective	Abramovich & Norton (2006) Abramovich & Erlich (2007)	Pre-service teachers were engaged with graphing software which allowed relations from any two-variable equation to graphed	2006: Participants better understood connections between geometry and algebra 2007: Visualization was fostered which provided conceptual insight.
The use of spreadsheets as cognitive tools	Alagic & Palenz (2006)	Used Microsoft Excel for professional development with middle school mathematics teachers. Teachers explored a real-world problem using multiple representations.	Teacher were able to explore many ideas in a small amount of time

Aspect of technology	Researchers	What was done	What they found
The use of spreadsheets as cognitive tools	Hsieh & Lin (2008)	Teaching experiment involving 8 sessions with three fifth grade students who needed mathematical remediation	Linked representations produced greater progress in understanding and knowledge was transferred to internal representations
The use of object oriented programming to build self-efficacy	Connell (1998)	52 caucasian lower-middle class elementary school students in student centered classrooms were taught object oriented programming language for mathematics – one class for presentation by the teacher and for exploration	Students benefitted from using the computer as a reflective tool which reacts to student input in a way that encourages accuracy
	Stevens, To, Harris, & Dwyer (2008)	Gifted children worked with LOGO computer software	Gifted children working with LOGO increased in creativity and verbal mathematical ability

Aspect of technology	Researchers	What was done	What they found
Exploring connections with graphing calculators and dynamic software	Hennessy, Fung, and Scanlon (2001)	Examined adults working with each other on activities involving graphing calculators	Advantages: speed, visualization, movement among representations, need to clarify ideas, pairs problem solving mediated Disadvantages: assessing mechanical vs. conceptual knowledge Paper and pencil use encouraged

Aspect of technology	Researchers	What was done	What they found
Exploring connections with graphing calculators and dynamic software	Santos-Trigo (2002)	25 first year university students in a calculus class for which calculators and dynamic computer software were available were interviewed and written reports were gathered related to three tasks	A visual approach helped the students understand the nature of the roots of equations. They were able to make connections between representations.
	Yerushalmy & Sternberg (20010	Created Function Sketcher software for use with seventh grade students to take them through algebraic, graphic, iconic, and symbolic phases to develop the concept of a function	Students could see how to translate a real-life event into a graphical representation and a graphical representation into a symbolic one

Aspect of technology	Researchers	What was done	What they found
Linked whole group study	Hegedus & Kaput, 2004	MathWorlds software used in teaching experiments in middle and high school classrooms in which each students or groups work could be uploaded and chosen for display to the class	Displaying student work helped focus attention on the underlying mathematical structure. Students felt personally connected to the work presented
The use of dynamic geometry environments to deepen thinking	Cuoco & Goldenberg, 1996	Provided samples of tasks from studies done in computational and dynamic geometry environments	Mathematical objects seemed to become real and the object of experimentation
	Falcade, Laboorde, and Mariotti, 2007	Used the trace tool in a dynamic geometry environment to introduce the concept of functions by looking at the covariation of dragged objects	The teacher's role is important in helping students make meaning. Students' work can form the basis for discussion

Recent Studies in Technology and Representation

Recent studies which focus on both technology and representation are numerous and varied in their approach. Following is a brief look at several studies conducted over the past ten years. The topics examined by these studies include the development of software to re-examine algebra through the graphing of relations, using spreadsheets as cognitive tools, the use of object oriented programming to develop logical thinking, looking at dynamic connections between representational forms with calculators and computers, the use of data-collecting laboratory tools, the use of technology to link a classroom of students in whole group study of their work, and the use of dynamic geometry software to deepen mathematical thinking.

Using relations to look at algebra from a geometric perspective. Abramovich's two studies, one with Norton (2006) and one with Erlich (2007) demonstrated the potential of technology to assist students in gaining a deeper understanding of inequalities. His work provided insight into errors through the use of visualization made possible through the use of graphing technology. Graphing software was developed which was able to graph a relation from any two-variable equation, as opposed to the standard calculator technology which requires the equation to be solved for the dependent variable. Both studies were done with pre-service teachers. The 2006 study allowed the participants to better understand the use of geometric ideas to understand algebraic relationships. The 2007 study showed that the graphing technology fostered visualization which gave the participants conceptual insight. They were able to see how algebraic manipulation affected the solution set of an inequality.

The use of spreadsheets as cognitive tools. Alagic and Palenz (2006) used Microsoft Excel (Excel) in a professional development setting involving middle school mathematics teachers. Their study focused on the development of conceptual understanding which can come from the use of multiple representations, using spreadsheets as cognitive tools. Their professional development model combined the immersion of teachers in the exploration of real-world problems and the connection of those activities to their classroom work. The teachers examined a problem involving exponential data from a story problem (The King's Chessboard). They looked at the data in tabular form, graphed it, and zoomed in and out to see how the appearance of the graph changed when the window was changed. The ability of teachers to distinguish between exponential and linear growth was increased as a result. The teachers then created their own stories and activities for their students. The technology allowed the teachers to explore a variety of instances of a mathematical idea in a smaller amount of time. Hsieh and Lin (2008) also used Excel for a study involving multiple representations. They conducted eight sessions of a teaching experiment with three fifth grade students who needed remediation in mathematics. Their subjects had no difficulty with reading, but did have difficulty decoding textual material related to mathematics. The Excel based lessons the researchers provided included textual, numerical, and graphical representations of word problems and provided students with instant feedback with which they could observe changes in representations resulting from their choices. The researchers found that such representations, when linked, resulted in greater progress in understanding. The students' knowledge was transferred to internal representations which allowed them to solve new problems (cf. p. 230 #6).

The use of object oriented programming to build self-efficacy. Connell (1998) discussed the use of an object oriented computer authoring language to create personally meaningful representations by means of computer based tools in a constructivist environment. Connell (1998) noted that the object oriented qualities of the software used in his study allowed the students to use powerful graphic tools such as drawing implements in a setting requiring relatively simple syntaxes to create their programs. 52 predominantly Caucasian lower-middle class elementary school students in two rural elementary school classrooms were the subjects of this study. Their teachers had been observed using constructivist methods (student centered, facilitative, encouraging problem solving) prior to the study's beginning and had had a year's experience with the materials used in the study. Though the technology was used in both classrooms, it was only used for student exploration in one of the classrooms. In the other it was used as a presentation tool. Students working with the technology in this study could create their own personal representations and tools, which required them to write programming scripts which would perform as desired. Results showed that the students benefited from computer use which goes beyond "delivery of static information" and acts as a reflective tool which reacts to student input in a way that encourages mathematical accuracy.

One report looked at the effect of study with LOGO computer software on seventh grade teachers' self-efficacy and self-determination (Stevens, To, Harris, & Dwyer, 2008). LOGO is computer software providing a graphics programming language which allows the student to write directions for programming activities and receive immediate visual feedback. It was chosen for its ability to take basic concepts to more complex levels. The problem solving process involved in using LOGO can offer a "window into the student's mind" (Stevens et al., 2008, p. 199). The student must apply logical reasoning to their programming. Researchers have recommended that the use of such software be facilitated by teachers who can help students make connections with mathematical ideas and recognize their own thinking processes. It has also been suggested that such software can help students learn to work through challenges with more confidence. Failure becomes an "opportunity to plan a new course of action" (Stevens et al., 2008, p. 200). Gifted children working with LOGO were shown to increase in creativity, and in verbal domains of mathematics. The teachers in this project appeared to be encouraged to implement technology into their classrooms and work through obstacles that might arise as they attempted to do so.

In a related effort Kynigos, Psycharis, and Moustaki (2010) conducted an experiment with eight 17 year old students studying mechanical engineering in a secondary technical and vocational school. The students were instructed in the use of MoPiX, a computer software environment in which formal algebraic structures were used to manipulate animated models of real-life situations. MoPiX provides the user with a library of equations which use verbal terms to identify the purpose of the equation, such as "greenColour(ME, t) = 100." This equation would make object "ME", the object to which the equation is assigned, 100% green at time t. Since t was not used in the equation in this instance, the color of the object would not have changed over time and the object would have remained 100% green. Other library equations were provided which allowed attributes to change over time. Two researchers, one a teacher at the school for several years, observed, circulated among, and questioned students as they worked in groups of two or three. The researchers also conducted whole class discussions. Students appeared to build connections between the formal equations and the behavior of the objects. It seemed that the equations became "tools for controlling and creating animated models", not just as they were given in the library, but as refined or newly constructed by the students (Kynigos et al., 2010).

Exploring connections with graphing calculators and dynamic software. Hennessy, Fung, and Scanlon (2001) examined adult students working with each other on activities involving graphing calculators. Advantages they noted in the use of the calculators included speed, visualization, seamless movement between representations, external reference to facilitate discourse, helping to make thinking explicit, and encouragement to clarify ideas. The seamless movement mediated the pairs’ problem solving. The use of the calculator also encouraged them to make their thinking explicit to each other and clarify their ideas. Disadvantages included the need to assess mechanical vs. conceptual knowledge produced, and the researchers encouraged the continued use of paper and pencil techniques to accompany the use of technology. Lopez (2001) study, in discussing problems used as part of an algebra in-service workshop funded by Casio, noted that used graphing calculators can be used "as a visualization tool to make connections between mathematical concepts" (p. 116). One such activity asked participants to draw the "golden arches" using functions. Another asked them to draw the outline of a stealth bomber. By working such problems, questions about the graphs involved force participants to "restructure their knowledge and connect it to the drawing that they are trying to display" (p. 118). Santos-Trigo (2002) used a series of tasks for which 25 first year university students taking a course in calculus had graphing calculators and dynamic computer software available. Data was gathered by means of student interviews and written reports. Three tasks were chosen to demonstrate different features of student interaction with mathematics that emerged. The first task involved examining a representation of quadratic equations as points in the plane, a quadratic of the form $y = x^2 + bx + c$ being represented by the point (b, c). The visual approach helped students understand the nature of the roots of such equations. This new type of representation showed the importance of looking at mathematics from new perspectives. The technology students used for the series of tasks assisted them in exploring connections among different representations.

Yerushalmy and Sternberg (2001) used their Function Sketcher software to take seventh grade students through three phases of learning algebra, graphic, iconic, and symbolic, in the development of the concept of function. The software was able to receive mouse input and allow students to draw on the xy plane. It also provided a "stair step" view which could be added to the graph to show the rates of change for various changes in x and made available sections of graphs with different characteristics, such as increasing slope, concavity, etc. It provided technological tools which allowed students to experiment with objects used in representations of functions. The continuous mouse input gave them a graphical representation. The graph sections tools were iconic in nature and the stair step view gave rise to symbolic representations. Students were able to see how to translate a real-life event into a graphical representation and how to use graphical representations to build symbolic ones.

Lapp and John (2009) examined the ways in which pre-service teachers' mathematical choices and conceptual understanding were affected by the use of "dynamically connected representations" (p. 37). The technology used was a prototype of Texas Instruments' (2006) TI-Nspire CAS™. The pre-service teachers were able to observe patterns that they probably would not have been able to see very easily without the use of technology. The researchers' hope was that experiencing technology as learners would encourage the pre-service teachers to be more likely to foster a student centered learning environment (Lapp & John, 2009).

Using technological laboratories to connect to real-life phenomenon. Microcomputer based-laboratories (MBL), calculator based laboratories (CBL), and calculator based rangers (CBR) allow the student a way to enter function information other than by equation. They also allow for different types of visual analysis than conventional representations (Yerushalmy & Shternberg, 2001). Students can study their own movement and discover relationships between the associated numerical, graphical and symbolic representations (Stylianou et al., 2005). Advantages of CBLs and MBLs include: multiple modalities, real events paired with their symbolic representations, scientific experience, elimination of mathematical drudgery, and the encouragement of collaboration. Appropriate curriculum is needed in order for CBL and MBL activities to be successful (Lapp & Cyrus, 2000)

The use of MBL activities was also examined by Hegedus and Kaput (2004), who showed how their computer software, SimCalc MathWorlds could be used in a linked classroom environment to promote discussion and engagement. The SimCalc project was based in part on the idea that the phenomenon being graphed is itself a fourth representation (the first three being equation, table, and graph.) An MBL could be used in conjunction with the MathWorlds software as a connection to the phenomenon being graphed. Some students need physical action as the fourth representation in order to understand the relationship of the graph to the real-world phenomenon and to begin to move flexibly among representations (Stylianou et al., 2005). A person's motion can serve as a semiotic embodiment when it is mathematical and facilitates the understanding of mathematical symbolism. Re-enacting the motion becomes an executable representation (Moreno-Armella et al., 2008).

SimCalc MathWorlds had the ability to take input from an MBL or to display an animated image of a virtual actor moving in the way the graph described. Kaput (1998) referred to the movement of the virtual actor as a "cybernetic phenomenon" (p. 273). He felt that the cybernetic or physical phenomenon should be at the center of the network of representations and that the other representations be used to understand that phenomenon.

Lapp and Cyrus (2000), when observing high school students working with CBR's at a Mathematics, Physics, and Advanced Technology Exploration Day found that students did not understand the graphical information during the activity, demonstrating common misconceptions. "To connect graphs with physical concepts, students need to see a variety of graphs representing different physical events" (p. 3-4). They described advantages of CBL's as including: multiple learning modalities addressed, real events paired with their symbolic representations, scientific experience gained, elimination of mathematical drudgery, and encouragement of collaboration.

Stylianou, Smith, and Kaput (2005) used CBRs with pre-service elementary teachers to help them develop mathematical understanding. 28 preservice teachers attending a mathematics course for elementary school teachers participated in a two week study in which they worked in groups of four or five. CBRs were used in conjuction with calculators equipped with MathWorlds software to allow the user to see the graph of the motion captured by the CBR and to replay that motion (Stylianouet al., 2005). Questions were asked to determine the participants' pre-existing understanding of graphs of motion. It was assumed that participants knew nothing about CBRs or graphing calculators and they were introduced to both of them. Students were asked to complete a task using the CBRs to collect data and represent mathematics in motion to help them understand a position graph. Pre and post tests were given. The researchers found that pre service teachers gained mathematical and pedagogical insights on graphs of functions when working with the CBR devices (Stylianouet al., 2005). Mathematical insights included facing their own misconceptions, realizing that graphs can be manipulated to allow for different views and arguments, and using graphs as a means for mathematical communication. Pedagogical insights included recognizing the value of building on students' kinesthetic experiences, recognizing the need to link concrete experience to symbolic representations of that experience, differentiating between local and global interpretation of graphs as tools for arguments and recognizing the need to provide learning environments that allow for discussion and communication about graphs (Stylianouet al., 2005).

Linked whole group study. The SimCalc project's goal was to democratize access to higher mathematical ideas (Hegedus & Kaput, 2004). Recent work added to their previous studies a look at the potential of hand held wireless devices linked to larger computers. They saw classroom connectivity (CC) as critical because of its potential to impact communication in the everyday classroom even more so than internet connectivity. Technology can serve to be more than just a medium for individuals, and become a "medium in which teaching and learning are instantiated in the social space of the classroom" (Hegedus & Kaput, 2004, p. 130). They saw this type of technology as aiding an epistemological shift in which technologically assisted mathematical learning situations evolve from participatory simulations to joint constructions of knowledge (Hegedus & Kaput, 2004).

Teaching experiments were conducted in middle and high school classrooms in Massachusetts and California. The project described the grouping of the students by two numbers (group number, count off number). The problem $y = 2x + b$, with "b" being the group number was assigned to each group. Each student could produce the function on their own device. The student's work was uploaded to the teacher, aggregated, chosen for display and discussed. Graphs for students in the same group should have overlapped and their animated objects should have moved alongside each other (Hegedus & Kaput, 2004). Organizing and displaying student work helped focus attention on the underlying mathematical structure. When using such an activity, before animating, asking what the race will look like will help students think about what the mathematical representation is telling them (Hegedus & Kaput, 2004). Triangulated data from pre and post test measures, video records, and field notes indicated that participating students' algebraic thinking improved.

Assigning the students a group and count-off number and asking that those numbers be incorporated into the function they are displaying gives students a personal connection to the work, as they look to see how their construction fits in with those of other people (Hegedus & Kaput, 2004). Combining dynamic representation with connectivity can help students understand important algebraic concepts (Hegedus & Kaput, 2004). When dynamic mathematics software is combined with digital networks, students' individual mathematical objects can interact in meaningful ways. Students can share mathematical experiences (Moreno-Armella et al., 2008).

The use of dynamic geometry environments to deepen thinking. Dynamic computer technology provides representations of mathematics which have not existed previously in the external environment (Moreno-Armella et al., 2008). Cuoco and Goldenberg (1996) provided examples of tasks from studies which had been done in both computational environments and dynamic geometry environments which showed how students can build mathematical habits of mind. Because computer environments perform exactly the instructions they are given, students using them must think about essential mathematical features. One activity used a dynamic geometry environment to examine a geometric construction to see how one segment changed when other features of the construction was changed. The purpose of the example was to show that technology can lead students to a style of thinking. Conjectures arose and were examined further using the technology. The mathematical objects became real and became the subject of experimentation. One concern was to determine whether or not the students were investigating the mathematics or the properties of the software. They believed that when students were engaged in the construction of the experiment, they would be more likely to feel that they were experimenting directly with mathematical objects (Cuoco and Goldenberg, 1996). .

Falcade, Laborde, and Mariotti (2007) used the trace tool in a dynamic geometry environment to introduce students in four 10th grade classes (15-16 year old students, two classes in France, two classes in Italy) to the concept of function by looking at covariation qualitatively. Considering functions from a standpoint of covariation means that variations in the independent variable (or input) and dependent variable (or output) are considered together. The dragging tool and trace tool were used in conjunction so that the user could experience and observe the combination of motions as an example of covariation. Students worked in pairs, but wrote individually about what they had learned. The writings were done in a setting detached from that in which the technological study took place. The students were also engaged in whole class discussion, which was important to the process of making meaning of what had been done, particularly in finding the search for a definition of function as it emerged from their technological work. The teacher redirected the discussion to the main objective, prompted the intervention of a student, repeated students' comments, pushed the discussion in important mathematical directions based on student input, tried to involve non-participating students, and orchestrated the formation of mathematical meaning using the contributions of students (Falcadeet al., 2007). Evidence highlighted the importance of the teacher's role in helping them to make meaning from what they were doing. Students needed the help of teacher facilitated discussion in moving from the technological experiences to a mathematical definition of function based on those experiences, however the ideas on which the discussion were based emerged from the students' work in the technological environment (Falcade et al., 2007).

This review has looked at several different ways that researchers have combined the study of technology and representation in mathematics education. A theoretical framework for the study will now be considered.

Theoretical framework: Constructivism

I have chosen to approach this study using a theoretical framework of constructivism. Constructivism emerged as a dominant theory in mathematics education in the 1980's (Lambdin & Walcott, 2007). Based on ideas of Piaget and Vygotsky, constructivism encouraged educators to create an atmosphere where students could work through cognitive conflict using their own strategies, and thus learning via problem solving (Lambdin & Walcott, 2007). Students then have the opportunity to construct their own knowledge (Silver, 1990).

In addition to offering an account of student learning, constructivism is also an epistemology and a research methodology (Ernest, 1998). Radical constructivism includes the ideas that knowledge is not passively received and that all knowledge is constructed and reveals nothing which can be applied with certainty to the world at large. Many researchers take issue with the second idea, believing that we "inhabit a knowable external reality" (Ernest, 1998, p. 29). Social constructivism includes the idea that the social dimension of students' worlds affects their learning, and that the knowledge constructed by the student is in response to socially situated experiences (Ernest, 1998). Social constructivism includes Vygotsky's zone of proximal development (ZPD), which refers to problems that a student may not be able to solve alone yet, but that they can solve with just a little assistance, such as a facilitating question, or a hint (Norton & D'Ambrosio, 2008). Norton and D'Ambrosio (2008) noted that Steffe defined a different zone, the zone of potential construction (ZPC). He defined ZPC as the changes students might make in their own understanding during or following mathematical interactions. The teacher considers what he or she knows about the student's current way of thinking and considers what might be changed about or added to that understanding. In addition to being a way of looking at student learning, constructivism can also be seen in the work of the researcher who constructs theory through the interpretation of data collected from subjects (Mills, Bonner, & Francis, 2006).

Schwandt (2007) has said that in order to make sense of constructivism, it is important to note what is being constructed. Constructivist frameworks allow me to make some sense of students' interactions with technology. Creating an atmosphere were students can work through cognitive conflict, observing the resulting changes students' make to their understanding, and constructing theory through interpretation of data allow me to see and understand more about their thinking and the internal representations they may be constructing. Such observations are possible because mathematical representation is a dynamic, active process, in which students re-present to themselves prior mathematical activities in ways crucial to the knowledge they are constructing (Cifarelli, 1998). Monk (2003) described seeing as a constructed activity, closely linked to thoughts and actions. Connell (1998) showed that students created personally meaningful representations using computer based tools for solving problems.

Ernest (1998) noted that constructivist methods must be approached cautiously, with the understanding that "there is no 'royal road' to knowledge" (p. 31). In addition, attention must be given to beliefs, conceptions, language, and shared meanings of the subject and researcher (Ernest, 1998). He contended that research methodology set in a constructivist epistemology be conducted with caution and humility. Researchers may interpret the actions and language of others but must remember that those others have their own realities. Qualitative researchers seek to understand the realities of others in company with their own, acknowledging that such realities can never be assumed to be fixed. The researcher is never external to the analysis (Ernest, 1998). In keeping with these cautions, issues of validity were important to this study and will be discussed in detail in chapter three.

Conclusions and Questions

Constructivist frameworks are being used in this study to examine issues surrounding the use of technology in mathematics education. Of particular concern are the internal representations which adult developmental mathematics students may develop though interactions with technology. There are both advantages and challenges to the use of technology in mathematics education. With the rapid pace of its development, research is continually needed in order to provide teachers of all ages with information to assist them in making wise choices (Atan et al., 2008; Hollenbeck & Fey, 2009). The issues surrounding adult developmental mathematics students are symptomatic of those challenges. Their presence in the educational system points to gaps in their learning. The combination of academic and personal challenges they face require adult developmental mathematics teachers to make careful choices about their use of educational resources (Qi & Polianskaia, 2007). Figure 2 summarizes ideas found in the literature and illustrates forces affecting adult learners. Technology has the potential to advance student learning, but there are challenges to its use. Research is needed into how it can be used beneficially.

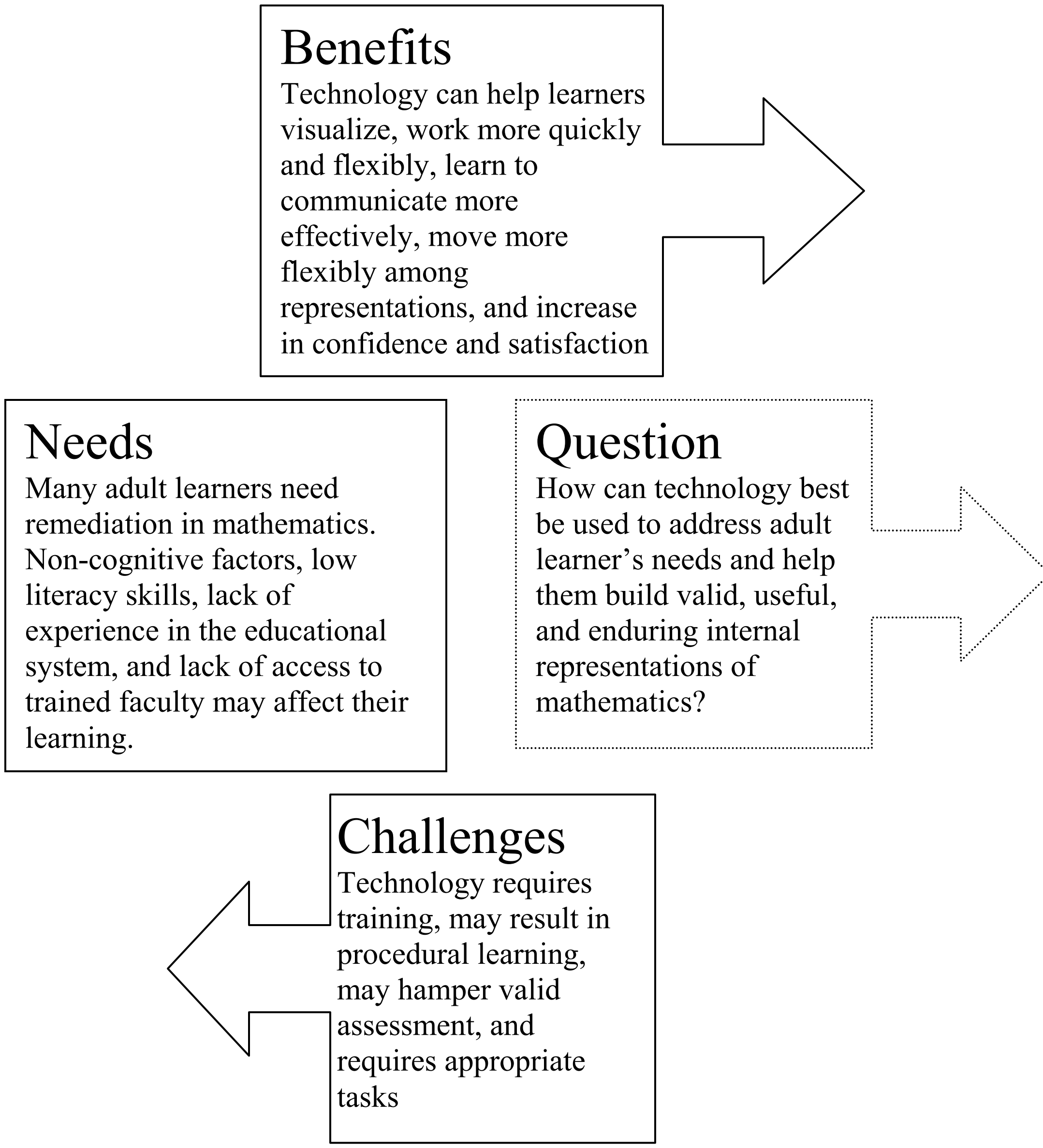

Figure 2: This diagram summarizes some of the issues adult learners face, benefits and challenges related to the use of technology, and the question this study seeks to address.

Technology has a transforming influence on the role of representation in school mathematics. Decisions about the choice of mathematics technology to use in the classroom are strengthened by an understanding of the role of mathematical representations in student learning. Technology can transform and add to the representations that are available to students. It can add to a student's ability to access, explore, analyze, and connect representations (National Council of Teachers of Mathematics, 2000; Yerushalmy & Shternberg, 2001). For the internal representations that students acquire through the use of technology to benefit them, those representations must be valid, useful, and enduring. The following research questions form the basis of the present study and address the issue of improving adult developmental mathematics students' learning through the informed use of technology

1. Following the introductory use of dynamic computer technology to explore mathematical concepts built upon previous knowledge, what internal representations of those concepts do developmental mathematics students possess?
2. What can be determined about the validity and usefulness of those representations?
3. How well do those representations endure over a period of time and in the company of tasks which build upon them?

This study was conducted to provide research-based evidence for developmental educators by allowing inquiry based learning to take place in the presence of technology, documenting students' thinking and learning in that setting, introducing students to accessible resources, and focusing on mathematics which they are responsible for learning. Developmental mathematics students need the increased opportunity that such a study provides. The following chapter will provide details as to the specific methodology used.

3. Methodology

This qualitative study was designed to gather knowledge about the effect of technological representations on developmental mathematics students' understanding of functions. A sequence of teaching interviews with developmental mathematics students was conducted, recorded, transcribed, and analyzed. It was an exploratory case study, conducted with an eye toward suggesting theory. Such theory generation does not require a large number of cases, since the suggestion of new theory, rather than the proof or verification of existing theory, is the goal (Glaser & Strauss, 1967). Further studies, some of which may take place over long periods of time, can build upon the theoretical suggestions which have arisen. Glaser and Strauss (1967) described theory as "an ever-developing entity" rather than "a perfected product" (p. 32). I chose qualitative research as the most viable method for an in depth examination of student thinking, and a teaching experiment as the most fruitful format in which such research could take place. I also chose to examine the emerging data using ideas from grounded theory, so that unexpected learning could be more readily and carefully examined. In order to provide a foundation for these choices, a look at qualitative research, qualitative research in mathematics education, and teaching experiments will begin the chapter. Grounded theory as an approach to data analysis will then be examined. This is followed by a look at what was learned from a pilot study. A look at the specific procedures used for the present study will follow, as will a look at my stance as a researcher, and an examination of issues of reliability of validity.

Theoretical foundations for qualitative research

It is helpful when considering the specifics of a study's methodology to consider some of the ideas from which those methods arise. A look at qualitative research in general will be followed by an examination of qualitative research in mathematics education. The specific ideas related to the conduct of teaching experiments as a form of qualitative research in mathematics education will then be examined. A look at the relationship of case studies to teaching experiments and tasks for teaching experiments will conclude this section.

Qualitative inquiry and foundations of thought. Qualitative data helps researchers understand underlying relationships (Pandit, 1996). An observed situation is conceptualized, specifics of the situation singled out within an analytical framework, a representational system is used to analyze those aspects, the analysis is interpreted and inferences are made about the original situation (Schoenfeld, 2007). It is an appropriate method to use when investigating internal representations, such as mental imageries (Presmeg, 2006). Important contributors to the field of qualitative research have been Eisner (1998) who discussed the idea of connoisseurship, and Lincoln and Guba (1985) who described the nature of naturalistic inquiry. The term qualitative has the advantage of encompassing many forms of human activity. It has also become part of educational discourse. Rather than coining new terms, Eisner (1998) chose to refine the discourse. As part of that refinement he referred to "qualitative inquiry", and indicated his belief that the word inquiry had a broader application than the words "research" or "evaluation" (Eisner, 1998, p. 6). His major focus was on educational connoisseurship and educational criticism. Both types of analysis focus on *qualities* and though they are commonly considered in relation to art, they can be applied to "educational phenomena" (p. 6). Connoisseurship requires high levels of "qualitative intelligence" (p. 64). Our knowledge about a situation influences our perception of it. When observing in a classroom for example, observers' perceptions would change depending on their knowledge of the teacher's level of experience. They would think differently of a first year teacher than they would of a veteran, though otherwise the situations in which those teachers were working were the same. Knowledge about a situation can provide a window, but it can also prove a hindrance. Labels and theories, for example, can promote expectations which get in the way of perception. "[A] way of seeing is also a way of not seeing" (Eisner, 1998, p. 67). For example, in order to combat that very idea, Edwards (1979) had her drawing students turn the photograph they were copying upside down so that prior knowledge of what they thought a subject should look like would not interfere with their current observations.

It is also helpful to consider the foundations of thought motivating methods of qualitative inquiry. Lincoln and Guba (1985) described three historical eras of thought. The division of those eras is centered in the emergence of positivism in the early nineteenth century, a movement which can be characterized by the belief that the scientific method can be used to study diverse topics and provide generalizable, exact, objective knowledge about those subjects. Before that emergence, the prepositivist era was characterized by passive observation, rather than active hands-on inquiry. In seeking to introduce the post-positivist era, characterized by a naturalist paradigm, Lincoln and Guba (1985) described the emergence of non-Euclidean geometry. They concluded, among other things, that "[d]ifferent axiom systems have different utilities depending on the phenomena to which they are applied" (p. 36). Euclidean geometry and Non-euclidean geometry are each preferred in different situations. In a similar way, a post-positivist or naturalist paradigm allows the researcher to understand aspects of knowledge that positivism inadequately addresses. Lincoln and Guba (1985) summarized the naturalist paradigm through five axioms, which they contrasted with positivist ideas. Those five axioms include the following ideas. There are multiple, holistic realities. The knower is inseparable from the known. Only idiographic (case specific) statements are possible, rather than universal generalizations. It is impossible to sort causes from effects because of mutual interdependence, and research is bound by values, including those of the researcher, the theories they employ, and the context within which they work (Lincoln & Guba, 1985). The present study will assume a post-positivist paradigm, noting that the research being conducted here is case specific, affected by the views and values of the researcher, and limited by the knowledge the researcher brought to the study regarding existing theories applicable to the research.

Qualitative research in mathematics education. Quantitative research methods have occasionally demonstrated some important relationships in mathematics education, but have rarely explained those relationships. Mathematics education researchers have turned increasingly to qualitative research (Silver & Herbst, 2007). The increase in qualitative research in mathematics education has raised the demand for theory on which to base such research. Theory can mediate the bidirectional relationships between problems, research and practices (Silver & Herbst, 2007). It mediates between problems and research by giving meaning to results of research studies, providing a lens with which to look at data, or providing a tool to describe a body of research. It mediates between research and practice by prescribing what educational practices should be like, helping researchers understand observed practices, providing language to describe practices, explaining causes for practices, and predicting aspects of practice. It mediates between practice and problems by providing a proposed solution to a problem, by establishing criteria by which different instances of problems can be compared, by identifying different types of problems, by identifying aspects of practice that pose or contribute to information about problems, by helping design new practices, and by helping justify choices in addressing problems (Silver & Herbst, 2007). Any suggested theory which may arise from the present study will be considered in light of such mediation. The specific approach taken in the present search for theory is a teaching experiment.

Specific approach: Teaching experiment. The decision as to the type of approach to be used in research is influenced by the goals of that research (Creswell, 2007). The goals of research in this case are to examine the effects of technological representations on developmental mathematics students' internal representations of functions. This requires the introduction of subjects to a technological mathematics environment and a deeply developed examination of their thinking. A teaching experiment provides a way to both introduce a teaching tool and examine student thinking. Researchers have used teaching experiments to investigate student thinking (Steele, 2008). As Lesh and Kelly (2000) have stated, teaching experiments can be used when it is desired that insight be gained into processes that bring student thinking from one state of knowledge to another. They allow conditions to be created which optimize the chance that change will occur while leaving open the direction in which the student's knowledge can develop (Lesh & Kelly, 2000). In their work examining the influence of technology, Falcade, Laborde, and Mariotti (2007) also noted that a teaching experiment is appropriate for introducing students to a mathematical concept.

Exploring student thinking. The ideas of valid, useful, and enduring internal representations developed from a review of the literature for this research study and were carried into the study as part of the a priori coding of events which would transpire there. A teaching experiment would be a fitting setting in which to explore such ideas. Enduring representations are those which stay with the student in various situations, are carried over to later work, deepened and built upon as the student progresses mathematically. The teaching experiment can be helpful in examining enduring representations, since it adds to the clinical interview by examining progress made over a period of time as opposed to just looking at the subject's current knowledge (Steffe & Thompson, 2000).

Valid representations have been defined by the researcher as those that accurately represent the mathematics they seek to represent. Useful representations are those which can be selected and applied to maximize problem solving, build new knowledge, and communicate mathematical ideas to others. A teaching experiment is useful for examining these two constructs, because of the deep exploration of the subject's thoughts and the development of those thoughts in various circumstances (Nemirovsky & Noble, 1997; Steffe & Thompson, 2000).

As the teacher-researcher observes the subject in a teaching experiment, attempts on the part of the student to resolve a mathematical problem may confirm the student's mathematical reality, showing how they think (Steffe & Thompson, 2000). This ability of a teaching experiment to enlighten the researcher as to student thinking also makes it a viable method for examining the impact of technology on a student's internal representations for mathematics, since, by definition, those representations exist in the student's mind (Goldin, 2003). Nemirovsky and Noble (1997) noted how the subject's thoughts developed, how she "came to recognize" certain mathematical behavior "by visual inspection" (p. 99). In their teaching experiment, in which they interviewed a subject for three one-hour sessions, they also noted her efforts to organize her thoughts as related to the visual experience with a computer based-tool which created graphs of height vs. distance and slope vs. distance (Nemirovsky & Noble, 1997). Their study demonstrates that teaching experiments allow the examination of teaching tools and the development of student knowledge within the environment that tool provides (Lesh & Kelly, 2000; Nemirovsky & Noble, 1997). The emphasis in teaching experiments on examining student thinking can also be seen in the selection of subjects for such a study, since in some cases students with particular thinking patterns were selected (Norton, 2008).

One way of looking at student thinking is to find out the schemas they develop for solving problems (Steele & Johanning, 2004). A schema is “a mechanism in human memory that allows for the storage, synthesis, generalization, and retrieval of similar experiences” (p. 66). Steele and Johanning (2004) cited Piaget as using the term “cognitive structure” (p. 66). It lets the learner recognize similar experiences. Abstraction and reflection are two vital mental processes in the development of schema. An experience may provide a memory upon which a schema can be built. That mental construction may be triggered when a new situation arises for the learner to process. Schema are not memorized in the traditional sense of memorizing a formula, but are built up, becoming deeper and more connected to other knowledge (Steele & Johanning, 2004). Steele and Johanning (2004) noted that one strength of a teaching experiment is the opportunity to engage in conceptual analysis, looking "behind what students say and do" (p. 70). Teaching experiments allow researchers to experience students' learning and reasoning firsthand. They test as well as generate hypotheses. The basic goal of a teaching experiment is to examine what students say and do while engaged in mathematical pursuits and to model students' mathematics (Steffe & Thompson, 2000). Part of understanding the students’ mathematics is understanding what they cannot do or understand. This also emerges in the teaching experiment. The researcher can then consider what rationality lies behind the students’ choices (Steffe and Thompson, 2000).

Case studies and teaching experiments. Suter (2005) noted that over 50% of researchers surveyed had used the case study method for their research. Educational case studies may involve teaching experiments, but they may also use other techniques, such as a focus on understanding students' existing knowledge or the effect of implementing a certain instructional approach.

What students do versus what they might do. Sometimes the researcher's aim is clearly stated as the understanding of existing student thinking, as in Sajka's (2003) work in which the object of the study was to examine an average student's understanding of function. The researcher in this case chose non-standard tasks which required the student to look at functions from a different perspective than the one to which they were probably accustomed. Sajka (2003) stated, "I was not interested in the pupil's ability to solve the problem on his or her own, but rather in observing the process of finding a solution" (p. 232). By observing problems the student encountered, the researcher gained information about the student's understanding. The student in this study had been learning about functions for three years, knew the formal definition, and was familiar with examples and representations (Sajka, 2003). The report was limited to an analysis of one dialogue which dealt with this task: "Give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x)+ f(y)$" (Sajka, 2003, p. 233). The questions used by the researcher during the course of the interview helped clarify the student's understanding. The resulting dialog provided the researcher with information about the student's understanding of function and its associated symbolism. Sajka (2003) was able to conclude that the subject's concept of function was not a "fully-fledged mathematical object" (p. 252). The choice of a task as one that was not conventional was a key factor, as it pointed also to "the influence of the typical nature of school tasks leading to standard procedures" (Sajka, 2003p. 253). In this way, the researcher, seeking to examine student thinking, also shed light upon an aspect of teaching.

Though examining student thinking occurs in this case, a case study investigation into how a student *does* think is different in its goals and activities from a teaching experiment which examines the way a student *may* think. Steffe (1991) stated that "[t]he constructivist teaching experiment is a technique that was designed to investigate children's mathematical knowledge and how it might be learned in the context of mathematics teaching" (Steffe, 1991, p. 177). The role of the researcher in a teaching experiment is more than that of an observer. The researcher becomes "an actor" constructing models of what is occurring in the student's mind as a result of the researcher's actions (Steffe, 1991, p. 177).

The choice to use a framework which considers all that students *may* think in regards to their experiences with technology permits the researcher to follow leads which may occur and turn the project into a truly exploratory case study from which possible theory may emerge. The possibilities for such theory are thus expanded. In a teaching experiment, as opposed to classic design, interactions that use the researcher's mathematical knowledge are allowed, and student sense-making may emerge (Steffe & Thompson, 2000). The "researcher acts as teacher" in an "interactive communication" with the subject with the goal of finding out what the subject *may* learn and what may foster that learning (Steffe, 1991, p. 177). Steffe (1991) described the teaching experiment as "an exploratory tool . . . aimed at investigating what might go on in children's heads" (p. 177).

Descriptions versus adjustments. In addition to adding to standard examinations of student knowledge by exploring what they may be able to learn, the exploratory and flexible nature of a teaching experiment also adds to standard explorations of instructional approaches in that the researcher's continued actions are based on the subject's actions. There is no predetermined way of solving the problem presented. The researcher bears the responsibility of making "on-the-spot" decisions based on what is happening in the experiment and the emerging model of the student's knowledge (Steffe, 1991, p. 177). In one teaching experiment, 6 pairs of children worked with a teacher/researcher for about 45 min. per week for about 75 weeks over a period of 3 years, outside their classroom, using computer tools which allowed the students to use discrete sets of objects and continuous line segments to model fraction concepts (Olive, 1999). The introduction of technology is an instructional approach and certain pedagogical approaches may be present with the use of technology that may not otherwise have been there (Kaput & Thompson, 1994). One of the tasks discussed in the study by Olive (1999) was to consider sharing part of a pizza among friends and find out how much of a whole pizza each friend would receive. The researchers made hypotheses during the experiment when they saw a difficulty a student was having. They introduced a constraint that might move the student forward by refocusing his attention on an overlooked aspect of the problem. The instructional approach was adjusted during the course of the experiment. The teacher's questions guided the student's thoughts (Olive, 1999).

Contrast this approach to a case study done by Butler, Beckingham, and Lauscher (2005) which looked at "the processes by which . . . students were supported to self-regulate their learning in mathematics more effectively" (p. 156). The instructional model used was Strategic Content Learning (SCL). Four SCL principles of instruction were described: "Integrate support for self-regulation into instruction . . . . Students as active interpreters . . . . Learning in mathematics as guided (re)construction" and "Learning in pursuit of a goal" (Butler, Beckingham, & Lauscher, 2005, p. 160). A formal evaluation of SCL had previously been done. This study looked at three eighth-grade students and asked how SCL achieved instructional goals, how the students' learning was mediated, and how SCL was used in responding to individual needs. Individualized Educational Plans (IEPs), psychoeducational assessments, and a Metacognitive Questionnaire gave the researchers information about the students (Butler et al., 2005). The researchers also conducted observations, collected teachers' written reflections, analyzed videotapes of instruction, and looked at student work samples and strategy sheets. They kept track of test performance, including an analysis of data from examinations given early and late in the project to provide pretest and posttest information. Their interpretation of data provided "a descriptive account of SCL intervention in relation to student learning" (Butler et al., 2005, p. 162). Researchers created narratives described how SCL worked for each case. They also completed a cross-case comparison.

Multiple data sources were used in the study by Butleret al. (2005) and a particular instructional model was examined, the work done was observational and descriptive of what was happening between the instructor and the student. The SCL model provided built in encouragement to adjust to student needs, and the teacher observed did so. The study provided a "rich description of instructional processes" (p. 172). The study by Butleret al. (2005) examined an instructional process and provided a *description* of it. The Olive (1999) study tried an instructional approach and *adjusted* it based on researcher observations during the course of the implementation. This quality of adjusting an approach based on researcher observations during the course of a study is characteristic of the teaching experiment, in which, as has been noted, the researcher is "an actor making 'on-the-spot' decisions in order to maximize the exploration" (Steffe, 1991, p. 177). This potential for adjustment to the teaching approach based on observations during the data collection also makes a teaching experiment more suitable for the current research study than a traditional examination of a teaching method, since, as has been noted, the study is an exploratory one, seeking to suggest theory. When potential theory development is a goal, the researcher must be allowed the freedom to follow such leads as may arise and question the subjects enough to develop the ideas that are emerging (Creswell, 2007). This pursuit of emerging ideas is consistent with the adjustable nature of the teaching experiment which has been described. A merely descriptive study would not provide sufficient flexibility to the researcher.

Tasks for teaching experiments. Rather than guiding the students toward a definite answer, teaching experiments present students with tasks that invoke in them a need to develop new interpretations or refine their thinking. The tasks can allow students to learn and to document their learning through built in descriptions and explanations (Lesh & Kelly, 2000). The tasks should allow the researcher to "bring forth and sustain students' independent mathematical activity" (Steffe & Thompson, 2000, p. 293). The questions accompanying the tasks should elicit conjectures based on the research questions or hypothesis (Norton, 2008).

To see how tasks for knowledge assessment and for assessing instructional approaches might differ from tasks used in a teaching experiment, consider the following task:

Describe the effect that changes to m and b have on the graph of the equation $y = mx + b$.

If the goal was to assess the student's knowledge about this topic, the task could be given to them as is, with no tools other than the knowledge possessed by the student and no prior interventions. If the researcher's goal was to test an instructional approach, the instructional approach would need to be implemented prior to administering the above task. For purposes of this discussion, suppose the instructional approach is the use of technology to teach this topic. The same task might be given to different groups, one group receiving it after the students have received instruction in the use of the technology in question, for example dynamic sketches in which they explored how changes in the parameters affected the graphs. The researcher might then administer the task and note the effect of the technological instruction on the subjects' knowledge. In contrast, a teaching experiment task should

- Provoke the student to develop and refine their thinking (Lesh & Kelly, 2000)
- Provide built in opportunities for them to describe and explain what they are doing (Lesh & Kelly, 2000)
- Allow the researcher to bring out mathematical activity in the student (Steffe & Thompson, 2000)
- Allow the researcher to follow up on research questions and hypotheses including those that arise during the course of the data collection (Norton, 2008)

Summarizing these ideas for practice, the task for a teaching experiment must be thought provoking, built on prior knowledge (so that it is accessible and thinking may be refined), open ended, provide prompts which encourage description and explanation, and be accompanied by a semi-structured interview protocol which is flexible enough to allow the researcher to make the necessary explorations for their study. Such ideas were taken into consideration in developing tasks for the current study.

Steffe (1991) has said, "a general goal of mathematics teaching is for teachers as well as students to learn, and the primary goal of the constructivist teaching experiment is but a microcosm of this general goal of mathematics teaching" (p. 192). The key quality indicated here is that the researcher's and subject's learning are intertwined and occurring simultaneously. This type of research will allow the investigator to examine the effect that the introduction of technological representations to the student has on their thinking (Falcade et al., 2007). The range of possible effects that might be discovered is widened in a teaching experiment with its flexible and exploratory nature over case studies which merely look at student's prior knowledge or the effect of a particular instructional approach. The teaching experiment looks at the student's knowledge as it develops and allows the adjustment of the instructional approach to permit the investigation of emerging theory (Steffe & Thompson, 2000). Teaching experiments generate detailed data in the interactions they track. Following is a discussion of theory which will help formulate a method for analyzing the data that has been generated.

Approach to data analysis: Grounded theory. The examination of data resulting from the teaching experiment conducted in the present study is inspired by the pattern described by Glaser and Strauss (1967) in their classic description of grounded theory. A researcher engaged in this method of analysis would first code his data into as many categories as are possible, taking care to compare different pieces of data in the same category. Theoretical properties of the category emerge from this comparison. As this coding continues, the researcher can periodically stop coding and use memoing to record thoughts and theoretical ideas (Glaser & Strauss, 1967). During these periods of reflection, care should be taken that any logic is grounded in the data and not in speculation. As more is learned about categories, different categories become integrated with each other when relationships between them become apparent. Questions may also emerge which may guide the subsequent collection of data. Glaser and Strauss (1967) noted that this continued coding and integration process leads to the reduction of categories to a "smaller set of higher level concepts" (p. 110). As this process moves forward, it is affected by what Strauss and Corbin (1990) referred to "theoretical sensitivity" which they described as the researcher's ability to notice the meaning in data.

They noted that a theoretically sensitive researcher will be able to separate important from unimportant qualities in the data. The ability to do so allows the researcher to formulate well-grounded theory more quickly than he or she otherwise would and give the data meaning which is "faithful to the reality" (Strauss & Corbin, 1990, p. 46). Their ideas are in keeping with Eisner’s (1998) notions of connoisseurship in qualitative inquiry. Theoretical sensitivity can be acquired by a study of the literature, through professional experiences in a particular field, through personal experiences related to the field of study, and as a by-product of the analytic process (Strauss & Corbin, 1990). The comparisons and ideas arising during analysis lead to other ideas which may result in a closer look at previously examined data and the discovery of new meanings.

Lincoln and Guba (1985) looked at the data processing aspects of Glaser and Strauss’s (1967) constant comparative method (p. 340). They expanded the information related to stage one of the process, "comparing incidents applicable to each category", noting that the emergence of categories involves more "effort, ingenuity, and creativity" than the statement that "categories 'emerge'" might imply (p. 340). Semantic relationships, for example, might be difficult to identify. A relationship might be inclusive, and be described as "x is a kind of y", or it might be sequential, allowing it to be referred to as "x is step (stage) in y" (p. 340). They stressed the importance of the analyst’s "tacit knowledge" as used on the first pass through the data, in which assignments to categories may be made based on what seems right to the analyst. Later analysis can clarify such understanding, but Lincoln and Guba (1985) claimed that such impressions may be hard to capture later. It is important that new incidents assigned to a category be compared with previous entries in the category. Thinking about such things assists in the process of refining categories. When conflicts arise, memos can help the analyst capture his or her thoughts, providing an outlet for any conflict he or she feels and helping him or her discover the category's properties. Stage two of the analysis finds the analyst shifting to a more rule-based system of classifying incidents. Subcategories may need to be formed or categories may need to be redefined. The analysis then comes closer to describing what is being studied. Lincoln and Guba (1985) noted that "fewer and fewer modifications will be required as more and more data are processed" and that as categories become clearer "options need no longer be held open" (p. 343). They said that "categories become *saturated*, that is, so well defined that there is no point in adding further exemplars to them" (p. 343-344). Saturation may also be described as that point at which "continuing data collection produces tiny increments of new information in comparison to the effort expended to get them" (Lincoln & Guba, 1985, p. 350). Specific methods for applying grounded theory techniques to the present teaching experiment will be discussed following the summary below.

Concluding ideas about methodological theory. This study is based in the ideas of constructivism. It assumes that both researcher and subject construct knowledge together during the course of the particular methodological approach taken here, the teaching experiment. Qualitative research methods allow the researcher to collect and examine data with the eye of a connoisseur and use the rigorous data analysis methods of grounded theory to construct ideas which are both truthful and important. The next portion of this chapter will describe some of what I learned from conducting a pilot study. The specific procedures which were used in the final study are then described. This will be followed by a look at my stance as a researcher, and issues of reliability and validity.

Methodology learned through the pilot study. Since I had not previously conducted a teaching experiment or tried the video techniques that would be used for this study, I conducted a pilot study. I recruited Steve[4], a male in his early 20's, who had been out of school for about 4 years and did not remember taking Algebra 2. He participated in 5 sessions for a total time of about 3 hours and 34 minutes.

Facilitating Steve's work gave me valuable experience in the type of progress adult developmental mathematics students might be able to make and the kind of questioning and facilitation it might require. I also gained practice in basing my actions on his actions and investigating the potential of his thinking, as described by Steffe (1991), rather than trying to guide Steve to a particular goal I had in mind. As I did this I found that Steve "came alive" when he was allowed to pursue his own ideas. One goal of a teaching experiment is to "bring forth and sustain students' independent mathematical activity" (Steffe & Thompson, 2000, p. 293).

In addition to gaining experience allowing him to work at his own pace, I learned more about providing the type of facilitation that is sometimes necessary to focus the subject's attention on pertinent mathematical relationships which will allow them to use mathematical ideas they already possess in new situations Olive's (1999). This occurred when I encouraged him to graph a vertical line so that he could visually follow the relationships of the x value to the point on the function he had graphed and the related y value more closely. As I worked with Steve I also found that the interview protocols I started with were too complex to use or follow well in a teaching experiment setting and was able to simplify them considerably for the final study.

A brief summary of the pilot study results, including the presentation of selected video clips from the pilot study, was presented to a group composed of two mathematics educators, a specialist in educational research, and a mathematician. Technical suggestions were made

[4] All names are pseudonyms

regarding the video set-up so that the appearance of the final product would be more easily seen and understood. Ideas for strengthening the study's theoretical investigation were also suggested. Particular attention was drawn to the nature of a teaching experiment and the perceived need to focus not on eliciting student understanding but on finding out how they come to understand.

One technique which was clarified through the conduct of a pilot study was the practice of taking into account what happened previously in preparation for each new interview session. Some things that might have been investigated were to note more fully the role of technology in enhancing the usefulness of the internal representations that Steve possessed. Another possible idea was to see how much the things Steve determined from his visual examination of the graphs and his examination of how they changed translated into increased algebraic understanding. Though such intriguing questions may arise in the researcher's mind, the pursuit of knowledge must be balanced with the pace at which the subject can be made to reveal information without imposing upon him or her the researcher's own thinking. A pilot study such as the one described here can be an invaluable opportunity, particularly for new researchers, to learn more about that balance.

Procedure

This discussion of specific techniques used for this study includes information about the selection of cases, instrumentation and data collection, data analysis, stance of the researcher, and issues of reliability and validity. It will then be followed by a brief examination of the pilot study which was conducted in preparation for the final teaching experiment.

Selection of Subjects. The following section starts with a look at how an institution appropriate to the needs of the study was selected. Following that, the process used to recruit the subjects of this study as well as the procedures used to decide the ordering of the work done with the subjects will be described.

Institutional selection. Subjects were recruited from and the data collection conducted at Harrisville State University[5] (HSU), a mid-sized university in the southern United States offering both undergraduate and advanced degrees. This institution was selected because of its proximity to my home and the presence of a large adult developmental mathematics population from which it was hoped that a sufficient number of subjects could be recruited. HSU has provided substantial support to returning students and others from the local community. That support has included a college within the university offering developmental courses, including two course offerings in mathematics[6], Math 98 and Math 99. Three faculty members at the college specialized in those developmental mathematics courses. In addition, a preparatory algebra course, Math 100, was also available to assist those students whose mathematics placement scores indicated the need for remediation, but who had not been required to test for learning support. This course was also considered to be at the developmental level, since it did not count for credit toward graduation. An inspection of course offerings during a recent spring semester including the two developmental mathematics courses and the preparatory course showed that 26 sections of mathematics at the developmental level were being offered, with a population of almost 600 students, and employing at least 15 different faculty members. This attention to the developmental mathematics population and the resulting presence of large pool of potential subjects in close proximity to my own location made this an ideal setting for the recruitment. The

[5] pseudonym

[6] Course designations have been changed

cooperation of personnel within the university in providing space and allowing recruitment also added to the benefits of doing the research at HSU.

Case selection. Both passive and active recruitment methods were used to find potential subjects for this study. With approval, the researcher entered developmental and preparatory course classrooms at the study site, and delivered a brief invitation to participate in the study. Flyers were provided at that time to potential participants, describing the study and including contact information. Flyers were posted in areas nearby those classrooms. Since this was an exploratory case study, two to three cases were deemed to be sufficient (Creswell, 2007). Five people responded to the recruitment to the extent of providing personal information and indicating an interest in the study. On one occasion I had appointments set up with three subjects in the same day. One of the subjects got lost, but called and was able to make the appointment. The other two did not show for their appointments. I was eventually able to get three initial interviews with three different subjects. Even though I had intended to select from a pool of up to 8 initial interview participants, the response was lower than expected. The three who did respond, however, were each enrolled in a different developmental mathematics course. Their initial interviews confirmed the notion that they represented a spectrum of mathematical experiences, and I decided to stop recruitment and use those three subjects. Recruitment and data collection took place during Fall Semester 2009.

The three subjects were Shirley, Marlon, and Marjorie[7]. Shirley was a 46 year old African American female enrolled in Math 98 who had been enrolled in the same course the previous Spring, but had not completed it. Previous mathematics courses she listed included general mathematics and Algebra 1. Marlon, a 53 year old African American male, was enrolled

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[7] All names are pseudonyms

in Math 99 and listed Math 98 as a previous mathematics course. Marjorie, a 36 year old African American female, was enrolled in Math 100 and could not remember a previous mathematics course, indicating that it had been 14 years since she had been in high school.

Miles and Huberman (1994) have noted that with a small number of cases, purposeful sampling can allow the sample to be chosen to fit the logic of the study. Random sampling would be less effective for the purpose of the study. Part of the purposefulness is in the setting of specific boundaries for the section of cases (Miles & Huberman, 1994). One boundary for this study is enrollment in developmental or preparatory mathematics at HSU. The cases selected provided a range of mathematical experiences, providing different perspectives for the study (Creswell, 2007). Although all of the subjects were African-Americans over age 35, they each showed different mathematical understanding in their initial interviews. They were also enrolled in different level courses of study for developmental mathematics students. This shows that although there were only two subjects, these two subjects represented different portions of the developmental mathematics population.

Since the idea of the study was to examine the influence of technology on the student's internal representations of mathematics, the previous experience of the potential subjects with technology in their mathematics learning was also one of the areas I examined through the initial interview. Both Shirley and Marlon indicated their use of the internet, particularly a computer aided instructional program used by their mathematics teacher. Marjorie indicated experience with calculators. Logically, the intervention used in the teaching interviews would have greater impact on inexperienced students' internal representations than it would for those who have previously internalized technological representations. Though they had some experience with technology, none of the subjects indicated any experience with the software to be used in the study, Geometer's Sketchpad. This added to their suitability as study subjects.

Marlon was selected to be the first participant in the study and Shirley and Marjorie were notified that they would be contacted later in the semester following my work with him. Since he was enrolled in the second developmental mathematics course, this would presumably allow a more central set of data to be collected and the two cases representing either extreme of the population could then be compared with his data. Unfortunately, due to illness and other challenges, Shirley was only able to attend the initial interview and one other session. Attempts were made to contact her early the following semester, but she did not make it to a follow up interview, and eventually stopped returning phone calls. Since two cases were deemed sufficient, the study was completed with Marlon's and Marjorie's cases.

Instrumentation and data collection. Interviews were the source of data for this study and the mode and structure of those interviews is described below. A discussion of the general interview technique is followed by specific looks at the initial interview and the series of teaching interviews which followed it.

Interview technique. Interview protocols are found in Appendix B. All interviews, both the initial interviews and the teaching experiment sequence, were semi-structured and in accordance with effective interview techniques, efforts were made at the *beginning* to put the subject at ease, gradually building to more direct examinations (Kvale, 1996). I designed questions which encouraged the subject to relax, to use language comfortable for them, to be descriptive, and to be specific about their experiences. At times I rephrased and repeated questions and encouraged them to explain what they were thinking to increase the validity of my interpretation of their responses (Kvale, 1996). Some brief field notes were collected during the interviews, but these were minimal as my attention was focused on conducting the teaching experiment and responding to circumstances. I kept more detailed notes in a descriptive word processed journal in which I recorded my reactions to sessions. Any reference to the subject in notes, journals, and transcriptions was made using a pseudonym.

Initial interview. Initial interviews can help researchers gain understanding of prior knowledge so that in addition to gaining other information for subject selection, they will gain knowledge about the subjects' experiences with similar mathematical tasks (Hollebrands, 2004). The initial interviews held in the present study lasted from about 35 to 45 minutes. Following appropriate measures to ensure informed consent of the subject, I collected data regarding the subject's demographics, times of availability, and educational background. Any reference to the subjects' real names was kept in a locked file box along with copies of their signed informed consent documents. I collected data regarding mathematical background by listening to the subject's anecdotal accounts of their educational experiences. I allowed them to share mathematics of their choice which they remembered, and presented them with a diagnostic task. Initial interview subjects were informed that they may or may not be selected for further participation. Note that provision was made for additional questions other than those listed on the protocol form.

Teaching experiment interviews and tasks. "Learning how to bring forth and sustain students' independent mathematical activity is a part of learning how to interact with students in a teaching experiment" (Steffe & Thompson, 2000, p. 293). Koichu and Harel (2007) suggested that interviewers encourage the interviewee in his or her thinking, that they not probe deeply early in the interview, and that they engage in a semi-structured conversation. With these things in mind, a sequence of tasks was designed to be presented to individual subjects. They were created to investigate and advance the subjects' understanding of different representations of functions at a level consistent with their exhibited prior knowledge. However, not all of the tasks prepared for the study were actually used. The tasks as designed can be seen in Appendix C.

The first tasks, designed to be used in the initial interview, consisted of the examination of patterns. After the subjects had been permitted to share some mathematics of their choice, they were presented with the task "Looking at patterns" and examined the sequence of shapes. They were then presented with the task "Looking at dot patterns" to see what patterns they could find there. Should it be needed, a third task, "Soda Cans" was also prepared, but it was not used in the actual study. The task "Another dot pattern" was used as a supplement to "Looking at dot patterns" and arose from work done in the pilot study. A sequence of tasks situated in technological settings was also prepared. The technological tasks were designed to allow both written and technological representations. The tasks were adaptable to the specific needs of the subject, allowing multiple entry levels, and open-ended responses reflecting student thinking. They were also designed to elicit information about the subjects' understanding of ideas and representations associated with functions. Probing and specifying questions were used as needed to elicit student thinking, as supplements to the protocol, consistent with semi-structured interviews.

Technological procedures. The mathematics software selected for the study was Geometer's Sketchpad v. 4.07s (Key Curriculum Press, 2006), which incorporates graphical, tabular, animated, and symbolic representations. This software was installed on a guest account on my laptop computer which I took to each session. I also provided a movable mouse for the subjects so they would not be hampered by the use of the laptop's touchpad mouse.

There were other technological considerations as well. Because the study was to be a teaching experiment, a record was needed of all of the pertinent actions and statements made so that they could then be analyzed. These actions included the subject's interactions with the software, the subject's creation of representations on paper, and the physical actions of the subject and interviewer. In order to capture all of these interactions, and at the suggestion of my advisor, three simultaneous recordings were made and those recordings later synchronized into one video production. Campbell (2003) described the use of "dynamic tracking . . . to capture a complete record of a learner's interactions with a [computer based learning environment] in real time" (p. 73). In order to capture the learner's interactions as completely as possible, Campbell (2003) made simultaneous video recordings of the learner and the learner's computer screen. The work done in the present study adds to these two the work done on paper as well. The synchronization required that either the paper view or physical movements view be synchronized first as a picture in picture (PIP) with the screen capture view and then the other view added as a PIP to the resulting video.

The three recordings were produced as follows. One recording was captured by software which was used to create a video record of the activities on the computer screen. A small camera was placed on a small tripod which sat on the table and faced downward to capture what was happening on paper. A blank piece of paper was taped to the table so the subject knew where to keep their written work so it would be in view of the small camera. Another camera was set up across the room on the other side of the table from the subject to capture the physical movements. Figure 3 is a still capture from one of the final synchronized videos.

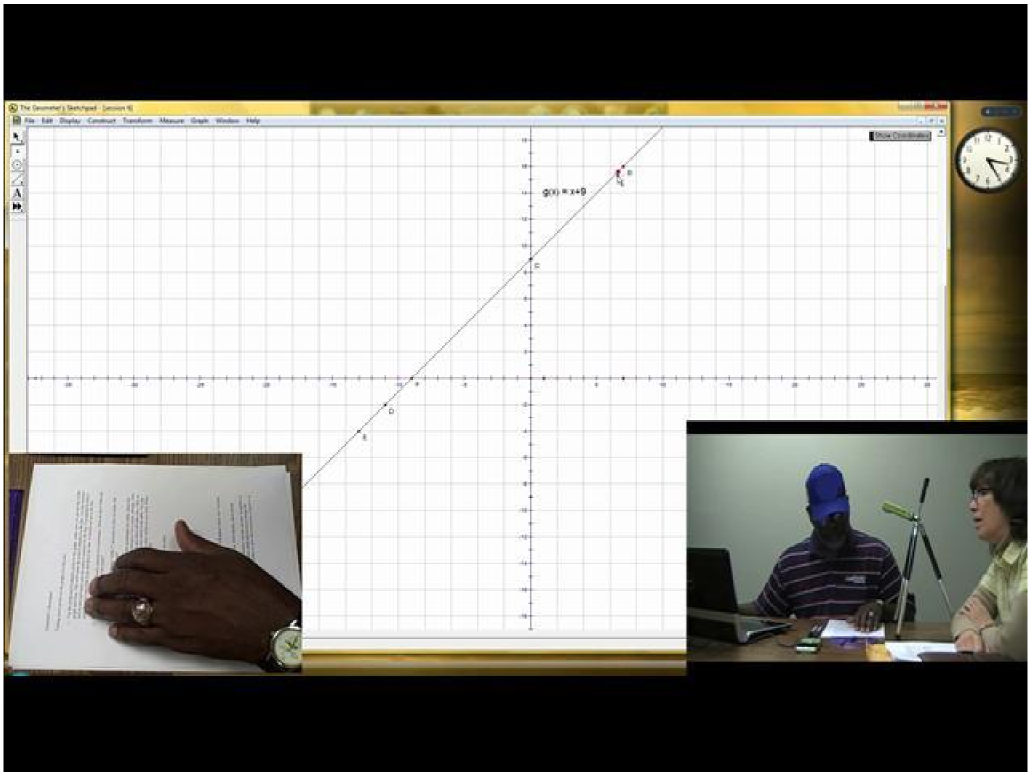


Figure 3. Still capture illustrating the camera set up and assembled video recordings

In the third session held with Marjorie, a technical error resulted in the loss of the recording of the computer screen, however sufficient data from the other two feeds provided insight into what happened during that session and implications were still possible through the conversations and paper representations which were preserved. A total of 7 sessions were held with Marlon for a total time of about 6 hours and 47 minutes. A total of 6 sessions were held with Marjorie for a total time of about 5 hours and 3 minutes. An examination of how the captured data was analyzed follows.

Data analysis. It is essential that those engaging in teaching experiments plan adequately for the labor-intensive activity of retrospective analysis, including a careful examination of the videotapes (Steffe & Thompson, 2000). In this case, videotapes included a record of the computer screen, the subject's reactions, and the subject’s written work, necessitating extra care in observations. As noted, video recordings were synchronized via computer software, so that interactions between researcher and subject, computer screen action, and the subject's written work were viewable simultaneously. Care was taken to note mouse movements along with transcription of the spoken word, and occasional notation was also included as to the timing of episodes within the session being transcribed. Transcription technique was inspired by Campbell (2003) and includes actions within braces {} and timing within brackets []. For example, the following transcription shows that Marjorie had the mouse pointing at the point (10, 6) on the xy-plane after she said the word "itself", moved it to (0, 5) before finishing her next statement, and then moved it so that it was at (0,0) before she said the word "So." It also shows that this statement ended at about 37 minutes and 56 seconds into the rendered video recording.

> MARJORIE: It's got to intersect with the graph itself {cursor now at (10, 6)} because it does not have any, um {cursor now at (0,5)} — its like to the exact — it is rounded up or its like rounded {cursor at (0,0)}. So it doesn't go to like any of the, the cents [37:56].

Following is a look at the coding techniques used in this study including a discussion of how emergent codes were addressed. One important emergent code is described.

Coding techniques. A unitizing and coding guide is provided in Appendix D for the reader's convenience and may be referred to during this discussion. This guide includes definitions for unitization, and a listing of families of codes with a priori and emergent codes noted as is appropriate within each family. Definitions and examples are given for each code.

Initial coding was done during the course of the data collection, to facilitate ongoing analysis (Miles & Huberman, 1994). Transcription and initial open, descriptive coding were usually done between sessions, along with such memoing as naturally arose (Creswell, 2007; Miles & Huberman, 1994). Frequent memoing allowed the researcher to capture emerging observations about what was happening while those thoughts were fresh. Coding was recorded using Atlas.ti data analysis software (Hewlett-Packard, 1993-2009). The unit of analysis was one or more sentences or paragraphs focused on a single topic. As many codes were attached to a unit as were deemed appropriate for an understanding of the important ideas present in the situation.

As the study progressed, it became clear that some codes needed refining. For example, at the beginning of the study, the a priori code “student’s mathematics” was used. It arose from the literature related to teaching experiments. When it became clear that many different forms of mathematics were being observed, and it would be more profitable to code these occurrences more specifically, that code was eliminated. Later on, the many codes which replaced it were reduced to the particular mathematical ideas listed in the final coding guide.

After the interviews were concluded, initial coding for the sessions was completed, including the addition of newer codes that had emerged during the course of the study. Transcriptions were examined and further coded based on what was learned during the course of the data collection, particularly with an eye to emergent ideas related to the subjects’ interactions with technology. Quotations were examined one by one to determine whether or not the coding which had been done appeared to be sufficient.

Axial coding was used to categorize data into families, and selective coding was used to connect cooccurring ideas and develop a view of what codes might be associated through theoretical hypotheses (Creswell, 2007). This follows the pattern described by Glaser and Strauss (1967) who noted that generation of conceptual categories is followed by hypotheses about the relationships among the categories. Co-occurring codes lists were created using Atlas.ti and examined to see which codes were associated with each other. Networks were created which could show the density or connectedness of particular codes. Such a process served to help

highlight emergent ideas. A one page code list was used at times to help with coding. The software was also used at times to filter the codes according to family when examining a quotation which seemed to contain information related to that family.
Queries done using the software helped create logical combinations of codes which were also helpful in analyzing the data.

Indicative movements. I noticed during the course of the study that gestures and mouse movements used to interact with the mathematical representations were more prevalent and apparently more connected to the subject's internal representations than I had anticipated. They were also connected to emergent ideas related to the influence the technology was having on student thinking. In order to go back and code for that emergent idea, I used the software's category searches feature, which allows the researcher to look for all instances of particular words in the transcripts. Gestures were considered to be occasions where the subject pointed to or indicated something by a physical movement in some way. A search was done through the transcriptions for forms of the word "indicate" or the word "gesture." Mouse movements were considered to be occasions where the subject's cursor movements could be tracked in some way. Such movements were usually indicated in the transcription by the word cursor or mouse or forms of the word trace or slide. Those words were used in a category search as well. Not every single occurrence in the above named searches was coded as such; for example if it was a movement of the interviewer and not the subject, it was not coded. In the final coding structure, the instances of gestures and mouse movements were combined into the code *indicative movements* which I felt more accurately described their role, as they seemed to be providing indications of student thinking in similar ways.

This review of the techniques used in the present study has included a discussion of case selection, interview techniques, teaching experiment procedures and tasks, and data analysis procedures. Attention will now be turned to the point of view I brought to the study as a researcher and to the methods used to ensure that the research that was done was reliable and valid.

My stance as researcher

It is helpful for readers to know any biases on the part of the researcher and to understand clearly the role of the researcher in the study he or she is conducting. It clarifies exactly what opinions of the researcher may have affected his or her objectivity or interpretation of data (Pratt, 2007). Readers of the present study need to know that I believe that it is essential that teachers today understand and incorporate into their classrooms the power that technology has to help students learn to love and understand mathematics. I have experience in the use of technology in the classroom which supports that belief.

I taught entry-level college mathematics students for five years, many of whom had just completed developmental mathematics courses. In this instruction I used technological representations on many occasions, gaining practical experience with such technology and its effect on student learning. I also used technology in teaching students in grades 8-10. At both the college and public school level I provided websites through which students could gain access to technological representations which would help them in their study of mathematics. I believe that technology has the power to engage both the adults and the younger students I observed with mathematics in a way that other techniques may not have.

In the public school setting in which I taught, computers were provided for every student in every classroom. No training was provided, however, as to how the teachers of each individual subject would incorporate those computers into their classrooms. I believe that a greater understanding of the importance of the use of technology in the mathematics classroom will help school stakeholders to make wise decisions about what resources they provide for teachers.

It was helpful that I had worked with adult learners previously, as is desired in a teaching experiment (Steffe & Thompson, 2000). I also had to observe what was happening and interact with students in a manner that would allow me to learn how students operate and put aside my own way of doing things as much as possible. I knew that there would be more happening than teaching. Because I was interested in watching the development of students' thinking, observing what I could of their internal representations, and watching the subjects' use of technology to solve problems, I responded to their confusions or misunderstandings with particular types of questioning rather than direct answers. I also at times allowed them to pursue mistaken ideas so that I could see how the use of technology could allow those mistaken ideas to be revealed and clarified. I made decisions as to what problems to present to the subjects so that emerging ideas could be pursued, as opposed to making choices based on the completion of curricular goals. All of these choices in the pursuit of emerging ideas about student thinking affected the course of the research. In addition to clarifying any bias I had in the conduct of this study, additional measures were taken to ensure its reliability and validity. A discussion of what was done and the importance of this follows.

Reliability and Validity

It is vital that any study presented to the academic community be evaluated as to the quality of the research which was done, in order for that research to be deemed worthy of consideration. Such qualities may be considered under the broad headings of reliability and validity. Reliability refers to the quality and clarity of the research techniques used. Validity includes the notions of internal validity or the authenticity of what it being said and external validity or the potential of the research to be applied to appropriate settings (Miles & Huberman, 1994). Following is a discussion of constructs used to address reliability and validity in qualitative research. After that the specific methods used in this study will be described.

Reliability and validity in qualitative research. Many constructs have been used to address these issues in qualitative research and have been aligned with or added to the above notions. Miles and Huberman (1994) provided a concise summary of many of those methods. They drew together ideas from researchers including Guba and Lincoln (1981) and Schwandt and Halpern (1988). Another researcher who drew from the work of Guba and Lincoln is Tuckett (2005) who provided practical examples as to how the ideals proposed by Guba and Lincoln might be enacted. He noted that Guba and Lincoln's (1989) trustworthiness criteria included the notions of credibility, transferability, dependability, and confirmability and that their evaluation criteria included credibility, fittingness, auditability, and confirmability. To help ensure credibility, Tuckett (2005) used journaling, recordings, member checking, and triangulation, among other things. To ensure transferability or fittingness, he included the use of thick description, and purposeful sampling. To help ensure dependability or auditablity, Tucket (2005) used field journaling, recordings, negative case analysis, and peer review. Field journaling also helped him ensure confirmability.

Geertz (1994) described "thick description" as providing "meaningful structures" through which people's actions may be better understood (p. 215). Thin description would merely describe the action. Thick description would give more than just a description of the action. It might provide, for example, information regarding the motivation for the action. Merely observing someone whose situation seems to fit the idea you are searching for does not give you "the thing entire" (Geertz, 1994, p. 226). Part of the task is to "uncover the conceptual structures that inform our subject's acts" (Geertz, 1994, p. 229). It is then hoped that this will generate useful analysis.

Miles and Huberman (1994) described reliability as comparable to dependability or auditability and said it was the quality which shows whether or not the process of the study was "consistent, stable over time, and across researchers and methods" and possessed "quality control" (p. 278). In exploratory studies such as the one reported here, an analyst applies global interpretations to the subject's mental processes, usually displaying transcript sections with his or her interpretations. This assists the reader in seeing how analytical decisions were made (Clement, 2000). Analysis must be conducted and reported in such a way that another researcher can follow decisions made by the author (Chiovitti & Piran, 2003).

Rather than relying on multiple independent coders, the exploratory method relies on an analyst who is sensitive to the subject and employs keen observation of detail (Clement, 2000). If the researcher works alone, he or she may present in detail his or her perspective on the data in such a way that readers can see things from the researcher's point of view, even if they do not agree with it (Kvale, 1996). Kvale (1996) noted that readers should be able to "retrace and check the steps of the analysis" (p. 209). Following is a description of measures taken to ensure the reliability and validity of the current study.

Measures taken in the present study. During the course of the present study, the following measures were taken which add to its reliability. A record was kept through memoing and journaling of some of the questions and thoughts that arose during the course of the study and its analysis. Multiple recordings were made of each session. Though only one researcher was involved in this study, triangulation is present in the form of visual, verbal, and written data all being collected from the subjects, providing multiple mediums through which information was conveyed. Peers provided feedback following the conclusion of the pilot study, a peer was consulted during the course of the study for purposes of discussion and feedback as to the progress of the study, and a peer was also consulted regarding the process of ensuring reliability.

Thick description and the inclusion of transcript portions are used in the description of the cases. Following are additional measures taken. Included are a sample of researcher questions, a look at the use of talk-aloud protocols, a description of how questioning was used to ensure internal validity, and a description of measures taken to ensure external validity.

Questioning during the course of the analysis. One way analytical decisions may be tracked is by knowing some of the questions which arose during the course of the research (Chiovitti & Piran, 2003). Questions which arose during the course of this study included the following examples, drawn from the researcher's reflective journal and from memos recorded in via Atlas.ti data analysis software (Hewlett-Packard, 1993-2009). Questions related to Marlon's case follow:

- Can technology help Marlon to see the connection between his idiosyncratic representation and the standard representation?
- How can an understanding of functions be built on the idea of counting?
- What other mathematics does he possess?
- What connections is he making?
- Could [a particular use of variables] be considered an idiosyncratic representation?
- What internal representations does he possess?

- [H]e did not think of these two numbers as being added together previously. Could this be related to his confusion about the coordinates of a point, which he sometimes represented as an addition problem?
- What learning came from the technology and what came from pattern examination? (Has he ever examined patterns like this before?)

Questions related to Marjorie's case follow:

- How much did the software really help her to get this idea?
- How important was her own discovery and investigation with the software to her understanding?
- How much information is there in what she says and does?
- What does this say about her internal representations?

These and other similar questions served to focus the researcher's thinking during the course of the study and afterwards as the analysis continued.

Inter-rater reliability. Unitizing was done semantically. Semantic units are chosen based on the meaning of the text, while syntactic units are chosen by "graphic convention" (Murphy, Ciszewska-Carr, & Manzanares, 2006, p. 3). Semantic units allow the researcher to encompass whatever he or she feels constitutes a "complete idea" (Murphy et al., 2006, p. 4). Semantic units allowed me to more clearly examine the meaning found in exchanges between interviewer and subject. Since this study was designed to investigate student thinking, I felt it was vital that meaning be a factor in the selection of units. Meetings were held with a knowledgeable peer to discuss issues of reliability.

Murphyet al. (2006) noted the challenges of working with semantic units, which require "interpretation and judgment on the part of coders" (p. 4). They said that "reliable and consistent interpretation and judgment between coders may not be possible in spite of training" (p. 4). Creswell (2007) stated that in obtaining inter-coder reliability, it is important to decide what it is that is to be agreed upon. It might be code names, code passages, or the selection of codes assigned to the same passages. He noted that "there is flexibility in the process" (p. 210). In an inter-coder agreement process he designed for data related to the HIPPA privacy act, he and those working with him determined that they would not seek unitizing or coding passage reliability. He said that, in the case of that particular research, to expect different coders to select the same passages "would be hard to achieve because some people code short passages and others longer passages" (p. 211). What they did do was look at passages that they had all coded and see how well the codes they selected for those passages matched (Creswell, 2007).

I did conduct unitizing reliability tests with two different trained persons. I found that the passages I selected were generally longer than the ones they selected, but there was some consistency as to the location of the breaks. Of the places where I deemed that a break between one unit and the next occurred, they also chose most of those same breaks (about 64%).

I also conducted inter-rater coding reliability tests, focusing on the four categories of codes: mathematical content and thinking processes, representational ideas and issues, influences and uses of technology, and other. I looked to see whether or not, given the same passage of transcript, a trained peer with experience coding qualitative data would find those same themes reflected in those passages. Using percentage of themes assigned by both coders that were matched, the calibration session produced an agreement of 88.46%. A first independent session produced an agreement of 77.78%. A second independent session by the same coder produced an agreement of 78.95%. Total agreement over the three sessions was 81.94%.

Internal validity. Creditability may also be described as trustworthiness or internal validity and relates to the faithfulness of the description of the phenomenon (Chiovitti & Piran, 2003). It speaks to whether or not one should believe what the author says, (Schoenfeld, 2007). It also indicates that the participants have in some way guided the process and that theory has been checked against participants' meanings (Chiovitti & Piran, 2003). Creditable research methods protect the researcher from inaccurately representing the subject's intended meanings (Chiovitti & Piran, 2003). In the present study, as noted above, a journal was kept. Some of these thoughts recorded following the sessions provide a fresh memory of what occurred during the session. The multiple video recordings also add to the believability of the study in addition to providing evidence of reliability. This is particularly helpful as it allows key portions of the resulting transcripts to be checked for accuracy. Triangulation, as noted above, also adds to internal validity. Following is a closer look at two other measures of internal validity used in this study, the use of talk-aloud protocols and careful questioning to elicit as accurately as possible the student's own thinking.

Talk-aloud protocol. Subjects were encouraged to talk out loud about what they were doing as they worked, to talk continually as if they were thinking out loud. Instructions given to subjects were adapted from ideas presented by Koichu and Harel (2007). At times the subjects had to be reminded of the idea of a talk-aloud protocol, but each subject provided narrations of their efforts during the course of their interactions with the technology. Such narrations, when they were more than just a sentence or two, were coded as verbal streams and were common in both Marlon's and Marjorie's cases. Campbell (2003) also referred to talk-aloud protocols and noted that such techniques along with "putting your mouse where your mind is" would help researchers better capture their subject's thinking (p. 74). Instructions were given to the subjects in the first session as to what they were expected to do. They were told to "read the directions and follow them . . . talk out loud about what you're doing as you work, talking continually as if you were thinking out loud." Other interviewer statements occurring during the course of the study to encourage this kind of talking were "What do you notice? Talk about what you're noticing" and "keep talking about what you're thinking so that I'll know how you're thinking about this."

Questioning for validity. Questions were also used which were designed to support the subject's sharing of their own thoughts. Kvale (1996) noted that careful questioning during an interview as to the meaning of what was said can help with validation. In this case, such questions were noted by the following four sentences which were used as codes. Why? Explain your meaning or choice. What do you see? What happened? Coding analysis shows that such questions were also common in both Marlon's and Marjorie's cases.

External validity. External validity is also referred to as transferability or fittingness (Miles & Huberman, 1994). Generality and importance ask how widely the research applies and whether or not it matters (Schoenfeld, 2007). Generality does not imply importance. A study may apply widely, but not contribute anything to our understanding of mathematics education (Schoenfeld, 2007). External validity is aided by a clear description of the scope of the study, the expectations for it, the subjects of the study, and by using thick description during the report of the study. The recruitment and sample used for this study have been clearly described. It has also been clarified that the purpose of this study is suggestive and not confirmative.

Any final decision regarding transferability rests with the reader (Chiovitti & Piran, 2003). The goal of theory development in mathematics education is to be able to communicate ideas about it to the educated world in ways no other academic field can. In order to do so, theories must be developed and used wisely (Silver & Herbst, 2007). It is not within the scope of this study to develop grand theory, but to formulate substantive theory related to the experiences of these students which may suggest further work (Schwandt, 1997).

Another way to increase external validity is through the use of thick descriptive data. This way judgments about fit can be more readily made by others (Lincoln & Guba, 2007). As has been noted, thick description refers to the interlacing of meaning into described actions which helps us to better understand those actions (Geertz, 1994). Case descriptions for this study will include connections to ideas about representation and technology and other emerging ideas which will help the reader to situate the subjects' actions theoretically.

Conclusion

In this chapter I have provided the reader with a description of the methods used for this study, beginning with a look at the theoretical basis for the methods chosen. I discussed qualitative research in general as well as qualitative research in mathematics education. I provided information to support my choice of a teaching experiment and showed how such a choice would allow me to learn more about the effect of the use of technology on student thinking. Grounded theory was discussed as a foundation for data analysis. Following that discussion, I presented a brief look at what I learned during the course of a pilot study. The specific procedures used in this study were described, including the selection of subjects, the instrumentation and data collection, and the technological procedures. I also discussed data analysis procedures, and defined my stance as a researcher. In addition, I showed how specific technological procedures in the form of a multiple camera technique allowed me to collect data so that students' statements, paper inscriptions, and technological choices could all be examined in detail. Finally, measures taken to ensure reliability and validity were described. A description of the results of the study follows.

4. Results

The purpose of this study was to determine the effect of the use of mathematics technology on adult developmental mathematics students' understanding of functions. Such understanding was to be characterized by the quality of internal representations those students appeared to be able to build as they interacted with technology. It was hoped that insight could be found which would enable teachers of adult developmental mathematics students to help those students overcome the challenges they may have, such as learning disabilities, a lack of self-efficacy (that is, a lack of a belief in their own ability to be effective learners) or a lack of understanding of what it is that is holding them back (Epper & Baker, 2009; Hall & Ponton, 2005; Wadsworth et al., 2007). It was also hoped that teachers of adult developmental mathematics students might be provided with information which could broaden their use of technology, which, as in other developmental situations, may not have been allowing the insight into student thinking that would help these teachers better serve their students (Kinney & Kinney, 2002). I conducted a qualitative case study in the form of a teaching experiment in the hopes that it would allow me the best opportunity to examine adult students' interactions with technology and study their thinking.

This chapter begins with an introduction to the subjects and some of their personal characteristics. This will be followed by a summary of what happened during the course of the teaching experiment. Following that summary the major theoretical ideas which arose will be examined and supported by examples from the collected data on Marlon's and Marjorie's experiences. The chapter will conclude with a summary of those ideas.

Description of Subjects

Marlon, a 53 year old African American male, entered the study desiring to learn and having persevered through personal challenges. He grew up in a large city in the northern United States where he had experienced the influence of gangs before moving to a different school. He dropped out of high school in his final year and then earned his GED while serving in the military. The exact reasoning for his dropping out of high school was not clear. He said, “I tried to take my subjects very seriously but . . . trying [to] raise kids at that time I just fell out.” He could not remember much about his high school mathematics classes, saying that he was “probably more into sports” at that time. He did say at one point that he had loved mathematics when he was growing up but that “you really have to practice it all the time.” He now found mathematics challenging, particularly after having been out of school for a while, which he said made learning harder. He had not passed the first developmental mathematics course when he took it during a recent spring semester, and stated that the heavy load from the English and reading classes he was taking at the same time made it more difficult to find the time to get his mathematics done. He took it again that summer and passed it. He has made some use of the mathematics tutoring lab available on campus. He also expressed a strong desire to have his own computer available at home. He had some experience with calculators and with software used by his teacher which he said had been helpful.

Marjorie was a 36 year old African American female with a military background who had left the military in order to pursue her education. Marjorie had apparently done well in school in her childhood having been on the honor roll until the 10th grade. While Marjorie was in middle school, her mother had become ill. For about 4 years the illness was not, apparently, life-threatening. Towards the end of that time, after they had returned to the United States from overseas, her mother's health deteriorated, and she passed away while Marjorie was in high school. This personal tragedy probably contributed to Marjorie's loss of interest in school for a time. Even so, she graduated from high school with a B average. She said that "I was able to catch up with everything . . . and I graduated . . . with like a 3.2 but it wasn't like . . . my 4.0's I was getting before." After high school she joined the military. She "didn't want to do anything with school" when she first got into the military, but about 3 years later she took some college level courses. As part of her military service she was sent overseas where there was less access to the courses she needed. She has continued to take classes, sometimes "sporadically" and wants to "get as much education as [she can]."

Table 10 provides a summary of the background information about the subjects of this study. Both Marlon and Marjorie had military backgrounds and had experienced difficulties in their childhood unrelated to their cognitive abilities which may have affected their academic progress. Such challenges are in keeping with the non-cognitive factors that often hamper adult developmental mathematics' students efforts in their return to school, such as personal demands on their time (Gerlaugh et al., 2007). Marlon seemed to become more easily discouraged than Marjorie, since he would often apologize for mistakes while Marjorie often asked to work on a problem further. He demonstrated more of the lack of self-efficacy described in the literature as being evident in some adult learners (Wadsworth et al., 2007). They both however, seemed to have a strong desire to learn.

Table 10

Subjects of Teaching Experiment

	Marlon	Marjorie
Demographics	53 year old African American Male Former military	36 year old African American female Former military
Secondary education	Dropped out of high school in his final year and earned his GED while serving in the military	Graduated from high school with about a 3.2 average
Post-secondary education	Had to take the first developmental mathematics course twice to pass it. Was enrolled in the second of three developmental mathematics courses during the study.	Had taken other college level courses. Was enrolled in the highest (third level) developmental mathematics course available during the study.
Attitude	Worked very hard to analyze what he saw. Often apologized for making mistakes.	Keen desire to understand everything. Wanted to work to solve problems beyond the designated time for the session.

	Marlon	Marjorie
Non-academic influences	Experienced the influence of gangs in his childhood	Lost her mother to cancer while she was in high school

The Teaching Experiment Sessions

Following is a summary of what occurred in the teaching experiment sessions. This summary will focus on basic events and choices made during the course of the experiment and not on the mathematical or theoretical results. Table 11, provided as a summary of the sessions for both subjects, will be followed by a narrative description of the sessions.

Table 11

Content of Teaching Experiment Sessions

	Marlon	Marjorie
Session 1	He told about his background and shared some mathematics he remembered. He examined "Looking at patterns" and "Looking at dot patterns"	She told about her background and shared some mathematics she remembered. She examined "Looking at patterns" and "Looking at dot patterns"

	Marlon	Marjorie
Session 2	He went back over his thinking for “Looking at dot patterns”. He was introduced to “Another dot pattern” and the software. He explored graphing coordinate points with the software.	She was introduced to the software, and after free exploration she then explored the xy plane, including the graphing of coordinate points.
Session 3	He continued to explore the graphing capabilities of the software. He described the patterns he saw in “Another dot pattern”, graphed the data points representing, and made observations and predictions about them.	She continued her exploration of graphing points with the software. She was reminded of her work with “Looking at dot patterns” and she continued that analysis.

	Marlon	Marjorie
Session 4	After about 22 minutes of work to remember what he had been doing, he was asked to make predictions about the number of dots in the 20th, nth or xth pattern. He was introduced to the functions menu and encouraged to try the functions menu, using x as a variable and to try creating a function which would pass through the data points.	She explored the function menu, and graphed some constant functions. She was encouraged to graph $h(x) = x$, put a measured sliding point on it, and asked her what it meant. She matched algebraic and graphical representations and looked for intersections of $h(x)$ with the constant functions.
Session 5	With facilitation, he explored as he tried to recall what we had done previously. He was encouraged to test ideas he was building about functions.	She was challenged to graph a function that would pass through "Looking at dot patterns." She explored the effect of the change of value of k in functions of the form $f(x) = kx$. With facilitation, she used the software to find the equation of the function passing through her data points.

	Marlon	Marjorie
Session 6	With facilitation, he placed a vertical line at x = 7 to help himself think about what was happening on the graph of the function g(x) = x + 9 with which he had chosen to work. Once those two were graphed, he was encouraged to examine their intersection.	She was presented with a new table of values, and asked to study it and use the software to explore it.
Session 7	Using written instructions, he created a dynamic representation which included a movable measured point attached to the graph of g(x) that generated a table of values.	(Only 6 sessions held with Marjorie)

Marlon's sessions. The purpose of the initial interview session was two-fold. First I asked him to talk about his background in a relaxed way. This provided me with information about him and allowed him to become comfortable with the setting and situation. Second, I asked him to analyze the patterns found in the two handouts "Looking at patterns" and "Looking at dot patterns", found in Appendix C. This allowed me to learn something about his mathematical thinking and to provide content which the software would be used to investigate in later sessions. Marlon attempted to remember some of the mathematics he was doing recently, but did not accurately remember how to use the devices of rote memorization he was familiar with such as the first outside inside last (FOIL) method of multiplying two binomials, which he misapplied. He looked at “Looking at patterns” and “Looking at dot patterns” during this session, taking pains to examine them closely.

During the second session, I allowed him to take considerable time going over his thinking for “Looking at dot patterns” so that I could understand his thinking better. I then gave him “Another dot pattern” which was a simpler growing pattern, and he seemed to work with it more fluidly. About 20 minutes into the second session, after his examination of “Another dot pattern” he was introduced to the software. He had not heard of Geometer’s Sketchpad. He was allowed to explore the tools, and then he was introduced to the graph menu. I asked him to choose graph and define coordinate system, asked him if he was familiar with it and what he could tell me about it. I could then choose steps which let him become familiar with the software and at the same time let me see him demonstrate and build understanding of graphing representations. I had him use the point tool to place a point on the plane, use the measure menu to find its coordinates, and use the selection arrow tool to move the point around so he could see how the coordinates changed, predict what the coordinates would be in certain locations on the plane, and test those predictions. This gave him practice selecting and moving objects and using the menus. Later I had him use the plot points menu, which places a fixed point at one location, so he could see the difference in the two types of graphing methods. I also introduced him to a method for changing the scale of the graph.

In session 3 I continued to allow him to explore the graphing capabilities of the software to try to recall what he had done the previous session for about 30 minutes. I then turned his attention to “Another dot pattern” before asking how we might use the software to explore the data found in the pattern. He described the patterns he saw. In his work on paper the data was already being represented as a table of values and with some scaffolding questions on my part, he graphed the data points with the step number as the value of x and the number of dots in that step as the value of y. Once the points were graphed, he made observations about why they fell where they did and what their relationship to each other was. He also made predictions about where additional points would fall. By having him graph the points that were in the table, I hoped to help him make the transition from a representation he seemed comfortable with (the table) to a different representation (the graph).

During the fourth session, Marlon initially still had trouble remembering how to graph with the technology. Helping him remember related mathematical vocabulary seemed to help him find the right menu choices. He first plotted random points, even though I put the work he had done on paper for “Another dot pattern” in front of him. After about 22 minutes of refamiliarization, we discussed how many dots would be in the 20th pattern and then in the nth pattern or the xth pattern. Following this discussion, I introduced him to the functions menu of the software. After some introductory explanation, I encouraged him to try creating some functions which used x as a variable. Even though I knew his understanding of these representations was weak, I wanted to see how much he could learn from using the software to explore such representations. He created and graphed some linear functions, and made some observations about them. I then challenged him to try creating a function that would pass through the data points he had graphed. With some scaffolding questions designed to help him observe what was happening in the things he was choosing to try, he was able to do so. I encouraged him to write down what he had found and to take notes on the other ideas he had explored. After this session I wondered how much he really understood about what he was seeing.

For the next session, the fifth with Marlon, I started the session with a file already open that showed coordinate points modeling data from “Another dot pattern” graphed on the xy-plane. The x-coordinate represented the step number, and the y-coordinate represented the number of dots in that pattern. I also encouraged him to use the display menu to change the colors of what he was creating so that the algebraic and graphical representations of the same function would be the same ocolor. I let him explore a little before giving him his notes from the previous session. He did not remember how we got the graph of the line that passed through the data points and created some other functions as he tried to remember. I focused my questioning on helping him think about what was happening and facilitating the explorations he was trying to make to build understanding. With this facilitation, he was able to reason his way back to the correct representation. I also encouraged him to test the ideas I saw him building about the representations of functions by trying other functions which were similar to or different from the ones he had been using.

In session 6, I wanted to challenge him to go beyond the understanding he seemed to have about the representations associated with functions. About 30 minutes into the session, after his explorations about the ideas he had been building, I facilitated his plotting of a vertical line at x=7 to draw his attention to the fact that the variable x in the function could take on many different values, such as 7. He had been thinking of x as zero. I hoped that looking at the intersection of $g(x) = x + 9$ and $x= 7$ would help him see that the function g(x) crossed over other places than the y-axis. I also hoped to give him entry into the meaning of the algebraic representation by allowing him to find points on g(x) other than the intercepts on which he had been focusing. In session 7, I gave him written instructions for creating a dynamic representation which would include a movable measured point attached to the graph of g(x) that generated a table of values. I hoped to see whether this would help him make the next conceptual step from noticing that the graph “crossed over” particular places to seeing it as the set of all such locations.

Marjorie’s sessions. As with Marlon the initial session with Marjorie allowed me to learn something about Marjorie’s background. I also allowed her to share some mathematics that she remembered and presented her with both the “looking at patterns” and “Looking at dot patterns” handouts. I found that she saw the pattern present in “Looking at dot patterns” more easily than the other two subjects. Because she was the second subject in the study, it was several weeks later before the second session with her was held.

For Marjorie’s second session I decided to start her with the introduction to the software, rather than looking back at her previous work as I had done with Marlon. As with Marlon, after some free exploration, I had her explore the xy plane in order that I might learn something about her understanding of it even as she built understanding and continued to learn about the software.

The screen shot of the third session was lost due to a technical error. It started with Marjorie looking at graphing points for a while until I felt she had done enough to be able to move ahead and go to the next task. About 15 minutes into the session, I asked her to think back to her work with “Looking at dot patterns”. She had seen that it went up by 3 each time. I added the idea of step number and asked her how else she might represent that data. After examining the graph of the dots pattern data with the technology, I asked her to tell me how many dots would be in the twentieth step. She made a general guess by looking at the technological graph and then extended the representation in the table of values to test her guess. I asked her how many dots would be in the 100th pattern and she used the paper representation of the table of values to solve this problem.

In the fourth session, I had her explore the function menu. She graphed some constant functions formed by entering a parameter in the formula. After she looked at the constant functions, I encouraged her to graph h(x) = x, put a measured sliding point on it, and asked her what it meant. I wanted her to see what understanding she had or could get from these explorations before we went back to the dot pattern and table of values. After encouraging her to match algebraic and graphical representations, I decided to let make her own choices as to what she would do next. She wanted to find the intersection of h(x) with the two constant functions. With minimal help she remembered what functionality and tools of the software would allow her to do that.

At the beginning of session 5, I asked her to remember what she had been doing and had two of her saved sketches ready to which she could refer. I challenged her to graph a function that would pass through the graphed data points from "Looking at dot patterns." I reminded her of the work she had done with the paper representations. Her choices led to an exploration of the effect of a change in slope on the graph of a linear equation. After she had made connections between what she found and the dot patterns, toward the end of the session, I assisted her in placing a line through the graphed points and asking the software to find the equation of the function for her and asked her to relate it to what she had already been doing. In session 6, I presented Marjorie with a table of values, and asked her to study it and use the software to explore it. Following these explorations, I asked her to give some concluding remarks about her experiences.

Major Themes Arising from the Study

Data analysis began with ideas from the literature related to teaching experiments and representation and a general search for places where technology use seemed to be important or to affect the student in some way. Through looking at the subjects' mathematical thinking, it became apparent that particular misconceptions were repeated. It also became apparent that the subjects had particular strengths in their mathematical thinking, such as the ability to observe patterns and to reason and make sense of things. Identifying certain types of mathematical thinking such as recursive thinking (noticing how patterns change from one to the next in a sequence of patterns) was also important. The types of mathematical thinking that were observed are collectively referred to by the category name *mathematical thinking processes.*

The importance of connecting multiple representations, the idea of a students' internal representations, and the ideas of validity, usefulness, and endurance were codes arising from the literature related to representation which were expected to occur. Mathematical language and visual observations were codes addressing behaviors related to verbal and visual representations. Idiosyncratic use of representations became apparent when the subjects used standard representations in unexpected ways. The idea of indicative movements arose from observations made about gestures used along paper representations and technological gestures made through the use of mouse movements. Some studies were found which helped to underscore the importance of these unexpected notions and assist in incorporating them into the present study (Campbell, 2003; Stevens et al., 2008). The category name *representational ideas and issues* is used to refer collectively to these constructs.

Some of the unexpected notions arising from an examination of representational issues also gave rise to notions relating to what was being learned about the use of technology. For example, examining indicative movements allowed the technology to become a window into the students' mind, an idea Stevenset al. (2008) described in relation to their work with LOGO. It also became apparent during the course of the study that technology was being used as an aid not only to reasoning, but also to mathematical communication, and to the use of standard representations. Technology also allowed misconceptions to be revealed and cleared up. In addition, the subjects were observed making choices which showed they were empowered by the use of technology. These ideas have been collected under the category name *influences and uses of technology.*

The reader may notice that there are additional codes listed in the coding guide. These codes were helpful during analysis in understanding aspects of some of the other codes which are used in the discussion of the results. They are not specifically addressed in this discussion, because the ideas they represent are encompassed by other codes. For example, the ideas of disequilibrium and equilibrium can be seen in the discussion of ways in which technology is used to reveal and clear up misconceptions. The codes listed under the category "other" were used to identify data which did not illuminate the current theoretical investigation. They were necessary because all data must be coded in some way.

Following is a more detailed look at the major ideas which arose from the study in each of these three categories and how those themes were evidenced in the data. A look at mathematical thinking processes will be followed by a look at representational ideas and issues. Finally, the influences and uses of technology will be examined. Each section will include a table which summarizes some of the data evidenced in the work of Marlon and Marjorie.

Mathematical thinking processes. As the study progressed, it became apparent that both Marlon's and Marjorie's mathematical thinking processes contained misconceptions and also showed evidence of their ability to reason. These ideas were important to understanding their interactions with mathematics technology. Weaknesses seen in the subject's mathematical thinking processes included algebraic misconceptions, function and coordinate point confusion, and graphical confusions. Strengths included observing patterns and problem solving, reasoning, and sense-making.

Algebraic misconceptions. Both Marlon and Marjorie showed evidence of algebraic misconceptions that appeared to be hampering their mathematical progress. Marlon examined the different representations carefully. At first he made connections and started to understand things and then a misconception interfered with the understanding he was building. It became apparent in the initial interview that Marlon had experienced some procedural instruction which did not provide him with useful and enduring mathematical representations. He stated, for example, that "I can always do this . . . this is the foil" referring to the FOIL (first, outside, inside, last) method of multiplying two binomials, but then he applied this knowledge to the problem $(4 \cdot 6)(4 \cdot 6)$. Figure 4 shows the representations he shared in the initial interview when asked to "show" and "tell . . . about" some mathematics of his choice that he remembered. His inability to properly use a procedure with these algebraic representations is evidence of his algebraic misconceptions.

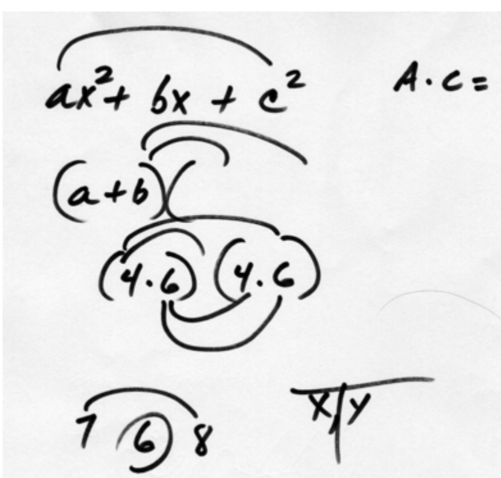


Figure 4. Representations Marlon created to show mathematics he remembered. Arcs indicate that the FOIL method was applied to the multiplication problem.

His lack of algebraic understanding could also be seen in his use of variables to explain his mathematical ideas. Marlon observed that the sequence “Another dot pattern” (Appendix C) added one to the step number to get the number of dots. When I asked him how many dots would be in the eighth step, he said, “I would just, in this case . . . I would just add one . . . if it was 8 it’d be 9.” He also said in a later session that “all I’m doing is actually adding a 1 to that and 10 it’s going to give me 11.” This was a correct description of the mathematics in the pattern. When I asked him to express the same idea using n as the step number, however, rather than giving the number of dots as n + 1, he reasoned that “I’m just looking at n representing a certain number, a pattern and o being the next letter in the alphabet. So it’s the same thing as far as the numerical pattern. I would assume that the letter is going to be in a alphabetical order.” His reasoning was logical and made sense in his mind, but showed that he lacked understanding of the use of variables in mathematics. He could describe the pattern but he could not represent it algebraically using a variable for the step number, which was further evidence of algebraic misconceptions.

Though Marjorie had showed some ability to use algebraic representations in the initial session, she also revealed weaknesses during the course of the study. She read f(x) = A as “f times x equals A.” Her use of the order of operations was weak, as she calculated $b \cdot 6 + 6$ by saying “6 plus 6 is 12 and you multiply that by B.” Her algebraic knowledge was also not strong enough to allow her to go from insightful reasoning about the function which represented the dot patterns to a corresponding algebraic representation. Although she noted that to get the number of dots you multiplied the step number by 3 and added 1, she could not represent that idea algebraically. As she considered such a possibility, she said, One times three is three but then you still gotta add and add one {she taps in the general direction of the table of values and it is uncertain as to whether she meant to indicate a certain value on the table} so if I was to try to do an algebraic expression, I guess I would do it like that — It'd be like

INTERVIEWER: Like what?

MARJORIE: One times x or um one x plus one equals y - something like that, because I'm not quite sure exactly how I would write it as an algebraic expression

She did say "the step" when asked what x would represent. Marjorie had the ability and desire to think logically and solve problems, but there was a gap between her ability to think and reason mathematically and her ability to use standard mathematical notation. This shows evidence of misconceptions as to the purpose and function of algebraic representations.

Function and coordinate point confusion. A particularly important misconception that Marlon brought to this study was his lack of understanding of the algebraic representation of a function. Marlon consistently confused the representations of coordinate points and the representations of functions. This was most commonly exhibited in his tendency to enter the function f(x) = a + b in his attempt to graph a function which would pass through the point (a, b). His confusion was also manifested in his representation of a coordinate point as a sum. When I asked him to represent the point (0, 9) after its location was identified on a graph, he wrote 0+9. When I asked him to write the same idea as if it were in a table, he said, "In a table . . . it'd be here, zero, plus nine" and entered 0 in the x column and +9 in the y column. Because of the uniqueness of the confusion he exhibited and the persistence of the misconception, I have kept this as a separate code from other forms of algebraic misconceptions, even though Marjorie did not exhibit the same tendency. It also bridges algebraic misconceptions and graphical confusion.

Graphical confusion. Marjorie exhibited considerable depth of observation and deduction, but when confronted with standard graphical representations she had difficulty working with them. As noted above, she did not seem to remember the idea of using a table of values for graphing. In addition to that, during the introduction of the software, additional weaknesses in Marjorie's graphical preparation appeared. Note that some of the misinterpretations may have come from misunderstandings about the terminology being used and a lack of understanding of what was being asked of her.

She had constructed a circle and point on the circle. She had also used the software to find the abscissa of the point on the circle, moved the point and noted that the value changed but stayed negative "because that's where the circle is, on the negative side." When I asked her to predict what the y-coordinate would be, she at first hesitated then decided that she thought she could. She said that "when you're doing graphing they always have those two, the x and the y: x I think is always the starting point and y is the endpoint." She seemed to envision the x and y values to be starting and ending points of a journey. After finding using the ordinate choice on the measure menu to find the y value of -8.22, she was unable to explain this number. She moved her cursor toward (-8.22, 0) looked for other coordinate points near (8, 0). Later in the same session, when she had a point with coordinates measured which she could drag around the xy plane to observe how the coordinates changed, she had difficulty finding a location other than (0,0) where both coordinates were the same. She was limiting herself in her search to the axes. Even after experimenting and studying the movement of a point and the change in its coordinates, she still described the x coordinate as "the starting point" and the y-coordinate as "where I want to get to . . . that's my second point, my y." When asked to explain what she meant soon after this, she noted locations on the plane where points were positive and negative, observed changes in the displayed coordinates of points A and B that she was dragging around the screen, but still could not really explain the meaning of what she was seeing. She compared it to "plotting coordinates on a map . . . . I know A is where you're at right now, and then B is where you want to get to." She explained her concerns this way:

> It is rather difficult to try to maybe explain what I do see . . . . I have an idea of why . . . it's negative, negative and it's positive, positive, but it's sort of a little difficult to try to . . . explain . . . what I'm thinking right now because I really don't know.

In session 4, she described the graphed points B = (-6, -6) and C = (18, 18) by saying "it gives me coordinates B and C and B is negative 6. {cursor to graphed point B which was at (-6,-6)}And C is 18, positive 18 {cursor to point C which was at (18,18) and then over to the coordinate point representations at the left of the screen}." When asked to tell why two numbers were listed for each point, she then indicated (0, 18) and (18, 0) with the cursor and said "that's where it actually meets." So at least by this time in the study, she seemed to understand what the two coordinate values meant, but the language she used to describe the points was still unconventional. Marlon also exhibited graphical confusion. For example, when asked to move the cursor to a location where both coordinates of a coordinate point were positive, he moved the cursor to the right along the x-axis.

Observing patterns, problem solving, reasoning, and sense-making. Both Marlon and Marjorie were able to examine patterns, and make some sense of them. In doing so, they showed their ability to reason and solve problems. Marlon's ability to think logically was also apparent in the initial interview, when he was asked to examine the pattern found in "Looking at dot patterns". He examined it carefully and determined that since the odd patterns had the leg of the T lined up with the center dot of the base but the even numbered pattern had the leg of the T lined up with the space between two dots in the base, that he would consider the odd and even steps separately as if they were two different patterns. He was confused about whether or not to count the center dot in the base of the odd patterns as part of the leg or not. His inconsistency in doing so caused confusion as he analyzed the odd patterns. Focusing on just the even patterns which he could more accurately analyze, he came up with a reasonable way of thinking about the number of dots in the legs of the even numbered patterns. He gave the base of the T in his fourth pattern 6 dots and the leg of the T in his fourth pattern 5 dots. He added 2 to the number of dots in the leg of the second pattern to get the number of dots in the leg of the fourth pattern. This reasoning does not fit the overall pattern as it would conventionally be analyzed, but it made sense to Marlon based on his observations. Figure 5 shows Marlon's work and to this has been added some of Marlon's statements about the even numbered steps as he saw them. Recall that he was only given steps 1, 2, and 3 in the pattern to begin with and the rest he created. He had at first "estimated" that step 10 would have 13 dots in its leg. He later reasoned logically to determine that he should remove the last two dots so that the leg would only have 11 dots.

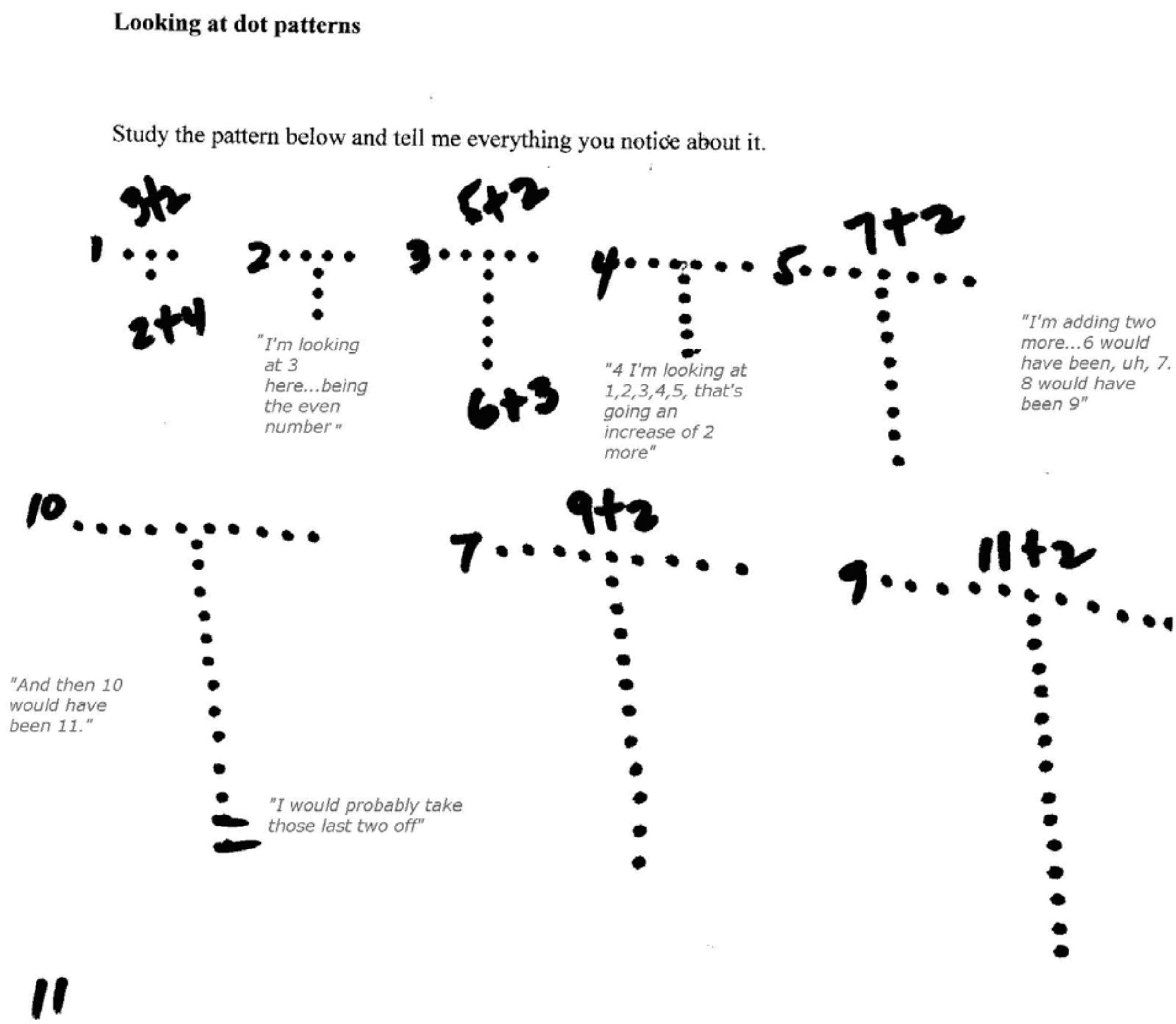


Figure 5. Marlon's work in analyzing the pattern given in the handout "Looking at dot patterns" is supplemented here by a record of some of the statements he made as he was working.

Marlon's focus on an aspect of the mathematics he could reason about amid a more complex and confusing situation, and his selection of a unique pattern that he observed is helpful background information in considering Marlon's interactions with technology. It shows his ability to observe patterns, solve problems, reason, and make sense of things.

When Marjorie examined "Looking at dot patterns", she first noticed the recursive relationship, observing fairly quickly that the number of dots in each pattern increased by 3 with each step. She wrote down the expected number of dots for several more steps in the pattern and drew a picture of the fourth pattern. With encouragement, she put a step number by each pattern and also wrote the number of dots in each pattern. In the third session, after having been introduced to the technology, she was asked to think again about the dot patterns and find the number of dots in the 20th pattern. She said she'd have to "count it out." She could not think of another way to represent the pattern that might be helpful. When asked if she remembered what a table of values was, she said "You mean (like) multiplication table(s)?" and when asked about a table used for graphing, she did not seem to remember it. She was given a blank table and asked to fill in the step number in the left column and the number of dots in the right column.

Once she had this representation to work with, she was asked again to find the number of dots in the 20th pattern. She found this by filling in the table using the recursive idea of an increase of 3 with each step which she had already noticed. She was later asked about the 100th pattern and then she talked about being "able to try to get there more quickly." She studied the table of values and noticed that there would be 61 dots in the 20th pattern. She said, "You know like 20 times three gives you 60, but that's 61." When asked to explain why she said that, she explained "I looked at the step {pointing to the 20 in the step number column} and then number of dots, {pointing to the 61 in the #dots column}." She was encouraged to write this idea down and after she did so she said, "It sort of works . . . because even with the ten . . . you can go . . . times . . . 3, but again its 31, not 30. So it ends up so I guess maybe if I do add – ooh . . . multiplying the step by 3 then add one." She looked at other entries and noticed that they were also 3 times the step number plus one. As she was describing the relationship, she wrote it down as "multiply step by 3 then add 1 = # dots." This episode shows the problem sovling, reasoning, and sense-making ability that Marjorie brought with her to the study and is important to consider when examining her interactions with the technology.

Summary of mathematical thinking processes. Even though Marlon and Marjorie were at different levels of developmental mathematics, there were commonalities in their mathematical thinking processes. Both of them had valid mathematical ideas that they were able to deduce from examining patterns in the data they saw. They were able to observe patterns, and to solve problems, reason, and make some sense of what they saw. Both of them, however, had difficulties expressing the mathematical relationships they saw algebraically. It is not known whether or not Marlon or Marjorie possessed a learning disability, a condition which is not uncommon to adult developmental mathematics students (Epper & Baker, 2009). It did seem to be clear that algebraic misconceptions appeared to be part of what was holding back their progress. Both of them also had difficulty at first in locating coordinate points, demonstrating graphical confusion. Marlon's mathematical thinking included unique a confusion about the representations associated with functions and coordinate points which Marjorie did not share. Table 12 provides a quick reference to some of the ways the constructs related to mathematical thinking processes were evidenced in Marlon's and Marjorie's work.

Table 12

Data Related to Mathematical Thinking Processes

	Marlon	Marjorie
Algebraic Misconceptions	Misuse of algorithm Inability to explain his valid mathematical ideas using algebraic notation	Misuse of the order of operations Challenges expressing her valid mathematical ideas algebraically
Function and coordinate point confusion	Lack of understanding of algebraic representation of function, confusing it with the representation for a coordinate point	She did not exhibit this particular misconception
Graphical confusion	Had difficulty when asked to find a location where both coordinates were positive, moving along the x-axis to the right.	Had difficulty when asked to find particular types of coordinate pairs, such as those whose coordinates were both the same

	Marlon	Marjorie
Observing Patterns Problem solving, reasoning, and sense making	Devised his own unique way of understanding “Looking at dot patterns”	Deduced the functional relationship in the table of values for “Looking at dot patterns”

Representational ideas and issues. Representational ideas and issues were important to understanding Marlon’s and Marjorie’s conceptions of mathematics. The ability of technology to assist teachers in understanding their student’s conceptions would be an important addition to the literature regarding the use of technology in teaching adults. A teacher’s current use of technology may not provide enough depth of insight into student thinking (Kinney & Kinney, 2002). Important representational issues in this study which were used in an examination of student thinking included the use of mathematical language, validity, usefulness, endurance, indicative movements, multiple representations, and internal representations.

Mathematical language. Marlon’s use of mathematical vocabulary was sometimes confused. On some occasions he used the word intercept to refer to the x and y coordinates of a point. When describing the position of the point (7, -11), he said, “I’m dropping down to a negative 11 for my y intercept – that’s where that point is located right there.” He also used the word formula when referring to the representation needed to plot a coordinate point. After entering 9 in the x column and -7 in the y column of a table, he said, “then what (would) I have to do is plot it (there) with the formula.” When I asked him to remember how he plotted points with the software, he brought up the “plot new function” menu rather than the “plot points” menu which was in the same list. He also sometimes had difficulty understanding questions, for example, when I asked him what the coordinates of an indicated point would be, he did not answer correctly until he was asked what it would look like if written in table of values. Marjorie also confused her mathematical vocabulary. She used the word coordinates on one occasion to refer to the name and value of a parameter. On another occasion she used it to refer to a point on the plane, saying that the software had given her “coordinates B and C” referring to points B and C. These episodes highlight the importance a student’s understanding of mathematical language has to their use of technology.

Validity and usefulness. It is also helpful to note which representations the subjects found to be valid or useful. Marlon, for example, seemed to find a table of values to be a useful representation. It allowed him to consider the two elements of a coordinate point using a standard representation rather than using a + b as representation of the point (a, b), as he sometimes did. It also allowed him to observe multiple patterns in the function represented by “Another dot pattern.” When asked what he would put on the next line of the table of values for “Another dot pattern” after he had made some entries, he noted he would put 6 in the left column and 7 in the right column, explaining that “looking at the pattern here everything is in numerical order, and I notice that the next one here {indicating the right hand column} follows 2 and it’s also starting from 2 in numerical order.” It is uncertain whether the statement transcribed as “follows 2” meant “follows the number 2” or “follows also.” It does seem clear that he was noticing that the right hand column started at 2 and that when looking down the column, the numbers were in numerical order. Later he gestured from the left to the right hand column in explaining why the eighth step would have 9 dots. These gestures, used to show where he was looking for his information, were followed by the statement, "It's just adding one." The tabular representation seemed to help him to see these relationships, and was therefore useful to him.

Marjorie's work demonstrated how valid and useful representations developed for her during the course of her work with the technology. At the end of session 5, after she had been examining slope and seen that the function s(x) = 3x came very close to passing through the coordinate points representing the dot patterns, Marjorie was shown how to construct a line through the dots and use the measure menu to find an equation for the constructed line. She noted that it was y = 3x + 1 and when asked how that related to what she had on paper, she said "Exact same thing, because no matter what it's a multiply of three and you're always going to add one to it."

When confronted with a table representing a different, unknown dot pattern in the last session she was able to describe its functional relationship fairly quickly. She noted that that there was a "difference of 2" and that "you multiply that step by the difference of the dots and add one." She still did not know how to describe the relationship algebraically. She did use the software, choosing from among ways that had been presented to her, and with some facilitation, found the algebraic representation and then described how that representation related to the functional relationship she had already described. This time she plotted the points for the pattern using the point tool instead of the plot points menu and so the resulting equation did not exactly correspond with the function describing the table of values. The software result was y = 2.02x + 1.04. She said "that is the equation" and that "it is letting me know . . . the difference between the different points which there is a difference of two add one." After I asked her what she would write down if she were to write down what the equation should be, she said she would write "y = 2x + 1", pointing at and gesturing towards the computer screen as she did so. She explained:

MARJORIE: Pretty much really exactly what um the equation is on there {pointing directly at the computer screen} I'd probably have ah x equals or I guess like that y equals 2 x plus one {she paused slightly before the words "plus one" and gestured in the air towards the computer screen as she said this}.

INTERVIEWER: Okay

MARJORIE: Because it's a matter of {she pointed to the table of values on paper from which the pattern arose} you're taking . . . how many steps you go {referring to paper} and {gesturing over the table of values left to right across the table} if you're going from like number five, five steps, its 11 dots {she pointed at the 5 and then the 11} if you go to six steps, its 13 dots, there's that difference of two {note that the table stopped at 5, there was a space and then step 20 was displayed - she gestured in the space below step 5} but the way you get it would be five times, well . . . five times ten, I mean five times two is ten add one is eleven. Six times two is 12 add one is 13. So it's multiplying {she now gestured in the air at the computer screen} two times the number add one.

She was making connections between representations. She saw the algebraic representation presented by the software as confirming her understanding of the mathematics in the functional relationship, and in that way it was valid and useful. It accurately reflected for her the mathematics in the dot patterns and helped her communicate those mathematical ideas. She understood that the coefficient of x in the technological representation told the difference between the number of dots from one step number to the next and that x represented the step number.

When I asked her to write on paper next to the table of values what the algebraic representation would be, she wrote $y = 5x + 1$ and said "but since I know what x is then". When I asked her what x was she said “x is two” and wrote y = 5(2) + 1 = 11. She was creating a separate expression for each line in the table and using x as if it represented the slope instead of the step number. For example, for step 5, she wrote y = 5x + 1, for step three she wrote y = 3x + 1 and explained that “since you know that the difference is two, x is two.” When I asked her why the software gave 2x + 1, she said “Because the difference is two . . . the x represents um I think the, the number or the step. 1, 2, 3, 4, 5, 6, .7. I think, I think that’s what the x represents here – the step . . . . The way I did it was the x represents the difference.” She knew that she and the computer had used the variable differently and was able to explain those different uses. It is uncertain why she used the variable on paper as she did, but the fact that she could clearly explain the difference indicates that there was validity and usefulness in those representations for her.

Endurance. Information about endurance can be seen when the subjects attempt to recall what they learned in earlier sessions. When Marjorie was asked to consider the work she had done in an earlier session, she was able to intelligently discuss the function she had created, f(x) = (A -5) + 20, but could not remember entering the one I had asked her to enter, h(x) = x. Regarding h(x) = x, she said “I don’t remember where that came from . . . . I don’t remember if I put that on there or not.”

When considering her work with f(x) however she said, “I already had an equation {cursor at f(x)=(A + 9) - 20} for um, f x and g x {moving cursor back and forth between those two algebraic representations}. And I have A equaling 5 and I have B equaling 2.” Earlier she had noted that entering 5 gave the function the value of -6 and that this number related to the location of the graph in some way. It appeared that that she had a clearer memory of the mathematical objects she created herself than the ones that I prescribed for her.

She also remembered from one session to next that the scale of the graph could be changed, although she could not always remember the technological procedures as to how this was done. In session 4, when she was trying to clarify the location of a graph, she wanted to change the scale. When I asked her what she was trying to do, she said, “trying to get what you had shown me about the moving it up and down . . . you can either make the [grid] squares larger or make them smaller . . . so I could see more” The dynamic qualities of the technological representation had endured for her in a manner that allowed her to call upon such an idea in order to solve a problem.

Marlon also remembered dynamic representations, and from one session to the next could remember how to change the scale of the graph. His verbal challenges, however, interfered with his ability to remember some of the other technological procedures. When asked to remember how to plot points, he said, “Let me just, let me just go up to these here functions again just to introduce myself again to them again {he moves the mouse along the top of the screen across the menus and settles on the graph menu} I can go to the um I can plot a new function or plot new function {he opens the new function window} just let play with it.” He continued to look at the list of menus, at one point opening the calculate menu. I had to facilitate his recall of the correct terminology in order for him to use the correct menus. When I told him the key word was “point” he opened the plot points menu, noted that it looked familiar, and was able to continue with his work.

Indicative movements and multiple representations. In response to my request that she “show me some mathematics” in the initial interview, Marjorie settled on factoring the sum of two cubes. In order to show this, she created her own prepared example by cubing 4 and 2 to get the expression $64x^3 + 8$. She recalled the formulas enough to produce $4x + 2(16x^2 + 8x + 4)$, which was not far from the actual solution of $(4x + 2)(16x^2 – 8x + 4)$. As she thought about it she at first wrote $4x + 2(x^2 + 8x + 4)$. She then gestured with the pen in a bouncing fashion between $4x$ and x^2 and then over to $64x^3$. Eventually she inserted the 16, saying "I knew I missed something there." The gestures may indicate that she was remembering that $4x$ should have been multiplied by the first term in the trinomial to obtain the x^3 term in the original binomial. Other events in the study showed how both gestural movements such as noted above, and movements made with the mouse provided insight into the subject's thinking beyond what their verbal statements alone might suggest. Some of this influence of indicative movements has already been noted in the discussion of the influences and uses of technology. Following are additional examples which highlight the use of indicative movements to make connections between multiple representations.

Marlon seemed to indicate that he was making connections between different representations as he worked to represent the dot pattern found on paper using the standard mathematical representations present in the technology. Figure 6 shows the situation at the time this incident occurred.

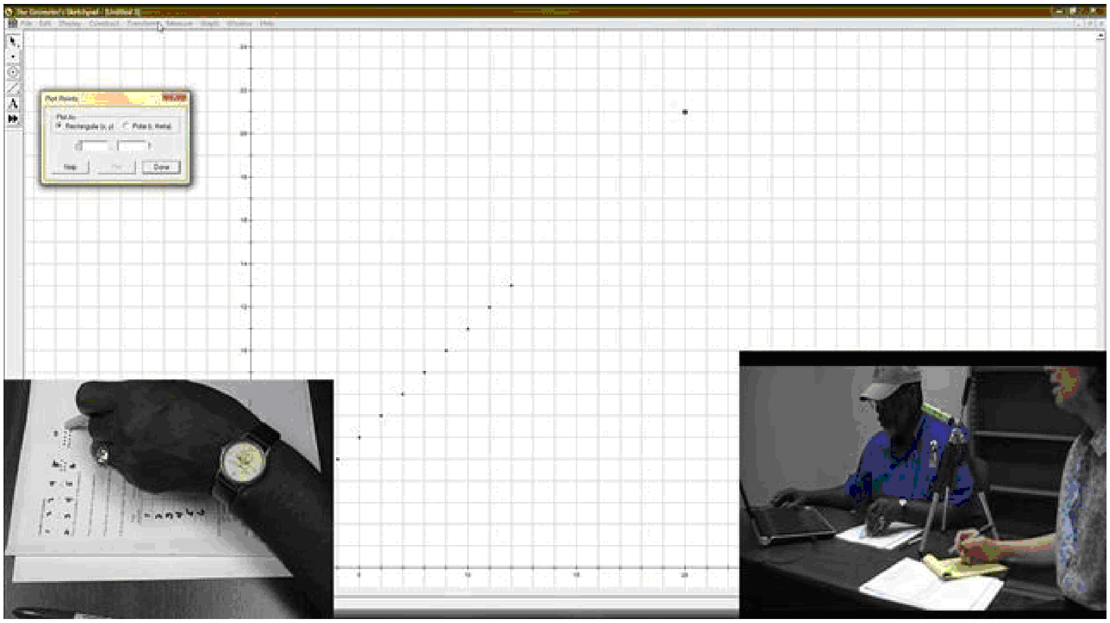


Figure 6: Marlon's situation at the time he was connecting multiple representations on paper and technology.

As he spoke about what he was seeing after predicting the location of (20, 21) and plotting it using the technology, he looked back and forth between the paper and the screen. He gestured with his pen along the paper representations and with his mouse along the technological representations. The coincidental nature of these gestures may indicate that he was connecting the different representations. A transcription of his dialog with actions is shown below.

> MARLON: So again I'm looking at a numerical sequence {looking at the paper and pointing with the pen from the step number 10 to the number of dots 11 written below it}. And again if I assume that that top number is my x-axis {looking at the paper and gesturing from the step number 10 written above the dot pattern to the space to the right of it, then looking up at the screen} because it is running across the x-axis {cursor moving along the x-axis} in a positive motion. My y is also in a positive motion being up here {he looks down at the paper and then up at the screen during this phrase and moves

the mouse to upper y-axis}, because these are also positive numbers going from the center all the way to the top {mouse moves from near origin all the way up the y-axis}.

Note particularly here his phrase “if I assume that that top number is my x-axis” and the indicative movements and statements that followed. He appeared to be connecting the step numbers used to label the dot patterns with the x-axis in the technological representation he was seeing on the screen.

Marjorie also connected multiple representations through the use of technology. One incidence of this occurred when she graphed f(x) = (A + 9) -20 and set A = 1. She noted that the graph was located at y = -10 and by examining the algebraic representation was able to make the connection that if A = 1, then (A + 9) - 20 = -10. In this way she connected the algebraic and graphical representations. This incident will be considered in more detail as an example of the way in which technology empowered Marjorie’s explorations.

Internal representations. One of the goals of the study was to determine the effect of technological representations on the subject’s internal representations of mathematics. As the study progressed, and it became clear that my time with Marlon was growing short, I decided to give him a more dynamic representation in the last session which might build on what he had been experiencing and effect his internal representations in some way. The activity presented him with a prescribed set of technological instructions, since this would introduce a new feature of the technology, and I wanted the technological steps to be clear. Even though the steps were described, he still needed some help in interpreting those instructions, for example confusing the selection arrow tool with the point tool (perhaps because it points to things).

The activity called for him to put a sliding point on the graph of f(x) = x + 9, create an electronic table of values to track where the point was located, and eventually animate the point.

Once the representation was created, he was free to study it and make observations. Marlon's language about what he was seeing seemed to change from the language he had previously been using. During the study he had at first focused on the y-intercept of the graph. He then noticed some other locations where the graph intercepted other parts of the xy plane. After seeing the dynamic representation, he said "when it was sliding up and down, it was actually giving me these different locations {cursor to table and then graph} where it was crossing over the line. Every single one was giving . . . in this case here x and y {cursor from line to table} . . . . So all of these here are actually on this particular connected." His mention of "different locations where it was crossing over" seemed to connect to his previous explorations. The rest of his meaning was unclear. When I asked him what he was trying to describe, he said "The whole line itself." It seemed that his understanding of the graph had moved to a new level. The sliding point and its accompanying table seemed to have helped him to consider the idea of the entire line – an idea conceptually beyond the multiple points he had noticed previously. This is supported by his statement that, "The whole line comes from actually . . . connecting all the different points in a straight line, connecting every last one . . . because they (were) plotted and they all . . . intersected each other." His internal representation seemed to have gone beyond a focus on the y-intercept and "intercepts" of other lines which the graph of f(x) crossed. He now seemed to consider the entire line as a collection of connected points. The phrase "every last one" in particular seems to indicate that his thinking may have been broadened and gone beyond the idea of "crossing over."

Insight into Marjorie's internal representations of graphing was gained when she used both verbal description and indicative movements to demonstrate how she found the location of a coordinate point by movement outward from the origin.

INTERVIEWER: So why does it list the numbers, why does it list two numbers for each of those?

MARJORIE: Because its going off of, for C, its actually um 18 and 18 {indicating (0,18) and (18.0)}and that like, that's where it actually meets I mean if you were at the zero {cursor at (0,0)} you would go to the right 18 {slides to the right along the x-axis} and you would go up 18 {goes up to (18,18)} and that's exactly where that point is.

Summary. Both Marlon and Marjorie had some difficulty with verbal mathematical representations, confusing words on various occasions. This is in keeping with the literature which tells us that adult learners' low literacy skills may hinder their ability to use technology effectively (Li & Edmonds, 2005). They both were able to use tables to see patterns and solve problems. Marjorie appeared to build validity and usefulness in her internal representation of a function, eventually being able to describe in her own words how it modeled the mathematics of the dot pattern it represented. Both Marlon and Marjorie remembered dynamic representations which they wished to use. Both of them also had some difficulty in remembering technological procedures, a representational issue, since recalling the verbal representation or mathematical language associated with the technological representation was a key to recalling how to produce it. Mathematical objects created by Marjorie endured better than those I prescribed for her. Indicative movements were used by both subjects to connect multiple representations of the same idea. Such movements also provided insight into their internal representations. These qualities became important in examining the influences and uses of technology. Table 13 provides a quick reference to some of the ways that constructs related to representational ideas and issues were evidenced in Marlon's and Marjorie's work.

Table 13

Data Related to Representational Ideas and Issues

	Marlon	Marjorie
Verbal: Mathematical language	Said “intercept” to refer to x and y coordinates of a point, referred to a set of coordinate points as a formula, and looked at the function menu when trying to plot points	Confused the use of the word coordinates, using it on one occasion to refer to the name and value of a parameter and at another occasion to refer to a point on the plane, e.g. “It gives me coordinates B and C” referring to points B and C.
Validity and usefulness	Found a table of values to be a useful representation	Built validity and usefulness in her internal algebraic representations of a function
Endurance	Marlon remembered how to change the scale of the graph, but misconceptions interfered with the endurance of other technological procedures	Mathematical objects she created herself seemed to endure better than mathematical objects I prescribed for her to enter. Dynamic qualities of helpful technological representations endured from one session to the next, but the technological procedures used did not.

	Marlon	Marjorie
Indicative movements and multiple representations	Gestured along both paper and technology, connecting the step numbers with the x-axis values	Gestured toward related algebraic representations as she worked on paper. Gestured to algebraic and graphical representations as she explained why the software placed the graph of f(x) = (A + 9) -20 at y = -10 when A was equal to 1.
Internal representations	Expressed a clearer vision of what a graph was representing following examination of a dynamic representation connecting table and graph	Demonstrated her understanding of graphing coordinate points as a movement from the origin. Also see validity and usefulness.

Influences and uses of technology. Marlon and Marjorie interacted with technology in particular ways that showed how technology can be useful and influential for adult developmental mathematics students in ways which add to its use in saving time, providing individualized attention, increasing confidence, and decreasing anxiety (Kinney & Kinney, 2002; Li & Edmonds, 2005; Taylor, 2008). Technology was used as an aid to mathematical communication and as an aid to reasoning. It was also useful for revealing and clearing up misconceptions. It provided a window into student thinking and aided them in the use of standard representations. It also empowered both Marlon and Marjorie mathematically. These ideas show how technology can be used in mathematics education and how it can influence student thinking.

Technology as an aid to mathematical communication. As he struggled with what was presented to him in the initial interview, Marlon gestured toward the representations he found and created on paper. Once introduced to the technological representations, he used the mouse to point just as he had used his hands. The use of mouse movements provided a bridge for him as he tried to communicate his thinking. An example of this occurred early in his struggles to find a way to use standard representations to analyze the data from “Another dot pattern.” During the second session, I gave him “Another dot pattern” to study and introduced him to the software. In the third session, I asked him to graph data points from his table of values for “Another dot pattern” so the pattern could be analyzed with the software using standard representations. Figure 7 shows a screen shot of the graphed points.

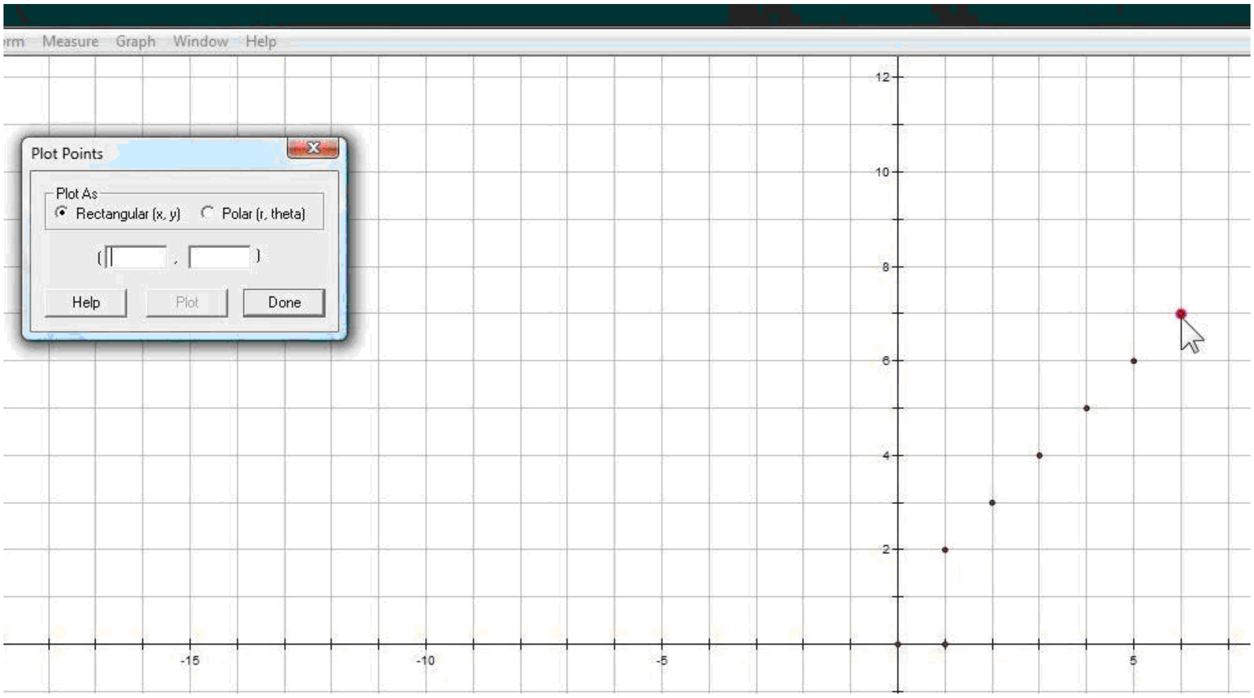


Figure 7. Marlon's graphed points which arose from the dot pattern shown in the handout "Another dot pattern"

Once the points were graphed he was asked if he noticed a relationship between them. He said that the points were "in a straight line" and "going actually on a 45 degree angle." When I asked him why he said that, he said "it's just completely straight on a 45 degree angle" and then he said, "Well now it's . . . I take that back." He then used the mouse to explain why he was changing his mind.

MARLON: Okay, here {tracing along the positive x-axis and then the positive y-axis} right now I'm dealing with a - I would say a 90 degree angle.

INTERVIEWER: Okay

MARLON: 45 would actually have been here {tracing along the path where the line y = x would be from the origin up to the right}. So it's right off a 45 degree angle. So it's not completely a 45 degree angle . . . it's a little bit off.

He traced the x and y axis and noted that they were at a 90 degree angle and that if the line going through the points had intersected the origin, then it would have been at a 45 degree angle, but since such a line would intersect at (0, 1), he said it was “just off of a 45 degree angle.” In this way he was able to demonstrate with the mouse what his language was not adequately expressing. He used the technology to make the genuine mathematical observation he was making understood.

In Marjorie’s case, technology could aid her communication. In session 2, she was observing how the movement of a point on the plane affected its coordinates. She moved the point around the plane and described what was happening. At one point she described the point A as being “in a negative spot.” At the time, A was at (-2.57, 7.20). The technological representation clarified her meaning, which otherwise would not have been clear. It thereby aided her mathematical communication. In each of these cases, Marlon and Marjorie were using the functionality of the software to accompany their description of mathematical ideas they were trying to communicate. The ideas were clarified, and in this way, technology became an aid to mathematical communication.

Technology as an aid to reasoning. Some of Marlon’s interactions with the technology in session 4 seemed to indicate that he was beginning to reason logically based on the representations he was seeing. He had graphed coordinate points for “Another dot pattern” up to (9, 10). When asked to explain what was happening and where the next point would be located, he correctly located the point (10, 11) before graphing it and without directly referring to the x and y-axis until asked how he knew where it would be. Soon after this, he described the pattern he was seeing and gave the location of the next point, (11, 12), before graphing it. He described what he understood about the pattern of points, accurately predicted the location of the next point in the pattern and graphed it using the technology, declaring, “And there it is.” Later when I asked him to predict where (20, 21) would go he went along the x-axis to 21 and then gestured with the mouse in the general area where the point would be and then made a precise prediction based on the locations of 20 and 21 on the x-axis and the y-axis. The use of technology allowed him to make and follow up on mathematical predictions and so aided him in his reasoning about mathematical patterns.

Marjorie’s work also showed the power of the technology to aid her in reasoning about mathematical ideas. During the fourth session, Marjorie graphed the coordinate points representing the sequence of dot patterns in “Looking at dot patterns.” In the fifth session she was asked to consider what function would result in the graph of a line which would pass through those coordinate points. She noted that in an algebraic expression “the letters represent numbers” and that “if we already know . . . what the numbers are that add up to the value . . . we would just go and just replace . . . maybe one of the values.” So she decided to “replace . . . one of . . . the numbers in the equation.” We had recently been discussing the number of dots in the 50th pattern. As she considered what to replace with a variable, she said “Probably the 3. Maybe like have . . . 50 times x equals 151? Or 150?” After being encouraged to try it, she used the new function menu to create q(x) = 50x. She wanted it to equal 150, but after some discussion, graphed it and said “It did something, but it didn’t . . . that don’t look right. That doesn’t look right at all.” Once she saw what that function did, she tried changing the 50 to different values to see what would happen. In this way the technology became an aid to her reasoning. She tried r(x) = 70x, then s(x) = 30x. She then observed that “the lower the number is, the more it moves away from the y-axis.” She then tried t(x) = 10x. She said, “Yes it does!” She moved the cursor back and forth from algebraic to graphical representations as she explained what she had observed.

Once she saw that a slope of 1 took her line beyond where she wanted it to be, she concluded that "I've got to . . . keep it between one and 10." Eventually she tried 3 times x. She had reasoned that the coefficient had to be between 1 and 5 and because of what she had seen in the pattern, she tried 3. Because of the scale of the graph at the time, the graph of the function appeared to land on the dots. As she discussed what had happened she mentioned that "It was a difference of three when [she] did the step and dots", referring to the rate of change in the pattern. The running conversation she kept up as part of the talk-aloud protocol helped provide insight into her thinking and demonstrate that she was engaging in reasoning and problem solving with the software. She was noticeably excited when it behaved the way she expected it to behave, and technology was an aid to her as she reasoned her way toward her conclusions.

Using technology to reveal and clear up misconceptions. Both Marlon and Marjorie experienced episodes where they discovered misconceptions through the use of the software and ideas were clarified. Two different episodes are used below to illustrate how this happened in Marlon's case. An episode from Marjorie's case follows.

Marlon's case. Because he confused the representations of functions and coordinate points, he used f(x) = a + b to try to graph a function which went through the point (a, b). During the fifth session, I asked him to recall the function he had previously found which, when graphed, produced a line which passed through his data points. He could not remember what that function was, and in his efforts to remember, his confusion over the representations for coordinate points and functions interfered. He opened the function menu. He knew that the function had to pass through the point (1, 2) and so he had graphed f(x) = 1 + 2. Since the graph did not travel in a diagonal line through the points, he tried again and looked at the currently graphed point which was farthest to the top and right of the graph at that time, (8,9), and graphed g(x) = 8 + 9. This graph of g(x) turned out to be out of the viewing window, and he had to change the scale of graph to see where it was. He did this himself with no facilitation after he had done some additional exploration which had produced h(x) = 8 + 9 and the equation 1 +2 = 3. When he finally saw the graph of g(x) he said "I'm getting a straight edge again here" and indicated 1+2 = 3 and then the graph of g(x) = 8 + 9 with his mouse. When I asked him where the graph of g(x) had come from, he at first replied "this last one I just put in" and indicated 1 + 2 = 3, then said "as you were" which was a phrase he commonly used when he realized something was wrong. He then counted to see that g(x) crossed the y-axis at 17. I asked him "Where might 17 come from?" This was genuinely puzzling to him, and he wondered aloud "How did I get 17 in there?" I asked him if there was anything on the screen that might give him 17. The situation at the time he responded is illustrated through the image presented in Figure 8, which is followed by his statements.

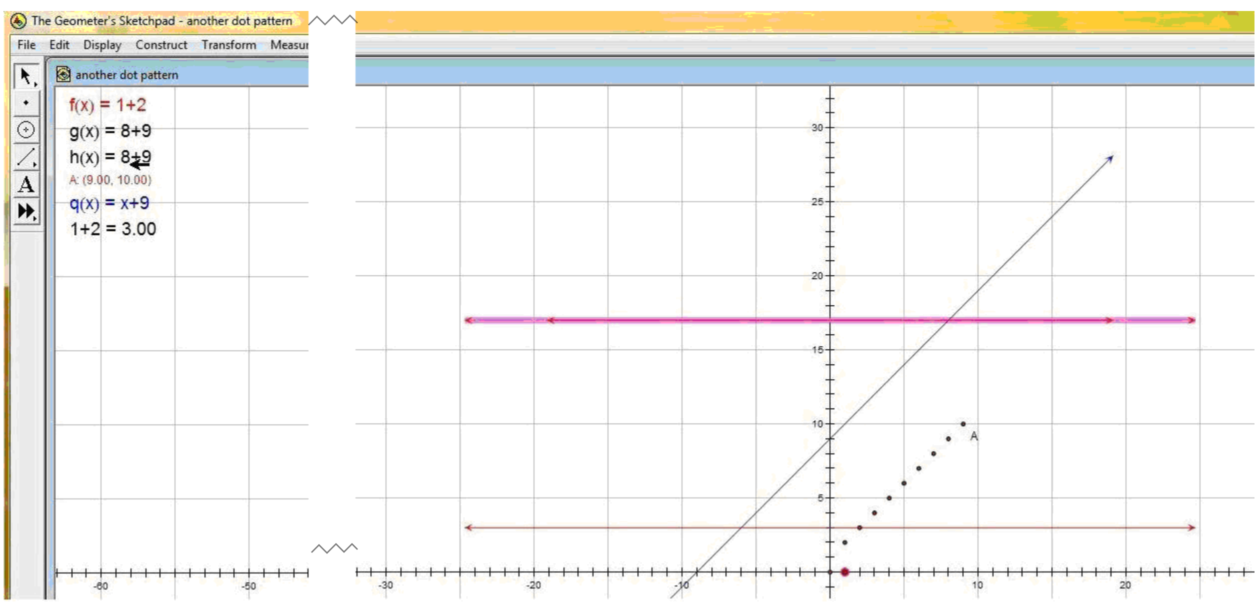


Figure 8: These portions of a screen shot, from which non-essential elements have been removed, show the situation at the time Marlon was asked to consider why the graph of h(x) = 8 + 9 might have been placed at y = 17.

MARLON: {he put the cursor at h(x) = 8+9 then moved it down toward the bottom of the list between q(x)=x+9 and 1+2=3} ,ummmm, {He moved the cursor back up the list and then to 1 + 2 = 3} I don't see, I mean, that would give me 17. How'd I get that one there? {cursor near 1+2=3 and q(x) = x + 9} And this is the, this is the same -similar to the one that I gave down here {cursor at (0,3)} through 3. Okay {cursor to 1 +2 =3} [18:48] {cursor up to h(x) = 8 + 9}. Oh, not unless these added together.

By looking at the multiple representations presented by the technology, he realized that in the functional notation f(x) = a + b, those two numbers were in fact added together to determine where the graph would be located, and were not representative of the two numbers describing a coordinate point, which was the way he had been trying to use them to graph the function. The location of the graph of h(x) = 8 + 9 in a different place than he expected it to be revealed the misconception and examining the different representations present helped clear up the confusion. Even though the representation 1 + 2 = 3 was not the functional representation which matched the graph which passed through (0, 3), the presence of that representation may have been important to his building an understanding that "these added together."

In session 6, I sought to challenge Marlon's understanding of functions of the form f(x) = x + b. He seemed to have an interpretation of them which was restricted to the location of the y-intercept. I had also learned in the pilot study that the use of a vertical line at a particular x value could help developmental mathematics students to visualize more clearly relationships in the graphs they are seeing. I facilitated Marlon's construction of the line x = 7 and asked him to notice where the graph of the function g(x) = x + 9 intersected that line. He found that it was (7, 16) by examining the graph and looking to see which x and y axis locations would give him a point at that intersection. He did not seem to understand the algebraic representation. When

asked where the 16 came from, he said that it came from the y-axis "where it intercepts . . . here again on the g of x", indicating the point (7, 16) and the algebraic representation of the function and the point again, but he did not say anything which indicated that he connected these representations with the idea of adding 9 to 7 to get 16. Earlier, I had asked him to think about some other points that would land on the graph and asked him to think about the algebra and consider what it represented. He said, "x is actually zero {cursor to (0,0)}
. . . and plus 9 is {cursor to algebra and back to (0,0) then up y-axis to (0,9)} plus 9 on my y a - intercept."

When I asked him what else we knew about the line, he pointed out the x-intercept. When asked to graph a point that would land there he tried (0, -9), decided he hadn't done something right and opened the plot points menu again. He began again by saying x was going to be zero and I asked him why. As he explained his thinking and moved the cursor to show that thinking, he was able to realize his own error.

INTERVIEWER: How do you know x is going to be zero?

MARLON: Well if - you know its, this is my x intercept again {cursor up and down the x-axis} so I know it's right there {indicating (0,0)}. And then if I go to a negative 9 . . . Oh {cursor down to (0, -9)}. Okay, um, {cursor back at (0,0}. If it's going to be x . . . is it going to be . . . {looking back at paper representation, pointing with pen to (0,-9)} I'm looking at this is going to be . . . {puts pen down and looks back at screen}. I'm trying to — I'm thinking that it's actually, in order for me to get it here {cursor at (-9,0)} (I'm on) a negative 9 on the x-axis. So if I give it x being zero its going to start here {cursor at (0,0)} and (then) say negative 9 it would have brought me down here {cursor at

(0, -9)}. So I'm thinking now I got to reverse it in order to get it over here. {cursor at (-9, 0)}.

Note that other than (0, -9), none of the other points he had written on his paper had an x-coordinate of zero. Because the technology graphed the point (0,-9) in its correct location, Marlon was able to recognize that his idea that it was the x-intercept was a misconception. He examined the representations further and was able to find the correct coordinate point value for the x-intercept, clearing up his misconception.

Marjorie's case. Marjorie's use of technology also allowed some of her misconceptions to be revealed and helped clear some of them up for her. This can be seen in her work in graphing coordinate points. I asked her to fill in a table of values on paper representing the sequence of dot patterns with the step number in the left column and the number of dots in the right column. She was then asked to graph those points. She dragged a point using the point tool to graph the point (1,4) and placed it at (4,1). I told her to use the "plot points" tool instead so that the points would stay where we wanted them to be. When she did so, the software placed the point in its correct location, and she was able to see her mistake. She was also able to give a clear description of what that mistake was. She said, "I went to the right hand side and . . . I just moved up from the center - I moved up one, and to the right, to the right four. But in actuality . . . I should have moved to the right first and then up four. So . . . it was right and up." In this way, the technology helped reveal and clarify a misconception.

Technology as a window into student thinking. An examination of Marjorie's graphing work also shows how technology can provide a window into student thinking. This adds to the idea of technology as an aid to mathematical communication by showing how technology can reveal internal representations and ways of thinking about mathematics that might not be apparent from the student's verbal descriptions alone. Stevenset al. (2008) in their work with LOGO, noted that examining student's problem solving processes can provide a "window in to the student's mind" (p. 199). Here, the window into student thinking provides a view of their internal representations of mathematics.

For example, when I asked Marjorie to find a point whose coordinates were the same, she moved the cursor from its current location at about (6, 9) over the y-axis and down to the origin. Later during the same session, when I asked her to find another such point she said, "Both the same, let's see. Hm. Actually I cannot. Not where they're both the same. Right there at the origin." Here is the same quotation with mouse movements inserted.

> Both the same, let's see {moves the cursor up to (0, 13), which was the maximum y-axis coordinate}. Hm. {down to (0, 0) and over to (-10,0) and to (22,0) which was the maximum x-axis coordinate, back to (0,0) and down to (0,-13) which was the minimum y-axis coordinate}. Actually, I cannot {cursor back to (0,13)}. Not where they're both the same. Right there at the origin. {cursor is now back at the origin}

The indicative movements described reveal that she was restricting her search to the axes, something that would not have been apparent from her words alone. In this way, the technology has provided a window into her thinking. The episode noted earlier in which Marlon is explaining what he means by "just off of a 45 degree angle" is also an example of how technology provides a window into his thinking. The indicative movements show what he meant and how he thought about the relationship of different angles beyond what his words alone would have told us. He chose aspects of technology to help him communicate, and this communication provided a window into how he thought about angles on the xy-plane.

Technology as an aid in the use of standard representations. After she failed to find a place other than the origin where both coordinates were the same, I asked Marjorie to find a point for which the right coordinate was bigger than the left. She said she could do that "By going into the negative" and moved the cursor to (-11, 0). After being asked "And what happens if you go up?" she went first to (-13, 9), observing that the right coordinate was increasing as she went up. She then began to explore more freely and moved the cursor to (-12, 12) without prompting, noting that "I'm at 12, 12. It's the exact same . . . ." After being prompted that this was true except for one thing, she said:

> MARJORIE: The negative and if I probably did it for the opposite side it'd be the exact same thing except for . . . the left it would be positive 12 and the right would be negative 12.
>
> INTERVIEWER: Why don't you try . . . {she moved the cursor to (12, 12)}
>
> MARJORIE: Yep
>
> INTERVIEWER: Now is the . . .
>
> MARJORIE: Ooooh, no! They're bo — okay, okay. They're both positive because I'm in the positive area {pointing to the screen}. If I go down it's negative {cursor at (12, -8)}.

These appeared to be genuine discoveries for her, even if as reminders of knowledge she possessed in the past. Note that once prompted to move off of the axes, she then made additional choices about what to explore. In addition to the interaction noted above, she made general observations about the xy plane, moving a point around the plane to confirm her observations, noting for example that "when I drag the A over to the left hand corner, the first number is negative." The technology had been an aid to her in the use of standard graphing representations.

Technology also helped Marlon in his use of standard representations. For example, when Marlon saw that h(x) = 8 + 9 passed through the y-axis at (0,17), he was able to make a connection between the algebraic and graphical representations. Later, he was able to create a function on his own using a similar representation.

Technology was also an aid in the use of standard representations in Marlon's work on an occasion when he was trying to find a function which would pass through the data points he had graphed for "Another dot pattern." He had tried r(x) = x + 2, and I asked him to think about what he had done.

MARLON: Okay so now what I did especially is I provided x plus 2, that's what gave me this {he traced that graph}.

INTERVIEWER: Okay

MARLON: And again, x , this is my x axle {indicating the x-axis} I mean axis and plus two gave me right here {indicated (0, 2)} [23:46]. Okay. So if I want to do this here I can actually say x + 1 {indicated (0, 1)} so if I go to . . . the graph a new function {he opened that menu} I can say x plus 1 {he entered it}.

He had not chosen plot new function. Once the function was plotted he said.

MARLON: There she go. x + 1 and the reason why I got it is because here again is my x-axis {traced the x-axis} and plus 1 is right here {he went up to (0,1) from (0,0)} and it intersected through those points that I graphed last week {he traced along the graph}.

Note that he used the cursor to indicate the location of the previous attempt and that once he graphed the new attempt he used the cursor to indicate and trace along key aspects of the new attempt. It may be that such movements serve to help the learners solidify in their minds the knowledge they are building.

Other instances are indicative of a focusing of the subject's mind as he or she examines representations. This could be seen when Marjorie changed the parameter A to equal 5 and then looked to see its effect. She moved her cursor along the algebraic representation containing A as she spoke aloud, saying "Okay, let me see . . . . 5 {cursor at algebraic representation, and she moved her cursor over it as she spoke} 14, and then I have to, because I did my parentheses first, it was 14, and 14 minus 20 would make that negative 6 because 14 minus 20 {looked up in the air in thought}, negative 6." She did look up in the air in thought at one point here, but she also moved her cursor along the representation in the process of trying to understand it. She then said, "And negative 6 is on the graph {cursor back and forth at the graph} negative 6." Here she was back at the technological representation making connections and building understanding of standard representations.

Empowerment through the use of technology. Marlon was making connections between different standard representations. He used the technology for his own explorations of ideas he was having about what was happening and worked to understand the functions he was creating. In this way technology empowered him mathematically. Two occurrences in session 5 demonstrate this.

Examples of empowerment from Marlon's work. Marlon's response to being asked to graph a function parallel to those he had seen (that is, a constant function) is a good example of the way he was using the technology to build on his own unique understanding. He decided that he wanted a function to go horizontally through the point (0, 14). Rather than entering f(x) = 14, even though he had seen that the numbers a + b in his other examples were "added together", he entered f(x) = 7 + 7. He said, "I would actually go, go to plot new function {which he did} and lets say I want to do 14. I would just say 7 + 7 again {he entered that expression into the function and hit okay} and it should give me straight across." He couldn't quite bring himself to abandon his internal representation of f(x) = a + b, but yet he understood something more than he had at the beginning. He decided himself that the horizontal line he wanted to graph would be at y = 14. He decided himself to use 7 + 7 (rather than something else for example such as 8 + 6), and he was able to see that what he thought would happen did in fact happen.

During session 4, I asked him to enter the variable x into the functions he was graphing. He graphed f(x) = x + 9 and g(x) = x - 9 and noticed that they crossed the y-axis at (0, 9) and (0, -9) respectively. He was building some understanding that functions of the form f(x) = x + b cross the y-axis at (0, b). He also observed that the two graphs were parallel. When I questioned him further, he was able to deduce and demonstrate that f(x) = x + 1 went through his graphed points. In session 5, he had reasoned his way back to that representation again after forgetting it.

Later in session 5, after some discussion, I asked him to try something that he hadn't done before. He entered v(x) = x – 9 and then said "Could I add more?" and I told him he could try that. The software automatically entered parentheses to what he entered to give the function v(x) = (x – 9) + 6. After giving the graph and equation matching colors, he moved the equation close to the graph, studied it and said, "Now how did I get that one?" I turned the question back over to him. During the exchange he moved the cursor from the algebraic representation to just below the y-intercept briefly, back to the algebra and then to a blank spot in the second quadrant.

INTERVIEWER: Think about what's happening with that one. Why is it going the way its going?

MARLON: Okay. x negative 9 - so - x negative 9 here - x negative 9 {cursor at algebra} — I'm coming across here to a 3, {cursor at (0,-3)} Ah! What's happening {cursor at algebra until the word "giving" when it moves to (0, -3)} is that it's subtracting {cursor moves back and forth along the algebra until "negative 3" when he moves it back to (0, -3)} the negative 9 from the 6 and it's giving me a negative 3 and that's the reason why it's intersecting here {indicates (0,-3)} because . . . its subtracting the -9 from a positive 6 which gives me actually negative 3 and again it is diagonal.

His cry of "Ah!" seemed to indicate confidence as did his clear explanation. Although there was most likely much he still did not understand, he understood a correct mathematical idea which made sense to him and was built on his own demonstrated prior knowledge. He had chosen the exploration himself, he had put his plan into action using the technology, and he had drawn a correct conclusion about the connection between the algebraic and graphical representations. Such an exploration may have been difficult for him to do without the use of technology, and in this way he was empowered by his use of technology.

Examples of empowerment from Marjorie's work. Technology also empowered Marjorie to explore her own mathematical ideas. This can be seen in her explorations of the function menu. One of the first things she did in this exploration was to create the parameter A = 1. She then used that in the function f(x) = (A + 9) – 20 and graphed that function. She noted that if one were to substitute in the value of 1 for A, f(x) would be -10. At first she didn't seem to notice the graph of the line. When I pointed it out to her she noticed how far to the left and right the line went. I asked her to consider where it was located "up and down" and she said "looks like negative 9" and wanted to change the scale of the graph, but couldn't remember how to do that as she had been previously instructed. She said that she was "trying to get what you had shown me . . . you can either make the squares larger or make them smaller." By "squares" she was referring the size of the grid spaces on the coordinate plane. In this instance, she knew what she wanted to do with the technology. She wanted to use a feature she had previously seen, changing the scale, in a way that would help her to understand the mathematics better, but she needed some facilitation. Once she was able to change the scale of the graph, she was then able to observe that “it’s at negative ten.” When I asked her why the graph was located there, she moved her cursor to the algebraic representation. Notice her statement and mouse movements as she thought about how the algebraic and graphical representations were related to each other.

> MARJORIE: Ah, you know I really don’t know. Because up here, {cursor running back and forth across f(x) = (A + 9) - 20} well, wait a minute, ten minus, I guess it would be negative ten. Yes it would be. {cursor near the A in that function}. Um, the equation for the function of x is negative 10 once the equation’s worked out. It is negative ten, {she has been moving the cursor along the algebraic representation of f(x) as she speaks and now moves it to (0, -10)} so that’s where it plotted . . . at negative ten. Let’s see {goes to graph menu}. Graph, plot new function {cursor to that choice, but doesn't open it}.

At first she said she didn’t know, then she said “wait a minute” and made a logical deduction. She appeared to consider and connect the different representations and then headed to the menu with which she could create another representation to check her understanding. Though invited to record on paper what she was experiencing, she instead stayed with the technology and created $b \cdot 6 + 6$, with B = 1. Even though she had some difficulty with order of operations, the fact that B was 1 in this instance concealed that weakness, preventing it from being revealed. The graph was at y = 12, where she expected it to be. She was making connections between different representations using her own mathematical choices which were empowered by the technology.

Later in the study, she changed the value of B and was able to see that something was wrong in her understanding. With B = 2, she expected the graph to be at y = 24, but it was at y =

18. After being asked to consider how that might happen, she noted that "you replace the B which equals 2, you're gonna put the 2 in there." After being asked what she would do then, she said, "I guess you could, two times 6 which is 12, and then add 6 to 12 and we get 18. Hm. Okay." Once she realized this, she again chose to explore further. She changed the parameter A to 5 so that she had f(x) = (A + 9) – 20 with A = 5. She said

> MARJORIE: Let's change that parameter for A . . . let's see, let's change that to 5. {after changing it to 5, she hovered the cursor over the "ok" button for 5 seconds, then clicked}. Okay, let me see if I (know), 5 {cursor was now at the algebraic representation and she moved her cursor over it as she spoke} 14, and then I have to, because I did my parentheses first, it was 14, and 14 minus 20 would make that negative 6 because 14 minus 20 {she looked up in the air in thought}, negative 6
>
> INTERVIEWER: Okay
>
> MARJORIE: And negative 6 is on the graph {cursor back and forth at graph} negative 6

Even though she was exploring and building some understanding, there were still limitations to what she knew about the representations, as revealed by her statement that "B is at 12, positive 12 and A is at negative ten because it's going off the equations." Rather than naming the functions f(x) and g(x), she was naming them by using the parameter which was included in their argument. She did, however, move her mouse between f(x) and g(x) and describe the parameters accurately. During the following session, when looking at her saved work from the session in which she had created f(x) = (A – 5) + 20 again, she was able to discuss with some understanding the function she had created, namely f(x) = (A – 5) + 20, but could not remember the function I had told her to enter, namely h(x) = x, which was in the same sketch. Her ability to chose an exploration, learn from that exploration, and then in a later session intelligently discuss her own creation is evidence of the empowerment technology is capable of providing.

Summary. Both Marlon and Marjorie used technology in their explanation and to reason about mathematical ideas. As they did so, misconceptions were revealed and cleared up, and they both learned more about standard representations. Their work provided a window into their thinking. They were also empowered in that they followed their own ideas, creating functions of their own choice for their investigations. Even though their work was at different levels, this empowerment could still be seen. Table 14 provides a summary as to some of the ways the influences and uses of technology were evidenced in Marlon's and Marjorie's work.

Table 14

Data Related to the Influences and Uses of Technology

	Marlon	Marjorie
As an aid to mathematical communication	Used cursor movements in his explanation of why he said the line through the points would be "just off of a 45 degree angle"	Used technology to explain the positive and negative aspects of the xy-plane allowing us to see what she meant when she said a point is "in a negative spot"
As an aid to reasoning	Deduced the location of additional coordinate points for "Another dot pattern"	Explored the influence of k in functions of the form f(x) = kx

	Marlon	Marjorie
To reveal and clear up misconceptions	Gained a better understanding of functions of the form f(x) = a + b. He also discovered the correct coordinates for (-9, 0), which he has mislabeled as (0, -9)	Discovered that she had misplaced the point (1,4) at (4,1) and found its correct location
As a window into student thinking	When communicating his idea that the graphed points were "just off of a 45 degree angle" his cursor movements allow us to see his thinking	Cursor movements showed she was only looking along the axes for a point of the form (a, a)
As an aid in the use of standard representations	Saw that h(x) = 8 + 9 passed through the y-axis at (0,17) and was able to make the connection between the two representations.	Found a point of the form (a,a) other than the origin and observed how the coordinates changes as she moved the point around the xy plane

	Marlon	Marjorie
Empowerment through the use of technology	Created the function f(x) = 7 + 7 to pass through (0,14) demonstrate his growing understanding. Created a function of the form f(x) = (x – a) + b and explained that it passed through the y-axis at –a+b	Created constant functions using parameters and explained how the value of the parameter affected the location of the graph

Chapter Summary

Three major categories of information emerged from this study: the importance of mathematical content and thinking processes in the use of technology, associated representational ideas and issues, and particular influences and uses of technology as related to student thinking. This chapter has examined some of the ways those themes emerged from the cases of Marlon and Marjorie. Even though their mathematical understandings differed in some ways coming into the study, they were able to use the technology in closely associated ways to build greater understanding of representations associated with functions. The themes which emerged in the study appeared to converge in many ways. Indicative movements provided insight into their internal representations. This was evidenced in the ways that Marlon and Marjorie appeared to think about what was happening, and what they appeared to understand. By considering the mathematical thinking and representational ideas present in the technological interactions, the issues and uses related to the use of technology could then be clarified.

Technology's use as an aid to reasoning and communication added meaning to the indicative movements. Those movements served a particular purpose. Technology became a window into students' thinking as they reasoned and communicated about the representations they were seeing and with which they were interacting.

In addition to examples previously given, the convergence of ideas can be seen in some additional examples. Marlon's illustrated what he was thinking about the function f(x) = x + 1 through the use of indicative movements. His efforts showed that he thought of the idea of x plus 1 in the function f(x) = x + 1 as a movement starting to the left of the origin on the x-axis, going to (0, 0) and then moving up and to (0,2) and ending at (0,1) as he spoke the words "x plus one." The insight this gives into the misconception of the role of x in the function might not have been obtained had he not chosen to clarify his thinking through the use of indicative movements. Here algebraic misconceptions, indicative movements, internal representations, technology as an aid to mathematical communication, and technology as a window into student thinking interact with each other.

Another convergence of themes can be seen in work Marjorie did while she was examining the idea of slope. She used mouse movements to punctuate her explanations, providing insight into her internal representations. These movements seemed to be motivated by the request to predict where the next graph was going to land before graphing it. When asked to make a prediction, she used indicative movements to aid in her explanations. Consider the following exchange. Figure 9 is provided below for your convenience in following her explanation.

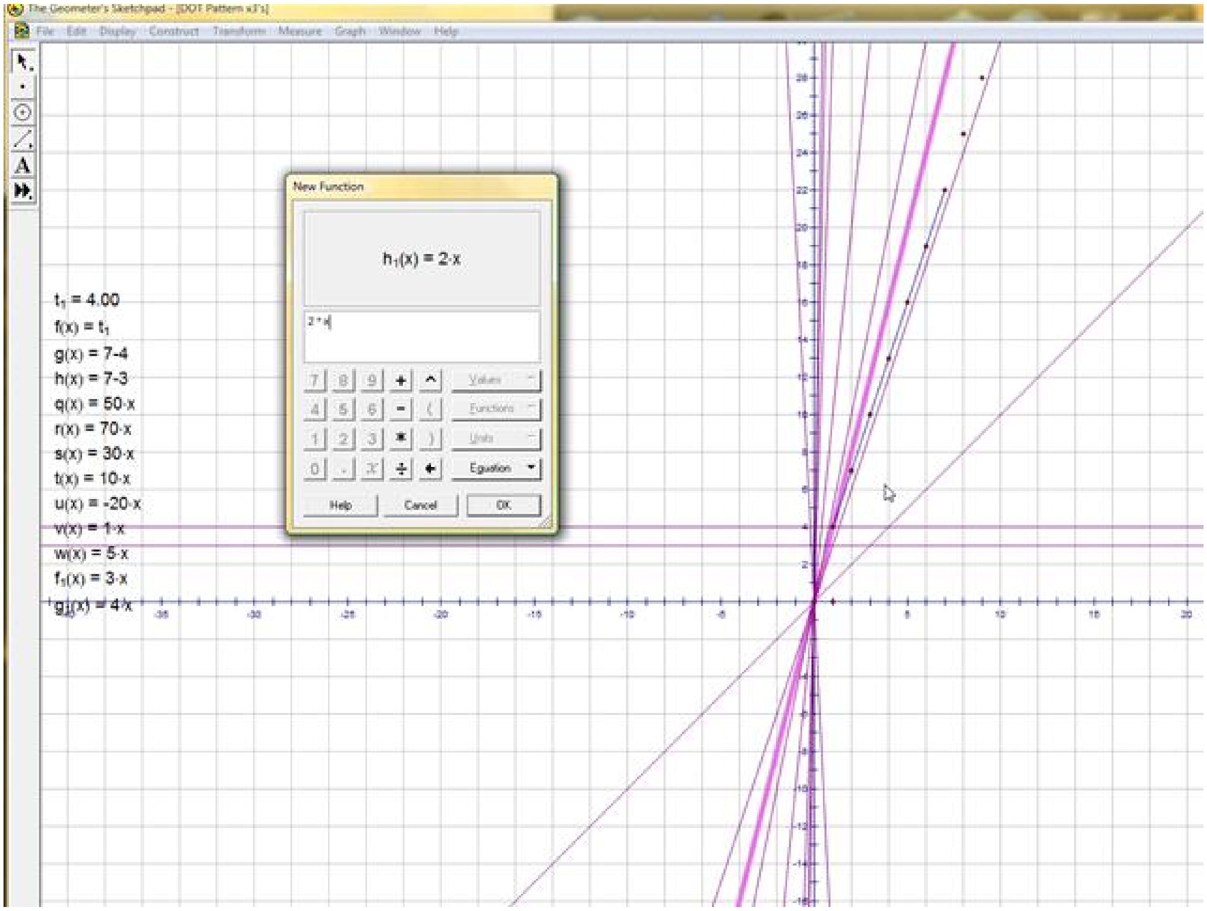


Figure 9: This screen shot shows the appearance of the screen at the time Marjorie provided her explanation of her prediction of where the graph of $h_1(x) = 2x$ would fall.

INTERVIEWER: Before you hit okay, where do you think . . . two times x is going to be — show me with your cursor where you think it's going to land.

MARJORIE: I think 2x might be right here {cursor near (4, 6)} [48:19]. Because the one and x was right there {cursor at about (6,6) on the graph of v(x) = 1x} that's the one and x.

INTERVIEWER: Okay

MARJORIE: And that's 4 and x {cursor at about (2.5, 10) on the graph of $g_1(x) = 4x$} [48:25] and that's 3 and x {cursor just below (3, 10) on a segment passing between plotted points near $f_1(x) = 3x$} so I think that 2 and x will probably be right there {moves cursor in area of first quadrant between graphs of $v(x) = 1x$ and $f_1(x)=3x$ back and forth along a short linear path near where the graph would land}

INTERVIEWER: Okay

MARJORIE: Really because that's 5, 4, 3,{cursor moves from one graph to the next as she speaks, hitting the segment when she gets to 3}. I think 2 will be (about) right there between 3 and 1 {cursor moving in the space in the first quadrant between the functions x and 3x along a short linear path near where the graph would land}, because I think no matter what I try or if I do it going this way {cursor in new function window} it's not going to put it directly on the dot, the step dot coordinates {cursor near the graph of 3x and then back to new function window}. Yeah. And I don't think so. So really 2 (is) probably a waste of time but I'll put 2 out there anyway just to see {she clicks okay to graph it}.

INTERVIEWER: Okay

MARJORIE: Yeah, I was right. {cursor at about (6, 12)} Yeah. 2 was - 2 was just as far out as 1 {cursor back and forth between 2x and 1x} maybe not as far out but yeah. Its definitely not {cursor back and forth between 2x and 3x} 2 or 1 definitely not them. Three is the closest {Cursor near about (5, 15), pointing at the segment joining two of the data points}.

Note that she purposefully moves the cursor from one example to the next and then indicates by cursor movements where she thinks the graph of $h_1(x) = 2x$ will land. Note also when she is analyzing the results, and saying "2 was just as far out as 1" that indicative movements gave clarity to the idea that was in her mind as she said this. It is clear that she knew they were two different graphs and that she knew where they were located, whereas her statement alone would not have provided that information. This example also shows how she was empowered by the technology. She created different functions based on her own explorations which helped to test and build on her own understanding. In this example we see patterning observations, problem solving, reasoning, and sense-making, internal representations, indicative movements, technology as a window into student thinking, technology as an aid to mathematical communication, and technology as an aid in the use of standard representations.

In addition to the convergence of ideas present in teaching experiment episodes, it is also evident that similar influences and uses of technology could be seen in the work of both subjects. This occurred even though they were working at different mathematical levels. Marlon worked with a simpler dot pattern. Marjorie had deduced the functional relationship in the more complex dot pattern. Marlon was confused about the representation associated with a coordinate point. Nevertheless, both Marlon and Marjorie built increased understanding of what they were studying and did so in part through their own mathematical choices. Such empowerment arising from their own choices may help adult learners to grow in their belief in their own effectiveness as learners of mathematics and as a result improve their achievement (Wadsworth et al., 2007). In the next chapter, a discussion will be presented as to how the study specifically addressed the research questions and what it showed about the potential of technology to help build valid, useful, and enduring internal representations of mathematics.

5. Discussion

Following a summary of chapters one through three and a description of the limitations of the study, I will examine the results of this study in light of the research questions. I will then discuss the implications of the study for adult developmental mathematics students, teachers of adult developmental mathematics students, the design of developmental mathematics programs, the general use of technology in mathematics education, and for further research.

Summary of chapters 1-3

The decision to address the problem of the effective use of technology for the mathematical education of adult learners arose in part from an examination of literature related to the use of technology in mathematics education, some of which presented a lofty vision of the potential of such technology use to improve students' learning (Epper & Baker, 2009; Hegedus & Kaput, 2004; Kaput, 1994). It was also clear that adult developmental mathematics students form a substantial population that has continuing and unique needs which require the attention of mathematics educators (Bryk & Treisman, 2010; Epper & Baker, 2009; Parsad & Lewis, 2003). Using technology in the classroom and doing so wisely has come to be expected of those aspiring to be excellent teachers of mathematics, and this includes teachers of adult learners (AMATYC, 2006; National Council of Teachers of Mathematics, 2000). Adult developmental mathematics students have specific needs that need to be addressed (Epper & Baker, 2009; Gerlaugh et al., 2007). Technology seems to have particular promise in impacting adult students' representations (Epper & Baker, 2009; Lapp & John, 2009).

As the National Council of Teachers of Mathematics has sought to improve the standards they promote and the resources they provide, appropriate use of representation emerged as one of the standards for mathematical teaching and thinking (NCTM, 2000). They have also noted technology's influence on the role of representation (NCTM, 2000). Some of the ideas examined with regards to representation have been representational systems, idiosyncratic representations, visualization, symbolization, the use of multiple representations to describe the same concept, modeling as a use of representation, and the mathematical idea of functions as a rich context for examining the use of representations. The ideas of valid, useful and enduring internal representations emerged as an interpretive framework from a study of the literature related to representation.

Benefits to the use of technology include the possibility of opening students' ability to conceptualize mathematical objects and to visualize as mathematicians do (Cuoco & Goldenberg, 1996; Moreno-Armella et al., 2008). Concerns include the promotion of conceptual over procedural learning (Fennell & Rowan, 2001). Other issues include the pace of progress in the cycle of invention, research, and implementation (Fey, 1984; Oncu, Delialioglu, & Brown, 2008). Since the form of technology provided for adult developmental mathematics students is often computer assisted instruction (CAI), the use of CAI for adult learners is important to consider. Its benefits include time saved and an increase in confidence level that some students experience. Challenges include the production of only superficial knowledge and the hindrance that low literacy skills can be in adult developmental mathematics students' use of CAI (Caverly et al., 2000; Li & Edmonds, 2005).

Several studies looking at various populations shed light on some of the connections between technology use and mathematical representations. Software has been developed by mathematics education researchers for specific purposes (Abramovich & Ehrlich, 2007; Yerushalmy & Shternberg, 2001). Studies involving spreadsheets, object oriented programming, graphing calculators, dynamic geometry software, and technological laboratories also provide insight. Issues such as the use of multiple representations, self-efficacy, gaining understanding of the mathematics involved in real-life situations, facing misconceptions, and deepening thinking are among the issues that research into technology and representation provides (Falcade et al., 2007; Hennessy et al., 2001; Stevens et al., 2008; Stylianou et al., 2005).

Research on mathematical representation suggests that students sometimes experience challenges in building their own conceptualizations of mathematical ideas because of the wide variety of situations, procedures, and symbols present in mathematics (Vernaud, 1998). This is particularly evident in adult developmental mathematics students, many of whom carry with them learning disabilities which may make navigating that variety of symbols more challenging, (Epper & Baker, 2009). Challenges to the implementation of the use of technology in adult developmental mathematics education also include student discipline and choice as well as faculty acceptance and perspectives (Epper & Baker, 2009; Kinney & Kinney, 2002). I hoped that by engaging adult developmental mathematics students in a teaching experiment, ideas might arise that would help in thinking about how to address some of these challenges. I sought insight regarding the following questions

1. Following the introductory use of dynamic computer technology to explore mathematical concepts built upon previous knowledge, what internal representations of those concepts do developmental mathematics students possess?

2. What can be determined about the validity and usefulness of those representations?
3. How well do those representations endure over a period of time and in the company of tasks which build upon them?

I decided that the study would be conducted following the qualitative research paradigm using a constructivist theoretical framework. I developed a teaching experiment in which I could construct knowledge of the use of technology in adult developmental mathematics education at the same time that the adult being examined was developing his or her own knowledge at a pace guided by their own zone of potential construction and my associated movements (Norton & D'Ambrosio, 2008). I used the ideas of grounded theory as a framework with which to examine the resulting video, audio, and hard copy data in this case study through the use of a priori, open, axial, and selective coding.

Limitations

This study is first and foremost limited by its nature, in that it was designed to suggest ideas for theory and was not intended as a verification of theory. No claims are made otherwise. In the summaries of the cases and in the conclusions given below, descriptions will be of what seemed to be happening. Suggestions for follow up will also be included. The study was also limited in that it is the interpretation of one person. Even though inter-rater reliability test results may show some consistency in the codes assigned to data, I selected those codes, and my choices are based on my interpretation of what is essential in the data. As discussed by Eisner (1998), the view of qualitative intelligence as connoisseurship includes the idea that a researcher's knowledge about a situation influences his or her perception of it. In this case, I had a favorable view of the potential of the use of technology in mathematics education and believed in the potential of each student, particularly the participants in the study, to learn from that technology. This bias may have affected my observations in some way. I also made some interpretations as to what the subject was thinking. Some of this was validated in by the subjects' own statements, but it nevertheless leaves room for uncertainty. As final conclusions are drawn, and existing theory connected, it is hoped that the proper place and power of this study will be clarified.

Conclusions

The specific purpose of this study was to find out how the use of dynamic computer technology to explore mathematical concepts affected the internal representations of mathematics possessed by adult developmental mathematics students, as reflected in the research questions. I hoped that such an investigation would provide information that would allow adult developmental mathematics educators to make wiser technological choices.

In considering the way in which internal representations might have been determined, Smith (2003) suggested that he learned more about student thinking through conversations with his subjects in which they discussed the development of their representations. Goldin (2003) also noted that research into students' internal representations relies on observations of students' interactions with and production of external representations and that he proposed that researchers use task-based interviews. I used teaching interviews in which I held conversations with the subjects about the development of the technological representations they were using. I also observed their interactions with and production of technological representations.

When I analyzed the results, the subject's indicative movements (gestures and mouse movements), technological choices, written work, and verbal descriptions were all tools which were available to help suggest student thinking. As noted in the description of cases, the subjects in this study appeared to make connections between multiple representations. They also appeared to find technology to be an aid in the use of standard representations and were in some instances mathematically empowered through the use of technology. During the course of their work, they also demonstrated the potential of technology as a means to facilitate communication and to provide a window into their thinking. In many instances, the insight gained into the subject's thinking was revealed through the interplay of their indicative movements and the things they said about what they were doing as a result of the talk-aloud protocol. Such observation and conversation is in keeping with the ideas suggested by Smith (2003) and Goldin (2003) for examining internal representations. A more detailed look at key results of the present study as related to the research questions follows.

What internal representations followed the use of technology? In considering what internal representations of mathematical concepts the subjects' possessed, it is first important to note that the use of dynamic computer technology allowed me to make inferences about those representations. Observations of indicative movements strengthened inferences about the student's internal representations, providing a window into their thinking (Campbell, 2003; Stevens et al., 2008; Yerushalmy & Shternberg, 2001).

For example, when Marjorie was searching for locations on the coordinate plane where the x and y coordinates were both the same, indicative movements allowed me to see that she was restricting her search to the x and y axes. When Marlon decided that the data points he had graphed to illustrate the mathematics in "Another dot pattern" was "just off" of a 45 degree angle, indicative movements allowed me to see that he considered passing through the origin to be a requirement of the pattern being exactly at a 45 degree angle. Marlon also revealed his thinking when he used indicative movements with both paper and technological representations coincidentally to show that he was connecting the step numbers listed on the paper with the x-axis. These and other incidences show that indicative movements can be used to better understand students' interactions with mathematical representations.

As suggested by the work of Yerushalmy and Sternberg (2001), free movement of the mouse is important and in the current study the mouse movements used by the subjects of the study were often not requested for a specific mathematical purpose, but were chosen freely by the subjects in order to communicate or reason mathematically. The current study also builds on the work done by Stevens et al. (2008) by demonstrating that a technological medium other than object oriented programming languages can serve as a "window into the student's mind" (p. 199). I extended the work done by Campbell (2003) in examining student's mouse movements in that paper and technological representations were connected and indicative movements examined in both media. I found that indicative movements recorded simultaneously in more than one media may provide additional insight and relate to and support each other. It also situates the use of recordings of both the learner and the learners' screen work in a teaching experiment in which the thinking of adult developmental mathematics students could be studied in depth. Hennessy, Fung, and Scanlon (2001) in their observations of students' work with graphing calculators noted the importance of the use of paper and pencil techniques to accompany technology. The current study adds to this finding by showing additional ways that the two mediums might work together.

The observations made of the subjects' internal representations showed them connecting multiple representations and building representations upon their own thinking. Each of these aspects will be discussed below.

Connecting multiple representations. In considering mental activity many studies have noted the importance of connecting multiple representations of the same mathematical concept to students' understanding (Abramovich & Norton, 2006; Pape, Bel, & Yetkin, 2003; Stylianou et al., 2005). The subjects of the current study were repeatedly observed to be making connections between multiple representations. This is demonstrated in Marjorie's investigations of slope, and Marlon's investigations of f(x) = a + b. As seen in figure 9, Marjorie had listed the algebraic forms of the functions electronically along with the graphs of those functions. She then described that algebraic information as she moved her mouse along the graphical representations, making statements such as "that's 4 and x" or "that's 5, 4, 3" referring to the value of k in f(x) = kx which went with that particular graph. Marlon was able to see that for functions of the form f(x) = a + b, he could find the sum a + b and that would tell him where the graph of that function would cross the y-axis. In this way he was connecting algebraic and graphical representations. In his final session, when he saw the animated table of values, the algebraic representation of the function, and the graph of the function, his understanding seemed to deepen as well. He saw the animated point, saw the different values that point was place in the table, and eventually made the statement that "the whole line itself" was being represented.

From this I inferred that the use of technology strengthens connections between algebraic, graphical, and numerical representations. Additional evidence for this can be seen in Marjorie's final session. She found an algebraic representation using technology for points she had plotted based on a new table of values. She was able to connect that algebraic representation with the mathematics she was seeing in the table of values, saying "it is letting me know . . . the difference between the different points which there is a difference of two add one." This statement seems to show that her experiences with the technology strengthened her understanding of how the algebraic representation related to the numerical information she saw in the table of values. She was connecting numerical, algebraic, and graphical representations.

Building representations based on their own thinking. Internal representations which were being built by the subjects of the study seemed to be based upon their own thinking. This can be seen in the earlier stages of their explorations when their understanding was weaker. Marlon's choice to try f(x) = 1 + 2 to pass through the point (1, 2) was based on his conception of a coordinate point as a movement to one side and then an upward motion symbolized by addition. This conception was evident from his mouse movements and written representations of coordinate points. He moved the mouse to the right and then up to show the locations of coordinate points. When asked to write down a representation of the point (0, 9), he wrote 0 + 9, and said, "It'd be here, zero, plus nine." In session 4, as he considered how to find a function to pass through his coordinate points, including the point (1, 2), he noted that he should have said 1 + 2 and moved the cursor from (0,0) to (1,0) to (1,2). His choice to try f(x) = 1 + 2 as a function which might pass through the data points for the pattern of dots found in "Another dot pattern" (see task in Appendix C) was based upon his idiosyncratic thinking about coordinate points.

Later in the study after he had begun to build some understanding, as shown I asked him to create a function of his choice. He had noticed that functions of the form f(x) = x + b passed through the y-axis at (0, b). He had already been working with g(x) = x – 9. When I asked him to create a function of his own, he decided to "add more" and created v(x) = (x – 9) + 6. He then observed that the function passed through the y-axis at -3 and was able to deduce correctly that the -9 and the +6 were combined to give the information as to where the graph crossed the y-axis. He was building upon his own understanding to understand a function he created.

Marjorie created three different functions which used constant parameters before she began to build an understanding of the use of x in a function. One of the functions with a constant parameter which she created was $f(x) = B \cdot 6 + 6$. At first she interpreted this representation as the quantity 6 plus 6 multiplied by B. When she tried different values for B, she was able to see that she had been mistaken about the order of operations that should be used, and saw the operation it represented as B times 6 followed by the addition of 6. This increased understanding of the order of operations was built on representations she had chosen to create. Later, she created graphs with different slopes based on the observations she was making about the effect of the parameter k in functions of the form $f(x) = kx$. Her choice to try $f(x) = 50x$ was based on her examination of the table of values she had created for the function she was trying to graph. When she saw that this function did not come close to her graphed points, she chose to try different values for k. The understanding she built about the impact of different values of k on the graphs was built on her choice to examine the representation $f(x) = 50x$ using the technology, a choice which arose from her own thinking about the table of values.

What was determined about validity and usefulness? Valid representations accurately reflect the mathematics students seek to reflect and are flexible enough to allow additional mathematical ideas to be built upon them. They are also accompanied by sound mathematical habits of mind. Useful representations are accessible for reasoning and sense-making, communication of mathematical ideas, and building new understanding.

While there was much about the representations they were working with that Marlon and Marjorie still did not understand, considered within their zone of potential construction (ZPC), they did build some validity and usefulness in those representations. Marlon's observation that functions of the form $f(x) = x + b$ cross the y-axis at the value b added validity to that representation. It also appeared to become a useful representation for him in his examination of functions of the form $f(x) = (x - a) + b$. This is indicated by his examination the y-intercept in order to understand $f(x) = (x – 9) + 6$.

The valid representation of the graphed points for "Looking at dot patterns" was useful to Marjorie in her exploration of the algebraic and graphical representations of that function. She knew that matching the graphed points was her goal in assessing the validity of the other types of representations. In addition, her valid conception of the pattern in the table of values included the understanding that the difference of the number of dots from one step to the next was 3. This became useful to her. She referred to that idea when she tried $f(x) = 3x$ as a function which might pass through the graphed points. She observed that three had been the difference between the number of dots from one step to the next.

The qualities of valid and useful representations can be seen in Marlon's and Marjorie's work. Marlon had built a sound habit of mind to accompany his internal algebraic representations of functions. Once he saw that the y-intercept correlated with the value b in $f(x) = x + b$, he examined the y-intercept in order to understand other representations of functions. In this way, he used his growing understanding of algebraic representations of functions to build new understanding. Once Marjorie noticed the pattern in the table of values, this added a habit of mind to her examinations of other representations. She carried that idea of the rate of change to her examination of algebraic and graphical representations and to the examination of a different table of values. When given a new table of values, she noted fairly quickly that the difference in the y-value from one step to the next was 2. She was able to make sense of this new table of values and also make sense of the algebraic representation the technology provided for her. Sound habits of mind, building new understanding, and making sense of things indicate the validity and usefulness that Marlon and Marjorie were building as they used technology to examine representations of functions.

How well did those representations endure over a period of time? The definition given of enduring internal representations is that they will remain with the student in various situations apart from the environment in which they were initially developed. They are also carried forward, built upon, and refined over a period of time. They become part of the student's "stored knowledge" (Rogers, 1999). They will become part of the set of mental objects available to students (Cifarelli, 1998).

Marlon's mathematical misconceptions interfered with the endurance of what he was learning. In examining his progress over the course of the study, it was apparent that his challenges with mathematical vocabulary interfered with the endurance of the verbal representations needed to use the software without facilitation. He also could not remember how the function $h(x) = x + 1$, which represented the dot patterns he had been studying, had been discovered. He did remember from one session to the next how to change the scale of the graph and used that feature to solve a problem in the fifth session, leading to his discovery that the constants a and b in a function of the form $f(x) = a + b$ are "added together" to give the y-intercept of that function. In the opening to session 6, when asked to show what he remembered, he was able to put an xy-plane on the sketch and graphed the function $f(x) = 5 – 9$, noting after observing the plot of the graph that "they wind up subtracting." Facilitation helped him to reason his way back through some of the thinking he had previously done during session 5 and call back to mind with some meaning the representations he had been studying, particularly the function which passed through the points representing the dot pattern. Marlon's handwritten graphs drawn from his technological work in a previous session were also a help to him in session 5.

Looking at previous work also helped representations endure for Marjorie, who was aided by referring back to a saved technological sketch. When considering the work done in this sketch, Marjorie was able to intelligently discuss the function she had created, namely f(x) = (A - 5) + 20, but could not remember entering the one I had asked her to enter, namely h(x) = x. She also remembered from one session to next that the scale of the graph could be changed, but not always how it was done.

These events indicate some possible conclusions about the endurance of internal representations arising from the use of technology. Saved technological representations which were created by the student and paper and pencil representations associated with them may help the student recall previous work more clearly. Dynamic movements of representations may remain in the student's mind as a tool for problem solving, even though the technological steps which produced them do not. Also, enduring representations may be more likely to be built through tasks in which students have control over their avenues of exploration and which are situated within their ZPC.

Implications

Though this was a teaching experiment conducted with only two subjects, those subjects represented a range of developmental mathematics experiences. The suggested theory may apply to students at more than just one level of developmental instruction. It has implications for adult developmental mathematics students, developmental mathematics teachers, for those who manage developmental mathematics programs, for the use of technology in mathematics education in general, for those interested in understanding student thinking, and for further research.

Adult developmental mathematics students. There are important implications of this study that might be shared with adult developmental mathematics students. This study reinforces the observation that some of the challenges adult learners face is their own lack of self-efficacy, lack of study skills, and learning experiences which may be procedurally rather then conceptually based (Epper & Baker, 2009; Wadsworth et al., 2007). Both subjects in this study found opportunities to make and check their own conjectures through the use of technology. Adult learners can consider that some of those investigations would have been difficult for them to engage in without technology. They can think how this type of empowering investigation may help them to make valid mathematical learning choices. Such experiences may help build their self-efficacy. Student centered investigations may also help adult developmental mathematics students to understand the learning process better in general and thus enhance their learning skills. In addition, they may see that being able to create and test their own conjectures can help them learn conceptually and see the advantage of learning conceptually. In the current study, for example, the subjects sometimes appeared to remember and discuss more clearly their own investigations.

Adult learners may be encouraged by the progress Marlon and Marorie made over the course of the study. Both Marlon and Marjorie grew over the course of the study. Marlon began with great confusion about what he was seeing, and an inability to properly represent coordinate points, representing them as a sum of the x coordinate plus the y coordinate. He did not understand what the functional notation the software required meant and saw it as just another way to graph coordinate points, entering $f(x) = a + b$ in an attempt to graph a line passing through the point (a, b). During the study he was able to discover that this was a misconception and that such functions create a horizontal line crossing the y-axis at $y = a + b$. By the end of the study, he had learned that functions of the form $f(x) = x + b$ create a diagonal line passing through the point $(0, b)$, and that such functions pass through other points on the plane. He learned to associate the step numbers for the table of values representing “Another dot pattern” with the numbers along the x-axis. He also seemed to begin to understand that a function is associated with points along the entire line, representing "the whole line itself."

Early in the study, Marjorie was able to deduce the functional relationship present in "Looking at dot patterns" merely by her analysis of the table of values representing the step number and its associated number of dots. She was, however, unable to use algebraic representations to describe this relationship. During the study she used the software to explore the xy-plane, learn about the effect of k in functions of the form f(x) = kx and find, recognize, and describe an algebraic representation of a table of values similar to the one for "Looking at dot patterns." Her description included the meaning of the variable x in that representation. Adult students can learn from the experiences of Marlon and Marjorie that by using technology in an interactive manner, they can make their own mathematical choices and investigations, and that they can understand mathematics at a deeper level than rote memorization.

Teachers of adult developmental mathematics students. Teachers of adult development students can note that the subjects of this study both brought mathematical misconceptions to the study which affected their progress. Teachers must do everything they can to understand the misconceptions adult learners may bring to their classrooms and to the use of mathematics technology. A better understanding of their students' misconceptions will allow adult developmental mathematics educators to recognize what their students' zones of potential construction (ZPCs) are and how they affect their use of technology (Norton & D'Ambrosio, 2008). The findings of this study imply that technology has the potential to help build learning within adult learners' ZPC.

The results of this study also imply that teachers of adult learners who are incorporating technology would benefit from having multiple forms of assessment they can draw upon. They can consider their students' indicative technological movements and verbal descriptions of what they are doing. They can also examine their students' use of both paper and technological representations in order to more clearly assess their understanding. Consider for example that Marjorie correctly described the algebraic representation for a table of values representing the function $f(x) = 2x + 1$ viewing the technology, but when asked to transfer that knowledge to paper, she used a different representation. When questioned she was able to tell what she meant by the representation she used on paper and to accurately describe different uses of the variable that were involved. Such understanding of Marjorie's thinking would be a great benefit to a teacher in deciding what step to take next to help build Marjorie's understanding of algebraic representations. Validity in making inferences about student understanding should include adequate relevant evidence (NCTM, 1995). Allowing students to represent mathematics in different media and asking students to explain those representations may allow teachers to make more valid inferences as to their students' learning.

Teachers should also take note that Marlon and Marjorie seemed to be able to build more enduring representations when they had some choice over the avenue of exploration and were building on those choices they had made. As noted above, teachers should choose tasks in which students have control over their avenues of exploration and which are situated with those students' ZPCs. In considering the challenges which adult learners may have in remembering technological procedures, teachers may also wish to provide clear facilitating tools to accompany the use of technology so that the dynamic representations and the empowerment they provide can be activated by the student for learning as easily as possible.

Design of developmental mathematics programs. Research suggests that developmental mathematics programs may employ teachers who lack appropriate professional development or have little experience dealing with the special needs of adult learners (Caverly et al., 2000). The teachers may also have beliefs which adversely affect the technological choices they make for their students (Caverly et al., 2000). Developmental mathematics programs should attend to the potential advantages of the appropriate use of technology which this study highlights, such as the mathematical empowerment of adult learners. Program directors may wish to make certain that dynamic interactive software such as Geometer's Sketchpad is available to their teachers and students and that their teachers receive some training in how to implement such technology into their classrooms. With appropriate training, teachers may be able to take greater advantage of time spent in computer laboratories as they learn to select appropriate tasks and assess their students' understanding. They can be trained to make valid assessments as they observe their students' interactions with technology.

They can also learn to question their students in ways that elicit valid information as they circulate among and interact with them. Teachers can learn to make appropriate inferences from the indicative movements they observe and the students' talk-aloud descriptions of their own work. In so doing, teachers of adult learners may be able to more easily uncover some of the misconceptions that their students possess, giving them greater power in serving those students (Li & Edmonds, 2005).

Directors of developmental mathematics programs should also note that there are indications in this study that adult learners may not be carrying into their current coursework the knowledge they are expected to have. Marlon was enrolled in the second level developmental mathematics course. According to the course description for the first level developmental mathematics course, which he had recently passed, he should have been introduced to algebra in a manner that included a discussion of linear functions, but he began the study with a lack of understanding of the representations associated with linear functions. Marjorie was in the third and highest level of developmental mathematics. It is not known whether or not she recently took the previous course, but her placement in the highest level course implied that she was expected to have greater knowledge of algebraic representations than she seemed to possess. The second level of developmental mathematics included basic algebra, linear and quadratic functions and she should have had that foundation, but had difficulty expressing ideas algebraically. Those who direct developmental mathematics programs may wish to examine the knowledge their students are carrying into subsequent levels of instruction, and examine the effectiveness of their programs. It may be that alternative solutions which include the use of dynamic mathematics technology would provide more enduring representations of mathematics and more lasting conceptual knowledge for these students.

The general use of technology in mathematics education. The use of technology as a window into student thinking, as an aid in the use of standard representations, as an aid to reasoning, as an aid to mathematical communication, to reveal and clear up misconceptions, and to empower students mathematically need not be restricted to the education of adult developmental mathematics students. These influences and uses may well occur with other populations, and teachers at all levels may wish to investigate how their own use of technology in the classroom may be expanded to include any or all of these uses not currently being realized. Middle school or secondary mathematics teachers, for example, may wish to allow their students to explain mathematical concepts using dynamic graphing software and indicative movements to show their thinking.

The power that is found in these potential effects of the use of technology is much greater than is often found in the ways that technology is actually used. In the present study, a dynamic interactive form of technology was used and the subjects made their own choices about their use of standard representations, receiving rapid feedback as to the effect of those choices. This feedback was embedded in multiple connected representations, sometimes dynamically animated so that the effect of changes in the representations could be seen. Such technology empowered these subjects in ways that some other forms of technology would not have been able to do. Forms of computer aided instruction (CAI) which do nothing more than provide electronic textbooks or multiple choice question and answer sessions may have their place in reinforcing what students have already learned, but would probably not provide the kind of cognitive power that dynamic connected interactive representations of equations, tables, and graphs was able to give these adult learners.

Further research. The current study opens up many avenues to potential further research (Epper & Baker, 2009). Potential areas of continuing research are the use of technological gestures, the interplay between previously held misconceptions and technological representations, the effect of technology on the internal representations of different populations of students, the use of indicative movements in a large group setting, the use of paper and pencil in combination with technology, and longitudinal work into the effects of dynamic mathematics technology in fostering enduring representations.

Additional research into the use of technological gestures may be able to provide teachers with practical ways to record and analyze such work. It may give them further support in using such representations to understand their student's thinking. Such gestures could be observed in various technological settings in addition to the one used here.

Research into how adult students' misconceptions interfere with their use of technology can provide teachers with information about how to facilitate the use of technology. Such research might show teachers how to diagnose difficulties students are having by observing particular technological behaviors. This might allow teachers to discover weaknesses they had not suspected, giving mathematics technology use additional power.

Future studies may include a similar examination of indicative movements and the resulting insight into student thinking that might be obtained with different populations of students. These might include elementary, middle, or secondary school students or college level students who do not require remediation. An examination of such populations may help generalize the influences of uses of technology noted. Younger students may also become more easily acquainted with new technologies. Observing indicative movements may give teachers at every level insight into their students' misconceptions, reasoning, and thinking. It may also provide students of all ages with a means to communicate mathematically which they had not previously considered. Teachers at all levels could conduct similar experiments with their own students in after school settings.

Campbell (2003) conducted his study in dynamic tracking in a large group setting with elementary pre-service teachers. Large group research settings are also possible with the methods used in the current study. Screen capture software could be installed in laboratory computers, and computer cameras used to record the subjects' expressions and actions. Students' paper artifacts could be recorded as well using additional small cameras or collected for examination. Observers could circulate and question the subjects and their field notes could be used as a basis for further examination and analysis of particular subjects' interactions.

In the current study, I presented the subjects with a paper and pencil activity on which the technological activity was built. During the course of the study, paper and pencil and technological interactions were both used, sometimes simultaneously, as when Marlon was connecting the step numbers on paper with the x-axis represented on screen. Marjorie wrote her thoughts about patterns represented by a table in her own words on paper and then connected those representations with technological ones. A subject of further study might be the interactions of paper and technological representations. How would the study have been different if the subjects had been presented with technological representations first and then asked to use paper and pencil to assist them in their investigation? How can teachers make decisions about the use of paper and pencil together with technology?

One of the ideas presented in this study was the possible endurance of technological representations beyond the use of technology. Because of the limited nature of the study, examination of endurance was limited. Further longitudinal studies could be conducted which look more closely at what students retain from dynamic interactive technological tasks designed to build conceptual understanding. For example, if such tasks were used in the first level of developmental mathematics, would students then retain more of what they had studied as they moved through the second level of developmental mathematics? Would summative paper and pencil assessments indicate that any of the internal representations they may have gained through the use of technology had endured with validity and usefulness?

Conclusion

Technology has the potential to empower adult developmental mathematics students to strengthen their internal representations of mathematics, allowing those representations to grow in validity and usefulness. Technology appears to have the potential to allow adult learners to build understanding by choosing their own avenues of exploration situated within their zone of potential construction (ZPC), although students may also bring misconceptions to these activities. Teachers seeking to provide such empowerment must first carefully consider the misconceptions related to the use of technology. Teachers need to keep these misconceptions in mind when assessing technological interactions. Such misconceptions as well as other aspects of student thinking may be revealed by indicative movements. Teachers must be ready to provide clear technical facilitation so that learners will be empowered by the capabilities of technology and not hampered by difficult technological procedures. Teachers may also benefit from making assessment inferences using multiple sources of data, not necessarily relying on technological interactions alone.

Marlon's cry of "Ah!" after he had chosen to graph $v(x) = (x – 9) + 6$, noticed where it crossed the y-axis, and connected that graph with the algebraic representation and his prior understanding is indicative of the empowerment technology can give adult learners. Such an exploration may have been difficult for him to do without the use of technology. There was still much he did not understand, but he did understand something more than he had previously understood. He made his own mathematical choices and used his own reasoning to analyze the results. Coming from a setting in which he was failing at his attempts to remember algorithms, this was mathematical empowerment. Technology has the potential to prevent developmental mathematics from being the "insurmountable barrier" it was recently described as being by Bryk and Treisman (2010), who noted that difficulties in developmental mathematics was "ending . . . aspirations for higher education" (p. 19). In the year 2008, a higher percentage of African Americans ages 25 to 34 were enrolled in some kind of schooling than were Whites, Hispanics, or the total population (Snyder & Dillow, 2010). Could they have been trying to make up for a lack of opportunity to learn earlier in their lives similar to that described by Tate (1995) in his discussion of African American students' experiences in urban schools? Adult developmental mathematics students need not be victims of traditional teaching methods which have been passed down as an artifact, but which have not been effective in meeting their learning needs (Tate, 1995). Proper use of technology can change adult developmental mathematics education for the better. Considering the efforts put forth by adult learners, the special needs they have which have affected their opportunities to learn, and the substantial population whose futures may be at stake, mathematics educators cannot afford to ignore this work.

References

Abramovich, S., & Ehrlich, A. (2007). Computer as a medium for overcoming misconceptions in solving inequalities. *Journal of Computers in Mathematics & Science Teaching, 26*(3), 181-196.

Abramovich, S., & Norton, A. (2006). Equations with parameters: A locus approach. *Journal of Computers in Mathematics & Science Teaching, 25*(1), 5-28.

Abrams, J. P. (2001). Teaching mathematical modeling and the skills of representation. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 269-282). Reston, VA: The National Council of Teachers of Mathematics.

Alagic, M., & Palenz, D. (2006). Teachers explore linear and exponential growth: Spreadsheets as cognitive tools. *Journal of Technology and Teacher Education, 14*(3), 633-649.

American Mathematical Association of Two-Year Colleges. (2006). *Beyond crossroads: Implementing mathematics standards in the first two years of college*. Memphis, TN: American Mathematical Association of Two-Year Colleges.

Atan, K. A. M., Suncheleev, R., Shitan, M., & Mustafa, M. S. B. (2008). Animated Introductory Calculus: Development and perception. *US-China Education Review, 5*(3), 1-3.

Aud, S., Hussar, W., Planty, M., Snyder, T., Bianco, K., Fox, M. A., et al. (2010). *The condition of education 2010*. Retrieved July 8, 2010. from http://nces.ed.gov/pubs2010/2010028.pdf.

Bridger, M., & Bridger, M. (2001). Mapping diagrams: Another view of functions. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 103-116). Reston, VA: The National Council of Teachers of Mathematics.

Brothen, T. (1998). Transforming instruction with technology for developmental students. *Journal of Developmental Education, 21*(3).

Bryk, A. S., & Treisman, U. (2010). Make math a gateway, not a gatekeeper. *The Chronicle of Higher Education [Section B]*, B19-B20.

Butler, D. L., Beckingham, B., & Lauscher, H. J. N. (2005). Promoting Strategic Learning by Eighth-Grade Students Struggling in Mathematics: A Report of Three Case Studies. *Learning Disabilities Research & Practice (Blackwell Publishing Limited), 20*(3), 156-174.

CAI. (2003). *McGraw-Hill dictionary of scientific and technical terms* Retrieved July 10, 2010, 2010, from Answers.com web site: http://www.answers.com/topic/computer-assisted-instruction

Campbell, S. R. (2003). Dynamic tracking of elementary preservice teachers' experiences with computer-based mathematics learning environments. *Mathematics Education Research Journal, 15*(1), 70-82.

Carnegie Learning. (2010). Retrieved July 10, 2010, from http://www.carnegielearning.com/

Caverly, D., Collins, T., DeMarais, L., Otte, G., & Thomas, P. (2000). *Technology and the "knowledge age"* Paper presented at the First Intentional Meeting on Future Directions in Developmental Education. from http://www.gen.umn.edu/research/crdeul

Chiovitti, R. F., & Piran, N. (2003). Rigour and grounded theory research. *Journal of Advanced Nursing, 44*(4), 427-435.

Cifarelli, V. V. (1998). The development of mental representations as a problem solving activity. *Journal of Mathematical Behavior, 17*(2), 239-264.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Collins, T., Bollman, L., Eaton, S., Otte, G., & Thomas, P. (2000). *Bridging the gap: High school to college matriculation*. Paper presented at the First Intentional Meeting on Future Directions in Developmental Education. from http://www.gen.umn.edu/research/crdeul

Connell, M. L. (1998). Technology in constructivist mathematics classrooms. *Journal of Computers in Mathematics & Science Teaching, 17*(4), 311-338.

Coulombe, W. N., & Berenson, S. B. (2001). Representations of patterns and functions: Tools for learning. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 166-172). Reston, Virginia: National Council of Teachers of Mathematics

Creswell, J., W. . (2007). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks, CA: Sage Publications.

Cunningham, D. J. (1992). Beyond educational psychology: Steps toward an educational semiotic. *Educational psychology review, 4*(2), 165-194.

Cuoco, A. A., & Curcio, F. R. (Eds.). (2001). *The role of representation in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Cuoco, A. A., & Goldenberg, E. P. (1996). A role for technology in mathematics education. *Journal of Education, 178*(2), 15.

Donovan II, J. E. (2006). Using the dynamic power of Microsoft Excel to stand on the shoulders of giants. *Mathematics Teacher, 99*(5), 334-339.

Duranczyk, I. M., Goff, E., & Opitz, D. L. (2006). Students' experiences in learning centers: Socioeconomic factors, grades, and perceptions of the Math Center. *Journal of College Reading & Learning, 36*(2), 39-49.

Duranczyk, I. M., & Higbee, J. L. (2006). Developmental mathematics in 4-year institutions: Denying access. *Journal of Developmental Education, 30*(1), 22-31.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1/2), 103-131.

Edwards, B. (1979). *Drawing on the right side of the brain*.

Eisner, E. W. (1998). *The enlightened eye: Qualitative inquiry and the enhancement of educational practice*. Upper Saddle River, NJ: Prentice-Hall.

Epper, R. M., & Baker, E. D. (2009). *Technology solutions for developmental math: An overview of current and emerging practices*. Seattle, WA: William and Flora Hewlett Foundation and the Bill and Melinda Gates Foundation.

Ernest, P. (1998). The epistemological basis of qualitative research in mathematics education: A postmodern perspective. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Falcade, R., Laborde, C., & Mariotti, M. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66*(3), 317-333.

Fennell, F., & Rowan, T. (2001). Representation: An important process for teaching and learning mathematics. *Teaching Children Mathematics, 7*(5), 288-292.

Fey, J. T. (Ed.). (1984). *Computing and mathematics: The impact on the secondary school curricula* College Park, Maryland: The University of Maryland.

Fey, J. T., Hollenbeck, R., & Wray, J. (2010). Technology and the mathematics curriculum. In *2009 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Fong, A. B., Huang, M., & Goel, A. M. (2008). *Examining thelinks between grade 12 mathematics and remedial courses in Nevada public colleges and universities*. Retrieved July 10, 2010. from http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=REL2008058.

Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in Algebra. In A. A. Cuoco & F. R. Curcio (Eds.), *The Role of Representation in School Mathematics* (pp. 173-185). Reston, VA: National Council of Teachers of Mathematics.

Gadanidis, G. (2008). *Rare events: Technology throughout the history of ICMI*. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI.

Gadanidis, G., Sedig, K., Liang, H., & Ning. (2004). Designing online mathematical investigation. *Journal of Computers in Mathematics & Science Teaching, 23*(3), 275-298.

Galbraith, M. W., & Jones, M. S. (2008). Experiential framework for teaching developmental mathematics. *The Community College Enterprise*, 23-36.

Gay, L. R. (1996). *Educational research: Competencies for analysis and replication*. Upper Saddle River, NJ: Prentice-Hall.

Geertz, C. (1994). Thick description: Toward an interpretive theory of culture. In M. Martin & L. C. McIntyre (Eds.), *Readings in the philosophy of social science* (pp. 213-231). Cambridge, MA: MIT Press.

Gerlaugh, K., Thompson, L., Boylan, H., & Davis, H. (2007). National study of developmental education II: Baseline data for community colleges. *Research in Developmental Education, 20*(4), 1-4.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago (1967) Aldine.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275-285). Reston, VA: National Council of Teachers of Mathematics.

Goldin, G. A., & Janvier, C. (1998). Representations and the psychology of mathematics education. *Journal of Mathematical Behavior, 17*(1), 1-4.

Hall, J. M., & Ponton, M. K. (2005). Mathematics self-efficacy of college freshmen. *Journal of Developmental Education, 28*, 26-30.

Hegedus, S. J., & Kaput, J. J. (2004). *An introduction to the profound potential of connected algebra activities: Issues of representation, engagement and pedagogy*: International Group for the Psychology of Mathematics Education.

Heid, M. K., & Blume, G. W. (2008). Algebra and function development. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 1. Research Syntheses* (Vol. 1, pp. 55-108). Charlotte, NC: Information Age Publishing.

Hennessy, S., Fung, P., & Scanlon, E. (2001). The role of the graphic calculator in mediating graphing activity. *International Journal of Mathematical Education in Science and Technology, 32*(2), 267-290.

Hewlett-Packard. (1993-2009). Atlas.ti: The knowledge workbench (Version 5.2.17). Berlin: Scientific Software Development GmbH.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123-134.

Hollebrands, K. F. (2004). High school students' intuitive understandings of geometric transformations. *Mathematics Teacher, 97*(3), 207-214.

Hollenbeck, R., & Fey, J. (2009). Technology and mathematics in the middle grades. *Mathematics Teaching in the Middle Grades*.

Hsieh, C., & Lin, S. (2008). Dynamic visual computer design for factors and multiples word problem learning. *International Journal of Mathematical Education in Science & Technology, 39*(2), 215-232.

Izsak, A., & Sherin, M. (2003). Exploring the use of [new?] representations as a resource for teacher learning. *School Science and Mathematics, 103*(1), 18-27.

Kaput, J. J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515-556). New York: Macmillan.

Kaput, J. J. (1994). Democratizing Access to Calculus: New Routes to Old Roots. In A. H. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 77-156). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (1998). Representations, inscriptions, descriptions, and learning: A kaleidoscope of windows *Journal of Mathematical Behavior, 17*(2), 265-281.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19-51). New York: Lawrence Erlbaum Associates.

Kaput, J. J., & Thompson, P. W. (1994). Technology in mathematics education research: The first 25 years in the JRME. *Journal for Research in Mathematics Education, 25*(6), 676.

Key Curriculum Press. (2006). The geometer's sketchpad: Dynamic geometry software for exploring mathematics (Version 4.07S). Emeryville, CA: KCP Technologies.

Kinney, D. P., & Kinney, L. S. (2002). Instructors perspectives of instruction in computer-mediated and lecture developmental mathematics classes. In J. L. Higbee, D. B. Lundell & I. M. Duranczyk (Eds.), *Developmental education: Policy and practice* (pp. 127-138). Auburn, CA: National Association for Developmental Education.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*(2), 129-164.

Koichu, B., & Harel, G. (2007). Triadic interaction in clinical task-based interviews with mathematics teachers. *Educational Studies in Mathematics, 65*, 349-365.

Kvale, S. (1996). *InterViews*. Thousand Oaks, CA: Sage Publications.

Kynigos, C., Psycharis, G., & Moustaki, F. (2010). Meanings Generated While Using Algebraic-Like Formalism to Construct and Control Animated Models. *International Journal for Technology in Mathematics Education, 17*(1), 17-32.

Lambdin, D. V., & Walcott, C. (2007). Changes through the years:Connections between psychological learning theories and the school mathematics curriculum. In W. G. Martin

& M. E. Strutchens (Eds.), *The Learning of Mathematics (69th Yearbook)* (pp. 3-25). Reston, VA: National Council of Teachers of Mathematics.

Lapp, D. A., & Cyrus, V. F. (2000). Using data-collection devices to enhance students' understanding. *Mathematics Teacher, 93*(6), 504-510.

Lapp, D. A., & John, D. S. (2009). Dynamically Connected Representations: A Powerful Tool for the Teaching and Learning of Mathematics. *International Journal for Technology in Mathematics Education, 16*(1), 37-44.

Lesh, R., & Kelly, A. (2000). Multitiered teaching experiments. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesik, S. A. (2007). Do developmental mathematics programs have a causal impact on student retention? An application of discrete-time survival and regression-discontinuity analysis. *Research in Higher Education, 48*(5), 583-608.

Li, Q., & Edmonds, K. A. (2005). Mathematics and at-risk adult learners: Would technology help? . *Journal of Research on Technology in Education, 38*(2), 143-166.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic Inquiry*. Beverly Hills, CA: Sage Publications.

Lincoln, Y. S., & Guba, E. G. (2007). But is it rigorous? Trustworthiness and authenticity in naturalistic evaluation. In *Growing issues in evaluation: The 20th anniversary collaboration between NDE and AEA* (pp. 15-25): Wiley.

Lingefjard, T. (2008). *Hundred years of technology in mathematics classrooms*. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of function on his implementation of a reform curriculum. *Journal for Research in Mathematics Education, 29*(3), 248-274.

Lopez Jr, A. M. (2001). A classroom note on: Making connections through geometric visualization. *Mathematics & Computer Education, 35*(2), 116-121.

Maximo, E. P., & Ceballos, T. R. (2004, 2004/10/21/2004 Annual Meeting, Toronto, CA). *Building up the notion of Dependence Relationship between Variables: A case study with 10-12 year old students working with SimCalc.* Paper presented at the Conference Papers -- Psychology of Mathematics & Education of North America.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. Thousand Oaks, CA: SAGE Publications.

Mills, J., Bonner, A., & Francis, K. (2006). The development of constructivist grounded theory. *International Journal of Qualitative Methods, 5*(1), 1-10.

Monk, S. (2003). Representation in school mathematics: Learning to graph and graphing to learn. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 250-262). Reston, VA: National Council of Teachers of Mathematics.

Moreno-Armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics: historical and representational perspectives. *Educational Studies in Mathematics, 68*(2), 99-111.

Murphy, E., Ciszewska-Carr, J., & Manzanares, M. A. R. (2006). Methodological issues in the content analysis of online asynchronous discussions: Unitizing, reliability, and latent content. *Journal for the Research Center for Educational Technology, 2*(2), 1-14.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nemirovsky, R., & Noble, T. (1997). On Mathematical Visualization and the Place Where We Live. *Educational Studies in Mathematics, 33*(2), 99-131.

Norton, A. (2008). Josh's Operational Conjectures: Abductions of a Splitting Operation and the Construction of New Fractional Schemes. *Journal for Research in Mathematics Education, 39*(4), 401-430.

Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of Teaching and Learning. *Journal for Research in Mathematics Education, 39*(3), 220-246.

O'Callaghan, B. R. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education, 29*, 21-40.

Obiekwe, J. C. (2000). *The Latent Structures of the Learning and Study Strategies Inventory (LASSI): A Comparative Analysis*.

Olive, J. (1999). From Fractions to Rational Numbers of Arithmetic: A Reorganization Hypothesis. *Mathematical Thinking & Learning, 1*(4), 279.

Oncu, S., Delialioglu, O., & Brown, C. A. (2008). Critical Components for Technology Integration: How do Instructors Make Decisions? *Journal of Computers in Mathematics and Science Teaching, 27*(1), 19.

Pandit, N. R. (1996). The creation of theory: A recent application of the grounded theory method. *The Qualitative Report, 2*(4).

Pape, S. J., Bel, C. V., & Yetkin, I. E. (2003). Developing Mathematical Thinking and Self-Regulated Learning: A Teaching Experiment in a Seventh-Grade Mathematics Classroom. *Educational Studies in Mathematics, 53*(3), 179.

Pape, S. J., & Tchoshanov, M. (2001). The role of representation(s) in developing mathematical understanding. *Theory into Practice, 40*(2), 118-127.

Parsad, B., & Lewis, L. (2003). *Remedial education at degree-granting postsecondary institutions in fall 2000*. Retrieved July 10, 2010. from http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2004010.

Payne, E. M., & Lyman, B. G. (1996). *Issues affecting the definition of developmental education*. Carol Stream, IL.

Plato Learning. (2004). PLATO interactive mathematics. Bloomington, MN: Plato Learning.

Pratt, M. G. (2007). Fitting ovals pegs into round holes: Tensions in evaluating and publishing qualitative research in top-tier North American journals. *Organizational Research Methods OnlineFirst*.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205-235). Dordrecht: Sense Publishers.

Qi, Z., & Polianskaia, G. (2007). A comparison of traditional lecture and computer-mediated instruction in developmental mathematics. *Research & Teaching in Developmental Education, 24*(1), 63-82.

Reinford, D. (1998). Problem solving with the rule of three. *Mathematics Teacher, 91*(7), 558-563.

Rogers, Y. (1999). What is different about interactive graphical representations? . *Learning and Instruction, 9*, 419-425.

Sajka, M. (2003). A secondary school student's understanding of the concept of function - A case study. *Educational Studies in Mathematics, 53*(3), 229.

Santos-Trigo, M. (2002). Students' use of mathematical representations in problem solving. *Mathematics and Computer Education, 36*(2), 101-114.

Saul, M. (2001). Algebra: What are we teaching? . In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics: 2001 yearbook* (pp. 35-43). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (2007). Method. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 69-107). Reston, VA: National Council of Teachers of Mathematics.

Schwandt, T. A. (1997). Notes for "The role of theory in interpretive inquiry, *Excerpts from, T. Schwandt, Qualitative Inquiry: A Dictionary of Terms. Sage Publications (forthcoming, March, 1997)* (pp. 10).

Schwandt, T. A. (2007). *The sage dictionary of qualitative inquiry*. Thousand Oaks, CA: Sage.

Schwarte, A. E. (2007). New standards for improving two-year mathematics instruction. *Education Digest, 73*(2), 39-42.

Sedig, K. (2008). From play to thoughtful learning: A design strategy to engage children with mathematical representations. *Journal of Computers in Mathematics and Science Teaching, 27*(1), 65.

Shaffer, D. W., & Kaput, J. J. (1998). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37*(2), 97-119.

Silver, E. A. (1990). Contributions of research to practice: Applying findings, methods and perspectives. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s, 1990 yearbook* (pp. 1-11). Reston, VA: National Council of Teachers of Mathematics. .

Silver, E. A., & Herbst, P. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 36-67). Reston, Va: National Council of Teachers of Mathematics

Smith, S. P. (2003). Representation in school mathematics: Children's representations of problems. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 263-274). Reston, VA: National Council of Teachers of Mathematics.

Snyder, T., & Dillow, S. A. (2010). *Digest of Education Statistics: 2009*. Retrieved July 10, 2010. from http://nces.ed.gov/programs/digest/d09/.

Steele, D. F. (2008). Seventh-grade students' representations for pictorial growth and change problems. *ZDM, 40*(1), 97-110.

Steele, D. F., & Johanning, D. I. (2004). A schematic-theoretic view of problem solving and development of algebraic thinking. *Educational Studies in Mathematics, 57*(1), 65-90.

Steffe, L. P. (1991). The constructivist teaching experiment. In E. Von Glasersfeld (Ed.), *Radical constructivism in mathematics education*. Dordrecht, The Netherlands: Kluwer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stevens, T., To, Y., Harris, G., & Dwyer, J. (2008). The LOGO project: Designing an effective continuing education program for teachers. *Journal of Computers in Mathematics & Science Teaching, 27*(2), 195-219.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.

Stylianou, D. A., Smith, B., & Kaput, J. J. (2005). Math in motion: Using CBRs to enact functions. . *Journal of Computers in Mathematics and Science Teaching, 24*(3), 229-324.

Tate, W. (1995). Mathematics communication: Creating opportunities to learn. *Teaching Children Mathematics*, 344-369.

Taylor, J. M. (2008). The effects of a computerized-algebra program on mathematics achievement of college and university freshmen enrolled in a developmental mathematics course. *Journal of College Reading & Learning, 39*(1), 35-53.

Trenholm, S. (2006). A study on the efficacy of computer-mediated developmental math instruction for traditional community college students. *Research & Teaching in Developmental Education, 22*(2), 51-62.

Trenholm, S. (2007). An investigation of assessment in fully asynchronous online math courses. *International Journal for Educational Integrity, 3*(2), 41-55.

Tuckett, A. G. (2005). Part II. Rigour in qualitative research: complexities and solutions. *Nurse Researcher, 13*(1), 29-42.

Van de Walle, J. A. (2007). *Elementary and middle school mathematics: Teaching developmentally*. Boston, MA: Pearson

Vernaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior, 17*(2), 167-181.

Villarreal, M. E. (2008). *Some reflections about old and new media in mathematics education*. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI.

Wadsworth, L. M., Husman, J., Duggan, M. A., & Pennington, M. N. (2007). Online Mathematics Achievement: Effects of Learning Strategies and Self-Efficacy. *Journal of Developmental Education, 30*(3), 6-14.

Yerushalmy, M., & Shternberg, B. (2001). Charting a visual course to the concept of function. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 251-267). Reston, VA: The National Council of Teachers of Mathematics.

Appendix A

Glossary of Terms

Glossary

Word	Definition
Affective domain	"The affective domain refers to feelings that pertain to mathematics, to the experiencing of mathematics, or to oneself in relation to mathematics" (Goldin, 2003, p. 280)
Behaviorist	Behaviorist teaching models may provide a stimulus and response approach without regard to conceptual understanding.
Computer assisted instruction	The use of software designed to tutor students, provide them with extra practice, and sometimes engage them in dialog is sometimes known as computer-assisted instruction (CAI) (Kinney & Kinney, 2002).
Concentration	Obiekwe (2000) in his discussion of the instrument used by Wadsworth et al. (2007) noted that concentration was thought of as a student's ability to give attention to an academic task.
Conceptual knowledge	Knowledge that is part of a network of connections to other ideas (Kinney & Kinney, 2002)
Constructivism	Based on ideas of Piaget and Vygotsky, constructivism encouraged educators to create an atmosphere where students could work through cognitive conflict using their own strategies, and thus learning via problem solving (Lambdin & Walcott, 2007). Students then have the opportunity to construct their own knowledge (Silver, 1990).
Covariation	Considering functions from a standpoint of covariation means that variations in the independent variable (or input) and dependent variable (or output) are considered together.
Developmental education	Developmental education, sometimes referred to as remedial education, refers to educational efforts which serve college students who need additional preparation in order to be successful (Payne & Lyman, 1996)
Direct instruction	Wadsworth et al., (2007) seem to use this term to indicate in-classroom instruction with a teacher present, as opposed to computer instruction students engage in independently outside of class.
Embodied, linguistic, formal, and internal representations	Embodied representations of mathematical ideas are external, physical situations in the environment. Linguistic representations are those in which the emphasis is on syntax and semantics. Formal systems use symbols, axioms, definitions, constructs, etc. Internal, individual systems describe thinking processes and are inferred from behavior or introspection (Goldin & Janvier,1998)

Word	Definition
Formative assessment	Assessment designed primarily to provide constructive feedback to the student so that he or she may improve
Function	A mathematical relationship in which one set of data is matched with another set of data so that each piece of data in the input set is matched to one and only one piece of data in the output set.
Function mappings	A function mapping diagram renders the x and y axes as parallel lines, with line segments connecting a point on the x-axis to its image on the y-axis. (Bridger, 2001)
Global	To deal with the function in a global way is to look at its overall behavior, such as when students sketch the graph of a function and look at its maximums and minimums and other characteristics. (Even, 1998)
Idiosyncratic representations	Those which are unique to the learner (Smith, 2003). Such personal forms of representation, which may be very meaningful to the student, but have little resemblance to those commonly used
Information processing	A student's ability to process knowledge. (Obiekwe, 2000)
Internal and External representations	The words external and internal refer to the relationship of that representation to the mind of the student. If the representation exists within the mind of the student, then it is an internal representation. If the representation is found in the environment outside of the student's mind, in a textbook, on a computer screen, or on a piece of paper for example, then it is considered to be an external representation. (Goldin, 2003)
Isomorphism	A one to one relationship between two sets of data preserving operations within the two sets. (Dictionary.com, 2010)
Manipulatives	Hand-held objects used to model mathematical ideas.
Model	A mathematical model is a form of representation which illustrates mathematical features of a complex phenomenon and is used to clarify situations and solve problems. (NCTM, 2000)
Non-cognitive factors	Non-cognitive factors include influences unrelated to the student's knowledge and may include such items as motivation and time management. (Gerlaugh, 2007)
Object oriented programming	As used in mathematics education, software which provides powerful tools such as drawing implements and other graphics in a setting requiring relatively simple syntaxes with which students can create their own programs. (Connell, 1998)

Word	Definition
Pointwise	To deal with functions in a pointwise way is to plot, read or deal with discrete points. (Even, 1998)
Reference field	A set of objects related to a set of representations. For example, one reference field for the set of nouns is the set of material objects. (Moreno-Armella et al., 2008)
Reification	The process by which something abstract becomes real to the learner and exists in his or her mind as a mental object. (O'Callaghan, 1998)
Representation	This term refers to both "the act of capturing a mathematical concept or relationship in some form" and "the form itself" (NCTM, 2000, p. 66). Cuoco and Curcio (2001), described a representation as a map of correspondence between a mathematical structure and a better understood structure.
Self-efficacy	A person's belief in their own ability to be effective in managing future situations. (Wadsworth, Husman, Duggan, and Pennington, (2007)
Self-testing	A student's ability to prepare for tests and classes. (Obiekwe, 2000)
Semiotic mediation	The process by which new signs are derived from the actions performed with another sign or symbol. It indicates an internalization process, producing a new internal tool. (Falcde, Laborde, & Mariotti, 2007)
Semiotics	Semiotics is the study of signs and their meanings. Cunningham (1992) described semiotics as "a way of thinking about the mind, and how we come to know and communicate knowledge" (p. 166).
Symbolization	The process by which symbolic structures re-design the architecture of the human mind and provide a meta-cognitive mirror in which our thought is reflected. (Moreno-Armella et al., 2008)
Technological representations	Charts, graphs, geometric shapes, algebraic equations, or other mathematical objects represented via mathematics software, internet sites, or on hand-held devices such as calculators.
Textual (descriptive) and visual (depictive)	Textual representations are semantically dense, and conveyed through rules. Visual representations are more analogical. (Sedig, 2008)
Thick description	Thin description would merely describe the action. Thick description would give more than just a description of the action. It might provide, for example, information regarding the motivation for the action. (Geertz, 1994)

Word	Definition
Title IV	Title IV institutions are those which participate in certain federal student aid programs. (Aud et al., 2010)
Visualization	The creation of a mental image to guide the representation of ideas. (Presmeg, 2006)

Appendix B

Interview Protocols

Interview Protocol: Initial Interview

Project: the Effect of Technological Representations of Developmental Mathematics Students' Understanding of Functions

Time of interview:

Date:

Place:

Interviewer:

Interviewee:

Obtaining of informed consent: Before anything else, I need to obtain your signature on this document. It officially lets you know about the study and what your rights and obligations are. *(point to first paragraph).* You can read here that the purpose of this project is to find out how using computer software to study mathematics affects how a person thinks about mathematics. Look over the rest of the document and let me know if you have any questions. *(Answer questions as needed, obtain signatures.)*

Obtaining personal information: *Ask subject to fill out the subject information sheet.*

Further description: This first interview will help me get to know you a little bit and find out what your experiences with math have been. I'll be selecting four people to continue on and participate in more interviews where we'll look at some math computer software and I can see how they think. No one will have to pass any kind of a test to be selected to continue. I'll be picking people to continue based on how they will fit in with what I want to learn and what I'm doing to do to learn it. Do you have any questions about the project?

Questions and Prompts:

1. Tell some things you remember about school when you were growing up.
2. Describe a *(another)* good experience you remember having in school.
3. *(If not already stated)* What helped you learn?
4. *(If not already stated)* What made learning harder?
5. Talk to me about your most memorable teacher.
    a. How did they teach?
    b. How did you feel about that class?
6. *(If not already mentioned)* Describe your most memorable math teacher.
    a. *(If not already stated)* How did they teach?
    b. *(If not already stated)* How did you feel about that class?
7. What about your other math classes?

8. What other things do you remember about your math classes?
9. Did your teachers ever use any kind of technology in teaching mathematics?
    a. That includes calculators, computers, and the internet . . .
    b. *(If so)* What did they use?
    c. How did they use it?
    d. How did you feel about using ________________ to study mathematics?
    e. Do you think ___________________ helped you learn?
10. *(Give them a blank sheet of paper and a pencil)* Show me and tell me about some mathematics you remember. It can be anything that you remember.
    a. Why did you pick this to share?
    b. *Additional probing questions might focus on the mathematics they share, but not for the purpose of measuring achievement, e.g.*
        i. Can you tell me anything else about ___________________ ?
        ii. *NOT* What's the answer to _________________ ?
11. I'm going to give you something to look at now, and I want you to just tell me everything you can about it. Don't worry about what the right answer is, or what math you're supposed to use or anything like that. Just look at this, read the directions, and follow them. One of the other things I want you to do is talk out loud about what you are doing as you work, talking continually, as if you were thinking out loud and I were listening in on your thoughts. *(Give them "Looking at patterns).*
    a. *Probing questions for this and number 12 might include:*
        i. How will the pattern continue? Can you draw the next shapes in the pattern?
        ii. What will the 20^{th} shape in the pattern be?
        iii. Can you tell what the 35^{th} shape in the pattern will be (*without drawing it?)* What about the 41^{st}? How do you know?

12. *(If this seems not to challenge them, you may continue with one or more of the following)* Here is another idea for you to think about. *(Give them "Looking at dot patterns" and/or"Soda cans").* Read the directions and follow them. Talk out loud about what you are doing as you work, talking continually, as if you were thinking out loud.

    a. What will the next pattern be?

    b. Can you tell me how many dots the 10^{th} pattern will have? How?

13. Thank you for participating. Is there anything else you'd like to say?

Teaching experiment interview protocol: Exploring patterns

Project: the Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions

Time of interview:

Date:

Place:

Interviewer:

Interviewee:

Questions and prompts:

Recalling the last session and examining the data:

1. What do you remember about this dot pattern that you saw last time?
   a. *Follow with probing questions, such as those listed below, to elicit student thinking*
      i. Why did you think that?
      ii. Is there anything else you've noticed about the pattern?
      iii. How do you know what the number of dots in the next pattern will be?
         1. Is there a way you can write down this information?
      iv. Do you know what a table of values is?
         1. *If they do:* Show me what you know about tables by creating a table of values showing what you know about the step numbers and numbers of dots.
            a. Tell me about your table.
         2. *If they don't or are unsure: then go ahead and give them page 1 of the exploration, so they can see the blank table.*

            a. Does this look familiar? What do the columns tell us?
            b. Enter what you know about the step numbers and number of dots.

2. Look at the information in your table and see if you notice any patterns. Record your thoughts and talk about them as you write.
   a. *Subject may also compare back to the dot pattern representation.*
   b. What is the number of dots in the _____ th pattern? (*several steps beyond the data they have).*
   c. Can you give me a rule for how many dots would be in any pattern?

i. *If they can, show them another pattern until you find one for which they can't. Finding the rule will be the overall question this teaching experiment seeks to answer.*

Introduction to the software

3. We are going to use mathematical software to study this question. The name of the software is the Geometer's Sketchpad. *(Open the software)* Have you ever heard of it?
   a. *(If so)* Open a new file and show me what you know about the Geometer's Sketchpad.
   b. *(If not)* We're going to explore it. Open a new file. Make it as large as you can.
      i. What do you notice?
      ii. Hold your cursor over the buttons at the left.
         1. What do you notice?
         2. What are the names of the buttons?
         3. What happens when you hold the buttons down?
      iii. Try some of these buttons and tell me what you notice.
   c. Open a new file. From the menu at the top, select *Graph, define coordinate system.*
      i. Does this remind you of anything?
      ii. *(If so)* What is it? What can you tell me about it?
      iii. *(If not, unsure, or more scaffolding is needed regarding the xy plane)* Use the point tool to put a point somewhere in the coordinate system. Use the measure menu, and measure the coordinates of the point. Use the selection arrow tool and move the point around. Talk about what you see. What are the coordinates telling you?
         1. *(If needed to elicit further observations).* Use the graph menu and choose snap points if you want the point to always land at an intersection. Look at the coordinates.
            a. What do you notice as you move the point around? Try this for a minute and as you do so, talk about what you see.
         2. Put the point at a place where[8]
            a. Both coordinates are the same
            b. The right is twice as much as the left
            c. The right is half as much as the left

[8] As suggested in Key Curriculum Press (2002)

d. The right equals the negative of the left or the left equals the negative of the right

d. Using the *selection arrow tool*, select the point at the middle of the crossed axes (called the origin) and move it to see how the coordinate grid moves around.
   i. What's happening to the point you put on the grid?
   ii. What about its coordinates?

e. Using the *selection arrow tool*, select the other point you see, the one at 1 on the x-axis and move it to see how the grid changes.
   i. Talk about what you see.

f. Using graph, plot points, enter some coordinates of your choice and see where those points lie. See if you can predict where they will be.
   i. Move these points around. How do they behave differently from the other points you have put on the graph?

Using the software to look at their data:

4. How can you use these tools to explore your data?
   a. What data do you have?
   b. What value in your data could be the left coordinate of a point? What could be the right coordinate to go with that point? Why?
5. *If they have graphed their points:* What pattern do your points form?
   a. (*If they think the points are in a straight line).* How could you find out for certain?
6. What if you had another step in the sequence of dot patterns, whose data is not already listed in your table? Where do you think the point representing that data would fall?

Additional questions may be asked, based on the student's thinking.

Teaching experiment interview protocol: Exploring functions

Project: the Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions

Time of interview:

Date:

Place:

Interviewer:

Interviewee:

Questions and prompts:

Recalling the last session and starting algebraic notation

1) Begin with GSP in its opening configuration
    a. Show me what you remember from last time. Talk about what you see
        i. What is the problem we're trying to solve?
        ii. What do you know about that problem?
        iii. What have we done so far to explore that problem with GSP?
        iv. What does the table tell us?
        v. What does the graph tell us?
        vi. If I asked you now how many dots were in a certain step number, how would you find out?
        vii. If we had a rule, how would we describe it?
            1. *Look at a simpler pattern for which a rule can be found and help the subject to see how to use algebraic notation to describe it.*

a. How is the number of dots related to the step number?

b. Can you write that in a sentence?

c. How can we write that in a simpler way?

    i. Use boxes to represent the step number if scaffolding to the use of variables is needed.

Using the software to explore algebraic notation

2) Go to graph, new function. In the rectangle, you can enter expressions, or rules.

    a. Enter the rule we found for our pattern

    b. Hit okay.

    c. Hit plot function.

    d. What do you notice?

    e. Why does the graph look the way it does?

        i. Explore some different expressions. You may change the colors to match using the display, color menu.

    f. What patterns do you notice?

Additional questions may be asked, based on the student's thinking.

Teaching experiment interview protocol: Exploring parameters

Project: the Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions

Time of interview:

Date:

Place:

Interviewer:

Interviewee:

Questions and prompts:

Recalling the last session

3) Begin with GSP in its opening configuration
   a. Show me what you remember from last time and talk about what you are doing.
   b. What did you find out about how the graphs of different equations behave?
   c. What is different about their . . .?
      i. Graphs
      ii. Tables
      iii. Equations

Using the software to explore parameters

4) Go to graph, new parameter. Enter a letter name (other than x or y) for the new parameter to replace the one that is there and hit okay.
5) Define and plot a new function which uses that parameter in its equation (find it under "values").
6) Double click on the parameter and change its value. Before you hit okay, predict how you think the graph will change.
   a. Try this until you can describe in general how this parameter affects this type of function.
7) Select the parameter and edit, action button, animate parameter. Predict how you think the graph will change as the parameter changes. Hit animate parameter and see what happens.
   a. Use the animation feature of GSP to further explore the equations of functions. Talk about what you are noticing.
8) What have you learned about how equations of functions behave?

Additional questions may be asked, based on the student's thinking and his or her progress to this point.

Teaching experiment interview protocol: Exploring applications

Project: the Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions

Time of interview:

Date:

Place:

Interviewer:

Interviewee:

Questions and prompts:

Recalling the last session

9) Begin with GSP in its opening configuration
    a. Show me what you remember from last time and talk about what you are doing.
    b. What did you find out about how parameters affect the way different functions behave?
    c. What have you learned about their . . .?
        i. Graphs
        ii. Tables
        iii. Equations

Using the software to explore parameters

10) Use the things you have learned so far to explore the problems on this handout. Talk about what you are thinking and doing as you explore.

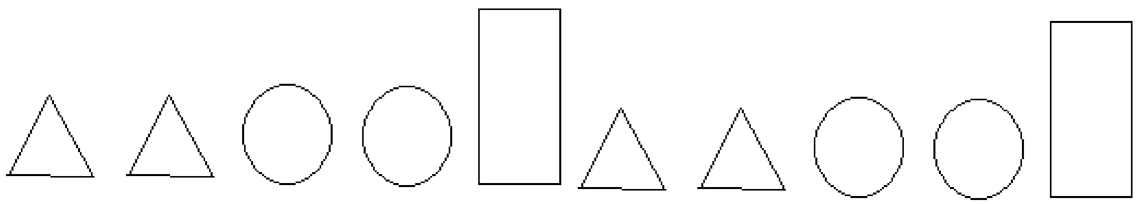
Additional questions may be asked, based on the student's thinking and his or her progress to this point.

Appendix C

Tasks

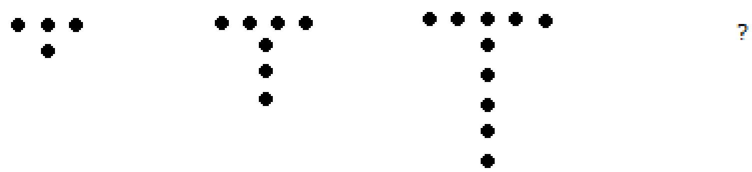
Looking at patterns

Study the pattern below and tell me everything you notice about it.



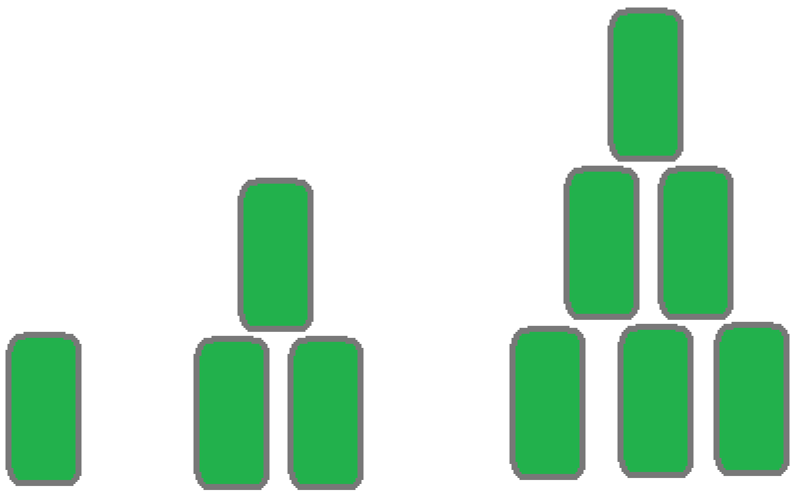
Looking at dot patterns

Study the pattern below and tell me everything you notice about it.



Soda Cans

How many soda cans will you put in the next stack if it follows the same pattern? Draw a picture to show the stack. What else can you say about this pattern? What if the manager of the store wanted 10 cans across the bottom of the stack? How many total cans would be in the stack?



Another dot pattern

Draw the next pattern in the sequence.

How many dots are in each pattern?

What else do you notice about the patterns?

Fill in the table below for the patterns

Step number	Number of dots

How many dots will be in the

10^{th} pattern?

25^{th} pattern?

107^{th} pattern?

How do you know that?

If I let x represent the step number, how many dots will be in the x^{th} pattern? Why?

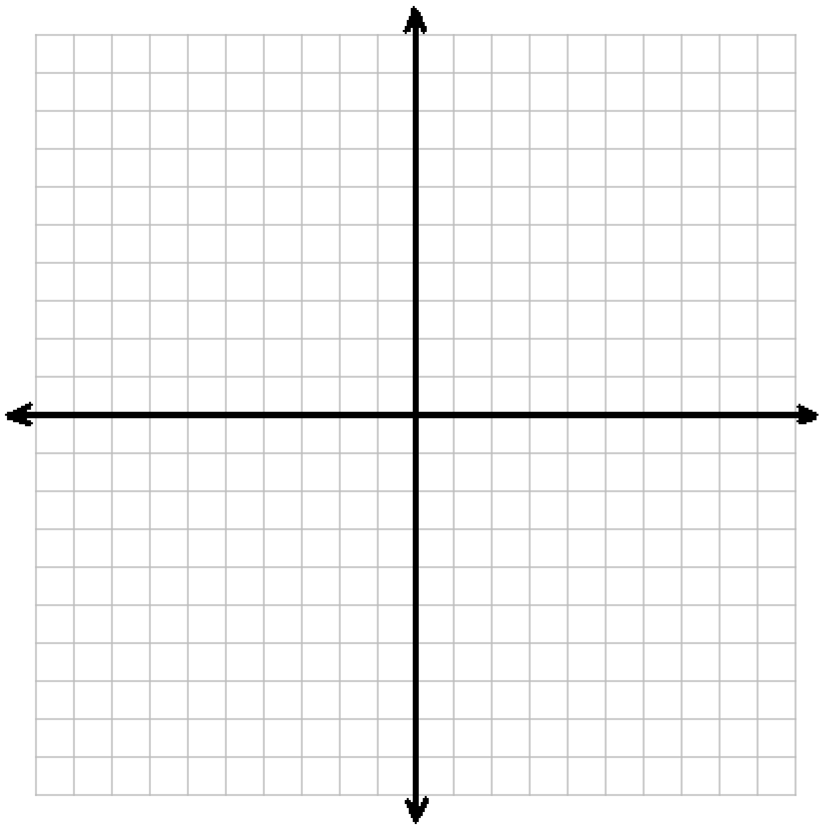
Exploring patterns

Step number	Number of dots

Enter what you know about the step numbers and number of dots for the pattern you are analyzing.

Look at the information and see if you notice any patterns. Record your thoughts below.

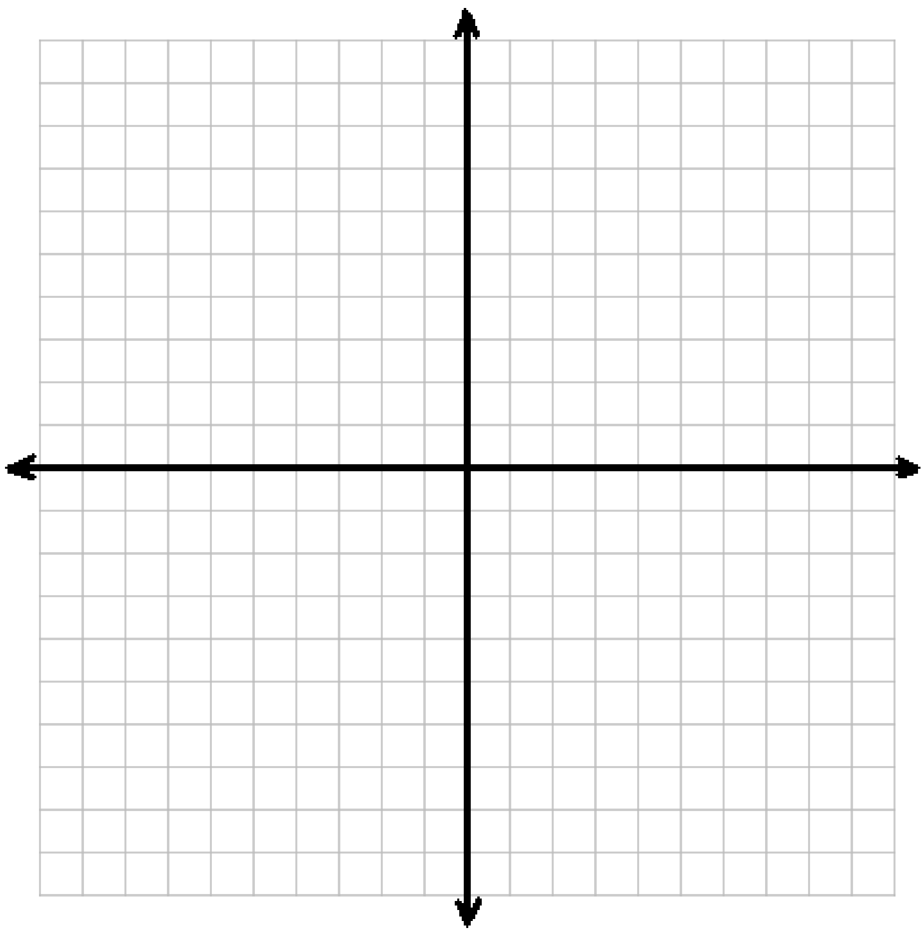
Use the grid and space below to record what you noticed about the things you explored today[9].



[9] Graph image found at http://faculty.matcmadison.edu/kmirus/GraphPaper20x20AxesUnits.bmp

Exploring Parameters[10]

Change the values of some of the parameters in a function you have used and observe what happens. Make notes about what you find.

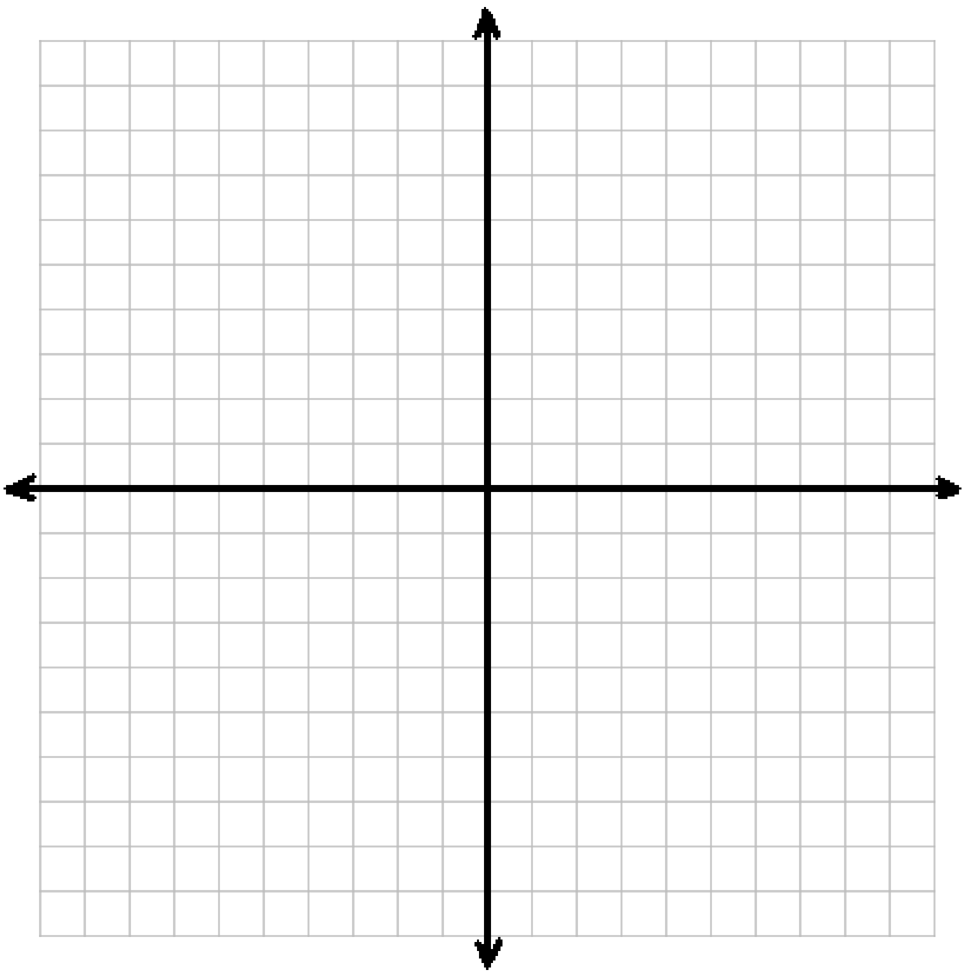


Notes about the effect of parameters:

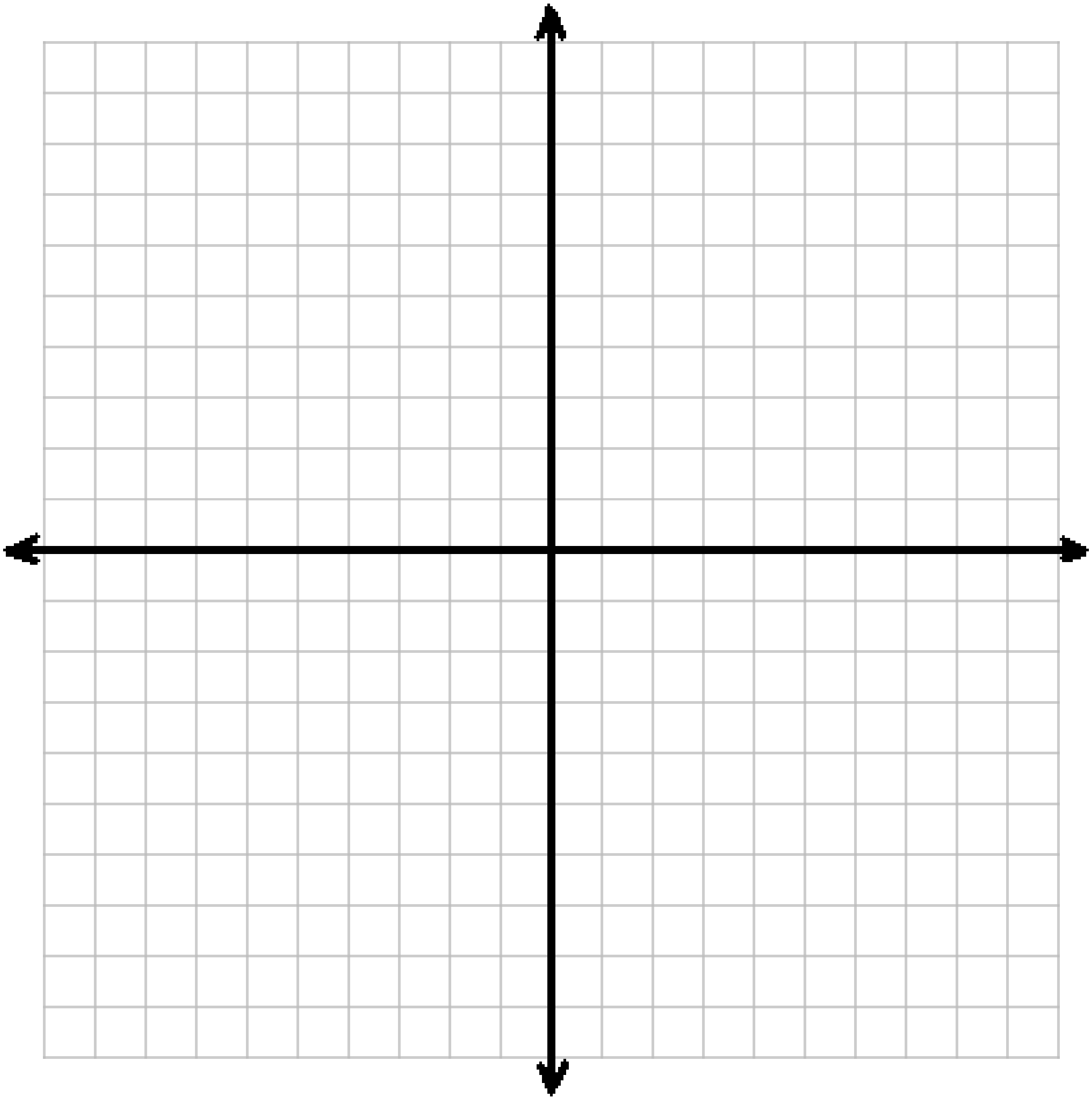
[10] Graph image found at http://faculty.matcmadison.edu/kmirus/GraphPaper20x20AxesUnits.bmp

Do you think those parameters would have the same effect on other functions? How could you find out?

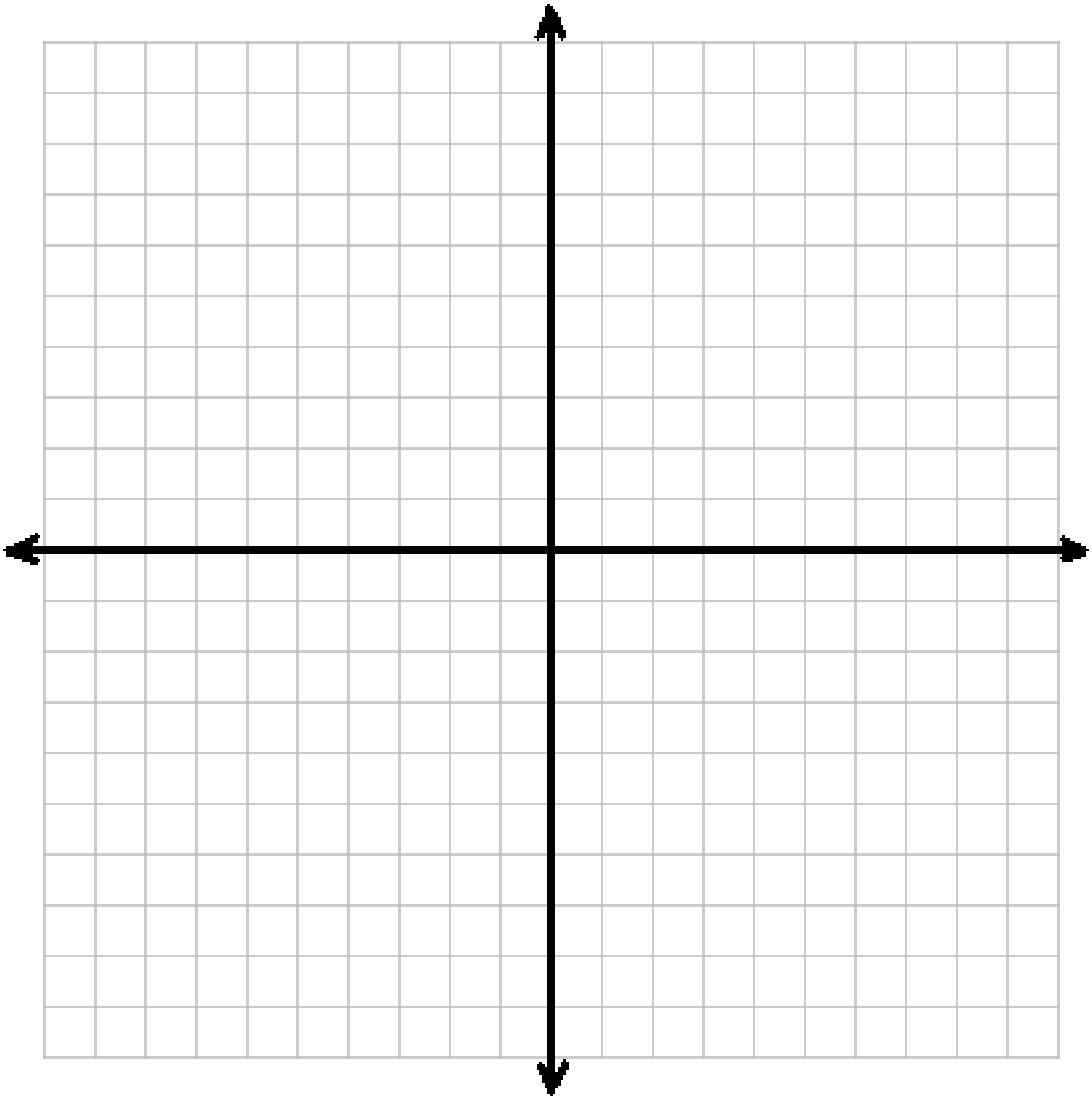
Use technology to investigate the effects of changing parameters on different types of functions.



Notes about the effect of parameters:



Notes about the effect of parameters:

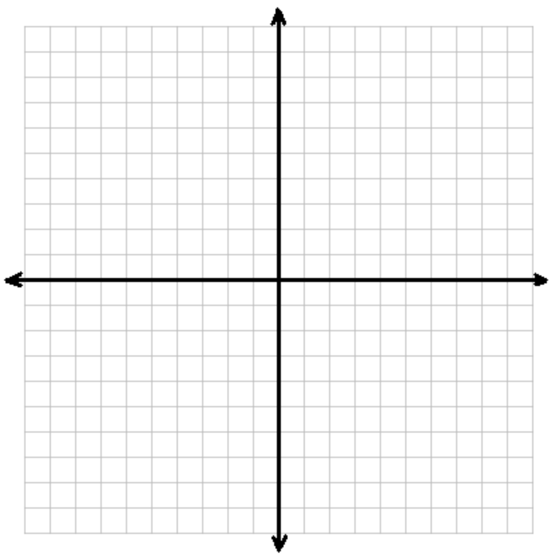


Notes about the effect of parameters:

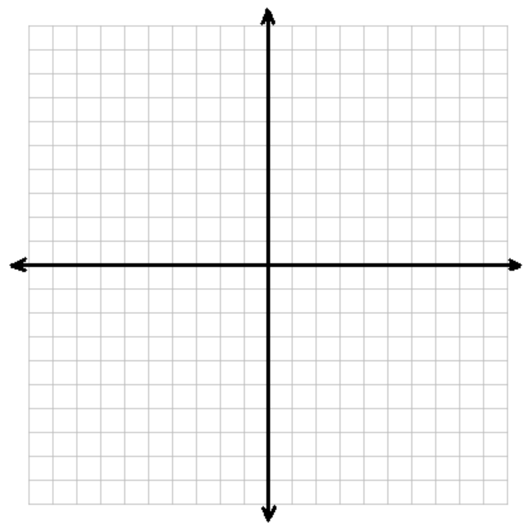
Summarize what you have noticed about the effect of parameters on functions:

Exploring functions
How does the equation for a function affect the way a function looks?

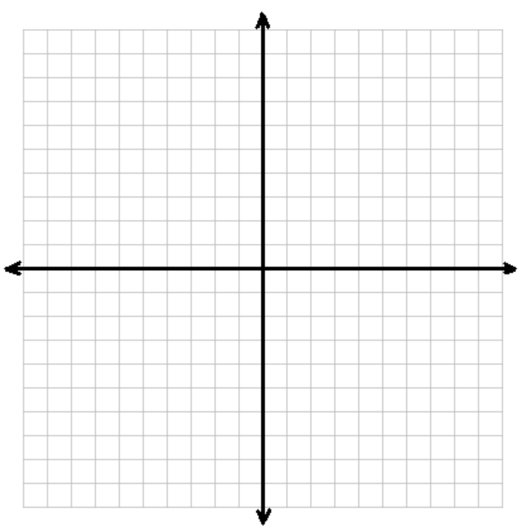
Part 1: Exploring equations and graphs
Use the Geometer's sketchpad to explore different equations of function. Make a record of your work by sketching a graph in each grid below and labeling the graph with the equation that goes with it.[11]



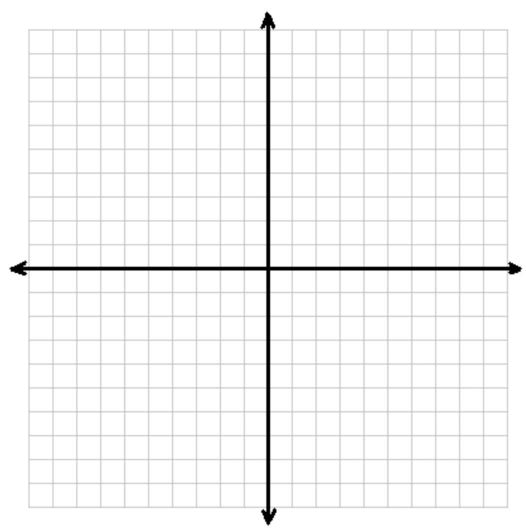
__________f(x) = ____________________________



______f(x) = ____________________________

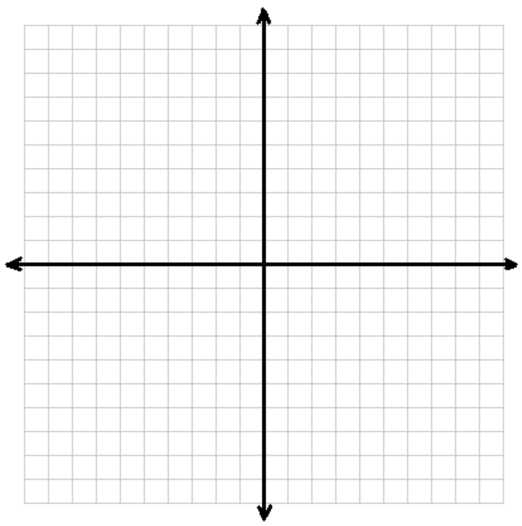


__________f(x) = ____________________________

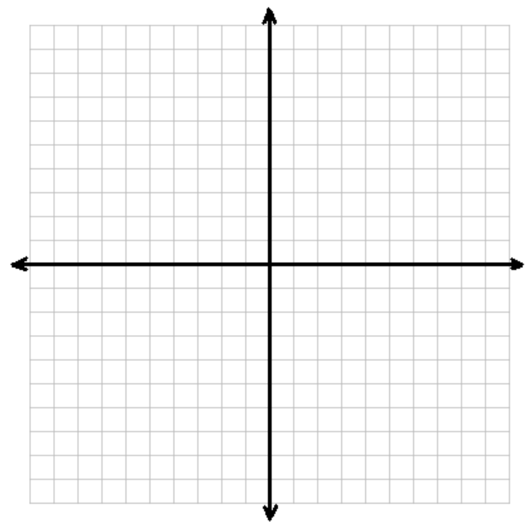


______f(x) = ____________________________

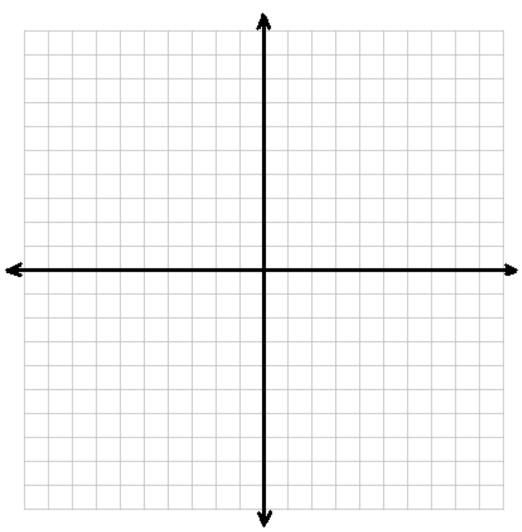
[11] Graph image found at http://faculty.matcmadison.edu/kmirus/GraphPaper20x20AxesUnits.bmp



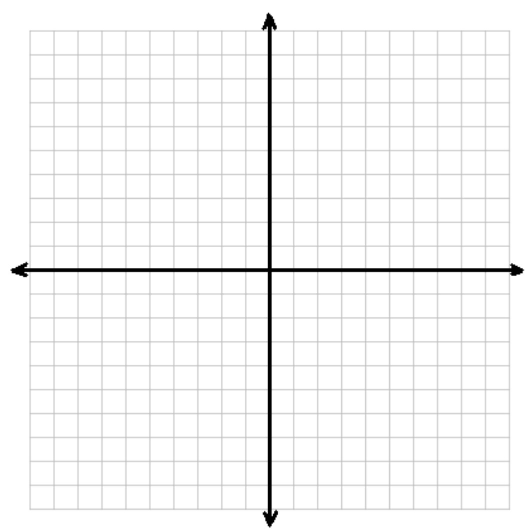
f(x) = ____________________



f(x) = ____________________



f(x) = ____________________

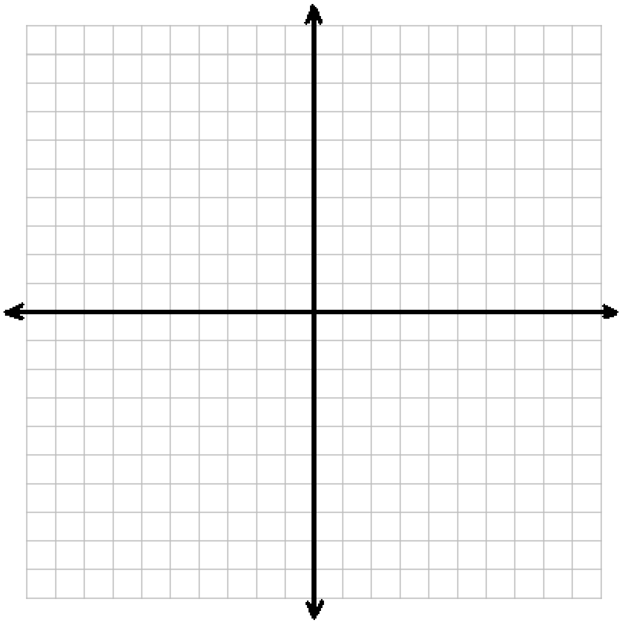


f(x) = ____________________

What did you find out about how the equations affect the graphs?

Part 2: Find an equation to go with points

Give the table a title and headings and record coordinate points in the table. Graph those points in the Geometer's Sketchpad and below. Use the Geometer's Sketchpad to find a function which will pass through most or all of your points. Record the equation and graph for that function.



____

_________f(x) = ________________________

How did you find the best function?

Exploring Different Applications of Functions

Part 1: Data for Albert and Carl

Open the sketch "Race with adjustable paths." Animate the points and see what happens. Drag the points back to the start.

1. Have Geometer's Sketchpad create a table of values which will keep track of Albert and Carl's distance from the finish line, updating the table every 10 seconds.

2. Run the race again.

    a. Where was Carl at 4 seconds?

    ____

    b. Where was Albert at 4 seconds?

    ____

3. Use the table you see in the Geometer's Sketchpad to fill in the table below.

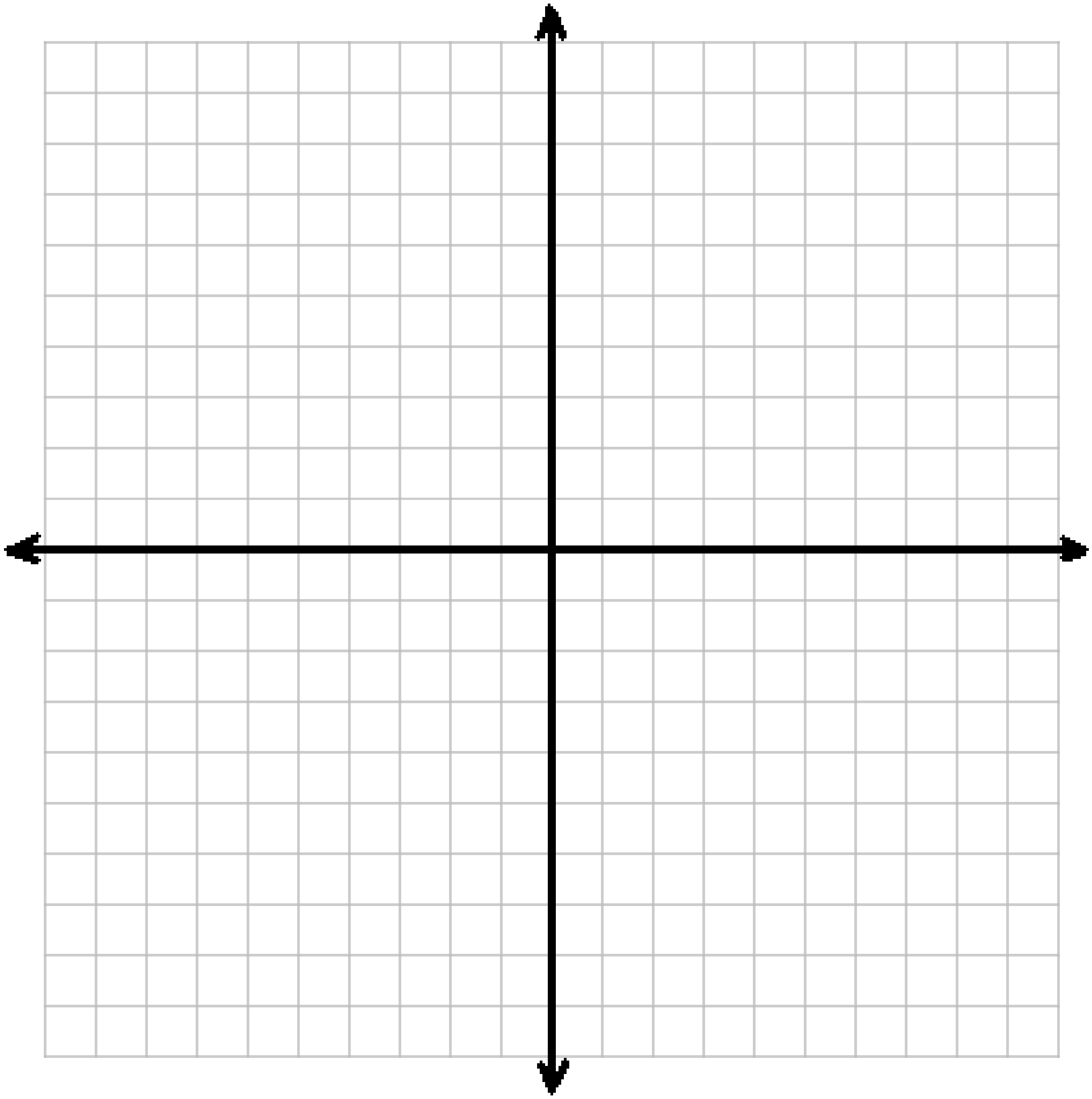
Seconds	Carl's distance from finish
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Seconds	Albert's distance from finish
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

4. Use what you have learned in the Geometer's sketchpad to find one function that shows us Carl's movement, and one function that shows us Albert's movement. You may open a new sketchpad file in which to work.

5. Sketch both graphs on a paper grid and label the grid[12].

[12] Graph image found at http://faculty.matcmadison.edu/kmirus/GraphPaper20x20AxesUnits.bmp



6. What can you tell me about Albert’s and Carl’s movement? How do their equations and graphs compare?

Part 2: Adjusting the Race

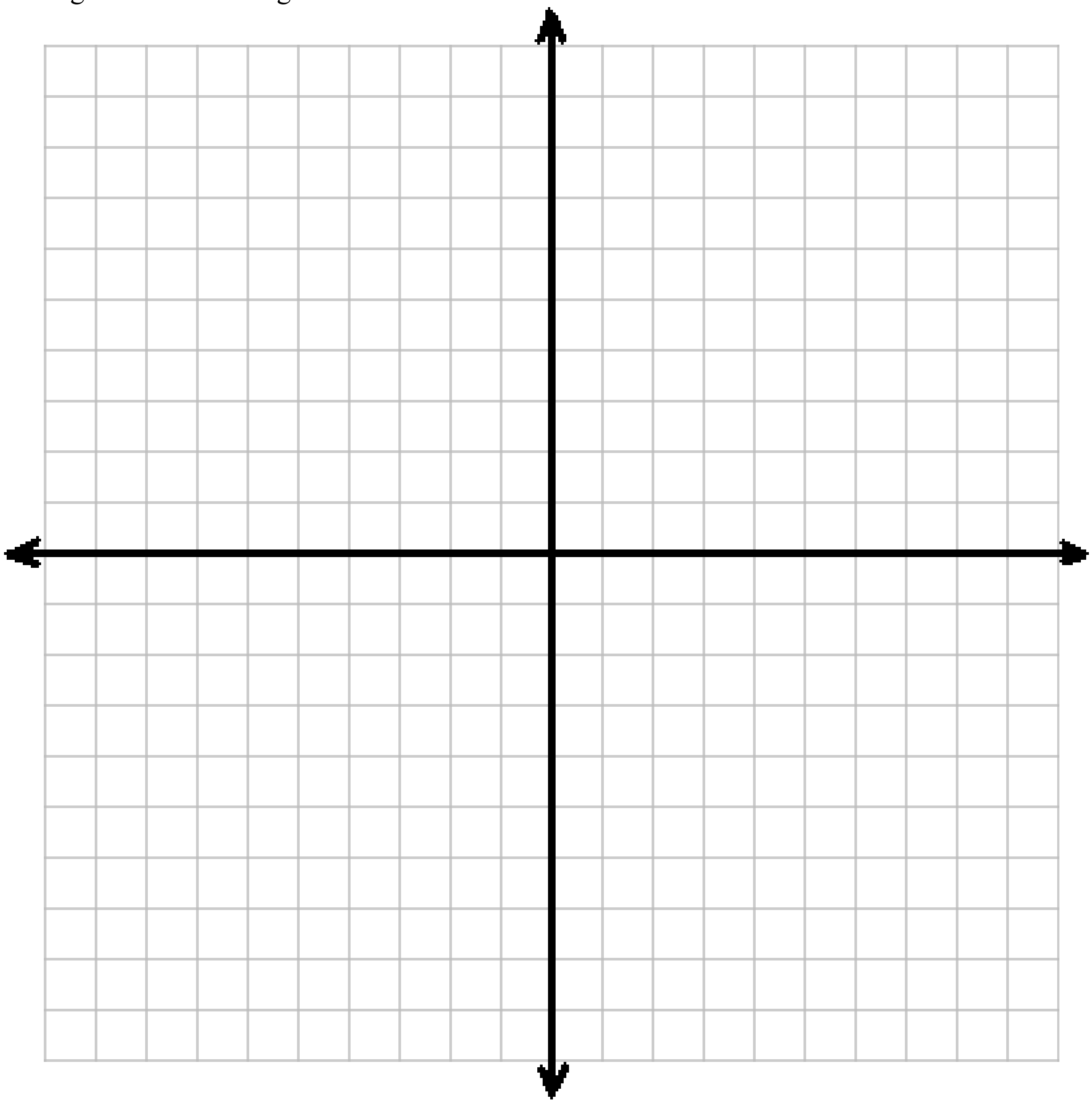
1. Drag Albert and Carl back to the start. Make an adjustment in the sketch so that Albert and Carl will finish at the same time.

2. Set up a table as you did before and run the race again so you can keep track of this data, showing where they are as each second passes.

Seconds	Carl's distance from finish
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Seconds	Albert's distance from finish
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

3. Find the functions to go with their new race and graph them. Sketch the graph on a grid and label the grid.



4. Describe how you adjusted the race and why and describe how you found the functions.

5. How do these equations and graphs compare to each other?

6. How do they compare to the equations and graphs for the first race?

Appendix D

Coding Guide

Coding Guide

The purpose of this guide is to explain the method that was used to unitize and code the transcriptions of both the initial interview and teaching experiment sessions held during the course of this study. Feeds from the computer screen, the work done on paper, and a view of participants were all coordinated together for the final video recording. Transcriptions were made of these video taped sessions. Notations were made of timing and actions of the participant in addition to what was said.

Unitizing

The table below shows the definitions used to describe units of analysis for this study.

Term	Definition
Topic	A complete idea, type of technological action, or topic of conversation upon which the attention of the speaker or speakers is focused.
Unit	A word, sentence, paragraph or several consecutive sentences or paragraphs focusing on the same topic.

Examples of Unitizing

Dividing a section into units

Below is a section of dialog which contains more than one unit of analysis

> INTERVIEWER: Okay. Could you tell me - can you tell me what the 35th shape in the pattern will be?
>
> MARJORIE: okay, 20, 22, 24, 26, 28 {she circulates back in her counting to the beginning of what she has illustrated} 30, 32, 34, a rectangle, 35.
>
> INTERVIEWER: Okay, and how did you know that? [31:17]
>
> MARJORIE: I just did the twos, I counted by twos.
>
> INTERVIEWER: Alright. What about the 41st? [31:28]
>
> MARJORIE: 41st, Okay so if that's {gesturing as she counts across the shapes} 35, 36, 37, 38, 39, 40, 41, triangle. And I counted that with, just counted one {laughs}

This section of dialog would be unitized as follows. Notice that each unit begins with a key word and a question on the part of the interviewer. Because finding the 41st pattern in a pattern which was five shapes long was qualitatively different from finding the 35th pattern, this question was considered to have started a new topic.

First unit	INTERVIEWER: Okay. Could you tell me - can you tell me what the 35th shape in the pattern will be? MARJORIE: okay, 20, 22, 24, 26, 28 {she circulates back in her counting to the beginning of what she has illustrated} 30, 32, 34, a rectangle, 35. INTERVIEWER: Okay, and how did you know that? [31:17] MARJORIE: I just did the twos, I counted by twos.
Second unit	INTERVIEWER: Alright. What about the 41st? [31:28] MARJORIE: 41st, Okay so if that's {gesturing as she counts across the shapes} 35, 36, 37, 38, 39, 40, 41, triangle. And I counted that with, just counted one {laughs}

Keeping a section in one unit

Below is a section of dialog which was kept as one unit. The rationale for doing so is also given. The boldfaced sentence might indicate a change of topics, but in this case I felt it was important to consider this passage as one unit.

Unit	Rationale
INTERVIEWER: Okay, so can you write on your paper what how you (would) write that in an algebraic expression? Can you write that down? MARJORIE: {writes y = 5 x + 1} but since I know what x is , then INTERVIEWER: What do you mean - what is x? MARJORIE: fi- x is two {writes y = 5(2) + 1 = 11} () gives you ll INTERVIEWER: **So what would the equation be for, for this step** MARJORIE: For step number three, it'd be the exact same thing you're just replacing the five with three. (It) would be three x plus one which would (get you) 7 and then since you know that the difference is two, x is two. 3 times 2 is 6 plus one equals 7. INTERVIEWER: Okay - MARJORIE: Hmm, mm hmm. Because when I first saw it, I was, I mean the one thing I did notice when I first saw it was that it was odd numbers {she gestures with the pen down the #dots column of the table} and I was like, oh, okay (something) there. But then once I started looking at it, and trying to you know, figure out what the difference was {she gestures back and forth from left to right entries in the table} because no matter what there's always, there's always got to be well not always but there seems to be a difference you know as far as these numbers just aren't randomly {gesturing down the #dots column} picked especially since I've been in this study. These numbers are not just randomly picked {some gesturing back and forth again as she speaks, INTERVIEWER: laughs} so its got to be a pattern and I just started you know, looking down the different steps till like {gesturing with pen down the step column}- okay and since um, and then when I didn't, when I couldn't see a correlation between the step and the dots, {gestures from left to right across table} I started looking at the dots {gestures with pen down the #dots column} because the steps were just going you know straight down {gestures down the step column} just like it was before. When I was looking I was like the dots {gestures down the #dots column}- what's the difference between the dots and then I just started counting and I got you know difference (from) when you add two to three you can get five if you added five, if you add two to five you get 7, and so forth {waves pen in air above the table}. I just went down. And then I noticed that pattern of two,{some gesturing towards #dots column} it was before it was three, but this one here is two.	I wanted to understand and I wanted her to think about her reasoning for saying 5(2) + 1 = 11 so the highlighted question is a continuation of the same topic: her reasoning for her paper representation being different from the representation given by the software. Note that following the boldfaced statement, she says "it'd be the exact same thing."

Coding

This study was entered into with a handful of a-priori codes, some of which evolved over the course of the study and were divided into codes which captured the nuances more clearly. Though specific names for codes may have changed over the course of the study, those listed as a-priori below reflect the ideas looked for going into the study. Some emergent codes were given names which were associated with the literature. Other unexpected ideas arose as well. Codes referring to characteristics which are beyond the scope of this study are not listed below.

Code Family: Mathematical Thinking Processes

Emergent codes

Code	Description	Example
Math-AM: Algebraic misconceptions	Quotations where it is evident that the subject doesn't understand the role, meaning, or purpose of an algebraic representation as commonly used or as related to other mathematical ideas.	“Okay, x + 9 so the x is equal to 9”
Math-FC: Function and Coordinate point confusion	Quotes where the representations and concepts related to functions and coordinate points seem to be becoming confused	“Do you remember the different ways you plotted those points? MARLON: Maybe, plot new function?”
Math-GC: Graphical confusion	Quotations where the subject exhibits Confusion about how points are located on the coordinate plane, the terminology used to describe them, or the nature and origin of graphed structures.	INTERVIEWER: Can you find one for me where the, the right coordinate is bigger than the left coordinate? MARJORIE: If I go to the right of the -and stay in the positive.{cursor at (21,0)}

A-priori codes

Code	Description	Example
Math-1: Disequilibrium	Quotes showing moments where the subject is "off-balance" mathematically - "Disequilibrium occurs when learners are presented with new information they must accommodate" (from my comments on Vander Zander 1989).	“Okay I’m confused now again”
Math-2: Equilibrium	Quotes where the subject appears to reach a point of understanding about something he or she found confusing	“[T]he reason why I can’t do it that way is because”
Math-3: Functional thinking	Coded where the subject directly connects the input with the output value of a function	“Because no matter what it’s a multiply of three and you’re always going to add one to it.”

Code	Description	Example
Math-4: Mathematical misconceptions	Quotations which reveal the subject's mathematical misunderstandings.	"and it has like f times x equals A {reading f(x) = A and cursor pointing along f(x)}"
Math-5: Observing patterns	Quotes which show the subject observing patterns.	"the pattern will continue with . . . two squares . . . two rectangles, two circles, and a rectangle."
Math-6: Problem Solving, reasoning, and sense-making	Quotes where the subject appears to be building knowledge by making conjectures, drawing logical conclusions, or connecting new topics with existing knowledge.	"INTERVIEWER . . . where do you think number 11 would go, step 11. Where would that point fall? If we graphed that - can you predict MARJORIE: Add 3 to that, should go to 34"
Math-7: Recursive thinking	Coded where the subject uses the rate of increase or decrease to describe a function	"I notice that each pattern is different between each one increasing at the base and also right down the middle and in between the middle it just changed each time."

Code Family: Representational Ideas and Issues

Emergent codes

Code	Description	Example
Rep-IM: Indicative movements	Gestures and mouse movements made by the subject as they interact with mathematical representations	"{gesturing from step one to step two}" *Also:* "{cursor waves back and forth between the x and y coordinates}"
Rep-IU: Idiosyncratic use of representations	Quotes in which the subject used standard representations in an idiosyncratic way which reflects the subject's mathematical thinking	"INTERVIEWER: How do you know what the - if we're just looking at that place where the arrow is pointing right now, that point right there, what would the table . . . MARLON: Oh, that would be zero, that would be zero, nine, so that would be zero (14:49). +9 {he writes 0+9}"

Code Family: Representational Ideas and Issues (cont.)

A-priori codes

Code	Description	Example
Rep-1: Endurance	Quotes showing places where a topic either did or did not endure for the subject from one session to the next or through the course of a session	"I can't remember what they actually do"
Rep-2: Internal representations	Quotes which seem to give possible insight into the subject's internal representations. Some ways this occurs are when the subject discusses the external representation so that we see how he or she uses that representation, or how he or she thinks about it.	"because I only have one point on there, so it's only going to give me . . . one measurement." "if I go over here [31:38] {Moves B to about (-4.5, 6.1)} I'm in a negative area here still in the positive area going upwards"
Rep-3: Multiple representations	Quotes in which the subject is working with different forms of representation of the same mathematical concept. These may be: • Pictures • Numbers • Graphs • Geometric figures • Tables • Algebraic Expressions or Equations • Coordinate pairs	INTERVIEWER: . . . Where would that point fall? If we graphed that – can you predict . . . MARJORIE: Add 3 to that, should go to 34 . . . . Plot and hit 11, and 34. Yep . . . {she enters 11, 34 in the table}
Rep-4: Usefulness	Quotes indicating whether or not a subject finds a particular representation to be useful.	"INTERVIEWER: How would you write it in a table? Write it as if it were in a table. MARLON: In a table so um it'd be here , zero , plus 9 {he creates an x, y table and enters 0 in the x column and +9 in the y column} That'd be my table"
Rep-5: Validity	Quotes which seem to show whether a representation is valid or not valid for the subject, that is whether or not it accurately represents the mathematics it seeks to represent and is flexible enough to allow additional mathematical ideas to be built upon it.	Because I think there was a difference of three. For each um. For at least the first three {points to patterns on the paper}, there was a difference of three. So I had just added three and just kept adding three. {points to pattern number 4 and beyond}

Code Family: Representational Ideas and Issues (cont.)

Code	Description	Example
Rep-6: Mathematical language	Quotes which give insight into the subject's use of verbal representations, such as his or her misuse or idiosyncratic use of terminology, difficulty in remembering mathematical terminology which affects his or her ability to use the software, or difficulty understanding what is being asked of him or her verbally by the interviewer. Idiosyncratic use would be non-standard, but make some sense to the subject (such as "axle" instead of "axis").	"I have the x and the new parameter and it looks like a pi sign and an e. So I click on new parameter .See what that does. Some coordinates are in there {she is referring to the name and value of the parameter as "coordinates"}"
Rep-7: Visual observation	Where the subject seems to see something in the way things look that affects his or her thinking. Among things to look for is the phrase "I noticed"	"they're all shaded"

Code Family: Influences and Uses of Technology

Emergent codes

Code	Description	Example
Tech-EM: Empowerment Through Technology	Quotes showing the subject making their own mathematical choices and/or discoveries using the technology	"Pretty neat. Let's see what measure is."
Tech-WT: Technology as a window into student thinking[13]	Quotes in which the subject's use of technology gives the observer insight into their mathematical thinking	"now on the right hand side of that is all positive {illustrates this by moving the cursor from the origin along the x-axis to the right}"
Tech-SR: Technology as an aid in the use of standard representations	Quotes which seem to show that the use of technology has aided the subject in the use of a standard representation with which they may have been struggling	"Where would that point fall? If we graphed that - can you predict MARJORIE: Add 3 to that, should go to 34 . . . Plot and hit 11, and 34.Yep."
Code	Description	Example
Tech-AR: Technology as an aid to reasoning	Quotes which show the subject using technology to reason about mathematics	"INTERVIEWER: Yeah. Did it do what you . . . thought it was going to do? MARLON: It's going to - it's actually 5, 14 now where it stopped it at . . . That would be um, {writes g(4) = 4 + 9 and then g(5) = 5 + 9}. Okay in this case here . . . 5 plus 9 is 14
Tech-MC: Technology as an aid to mathematical communication	Quotes in which the subject uses or expresses a desire to use the software to communicate particular mathematical ideas.	Okay same thing (here) if I go over here [31:38] {Moves B to about (-4.5, 6.1)} I'm in a negative area here still in the positive area going upwards. So this x and y, okay. Same thing down here {moves B to 3rd quadrant}.
Tech-CM: Using technology to reveal and clear up misconceptions	Quotes in which the use of technology helps identify and/or clear up a misconception	"and hit okay . . . Oop. Okay I did it wrong"

[13] Stevens, To, Harris, and Dwyer (2008) spoke of LOGO as giving a "window into the student's mind" (p. 199).

Code Family: Other

These codes are not part of the final theory, but were used to note characteristics of data and were provided so that inter-coder reliability tests could be accurately conducted.

Code	Description	Example
Other-1: Characteristics of the learner	Quotations which give information about the background or attitude of the subject.	"I used to be in the military"
Other-2: Clerical and technical procedures of the Study	Portions where papers are being signed, the use of cameras and other arrangements are being described, and the methodology of the study is being explained.	"move this motion controller up, because when I put the films together your paper will be down there."
Other-3: Incidental conversation	Casual conversation such as discussions about the weather, the temperature of the room, greetings, good-byes. Conversation not related to the topic of the study or how it is being conducted.	"I'm sorry it was so hot in here"
Other-4: Narrative notes about the study	Not part of the transcription of what happened during the study – environmental or background information added in later.	Marjorie did not create or interact with any mathematical representations on paper during this session.
Other-5: Scaffolding	Places where particular statements or actions on the part of the interviewer are influencing the actions of the subject.	"And what else do you see happening in that picture?"

Example of coding

Unit	Codes	Reasoning
INTERVIEWER: Okay, so you don't have any existing values in your sketch. So you can click on those menus and see what you find.	Indicative movements	The cursor movements which are present
MARJORIE: Okay, click on values {she does} INTERVIEWER: Go ahead (values)	Mathematical misconception	She confuses the name and value of a parameter with "coordinates"
MARJORIE: I have the x and the new parameter and it looks like a pi sign and an e {stated as cursor moves down the list of these items}. So I click on new parameter. See what that does. Some coordinates are in there {cursor moves along the name and value of the parameter as she says "coordinates"} um, under the name, it looks like they have a bracket and one and then under it says equal value one point	Scaffolding	The interviewer suggests actions on the part of the subject
	Verbal/mathematical vocabulary/language	Misuse of the word "coordinates"

Appendix E

Consent Forms

IRB approval notice

MEMORANDUM TO: Lauretta Garrett
Curriculum & Teaching

PROTOCOL TITLE: "The Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions"

IRB FILE NO.: 09-164 EX 0907

APPROVAL DATE: July 3, 2009
EXPIRATION DATE: July 2, 2010

The referenced protocol was approved "Exempt" on July 3, 2009 under 45 CFR 46.101 (b) (2):

> "Research involving the use of educational tests (cognitive, diagnostic, aptitude, achievement), survey procedures, interview procedures or observation of public behavior, unless:
> (i) information obtained is recorded in such a manner that human subjects can be identified, directly or through identifiers linked to the subjects; and
> (ii) any disclosure of the human subjects' response outside the research could reasonably place the subjects at risk of criminal or civil liability or be damaging to the subjects' financial standing, employability, or reputation."

You should retain this letter in your files, along with a copy of the revised protocol and other pertinent information concerning your study. If you should anticipate a change in any of the procedures authorized in this protocol, you must request and receive IRB approval prior to implementation of any revision. Please reference the above IRB file number in any correspondence regarding this project.

If you will be unable to file a Final Report on your project before July 2, 2010, you must submit a request for an extension of approval to the IRB no later than June 15, 2010. If your IRB authorization expires and/or you have not received written notice that a request for an extension has been approved prior to July 2, 2010 you must suspend the project immediately and contact the Office of Human Subjects Research for assistance.

A Final Report will be required to close your IRB project file. You are reminded that you must use the stamped, IRB-approved informed consent when you consent your participants. Please remember that signed consent forms must be retained at least three years after completion of your study.

If you have any questions concerning this Board action, please contact the Office of Human Subjects Research at

COLLEGE OF EDUCATION

CURRICULUM AND TEACHING

(NOTE: DO NOT SIGN THIS DOCUMENT UNLESS AN IRB APPROVAL STAMP WITH CURRENT DATES HAS BEEN APPLIED TO THIS DOCUMENT.)

INFORMED CONSENT
for a Research Study entitled
"The Effect of Technological Representations on Developmental Mathematics Students' Understanding of Functions"

You are invited to participate in a research study to find out how using computer software to study mathematics affects how a person thinks about mathematics. The study is being conducted by Lauretta Garrett, graduate student, under the direction of Dr. professor in the University Department of Curriculum and Teaching. You were selected as a possible participant because you are a developmental mathematics student and are age 18 or older.

What will be involved if you participate? If you decide to participate in this research study, you will be asked to participate in a sequence of video taped interviews. Your total time commitment in addition to the initial interview will be 30 to 45 minutes per interview, for four to eight interviews over a period of 4 to 8 weeks. In addition to the researcher, the tapes may be viewed by the researcher's advisor.

Are there any risks or discomforts? The risks associated with participating in this study are that others may find out your personal information, connect you personally with the data collected, or that you may experience some discomfort at recalling unpleasant prior experiences with mathematics. To minimize these risks, the researcher will keep your personal information in a locked file box and it will eventually be destroyed. In addition, you will have the right to decide who, other than the researcher and her advisor, may view the videotapes, and this will be documented in writing. You will also have the right to withdraw from the study at any time.

Are there any benefits to yourself or others? If you participate in this study, you can expect to learn about a type of technology used for mathematics education. You will also receive personal instruction in a topic that is part of the curriculum for the course in which you are currently enrolled. You can expect that these things may benefit you in future mathematics courses. Results of this study will help researchers understand the effects of technology better so that future teachers can use technology for teaching mathematics more effectively. I cannot promise you that you will receive any or all of the benefits described.

Will you receive compensation for participating? To thank you for your time, you will be given a $30 gift certificate if you complete at least 4 interviews and an additional $30 gift certificate if you remain in the study until it is completed.

Participant's initials ______

Are there any costs? If you decide to participate, you will need to be present at the interviews and provide your time over the period mentioned above.

If you change your mind about participating, you can withdraw at any time during the study. Your participation is completely voluntary. If you choose to withdraw, your data can be withdrawn as long as it is identifiable. Your decision about whether or not to participate or to stop participating will not jeopardize your future relations with Jniversity, the Department of Curriculum and Teaching, or University.

Your privacy will be protected. Any information obtained in connection with this study will remain confidential. Information obtained through your participation may be used to fulfill the requirements for an advanced degree, published in a professional journal, presented at a professional meeting, used for educational purposes, etc.

If you have questions about this study, please ask them now or contact

A copy of this document will be given to you to keep.

If you have questions about your rights as a research participant, you may contact the University Office of Human Subjects Research or the Institutional Review Board by
IRBChair

HAVING READ THE INFORMATION PROVIDED, YOU MUST DECIDE WHETHER OR NOT YOU WISH TO PARTICIPATE IN THIS RESEARCH STUDY. YOUR SIGNATURE INDICATES YOUR WILLINGNESS TO PARTICIPATE.

Participant's signature Date   Investigator obtaining consent Date

Printed Name   Printed Name

Additional video tape use:
By signing below, I am saying YES, I agree to allow the video tapes of myself which are taken during this study to be used for additional purposes (*education, presentations, training, etc.*) beyond the immediate needs of this study. I understand that these video tapes will not be destroyed at the end of this research, but will be retained indefinitely.

Participant's signature Date

Printed Name

RECRUITMENT SCRIPT (verbal, in person)
(*This should be a brief version of the consent document.*)

My name is Lauretta Garrett, a doctoral student from the Department of Curriculum and Teaching at . I would like to invite you to participate in my research study to find out how using computer software to study mathematics affects how a person thinks about mathematics. Anyone may participate if they are enrolled in Math 098, Math 099, or Math 100 and age 18 or older. Please do not participate if you are under age 18.

As a participant, you will be asked to participate in a sequence of video taped interviews. Your total time commitment after the initial interview will be approximately 30 to 45 minutes per interview, for four to eight interviews over a period of about 4 to 8 weeks.

(*Briefly discuss any risks, compensation or benefits, costs, privacy issues, or other information that would likely influence the participant's interest in the study*)
By participating, you risk the possibility of your personal information being known to others, but I will take steps to protect that data. You also risk some discomfort if you suffer from mathematics anxiety, but you will be able to work at your own pace. You do have the opportunity to learn about mathematics software and mathematics topics from your course. Your personal information will be kept secure during the study and will be destroyed when the research is completed You can withdraw at any time. You can decide who, in addition to me and possibly one other person involved with the research, will see the videotapes.

If you would like to participate in this research study, contact me at the email address or phone number listed on this flyer (pass out flyers).

Do you have any questions now?

If you have questions later, my contact information is on the flyer.

Technology Study

HAVE FUN! LEARN! PREPARE FOR THE FUTURE!

Are you enrolled in Math 98, Math 99, or Math 100 and age 18 or older?

Do you want to learn about some mathematics software that might help you to be successful in mathematics?

If you answered ***YES*** to these questions, you may be eligible to participate in a mathematics education research study.

The purpose of this research study is to find out how using computer software affects the way people think about mathematics. Benefits include learning about a type of technology used for mathematics education and receiving personal instruction in a topic that is part of the curriculum for the course in which you are currently enrolled. **Participants will receive gift certificates in appreciation for their participation.**

Anyone 18 or older enrolled in Math 97, Math 98, or Math 100 is eligible.

This study is being conducted by a doctoral student in the Department of Curriculum and Teaching at . Interviews will take place at .

Please contact Lauretta Garrett at or for more information.

PLEASE RESPOND BY SEPTEMBER 25th