

**Simplification and Order Reduction of Parametrically Excited
Nonlinear Dynamical Systems**

by

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Abstract

System simplification is, by far, the most common theme behind various techniques available for the analysis and controller design of nonlinear dynamic systems. One such technique that is widely used by engineers is the direct linearization about an equilibrium point. Most other techniques, however, involve carefully designed state transformations so that the system in the transformed domain preserves nonlinear characteristics of the original system and facilitates efficient analysis and controller design. This dissertation presents a few of such methodologies for analysis and control of nonlinear dynamic systems. However, the emphasis is on nonlinear systems with parametric excitations.

The main focus is on the development of techniques for reduced order modeling of nonlinear dynamic systems that are influenced by certain external inputs. These reduced order models accurately approximate the dominant dynamics of original systems and are much simpler to analyze and control. Two types of inputs are considered; namely, external periodic excitations and a nonlinear, stabilizing state feedback control. The approach is based on the construction of an invariant subspace such that any motion initiated on this subspace, often called an invariant manifold, remains confined there for all time, t . As a result, the system dynamics can be modeled by a smaller number of differential equations which govern system evolution on the manifold, alone. Techniques to identify suitable state variables which span these manifolds are suggested and approximate solutions of the resulting partial differential equations, which govern the geometry of such manifolds, are presented. Reduced order models for systems with periodic excitation inputs can be integrated directly to analyze the dynamics of original systems. For

situations where there is a state feedback input, nonlinear controllers can be designed in the reduced order domain. A technique to synthesize such controllers is also discussed in this work.

Another important simplification technique proposed in this work is the construction of normal forms (simplest forms) of nonlinear systems by direct application of time-periodic near-identity transformations and Poincaré normal form theory. This is an independent analysis technique for systems subjected to periodic forces. It also provides a basis for construction of time-dependent invariant manifolds for reduced order modeling of such systems. Some typical dynamic systems of engineering interest are considered and the effectiveness of proposed methodologies is illustrated.

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Table of Contents

Abstract	ii	
Acknowledgments	iv	
List of Figures	vii	
Chapter 1	Introduction	1
	1.1 Motivation	1
	1.2 Model Order Reduction Background	2
	1.3 Contributions of this Dissertation	7
Chapter 2	Analysis of Nonlinear Systems Subjected to External Periodic Excitations via Normal Forms	11
	2.1 Introduction	11
	2.2 Mathematical Background	12
	2.2.1 Time Independent Normal Forms (TINF)	12
	2.2.2 Time Dependent Normal Forms (TDNF)	15
	2.3 Construction of Normal Forms for Forced Nonlinear Systems	19
	2.3.1 Systems with Constant Coefficients	19
	2.3.2 Systems with Time-Periodic Coefficients	24
	2.3.3 An Alternate Approach for Non-resonant Case	26
	2.4 Applications.....	28
	2.4.1 Forced Duffing's Equation	29
	2.4.2 Forced Mathieu-Duffing Equation	35
Chapter 3	Order Reduction of Nonlinear Systems Subjected to External Periodic Excitations	44

3.1	Introduction	44
3.2	Order reduction of Systems Subjected to Periodic Excitations	46
3.2.1	Systems with Constant Coefficients	46
3.2.2	Systems with Time-Periodic Coefficients	54
3.3	Applications	57
3.3.1	A 2-DOF Spring-Mass-Damper System	57
3.3.2	A 2-DOF Coupled Inverted Pendulum	62
Chapter 4	Reduced Order Controller Design for Nonlinear Systems	76
4.1	Introduction	76
4.2	Controller Design for Linear Time-Periodic Systems	77
4.3	Reduced Order Nonlinear Controller Design	78
4.3.1	Systems with Time-Periodic Coefficients	78
4.3.2	Systems with Constant Coefficients	89
4.4	Applications	91
4.4.1	A 4-DOF Coupled Inverted Pendulum	91
4.4.2	A 2-DOF Spring-Mass-Damper System	97
Chapter 5	Discussion and Conclusions	110
5.1	Summary of Work	110
5.2	Suggestions for Future Work	114
References	117
Appendix A	Floquet Theory and Lyapunov-Floquet Transformation	122
Appendix B	Solution of Time-periodic Sylvester Differential Equation	131
Appendix C	Computer Codes	136

List of Figures

2.1	System response for forced Duffing's equation (non-resonant forcing)	38
2.2	System response for Forced Duffing's equation ($3\Omega \approx \omega_n$)	39
2.3	System response for forced Duffing's equation ($\Omega \approx 3\omega_n$)	40
2.4	System response for forced Mathieu-Duffing equation (Floquet exponent has negative real parts)	41
2.5	System response for forced Mathieu-Duffing equation (Floquet exponent are purely imaginary)	42
2.6	System response for forced Mathieu-Duffing equation ($a \equiv 0$, no generating solution exists)	43
3.1	A 2-DOF spring-mass-damper	65
3.2	System response for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$)	66
3.3	Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$) ...	67
3.4	Amplitude variation vs excitation frequency for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$)	68
3.5	System response for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, 3\Omega_2 \approx \omega_2$)	69
3.6	Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, 3\Omega_2 \approx \omega_2$)	70
3.7	System response for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, \Omega_2 \approx 3\omega_2$)	71
3.8	Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, \Omega_2 \approx 3\omega_2$)	72
3.9	A 2-dof coupled inverted pendulum	73
3.10	System response for 2-DOF system with periodic coefficients	74

3.11	Fast Fourier transforms for 2-DOF system with periodic coefficients)	75
4.1	A 4-DOF coupled inverted pendulums	100
4.2.	Uncontrolled system response for system with periodic coefficients	101
4.3	Controlled system response for system with periodic coefficients	102
4.4	Control effort for system with periodic coefficients	103
4.5	Regions of attraction for controlled system (system with periodic coefficients)	104
4.6	2-DOF spring-mass-damper system	105
4.7	Uncontrolled system response for system with constant coefficients	106
4.8	Controlled system response for system with constant coefficients	107
4.9	Control effort for system with constant coefficients	108
4.10	Regions of attraction for controlled system (System with constant coefficients)	109

Chapter 1

Introduction

1.1 Motivation

Dynamics of many engineering systems is often represented by a set of large number of nonlinear ordinary differential equations (ODEs). These equations may result from lumped modeling of systems or finite element discretization of partial differential equations (PDEs) associated with the system dynamics. In some cases a system itself may have a discrete representation with a very large number of degrees of freedom. An important class of systems also results into nonlinear ODEs with periodic coefficients. Systems with structural members subjected to in-plane loads, torsional systems driven by a Hook's joint, asymmetric rotor bearing systems, etc. are few examples which yield equations of motion of this kind. Analysis and simulations of such large dimensional nonlinear systems, with constant or periodic coefficients, are costly to perform in terms of computer time and storage. More importantly, it is difficult to grasp interrelationships among various system parameters and their influence on system dynamics as these interactions are often too complex due to system nonlinearities; which is true even for a small scale systems. At the same time, the task of designing nonlinear feedback controllers for such systems poses serious challenges. Because the nonlinear controller design techniques typically require symbolic computations (e.g., Deshmukh (2000), Zhang (2007)), the high dimensionality of the phase-space of large nonlinear systems presents significant challenges to these efforts. Construction of reduced order models, which are represented by a lesser number

of state variables, may overcome some of these difficulties. The main objective of this dissertation is to address such model order reduction strategies for nonlinear dynamic systems with constants as well as periodic coefficients which are subjected to external periodic or stabilizing control input. The approach is based on the construction of an invariant subspace on which the system dynamics is represented by a fewer number of state equations.

Another simplification technique proposed in this work is aimed at construction of normal forms for nonlinear systems subjected to external periodic excitation. However, unlike order reduction, in this case the system dimensions remain the same as that of the original system in its simplified form. Here also, systems with constant as well as periodic coefficients are considered.

1.2 Model Order Reduction Background

Consider a dynamic system represented by a set of first order ordinary differential and algebraic equations given as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} &= \boldsymbol{\chi}(\mathbf{x}, t)\end{aligned}\tag{1.1}$$

where $\mathbf{x} \in R^n$ is the state vector, $\mathbf{y} \in R^m$ is the output, $\mathbf{u}(\mathbf{u} \in R^s, s \leq n)$ is an input or excitation function, while $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ and $\boldsymbol{\chi}(\mathbf{x}, t)$ are nonlinear vector fields of an appropriate dimension. The complexity of such a system is usually indicated by the number of state variables that comprise the system. Model order reduction in such cases implies construction of a dynamic system of the form

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, t) \\ \tilde{\mathbf{y}} &= \tilde{\chi}(\tilde{\mathbf{x}}, t)\end{aligned}\tag{1.2}$$

such that $\tilde{\mathbf{x}} \in R^p$ ($p < n$) and the output of the reduced order model (1.2) $\tilde{\mathbf{y}}(\tilde{\mathbf{y}} \in R^m)$ is a good approximation of the original output (\mathbf{y}). A typical model order reduction method requires one to define a transformation such that the less important (non-dominant) states of the system are expressed as functions of the important (dominant) states. This transformation can then be substituted into the original system to eliminate the less important states. Further, apart from accurately estimating the dynamics of the original system, a reduced order system may also need to satisfy some of the following conditions.

1. Stability and bifurcation characteristics of the original model should be preserved.
2. In case the system is subjected to a state feedback, it should be feasible to design a controller in the reduced order domain.
3. The procedure should be computationally stable and efficient.

The problem outlined above is a generalized order reduction problem and more specific cases of this problem have been addressed by various researchers in the past. Among them, the case of linear time-invariant systems (LTI) has received a considerable amount of attention in the literature. These systems result from the direct linearization of (1.1) about the equilibrium point when the equation do not contain any time-varying terms. A comprehensive survey of the recent work on this topic can be found in Antoulas (2005), Antoulas et al. (2004), Benner (2005, 2006) and Benner et al. (2006). These order reduction techniques, which are based on the idea of linear

projection of system's original state-space onto a smaller dimensional space, fall into three main categories (Benner, 2006) namely; balanced truncation, moment matching and modal truncation.

The idea behind balanced truncation is to eliminate those states which are difficult to control and observe. This is achieved by transforming the system equation into the balanced form for which the controllability and observability grammians are equal and in the diagonal form. States corresponding to the small diagonal elements are nearly uncontrollable and unobservable and, therefore, can be eliminated. However, this approach preserves the stability and provides bounds on approximation error when applied to stable systems. Next, the moment matching is a frequency domain based order reduction technique. In this technique the transfer functions of a large system and a reduced order system are expanded into a power series and first k coefficients of the power series are matched to find parameters of the reduced order system. Coefficients of the power series are called 'moments' and k is the order of the reduced system. Modal truncation technique is, by far, the simplest of the three. The principle of this method is to project the state space onto the subspace spanned by eigenvectors corresponding to the dominant eigenvalues of linear state matrix. Thus, the dominant behavior of the original system is preserved in a reduced order system. Typically, eigenvalues which are very close to the stability boundary or those which are in resonance with external excitations are considered the dominant.

Further, these techniques have also been generalized for the linear time-varying (LTV) systems. Among these, balanced truncation is the most widely used techniques for order reduction of the LTV systems. Some of the early work and references on balanced truncation of the LTV systems can be found in Shokoohi et al. (1983) and Verriest and Kailath (1983). Recently the moment matching type techniques have also been used for order reduction of the LTV systems arising from large electronic circuits (Phillips, 1998; Roychowdhury, 1999; Wan

and Roychowdhury, 2005). However, a direct modal truncation for LTV systems is not possible due to the fact that unlike LTI systems, the modes of LTV systems are not defined, in general. For linear systems with time-periodic coefficients, one may employ the Lyapunov-Floquet (L-F) transformation which yields time-invariant systems. However, not only the calculation of the L-F transformation matrix is difficult, it also renders a time-periodic/quasiperiodic input gain matrix. To author's best knowledge there is only one reference (Deshmukh et al., 2000), so far, which has explored this possibility.

Although there exists an abundant amount of literature on the order reduction of linear systems, there is not as much work done for nonlinear systems; more so, for nonlinear systems with time-varying parameters. The existing nonlinear order reduction techniques can be divided into two main categories; namely, 1) linear projection based techniques adapted for nonlinear model reduction and 2) nonlinear projection based techniques which are intrinsically nonlinear in nature. The former category typically employs either the Guyan or proper orthogonal decomposition (POD) like order reduction techniques to nonlinear dynamic systems. Some recent examples of the application of these techniques to nonlinear order reduction problem can be found in Friswell et al. (1995), Burton and Rhee (2000), Lenaerts et al. (2001), Kerschen et al. (2005), and Kumar and Burton (2007, 2009). Although, these techniques are computationally efficient and easy to implement, they do not yield various reducibility conditions in a closed form. These reducibility conditions, is an important tool in understanding interactions between system's modes and their interactions with external and/or parametric excitations. Moreover, POD based techniques require *a priori* simulation of the original large system to find principle orthogonal modes.

The nonlinear projection based order reduction techniques can be broadly classified into three categories; namely, center manifold, nonlinear normal modes (NNMs) and singular perturbation. A common thread which relates these three techniques is the construction of an invariant subspace in systems phase-space such that any motion initiated on this invariant subspace, often called an invariant manifold, remains on the subspace for all time t . The system dynamics on this subspace is governed by a fewer number of equations. First technique, the ‘center manifold’, is a useful order reduction tool when the linear system matrix has a special spectral decomposition; wherein, some of the eigenvalues of the system lie on the imaginary axis (critical modes) while the remaining are on the left half of the complex plane (stable modes). In such systems, the evolution of the dynamics is governed by the critical modes alone and damped modes follow this ‘dominant dynamics’ passively. As a result, it is possible to construct a minimal sized model which depends only on the critical modes of the system. A detailed discussion of the center manifold technique can be found in Carr (1981) and Guckenheimer and Holmes (1983). The second technique, NNMs (Nonlinear Normal Modes), was originally introduced by Rosenberg (1966). He defined NNMs for autonomous, conservative nonlinear systems as a vibration in unison with fixed relations between generalized coordinates. Later, Shaw and Pierre (1993) generalized the concept of NNMs to non-conservative systems. They used invariant manifold approach in which NNMs are assumed to be invariant subspaces which are non-planer and tangential to corresponding normal-modes of the linearized system at the equilibrium. This nonlinear extension of the modal analysis allows one to generate reduced order models by projecting non-essential modes on the invariant manifold which is parameterized by a set of desired modes. NNMs approach has gained popularity among structural/mechanical engineers in recent years. Agnes and Inman (2001) and Pesheck et al. (2002) have applied this technique to reduce the order of real engineering problems. The third technique, ‘singular

perturbation', is more popular in the control engineering community where it has been used effectively to design reduced order controllers for various nonlinear systems. Systems which come under the premise of singular perturbation admit (or can be transformed into) a special form in the state-space. In this form some of the derivatives of states are multiplied by a small positive parameter ε . Such systems typically display a two time-scale phenomenon; by which, the fast subsystem approaches a steady state more quickly than the slow subsystem. Consequently, the long term behavior of such systems can be approximated by the slow subsystems, alone, on a lower dimensional subspace. The singular perturbation provides a theoretical frame work to decompose such large systems into two subsystems; the slow subsystems representing the reduced order models. A detailed discussion on the subject can be found in books by Kokotović et al. (1999) and Khalil (2002). Among the techniques discussed above, the center manifold technique has been extended to systems with periodic coefficients (Malkin, 1962). An interesting account of the application of time-periodic center manifold theory for the bifurcation analysis of parametrically excited systems can be found in Pandiyan and Sinha (1995) and Dávid and Sinha (2003). Further, Sinha et al. (2004) have also generalized the NNM technique, introduced by Shaw and Pierre (1993), for systems with periodic coefficients. The treatment for singular perturbation technique also accommodates time-varying parameters in some of the literature. However, to the author's best knowledge, the practical implementation of this technique to general time-varying nonlinear systems has not been reported.

1.3 Contributions of this Dissertation

The primary goal of this dissertation is to develop some new strategies as well as practical algorithms for the order reduction of nonlinear dynamic systems which are subjected to certain external inputs. Two types of inputs are considered; namely, external periodic excitations

and a nonlinear, stabilizing state feedback control. The techniques proposed here are based on the construction of invariant manifolds such that the system dynamics projected on these manifolds is represented by a smaller number of equations. Another important contribution of this dissertation is the development of Poincaré normal form technique for nonlinear systems subjected to external periodic excitations. Apart from being an independent analysis technique, this work also provides a basis for the construction of time-dependent invariant manifolds for systems subjected to periodic inputs. The rest of the dissertation is arranged in the following order.

In Chapter 2, a direct application of the method of normal forms for the analysis of nonlinear systems subjected to a periodic excitation is discussed. The direct approach presented here is in contrast to the procedures where such systems are transformed into higher order homogenous forms by defining additional coordinates. A set of time-periodic transformations, called near-identity transformations, are applied such that the resulting system of equations is of simpler form, or even linear if all nonlinear terms can be eliminated. The solvability conditions for such reductions are also derived in a closed form. Systems with time-periodic coefficients are handled via an application of the L-F transformation that converts the linear parts of systems into time-invariant forms. The technique has distinct advantage over averaging or perturbation techniques because periodic coefficient of the linear term, in this case, does not require a small parameter assumption. In addition, the question of existence of so called ‘generating solution’ does not arise in this approach. The usefulness of the method is demonstrated by applications to periodically forced time-invariant as well as time-periodic systems. Comparisons with numerical solutions show that the proposed techniques provide accurate results for all cases considered. This is a stand-alone analysis technique for periodically forced nonlinear problems; however, results obtained in this work also play an important role in the development of methodology for the construction of time-dependent invariant manifolds.

In Chapter 3, an order reduction technique for nonlinear systems subjected to external periodic excitations is proposed via construction of time-dependent invariant manifolds. The geometry of invariant manifolds is governed by a PDE which is nonlinear and contains time-periodic coefficients. An approximate solution of this PDE is suggested in terms of a multivariable Taylor-Fourier series. It is shown that in the case when the external excitations projected on the dominant dynamics are small, the equation governing the manifold can be reduced to a set of equations which are simpler to solve and yield the reducibility conditions in a closed form. These reducibility conditions, which represent interactions among dominant, non-dominant states and the excitation terms, are indicators of the feasibility and extent of accuracy of such order reductions. The proposed technique does not put any constraint on the selection of dominant and non-dominant states; provided that the chosen partitioning satisfies all reducibility conditions. Therefore, an investigator is free to choose any set of states of practical importance as dominant states. The resulting reduced order models can be numerically integrated or can be analyzed using traditional nonlinear analysis techniques. Systems with time-periodic coefficients are handled via an application of the L-F transformation. Due to the periodic nature of the L-F transformation matrix, however, the system nonlinearities in the transformed domain contain additional time-periodic coefficients which warrant a modification in the structure of the invariant manifold. The reducibility conditions are obtained in terms of Floquet exponents in this case. The effectiveness of the method is demonstrated by constructing reduced order models for some typical engineering systems. Results obtained from reduced order models are compared with numerical simulations of full order models. Based on these results, it is concluded that the proposed technique provides accurate reduced order models for nonlinear systems of fairly general form provided that the reducibility conditions are satisfied.

In Chapter 4, a technique for the design of reduced order controllers for parametrically excited nonlinear systems is described first. The approach is restricted to the class of local controllers where large scale nonlinear systems may need to be stabilized about an equilibrium point, a periodic motion or desired to be driven to a periodic orbit. Since only the stabilization problem is addressed in this work, unstable states (Floquet multipliers outside the unit circle) are considered to be the dominant states and stable states (Floquet multipliers inside the unit circle) are considered as the non-dominant states. Techniques similar to the Poincaré normal form theory is used to transform the reduced order nonlinear closed-loop system to a time-periodic linear system. The linear controller is designed using a symbolic approach that allows one to place the Floquet multipliers within the unit circle at the desired locations. After the necessary transformations, a controller can be obtained in the original states. The case of systems with constant coefficients is treated as a special case of its time-periodic counterpart.

In Chapter 5, a summary of this dissertation is presented. Some suggestions for future research activities in this area are also outlined.

Chapter 2

Analysis of Nonlinear Systems Subjected to External Periodic Excitations via Normal Forms

2.1 Introduction

The theory of normal forms, originally introduced by Poincaré, is a powerful tool in simplification and analysis of complex nonlinear systems. Originally the normal form theory was developed for time-invariant systems only. Some of the early discussions on time-invariant normal form theory can be found in Birkhoff (1927), Moser (1928) and Arnold (1978). Later, the technique was extended to systems with parametric forcing (Arnold 1983; Arrowsmith and Place 1990), where the Lyapunov-Floquet theory was used to transform the time-periodic system into a form where the linear part of the system becomes time invariant. However, the applications to real engineering problems remained unsuccessful until Sinha and Pandiyan (Sinha and Pandiyan, 1994; Pandiyan and Sinha, 1995) developed a practical technique for the analysis of such systems via an efficient computation of L-F transformation and use of the time dependent normal form theory. However, these techniques are not directly applicable when systems are subjected to an external periodic forcing. Nayfeh (1993) analyzed Duffing's oscillator with external forcing using the method of normal forms, where an additional state-variable was introduced to convert the system into a homogenous form. However, to author's knowledge, to date, there is no normal form formulation available for the analysis of nonlinear dynamic systems with time varying coefficients and external forcing.

The work presented in this chapter deals with a quantitative analysis of nonlinear forced problems via a direct application of near-identity transformations and normal forms. Systems with constant as well as periodic coefficients are considered. In the former case, the suggested transformation is applied directly; while, for the latter case, first, the L-F transformation is used to convert a system into the form where linear part becomes time-invariant. In the following section the general frame work of the normal form technique for homogeneous systems is discussed.

2.2 Mathematical Background

2.2.1 Time Independent Normal Forms (TINF)

The idea behind the normal form reduction is to simplify nonlinear equations by eliminating as many nonlinear terms as possible, while keeping the linear part unchanged. This is achieved by application of successive ‘*near-identity*’ transformations to the original nonlinear system, beginning with the lowest order transformation. Consider a nonlinear dynamic system given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \bar{\mathbf{f}}_2(\mathbf{x}) + \bar{\mathbf{f}}_3(\mathbf{x}) + \cdots + \bar{\mathbf{f}}_k(\mathbf{x}) + \mathbf{O}\left(|\mathbf{x}|^{k+1}\right) \quad (2.1)$$

Typically, such systems results form a Taylor series expansion of nonlinear vector fields about the equilibrium point. In the above equation \mathbf{A} is an $n \times n$ constant matrix, $\bar{\mathbf{f}}_r(\mathbf{x})$ are $n \times 1$ vectors of homogeneous monomials in \mathbf{x} of order r ($r = 2, 3, \dots, k$). An application of the modal transformation ($\mathbf{x} = \mathbf{M}\mathbf{z}$) to above system yields

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{M}^{-1}\bar{\mathbf{f}}_2(\mathbf{M}\mathbf{z}) + \mathbf{M}^{-1}\bar{\mathbf{f}}_3(\mathbf{M}\mathbf{z}) + \cdots + \mathbf{M}^{-1}\bar{\mathbf{f}}_k(\mathbf{M}\mathbf{z}) + \mathbf{M}^{-1}\mathbf{O}\left(|\mathbf{M}\mathbf{z}|^{k+1}\right) \quad (2.2)$$

where $n \times n$ matrix \mathbf{J} is in the Jordan canonical form and \mathbf{M} is the modal transformation matrix.

Further, an application of the ‘near-identity’ transformation of the form

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_2(\mathbf{v}) \quad (2.3)$$

where $\mathbf{h}_2(\mathbf{v}, t)$ is an $n \times 1$ homogeneous vector of monomials in \mathbf{v} of order 2, to equations (2.2)

yields

$$\dot{\mathbf{v}} = \mathbf{J}\mathbf{v} - \left\{ \frac{\partial \mathbf{h}_2(\mathbf{v})}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_2(\mathbf{v}) - \mathbf{f}_2(\mathbf{v}) \right\} + \mathbf{f}_3(\mathbf{v}) + \dots + \mathbf{f}_k(\mathbf{v}) + \mathbf{O}(|\mathbf{v}|^{k+1}) \quad (2.4)$$

Thus, if one wants to eliminate all the quadratic terms from equation (2.4) the ‘near-identity’ transformation (2.3) must satisfy a PDE given by

$$\frac{\partial \mathbf{h}_2(\mathbf{v})}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_2(\mathbf{v}) - \mathbf{f}_2(\mathbf{v}) = 0 \quad (2.5)$$

Above equation is known as the ‘*homological equation*’ associated with $\mathbf{h}_2(\mathbf{v})$.

Similarly, it can be shown that the higher order nonlinearities can be eliminated by successive application of near-identity transformations of the respective degree. The resulting general form for the homological equation can be written as

$$\frac{\partial \mathbf{h}_r(\mathbf{v})}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_r(\mathbf{v}) - \mathbf{f}_r(\mathbf{v}) = 0 \quad (2.6)$$

It should be noted that for $r > 3$, $(\mathbf{f}_r(\mathbf{v}))$ is expressed in terms of the solution of homological equation of degree $r-1$. In general, it is not possible to determine an exact solution for the homological equation (2.6). However, one can obtain an approximate solution in terms of series expansion. Accordingly, $\mathbf{h}_r(\mathbf{v})$ and $\mathbf{f}_r(\mathbf{v})$ are expressed as

$$\begin{aligned}\mathbf{h}_r(\mathbf{v}) &= \sum_{j=1}^n \sum_{\mathbf{m}_r} h_{r,j,\mathbf{m}_r} |\mathbf{v}|^{\mathbf{m}_r} \mathbf{e}^j \\ \mathbf{f}_r(\mathbf{v}) &= \sum_{j=1}^n \sum_{\mathbf{m}_r} f_{r,j,\mathbf{m}_r} |\mathbf{v}|^{\mathbf{m}_r} \mathbf{e}^j\end{aligned}\tag{2.7}$$

where $\mathbf{m}_r = (m_1, m_2, \dots, m_n)$, $\sum_{i=1}^n m_i = r$ ($r = 2, 3, \dots, k$), $|\mathbf{v}|^{\mathbf{m}} = \mathbf{v}_1^{m_1} \mathbf{v}_2^{m_2} \dots \mathbf{v}_n^{m_n}$ and \mathbf{e}^j is the j^{th} member of the natural basis. Then, substitution of (2.7) into equation (2.6) and comparison of coefficients of the similar terms on both sides yields

$$h_{r,j,\mathbf{m}_r} = \frac{f_{r,j,\mathbf{m}_r}}{\mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j}\tag{2.8}$$

which are coefficients of any r^{th} order near-identity transformation. In above equation $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ is a vector containing eigenvalues of the linear matrix \mathbf{A} . It should be noted that all coefficients of the r^{th} order near-identity transformation can be found only if the following solvability condition is satisfied

$$\mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j \neq 0\tag{2.9}$$

If the solvability condition is satisfied then equation (2.4) can be reduced to its equivalent linear form. In case, if the condition is not satisfied then it is called as the ‘*resonant*’ case and the corresponding resonant terms remain in the transformed equation. The resulting equation can be written in its simplest nonlinear form as

$$\dot{\mathbf{v}} = \mathbf{J}\mathbf{v} + \sum_{r=2}^k \mathbf{f}_r^*(\mathbf{v}) + \mathbf{O}\left(|\mathbf{v}|^{k+1}\right)\tag{2.10}$$

where $\mathbf{f}_r^*(\mathbf{v})$ are the resonant nonlinear terms.

One can also define a linear operator

$$L_A(\mathbf{h}_r) = \frac{\partial \mathbf{h}_r}{\partial \mathbf{v}} \mathbf{J} \mathbf{v} - \mathbf{J} \mathbf{h}_r, \quad r = 2, \dots, k \quad (2.11)$$

which carries homogeneous vector polynomials over the vector polynomials of the same degree.

If the set of eigenvalues of L_A does not contain any zeros then L_A is invertible, and equation (2.6) can be solved.

2.2.2 Time Dependent Normal Forms (TDNF)

Next, consider a nonlinear dynamic system with time-periodic coefficients given as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \bar{\mathbf{f}}_2(\mathbf{x}, t) + \bar{\mathbf{f}}_3(\mathbf{x}, t) + \dots + \bar{\mathbf{f}}_k(\mathbf{x}, t) + \mathbf{O}(|\mathbf{x}|^{k+1}, t) \quad (2.12)$$

where $n \times n$ matrix $\mathbf{A}(t)$ is T periodic such that $\mathbf{A}(t) = \mathbf{A}(t+T)$. $n \times 1$ nonlinear vectors $\bar{\mathbf{f}}_r(\mathbf{x}, t)$ are homogenous monomials in \mathbf{x} with T periodic coefficients. Typically, such equations arise due to Taylor series expansion of time-periodic systems about an equilibrium point. They may also arise in the study of stability and bifurcation of periodic orbits which may represent steady-state solutions of autonomous/nonautonomous systems. Since the matrix $\mathbf{A}(t)$ in equation (2.12) is time dependent, the direct application of normal form theory is not possible. However, if one can find the L-F transformation matrix, which transforms equation (2.12) into its equivalent dynamic form where the linear part of the transformed equation is time-invariant (Refer appendix A), an application of time dependent normal forms is possible as shown by Arnold (1983). For certain class of systems, called commutative systems, it is possible to obtain the L-F transformation in a closed form. However, for general periodic systems computation of such L-F transformation matrices is a difficult task. In this case, one has to compute the state

transition matrix (STM) as an explicit function of time. Sinha and co-workers (Sinha and Wu, 1991; Sinha et al., 1993; Wu and Sinha, 1994; Sinha and Pandiyan, 1994; Pandiyan and Sinha, 1995; Sinha et al., 1996) introduced one such technique which uses Chebyshev expansion for this purpose. The technique is not only computationally efficient, it is also suitable for relatively large problems and it has been used, through out this work, for successful computation of the L-F transformation matrix.

Application of L-F transformation ($\mathbf{L}(t)$) to system represented by equation (2.12) yields

$$\begin{aligned} \dot{\mathbf{y}} = & \mathbf{R}\mathbf{y} + \mathbf{L}^{-1}(t)\bar{\mathbf{f}}_2(\mathbf{L}(t)\mathbf{y},t) + \mathbf{L}^{-1}(t)\bar{\mathbf{f}}_3(\mathbf{L}(t)\mathbf{y},t) + \dots \\ & + \mathbf{L}^{-1}(t)\bar{\mathbf{f}}_k(\mathbf{L}(t)\mathbf{y},t) + \mathbf{L}^{-1}(t)\mathbf{O}\left(|\mathbf{L}(t)\mathbf{y}|^{k+1},t\right) \end{aligned} \quad (2.13)$$

where $\mathbf{x} = \mathbf{L}(t)\mathbf{y}$ and \mathbf{R} is a $(n \times n)$ constant real matrix which is, in general, complex. However, it is always possible to find a $2T$ periodic L-F transformation which results in a real matrix \mathbf{R} . Above form of the equation is amenable to the direct application of the time dependent normal form technique.

Equation (2.13) can be written in the Jordan canonical form as

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{f}_2(\mathbf{z},t) + \mathbf{f}_3(\mathbf{z},t) + \dots + \mathbf{f}_k(\mathbf{z},t) + \mathbf{O}\left(|\mathbf{z}|^{k+1},t\right) \quad (2.14)$$

where \mathbf{J} is Jordan canonical form of matrix \mathbf{R} , $\mathbf{f}_r(\mathbf{z},t)$ contain homogeneous monomials in \mathbf{z} of order r and their periodic coefficients have commensurate periods. Then, a sequence of time-periodic near-identity coordinate transformation can be applied, successively, in order to remove nonlinear terms. A general form for this time-periodic near-identity transformation can be given as

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_r(\mathbf{v}, t) \quad (2.15)$$

where $\mathbf{h}_r(\mathbf{v}, t)$ is a formal power series in \mathbf{v} of degree r . The unknown periodic coefficients of the near-identity transformation are assumed to have periods commensurate with T . Application of the near-identity transformation (2.15) to equation (2.14) yields

$$\begin{aligned} \dot{\mathbf{v}} = \mathbf{J}\mathbf{v} + \mathbf{f}_{r-1}^*(\mathbf{v}, t) - \left\{ \frac{\partial \mathbf{h}_r(\mathbf{v}, t)}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_r(\mathbf{v}, t) + \frac{\partial \mathbf{h}_r(\mathbf{v}, t)}{\partial t} - \mathbf{f}_r(\mathbf{v}, t) \right\} \\ + \mathbf{f}_{r+1}(\mathbf{v}, t) + \cdots + \mathbf{O}\left(|\mathbf{v}|^{k+1}, t\right) \end{aligned} \quad (2.16)$$

Therefore, the homological equation in this case is defined as

$$\frac{\partial \mathbf{h}_r(\mathbf{v}, t)}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_r(\mathbf{v}, t) + \frac{\partial \mathbf{h}_r(\mathbf{v}, t)}{\partial t} - \mathbf{f}_r(\mathbf{v}, t) = 0 \quad (2.17)$$

Similar to the time-invariant case, one can solve these time-periodic homological equations (Arnold, 1983; Arrowsmith and Place, 1990; Sinha and Pandiyan, 1994; Pandiyan and Sinha, 1995; Sinha et al., 1996; Butcher and Sinha, 2000; Wooden and Sinha, 2007) by expanding known ($\mathbf{f}_r(\mathbf{v}, t)$) and unknown ($\mathbf{h}_r(\mathbf{v}, t)$) terms into a multivariable Taylor-Fourier series as

$$\begin{aligned} \mathbf{f}_r(\mathbf{v}, t) &= \sum_{j=1}^n \sum_{\mathbf{m}_r} \sum_{l=-k}^k f_{r,j,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}|^{\mathbf{m}_r} \mathbf{e}^j \\ \mathbf{h}_r(\mathbf{v}, t) &= \sum_{j=1}^n \sum_{\mathbf{m}_r} \sum_{l=-k}^k h_{r,j,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}|^{\mathbf{m}_r} \mathbf{e}^j \end{aligned} \quad (2.18)$$

where $i = \sqrt{-1}$, $\omega = \pi/T$ and all other symbols carry same meaning as before. A term by term comparison of the Taylor-Fourier coefficients of the monomials yields coefficients of $\mathbf{h}_r(\mathbf{v}, t)$ as

$$h_{r,j,\mathbf{m}_r,l} = \frac{f_{r,j,\mathbf{m}_r,l}}{il\omega + \mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j} \quad (2.19)$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ and λ 's are eigenvalues of matrix \mathbf{J} and referred as the Floquet exponents of the system. Clearly, the solvability condition, $il\omega + \mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j \neq 0$, must be satisfied to determine all the coefficients of the near-identity transformation. Otherwise corresponding resonant terms will remain in the transformed equation and it can be written in its simplest nonlinear form as

$$\dot{\mathbf{v}} = \mathbf{J}\mathbf{v} + \sum_{r=2}^k \mathbf{f}_r^*(\mathbf{v}, t) + \mathbf{O}(|\mathbf{v}|^{k+1}, t) \quad (2.20)$$

where $\mathbf{f}_r^*(\mathbf{v}, t)$ are the resonant nonlinear terms. It is interesting to note the dependence of the resonance condition on the Fourier harmonics of periodic coefficients in the nonlinear terms. A detailed discussion on the structure of these resonance conditions for certain periodic and periodic-quasiperiodic systems can be found in Butcher and Sinha (2000) and Wooden and Sinha (2007), respectively.

It can be seen from the above discussion that solution of time-dependent homological equation requires solution of a large set of algebraic equations, which make application of TDNF computationally intensive. It has been shown by Rosenblat and Cohen (1980, 1981) that an approximation can be made, with out any significant compromise in quality of solution, by retaining only constant terms in $\mathbf{f}_r(\mathbf{z}, t)$ and neglecting all time-periodic terms in equation (2.14). Since the resulting equation is autonomous, the analysis can be carried out via the time independent normal form theory as described before.

In many cases, the resulting normal form equation can be solved analytically and the stability characteristics can be analyzed. Also, for these simpler equations, some algebraic manipulations and/or transformation to polar co-ordinate may render closed form solution. This final solution can be transformed back into original coordinate system by substituting back all the transformations.

2.3 Construction of Normal Forms for Forced Nonlinear Systems

This section discusses the quantitative analysis of nonlinear forced problems via a direct application of near-identity transformations and normal form theory. In the case of systems with constant coefficients, the suggested transformation is applied directly; while, for time varying systems, first, the L-F transformation is used to convert the system into the form where linear part becomes time-invariant. The transformation yields an expression that presents the solution as a superposition of steady state and transient solutions. Desired steady state solutions are obtained by the method of harmonic balance. Transient solutions are obtained by solving the homological equations and the resulting normal form equations.

2.3.1 Systems with Constant Coefficients

Consider the nonlinear system given by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \bar{\mathbf{F}}(t) + \varepsilon \bar{\mathbf{f}}_2(\mathbf{x}) + \varepsilon^2 \bar{\mathbf{f}}_3(\mathbf{x}) + \cdots + \varepsilon^{k-1} \bar{\mathbf{f}}_k(\mathbf{x}) + \varepsilon^k O(|\mathbf{x}|^{k+1}) \quad (2.21)$$

where ε is a *small* bookkeeping parameter, $\bar{\mathbf{F}}(t)$ is a periodic force vector with period T and all other terms carry similar meaning as described before. It is important to note that, generally speaking, there is no small parameter restriction on the forcing term $\bar{\mathbf{F}}(t)$.

Application of modal transformation $\mathbf{x} = \mathbf{M}\mathbf{z}$ to equation (2.21) puts the linear part of the equation in Jordan canonical form as

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{F}(t) + \varepsilon \mathbf{f}_2(\mathbf{z}) + \varepsilon^2 \mathbf{f}_3(\mathbf{z}) + \cdots + \varepsilon^{k-1} \mathbf{f}_k(\mathbf{z}) + \varepsilon^k \mathbf{O}(|\mathbf{z}|^{k+1}) \quad (2.22)$$

where \mathbf{J} is the Jordan form of matrix \mathbf{A} , $\mathbf{M}^{-1}\bar{\mathbf{f}}_r(\cdot) = \mathbf{f}_r(\cdot)$ and $\mathbf{M}^{-1}\bar{\mathbf{F}}(t) = \mathbf{F}(t)$.

As discussed before, the basic principle behind the method of normal forms is to apply a series of near-identity transformations in order to reduce a system to its simplest form. Following this general idea, a sequence of transformations is constructed, beginning with the lowest order nonlinearity, to successively remove nonlinear terms from equation (2.22). In general, in order to remove the nonlinear term of order r , a near-identity transformation of the form

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1}(\mathbf{h}_{r1}(\mathbf{v}, t) + \mathbf{h}_{r2}(\mathbf{v}, t) + \cdots + \mathbf{h}_{rr-1}(\mathbf{v}, t) + \mathbf{h}_{rr}(\mathbf{v})); \quad r \geq 2 \quad (2.23)$$

is applied, where unknown $\mathbf{h}_0(t)$ is a function of time alone, $\mathbf{h}_{rs}(\mathbf{v}, t)$ constitutes of monomials in \mathbf{v} of degree s ($s = 1, 2, \dots, r-1$) with periodic coefficients, while $\mathbf{h}_{rr}(\mathbf{v})$ are monomials in \mathbf{v} of degree r with constant coefficients. Applying transformation (2.23) to equation (2.22) and assuming $|\varepsilon| < 1$, one obtains

$$\begin{aligned} \dot{\mathbf{v}} = & \left(I - \varepsilon^{r-1} \left(\frac{\partial \mathbf{h}_{r1}(\mathbf{v}, t)}{\partial \mathbf{v}} + \frac{\partial \mathbf{h}_{r2}(\mathbf{v}, t)}{\partial \mathbf{v}} + \cdots + \frac{\partial \mathbf{h}_{rr}(\mathbf{v})}{\partial \mathbf{v}} \right) \right) \\ & \times \left(\mathbf{J}(\mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1}(\mathbf{h}_{r1}(\mathbf{v}, t) + \cdots + \mathbf{h}_{rr}(\mathbf{v}))) + \mathbf{F}(t) \right. \\ & + \varepsilon^{s-1} \mathbf{f}_r(\mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1}(\mathbf{h}_{r1}(\mathbf{v}, t) + \cdots + \mathbf{h}_{rr}(\mathbf{v})), t) - \frac{d\mathbf{h}_0(t)}{dt} \\ & \left. - \varepsilon^{r-1} \left(\frac{\partial \mathbf{h}_{r1}(\mathbf{v}, t)}{\partial t} - \frac{\partial \mathbf{h}_{r2}(\mathbf{v}, t)}{\partial t} - \cdots - \frac{\partial \mathbf{h}_{rr-1}(\mathbf{v}, t)}{\partial t} \right) \right) \end{aligned} \quad (2.24)$$

Expanding equation (2.24) and collecting terms of like powers in \mathbf{v} results in

$$\begin{aligned}
\dot{\mathbf{v}} = & \mathbf{J}\mathbf{v} + \left(\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0}{dt} + \mathbf{F}(t) + \varepsilon^{s-1}\mathbf{f}_{r_0}(t) \right) \\
& - \varepsilon^{r-1} \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial \mathbf{v}} \left(\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0}{dt} + \mathbf{F}(t) + \varepsilon^{r-1}\mathbf{f}_{r_0}(t) \right) \\
& - \varepsilon^{r-1} \left(L_A(\mathbf{h}_{r_1}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{r_1}(\mathbf{v}, t) \right) \\
& + \varepsilon^{2(r-1)} \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial \mathbf{v}} \left(L_A(\mathbf{h}_{r_1}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{r_1}(\mathbf{v}, t) \right) \\
& - \varepsilon^{r-1} \frac{\partial \mathbf{h}_{r_2}(\mathbf{v}, t)}{\partial \mathbf{v}} \left(\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0}{dt} + \mathbf{F}(t) + \varepsilon^{r-1}\mathbf{f}_{r_0}(t) \right) \\
& \quad \vdots \\
& \quad \vdots \\
& - \varepsilon^{r-1} \left(L_A(\mathbf{h}_{rr}(\mathbf{v})) - \mathbf{f}_{rr}(\mathbf{v}) \right) + \varepsilon^{2(r-1)} \left(L_A(\mathbf{h}_{rr}(\mathbf{v})) - \mathbf{f}_{rr}(\mathbf{v}) \right) \\
& + \varepsilon^{2(r-1)} \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial \mathbf{v}} \left(L_A(\mathbf{h}_{rr}(\mathbf{v})) - \mathbf{f}_{rr}(\mathbf{v}, t) \right) + \dots \\
& \dots + \varepsilon^{2(r-1)} \frac{\partial \mathbf{h}_{r(r-1)}(\mathbf{v}, t)}{\partial \mathbf{v}} \left(L_A(\mathbf{h}_{r_2}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{r_2}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{r_2}(\mathbf{v}, t) \right) \\
& + \varepsilon^{2(r-1)} \frac{\partial \mathbf{h}_{rr}(\mathbf{v})}{\partial \mathbf{v}} \left(L_A(\mathbf{h}_{r_1}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{r_1}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{r_1}(\mathbf{v}, t) \right) \\
& + \varepsilon^r \mathbf{f}_{r+1}(\mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1}(\mathbf{h}_{r_1}(\mathbf{v}, t) + \mathbf{h}_{r_2}(\mathbf{v}, t) + \dots + \mathbf{h}_{rr}(\mathbf{v})), t) + \dots
\end{aligned} \tag{2.25}$$

where

$$L_A(\mathbf{h}_{rs}(\mathbf{v}, t)) = \frac{\partial \mathbf{h}_{rs}(\mathbf{v}, t)}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_{rs}(\mathbf{v}, t); (s = 1, 2, \dots, r-1)$$

$$L_A(\mathbf{h}_{rr}(\mathbf{v})) = \frac{\partial \mathbf{h}_{rr}(\mathbf{v})}{\partial \mathbf{v}} \mathbf{J}\mathbf{v} - \mathbf{J}\mathbf{h}_{rr}(\mathbf{v})$$

are the *Lie operators* which carries homogeneous vector polynomials over to vector polynomials of the same degree. $\mathbf{f}_{rs}(\mathbf{v}, t)$ are ε^{r-1} order terms of $\mathbf{f}_r(\mathbf{z})$ after the transformation and s is the degree of \mathbf{v} . Thus, $\mathbf{f}_{r0}(t)$ are pure time terms of $\mathbf{f}_r(\mathbf{z})$ after the transformation, $\mathbf{f}_{r1}(t)$ are linear in \mathbf{v} and so on.

It is seen from equation (2.25) that nonlinear terms of degree r can be eliminated if following set of equations is satisfied.

$$\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0(t)}{dt} + \mathbf{F}(t) + \varepsilon^{r-1}\mathbf{f}_{r0}(\mathbf{h}_0(t)) = 0 \quad (2.26.a)$$

$$\begin{aligned} L_A(\mathbf{h}_{r1}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{r1}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{r1}(\mathbf{v}, \mathbf{h}_0(t)) &= 0 \\ \vdots & \\ L_A(\mathbf{h}_{rr-1}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{rr-1}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{rr-1}(\mathbf{v}, \mathbf{h}_0(t)) &= 0 \\ L_A(\mathbf{h}_{rr}(\mathbf{v})) - \mathbf{f}_{rr}(\mathbf{v}) &= 0 \end{aligned} \quad (2.26.b)$$

Equation (2.26.a) involves only temporal terms and the desired steady state solution of this equation is obtained by the method of harmonic balance. This permits an investigator to choose the steady state solution of the system in the desired form (i.e., fundamental, sub or super harmonic). Quasiperiodic solutions are not included in this study.

Equation set (2.26.b) is a set of homological equations with periodic coefficients except for the last equation, which has constant coefficients. These equations are independent of each other and can be solved if the set of eigenvalues of $(L_A + \partial/\partial t)$ does not contain any zeros. To solve homological equations with time varying coefficients, approximately, a similar procedure as

described in the earlier section can be followed. For this purpose, known $\mathbf{f}_{rs}(\mathbf{v}, t)$ and unknown $\mathbf{h}_{rs}(\mathbf{v}, t)$ monomials are expressed in a multivariable Taylor-Fourier series as

$$\begin{aligned}\mathbf{f}_{rs}(\mathbf{v}, t) &= \sum_{j=1}^n \sum_{\mathbf{m}_s} \sum_{l=-k}^k f_{r,j,\mathbf{m}_s,l} e^{il\omega} |\mathbf{v}|^{\mathbf{m}_s} \mathbf{e}^j \\ \mathbf{h}_{rs}(\mathbf{v}, t) &= \sum_{j=1}^n \sum_{\mathbf{m}_s} \sum_{l=-k}^k h_{r,j,\mathbf{m}_s,l} e^{il\omega} |\mathbf{v}|^{\mathbf{m}_s} \mathbf{e}^j\end{aligned}\quad (2.27)$$

where $s = 1, 2, \dots, r-1$, $\mathbf{m}_s = (m_1, m_2, \dots, m_n)$, $\sum_{i=1}^n m_i = s$ ($s = 1, 2, \dots, r-1$) and ω is the principal frequency of $\mathbf{h}_0(t)$.

Substituting these expressions into homological equations (2.26.b) and comparing Taylor-Fourier coefficients of the expansion, term-by-term, one obtains a set of linear algebraic equations, which can be solved to determine unknown coefficients of $\mathbf{h}_{rs}(\mathbf{v}, t)$ as

$$h_{r,j,\mathbf{m}_s,l} = \frac{f_{r,j,\mathbf{m}_s,l}}{il\omega + \mathbf{m}_s \cdot \boldsymbol{\lambda} - \lambda_j} \quad (2.28)$$

where, again, $\boldsymbol{\lambda} = (\lambda_1 \lambda_2 \dots \lambda_n)^T$ are the eigenvalues of Jordan matrix \mathbf{J} . Thus when the solvability condition $il\omega + \mathbf{m}_s \cdot \boldsymbol{\lambda} - \lambda_j \neq 0$ is satisfied, then the corresponding term in $\mathbf{f}_r(\mathbf{z}, t)$ can be eliminated. Otherwise the resonant term will remain in the reduced equation.

Homological equation for the term $\mathbf{h}_{rr}(\mathbf{v})$ is time-independent. Following the solution procedure discussed earlier, one can find

$$h_{r,j,\mathbf{m}_r} = \frac{f_{r,j,\mathbf{m}_r}}{\mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j} \quad (2.29)$$

where $\mathbf{m}_r = (m_1, m_2, \dots, m_n)$ and $\sum_{i=1}^n m_i = r$.

Equation (2.29) can also be regarded as a special case of equation (2.28) where $l = 0$, meaning coefficients of terms in \mathbf{h} are constant. The solvability condition $\mathbf{m}_r \cdot \boldsymbol{\lambda} - \lambda_j \neq 0$ represents the case of internal resonance (Nayfeh and Mook, 1979, Nayfeh, 1985).

After the transformation, the subsequent reduced equation takes the form

$$\begin{aligned} \dot{\mathbf{v}} = & \mathbf{J}\mathbf{v} + \varepsilon^{r-1}\mathbf{f}_{r1}^*(\mathbf{v}, t) + \varepsilon^{r-1}\mathbf{f}_{r2}^*(\mathbf{v}, t) + \dots + \varepsilon^{r-1}\mathbf{f}_{rr}^*(\mathbf{v}) + \varepsilon^r\mathbf{f}_{r+1}(\mathbf{v}) + \dots \\ & + \varepsilon^{k-1}\mathbf{f}_k(\mathbf{v}) + \varepsilon^k O(|\mathbf{v}|^{k+1}) \end{aligned} \quad (2.30)$$

where \mathbf{f}^* contains only the resonating terms. Next, $(r+1)^{th}$ order terms can be removed in a similar fashion.

2.3.2 Systems with Time-Periodic Coefficients

Consider a nonlinear system represented by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \bar{\mathbf{F}}(t) + \varepsilon\bar{\mathbf{f}}_2(\mathbf{x}, t) + \varepsilon^2\bar{\mathbf{f}}_3(\mathbf{x}, t) + \dots + \varepsilon^{k-1}\bar{\mathbf{f}}_k(\mathbf{x}, t) + \varepsilon^k\mathbf{O}(|\mathbf{x}|^{k+1}, t) \quad (2.31)$$

where ε , once again, is a *small* bookkeeping parameter, $n \times n$ matrix $\mathbf{A}(t)$ is T periodic such that $\mathbf{A}(t) = \mathbf{A}(t+T)$. $n \times 1$ nonlinear vector $\bar{\mathbf{f}}_k(\mathbf{x}, t)$ contains homogenous monomials in \mathbf{x} with $k_1 T$ periodic coefficients and $\bar{\mathbf{F}}(t)$ is a force vector with period $k_2 T$. k_1 and k_2 are assumed to be integers.

Following the approach summarized in appendix A to calculate the real L-F transformation matrix $\mathbf{L}(t)$ and applying the transformation $\mathbf{x}=\mathbf{L}(t)\mathbf{y}$ to equation (2.31) produces

$$\begin{aligned} \dot{\mathbf{y}} = & \mathbf{R}\mathbf{y} + \mathbf{L}^{-1}(t)\bar{\mathbf{F}}(t) + \varepsilon\mathbf{L}^{-1}(t)\bar{\mathbf{f}}_2(\mathbf{L}(t)\mathbf{y},t) + \varepsilon^2\mathbf{L}^{-1}(t)\bar{\mathbf{f}}_3(\mathbf{L}(t)\mathbf{y},t) + \dots \\ & + \varepsilon^{k-1}\mathbf{L}^{-1}(t)\bar{\mathbf{f}}_k(\mathbf{L}(t)\mathbf{y},t) + \varepsilon^k\mathbf{L}^{-1}(t)\mathbf{O}\left(|\mathbf{L}(t)\mathbf{y}|^{k+1},t\right) \end{aligned} \quad (2.32)$$

where $\mathbf{L}(t)$ is typically $2T$ periodic and \mathbf{R} is a real valued $n \times n$ constant matrix. This equation may be written as

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{F}(t) + \varepsilon\mathbf{f}_2(\mathbf{z},t) + \varepsilon^2\mathbf{f}_3(\mathbf{z},t) + \dots + \varepsilon^{k-1}\mathbf{f}_k(\mathbf{z},t) + \varepsilon^k\mathbf{O}\left(|\mathbf{z}|^{k+1},t\right) \quad (2.33)$$

where \mathbf{J} is the Jordan canonical form of \mathbf{R} , $\mathbf{M}^{-1}\mathbf{L}^{-1}\bar{\mathbf{F}}(t) = \mathbf{F}(t)$ and $\mathbf{M}^{-1}\mathbf{L}^{-1}\bar{\mathbf{f}}_r(\mathbf{L}\mathbf{M}\mathbf{z},t) = \mathbf{f}_r(\mathbf{z},t)$.

Similar to the time invariant case, in order to remove the r^{th} order nonlinearity from equation (2.33), the following transformation is applied

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1}\left(\mathbf{h}_{r1}(\mathbf{v},t) + \mathbf{h}_{r2}(\mathbf{v},t) + \dots + \mathbf{h}_{rr}(\mathbf{v},t)\right) \quad (2.34)$$

where $\mathbf{h}_0(t)$ is a function of t alone, $\mathbf{h}_{rs}(\mathbf{v},t)$ constitutes of monomials in \mathbf{v} of degree s with, yet unknown, periodic coefficients. Applying above transformation to equation (2.33) and collecting terms of like of powers of \mathbf{v} , one obtains a similar set of equations as in the time-invariant case (*c.f.*, equations 2.26.a and 2.26.b). The only difference being that the last equation in (2.26.b) now has both spatial and temporal terms and is given by

$$L_A(\mathbf{h}_{rr}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{rr}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{rr}(\mathbf{v}, t) = 0 \quad (2.35)$$

The coefficients of $\mathbf{f}_{rr}(\mathbf{v}, t)$ are periodic and the solution of equation (2.35) follows a similar general procedure as described for periodic equations in (2.26.b).

Finally the transformed equation takes the form

$$\begin{aligned} \dot{\mathbf{v}} = & \mathbf{J}\mathbf{v} + \varepsilon^{r-1} \mathbf{f}_{r1}^*(\mathbf{v}, t) + \varepsilon^{r-1} \mathbf{f}_{r2}^*(\mathbf{v}, t) + \dots + \varepsilon^{r-1} \mathbf{f}_{rr}^*(\mathbf{v}, t) + \varepsilon^r \mathbf{f}_{r+1}(\mathbf{v}, t) + \dots \\ & + \varepsilon^{k-1} \mathbf{f}_k(\mathbf{v}, t) + \varepsilon^k \mathbf{O}(|\mathbf{v}|^{k+1}, t) \end{aligned} \quad (2.36)$$

where \mathbf{f}_s^* contain the resonating term. This procedure is continued until all nonlinearities are eliminated. Later, while discussing an example, it is shown that for a single degree of freedom system with damping all nonlinearities can be eliminated and the resulting normal form has a closed form solution.

2.3.3 An Alternate Approach for Non-Resonant Case

In the case, $\mathbf{h}_0(t)$ is limited to be the non-resonant fundamental solution, a simple modification in the near-identity transformation yields considerable simplification. This alternative approach results into two linear differential equations instead of a nonlinear equation (2.26.a). These two equations can be solved easily using convolution integral and the need for using harmonic balance solution is obviated. In order to remove the r^{th} order nonlinearity from equation (2.22), a near-identity transformation of the form

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_0(t) + \varepsilon^{r-1} (\mathbf{h}_{r0}(t) + \mathbf{h}_{r1}(\mathbf{v}, t) + \mathbf{h}_{r2}(\mathbf{v}, t) + \dots + \mathbf{h}_{rr}(\mathbf{v})) \quad (2.37)$$

is applied, where an additional term $\mathbf{h}_{r,0}(t)$ is purely time dependent. Applying this transformation to equation (2.22) and collecting terms of the order ε^0 yield

$$\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0(t)}{dt} + \mathbf{F}(t) = 0 \quad (2.38)$$

Equation (2.38) is a linear differential equation involving only temporal arguments. The solution can be determined using the convolution integral as

$$\mathbf{h}_0(t) = e^{\mathbf{J}t}\mathbf{h}_0(0) + \int_0^t e^{\mathbf{J}(t-\tau)}\mathbf{F}(\tau)d\tau \quad (2.39)$$

Expressing $\mathbf{F}(t)$ in terms of finite Fourier series as

$$\mathbf{F}(t) = \sum_{j=1}^n \sum_{l=-k}^k c_{j,l} e^{il\omega t} \mathbf{e}_j \quad (2.40)$$

where ω is the principal frequency of forcing, the solution of integral equation (2.39) (with $\mathbf{h}_0(0) \equiv 0$) can be written as

$$\mathbf{h}_0(t) = \sum_{j=1}^n \sum_{l=-k}^k c_{j,l} \frac{e^{il\omega t}}{il\omega - \lambda_j} \mathbf{e}_j - \sum_{j=1}^n \sum_{l=-k}^k c_{j,l} \frac{e^{\lambda_j t}}{il\omega - \lambda_j} \mathbf{e}_j \quad (2.41)$$

where $\lambda_j; j=1,2,\dots,n$, are the eigenvalues of \mathbf{J} . If all eigenvalues of \mathbf{J} are purely imaginary then it can be seen that, if $l\omega = \lambda_j$ for any l , $\mathbf{h}_0(t)$ cannot be determined and the system is said to be in the main resonance.

Collecting the terms of the order ε^{r-1} yields

$$\mathbf{J}\mathbf{h}_{r_0}(t) - \frac{d\mathbf{h}_{r_0}(t)}{dt} + \mathbf{f}_{r_0}(t) = 0 \quad (2.42)$$

Equation (2.42) is again a linear differential equation which can be solved using convolution integral as discussed above. The forcing term $\mathbf{f}_{s_0}(t)$ consists of all temporal terms of order ε^{r-1} in $\mathbf{f}_s(\mathbf{z})$ after the transformation. In addition, one gets a set of homological equations similar to (2.26.b), the solution for which has already been discussed (*c.f.*, section 2.3.1). This methodology, in general, is applicable for time varying case also.

A similar approach has been used by Wooden (2002) to solve forced nonlinear equations with quasiperiodic coefficients. Wherein, a near-identity transformation of the form

$$\mathbf{z} = \mathbf{v} + \varepsilon^{r-1}(\mathbf{h}_0(t) + \mathbf{h}_{rr}(\mathbf{v}, t)) \quad (2.43)$$

has been used to reduce the problem to its normal form. Resulting equations, which need to be satisfied, are equation (2.35) and (2.38). Again, this transformation holds well only if the desired steady state solution is non-resonant. Moreover, it does not have any provision to include superharmonic or subharmonic component in the solution.

2.4 Applications

To illustrate applications and demonstrate effectiveness of the suggested techniques, two examples are considered. First, a forced Duffing's equation with constant coefficients is considered. This equation can be solved through a direct application of the proposed technique suggested in section 2.3.1. In the following, known functions $\mathbf{f}_{rs}(\mathbf{v}, t)$ are expressed in a closed form and all possible resonance conditions are derived. It is shown that for time-independent resonance, the resulting normal form equation is still solvable. For the case of systems with time

varying coefficients, a Mathieu-Duffing type equation is considered. Using the L-F transformation, the system is first reduced to a form where the linear part is time invariant and then the procedure outlined in section 2.2.3 is applied to obtain an approximate solution. In each case, normal form solutions are compared with numerical solutions.

2.4.1 Forced Duffing's equation

Consider a forced Duffing's equation given by

$$\begin{cases} \dot{x}_1 \\ \dot{x}_2 \end{cases} = \begin{bmatrix} 0 & 1 \\ -a & -d \end{bmatrix} \begin{cases} x_1 \\ x_2 \end{cases} - \varepsilon \begin{cases} 0 \\ cx_1^3 \end{cases} + F \begin{cases} 0 \\ \cos(\Omega t) \end{cases} \quad (2.44)$$

where a, d, c and ε are system parameters. It is to be noted here that damping and forcing term do not contain any small parameter. After applying modal transformation ($\mathbf{x} = \mathbf{M}\mathbf{z}$) the system becomes

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} - \varepsilon \mathbf{M}^{-1} \begin{cases} 0 \\ c(M_{11}z_1 + M_{12}z_2)^3 \end{cases} + \mathbf{M}^{-1}F \begin{cases} 0 \\ \cos(\Omega t) \end{cases} \quad (2.45)$$

where \mathbf{J} is in the Jordan canonical form. The above equation is of the form

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{F}(t) + \varepsilon \mathbf{f}_3(\mathbf{z}) \quad (2.46)$$

where

$$\begin{aligned}
\mathbf{F}(t) &= \begin{Bmatrix} M_{12}^{-1} F \cos(\Omega t) \\ M_{22}^{-1} F \cos(\Omega t) \end{Bmatrix} \\
\mathbf{f}_3(\mathbf{z}) &= - \begin{Bmatrix} \bar{f}_{3,1,(3,0)} z_1^3 + \bar{f}_{3,1,(2,1)} z_1^2 z_2 + \bar{f}_{3,1,(1,2)} z_1 z_2^2 + \bar{f}_{3,1,(0,3)} z_2^3 \\ \bar{f}_{3,2,(3,0)} z_1^3 + \bar{f}_{3,2,(2,1)} z_1^2 z_2 + \bar{f}_{3,2,(1,2)} z_1 z_2^2 + \bar{f}_{3,2,(0,3)} z_2^3 \end{Bmatrix}, \\
\bar{f}_{3,1,(3,0)} &= cM_{12}^{-1} M_{11}^3; \quad \bar{f}_{3,1,(2,1)} = 3cM_{12}^{-1} M_{11}^2 M_{12}; \\
\bar{f}_{3,1,(1,2)} &= 3cM_{12}^{-1} M_{11} M_{12}^2; \quad \bar{f}_{3,1,(0,3)} = cM_{12}^{-1} M_{12}^3; \\
\bar{f}_{3,2,(3,0)} &= cM_{22}^{-1} M_{11}^3; \quad \bar{f}_{3,2,(2,1)} = 3cM_{22}^{-1} M_{11}^2 M_{12}; \\
\bar{f}_{3,2,(1,2)} &= 3cM_{22}^{-1} M_{11} M_{12}^2; \quad \bar{f}_{3,2,(0,3)} = cM_{22}^{-1} M_{12}^3;
\end{aligned} \tag{2.47}$$

Following equation (2.23), in order to remove cubic nonlinearity from the above equation, a near-identity transformation of the form

$$\begin{aligned}
\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} &= \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{Bmatrix} \bar{h}_{3,1,(0)} \\ \bar{h}_{3,2,(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{h}_{3,1,(1,0)} v_1 + \bar{h}_{3,1,(0,1)} v_2 \\ \bar{h}_{3,2,(1,0)} v_1 + \bar{h}_{3,2,(0,1)} v_2 \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} \bar{h}_{3,1,(2,0)} v_1^2 + \bar{h}_{3,1,(1,1)} v_1 v_2 + \bar{h}_{3,1,(0,2)} v_2^2 \\ \bar{h}_{3,2,(2,0)} v_1^2 + \bar{h}_{3,2,(1,1)} v_1 v_2 + \bar{h}_{3,2,(0,2)} v_2^2 \end{Bmatrix} \\
&+ \varepsilon^3 \begin{Bmatrix} \bar{h}_{3,1,(3,0)} v_1^3 + \bar{h}_{3,1,(2,1)} v_1^2 v_2 + \bar{h}_{3,1,(1,2)} v_1 v_2^2 + \bar{h}_{3,1,(0,3)} v_2^3 \\ \bar{h}_{3,2,(3,0)} v_1^3 + \bar{h}_{3,2,(2,1)} v_1^2 v_2 + \bar{h}_{3,2,(1,2)} v_1 v_2^2 + \bar{h}_{3,2,(0,3)} v_2^3 \end{Bmatrix}
\end{aligned} \tag{2.48}$$

is applied. Where, $\bar{h}_{r,j,(m_1,m_2)}$ are time periodic and $\bar{h}_{r,j,(m_1,m_2)}^*$ are constant coefficients. Substituting this transformation in equation (2.46) and collecting terms of like powers of \mathbf{v} , one obtains the following set of equations

$$\mathbf{J}\mathbf{h}_0(t) - \frac{d\mathbf{h}_0(t)}{dt} + \mathbf{F}(t) + \varepsilon \mathbf{f}_{30}(t) = 0; \quad \mathbf{h}_0 = \begin{Bmatrix} \bar{h}_{3,1,(0)} \\ \bar{h}_{3,2,(0)} \end{Bmatrix} \tag{2.49}$$

$$L_A(\mathbf{h}_{31}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{31}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{31}(\mathbf{v}, t) = 0; \quad \mathbf{h}_{31} = \begin{Bmatrix} \bar{h}_{3,1,(1,0)} v_1 + \bar{h}_{3,1,(0,1)} v_2 \\ \bar{h}_{3,2,(1,0)} v_1 + \bar{h}_{3,2,(0,1)} v_2 \end{Bmatrix} \tag{2.50}$$

$$L_A(\mathbf{h}_{32}(\mathbf{v}, t)) + \frac{\partial \mathbf{h}_{32}(\mathbf{v}, t)}{\partial t} - \mathbf{f}_{32}(\mathbf{v}, t) = 0; \quad \mathbf{h}_{32} = \begin{Bmatrix} \bar{h}_{3,1,(2,0)} v_1^2 + \bar{h}_{3,1,(1,1)} v_1 v_2 + \bar{h}_{3,1,(0,2)} v_2^2 \\ \bar{h}_{3,2,(2,0)} v_1^2 + \bar{h}_{3,2,(1,1)} v_1 v_2 + \bar{h}_{3,2,(0,2)} v_2^2 \end{Bmatrix} \tag{2.51}$$

$$L_A(\mathbf{h}_{33}(\mathbf{v})) - \mathbf{f}_{33}(\mathbf{v}) = \mathbf{0}; \quad \mathbf{h}_{33} = \left\{ \begin{array}{l} \bar{h}_{3,1,(3,0)}^* v_1^3 + \bar{h}_{3,1,(2,1)}^* v_1^2 v_2 + \bar{h}_{3,1,(1,2)}^* v_1 v_2^2 + \bar{h}_{3,1,(0,3)}^* v_2^3 \\ \bar{h}_{3,2,(3,0)}^* v_1^3 + \bar{h}_{3,2,(2,1)}^* v_1^2 v_2 + \bar{h}_{3,2,(1,2)}^* v_1 v_2^2 + \bar{h}_{3,2,(0,3)}^* v_2^3 \end{array} \right\} \quad (2.52)$$

Equation (2.49) is a nonlinear differential equation with

$$\mathbf{f}_{30}(t) = - \left\{ \begin{array}{l} \bar{f}_{3,1,(2,1)} \bar{h}_{3,1,(0)}^3 + \bar{f}_{3,1,(2,1)} \bar{h}_{3,1,(0)}^2 \bar{h}_{3,2,(0)} + \bar{f}_{3,1,(1,2)} \bar{h}_{3,1,(0)} \bar{h}_{3,2,(0)}^2 + \bar{f}_{3,1,(0,3)} \bar{h}_{3,2,(0)}^3 \\ \bar{f}_{3,2,(2,1)} \bar{h}_{3,1,(0)}^3 + \bar{f}_{3,2,(2,1)} \bar{h}_{3,1,(0)}^2 \bar{h}_{3,2,(0)} + \bar{f}_{3,2,(1,2)} \bar{h}_{3,1,(0)} \bar{h}_{3,2,(0)}^2 + \bar{f}_{3,2,(0,3)} \bar{h}_{3,2,(0)}^3 \end{array} \right\} \quad (2.53)$$

This equation is solved using the harmonic balance method which yields the steady state value of $\mathbf{h}_0(t)$.

Equations (2.50) and (2.51) are homological equations with time varying coefficients for which terms of the form $f_{r,j,m_s,i} e^{il\omega t} v_1^{m_1} v_2^{m_2} \mathbf{e}_j$ remain when the following resonance conditions is satisfied

$$il\omega + m_1\lambda_1 + m_2\lambda_2 - \lambda_j = 0 \quad (2.54)$$

where $m_1 + m_2 = s$, ω is principal frequency of $\mathbf{h}_0(t)$ and \mathbf{e}_j is the j^{th} member of the natural basis. It is apparent that time-dependent resonance ($l \neq 0$) occurs only if eigenvalues of matrix \mathbf{J} are purely imaginary, and therefore only systems with no damping are relevant in this case. Time-independent resonance occurs when $l = 0$.

For equation (2.50), $\mathbf{f}_{31}(\mathbf{v}, t)$ is given as

$$\begin{aligned}
\mathbf{f}_{31}(\mathbf{v}, t) = & \left\{ \begin{aligned} & \left(3\bar{f}_{3,1,(3,0)}\bar{h}_{3,1,(0)}^2 + 2\bar{f}_{3,1,(2,1)}\bar{h}_{3,1,(0)}\bar{h}_{3,2,(0)} + \bar{f}_{3,1,(1,2)}\bar{h}_{3,2,(0)}^2 \right) v_1 \\ & \left(3\bar{f}_{3,2,(3,0)}\bar{h}_{3,1,(0)}^2 + 2\bar{f}_{3,2,(2,1)}\bar{h}_{3,1,(0)}\bar{h}_{3,2,(0)} + \bar{f}_{3,2,(1,2)}\bar{h}_{3,2,(0)}^2 \right) v_1 \\ & + \left(3\bar{f}_{3,1,(0,3)}\bar{h}_{3,2,(0)}^2 + 2\bar{f}_{3,1,(1,2)}\bar{h}_{3,1,(0)}\bar{h}_{3,2,(0)} + \bar{f}_{3,1,(2,1)}\bar{h}_{3,1,(0)}^2 \right) v_2 \\ & + \left(3\bar{f}_{3,2,(0,3)}\bar{h}_{3,2,(0)}^2 + 2\bar{f}_{3,2,(1,2)}\bar{h}_{3,1,(0)}\bar{h}_{3,2,(0)} + \bar{f}_{3,2,(2,1)}\bar{h}_{3,1,(0)}^2 \right) v_2 \end{aligned} \right\} \quad (2.55)
\end{aligned}$$

and $\mathbf{m}_1 = (1,0), (0,1)$. $f_{31}(\mathbf{v}, t)$ contains quadratic terms in $\mathbf{h}_0(t)$, and therefore time-independent resonance ($l=0$) is always present in this case and the corresponding term in $f_{31}(\mathbf{v}, t)$ can not be eliminated. The resulting normal form equation can be written as

$$\begin{aligned}
\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = & \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} f_{3,1,0,(1,0)}v_1 \\ f_{3,2,0,(0,1)}v_2 \end{Bmatrix} \quad (2.56)
\end{aligned}$$

The resonant terms, being linear in \mathbf{v} , can be added to matrix \mathbf{J} which remains in the diagonal form and the two independent equations can be solved in a closed form. For time-independent resonances, $l \neq 0$, and the corresponding normal form is given by

$$\begin{aligned}
\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = & \begin{bmatrix} \lambda_1 + \varepsilon f_{3,1,0,(1,0)} & 0 \\ 0 & \lambda_2 + \varepsilon f_{3,2,0,(0,1)} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} f_{3,1,l,(0,1)}e^{il\omega}v_2 \\ f_{3,2,-l,(1,0)}e^{-il\omega}v_1 \end{Bmatrix} \quad (2.57)
\end{aligned}$$

where $il\omega = 2\lambda_1$.

For equation (2.51), $\mathbf{f}_{32}(\mathbf{v}, t)$ can be written as

$$\mathbf{f}_{32}(\mathbf{v}, t) = \left\{ \begin{aligned} & \left(3\bar{f}_{3,1,(3,0)}\bar{h}_{3,1,(0)} + \bar{f}_{3,1,(2,1)}\bar{h}_{3,2,(0)} \right) v_1^2 + \left(2\bar{f}_{3,1,(2,1)}\bar{h}_{3,1,(0)} + 2\bar{f}_{3,1,(1,2)}\bar{h}_{3,2,(0)} \right) v_1 v_2 \\ & \left(3\bar{f}_{3,2,(3,0)}\bar{h}_{3,1,(0)} + \bar{f}_{3,2,(2,1)}\bar{h}_{3,2,(0)} \right) v_1^2 + \left(2\bar{f}_{3,2,(2,1)}\bar{h}_{3,1,(0)} + 2\bar{f}_{3,2,(1,2)}\bar{h}_{3,2,(0)} \right) v_1 v_2 \\ & + \left(\bar{f}_{3,1,(1,2)}\bar{h}_{3,1,(0)} + 3\bar{f}_{3,1,(0,3)}\bar{h}_{3,2,(0)} \right) v_2^2 \\ & + \left(\bar{f}_{3,2,(1,2)}\bar{h}_{3,1,(0)} + 3\bar{f}_{3,2,(0,3)}\bar{h}_{3,2,(0)} \right) v_2^2 \end{aligned} \right\} \quad (2.58)$$

and $\mathbf{m}_2 = (2,0), (1,1), (0,2)$. Time-independent resonance is absent in this case as periodic coefficients in $\mathbf{f}_{32}(\mathbf{v}, t)$ are linear in $\mathbf{h}_0(t)$ which eliminate the case $l=0$. Time-dependent resonance occurs for Fourier terms of $\mathbf{f}_{32}(\mathbf{v}, t)$ that satisfy the following conditions.

$$\begin{aligned} i l \omega &= \lambda_1 \\ i l \omega &= 3\lambda_1 \end{aligned} \quad (2.59)$$

Homological equation (2.52) is time independent and solution of which is widely discussed in the literature (Arnold, 1983; Arrowsmith and Place, 1990). In the present example $\mathbf{f}_{33}(\mathbf{v}, t)$ has the form

$$\mathbf{f}_{33}(\mathbf{v}) = \left\{ \begin{aligned} & \bar{f}_{3,1,(3,0)} v_1^3 + \bar{f}_{3,1,(2,1)} v_1^2 v_2 + \bar{f}_{3,1,(1,2)} v_1 v_2^2 + \bar{f}_{3,1,(0,3)} v_2^3 \\ & \bar{f}_{3,2,(3,0)} v_1^3 + \bar{f}_{3,2,(2,1)} v_1^2 v_2 + \bar{f}_{3,2,(1,2)} v_1 v_2^2 + \bar{f}_{3,2,(0,3)} v_2^3 \end{aligned} \right\} \quad (2.60)$$

where $\mathbf{m}_3 = (3,0), (2,1), (1,2), (0,3)$. For purely imaginary eigenvalues of the system the resonance terms remain, resulting in the normal form (Sinha and Pandiyan, 1994; Pandiyan and Sinha, 1995; Butcher and Sinha, 2000]

$$\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{f}_{3,1,(2,1)} v_1^2 v_2 \\ \bar{f}_{3,2,(1,2)} v_1 v_2^2 \end{Bmatrix} \quad (2.61)$$

Resonant terms in the above equation are in addition to the terms shown in equation (2.56) and (2.57).

Above analysis shows that when damping is present in the system, terms that cannot be eliminated are the linear resonant terms in $\mathbf{h}_{31}(\mathbf{v}, t)$. The resulting normal form equation is given by equation (2.56). However, resonant terms being linear in \mathbf{v} with constant coefficients, solution can be obtained in a closed form. Resulting solution vector \mathbf{v} is transformed back to (x_1, x_2) by back substitution of all the transformations.

In the following, some case studies for various values of a, b, d, c and ε are presented for the forced Duffing's equation. For parameter set $a = 3, d = 0.3, c = 1, F = 2, \Omega = 4$ and $\varepsilon = 0.5$, the undamped natural frequency of the system is 1.732 and thus it is the case of a non-resonant excitation. After applying the transformations, terms that can not be eliminated are $0.00676442iv_1 \mathbf{e}_1$ and $-0.00676442iv_2 \mathbf{e}_2$. These are linear terms with constant coefficients and can be added to the corresponding terms in matrix \mathbf{J} . It is to be noted that these terms, being purely imaginary, do not affect the stability of the system. Fig. 2.1 shows variations in states (x_1, x_2) with respect to time. It is seen that the approximate solutions obtained by the proposed technique matches almost exactly with the numerical solutions, including the transient response.

Next the case where superharmonic component is dominant in the solution is considered. For this situation, the parameter set is $a = 5, d = 0.3, c = 1, F = 4, \Omega = 0.745$ and $\varepsilon = 0.5$ which is a superharmonic resonance case of order 3 since the natural frequency $\omega_n = \sqrt{5} = 2.236$. All homological equations are still solvable since the eigenvalues of the system are complex. Time-independent resonance terms are $0.245095iv_1 \mathbf{e}_1$ and $-0.245095iv_2 \mathbf{e}_2$. Fig. 2.2 shows a close agreement of the proposed solution with the numerical solution.

For a typical set of system parameters $a = 4, d = 0.1, c = 7.972, F = 10, \Omega = 6.6$ and $\varepsilon = 0.5$, response is dominated by the subharmonic component of order 1/3 (note that $\omega_n = 2$). The linear leftover terms are $0.96214iv_1 \mathbf{e}_1$ and $-0.96214iv_2 \mathbf{e}_2$. Fig. 2.3 shows presence of subharmonic component. The results from the proposed method follow the transient solution fairly well and the steady state motion matches almost exactly with the numerical solution.

2.4.2 Forced Mathieu-Duffing equation

Next, a Mathieu-Duffing type equation is analyzed where the linear as well as nonlinear terms carry periodic coefficients. Consider

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a + b \cos(2\pi t)) & -d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - \varepsilon \cos(\pi t) \begin{Bmatrix} 0 \\ x_1^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \cos(4\pi t) \end{Bmatrix} \quad (2.62)$$

where a, b, d and ε are system parameters. In order to apply a near-identity transformation the system is first transformed to the form where the linear part is time-invariant. This is accomplished by an application of the L-F transformation. Accordingly, a $2T$ -periodic real L-F transformation matrix $\mathbf{L}(t)$ is constructed. After applying L-F and modal transformations, the system becomes

$$\begin{aligned} \dot{\mathbf{z}} = & \mathbf{J}\mathbf{z} - \varepsilon \cos(\pi t) \mathbf{M}^{-1} \mathbf{L}^{-1} \left\{ \begin{array}{c} 0 \\ \left((M_{11}(Q_{11}z_1 + Q_{12}z_2) + M_{12}(Q_{21}z_1 + Q_{22}z_2))^3 \right) \end{array} \right\} \\ & + \mathbf{M}^{-1} \mathbf{L}^{-1} \left\{ \begin{array}{c} 0 \\ F \cos(4\pi t) \end{array} \right\} \end{aligned} \quad (2.63)$$

where \mathbf{J} is in the Jordan canonical form. The above equation is of the form

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{F}(t) + \varepsilon \mathbf{f}_3(\mathbf{z}, t) \quad (2.64)$$

A near-identity transformation of the form

$$\begin{aligned} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} &= \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{Bmatrix} \bar{h}_{3,1,(0)} \\ \bar{h}_{3,2,(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{h}_{3,1,(1,0)}v_1 + \bar{h}_{3,1,(0,1)}v_2 \\ \bar{h}_{3,2,(1,0)}v_1 + \bar{h}_{3,2,(0,1)}v_2 \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} \bar{h}_{3,1,(2,0)}v_1^2 + \bar{h}_{3,1,(1,1)}v_1v_2 + \bar{h}_{3,1,(0,2)}v_2^2 \\ \bar{h}_{3,2,(2,0)}v_1^2 + \bar{h}_{3,2,(1,1)}v_1v_2 + \bar{h}_{3,2,(0,2)}v_2^2 \end{Bmatrix} \\ &+ \varepsilon^3 \begin{Bmatrix} \bar{h}_{3,1,(3,0)}v_1^3 + \bar{h}_{3,1,(2,1)}v_1^2v_2 + \bar{h}_{3,1,(1,2)}v_1v_2^2 + \bar{h}_{3,1,(0,3)}v_2^3 \\ \bar{h}_{3,2,(3,0)}v_1^3 + \bar{h}_{3,2,(2,1)}v_1^2v_2 + \bar{h}_{3,2,(1,2)}v_1v_2^2 + \bar{h}_{3,2,(0,3)}v_2^3 \end{Bmatrix} \end{aligned} \quad (2.65)$$

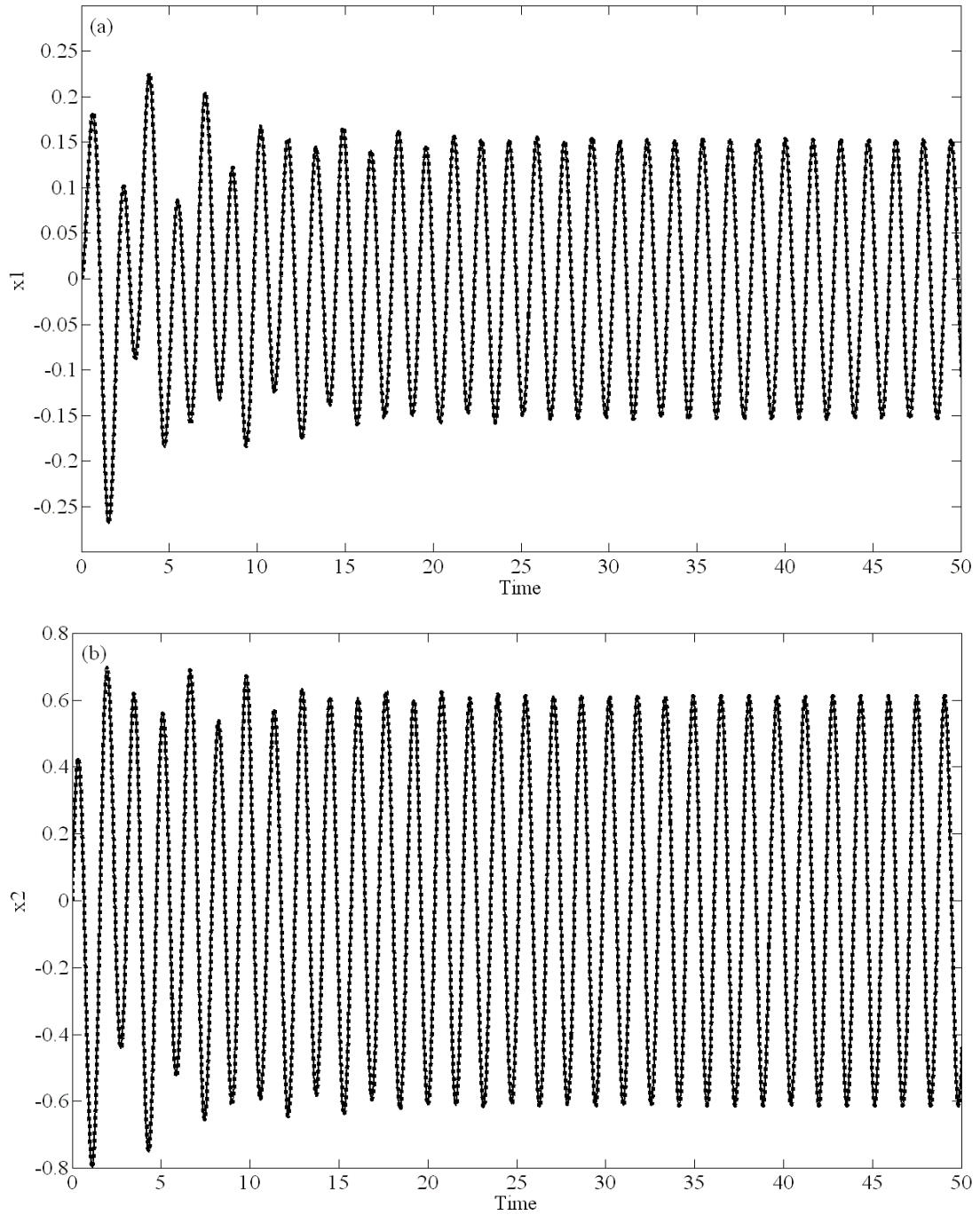
is applied. It is to be noted that all coefficients $\bar{h}_{3,j,(m_1,m_2)}$ are time periodic. Upon substitution of this transformation into equation (2.64), a similar set of equations as given by (2.49), (2.50), (2.51) and (2.52) is obtained where the last homological equation (equation (2.52)) now contains time periodic coefficients. Solution of equations (2.49, 2.50 and 2.51) in this case follows the same procedure as discussed before. Following a similar solution procedure for equation (2.52) leads to a specific structure of resonance conditions. Readers are directed to reference Butcher and Sinha (2000) for a detailed analysis of these resonance sets.

In the following, some case studies for various values of a, b, d and ε are presented. Fig. 2.4 shows a state response of the system when system parameters are $a = 3, b = 1, d = 0.3, F = 1$ and $\varepsilon = 0.9$. For these parameter values, the eigenvalues of \mathbf{R} (Floquet exponents) are $-0.15 \pm 1.41079i$. These results are compared with numerical results. In both cases an excellent comparison was observed. It is interesting to note that in Fig. 2.4, the value of ε is 0.9, implying that the nonlinearity is not small, and yet the accuracy is good.

For system parameters $a = 3, b = 1, d = 0, F = 1$ and $\varepsilon = 0.5$, the eigenvalues of matrix \mathbf{J} are $\pm 1.40429i$. Since eigenvalues of \mathbf{J} are purely imaginary and non-commensurate with the excitation frequency, the system solution is quasiperiodic in nature. Fig. 2.5 shows system

response as calculated from the proposed methodology and numerical integration which show excellent agreement.

Fig. 2.6 shows state trajectories when the time-invariant part of the linear term vanishes and thus the system does not have a generating solution (i.e., $a \equiv 0$). Perturbation and averaging techniques are not applicable in this situation. For this case also the proposed technique shows a close agreement with numerical solutions.



**Fig. 2.1 System response for forced Duffing's equation (non-resonant forcing).
 — , numerical solution; ····· , normal form solution.**

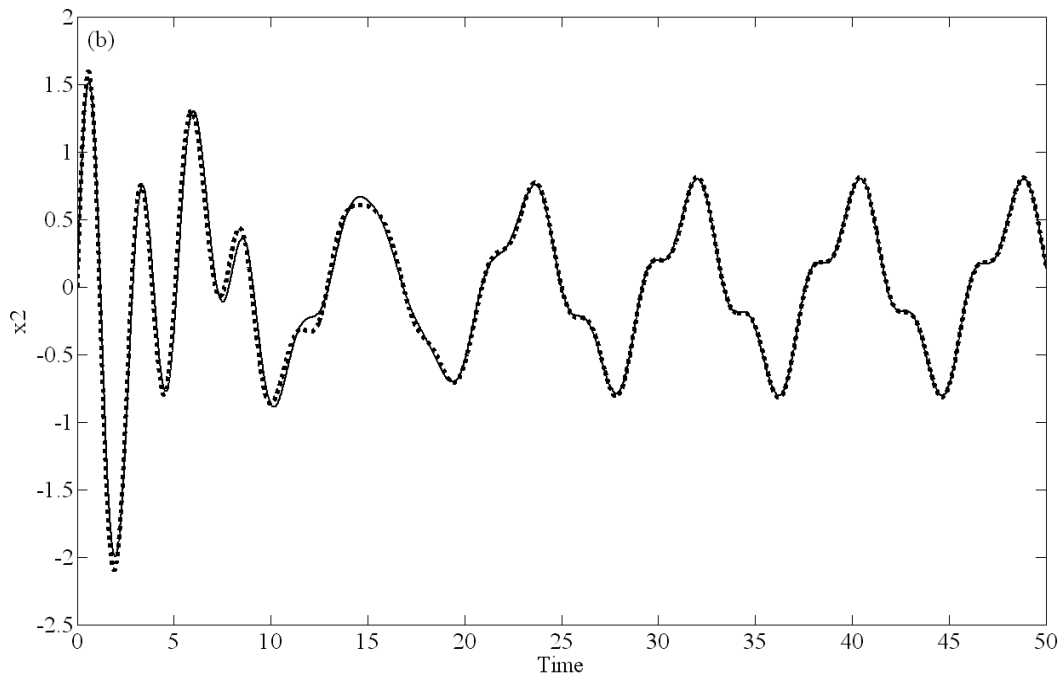
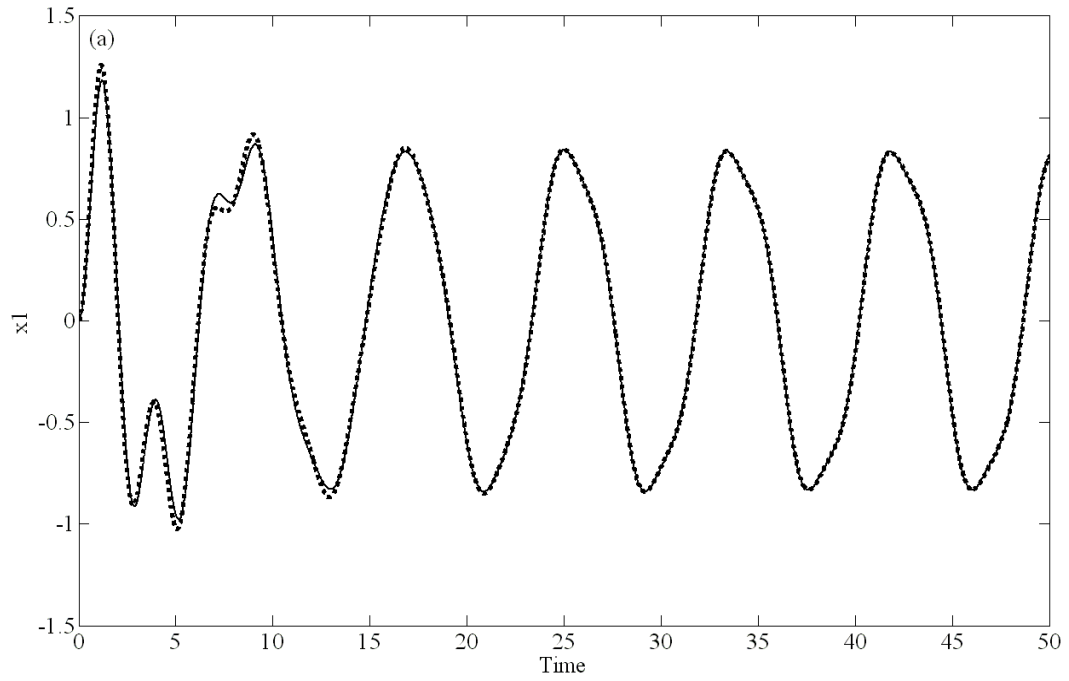


Fig. 2.2 System response for Forced Duffing's equation ($3\Omega \approx \omega_n$).
———— , numerical solution; **.....** , normal form solution.

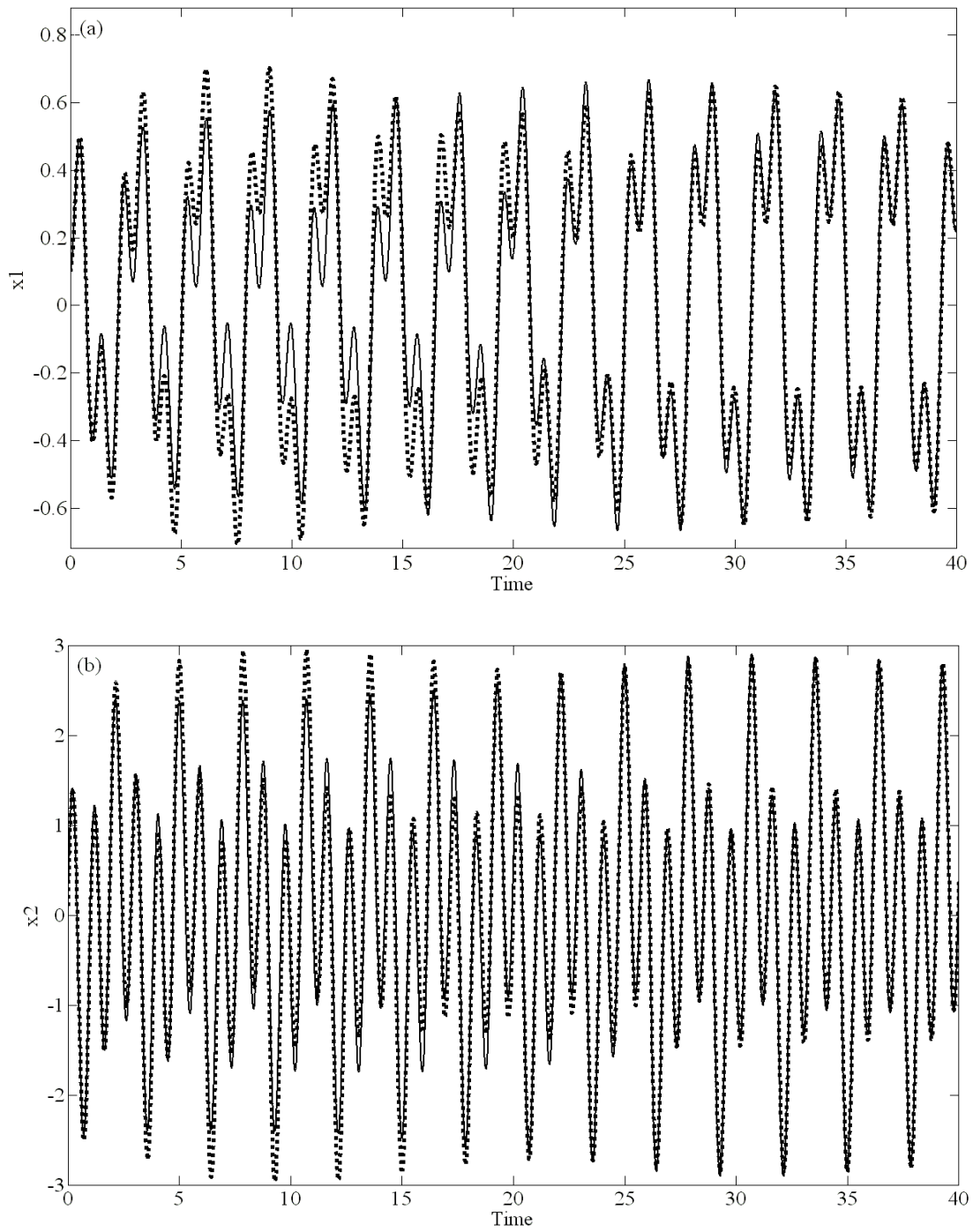


Fig. 2.3 System response for forced Duffing's equation ($\Omega \approx 3\omega_n$).
— , numerical solution; ······ , normal form solution.

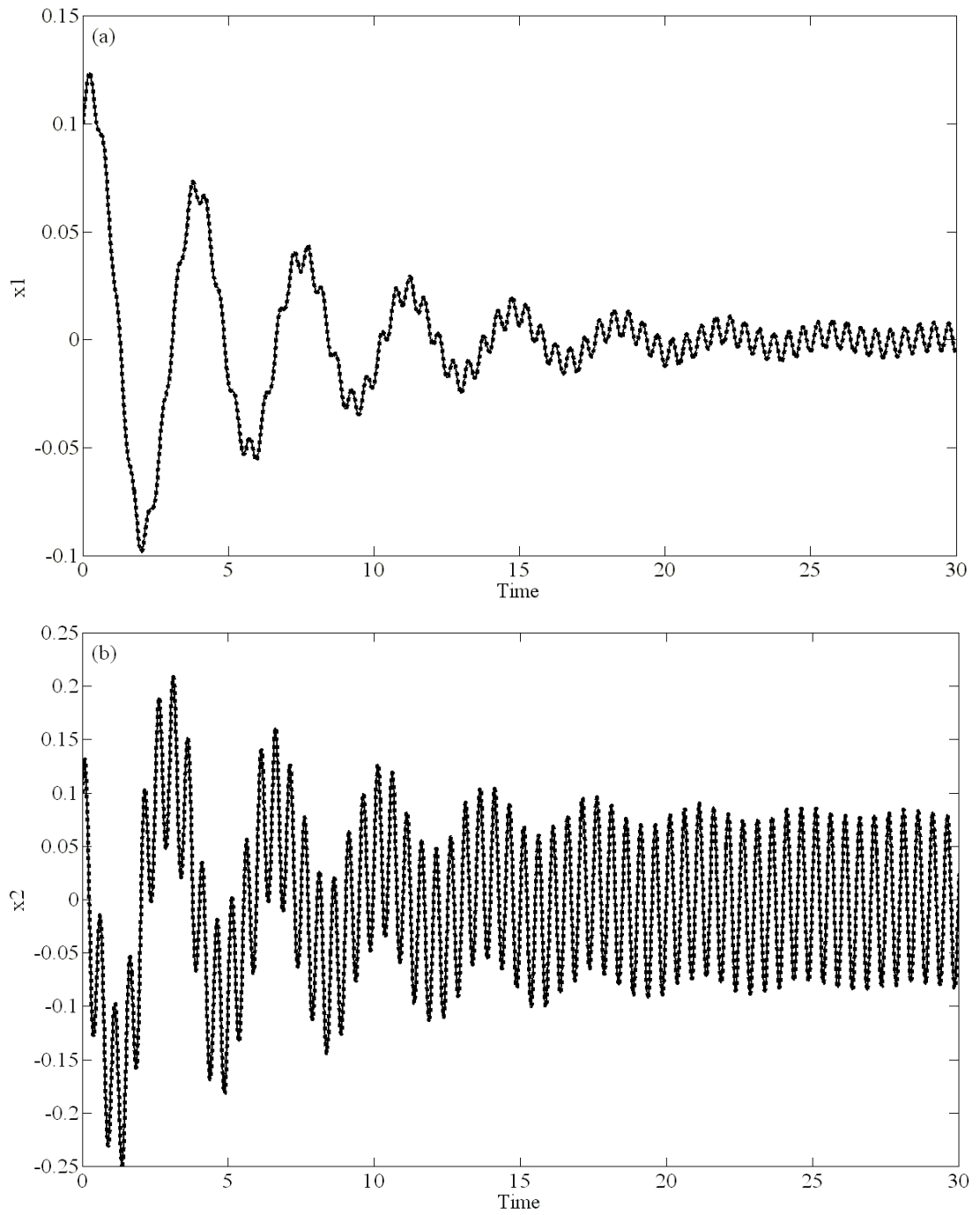


Fig. 2.4 System response for forced Mathieu-Duffing equation (Floquet exponent has negative real parts).

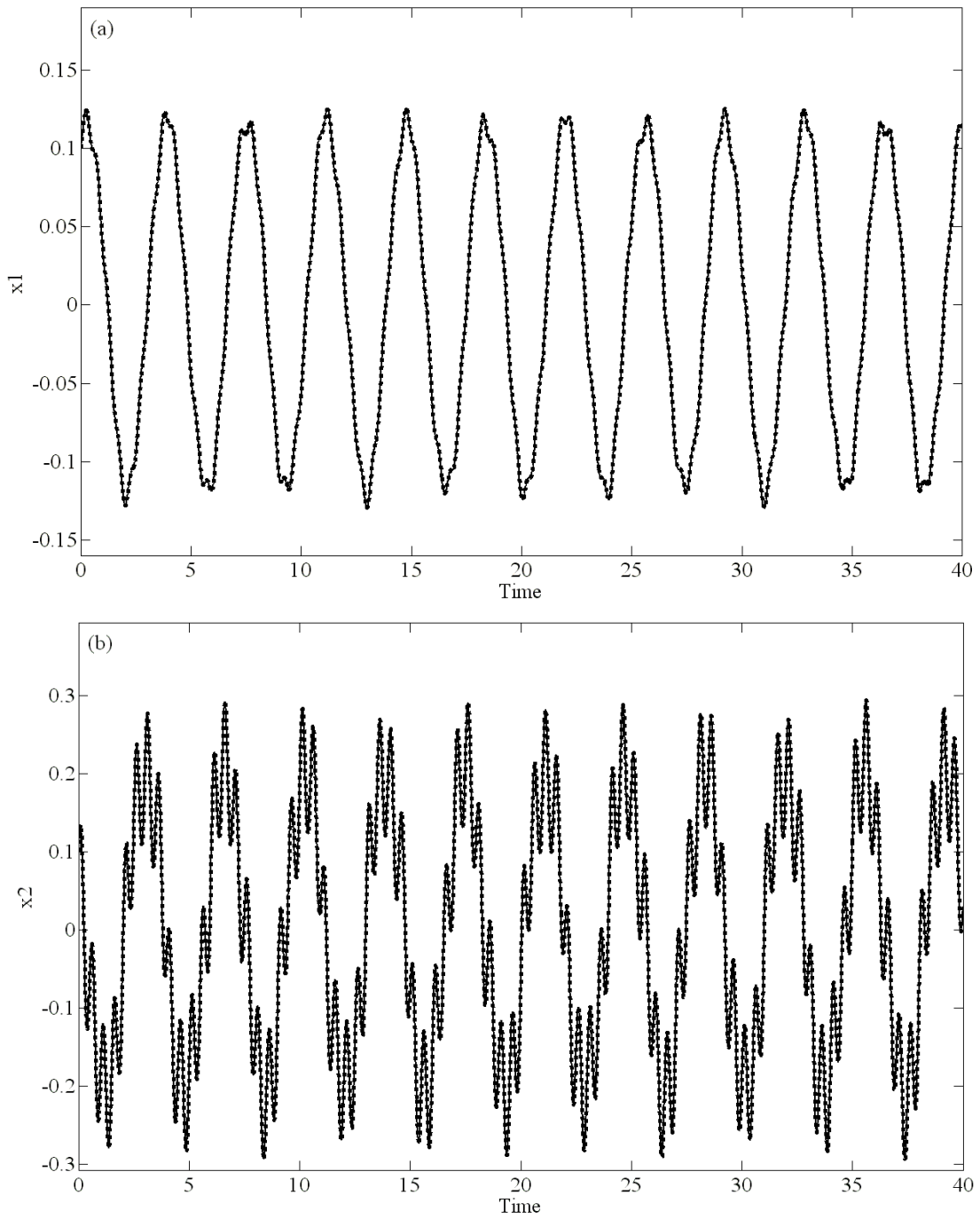


Fig. 2.5 System response for forced Mathieu-Duffing equation (Floquet exponent are purely imaginary).

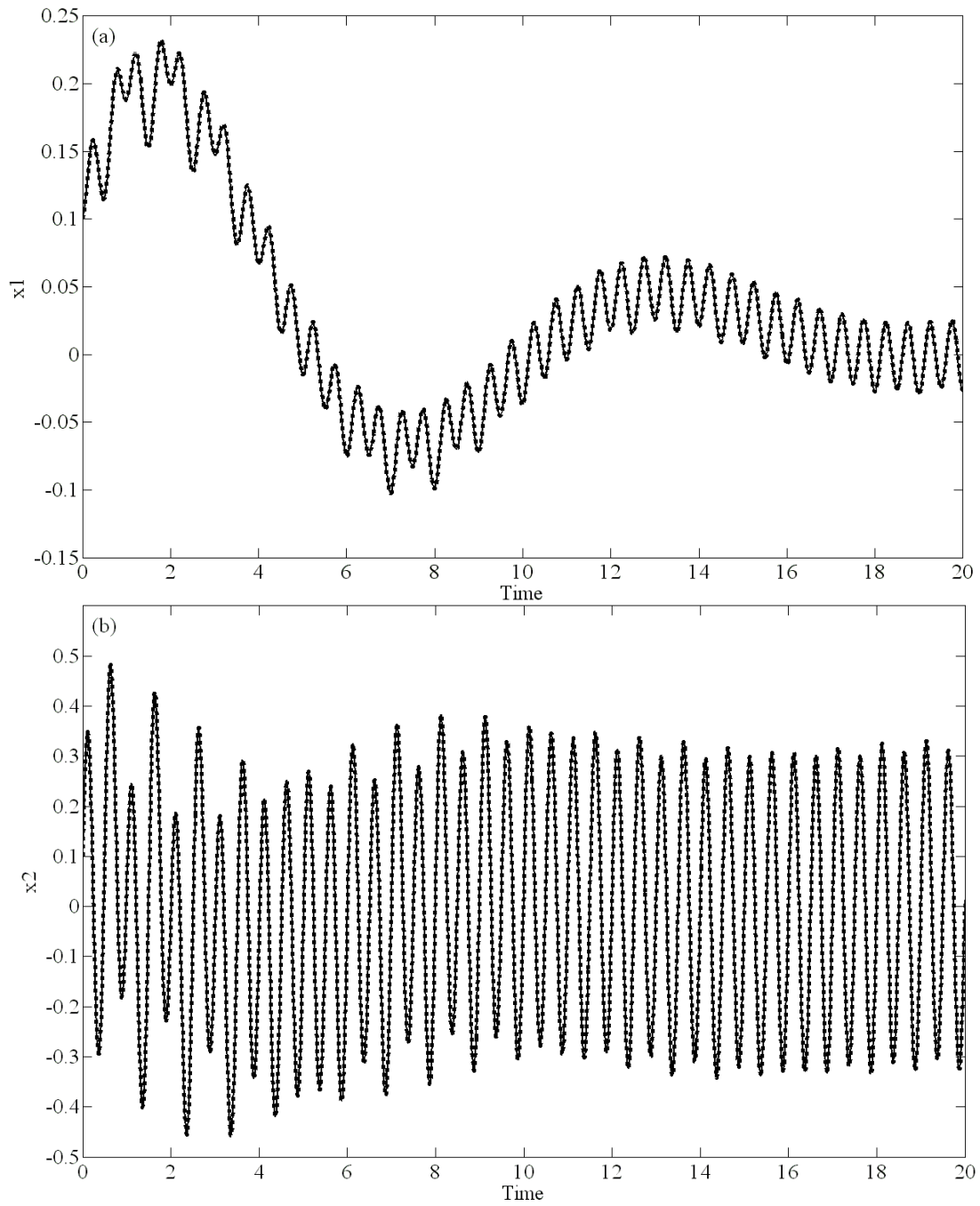


Fig. 2.6 System response for forced Mathieu-Duffing equation ($a \equiv 0$, no generating solution exists).

Chapter 3

Order Reduction of Nonlinear Systems Subjected to External Periodic Excitations

3.1 Introduction

This chapter discusses a methodology for determining reduced order models of periodically excited nonlinear systems with constant as well as periodic coefficients. Approach is based on construction of a time-periodic invariant manifold such that the dynamics of a system, when projected on the invariant manifold, is governed by a fewer number of ordinary differential equations. The idea of using invariant manifolds for model order reduction of nonlinear homogeneous systems has been exploited by quite a few researchers, recently. As discussed in Chapter 1, these techniques can be broadly classified into three categories; namely, center manifold, nonlinear normal modes (NNMs) and singular perturbation. A detailed discussion and comparison of some these techniques can be found in Steindl and Troger (2001) and Rega and Troger (2005).

However, nonlinear systems subjected to external excitations require a special attention as none of the above techniques can directly be applied to this class of problems. Following the general idea of introducing a new state variable to represent the forcing term, such that the augmented system is homogeneous, Shaw et al. (1999) first applied the concept of NNMs to reduce the order of periodically excited autonomous systems. They proposed polynomial expansion to approximate the solution to a set of PDEs that governs the geometry of the invariant manifold. Later, Jiang et al. (2005) used a similar approach and employed Galerkin based

numerical solution procedure to solve these PDEs. Their objective was to improve the validity of reduced order models in the large amplitude region. Further, Agnes and Inman (2001) used the method of multiple scales to construct NNMs for such systems; however, they also employed augmented state-space to deal with the periodic forcing. The idea of introducing new state variables to transform the system into a homogeneous form, however, is not very appealing for order reduction of systems which are subjected to multifrequency external and/or parametric excitations. Since every excitation frequency, present in the system, increases the degrees of freedom of the equivalent homogenous system by one. This may make the construction of the invariant manifold a cumbersome process and may outweigh the advantages of order reduction all together. To overcome this drawback, recently, Redkar and Sinha (2008) proposed an innovative approach where they directly constructed a time-varying invariant manifold in system's modal coordinates. They assumed the invariant manifold as time modulated nonlinear functions of the dominant states which has a certain structure in terms of spatial and temporal terms. Their approach is quite general, in the sense that it can also be used for systems with periodic coefficients after the application of L-F transformation (Redkar, 2005; Redkar and Sinha, 2009). However, they observed that the formulation does not yield promising results when the unmodeled dynamics of the systems is near secondary resonance with external excitation frequencies. The work presented here draws on the idea of Redkar and Sinha (Redkar, 2005; Redkar and Sinha, 2008; Redkar and Sinha, 2009) but significantly differs in the way the time-varying manifolds are constructed. It is expected that reduced order models obtained with the proposed methodology will accurately estimate the response of the original system even if the unmodeled dynamics of the system is near primary or secondary resonance with external excitations. In the following section, the construction of such invariant manifolds and ensuing model order reduction for systems with constant as well as periodic coefficients is discussed.

3.2 Order Reduction of Systems Subjected to Periodic Excitations

3.2.1 Systems with Constant Coefficients

It is proposed to construct an approximation of the forced nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{Ax} + \varepsilon \bar{\mathbf{f}}_r(\mathbf{x}) + \bar{\mathbf{F}}(t) \quad (3.1)$$

by a set of differential equations with smaller dimension. In the above equation ε is a small positive number, \mathbf{A} is an $n \times n$ time-invariant matrix, $\bar{\mathbf{f}}_r(\mathbf{x})$ is an n dimensional vector of nonlinear terms containing monomials in \mathbf{x} up to order r and $\bar{\mathbf{F}}(t)$ is a T periodic force vector of commensurate dimension.

The proposed technique follows a two step procedure in which the system is first transformed into a state-space realization in which the new state variables can be grouped according to some measure of importance. The second step involves truncation of less important state variables in order to get a lower dimensional model. Starting with the modal transformation $\mathbf{x} = \mathbf{Mz}$, equation (3.1) becomes

$$\dot{\mathbf{z}} = \mathbf{Jz} + \varepsilon \mathbf{f}_r(\mathbf{z}) + \mathbf{F}(t) \quad (3.2)$$

where \mathbf{J} is in Jordan canonical form, $\mathbf{f}_r(\mathbf{z}) = \mathbf{M}^{-1} \bar{\mathbf{f}}_r(\mathbf{Mz})$ and $\mathbf{F}(t) = \mathbf{M}^{-1} \bar{\mathbf{F}}(t)$. Without any loss of generality, Equation (3.2) can be partitioned as

$$\begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{z}}_q \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_p & 0 \\ 0 & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{z}_q \end{Bmatrix} + \varepsilon \begin{Bmatrix} \mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{z}_q) \\ \mathbf{f}_{qr}(\mathbf{z}_p, \mathbf{z}_q) \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}_p(t) \\ \mathbf{F}_q(t) \end{Bmatrix} \quad (3.3)$$

where \mathbf{J}_p is a $p \times p$ ($p \ll n$) Jordan block associated with the dominant states (\mathbf{z}_p). These are the states to be retained in the reduced order model. \mathbf{J}_q is $q \times q$ ($q = n - p$) Jordan block associated with the non-dominant states (\mathbf{z}_q). These are the states to be eliminated from the reduced order model. The proposed technique does not put any restrictions on the selection of the dominant and non-dominant states provided that they satisfy certain reducibility conditions, which are derived later in the discussion. As a result, one can choose any set of states, which are of practical interest, as the dominant states. For systems which are subjected to periodic excitations, the response is usually dominated by the evolution of states which are in or near resonance (primary or secondary) with the external excitations or those which corresponds to eigenvalues of \mathbf{J} with small negative real parts. Therefore, these states usually become natural choice of the dominant states.

Once the dominant and non-dominant dynamics of the system is identified in Equation (3.3), the non-dominant states can be eliminated from the dominant dynamics by expressing them in terms of the dominant states. A nonlinear, time-varying constraint relationship between the dominant and non-dominant states is proposed in form:

$$\mathbf{z}_q = \mathbf{H}(\mathbf{z}_p, t) = \mathbf{h}_0(t) + \varepsilon(\mathbf{h}_{r_1}(\mathbf{z}_p, t) + \mathbf{h}_{r_2}(\mathbf{z}_p, t) + \cdots + \mathbf{h}_{r_{r-1}}(\mathbf{z}_p, t) + \mathbf{h}_{rr}(\mathbf{z}_p)) \quad (3.4)$$

where the unknown $\mathbf{h}_0(t)$ is assumed to be a function of time alone, $\mathbf{h}_{rs}(\mathbf{z}_p, t)$ constitutes of monomials in \mathbf{z}_p of degree s ($s = 1, \dots, r - 1$) with unknown time-periodic coefficients, while $\mathbf{h}_{rr}(\mathbf{z}_p)$ are monomials in \mathbf{z}_p of degree r with unknown constant coefficients. The constraint relationship represents a p dimensional manifold. The idea of constructing the manifold in this specific structural form emanates from the construction of near-identity transformation for the

forced nonlinear systems as described in the previous chapter (*c.f.*, Equation (2.23)). There it has been shown that the near-identity transformation with the particular structural form facilitates an investigator to correctly express the system response in the desired form (i.e., fundamental, sub or super harmonic solution). Similarly, here the invariant manifold of this special structural form allows an investigator to correctly express the interaction between the external forcing terms with the non-dominant dynamics of the system. This is why the proposed methodology accurately estimates the response of the original system even if the unmodeled dynamics of the system is near primary or secondary resonance with external excitations.

For the manifold (Equation (3.4)) to be invariant (i.e., given $\mathbf{z}_q(0) = \mathbf{H}(\mathbf{z}_p(0), 0)$, the solution $(\mathbf{z}_p(t), \mathbf{z}_q(t))$ lies on the manifold $\forall t \geq 0$), it must satisfy Equation (3.3). Substituting equation (3.4) into equation (3.3) yields

$$\dot{\mathbf{z}}_p = \mathbf{J}_p \mathbf{z}_p + \varepsilon \mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{H}(\mathbf{z}_p, t)) + \mathbf{F}_p(t) \quad (3.5.a)$$

$$\frac{\partial \mathbf{H}(\mathbf{z}_p, t)}{\partial t} + \frac{\partial \mathbf{H}(\mathbf{z}_p, t)}{\partial \mathbf{z}_p} \dot{\mathbf{z}}_p = \mathbf{J}_q \mathbf{z}_q + \varepsilon \mathbf{f}_{qr}(\mathbf{z}_p, \mathbf{H}(\mathbf{z}_p, t)) + \mathbf{F}_q(t) \quad (3.5.b)$$

Dynamics of the system on the invariant manifold is described by p ($p < n$) dimensional differential system (3.5.a), which is a desired reduced order model of the original system (3.1). The geometry of the invariant manifold is governed by PDE (3.5.b). Thus, the solution of equation (3.5.b) is essential for a successful order reduction which is a non-homogenous, nonlinear PDE with time-varying coefficients. To author's best knowledge this PDE is not tractable analytically and one has to use numerical techniques or a series expansion to solve this equation approximately. In this work, the series expansion is proposed for this purpose.

Since the invariant manifold is assumed in a specific temporal and spatial form (3.4), a combined Fourier and multivariable Taylor-Fourier series is used. This technique can not only be easily automated for a nonlinearity of any degree, it also yields various reducibility conditions in a closed form in certain cases. However, it should be noted that such an approximation is only valid for systems with weak nonlinearities and small amplitudes of motion.

Accordingly, expansion of nonlinear terms in equation (3.5.b) and collection of terms which are alike in powers of \mathbf{z}_p yields

$$\begin{aligned}
\mathbf{J}_q \mathbf{h}_0(t) - \frac{d\mathbf{h}_0(t)}{dt} + \mathbf{F}_q(t) + \varepsilon \mathbf{f}_{q^0}(\mathbf{h}_0(t)) - \varepsilon \frac{\partial \mathbf{h}_{r1}(\mathbf{z}_p, \mathbf{h}_0(t))}{\partial \mathbf{z}_p} \mathbf{F}_p(t) &= 0 \\
L(\mathbf{h}_{r1}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) + \frac{\partial \mathbf{h}_{r1}(\mathbf{z}_p, t)}{\partial t} - \mathbf{f}_{q^1}(\mathbf{z}_p, \mathbf{h}_0(t)) + \frac{\partial \mathbf{h}_{r2}(\mathbf{z}_p, t)}{\partial \mathbf{z}_p} \mathbf{F}_p(t) &= 0 \\
&\vdots \\
L(\mathbf{h}_{rr-1}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) + \frac{\partial \mathbf{h}_{rr-1}(\mathbf{z}_p, t)}{\partial t} - \mathbf{f}_{q^{r-1}}(\mathbf{z}_p, \mathbf{h}_0(t)) + \frac{\partial \mathbf{h}_{rr}(\mathbf{z}_p, t)}{\partial \mathbf{z}_p} \mathbf{F}_p(t) &= 0 \\
L(\mathbf{h}_{rr}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) - \mathbf{f}_{q^r}(\mathbf{z}_p) &= 0
\end{aligned} \tag{3.6}$$

where terms up to the order of ε are retained in the expansion. *Lie operators* ($L(\cdot)$) are defined as

$$L(\mathbf{h}_{rs}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) = \frac{\partial \mathbf{h}_{rs}(\mathbf{z}_p, t)}{\partial \mathbf{z}_p} \mathbf{J}_p \mathbf{z}_p - \mathbf{J}_q \mathbf{h}_{rs}(\mathbf{z}_p, t); (s = 1, 2, \dots, r-1)$$

$$L(\mathbf{h}_{rr}(\mathbf{z}_p), \mathbf{J}_p, \mathbf{J}_q) = \frac{\partial \mathbf{h}_{rr}(\mathbf{z}_p)}{\partial \mathbf{z}_p} \mathbf{J}_p \mathbf{z}_p - \mathbf{J}_q \mathbf{h}_{rr}(\mathbf{z}_p)$$

and $\mathbf{f}_{q_s}(\cdot)$ represents s^{th} degree terms in \mathbf{z}_p arising from the non-dominant dynamics in equation (3.3).

The elements of invariant manifold are expanded as

$$\begin{aligned}\mathbf{h}_0(t) &= \sum_{j=1}^q \sum_{l=-k}^k h_{j,l} e^{il\omega t} \mathbf{e}^j \\ \mathbf{h}_{rs}(\mathbf{z}_p, t) &= \sum_{j=1}^q \sum_{\mathbf{m}_s} \sum_{l=-k}^k h_{r,j,\mathbf{m}_s,l} e^{il\omega t} |\mathbf{z}_p|^{\mathbf{m}_s} \mathbf{e}^j \\ \mathbf{h}_{rr}(\mathbf{z}_p, t) &= \sum_{j=1}^q \sum_{\mathbf{m}_r} h_{r,j,\mathbf{m}_r} |\mathbf{z}_p|^{\mathbf{m}_r} \mathbf{e}^j\end{aligned}\tag{3.7}$$

where $s = 1, 2, \dots, r-1$, $\mathbf{m}_s = (m_1, m_2, \dots, m_p)$, $\sum_{i=1}^p m_i = s$ ($s = 1, 2, \dots, r$), $|\mathbf{z}_p|^{\mathbf{m}_s} = \mathbf{z}_1^{m_1} \mathbf{z}_2^{m_2} \dots \mathbf{z}_p^{m_p}$,

$i = \sqrt{-1}$, $\omega = \pi/T$ and \mathbf{e}^j is the j^{th} member of the natural basis. Substituting equation (3.7) into equation set (3.6) and comparing coefficients of similar terms of the expansion one gets a set of coupled nonlinear algebraic equations. These equations then can be solved numerically to obtain the Fourier and Taylor-Fourier coefficients of (3.7).

However, depending upon the dimension of original system, the type of external excitation (single frequency/multi-frequency) and the degree of the nonlinearity, finding a solution to this set of coupled, nonlinear, algebraic equations may prove to be a difficult task. Further, these algebraic equations typically have multiple solutions which give rise to multiple invariant manifolds. As a result, an investigator faces the challenge of choosing one of these manifolds (out of many) that yields the most accurate reduced order model of the system. Moreover, this technique does not reveal the reducibility conditions; which, if not satisfied can lead to singularities in the solution of the nonlinear algebraic equations. In the case, however,

when the external periodic excitations projected on the dominant dynamics are small ($\mathbf{F}_p(t) \equiv \varepsilon \mathbf{F}_p(t)$), the above drawbacks can be overcome as described next.

Assuming $\mathbf{F}_p(t) \equiv \varepsilon \mathbf{F}_p(t)$, segregation of terms in equation (3.5.b) according to their degree in \mathbf{z}_p yields

$$\mathbf{J}_q \mathbf{h}_0(t) - \frac{d\mathbf{h}_0(t)}{dt} + \mathbf{F}_q(t) + \varepsilon \mathbf{f}_{qr0}(\mathbf{h}_0(t)) = 0 \quad (3.8.a)$$

$$\begin{aligned} L_A(\mathbf{h}_{r1}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) + \frac{\partial \mathbf{h}_{r1}(\mathbf{z}_p, t)}{\partial t} - \mathbf{f}_{qr1}(\mathbf{z}_p, \mathbf{h}_0(t)) &= 0 \\ \vdots & \\ L_A(\mathbf{h}_{rr-1}(\mathbf{z}_p, t), \mathbf{J}_p, \mathbf{J}_q) + \frac{\partial \mathbf{h}_{rr-1}(\mathbf{z}_p, t)}{\partial t} - \mathbf{f}_{qrr-1}(\mathbf{z}_p, \mathbf{h}_0(t)) &= 0 \end{aligned} \quad (3.8.b)$$

$$L(\mathbf{h}_{rr}(\mathbf{z}_p), \mathbf{J}_p, \mathbf{J}_q) - \mathbf{f}_{qrr}(\mathbf{z}_p) = 0 \quad (3.8.c)$$

where, again, terms up to order of ε are retained in the expansion. One can notice that this set of equations is very similar to the set obtained in Chapter 2 which determines coefficients of the near-identity transformation (*c.f.*, equations 2.26.a and 2.26.b).

In the above set of equations, equation (3.8.a) involves only temporal terms ($\mathbf{h}_0(t)$) and is decoupled from rest of the equations. Therefore, any conventional technique such as perturbation, averaging, normal form or harmonic balance can be used to obtain a solution to this non-homogeneous nonlinear differential equation. It also permits an investigator to choose the form of desired solution, i.e., fundamental, sub or superharmonic to correctly express the interaction between non-dominant dynamics and the external excitations. In the present work the

method of harmonic balance is used for this purpose and the discussion is limited to the periodic solutions only.

Substitution of $\mathbf{h}_0(t)$ in equation set (3.8.b) yields a set of decoupled equations with periodic coefficients. To solve these equations approximately, the known $(\mathbf{f}_{qrs}(\mathbf{z}_p, \mathbf{h}_0(t)))$ and the unknown $(\mathbf{h}_{rs}(\mathbf{z}_p, t))$ terms are expanded in multivariable Taylor-Fourier series and the coefficients of the expansion are compared term-by-term. Equation (3.8.c) is time independent and similar to homological equations associated with the standard Poincaré Normal form theory. Solution of this equation can be approximated by a polynomial expansion of known and unknown terms and a comparison of coefficients of the similar terms as discussed in earlier chapter. In either case a set of linear algebraic equations are obtained which are solvable only if the corresponding linear coefficient matrices are invertible. It is important to note that this solution technique presents a unique solution to PDE (3.5.b) for any chosen solution of (3.8.a). It also provides a greater control over the way an investigator want to describe the interaction between external excitations and non-dominant dynamics.

If matrix \mathbf{A} has n linearly independent eigenvectors, matrix \mathbf{J} and linear coefficient matrices corresponding to (3.8.b) and (3.8.c) can be obtained in a diagonal form. Consequently, a general forms for the Taylor-Fourier coefficients of $\mathbf{h}_{rs}(\mathbf{z}_p, t)$ and the polynomial coefficients of $\mathbf{h}_{rr}(\mathbf{z}_p)$ are given as

$$h_{r,j,m_s,l} = \frac{f_{q,r,j,m_s,l}}{il\omega + \mathbf{m}_s \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j} \quad (3.9.a)$$

$$h_{r,j,m_r} = \frac{f_{q^r,j,m_r}}{\mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j} \quad (3.9.b)$$

where $\boldsymbol{\lambda}_{\text{dom}} = (\lambda_1 \lambda_2 \cdots \lambda_p)^T$ is a vector of eigenvalues of the Jordan matrix \mathbf{J}_p , λ_j ($j=1,2,\dots,q$)

are eigenvalues of the Jordan matrix \mathbf{J}_q , $\mathbf{f}_{q^s}(\mathbf{z}_p, \mathbf{h}_0(t)) = \sum_{j=1}^q \sum_{\mathbf{m}_s} \sum_{l=-k}^k f_{q^s,j,m_s,t} e^{il\omega t} |\mathbf{z}_p|^{\mathbf{m}_s} \mathbf{e}^j$ and

$\mathbf{f}_{q^r}(\mathbf{z}_p) = \sum_{j=1}^q \sum_{\mathbf{m}_r} f_{q^r,j,m_r} |\mathbf{z}_p|^{\mathbf{m}_r} \mathbf{e}^j$. It can be easily noticed from equations (3.9.a) and (3.9.b) that

one can find all Taylor-Fourier and polynomial coefficients of the expansion (or the linear coefficient matrices are invertible) only if the following *reducibility conditions* are satisfied.

$$il\omega + \mathbf{m}_s \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \neq 0 \quad (3.10.a)$$

$$\mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \neq 0 \quad (3.10.b)$$

In the event when condition (3.10.a) and/or (3.10.b) are not satisfied, it is called a ‘*resonant*’ case and the corresponding non-dominant states can not be expressed in terms of the dominant states.

Reducibility condition (3.10.a), based on the value of l , represents either the case of ‘*true internal resonance*’ ($l=0$) or ‘*true combination resonance*’ ($l \neq 0$); while, the reducibility condition (3.10.b) always represents the case of ‘*true internal resonance*’. In the event of a true internal resonance, eigenvalues corresponding to the states involved in the resonance satisfy an integral relationship governed by (3.10.a) or (3.10.b). However, in the event of true combination resonance, these eigenvalues satisfies an integral relationship governed by (3.10.a) which includes one of the external excitation frequencies. In such cases the proposed dominant and non-

dominant dynamics can not be separated by the suggested constraint relationship. Near resonance cases are also not desirable as they yield excessively large coefficients in the series expansion. Nevertheless, for ‘resonance’ and ‘near resonance’ cases one can always construct a reduced order model by making all resonating states a part of either the dominant or the non-dominant state vector, in its entirety. In doing so, the unsatisfied reducibility conditions which arise due to the particular choice of dominant and non-dominant states, can be avoided.

Once the constraint relationship between dominant and non-dominant states is obtained, it can be substituted into equation (3.5.a) to get the desired reduced order model. The reduced order model can then be solved by any conventional nonlinear solution technique such as normal form, perturbation etc., or can be integrated numerically. Using the modal transformation the state response in the reduced order domain can be transformed into the original domain. It should be noted that if the external excitations are absent and the eigenvalues of \mathbf{J}_p are critical, then the formulation reduces to the well known center manifold reduction.

3.3.2 Systems with Time-Periodic Coefficients

Next, consider an order reduction problem for nonlinear systems with periodic external and parametric excitations given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \varepsilon \bar{\mathbf{f}}_r(\mathbf{x}, t) + \bar{\mathbf{F}}(t) \quad (3.11)$$

In the above equation, ε is a small positive parameter, $n \times n$ matrix $\mathbf{A}(t)$ is T periodic such that $\mathbf{A}(t) = \mathbf{A}(T+t)$. $\bar{\mathbf{f}}_r(\mathbf{x}, t)$ is an n dimensional vector of nonlinear terms containing monomials in \mathbf{x} up to the order r with $k_1 T$ periodic coefficients and $\bar{\mathbf{F}}(t)$ is a $k_2 T$ periodic force vector of appropriate dimensions. k_1 and k_2 are assumed to be integers. However, the

technique is not limited by this assumption and can be extended to systems with non-commensurate excitations, also.

Proceeding along similar lines as for center manifold and Poincaré normal form theories for periodic systems (Malkin, 1962; Arnold, 1983; Arrowsmith and Place, 1990; Sinha and Pandiyan, 1994; Pandiyan and Sinha, 1995), the system is first transformed into its equivalent dynamic form for which the linear part of the system is time-invariant.

Application of the transformation $\mathbf{x} = \mathbf{L}(t)\mathbf{y}$ to (3.11) yields

$$\dot{\mathbf{y}} = \mathbf{R}\mathbf{y} + \varepsilon\mathbf{L}^{-1}(t)\bar{\mathbf{f}}_r(\mathbf{L}(t)\mathbf{y}, t) + \mathbf{L}^{-1}(t)\bar{\mathbf{F}}(t) \quad (3.12)$$

where \mathbf{R} is a real valued $n \times n$ constant matrix. For this system a modal transformation can be found such that the linear part of (3.12) takes the Jordan canonical form.

The resulting equation can be partitioned similar to equation (3.3) and written as

$$\begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{z}}_q \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_p & 0 \\ 0 & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{z}_q \end{Bmatrix} + \varepsilon \begin{Bmatrix} \mathbf{f}_{p,r}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{f}_{q,r}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}_p(t) \\ \mathbf{F}_q(t) \end{Bmatrix} \quad (3.13)$$

where \mathbf{z}_p is a p dimensional vector of dominant states and \mathbf{z}_q is a q dimensional ($p + q = n$) vector of non-dominant states. Selection of dominant states, in this case, essentially follows similar guidelines as described in the previous section. For this purpose, eigenvalues of matrix \mathbf{R} , which have similar properties as ‘‘Floquet exponents’’, can be used. Accordingly, one can select these states based on to the relative magnitudes of the real part of Floquet exponents, or their resonances with external excitations. However, unlike systems with constant coefficients, in this case, the nonlinearities contain periodic coefficients. Therefore, in addition to the primary

and secondary resonances one has to check for the possibility of combination resonances, too (Deshmukh et al., 2006). Moreover, the external excitations are inherently multifrequency due to the periodic nature of $\mathbf{L}(t)$.

Equation (3.13) differs from the (3.3) in nonlinear terms $\mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{z}_q, t)$ and $\mathbf{f}_{qr}(\mathbf{z}_p, \mathbf{z}_q, t)$, which now contains time periodic coefficients. As a result, a constraint relationship between dominant and non-dominant states is proposed in the following form

$$\mathbf{z}_q = \mathbf{H}(\mathbf{z}_p, t) = \mathbf{h}_0(t) + \varepsilon(\mathbf{h}_{r1}(\mathbf{z}_p, t) + \mathbf{h}_{r2}(\mathbf{z}_p, t) + \dots + \mathbf{h}_{rr-1}(\mathbf{z}_p, t) + \mathbf{h}_{rr}(\mathbf{z}_p, t)) \quad (3.14)$$

in which all the terms are similar to the one represented by (3.4) except the r^{th} degree terms in \mathbf{z}_p ; which are assumed to have time-periodic coefficients. This constraint relationship is a p dimensional invariant manifold and must also satisfy the PDE (3.5.b). The same techniques are used as described before to solve this PDE, approximately. However, in this case, segregation of equation (3.5.b), in terms of power of \mathbf{z}_p , yields equation with time-periodic coefficients for the r^{th} degree terms. In the case, when external excitation terms projected on the dominant dynamics are small ($\mathbf{F}_p(t) \equiv \varepsilon \mathbf{F}_p(t)$) one can approximate PDE (3.5.b) by equations similar to (3.8.a), (3.8.b) and (3.8.c); except that, equation (3.8.c) now contains time-periodic terms. Consequently, the reducibility conditions (3.10.a) and (3.10.b) can be represented by a single generalized reducibility condition as

$$i\omega + \mathbf{m}_s \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \neq 0 \quad (3.15)$$

where $\mathbf{m}_s = (m_1, m_2, \dots, m_p)$, $\sum_{i=1}^p m_i = s$ ($s = 1, 2, \dots, r$). If condition (3.15) is not satisfied, it corresponds to the ‘resonant’ case and states involved can not be decoupled. Nonetheless, in certain cases, one can avoid the resonance by proper choice of dominant and non-dominant state vectors as pointed out earlier.

3.3 Applications

3.3.1 A 2-DOF Spring-Mass-Damper System

To demonstrate possible applications, first, a reduced order model is constructed for a 2-DOF spring-mass-damper system with nonlinear springs, as shown in Fig. 3.1. The equations of motion (with constant coefficients) can be written in the state-space form as

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1 + k)/m_1 & k/m_1 & -c_1/m_1 & 0 \\ k/m_2 & -(k_2 + k)/m_2 & 0 & -c_2/m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \\ &+ \varepsilon \begin{Bmatrix} 0 \\ 0 \\ b/m_1(x_2 - x_1)^3 - b_1/m_1 x_1^3 \\ -b/m_2(x_2 - x_1)^3 - b_2/m_1 x_2^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ f_1 \cos(\Omega_1 t) \\ f_2 \cos(\Omega_2 t) \end{Bmatrix} \end{aligned} \quad (3.16)$$

The nonlinearities are assumed to be of the cubic form. A similar system was considered by Jiang et al. (2005) for order reduction via construction of NNM. It involved augmentation of the system to its higher order homogenous form before the order reduction procedure can be applied. Moreover, the analysis was limited to a single frequency excitation ($f_2 = 0$) and they considered only the case of primary resonance near the linear modes. In the present example, a two dimensional time-varying invariant manifold is directly constructed using the proposed technique

and the 2-DOF system is approximated by a single-DOF system. Cases of multifrequency excitations are also considered which include secondary resonances near one of the linear modes.

Application of modal transformation ($\mathbf{x} = \mathbf{Mz}$) to equation (3.17) yields an equation similar to (3.3). For the present case, $n = 4$, $r = 3$, and $\mathbf{F}_p(t)$ and $\mathbf{F}_q(t)$ are given by

$$\mathbf{F}_p(t) = \begin{Bmatrix} M_{13}^{-1} f_1 \cos(\Omega_1 t) + M_{14}^{-1} f_2 \cos(\Omega_2 t) \\ M_{23}^{-1} f_1 \cos(\Omega_1 t) + M_{24}^{-1} f_2 \cos(\Omega_2 t) \end{Bmatrix} \text{ and } \mathbf{F}_q(t) = \begin{Bmatrix} M_{33}^{-1} f_1 \cos(\omega_1 t) + M_{34}^{-1} f_2 \cos(\Omega_2 t) \\ M_{43}^{-1} f_1 \cos(\omega_1 t) + M_{44}^{-1} f_2 \cos(\Omega_2 t) \end{Bmatrix}.$$

M_{ab}^{-1} represents elements \mathbf{M}^{-1} .

Since the system contains cubic nonlinearity, an invariant manifold of the following form (*c.f.*, equation (3.4)) is proposed.

$$\mathbf{z}_q = \mathbf{H}(\mathbf{z}_p, t) = \mathbf{h}_0(t) + \varepsilon \left(\mathbf{h}_{31}(\mathbf{z}_p, t) + \mathbf{h}_{32}(\mathbf{z}_p, t) + \mathbf{h}_{33}(\mathbf{z}_p) \right) \quad (3.17)$$

As discussed before, equation (3.17) must satisfy equation set (3.6) or (3.8). If all elements of (3.17) can be determined, then a reduced order model of the system can be obtained by replacing the non-dominant states from the dominant dynamics (3.5.a) as functions of dominant states, alone.

In the following, some case studies for various system parameter values are presented. Parameters are chosen such that the system satisfies all the reducibility conditions and the order reduction using proposed constraint relationship is possible. In contrast to guidelines proposed earlier for the selection of dominant/ non-dominant states, states which are near resonance (primary or secondary) with the external excitation are assumed as the non-dominant states. This

is intentionally done in order to highlight the effectiveness of the technique in expressing unmodeled dynamics of the system via reduced order models.

First a case of primary resonance is considered. For system parameters $m_1 = m_2 = 1$, $k = 1$, $k_1 = 4$, $k_2 = 13$, $c_1 = 0.2$, $c_2 = 0.2$, $b = 1$, $b_1 = 1$, $b_2 = 5$, $f_1 = 0.5$, $f_2 = 1.1$, the undamped natural frequencies of the system are $\omega_1 = 2.2114$ rad/sec, $\omega_2 = 3.7563$ rad/sec and matrix \mathbf{J} in equation (3.2) can be written as $diag[-0.1 + 2.2091i, -0.1 - 2.2091i, -0.1 + 3.7550i, -0.1 + 3.7550i]$. The frequency of external excitation chosen as 4.25 rad/s ($\Omega_1 = \Omega_2 = 4.25$ rad/sec). The states corresponding to eigenvalues $-0.1 \pm 2.2091i$ are taken as the dominant states while the states corresponding to eigenvalues $-0.1 \pm 3.7550i$ are assumed as the non-dominant states ($p = q = 2$) and the original 2-DOF system is approximated by a 1-DOF system. Invariant manifolds are calculated by solving, both, equation set (3.6) as well as (3.8) using proposed techniques. Since frequency of excitation (4.25 rad/sec) is away from the $3\omega_2$ and $\omega_2/3$ terms, a fundamental solution of equation (3.8.a) is sought, which is found by using the method of harmonic balance. Of course, one can use any other analytical method such as perturbation, averaging or normal form to find solution of (3.8.a). The reduced order model is integrated numerically and using the modal transformation all states in \mathbf{x} domain are recovered. The original system is also integrated numerically for a comparison. The initial conditions are taken such that the system's trajectories are initiated on the invariant manifold (i.e., $\mathbf{x}(0) = \mathbf{M} \cdot \{\mathbf{z}_p(0), \mathbf{H}(\mathbf{z}_p(0), 0)\}^T$). Fig. 3.2 shows time traces for states x_1 and x_2 as obtained from reduced and original models which show very good agreement. In order to gain insight into the frequency content of the response, fast Fourier transforms (FFTs) of the time traces of the original and reduced order systems are also presented (Fig. 3.3). In this case the total response is dominated by the fundamental solution of the system,

alone. Therefore, only one dominant peak is observed (frequency 4.25 rad/sec) in the FFTs. Variation in amplitude of stable, steady-state response of the system near main resonance frequency is also shown in Fig. 3.4, which shows close agreement between the reduced and full-order model response. Thus both reduced order models, as derived by solving equation set (3.6) and (3.8), are found to be equally successful in approximating response of the original model in this case.

For parameter values $m_1 = m_2 = 1$, $k = 0$, $k_1 = 4$, $k_2 = 10$, $c_1 = 0.2$, $c_2 = 0.2$, $b = 4$, $b_1 = 2$, $b_2 = 3$, $f_1 = 0.5$, $f_2 = 3.4$, the undamped natural frequencies of the system are $\omega_1 = 2$ rad/sec, $\omega_2 = 3.1623$ rad/sec and matrix \mathbf{J} can be written in the form $diag[-0.1 + 1.9975i, -0.1 - 1.9975i, -0.1 + 3.1607i, -0.1 - 3.1607i]$. The external excitation frequencies are chosen as $\Omega_1 = 3.0$ rad/sec and $\Omega_2 = 1.2$ rad/sec ($3\Omega_2 \approx \omega_2$) such that the system response contains a dominant superharmonic component of order 3. States corresponding to eigenvalues $-0.1 + 1.9975i$ are taken as the dominant states and to those corresponding to $-0.1 + 3.1607i$ as the non-dominant states. The non-dominant dynamics of the system is near super-harmonic resonance with the external excitation. Therefore, a third order superharmonic solution is sought for equation (3.8.a), which is obtained using the method of harmonic balance. It should be noted that the use of equation (3.6) and the following numerical solution technique, does not offer such control on the construction of the invariant manifold. Fig. 3.5 and 3.6 show time traces and FFTs, respectively, of the system response as obtained from simulation of the reduced and full order models.

It is observed from Fig. 3.6 that the system response is dominated by three distinct harmonics at 1.2 rad/sec, 3 rad/sec and 3.6 rad/sec. The first two harmonics corresponds to the

external excitation frequencies while the third harmonic is due to superharmonic resonance of non-dominant states with the external excitation. It is evident that the both reduced order models capture these components of the system dynamics successfully. Fractional harmonics at 0.6 rad/sec and 1.8 rad/sec are also rendered well by these models. The reduced order model obtained by assuming small dominant forcing, however, overestimates harmonics at 2.4 rad/sec in \mathbf{x}_1 and at 4.8 rad/sec and 5.4 rad/sec in \mathbf{x}_2 . Since amplitudes of these harmonics are much smaller compared to the dominant one it has negligible effect on the overall system response.

Next, the parameter values $m_1 = m_2 = 1$, $k = 0$, $k_1 = 43$, $k_2 = 4$, $c_1 = 0.1$, $c_2 = 0.1$, $b = 3$, $b_1 = 3$, $b_2 = 7.972$, $f_1 = 2$, $f_2 = 7.5$ are chosen such that the undamped natural frequencies of the system are $\omega_1 = 6.5574$, $\omega_2 = 2$ and the matrix \mathbf{J} can be written in the form $diag[-0.05 + 6.5572i, -0.05 - 6.5572i, -0.05 + 1.9994i, -0.05 - 1.9994i]$. The forcing frequencies are chosen as $\Omega_1 = 4.4$ rad/sec and $\Omega_2 = 6.6$ rad/sec ($\Omega_2 \approx 3\omega_2$), which, for the above parameter values, introduce subharmonic components of order $\frac{1}{3}$ in the system response. The states corresponding to the eigenvalues $-0.05 \pm 6.5572i$ are assumed as the dominant states and those corresponding to $-0.05 \pm 1.9994i$ are assumed as the non-dominant states and once again the system is approximated by a 1-DOF system. Since the non-dominant undamped natural frequency ω_2 is close to $\frac{\Omega_2}{3}$, a sub harmonic solution of order $\frac{1}{3}$ is sought for equation (3.8.a). Fig. 3.7 shows the system response as calculated from the original and reduced order models for the current parameter values, while Fig. 3.8 shows FFTs of the steady state response. In this case, too, both reduced order models estimates response of the original system quite well.

3.3.2 A 2-DOF Coupled Inverted Pendulum

Next the reduced order modeling of a dynamic system with periodic coefficients is considered. Fig. 3.9 shows a 2-DOF coupled inverted pendulum in the horizontal plane. The pendulum is subjected to periodic loading and torques as shown in the figure. The equations of motion for this system, when expanded up to the cubic terms, about the fixed point

$\{\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\} = \{0, 0, 0, 0\}$ can be shown to be (Sinha et al., 2005)

$$\begin{aligned}
 & \ddot{\theta}_1 + \frac{h_1}{ml^2} \dot{\theta}_1 + \frac{k_{t1}}{ml^2} \theta_1 + \frac{k}{4m} \left[c_{11} (\theta_1 - \theta_2) + c_{12} (\theta_1 - \theta_2)^2 + c_{13} (\theta_1 - \theta_2)^3 \right] \\
 & - \frac{P_1(t)}{ml} \left(\theta_1 - \frac{\theta_1^3}{6} \right) = M_1 \cos(\Omega_1 t) \\
 & \ddot{\theta}_2 + \frac{h_2}{ml^2} \dot{\theta}_2 + \frac{k_{t2}}{ml^2} \theta_2 + \frac{k}{4m} \left[c_{11} (\theta_2 - \theta_1) + c_{12} (\theta_2 - \theta_1)^2 + c_{13} (\theta_2 - \theta_1)^3 \right] \\
 & - \frac{P_2(t)}{ml} \left(\theta_2 - \frac{\theta_2^3}{6} \right) = M_2 \cos(\Omega_2 t)
 \end{aligned} \tag{3.18}$$

where $P_i(t) = P_{i1} + P_{i2} \cos(\omega t)$. k_{ti} and h_i denote torsional stiffness and damping of the system, respectively ($i=1,2$). Above set of equations contains time-periodic coefficients in linear as well as nonlinear terms. Typically, nonlinear decoupling of its states using traditional NNM technique requires transformation of this system into a higher order system which is homogenous as well as autonomous. With proposed techniques, however, one can directly construct a 2-dimensional, time-varying invariant manifold and above 2-DOF time-periodic system can be approximated by a 1-DOF system as illustrated below.

For a typical parameter set, equation (3.18) can be written in the state-space form as

$$\begin{aligned}
\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(1 + \cos(2\pi t)) & 0 & -1 & 0 \\ 0 & -(3 + 2\cos(2\pi t)) & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \\
&+ \begin{Bmatrix} 0 \\ 0 \\ -\cos(2\pi t)x_1^3/6 - 2.5(x_1 - x_2)^3 \\ -2\cos(2\pi t)x_2^3/6 + 2.5(x_1 - x_2)^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0.5\cos(\pi t) \\ 0.8\cos(0.5\pi t) \end{Bmatrix}
\end{aligned} \tag{3.19}$$

where $\{x_1, x_2, x_3, x_4\}^T = \{\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\}^T$, $P_{i1} = 0$ and $c_{i2} = 0$. As described in section 3.3.2, the L-F transformation ($\mathbf{x} = \mathbf{L}(t)\mathbf{y}$) is applied first such that the linear part becomes time-invariant. Application of the modal transformation ($\mathbf{y} = \mathbf{M}\mathbf{z}$) to this equation transforms it into the form represented by equation (3.13) as shown below.

$$\begin{aligned}
\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix} &= \begin{bmatrix} -0.5 & -0.8739 & 0 & 0 \\ 0.8739 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & -1.4621 \\ 0 & 0 & 1.4621 & -0.5 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \\
&+ \begin{Bmatrix} -1.4111z_2^3 - 1.2409z_2z_2^2 - 1.0020z_2z_4^2 + 3.1826z_2^2z_2\cos(\pi t) + \dots \\ 0.0106z_2^2z_4\cos(\pi t) - 0.0101z_2^2z_4\cos(3\pi t) + 0.0115z_2^2z_3\sin(\pi t) + \dots \\ 0.9912z_2^2z_4 + 0.1496z_3^2z_4 + 0.1131z_4^3 + 0.3610z_2^3\cos(\pi t) - 0.8672z_2^3\sin(\pi t) + \dots \\ -1.2276z_2^2z_3 - 0.1892z_3^3 - 0.1495z_2z_4^2 + 1.0494z_2^3\cos(\pi t) + 0.2983z_2^3\sin(\pi t) + \dots \end{Bmatrix} \\
&+ \begin{Bmatrix} 0.8296\cos(\pi t) + 0.0113\cos(3\pi t) + \dots \\ 0.0031\sin(\pi t) + 0.0031\sin(3\pi t) + \dots \\ 0.1615\cos(0.5\pi t) + 0.1615\cos(1.5\pi t) - 0.4212\sin(0.5\pi t) + \dots \\ 0.4696\cos(0.5\pi t) + 0.4696\cos(1.5\pi t) + 0.1449\sin(0.5\pi t) + \dots \end{Bmatrix}
\end{aligned} \tag{3.20}$$

The approach described in the appendix A is used to calculate the real L-F transformation matrix $\mathbf{L}(t)$ which is $2T$ periodic in this case. The states corresponding to exponents (eigenvalues of matrix \mathbf{R}) $-0.5 \pm 0.8739i$ are assumed as the dominant while those

corresponding to $-0.5 \pm 1.4621i$ are assumed as the non-dominant. Due to the cubic nature of the nonlinearity, a constraint relationship between the dominant and non-dominant states is proposed in a form similar to (3.17) with the exception of last term $\left(\mathbf{h}_{33}(\mathbf{z}_p)\right)$ which, in this case, is assumed to have time-periodic coefficients.

As described in section 3.3.2, this constraint relationship is governed by equations similar to equation set (3.6) or (3.8). In the present example, the invariant manifolds are constructed by solving both sets of governing equations and thus two separate reduced order models are obtained. equation (3.8.a) is solved using the method of harmonic balance where the solution is expanded up to 10 harmonic terms. Simulated response of the original system and its reduced order models in shown in Fig. 3.10. Fig. 3.11 shows FFTs of the steady state trajectories.

It is interesting to note that, in this case, a typical multivariable Taylor-Fourier expansion of the governing PDEs similar to (3.6), in which 10 Fourier terms are kept in the expansion, yields a set of 420 coupled nonlinear algebraic equations. An identical expansion of the terms, when used in equation set (3.8), results in 42 coupled nonlinear algebraic equations and 378 linear algebraic equations, where the nonlinear equations are decoupled from the linear equations. Thus, if $\mathbf{F}_p(t)$ is assumed to be small, the effort required to construct the invariant manifold is considerably smaller. However, such simplification may come with a tradeoff (as seen in this example), where the reduced model calculated assuming small $\mathbf{F}_p(t)$ overestimates the 2π frequency component in x_2 .

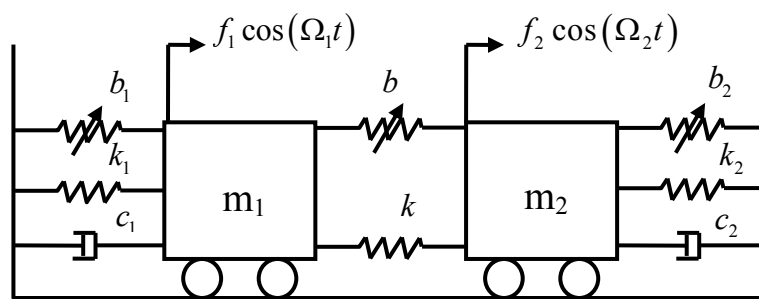


Fig. 3.1 A 2-DOF spring-mass-damper.

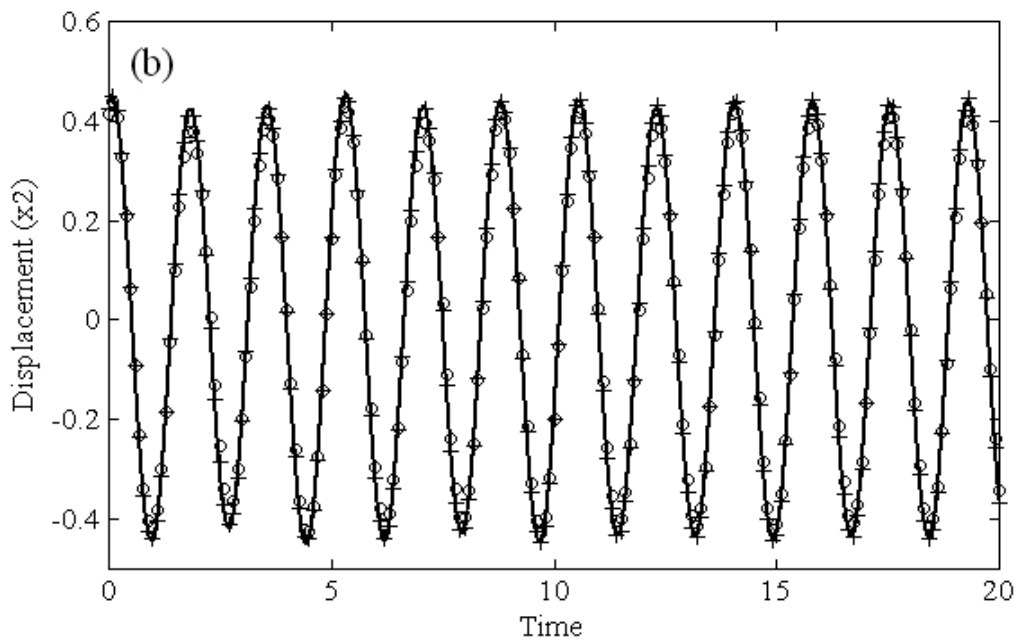
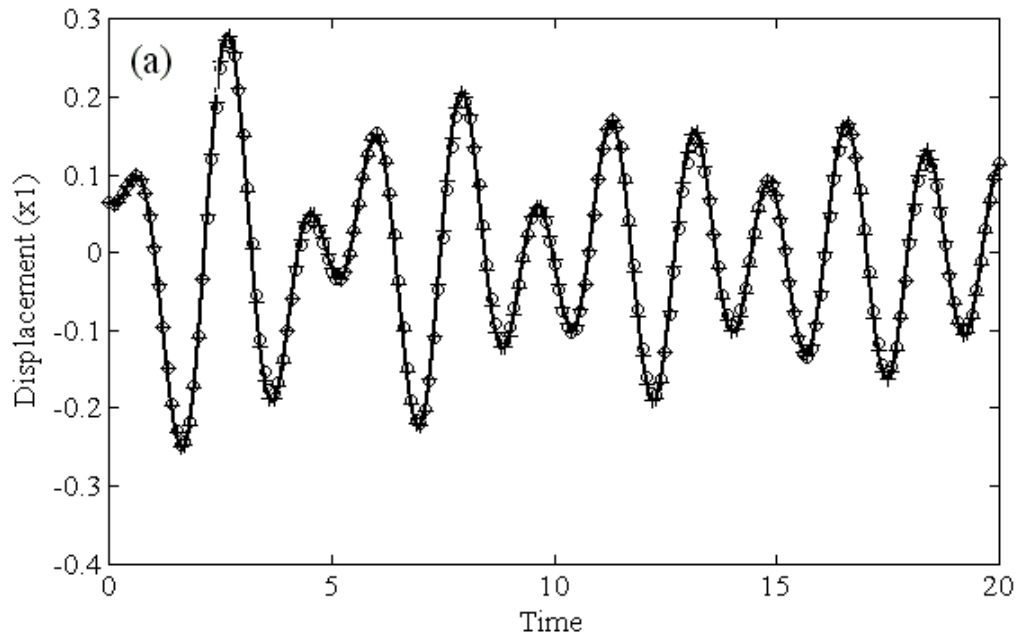


Fig. 3.2 System response for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$).

—, full order model; +, reduced order model; o, reduced order model assuming F_p to be small.

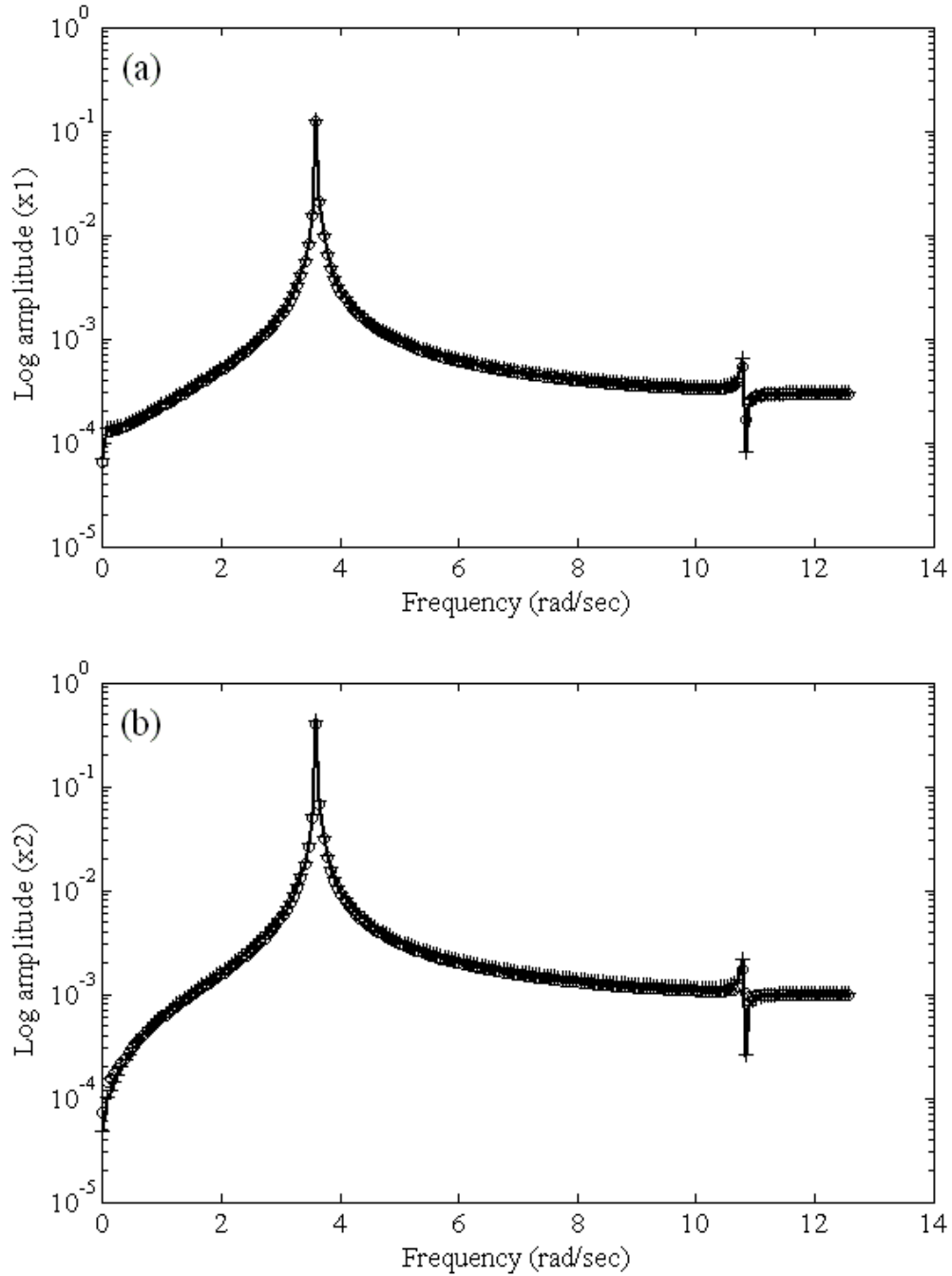


Fig. 3.3 Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$).

—, full order model; +, reduced order model; o, reduced order model assuming F_p to be small.

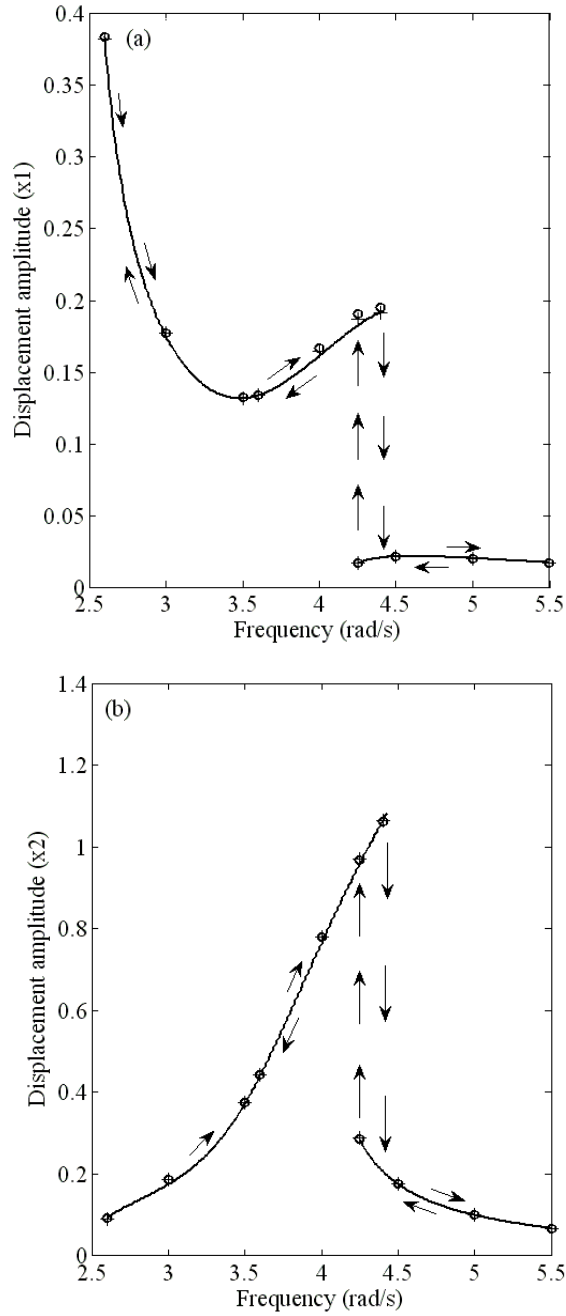


Fig. 3.4 Amplitude variation vs excitation frequency for 2-DOF system with constant coefficients ($\Omega_1 = \Omega_2 \approx \omega_2$).

—, full order model; +, reduced order model; o, reduced order model assuming F_p to be small.

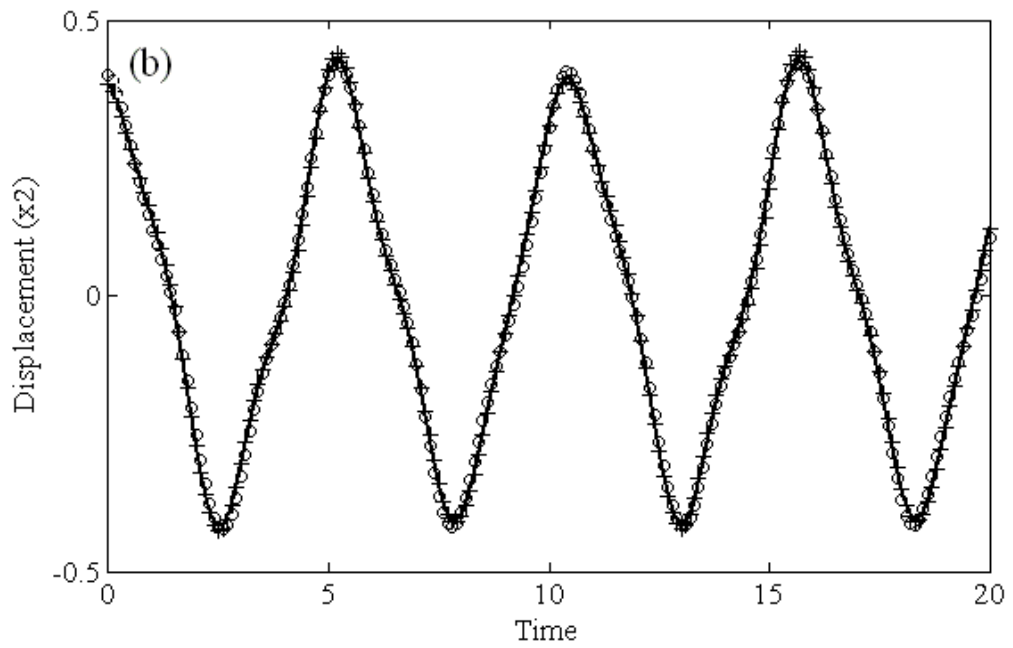
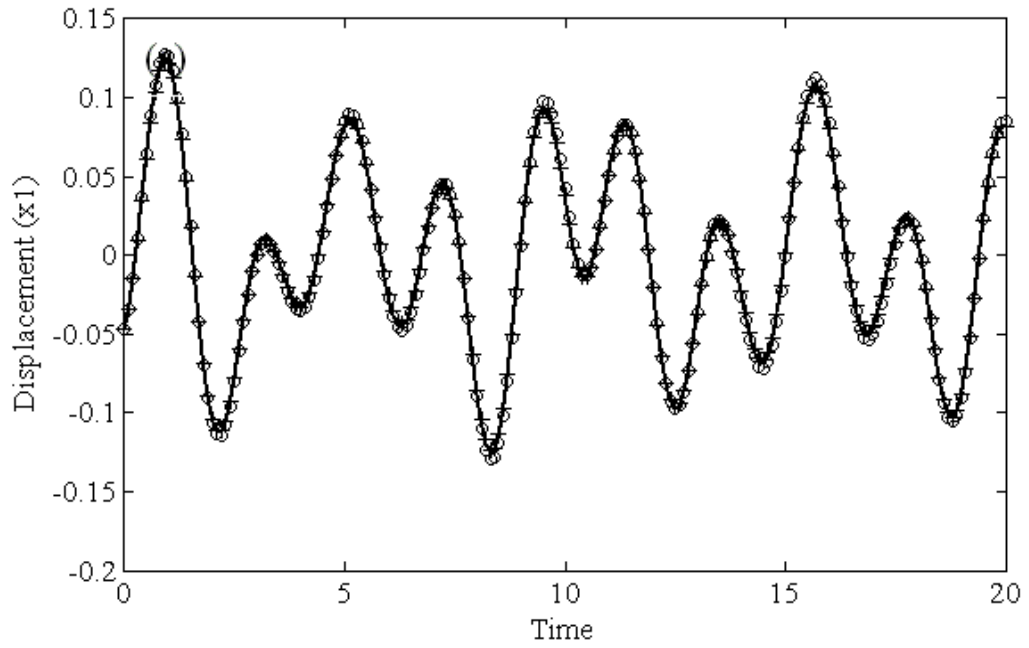


Fig. 3.5 System response for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, 3\Omega_2 \approx \omega_2$).

—, full order model; **+**, reduced order model; **o**, reduced order model assuming F_p to be small.

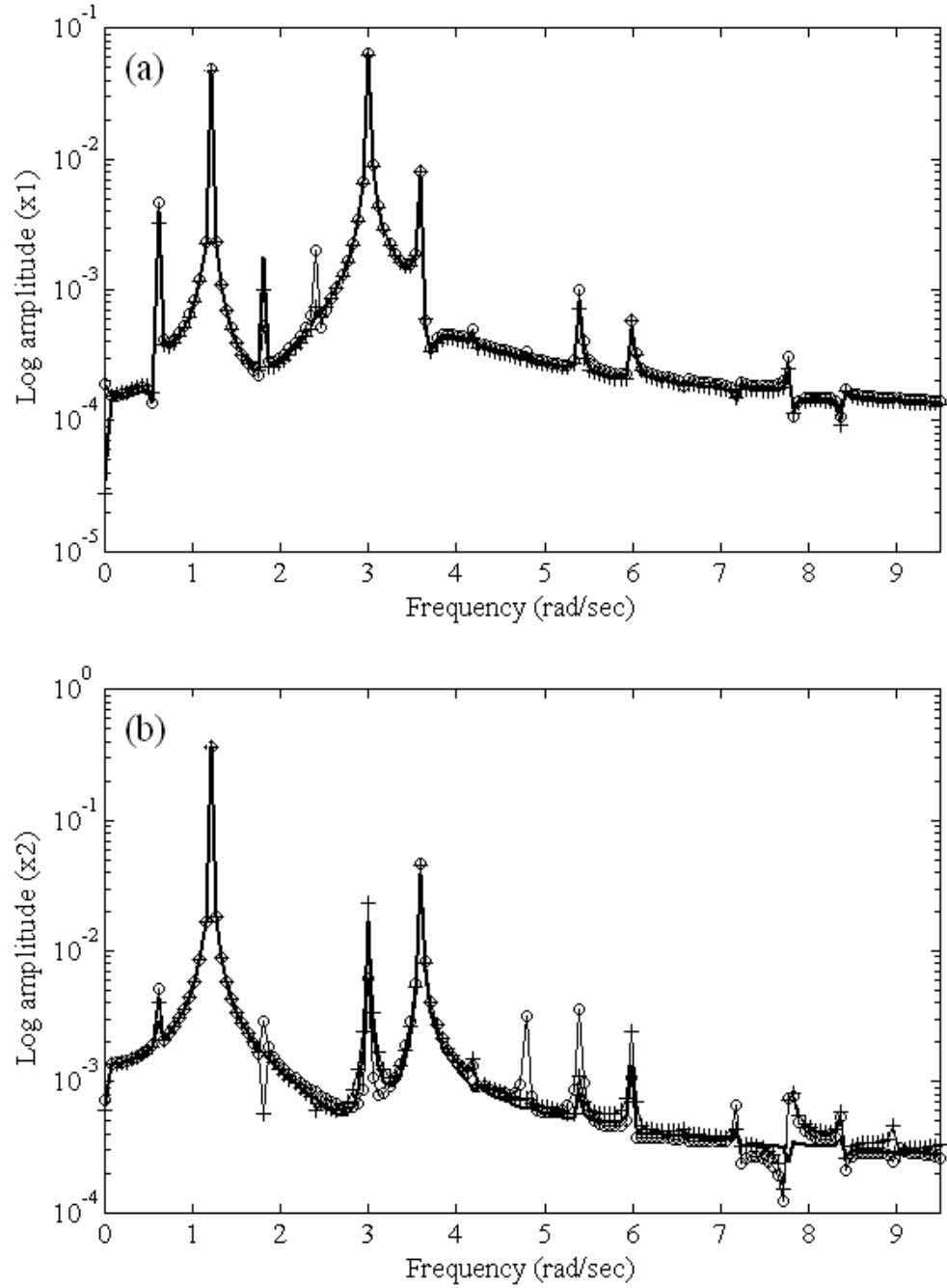


Fig. 3.6 Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, 3\Omega_2 \approx \omega_2$).

—, full order model; +, reduced order model; o, reduced order model assuming F_p to be small.

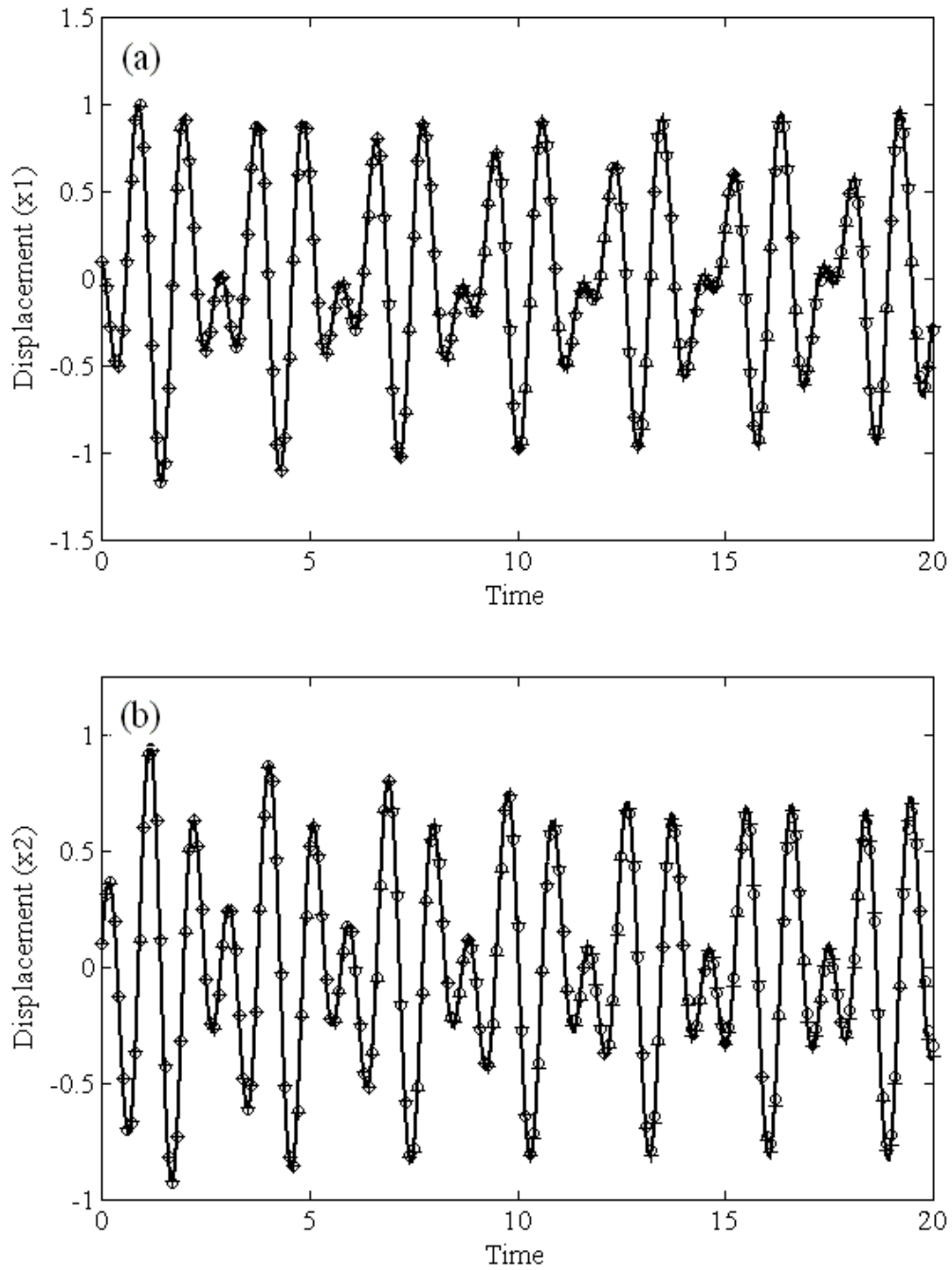


Fig. 3.7 System response for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, \Omega_2 \approx 3\omega_2$).

—, full order model; **+**, reduced order model; **o**, reduced order model assuming F_p to be small.

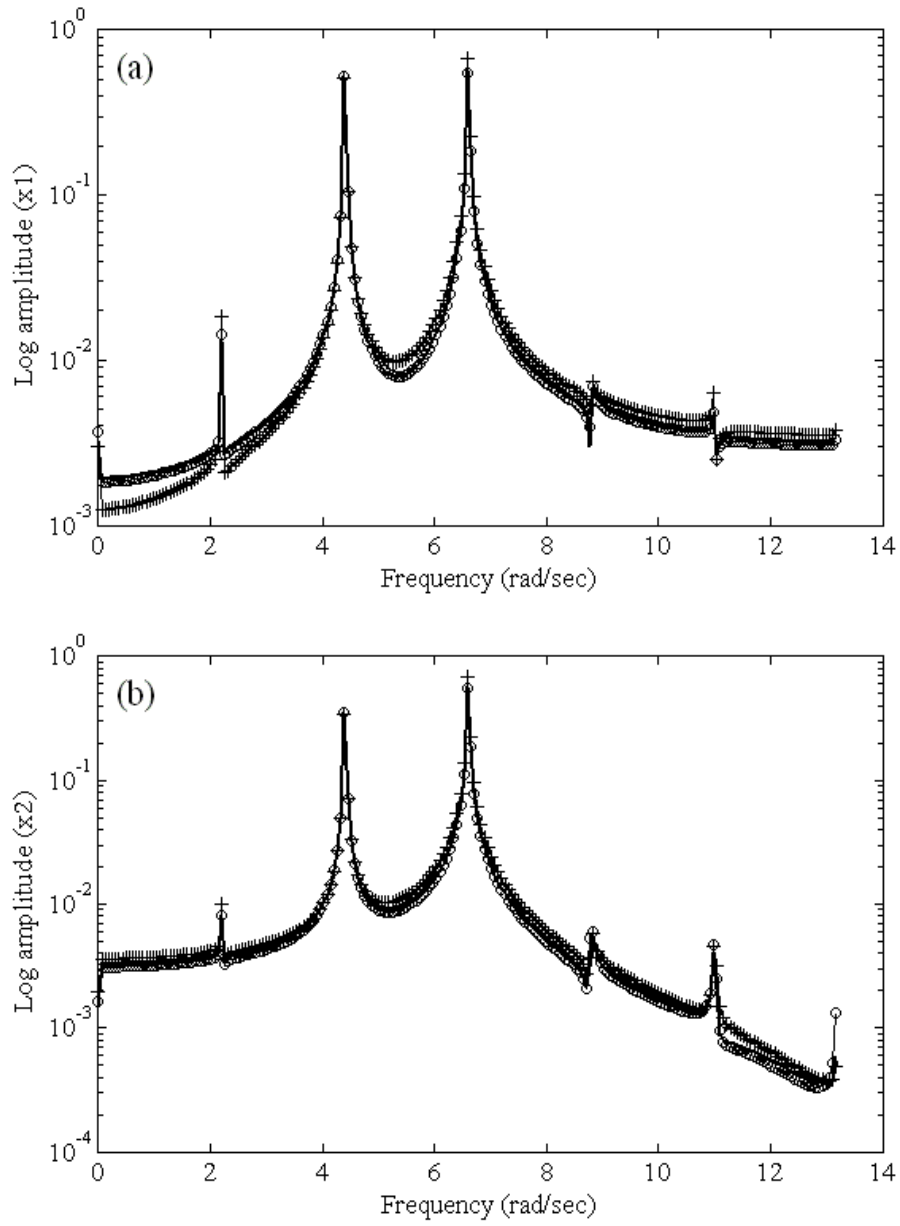


Fig. 3.8 Fast Fourier transforms for 2-DOF system with constant coefficients ($\Omega_1 \neq \Omega_2, \Omega_2 \approx 3\omega_2$).
—, full order model; +, reduced order model; o, reduced order model assuming F_p to be small.

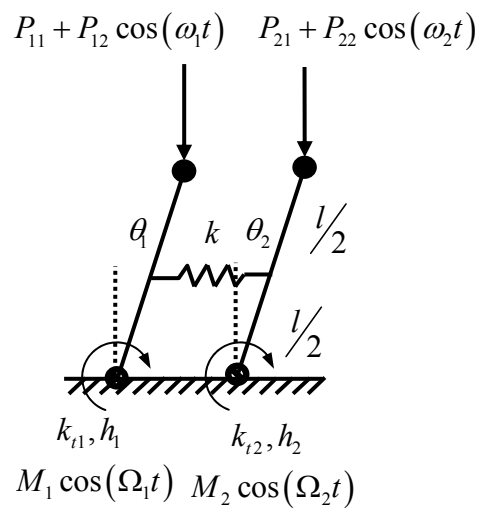


Fig. 3.9 A 2-dof coupled inverted pendulum.

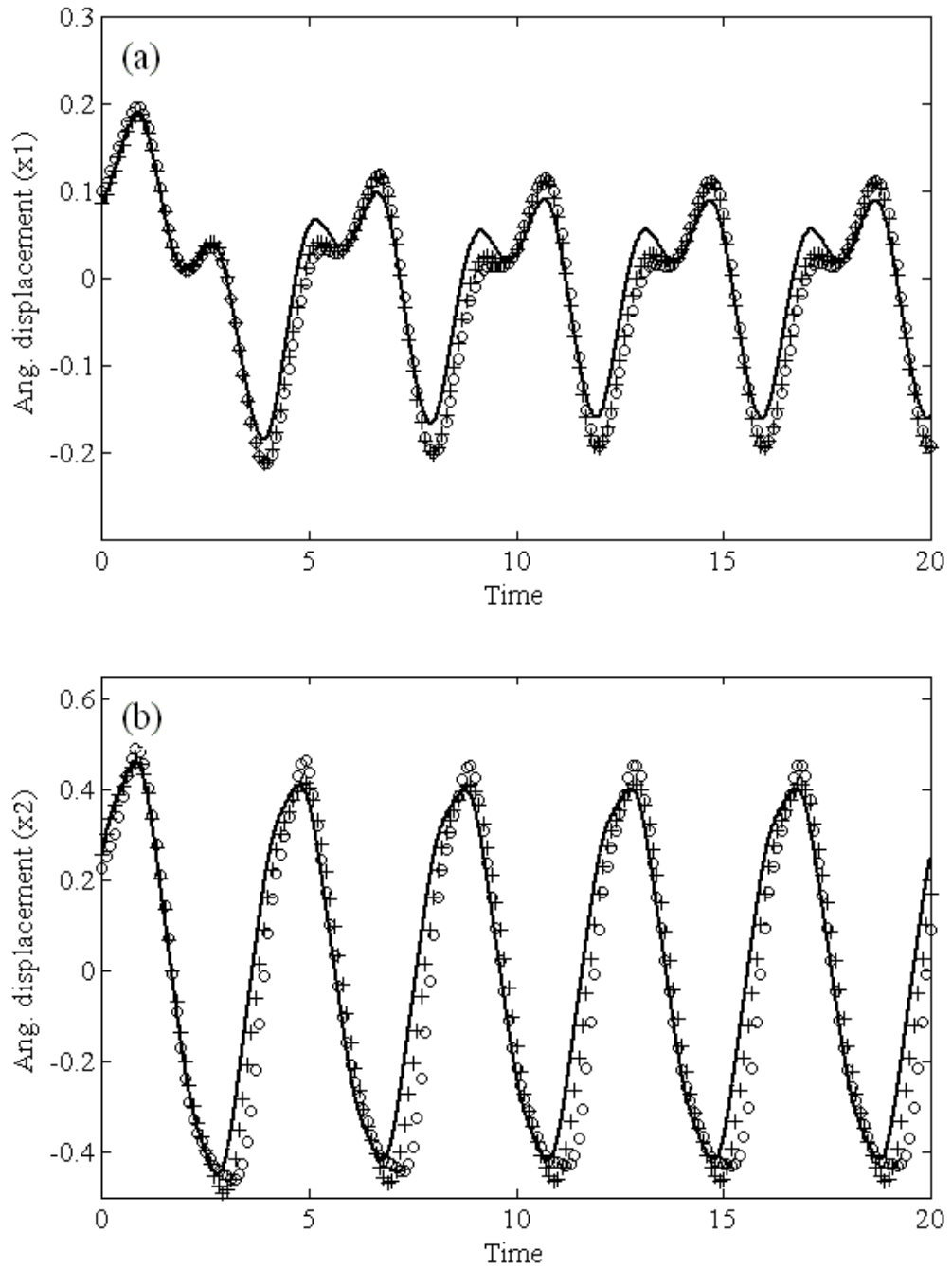


Fig. 3.10 System response for 2-DOF system with periodic coefficients. **—**, full order model; **+**, reduced order model; **o**, reduced order model assuming F_p to be small.

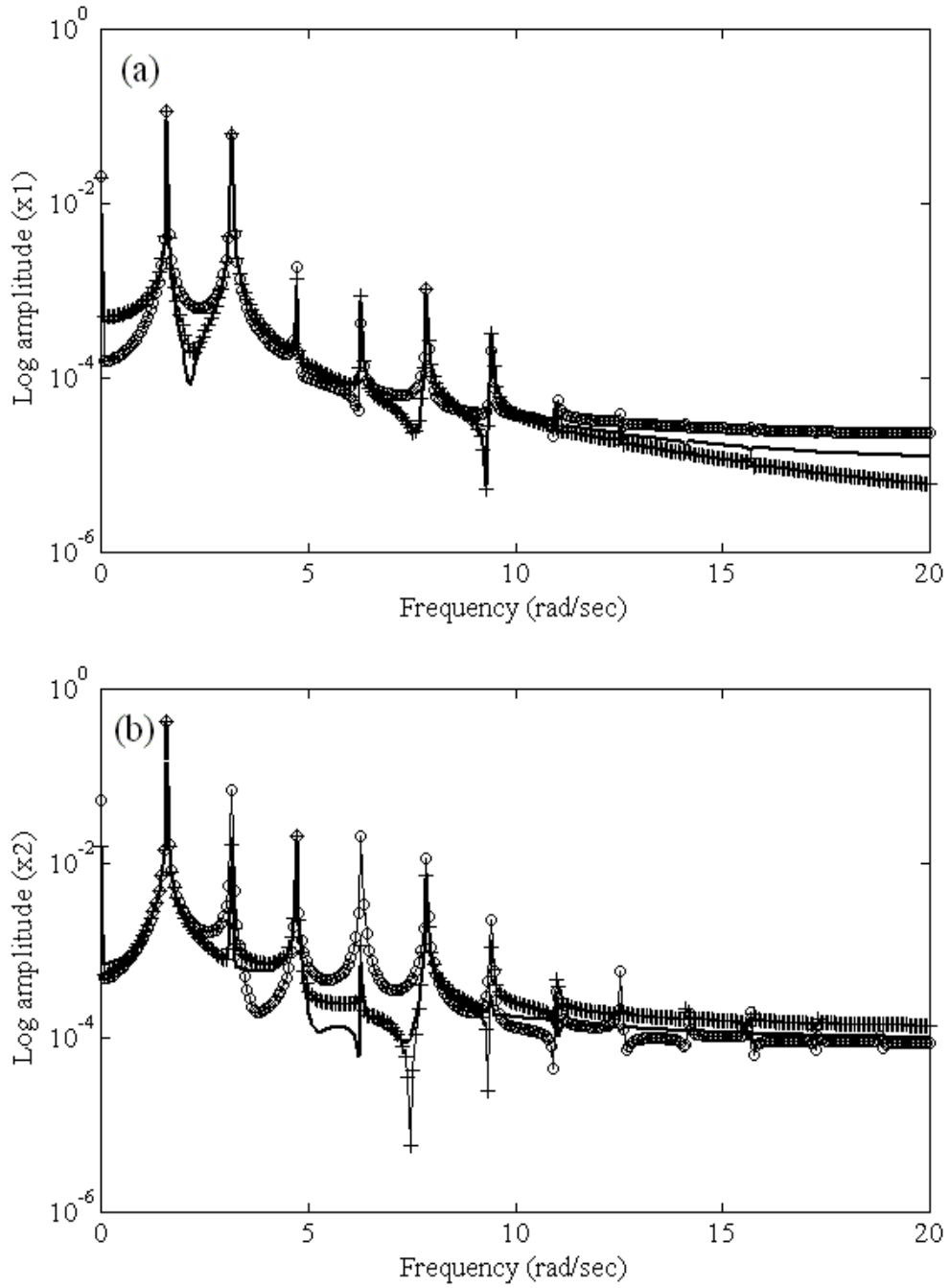


Fig. 3.11 Fast Fourier transforms for 2-DOF system with periodic coefficients).
—, full order model; **+**, reduced order model; **o**, reduced order model assuming F_p to be small.

Chapter 4

Reduced Order Controller Design for Nonlinear Systems

4.1 Introduction

In this chapter a methodology for designing a reduced order controller for nonlinear dynamic systems with constant as well as periodic coefficients is discussed. The approach is based on a similar concept as for the construction of invariant manifolds discussed in the last chapter. However, compared to the open-loop order reduction, the problem of closed-loop order reduction is much more complex in nature. This is because, any nonlinear controller, which is designed in the reduced order domain, in turn, alters the geometry of the invariant manifold, itself. As a result, one has to construct an invariant manifold which is parameterized in terms of nonlinear gains. The complexity of the problem becomes multifold for systems with time-periodic coefficients, as invariant manifolds for such systems are typically time-varying.

A class of systems which requires a stabilizing controller design is considered in this work. The system equations are represented by quasi-linear differential equations in state space. The controller is assumed to be a scalar; however, it is not a limitation on the methodology and one can extend the technique to systems with multiple inputs. Since the readers are already acquainted with the application of L-F transformation and construction of time-periodic invariant manifolds, this chapter addresses the case of time-periodic systems in detail, first. The case of systems with constant coefficients is treated as a subset of its periodic counterpart.

4.2 Controller Design for Linear Time-Periodic Systems

In this section, a symbolic controller design technique for linear time-periodic systems as introduced by Sinha et al. (2005) is outlined, first. Consider an n -dimensional linear periodic system of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \quad (4.1)$$

where $n \times n$ matrix $\mathbf{A}(t)$ is T periodic such that $\mathbf{A}(t) = \mathbf{A}(T+t)$. Let $\Phi(t)$ be the state transition matrix (STM) and $\Phi = \Phi(T)$ be the Floquet transition matrix (FTM) of the linear time-periodic system (4.1). Then, according to the Floquet theory (appendix A), the stability of equation (4.1) can be completely characterized by the FTM of the system. Moreover, system (4.1) is asymptotically stable if all eigenvalues of the FTM, called the Floquet multipliers, lie within the unit circle of the complex plane. The system is unstable if at least one of the eigenvalues of the FTM has magnitude greater than one. Floquet theory can effectively be used for design of a linear controller for systems with periodic coefficients. To illustrate this let us consider a full state feedback control problem,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)u \quad (4.2)$$

where $\mathbf{A}(t+T) = \mathbf{A}(t)$ and $\mathbf{B}(t+T) = \mathbf{B}(t)$. With $u = -\mathbf{K}(t)\mathbf{x}$, assuming (4.2) is controllable, the closed-loop system is given by

$$\dot{\mathbf{x}} = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\mathbf{x} \quad (4.3)$$

where $\mathbf{K}(t)$ is a time-periodic matrix with unknown gains k_1, k_2, \dots, k_m . Since equation (4.3) is periodic, the FTM is computed symbolically using the method introduced by Sinha and Butcher

(1997). Generally speaking, such FTMs contain high degree polynomial expressions of k_1, k_2, \dots, k_m . Eigenvalues of the FTM (Floquet multipliers) can then be placed at desired locations in the unit circle with a technique similar to the “Pole placement method”, as described in the following.

The characteristic polynomial of Φ is given by,

$$\text{Det}(\Phi - \rho \mathbf{I}) = \prod_{i=1}^n (\rho - \rho_i) \quad (4.4)$$

where ρ_i are the desired Floquet multipliers. The coefficients of ρ^k in the left-hand side of equation (4.4) are dependent on the control gains k_1, k_2, \dots, k_m and coefficient of ρ^k in the right-hand side are known. A term by term comparison of the coefficients provides the values of unknown gains.

4.3 Reduced Order Nonlinear Controller Design

4.3.1 Systems with Time-Periodic Coefficients

The idea is based on the premise that under certain conditions, a nonlinear, time-periodic system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t)\bar{u} \quad (4.5)$$

can be transformed into a reduced order linear time-periodic system

$$\dot{\mathbf{y}} = \mathbf{J}_p \mathbf{y} + \mathbf{g}_{p,0}(t)\bar{v} \quad (4.6)$$

through the construction of an invariant manifold by finding a suitable state feedback and coordinate transformations. In above equations $\mathbf{x} \in R^n$, $\mathbf{y} \in R^p$, $\mathbf{f}(\mathbf{x}, t+T) = \mathbf{f}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t+T) = \mathbf{g}(\mathbf{x}, t)$ are $n \times 1$ nonlinear vector fields. It is assumed that $\mathbf{x} = 0$ and $\bar{u} = 0$ is the equilibrium point. \mathbf{J}_p is a $p \times p$ constant matrix and $\mathbf{g}_{p,0}(t+T) = \mathbf{g}_{p,0}(t)$ is a $p \times 1$ control influence matrix for the reduced order system ($p \ll n$). For simplicity, \bar{u} and \bar{v} are restricted to be scalars; the generalization to the vector case is straight forward. A linear time-periodic controller can be designed for system (4.6) by the symbolic controller design technique discussed in section 4.2 which guaranties asymptotic stability.

The technique described here is local in nature, and assuming sufficient smoothness equation (4.5) can be expanded in into Taylor series about the equilibrium point as

$$\begin{aligned} \dot{\mathbf{x}} = & \mathbf{A}(t)\mathbf{x} + \bar{\mathbf{f}}_2(\mathbf{x}, t) + \bar{\mathbf{f}}_3(\mathbf{x}, t) + \dots + \bar{\mathbf{f}}_r(\mathbf{x}, t) + \dots \\ & + [\bar{\mathbf{g}}_0(t) + \bar{\mathbf{g}}_2(\mathbf{x}, t) + \dots + \bar{\mathbf{g}}_{r-1}(\mathbf{x}, t) + \dots] \bar{u} \end{aligned} \quad (4.7)$$

where $\bar{\mathbf{f}}_a$ and $\bar{\mathbf{g}}_b$ are a^{th} and b^{th} order monomial forms of \mathbf{x} with T periodic coefficients, respectively; and \mathbf{x} is the perturbation variable. Alternatively, the system represented by equation (4.7) may also result if autonomous or time-periodic system is driven to a desired periodic orbit.

Following the approach described in appendix A to calculate the L-F transformation matrix $\mathbf{L}(t)$ and applying the transformation $\mathbf{x} = \mathbf{L}(t)\mathbf{Mz}$ to (4.7) produces

$$\dot{\mathbf{z}} = \mathbf{Jz} + \mathbf{f}_2(\mathbf{z}, t) + \mathbf{f}_3(\mathbf{z}, t) + \dots + \mathbf{f}_r(\mathbf{z}, t) + \dots + [\mathbf{g}_0(t) + \mathbf{g}_2(\mathbf{z}, t) + \dots + \mathbf{g}_{r-1}(\mathbf{z}, t) + \dots] u \quad (4.8)$$

where \mathbf{M} is the modal matrix and \mathbf{J} is in the Jordan canonical form. Equation (4.8) can be partitioned and written in a block diagonal form as

$$\begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{z}}_q \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_p & 0 \\ 0 & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{z}_q \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{p,r}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{f}_{q,r}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{g}_{p0}(t) + \mathbf{g}_{p,r-1}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{g}_{q0}(t) + \mathbf{g}_{q,r-1}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} u \quad (4.9)$$

where \mathbf{J}_p is a $p \times p$ ($p \ll n$) Jordan block associated with the dominant states (\mathbf{z}_p). These are the states to be kept in the reduced order model. \mathbf{J}_q is $q \times q$ ($q = n - p$) Jordan block associated with the non-dominant states (\mathbf{z}_q). These are the states to be eliminated from the reduced order model. As shown earlier, construction of an invariant manifold and application of a near-identity transformation do not affect the lower order terms. Therefore, without any loss of generality, only r^{th} order nonlinearity is kept in the equation (4.9). In practice, one can start with the lowest order nonlinearity in equation (4.8) and construct the invariant manifolds and apply the near-identity transformations sequentially to eliminate higher order nonlinearities, successively.

Selection of dominant and non-dominant states can be arbitrary, in general. However, for the stabilizing controller design, unstable states of a system should be chosen as the dominant states and stable states as the non-dominant states. For time-periodic systems, this implies that states corresponding to Floquet multipliers outside the unit circle are the dominant states while those corresponding to inside the unit circle are the non-dominant states. However, in the case, one would like to increase the stability margin of a system because some of the Floquet multipliers inside the unit circle but close to the boundary, then those states could also be chosen as the dominant states.

As stated earlier, in closed-loop systems the nonlinear state feedback affects the geometry of the invariant manifold. To account for this effect a controller in the following specific form is proposed

$$u = v + u_{nl}(\mathbf{z}_p, t) \quad (4.10)$$

where

$$u_{nl} = \mathbf{K}_p(t) \sum_{l=0}^r \boldsymbol{\Psi}_{r+l(r-1)}(\mathbf{z}_p, \psi) + \sum_{l=0}^r \boldsymbol{\alpha}_{r+l(r-1)}(\mathbf{z}_p, \alpha) + \sum_{l=0}^r \boldsymbol{\beta}_{r+l(r-1)-1}(\mathbf{z}_p, \beta) v \quad (4.11)$$

Later, the invariant manifold is constructed in terms of the linear and nonlinear control gains. In equation (4.10) the linear control term $v = \mathbf{K}_p(t) \mathbf{z}_p$ is such that the Floquet multipliers of $(\mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t))$ can be placed at desired locations to guarantee asymptotic stability. $\boldsymbol{\Psi}_a(\mathbf{z}_p, \psi(t))$ is a $p \times 1$ vector of a^{th} order monomials in \mathbf{z}_p with unknown time-periodic coefficients. $\boldsymbol{\alpha}_a(\mathbf{z}_p, \alpha(t))$ and $\boldsymbol{\beta}_b(\mathbf{z}_p, \beta(t))$ are a^{th} and b^{th} order scalar monomials in \mathbf{z}_p with unknown time-periodic coefficients, respectively. It should be noted that the proposed controller contains only the dominant states.

The resulting closed-loop equation takes the form

$$\begin{aligned} \begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{z}}_q \end{Bmatrix} &= \begin{bmatrix} \mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t) & 0 \\ \mathbf{g}_{q,0}(t) \mathbf{K}_p(t) & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{z}_q \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{p^r}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{f}_{q^r}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} \\ &+ \begin{Bmatrix} \mathbf{g}_{p^{r-1}}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{g}_{q^{r-1}}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} v + \begin{Bmatrix} \mathbf{g}_{p0}(t) + \mathbf{g}_{p^{r-1}}(\mathbf{z}_p, \mathbf{z}_q, t) \\ \mathbf{g}_{q0}(t) + \mathbf{g}_{q^{r-1}}(\mathbf{z}_p, \mathbf{z}_q, t) \end{Bmatrix} u_{nl} \end{aligned} \quad (4.12)$$

Above equation is coupled in linear terms; moreover, the linear part now contains time-periodic terms. As a result, equation (4.12) is not amenable for the construction of an invariant manifold. A linear state transformation, $\mathbf{z}_q = \mathbf{v}_q + \mathbf{P}(t) \mathbf{z}_p$, decouples the linear part, provided that $\mathbf{P}(t)$ satisfies the following time-periodic Sylvester differential equation.

$$\dot{\mathbf{P}}(t) = \mathbf{J}_q \mathbf{P}(t) - \mathbf{P}(t) (\mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t)) + \mathbf{g}_{q,0}(t) \mathbf{K}_p(t) \quad (4.13)$$

Equation (4.12) then yields

$$\begin{aligned} \begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{v}}_q \end{Bmatrix} &= \begin{bmatrix} \mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t) & 0 \\ 0 & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{v}_q \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{v}_q, t) \\ \tilde{\mathbf{f}}_{qr}(\mathbf{z}_p, \mathbf{v}_q, t) \end{Bmatrix} \\ &+ \begin{Bmatrix} \mathbf{g}_{pr-1}(\mathbf{z}_p, \mathbf{v}_q, t) \\ \tilde{\mathbf{g}}_{qr-1}(\mathbf{z}_p, \mathbf{v}_q, t) \end{Bmatrix} \mathcal{V} + \begin{Bmatrix} \mathbf{g}_{p,0}(t) + \mathbf{g}_{pr-1}(\mathbf{z}_p, \mathbf{v}_q, t) \\ \tilde{\mathbf{g}}_{q,0}(t) + \tilde{\mathbf{g}}_{qr-1}(\mathbf{z}_p, \mathbf{v}_q, t) \end{Bmatrix} \mathcal{U}_{nl} \end{aligned} \quad (4.14)$$

where $\tilde{\mathbf{f}}_{qr}(\mathbf{z}_p, \mathbf{v}_q, t) = \mathbf{f}_{qr}(\mathbf{z}_p, \mathbf{v}_q, t) - \mathbf{P}(t) \mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{v}_q, t)$, $\tilde{\mathbf{g}}_{q,0}(t) = \mathbf{g}_{q,0}(t) - \mathbf{P}(t) \mathbf{g}_{p,0}(t)$ and $\tilde{\mathbf{g}}_{qr-1}(\mathbf{z}_p, \mathbf{v}_q, t) = \mathbf{g}_{qr-1}(\mathbf{z}_p, \mathbf{v}_q, t) - \mathbf{P}(t) \mathbf{g}_{pr-1}(\mathbf{z}_p, \mathbf{v}_q, t)$. A detailed discussion on the solution of equation (4.13) is given in appendix B.

Application of transformation $\mathbf{z}_p = \mathbf{L}_p(t) \mathbf{M}_p \mathbf{v}_p$ to (4.14) produces,

$$\begin{aligned} \begin{Bmatrix} \dot{\mathbf{v}}_p \\ \dot{\mathbf{v}}_q \end{Bmatrix} &= \begin{bmatrix} \bar{\mathbf{J}}_p & 0 \\ 0 & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{v}_p \\ \mathbf{v}_q \end{Bmatrix} + \begin{Bmatrix} \bar{\mathbf{f}}_{pr}(\mathbf{v}_p, \mathbf{v}_q, t) \\ \bar{\mathbf{f}}_{qr}(\mathbf{v}_p, \mathbf{v}_q, t) \end{Bmatrix} \\ &+ \begin{Bmatrix} \bar{\mathbf{g}}_{pr-1}(\mathbf{v}_p, \mathbf{v}_q, t) \\ \bar{\mathbf{g}}_{qr-1}(\mathbf{v}_p, \mathbf{v}_q, t) \end{Bmatrix} \bar{\mathcal{V}} + \begin{Bmatrix} \bar{\mathbf{g}}_{p,0}(t) + \bar{\mathbf{g}}_{pr-1}(\mathbf{v}_p, \mathbf{v}_q, t) \\ \bar{\mathbf{g}}_{q,0}(t) + \bar{\mathbf{g}}_{qr-1}(\mathbf{v}_p, \mathbf{v}_q, t) \end{Bmatrix} \bar{\mathcal{U}}_{nl} \end{aligned} \quad (4.15)$$

where $\mathbf{L}_p(t)$ and \mathbf{M}_p are the L-F and modal transformations for the matrix $\mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t)$, respectively, $\bar{\mathbf{J}}_p$ is in the Jordan canonical form. Expressions for the terms with over-bars are obvious and are omitted here for brevity. Equation (4.15) is in the form where construction of an invariant manifold is possible. As shown by Sinha et al. (2004), a nonlinear relationship (invariant manifold) between dominant and the non-dominant states is given by

$$\mathbf{v}_q = \mathbf{h}_r(\mathbf{v}_p, t) \quad (4.16)$$

where, $\mathbf{h}_r(\mathbf{v}_p, t)$ is a $q \times 1$ vector of r^{th} order monomials in \mathbf{v}_p with unknown time-periodic coefficients.

Substituting equation (4.16) into equation (4.15) yields the manifold equation

$$L(\mathbf{h}_r(\mathbf{v}_p, t), \bar{\mathbf{J}}_p, \mathbf{J}_q) + \frac{\partial \mathbf{h}_r(\mathbf{v}_p, t)}{\partial t} - \bar{\mathbf{f}}_{qr}(\mathbf{v}_p, t) - \bar{\mathbf{g}}_{q0}(t) \bar{u}_{nl} = 0 \quad (4.17)$$

where *Lie operator* ($L(\cdot)$) is defined as

$$L(\mathbf{h}_r(\mathbf{v}_p, t), \bar{\mathbf{J}}_p, \mathbf{J}_q) = \frac{\partial \mathbf{h}_r(\mathbf{v}_p, t)}{\partial \mathbf{x}} \bar{\mathbf{J}}_p \mathbf{v}_p - \mathbf{J}_q \mathbf{h}_r(\mathbf{v}_p, t) \quad (4.18)$$

A formal power series solution for the above PDE (4.17) is sought by expanding the known and unknown terms into a multivariable Taylor-Fourier series and comparing coefficients of the similar terms. The known and unknown terms are expanded as

$$\begin{aligned} \mathbf{h}_r(\mathbf{v}_p, t) &= \sum_{j=1}^q \sum_{\mathbf{m}_r} \sum_{l=-k}^k h_{r,j,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_r} \mathbf{e}^j \\ \bar{\mathbf{f}}_{qr}(\mathbf{v}_p, t) &= \sum_{j=1}^q \sum_{\mathbf{m}_r} \sum_{l=-k}^k \bar{f}_{qr,j,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_r} \mathbf{e}^j \\ \bar{\mathbf{g}}_{q,0}(t) &= \sum_{j=1}^q \sum_{l=-k}^k \bar{g}_{j,l} e^{il\omega t} \mathbf{e}^j \\ \boldsymbol{\psi}_r(\mathbf{v}_p, t) &= \sum_{\mathbf{m}_r} \sum_{l=-k}^k \psi_{r,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_r} \\ \boldsymbol{\alpha}_r(\mathbf{v}_p, t) &= \sum_{\mathbf{m}_r} \sum_{l=-k}^k \alpha_{r,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_r} \\ \boldsymbol{\beta}_{r-1}(\mathbf{v}_p, t) &= \sum_{\mathbf{m}_{r-1}} \sum_{l=-k}^k \beta_{r-1,\mathbf{m}_{r-1},l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_{r-1}} \end{aligned} \quad (4.19)$$

where $\mathbf{m}_s = (m_1, m_1, \dots, m_p)$, $\sum_{l=1}^p m_l = s$ ($s = r-1, r$), $|\mathbf{v}_p|^{\mathbf{m}_s} = v_1^{m_1} v_2^{m_2} \dots v_p^{m_p}$, $\omega = \pi/T$, $i = \sqrt{-1}$

and \mathbf{e}_j is the j^{th} member of the natural basis. If $\bar{\mathbf{J}}_p$ and \mathbf{J}_q are diagonal, the general form for the coefficients of $\mathbf{h}_r(\mathbf{v}, t)$ is given as

$$h_{r,j,\mathbf{m}_r,l} = \frac{w_{r,j,\mathbf{m}_r,l}}{il\omega + \mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j} \quad (4.20)$$

where $\boldsymbol{\lambda}_{\text{dom}} = (\lambda_1 \lambda_2 \dots \lambda_p)^T$ is a vector of eigenvalues of the Jordan matrix $\bar{\mathbf{J}}_p$, λ_j ($j = 1, 2, \dots, q$)

are eigenvalues of the Jordan matrix \mathbf{J}_q and $\bar{\mathbf{f}}_{q^r}(\mathbf{v}_p, t) + \bar{\mathbf{g}}_{q^0}(t)\bar{u}_{nl} = \sum_{j=1}^q \sum_{\mathbf{m}_r} \sum_{l=-k}^k w_{r,j,\mathbf{m}_r,l} e^{il\omega t} |\mathbf{v}_p|^{\mathbf{m}_r} \mathbf{e}^j$.

It is rather obvious from equation (4.20) that in order to find coefficients of the invariant manifold the following ‘*reducibility condition*’ must be satisfied.

$$il\omega + \mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \neq 0 \quad (4.21)$$

In the event equation (4.21) is not satisfied then it is called a ‘*resonant*’ case and the corresponding non-dominant states can not be expressed in terms of the dominant states. Depending upon the value of k , it is either called a ‘*true internal resonance*’ ($k = 0$), or a ‘*true combination resonance*’ ($k \neq 0$). Also, a near resonance condition is not desirable as it results in very large values of the manifold coefficients ($il\omega + \mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \rightarrow 0, h_{r,j,\mathbf{m}_r,l} \rightarrow \infty$). While, the resonant and near resonant cases can pose problems in the order reduction of open-loop systems, the problem may be avoided here by changing the eigenvalues of $\bar{\mathbf{J}}_p$ by the linear feedback $\mathbf{K}_p(t)\mathbf{z}_p$. In doing so, the denominator of equation (4.20) is moved away from zero. The resonant case can also be avoided by retaining the resonant terms in the reduced order model.

Thus one can still successfully reduce the order of the system; however, the reduced order model will be of higher dimension in this case. For no resonance, the invariant manifold can be constructed, and it is in terms of the unknown time-periodic nonlinear gains $\psi_{r,m_r,l} e^{il\omega t}$, $\alpha_{r,m_r,l} e^{il\omega t}$ and $\beta_{r-1,m_{r-1},l} e^{il\omega t}$. After the inverse transformation $\mathbf{z}_p = \mathbf{M}_p^{-1} \mathbf{L}_p^{-1}(t) \mathbf{z}_p$, the resulting reduced order equation can be obtained as

$$\begin{aligned} \dot{\mathbf{z}}_p = & \mathbf{J}_p \mathbf{z}_p + \sum_{l=0}^r \mathbf{f}_{p^{r+l(r-1)}}(\mathbf{z}_p, t, \psi_{r,m_r,l} e^{il\omega t}, \alpha_{r,m_r,l} e^{il\omega t}, \beta_{r-1,m_{r-1},l} e^{il\omega t}) \\ & + \left(\mathbf{g}_{p,0}(t) + \sum_{l=0}^r \mathbf{g}_{p^{r+l(r-1)-1}}(\mathbf{z}_p, t, \psi_{r,m_r,l} e^{il\omega t}, \alpha_{r,m_r,l} e^{il\omega t}, \beta_{r-1,m_{r-1},l} e^{il\omega t}) \right) (\mathbf{v} + \mathbf{u}_{nl}) \end{aligned} \quad (4.22)$$

The equation is nonlinear and contains time-periodic terms. Designing a nonlinear controller for this system is still a challenging task. However, system (4.22) has a special structure and lends itself to application of a series of near-identity transformations and the Poincaré normal form theory which can transform it into a linear, time-periodic, closed-loop system given by equation (4.6). A stabilizing controller for this linear time-periodic system can be designed using the Floquet theory. A similar technique was originally introduced by Zhang and Sinha (2007) as a feedback linearization methodology for parametrically excited nonlinear systems.

Application of the near identity transformation

$$\mathbf{z}_p = \mathbf{y} + \sum_{l=0}^r \boldsymbol{\Phi}_{r+l(r-1)}(\mathbf{y}, t) \quad (4.23)$$

(where $\boldsymbol{\Phi}_a(\mathbf{y}, t)$ are a^{th} order monomial in \mathbf{y} with unknown time-periodic coefficients) to equation (4.22) and collecting terms of like powers of \mathbf{y} results in

$$\begin{aligned}
\dot{\mathbf{y}} = & \mathbf{J}_p \mathbf{y} + \mathbf{g}_{p0}(t) \bar{\mathbf{v}} + \left(\mathbf{f}_{pr}(\mathbf{y}, t) + \mathbf{g}_{p0}(t) \boldsymbol{\alpha}_r(\mathbf{y}, t) - L_A(\boldsymbol{\varphi}_r(\mathbf{y}, t)) - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial t} \right) \\
& + \left(\mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) + \mathbf{g}_{pr-1}(\mathbf{y}, t) - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \right) \bar{\mathbf{v}} \\
& + \left(\mathbf{f}_{p2r-1}(\mathbf{y}, t, \boldsymbol{\psi}_r, \boldsymbol{\alpha}_r, \boldsymbol{\beta}_{r-1}) + \mathbf{g}_{p0}(t) \boldsymbol{\alpha}_{2r-1}(\mathbf{y}, t) - L_A(\boldsymbol{\varphi}_{2r-1}(\mathbf{y}, t)) \right. \\
& \left. - \frac{\partial \boldsymbol{\varphi}_{2r-1}(\mathbf{y}, t)}{\partial t} - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \boldsymbol{\varphi}_r(\mathbf{y}, t) + \mathbf{f}_{pr}(\mathbf{y}, t) - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial t} \right) \right) \\
& + \left(\mathbf{g}_{p0}(t) \boldsymbol{\beta}_{2r-2}(\mathbf{y}, t) + \mathbf{g}_{p2r-2}(\mathbf{y}, t) - \frac{\partial \boldsymbol{\varphi}_{2r-1}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \right. \\
& \left. - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) \right) \bar{\mathbf{v}} + \dots \\
& + \left(\mathbf{f}_{pr^2}(\mathbf{y}, t, \boldsymbol{\psi}_r, \boldsymbol{\alpha}_r, \boldsymbol{\beta}_{r-1}) + \mathbf{g}_{p0}(t) \boldsymbol{\alpha}_{r^2}(\mathbf{y}, t) - L_A(\boldsymbol{\varphi}_{2r-1}(\mathbf{y}, t)) - \frac{\partial \boldsymbol{\varphi}_{r^2}(\mathbf{y}, t)}{\partial t} \right. \\
& \left. - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \boldsymbol{\varphi}_{r^2-r+1}(\mathbf{y}, t) + \mathbf{f}_{pr^2-r+1}(\mathbf{y}, t, \boldsymbol{\psi}_r, \boldsymbol{\alpha}_r, \boldsymbol{\beta}_{r-1}) - \frac{\partial \boldsymbol{\varphi}_{r^2-r+1}(\mathbf{y}, t)}{\partial t} \right) - \dots \right. \\
& \left. - \frac{\partial \boldsymbol{\varphi}_{r^2-r+1}(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \boldsymbol{\varphi}_r(\mathbf{y}, t) + \mathbf{f}_{pr}(\mathbf{y}, t) - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial t} \right) \right) \\
& + \left(\mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r^2-1}(\mathbf{y}, t) + \mathbf{g}_{pr^2-1}(\mathbf{y}, t) - \frac{\partial \boldsymbol{\varphi}_{r^2}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \right. \\
& \left. - \frac{\partial \boldsymbol{\varphi}_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r^2-r}(\mathbf{y}, t) - \dots - \frac{\partial \boldsymbol{\varphi}_{r^2-r+1}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) \right) \bar{\mathbf{v}}
\end{aligned} \tag{4.24}$$

where the *Lie operator* $(L_A(\cdot))$ is defined as

$$L_A(\boldsymbol{\varphi}_s(\mathbf{y}, t)) = \frac{\partial \boldsymbol{\varphi}_s(\mathbf{y}, t)}{\partial \mathbf{v}} \mathbf{J}_p \mathbf{y} - \mathbf{J}_p \boldsymbol{\varphi}_s(\mathbf{y}, t); (s = r, 2r-1, \dots, r^2)$$

and $\boldsymbol{\psi}_{r, \mathbf{m}_r, l} e^{il\omega t}$, $\boldsymbol{\alpha}_{r, \mathbf{m}_r, l} e^{il\omega t}$, $\boldsymbol{\beta}_{r-1, \mathbf{m}_{r-1}, l} e^{il\omega t}$ are shown as $\boldsymbol{\psi}_r$, $\boldsymbol{\alpha}_r$ and $\boldsymbol{\beta}_{r-1}$ for brevity.

$\bar{\mathbf{v}} = \mathbf{K}_p(t) \mathbf{y}$ and $\sum_{l=0}^r \boldsymbol{\psi}_{r+l(r-1)}(\mathbf{z}_p, t)$ is chosen such that it satisfies

$$\mathbf{y} = \mathbf{z}_p + \sum_{l=0}^r \Psi_{r+l(r-1)}(\mathbf{z}_p, t) \quad (4.25)$$

It can be seen from equation (4.24) that equation (4.22) can be reduced to equation (4.6) if the following sets of equations are satisfied

$$\begin{aligned} L_A(\Phi_r(\mathbf{y}, t)) + \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial t} - \mathbf{f}_{pr}(\mathbf{y}, t) - \mathbf{g}_{p0} \mathbf{a}_r(\mathbf{y}, t) &= 0 \\ \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) - \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) - \mathbf{g}_{pr-1}(\mathbf{y}, t) &= 0 \end{aligned} \quad (4.26.1)$$

$$\begin{aligned} L_A(\Phi_{2r-1}(\mathbf{y}, t)) + \frac{\partial \Phi_{2r-1}(\mathbf{y}, t)}{\partial t} - \mathbf{f}_{p2r-1}(\mathbf{y}, t, \psi_r, \alpha_r, \beta_r) \\ - \mathbf{g}_{p0}(t) \mathbf{a}_{2r-1}(\mathbf{y}, t) + \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \Phi_r(\mathbf{y}, t) + \mathbf{f}_{pr}(\mathbf{y}, t) - \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial t} \right) &= 0 \end{aligned} \quad (4.26.2)$$

$$\frac{\partial \Phi_{2r-1}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) - \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{2r-2}(\mathbf{y}, t) - \mathbf{g}_{p2r-2}(\mathbf{y}, t) + \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) = 0$$

⋮
⋮

$$\begin{aligned} L_A(\Phi_{r^2}(\mathbf{y}, t)) + \frac{\partial \Phi_{r^2}(\mathbf{y}, t)}{\partial t} - \mathbf{f}_{pr^2}(\mathbf{y}, t, \psi_r, \alpha_r, \beta_r) - \mathbf{g}_{p0}(t) \mathbf{a}_{r^2}(\mathbf{y}, t) \\ + \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \Phi_{r^2-r+1}(\mathbf{y}, t) + \mathbf{f}_{pr^2-r+1}(\mathbf{y}, t, \psi_r, \alpha_r, \beta_r) - \frac{\partial \Phi_{r^2-r+1}(\mathbf{y}, t)}{\partial t} \right) + \dots \\ + \frac{\partial \Phi_{r^2-r+1}(\mathbf{y}, t)}{\partial \mathbf{y}} \left(\mathbf{J}_p \Phi_r(\mathbf{y}, t) + \mathbf{f}_{pr}(\mathbf{y}, t) - \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial t} \right) &= 0 \end{aligned} \quad (4.26.r)$$

$$\begin{aligned} \frac{\partial \Phi_{r^2}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) - \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r^2-1}(\mathbf{y}, t) - \mathbf{g}_{pr^2-1}(\mathbf{y}, t) \\ + \frac{\partial \Phi_r(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r^2-r}(\mathbf{y}, t) + \dots + \frac{\partial \Phi_{r^2-r+1}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{g}_{p0}(t) \boldsymbol{\beta}_{r-1}(\mathbf{y}, t) &= 0 \end{aligned}$$

Finding an analytical solution for the above sets of coupled PDEs is a difficult task, to say the least. Therefore, once again an approximate analytical solution for these equations are attempted by a multivariable Taylor-Fourier series approximation. Further, it can be noticed that equation set (4.26.1) is independent of sets (4.26.2-26.r); while equation set (4.26.2) is independent of the higher order sets. Therefore, an approximate solution to these PEDs can be found by solving these equation sets sequentially starting with the lowest order set (4.26.1). The first equation in (4.26.1) is identical to equation (4.17) whose solution can be approximated by expanding known and unknown terms into a multivariable Taylor-Fourier series and comparing coefficients of the expansion. It also yields a solvability condition similar to (4.21). This solution, which is in terms of the unknown α_r , then can be substituted in the second equation of (4.26.1). Upon expanding the terms of this equation into a multivariable Taylor-Fourier series and comparing coefficients of the expansion, typically, a set of underdetermined algebraic equations is obtained. Generally speaking, such systems may have no solutions or have an infinite number of solutions. If a solution can be found, then equation set (4.26.1) is solved. The remaining PDEs can be solved in a similar way and the possible solution contains coefficients of the near-identity transformations and nonlinear feedback gains. Even in the case of resonance, it may be possible to find $\alpha_{r+l(r-1)} : l \rightarrow 0, \dots, r$ that cancel the resonant terms via feedback. Also, depending upon the degree of sophistication needed in the controller design, one may choose only first few terms of the controller and the near-identity transformation ($0 \leq l \leq r$) in equation (4.11) and (4.23), thereby reducing the number of PDEs in (4.26) to be solved. Since equation (4.25) is polynomial inversion of the near-identity transformation (4.23), any conventional algorithm for the inversion of formal power series (e.g., Arfken (1996)) can be adopted for finding the inverse of multivariable truncated Fourier series. The original controller \bar{u} in equation (4.5) is obtained by substituting the L-F and modal transformations.

4.3.2 System with Constant Coefficients

The technique described above can be easily adapted for systems with constant coefficients. In this case, of course, one does not have to use the L-F transformation. Moreover, the controller u in equation (4.10) contains only the spatial terms. For brevity, the entire flow of transformations for the time-invariant case is not repeated but only the important points are noted below.

For the time-invariant case, equation (4.12) takes the following form

$$\begin{aligned} \begin{Bmatrix} \dot{\mathbf{z}}_p \\ \dot{\mathbf{z}}_q \end{Bmatrix} &= \begin{bmatrix} \mathbf{J}_p + \mathbf{g}_{p0}\mathbf{K}_p & 0 \\ \mathbf{g}_{q0}\mathbf{K}_p & \mathbf{J}_q \end{bmatrix} \begin{Bmatrix} \mathbf{z}_p \\ \mathbf{z}_q \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{pr}(\mathbf{z}_p, \mathbf{z}_q) \\ \mathbf{f}_{qr}(\mathbf{z}_p, \mathbf{z}_q) \end{Bmatrix} \\ &+ \begin{Bmatrix} \mathbf{g}_{pr-1}(\mathbf{z}_p, \mathbf{z}_q) \\ \mathbf{g}_{qr-1}(\mathbf{z}_p, \mathbf{z}_q) \end{Bmatrix} \mathbf{v} + \begin{Bmatrix} \mathbf{g}_{p0} + \mathbf{g}_{pr-1}(\mathbf{z}_p, \mathbf{z}_q) \\ \mathbf{g}_{q0} + \mathbf{g}_{qr-1}(\mathbf{z}_p, \mathbf{z}_q) \end{Bmatrix} \mathbf{u}_{nl} \end{aligned} \quad (4.27)$$

where \mathbf{K}_p is a constant gain matrix that places eigenvalues of $\mathbf{J}_p + \mathbf{g}_{p0}\mathbf{K}_p$ at the desired locations in the complex plane. A linear transformation, $\mathbf{z}_q = \mathbf{v}_q + \mathbf{P}\mathbf{z}_p$, decouples the linear part, where \mathbf{P} must satisfy the algebraic Sylvester equation given by

$$\mathbf{J}_q\mathbf{P} - \mathbf{P}(\mathbf{J}_p + \mathbf{g}_{p0}\mathbf{K}_p) = -\mathbf{g}_{q0}\mathbf{K}_p \quad (4.28)$$

Algorithms to find a solution for the algebraic Sylvester equation have been widely discussed in the literature; some of the important references are Gantmacher (1959) and Bartels and Stewart (1972). Further, commercially available software MATLAB[®] has a routine to solve the Sylvester equation. It should be noted that equation (4.28) has a unique solution only if $\lambda_i \neq \lambda_j$, where λ_i ($i=1,2,\dots,q$) are the eigenvalues of \mathbf{J}_q and λ_j ($j=1,2,\dots,p$) are the

eigenvalues of $\mathbf{J}_p + \mathbf{g}_{p0}\mathbf{K}_p$. This provides a condition under which such a decoupling transformation is possible. The resulting equation, which is decoupled in the linear terms, can be used to construct an invariant manifold. Here, an invariant manifold of the form

$$\mathbf{v}_q = \mathbf{h}_r(\mathbf{z}_p) \quad (4.29)$$

is proposed, where, $\mathbf{h}_r(\mathbf{z}_p) = \sum_{j=1}^q \sum_{\mathbf{m}_r} h_{j,\mathbf{m}_r} |\mathbf{z}_p|^{\mathbf{m}_r} \mathbf{e}_j$, $\mathbf{m} = (m_1, m_2, \dots, m_p)$, $\sum_{l=1}^p m_l = r$, $|\mathbf{z}_p|^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \dots z_p^{m_p}$, and \mathbf{e}_j is j^{th} member of natural basis. The manifold in this case must satisfy the following equation.

$$L(\mathbf{h}_r(\mathbf{z}_p), \bar{\mathbf{J}}_p, \mathbf{J}_q) - \bar{\mathbf{f}}_{qr}(\mathbf{z}_p) - \bar{\mathbf{g}}_{q0} u_{nl} = 0 \quad (4.30)$$

which is time independent. An approximate solution for above PDE can be obtained by a term-by-term comparison of coefficients of the monomials. It can also be noticed that, in order to express the non-dominant states in terms of the dominant states, the following *reducibility condition* must be satisfied.

$$\mathbf{m}_r \cdot \boldsymbol{\lambda}_{\text{dom}} - \lambda_j \neq 0 \quad (4.31)$$

where $\boldsymbol{\lambda}_{\text{dom}} = (\lambda_1 \lambda_2 \dots \lambda_p)^T$ is a vector of eigenvalues of the Jordan matrix $\bar{\mathbf{J}}_p$ and λ_j ($j=1,2,\dots,q$) are eigenvalues of the Jordan matrix \mathbf{J}_q . If an invariant manifold can be constructed, then a reduced order model can be successfully obtained. The procedure for transforming this equation into a linear closed-loop equation essentially follows the same steps as described before and the resulting linear closed-loop equation takes the following form

$$\dot{\mathbf{y}} = \mathbf{J}_p \mathbf{y} + \mathbf{g}_{p0} \bar{\mathbf{V}} \quad (4.32)$$

Any traditional linear method can be used for designing a controller for the above equation and the results may be transformed back into the original coordinates.

4.4 Applications

The technique proposed above can be effectively applied to design a reduced order controller for practical engineering structures modeled by a set of nonlinear differential equations with or without periodic coefficients. To demonstrate possible applications, a reduced order controller is designed for a system consisting of four inverted coupled pendulums moving in the horizontal plane with time dependent loads acting on each pendulum. Such a system yields equations of motion with periodic coefficients. A 2-DOF spring-mass-damper system with nonlinear springs is also considered for which the equations of motion have constant coefficients.

4.4.1 A 4-DOF Coupled Inverted Pendulum

The structural diagram of the inverted pendulums with axial periodic loads is shown in Fig. 4.1. The equations of motion for this system, when expanded up to cubic terms, about the fixed point $(\theta_1, \theta_2, \theta_3, \theta_4, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4) = (0, 0, 0, 0, 0, 0, 0, 0)$ can be shown to be

$$\begin{aligned}
\ddot{\theta}_1 + \frac{h_1}{ml^2} \dot{\theta}_1 + \frac{k_{t1}}{ml^2} \theta_1 + \frac{k}{4m} [c_{11}(\theta_1 - \theta_2) + c_{12}(\theta_1 - \theta_2)^2 + c_{13}(\theta_1 - \theta_2)^3] - \frac{P_1(t)}{ml} \left(\theta_1 - \frac{\theta_1^3}{6} \right) &= 0 \\
\ddot{\theta}_2 + \frac{h_2}{ml^2} \dot{\theta}_2 + \frac{k_{t2}}{ml^2} \theta_2 + \frac{k}{4m} [c_{11}(\theta_2 - \theta_1) + c_{12}(\theta_2 - \theta_1)^2 + c_{13}(\theta_2 - \theta_1)^3 \\
+ c_{21}(\theta_2 - \theta_3) + c_{22}(\theta_2 - \theta_3)^2 + c_{23}(\theta_2 - \theta_3)^3] - \frac{P_2(t)}{ml} \left(\theta_2 - \frac{\theta_2^3}{6} \right) &= \frac{\bar{u}(t)}{ml^2} \\
\ddot{\theta}_3 + \frac{h_3}{ml^2} \dot{\theta}_3 + \frac{k_{t3}}{ml^2} \theta_3 + \frac{k}{4m} [c_{21}(\theta_3 - \theta_2) + c_{22}(\theta_3 - \theta_2)^2 + c_{23}(\theta_3 - \theta_2)^3 \\
+ c_{31}(\theta_3 - \theta_4) + c_{32}(\theta_3 - \theta_4)^2 + c_{33}(\theta_3 - \theta_4)^3] - \frac{P_3(t)}{ml} \left(\theta_3 - \frac{\theta_3^3}{6} \right) &= 0 \\
\ddot{\theta}_4 + \frac{h_4}{ml^2} \dot{\theta}_4 + \frac{k_{t4}}{ml^2} \theta_4 + \frac{k}{4m} [c_{31}(\theta_4 - \theta_3) + c_{32}(\theta_4 - \theta_3)^2 + c_{33}(\theta_4 - \theta_3)^3] - \frac{P_4(t)}{ml} \left(\theta_4 - \frac{\theta_4^3}{6} \right) &= 0
\end{aligned} \tag{4.33}$$

where $P_i(t) = P_{i1} + P_{i2} \cos(\omega t)$. m , l , k_{t_i} , k and h_i denote mass, length, torsional stiffness, coupling stiffness and torsional damping, respectively ($i=1,2,3,4$), and $\bar{u}(t)$ is the scalar controller. Setting $P_{i1} = 0$, $c_{i2} = 0$ and selecting some typical parameter values, equation (4.33) can be written in the state space form as

$$\begin{aligned}
\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(1-16.5 \cos(2\pi t)) & 1 & 0 & 0 \\ 1 & -(20-2 \cos(2\pi t)) & 1 & 0 \\ 0 & 1 & -(21-\cos(2\pi t)) & 1 \\ 0 & 0 & 1 & -(19-2.5 \cos(2\pi t)) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3.3(x_1 - x_2)^3 - 2.75 \cos(2\pi t)x_1^3 + u \\ 2(x_2 - x_3)^3 + 3.3(x_1 - x_2)^3 - 0.33 \cos(2\pi t)x_2^3 \\ 2(x_3 - x_4)^3 - 2(x_2 - x_3)^3 - 0.17 \cos(2\pi t)x_3^3 \\ -2(x_3 - x_4)^3 - 0.42 \cos(2\pi t)x_4^3 \end{Bmatrix} \tag{4.34}
\end{aligned}$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}^T = \{\theta_1, \theta_2, \theta_3, \theta_4, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4\}^T$ and $u = \bar{u}(t) / ml^2$.

After applying the L-F transformation to equation (4.34) one gets a dynamically equivalent system whose linear part is time-invariant. The eigenvalues of this time-invariant matrix (Floquet exponents) are obtained as

$$(0.2656, -0.3355, -3.5214 \pm 0.4108i, -3.5809 \pm 0.1825i, -3.6376 \pm 0.7214i)$$

and the corresponding Floquet multipliers are

$$(-1.3043, -0.7149, -0.0271 \pm 0.0118i, -0.0273 \pm 0.0051i, -0.0197 \pm 0.0174i).$$

The states corresponding to Floquet multipliers $(-1.3043, -0.7149)$ are chosen as the dominant states to approximate the closed-loop system dynamics. For the resulting 2×2 system a stabilizing feedback controller is designed. As discussed in section 4.3.1, selection of the dominant states is governed by the relative location of corresponding Floquet multipliers in the unit circle. In the present case the first Floquet multiplier lies outside the unit circle, which makes the system unstable. Therefore, the state corresponding to this Floquet multiplier is chosen as one of the dominate states. The second Floquet multiplier, though inside the unit circle, is very close to the boundary. The state corresponding to this Floquet multiplier, if not accounted in the reduced order model, will results in a very limited region of asymptotic stability. Therefore the state corresponding to Floquet multiplier -0.7149 is chosen as the second dominant state. Following section 4.3.1, the linear and nonlinear feedback controllers are selected in the following forms

$$\begin{aligned}
v &= \{K_1(t), K_2(t)\} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \\
u_{nl} &= \{K_1(t), K_2(t)\} \left\{ \begin{aligned} &\psi_{3,1,(3,0),0} z_1^3 + \psi_{3,1,(2,1),0} z_1^2 z_2 + \dots + \psi_{3,1,(3,0),1} e^{i\omega t} z_1^3 + \dots + \psi_{3,1,(0,5),0} z_2^5 \\ &\psi_{3,2,(3,0),0} z_1^3 + \psi_{3,2,(2,1),0} z_1^2 z_2 + \dots + \psi_{3,2,(3,0),1} e^{i\omega t} z_1^3 + \dots + \psi_{3,2,(0,5),0} z_2^5 \end{aligned} \right\} \quad (4.35) \\
&\quad + \alpha_{3,(3,0),0} z_1^3 + \alpha_{3,(2,1),0} z_1^2 z_2 + \dots + \alpha_{3,(3,0),1} e^{i\omega t} z_1^3 + \dots + \alpha_{3,(0,5),0} z_2^5 \\
&\quad + \left(\beta_{2,(2,0),0} z_1^2 + \beta_{2,(1,1),0} z_1 z_2 + \dots + \beta_{2,(2,0),1} e^{i\omega t} z_1^2 + \dots + \beta_{2,(0,4),0} z_2^4 \right) v
\end{aligned}$$

where z_1 and z_2 are the dominant states. Nonlinear feedback up to 5^{th} ($l=1$) order is used in this example. The linear control gains $K_1(t)$ and $K_2(t)$ are chosen such that the Floquet multipliers of the dominant system are placed at the desired locations inside the unit circle (*c.f.*, section 4.2).

For the given example

$$\mathbf{J}_p = \begin{bmatrix} 0.2656 & 0 \\ 0 & -0.3355 \end{bmatrix} \quad \text{and}$$

$$\mathbf{g}_{p0}(t) = \begin{Bmatrix} -0.2883 \cos(\pi t) + 0.0416 \cos(3\pi t) + 2.4594 \sin(\pi t) - 0.2293 \sin(3\pi t) + \dots \\ 0.2974 \cos(\pi t) - 0.0434 \cos(3\pi t) + 2.4694 \sin(\pi t) - 0.2301 \sin(3\pi t) + \dots \end{Bmatrix}$$

Choosing $K_1(t) = 1.7184 \cos(\pi t) - 0.4807 \cos(3\pi t) + \dots - 0.2006 \sin(\pi t) + 0.0871 \sin(3\pi t) + \dots$ and $K_2(t) = -1.7511 \cos(\pi t) + 0.4841 \cos(3\pi t) + \dots + 0.1205 \sin(\pi t) + 0.0599 \sin(3\pi t) + \dots$ the linear feedback places the Floquet multipliers of the dominant system at $0.6657 \pm 0.2163i$ (Floquet exponent at $-0.3566 \pm 0.3141i$) which guarantees the linear asymptotic stability. After decoupling the linear part of the system, an invariant manifold of the form given by equation (4.16) is assumed. For this particular example, $\lambda_{\text{dom}} = \{-0.3566 + 0.3141i, -0.3566 - 0.3141i\}$, $\lambda_j = -3.5221 \pm 0.4115i, -3.5826 \pm 0.1826i, -3.6381 \pm 0.7213i$ and $\mathbf{m} = (3,0), (2,1), (1,2), (0,3)$ therefore, the system does not undergo ‘true internal resonance’ (*c.f.*, equation (4.21)). Also,

since none of the eigenvalues of $\bar{\mathbf{J}}_p$ and \mathbf{J}_q are purely imaginary the system does not undergo ‘true combination resonance’, either (it can be seen from equation (4.21) that in order to have a true combination resonance at least one of the eigenvalue of $\bar{\mathbf{J}}_p$ and \mathbf{J}_q must be purely imaginary). Therefore, an approximate solution for the PDE (4.17) can be successfully obtained.

The resulting reduced order closed-loop equation is obtained in the following form

$$\begin{aligned} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = & \begin{bmatrix} 0.2656 & 0 \\ 0 & -0.3355 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} f_{3,1}(z_1, z_2, t) \\ f_{3,2}(z_1, z_2, t) \end{Bmatrix} + \begin{Bmatrix} f_{5,1}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \\ f_{5,2}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \end{Bmatrix} \\ & + \begin{Bmatrix} f_{7,1}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \\ f_{7,2}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \end{Bmatrix} + \begin{Bmatrix} f_{9,1}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \\ f_{9,2}(z_1, z_2, \psi_3, \alpha_3, \beta_2, t) \end{Bmatrix} + \begin{Bmatrix} g_{0,1}(t) \\ g_{0,2}(t) \end{Bmatrix} (v + u_{nl}) \end{aligned} \quad (4.36)$$

The symbolic computations were carried out using MathematicaTM and the explicit long expressions are omitted here for brevity. Next, a near-identity transformation of the form

$$\begin{aligned} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = & \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} h_{3,1,(3,0),0} \mathcal{Y}_1^3 + h_{3,1,(2,1),0} \mathcal{Y}_1^2 y_2 + \cdots + h_{3,1,(3,0),1} e^{i\omega t} y_1^3 + \cdots \\ h_{3,2,(3,0),0} \mathcal{Y}_1^3 + h_{3,2,(2,1),0} \mathcal{Y}_1^2 y_2 + \cdots + h_{3,2,(3,0),1} e^{i\omega t} y_1^3 + \cdots \end{Bmatrix} \\ & + \begin{Bmatrix} h_{5,1,(5,0),0} \mathcal{Y}_1^5 + h_{5,1,(4,1),0} \mathcal{Y}_1^4 y_2 + \cdots + h_{5,1,(5,0),1} e^{i\omega t} y_1^5 + \cdots \\ h_{5,2,(5,0),0} \mathcal{Y}_1^5 + h_{5,2,(4,1),0} \mathcal{Y}_1^4 y_2 + \cdots + h_{5,2,(5,0),1} e^{i\omega t} y_1^5 + \cdots \end{Bmatrix} \end{aligned} \quad (4.37)$$

is used such that the nonlinear terms up to order five are eliminated from the reduced order model (4.36). This is consistent with the order of the nonlinear controller chosen. Application of the near-identity transformation (4.37) to equation (4.36) produces two sets of PDEs similar to equations (4.26.1) and (4.26.2), which are solved by expanding the known and unknown terms into a multivariable Taylor-Fourier series and comparing Taylor-Fourier coefficients term-by-term. For the present set of eigenvalues of \mathbf{J}_p , the system does not undergo any kind of resonance and corresponding nonlinear controller is obtained as

$$\begin{aligned}
u_{nl} = & -0.0231\cos(\pi t)z_1^3 - 0.0622\cos(\pi t)z_1^2z_2 + \dots + 0.0104\sin(\pi t)z_1^3 - 0.0336\sin(\pi t)z_1^2z_2 + \dots \\
& - 0.0136\cos(\pi t)z_1^5 + 0.0355\cos(\pi t)z_1^4z_2 + \dots + 0.0116\sin(\pi t)z_1^5 - 0.0487\sin(\pi t)z_1^4z_2 + \dots \\
& - 0.2961\cos(\pi t)z_1^3 + 0.5631\cos(\pi t)z_1^2z_2 + \dots + 0.5761\sin(\pi t)z_1^3 - 1.7708\sin(\pi t)z_1^2z_2 + \dots \\
& - 0.0611\cos(\pi t)z_1^5 + 0.4504\cos(\pi t)z_1^4z_2 + \dots + 0.0877\sin(\pi t)z_1^5 - 0.1672\sin(\pi t)z_1^4z_2 + \dots \quad (4.38) \\
& + (0.0498z_1^2 - 0.0762z_1z_2 + 0.0317z_2^2 + 0.0349\cos(2\pi t)z_1^2 \dots + 0.0737\sin(2\pi t)z_1^2 + \dots \\
& \dots - 0.0145z_1^4 - 0.1367z_1^3z_2 + \dots + 0.0594\cos(2\pi t)z_1^4 + 0.0334\cos(2\pi t)z_1^3z_2 + \dots \\
& + 0.1121\sin(2\pi t)z_1^4)v
\end{aligned}$$

The controller is then transferred back into the original coordinates. The states for the uncontrolled and controlled dynamics are shown in Fig. 4.2 and 4.3, respectively. Fig. 4.4 shows the control effort in the original (\mathbf{x}) coordinates. Dotted lines in Fig. 4.3 and 4.4 show the state response and control effort, respectively, when only the linear controller (v) is applied to the system. It is evident from the figures, that the proposed dual controller performs better than the linear controller in terms of time required for convergence to the steady state as well as the amount of control effort required. Further, Fig. 4.5 shows the region of attraction (Ω) for the linear and dual controller on $x_1 \sim x_5(\dot{x}_1)$ plane. Ω is defined as $\Omega = \{ \mathbf{x}_0 \in R^n \mid \phi(t, \mathbf{x}_0) \rightarrow 0 \text{ as } t \rightarrow \infty \}$, where $\phi(t, \mathbf{x}_0)$ is the solution of the closed-loop system that starts at initial state \mathbf{x}_0 at time $t=0$. The regions of attraction were found by numerical integration of the closed-loop systems at various initial conditions and observing the state trajectories. The stability of closed-loop system is most sensitive to the initial angular displacement and velocity of the first pendulum (this is the pendulum which causes instability of the system). Therefore, the states x_1 and $x_5(\dot{x}_1)$ are chosen to find the regions of attraction while all other states are fixed at 0.5. It can be seen that use of the nonlinear controller greatly increases the region of attraction for the closed-loop system.

4.4.2 A 2-DOF Spring-Mass-Damper System

Fig. 4.5 shows a 2-DOF spring-mass-damper system with cubic stiffness terms. For this system the equation of motion in state-space form can be written as

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k)/m_1 & k/m_1 & -c_1/m_1 & 0 \\ k/m_2 & -(k_2+k)/m_2 & 0 & -c_2/m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \\ &+ \begin{Bmatrix} 0 \\ 0 \\ -b/m_1(x_2-x_1)^3 + b_1x_1^3 \\ -b/m_2(x_2-x_1)^3 + b_2x_2^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} u \end{aligned} \quad (4.39)$$

Typically, systems modeled by these equations may have very small damping and the main purpose of a controller in such cases is vibration suppression. However, since the problem under consideration involves stabilizing controller design some negative damping is introduced in the system, without any loss of generality. For some typical parameter values, $m_1 = m_2 = 1$, $k = 3$, $k_1 = 2$, $k_2 = 11$, $c_1 = -1$, $c_2 = 7$, $b = 4$, $b_1 = 2$, $b_2 = -3$, the eigenvalues of linear part of the equation are $0.4189 \pm 2.1203i$ and $-3.4189 \pm 1.1700i$. Therefore, the 2-DOF system is approximated by a single-DOF system; where, the states belonging to unstable eigenvalues are assumed to be the dominant states. The controller is still in the form of equation (4.10) but contains only time-invariant terms. By taking $l=1$, a 5^{th} order nonlinear controller is proposed. Linear control gains $\{K_1, K_2\} = \{-1.4617, -2.2102\}$ place the eigenvalues of the dominant states at -0.9 and -1 . After finding decoupling transformation between the dominant and non-dominant states by the methodology suggested earlier, the resulting reduced order model is obtained as

$$\begin{aligned}
\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} &= \begin{bmatrix} 0.4189 & -2.1203 \\ 2.1203 & 0.4189 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} -0.1160z_1^3 + 0.1011z_1^2z_2 + \dots \\ -0.3909z_1^3 + 0.3258z_1^2z_2 + \dots \end{Bmatrix} \\
&+ \left\{ \begin{aligned} &\left(0.0192 - 0.0248\alpha_{3,(3,0)} + 0.0186\beta_{2,(2,0)} + 0.0117\psi_{3,(2,1),1} + \dots \right) z_1^5 + \dots \\ &\left(0.0522 + 0.0189\alpha_{3,(0,3)} - 0.0225\beta_{2,(0,2)} - 0.0277\psi_{3,(0,3),1} + \dots \right) z_1^5 + \dots \end{aligned} \right\} \\
&+ \begin{Bmatrix} 0.2641 \\ 1.1545 \end{Bmatrix} (v + u_{nl})
\end{aligned} \tag{4.40}$$

It should be noted that for the present set of eigenvalues of the system, \mathbf{P} in equation (4.28) has a unique solution. Also, the system reducibility condition (4.31) is satisfied. In order to transform equation (4.40) into its equivalent linear form, a near-identity transformation of the form

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} h_{3,1,(3,0)}y_1^3 + h_{3,1,(2,1)}y_1^2y_2 + \dots \\ h_{3,2,(3,0)}y_1^3 + h_{3,2,(2,1)}y_1^2y_2 + \dots \end{Bmatrix} + \begin{Bmatrix} h_{5,1,(5,0)}y_1^5 + h_{5,1,(4,1)}y_1^4y_2 + \dots \\ h_{5,2,(5,0)}y_1^5 + h_{5,2,(4,1)}y_1^4y_2 + \dots \end{Bmatrix} \tag{4.41}$$

is applied. Substituting equation (4.41) into equation (4.40) and collecting the terms of like powers of \mathbf{y} result in PDEs similar to equations (4.26.1) and (4.26.2), which are time-invariant. Approximate solutions are obtained in the form of power series expansion. Once again, since the dominant eigenvalues are complex, the reduced order system does not exhibit the resonance and a nonlinear feedback and coordinate transformation (near-identity transformation) can be found to transform equation (4.40) into a dynamically equivalent linear equation. The corresponding of nonlinear feedback is obtained as

$$\begin{aligned}
u_{nl} &= 0.3660z_1^3 - 0.4400z_1^2z_2 + 0.2254z_1z_2^2 - 0.0336z_2^3 - 0.0244z_1^5 + \dots \\
&+ \left(0.0297z_1^2 - 0.0326z_1z_2 + 0.0215z_2^2 - 0.0034z_1^2 + \dots \right) v
\end{aligned} \tag{4.42}$$

The uncontrolled and controlled response of the system is shown in Fig. 4.7 and 4.8, respectively. It is evident from Fig. 4.8 that the duel controller performs better than the linear controller (v), alone, in terms of the speed of convergence to the steady state. Fig. 4.9 shows control efforts by the linear and duel controller. Again, the duel controller needs much less effort compared to the purely linear controller. Fig. 4.10 shows the region of attraction (Ω) for the two controllers on $x_1 \sim x_3(\dot{x}_1)$ plane for some fixed initial values of x_2 and x_4 . In this case too, the nonlinear controller has a greater region of attraction as compared to the linear controller.

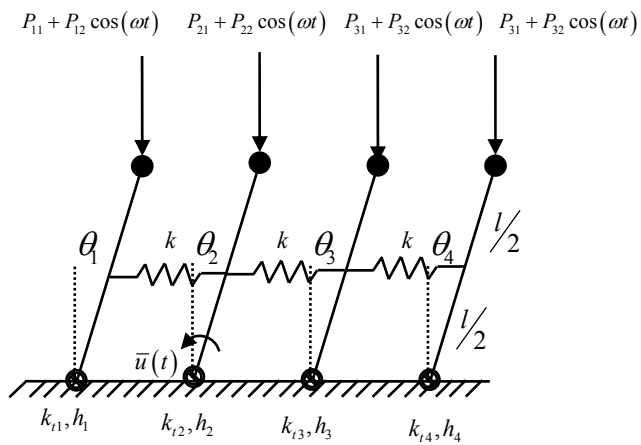


Fig. 4.1 A 4-DOF coupled inverted pendulums.

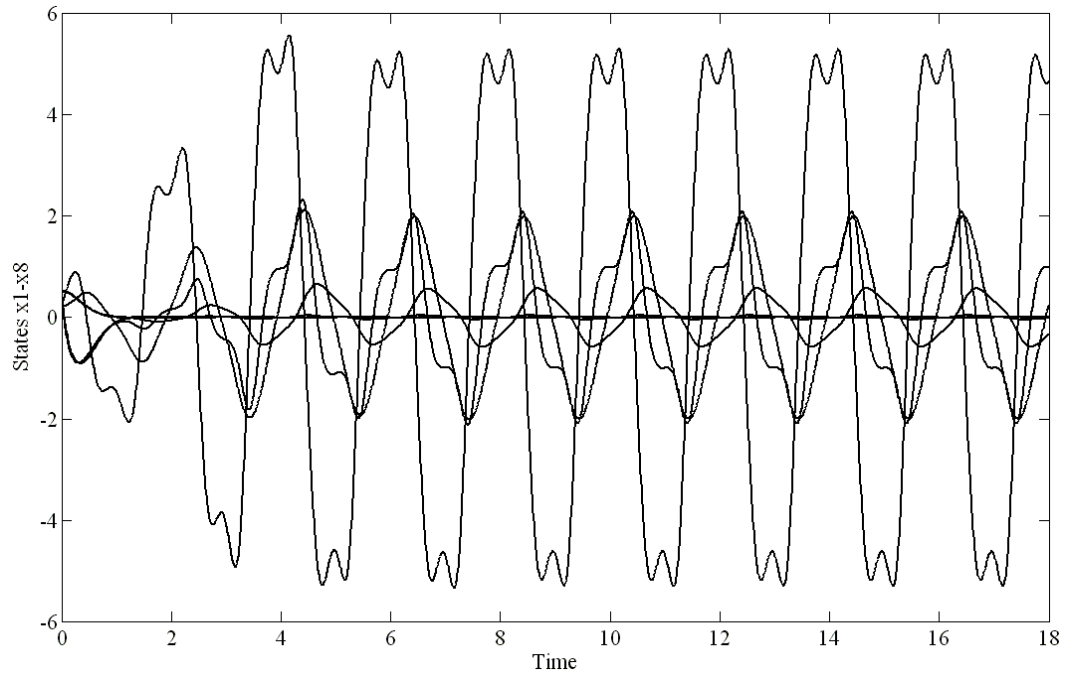


Fig. 4.2. Uncontrolled system response for system with periodic coefficients.

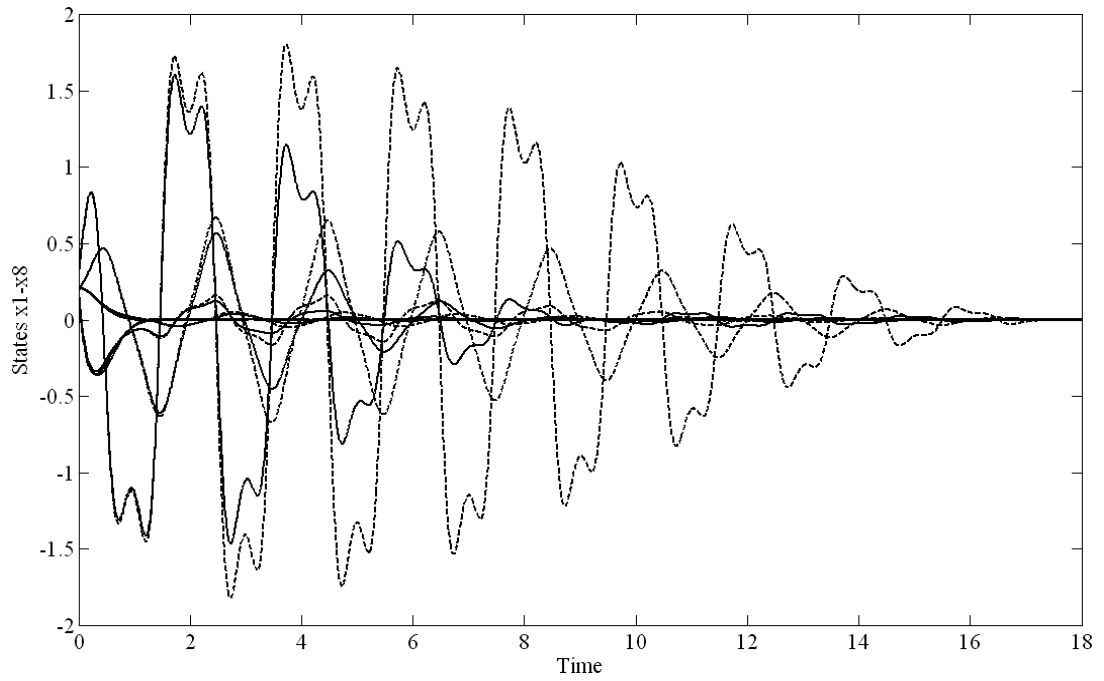


Fig. 4.3 Controlled system response for system with periodic coefficients.
—— , linear + nonlinear control; **.....** , linear control

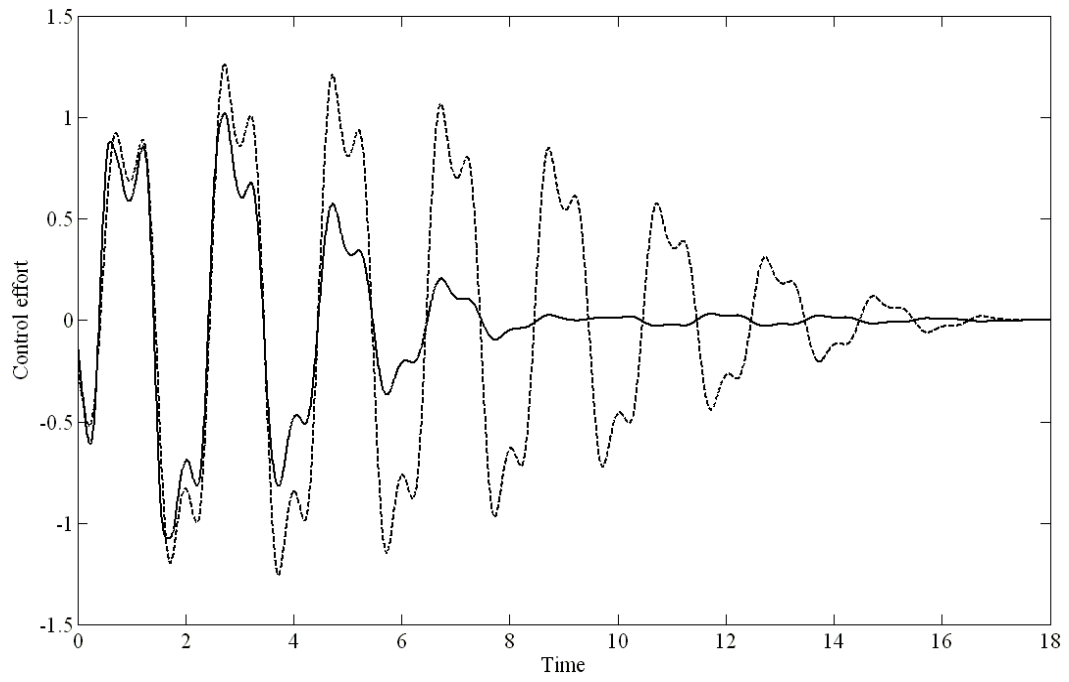


Fig. 4.4 Control effort for system with periodic coefficients.
—— , linear + nonlinear control; , linear control

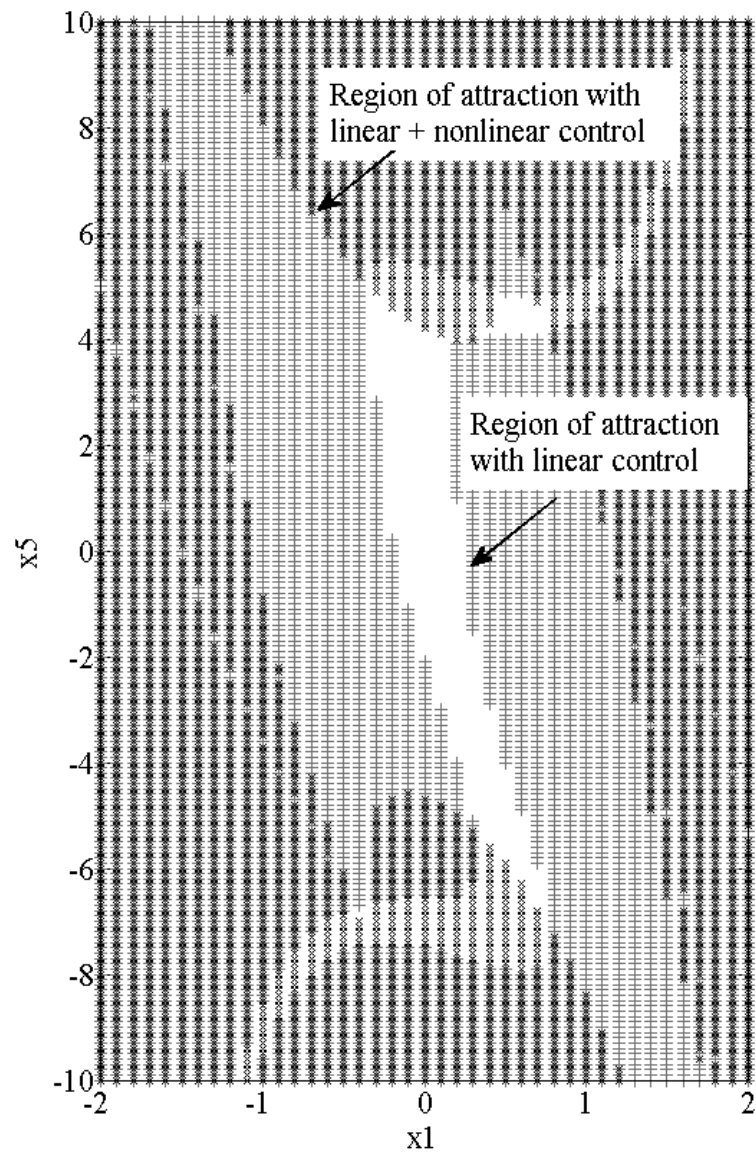


Fig. 4.5 Regions of attraction for controlled system (system with periodic coefficients).

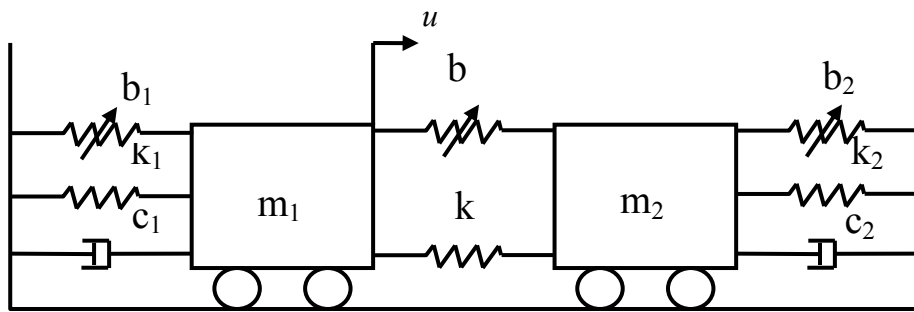


Fig. 4.6 A 2-DOF spring-mass-damper system.

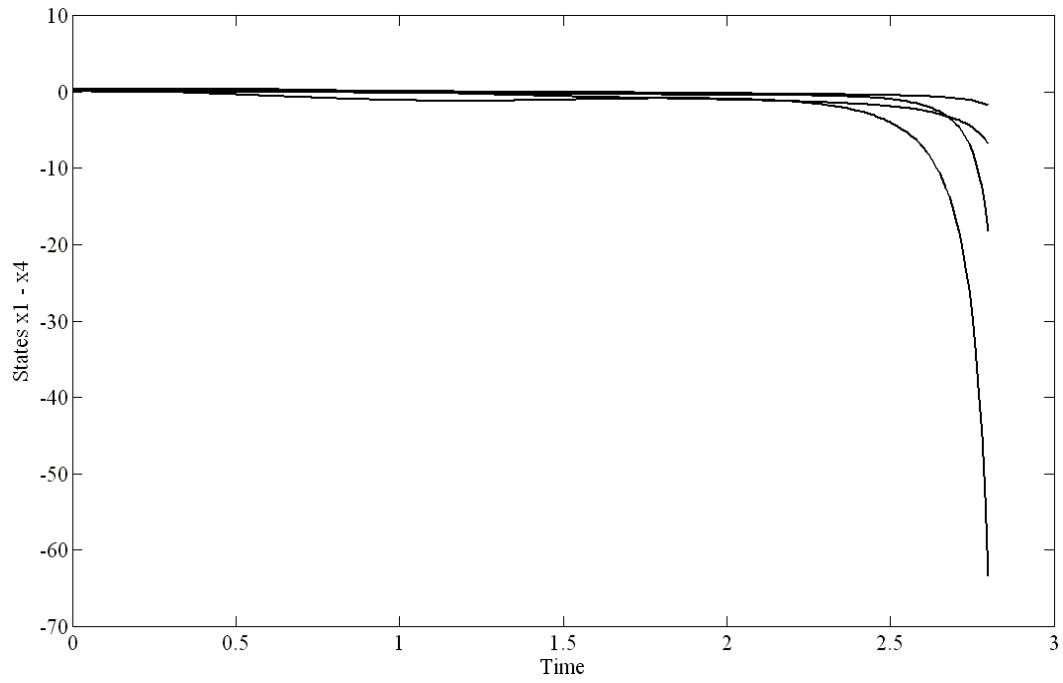


Fig. 4.7 Uncontrolled system response for system with constant coefficients.

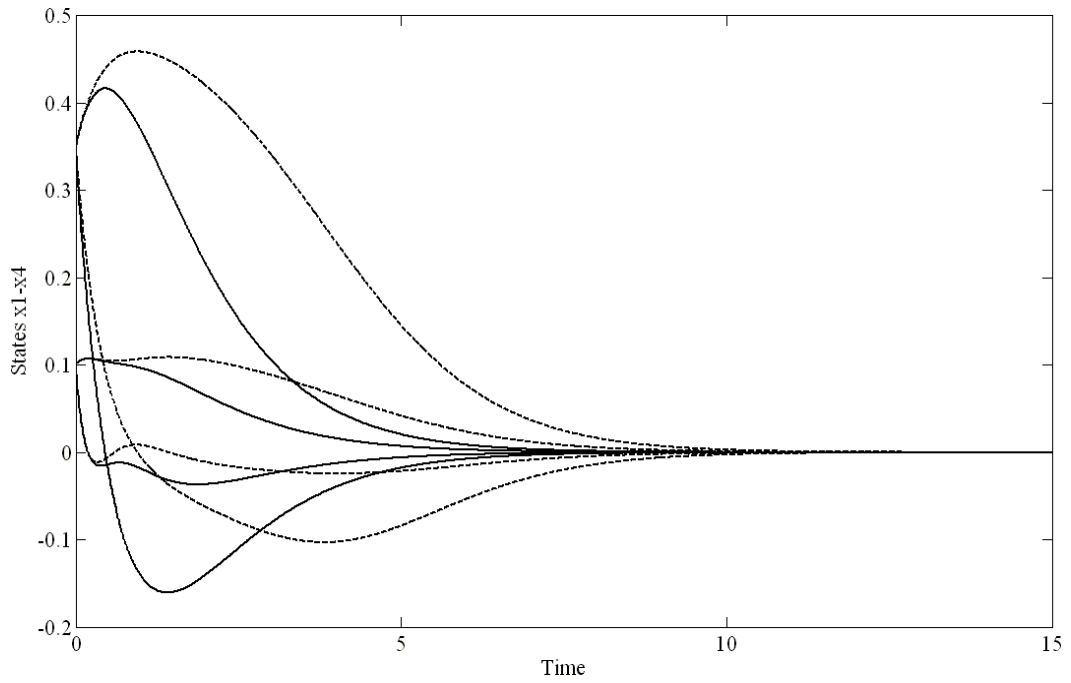


Fig. 4.8 Controlled system response for system with constant coefficients.
—— , linear + nonlinear control; **.....** , linear control

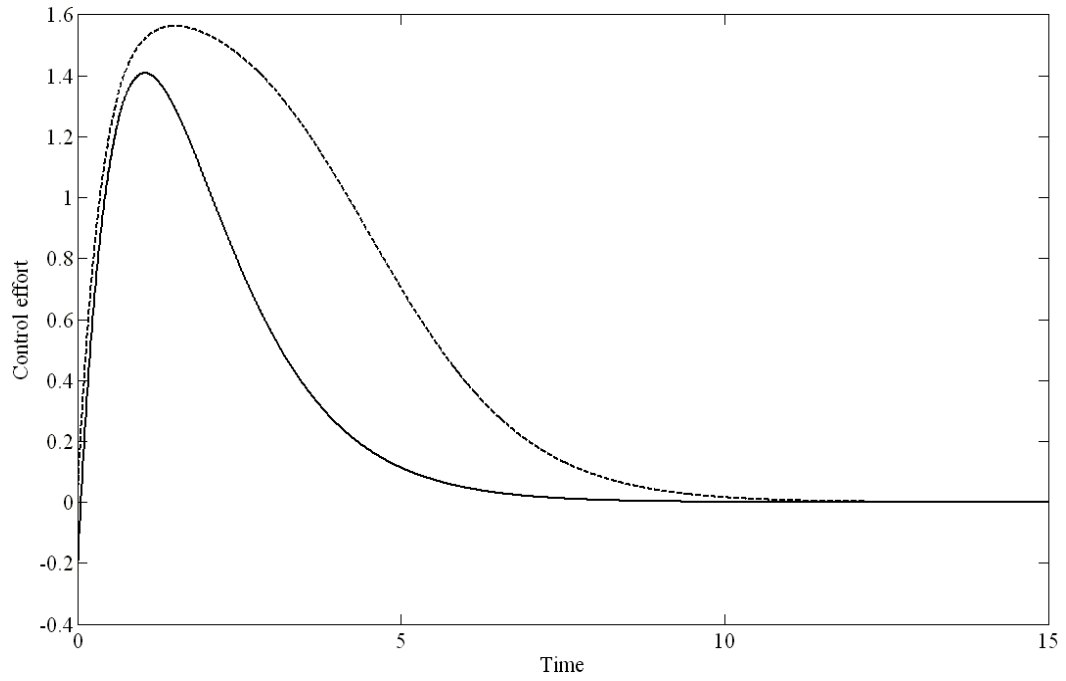


Fig. 4.9 Control effort for system with constant coefficients.
—— , linear + nonlinear control; **.....** , linear control

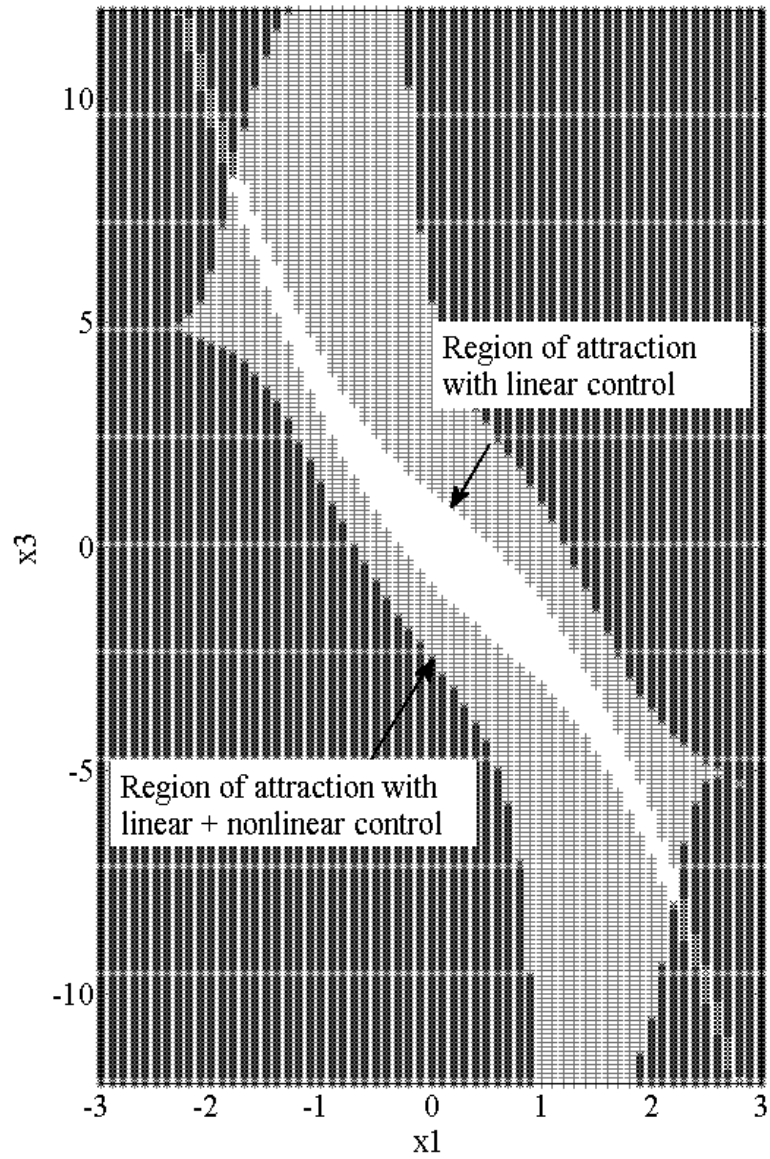


Fig. 4.10 Regions of attraction for controlled system (System with constant coefficients).

Chapter 5

Discussion and Conclusions

5.1 Summary of Work

This dissertation proposes some new methodologies for simplification of nonlinear dynamic systems such that the analysis and control problems can be carried out in a simpler and efficient way. The systems under consideration may include parameters that are constant or periodically excited and they are subjected to either external periodic excitations or stabilizing state feedback. The main focus of this work is on order reduction of nonlinear dynamical systems which is accomplished through the construction of invariant manifolds. Another important simplification technique presented in this work is a direct application of near-identity transformations and construction of the normal forms for nonlinear systems subjected to external periodic excitations. It is rather important to note that the methodologies developed here are equally applicable to nonlinear systems with constant as well as periodic coefficients. Moreover, for systems with periodic coefficients, the techniques are not limited by the size of time-periodic component which is a distinct advantage over the traditional approaches.

Chapter 2 presents a direct methodology for a quantitative analysis of nonlinear dynamic systems with external periodic forcing via an application of the theory of normal forms. Rather than introducing a new state variable to reduce the problem to a homogenous one, a set of time-dependent near-identity transformations is applied to construct the normal forms. In the process the total response of the system is expressed as superposition of a steady state solution and a

transient solution. A steady state solution of the system is obtained by the method of harmonic balance and the transient solution is obtained by solving a set of time periodic homological equations. After discussing the time-invariant case, the methodology is extended to systems with time-periodic coefficients. The case of time periodic systems is handled through an application of the L-F transformation. Application of the L-F transformation produces a dynamically equivalent system in which the linear part of the system is time-invariant, making the system amenable to near-identity transformations. An example for each type of system, *viz.*, constant coefficients and time-varying coefficients, is included to demonstrate effectiveness of the method. Various resonance conditions are discussed. It is observed that the linear parametric excitation term need not be small as generally assumed in perturbation and averaging techniques. Results obtained by proposed methods are compared with numerical solutions. Close agreements are found in some typical applications.

In Chapter 3, a methodology for determining reduced order models of periodically excited nonlinear systems is presented. The approach is based on the construction of an invariant manifold. Due to the existence of external and parametric periodic excitations, however, the geometry of the manifold varies with time. As a result, the manifold is constructed in terms of temporal and dominant state variables. The governing PDE for the manifold is nonlinear and contains time-varying coefficients. An approximate technique to find a solution of this PDE using a multivariable Taylor-Fourier series is suggested. It is shown that, in certain cases, it is possible to obtain various reducibility conditions in a closed form. The case of time-periodic systems is handled through the use of L-F transformation. Application of the L-F transformation produces a dynamically equivalent system in which the linear part of the system is time-invariant, however, the nonlinear terms get multiplied by a truncated Fourier series containing multiple parametric excitation frequencies. This warrants some structural changes in the proposed manifold, but the

solution procedure remains the same. Two examples; namely, a 2-DOF mass-spring-damper system and an inverted pendulum with periodic loads, are used to illustrate applications of the technique for systems with constant and periodic coefficients, respectively. Results show that the dynamics of these 2-DOF systems can be accurately approximated by equivalent 1-DOF systems using the proposed methodology.

Chapter 4 provides a methodology for reduced order stabilizing controller design for nonlinear dynamic systems. The equations of motion are represented by quasi-linear differential equations in state space, containing a time-periodic linear part and nonlinear monomials of states with periodic coefficients. The L-F transformation is used to transform the time-varying linear part of the system into a time-invariant form. Eigenvalue decomposition of the time-invariant linear part can then be used to identify the dominant/ non-dominant dynamics of the system. The non-dominant states of the system are expressed as a nonlinear, time-periodic, manifold relationship in terms of the dominant states. As a result, the original large system can be expressed as a lower order system represented only by the dominant states. A reducibility condition is derived to provide conditions under which a nonlinear order reduction is possible. Then a proper coordinate transformation and state feedback can be found under which the reduced order system is transformed into a linear, time-periodic, closed-loop system. This permits the design of a time-varying feedback controller in linear space to guarantee the stability of the system. The case of systems with constant coefficients is treated as a subset of its periodic counterpart. The proposed methodology is illustrated by designing a two dimensional reduced order controller for a 4-degrees of freedom (DOF), inverted pendulum subjected to a periodic force for which the equations of motion are time-periodic. An example involving a 2-DOF spring-mass-damper system is also presented which yields the equations of motion with constant coefficients.

In some sense, the order reduction technique presented here is a generalization of the center manifold theory. Similar to the center manifold theory, where stable states are expressed as nonlinear functions of critical states, in this approach the stable states are expressed as nonlinear time-periodic functions of the unstable states. From the results obtained by numerical simulation of the 4-DOF system with periodic coefficients, it is concluded that a satisfactory control system may be designed using a single-DOF model. Advantages of using the nonlinear controller over the linear controller are evident from the improved system response (in terms of time required for convergence to the steady state) and the lesser amount of control effort needed. It should also be noted from Figures 4.5 and 4.10 that the regions of attraction for the examples with proposed nonlinear controllers are much larger as compared to those with the linear controllers.

In summary, the main contributions and salient features of this work can be listed as follows:

- Development of a methodology for the construction of normal forms for nonlinear systems with parametric and external periodic excitations via direct application of near-identity transformations and the Poincaré normal forms theory.
- Development of an invariant manifold based methodology for the construction of reduced order models for nonlinear systems with parametric and external periodic excitations.
- Development of an invariant manifold based methodology for the construction of reduced order models for parametrically excited nonlinear systems subjected to a nonlinear stabilizing state feedback.

- For systems subjected to external periodic excitations, the proposed methodologies accurately capture interactions between external excitations and systems' nonlinearities.
- A transformation for the diagonalization of linear time-periodic systems with lower triangular linear matrices is proposed. The proposed transformation is governed by a time-periodic Sylvester differential equation. A methodology for the construction of a solution for this equation is also proposed.
- It is demonstrated that normal form equations and reduced order models derived using proposed methodologies capture the important dynamics of original systems accurately. Also, nonlinear controllers developed for reduced order models have distinct advantages over linear controllers.

Suggestions for Future Work

There are several opportunities to extend the work presented here beyond its current precincts and some of those are highlighted here.

- 1) In this work, reduced order models are obtained for small to medium order systems. Although techniques presented here are not limited by the size of a system (number of DOF), their effectiveness for large systems needs to be assessed.
- 2) Nonlinear dynamic systems with periodic-quasiperiodic coefficient appear naturally in various branches of science and engineering. For example, a continuous cantilever beam subjected to periodic axial and transverse load with dissimilar frequencies yields such equations. The techniques presented in this work can be adapted for simplified analysis and

controller design for such systems. Work by Wooden and Sinha (2007) and Redkar (2005) are few such examples which propose normal form and order reduction methodologies for periodic-quasiperiodic, homogeneous systems. Techniques presented can be employed for periodic-quasiperiodic systems subjected to external inputs.

- 3) Many mechanical and structural dynamic systems are usually modeled as a set of second order differential equations given by $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{F}(t)$. Where, \mathbf{M} , \mathbf{C} , \mathbf{K} are mass, damping and stiffness matrix, respectively and \mathbf{x} is the displacement vector. Rather than transforming these equations into a state-space form, developing order reduction techniques in second order form is more intuitive in nature. References Burton and Rhee (2000) and Deshmukh et al. (2006) are few such examples, among others, which use second order formulation for nonlinear order reduction. The class of problems considered in this work yet needs to be explored for second order formulation.
- 4) Reduced order controller design methodology proposed in the Chapter 5 assumes availability of all system's states. In practice, however, it may not be always possible to obtain all states of a dynamic system using direct measurements. Therefore, there is a great potential for designing reduced order observers which can estimate the desired states of a system.
- 5) The author believes that the work described here lays foundation for the development of a reduced order controller design methodology for systems subjected to periodic disturbances. In the case, if system is subjected to known periodic disturbance the techniques described in Chapter 3 and 4 can be used directly, in conjunction, with the reduced order controller design. However, in the case of unknown periodic disturbance one has to devise a methodology to estimate the disturbance. This could be a challenging task but has tremendous potential for applications to some real life control problems.

- 6) In the present work, regions of attraction for the controlled systems are obtained by actual numerical simulation of systems in the domain of interest. Development of an analytical technique for the estimation of such regions of attraction is something the future work should look into.

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Appendix A

Floquet Theory and Lyapunov-Floquet Transformation

A.1 Floquet Theory

Floquet theory is an important tool in predicting stability and response of linear differential equations with periodic coefficients. The idea is based on the fact that, once the solution of problem is known for the principal period, it is known for all time t . Stability of the system is determined by eigenvalues of the State Transition Matrix (STM) calculated at the end of one period. These eigenvalues are called Floquet multipliers of the system and the STM evaluated at the end of the principle period is called as Floquet Transition Matrix (FTM). Another important outcome of the Floquet theory is Lyapunov-Floquet (L-F) transformation which essentially gives methodology to convert time-periodic linear system into equivalent time independent linear system.

Consider a linear time-periodic system given as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \quad (\text{A.1})$$

where $n \times n$ matrix $\mathbf{A}(t)$ is a T periodic such that $\mathbf{A}(t+T) = \mathbf{A}(t)$. Let $\Phi(t)$ be the state transition matrix of (A.1) so that $\Phi(0) = \mathbf{I}$, where \mathbf{I} is the identity matrix. Then, according to the Floquet theory

1. $\Phi(t+T) = \Phi(t)\Phi(T)$ $0 \leq t \leq T$ and hence

$$2. \quad \Phi(t+nT) = \Phi(t)\Phi(T)^n \quad 0 \leq t \leq T, n = 2, 3, \dots$$

$$3. \quad \mathbf{x}(t) = \Phi(t)\mathbf{x}(0) \quad t \geq 0$$

This implies that if the solution is known over the principal period then it can be constructed for all time t .

Floquet theory also addresses the stability of linear time periodic system in terms of eigenvalues of FTM which are called characteristic multipliers. Let $\xi_k (k=1, \dots, n)$ denote eigenvalues of $\Phi(T)$. Then system (A.1) is asymptotically stable if all ξ_k lie inside the unit circle of the complex plane. The system is unstable if at least one of the eigenvalues of the FTM has magnitude greater than one. Alternatively, stability can also be expressed in terms of characteristic exponents of the system. If $\xi_k = \xi_{kR} + i\xi_{kI}$ are characteristic multipliers of the system, which, in general, are complex. Then, characteristic exponents of the system are defined as $\alpha_k \pm i\beta_k$ where

$$\begin{aligned} \alpha_k &= \frac{1}{T} \ln |\xi_k| \\ \beta_k &= \frac{1}{T} \tan^{-1} \left(\frac{\xi_{kI}}{\xi_{kR}} \right) \end{aligned} \tag{A.2}$$

The system is stable if $\alpha_k < 0$ for $k=1, 2, \dots, n$. The detailed treatment of Floquet theory can be found in Yakubovich and Starzhinskii (1975).

Another important result of Floquet theory is the Liapunov-Floquet theorem (Yakubovich and Starzhinskii 1975)

Theorem: Each $n \times n$ fundamental matrix $\Phi(t)$ of equation (A.1) can be written as the product of two $n \times n$ matrices as

$$\Phi(t) = \mathbf{Q}(t)e^{\mathbf{C}t} \quad (\text{A.3})$$

where $\mathbf{Q}(t)$ is T periodic and \mathbf{C} is a constant matrix. $\mathbf{Q}(t)$ and \mathbf{C} , in general, are complex.

Corollary-1: Each fundamental matrix $\Phi(t)$ can also be factored as

$$\Phi(t) = \mathbf{L}(t)e^{\mathbf{R}t} \quad (\text{A.4})$$

where $\mathbf{L}(t)$ is real and periodic with period $2T$ and \mathbf{R} is an appropriate real matrix.

Corollary-2: The Liapunov-Floquet transformation

$$\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t) \quad (\text{A.5})$$

reduces the original system (A.1) to

$$\dot{\mathbf{z}}(t) = \mathbf{C}\mathbf{z}(t) \quad (\text{A.6})$$

Moreover, the $2T$ periodic transformation

$$\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t) \quad (\text{A.7})$$

produces a real representation given by

$$\dot{\mathbf{z}}(t) = \mathbf{R}\mathbf{z}(t) \quad (\text{A.8})$$

However, though the Liapunov-Floquet transformation reduces a non-autonomous linear differential equation to an autonomous one, there is no simple way to construct either $\mathbf{Q}(t)$

or $\mathbf{L}(t)$; exception being to the case of commutative system for which there is analytical procedure to calculate L-F transformation matrix in close form (Lukes, 1982).

A.2 Calculation of STM and L-F Transformation Matrix

It has been shown by Sinha and associates (Sinha & Wu 1991; Sinha *et al.* 1993; Wu & Sinha 1994) that the STMs for linear periodic systems can be obtained in terms of the shifted Chebyshev polynomials of the first kind. This procedure is suitable for calculating STM for any general time-periodic matrix $\mathbf{A}(t)$. The STM is expressed in terms of time t and Liapunov-Floquet theory can be applied further to find the L-F transformation matrix for the system. The main idea of this technique is to expand the solution vector $\mathbf{x}(t)$ and time periodic matrix $\mathbf{A}(t)$ of the linear time-periodic system in terms of the shifted Chebyshev polynomials in the interval $[0, T]$. Then using recursive integration and multiplication properties of Chebyshev polynomials, original time periodic differential equation can be reduced to a set of algebraic equations with unknown Chebyshev coefficients which can be solved to find the solution of the system.

$\mathbf{x}(t)$ and $\mathbf{A}(t)$ are expanded in terms of Chebyshev polynomials in the interval $[0, T]$ as

$$\begin{aligned} \mathbf{x}_i(t) &\approx \sum_{r=0}^{m-1} b_r^i s_r^*(t) \equiv \mathbf{s}^{*T}(t) \mathbf{b}^i, \quad i = 1, 2, \dots, n \\ A_{ij}(t) &\approx \sum_{r=0}^{m-1} d_r^{ij} s_r^*(t) \equiv \mathbf{s}^{*T}(t) \mathbf{d}^{ij}, \quad i, j = 1, 2, \dots, n \end{aligned} \tag{A.9}$$

where

$\mathbf{b}^i = \{b_0^i \ b_1^i \ \dots \ b_{m-1}^i\}^T$, $\mathbf{d}^{ij} = \{d_0^{ij} \ d_1^{ij} \ \dots \ d_{m-1}^{ij}\}^T$, $\mathbf{s}^{*T}(t) = \{s_0^*(t) s_1^*(t) \ \dots \ s_{m-1}^*(t)\}$ and m is number of terms in Chebyshev expansion.

Here b_r^i are unknown expansion coefficients of $x_i(t)$, d_r^{ij} are known expansion coefficients of $A_{ij}(t)$ and $s_r^*(t)$ are the shifted Chebyshev polynomials of the first kind. Let Chebyshev polynomial matrix be defined as

$$\hat{\mathbf{S}}(t) = \mathbf{I} \otimes \mathbf{s}^{*T}(t) \quad (\text{A.10})$$

where \otimes represents Kronecker product, and \mathbf{I} is $(n \times n)$ identity matrix. Using equation (A.9) and (A.10), $\mathbf{x}(t)$ and $\mathbf{A}(t)$ can be written as

$$\mathbf{x}(t) = \hat{\mathbf{S}}(t)\bar{\mathbf{b}}; \quad \mathbf{A}(t) = \hat{\mathbf{S}}(t)\mathbf{D} \quad (\text{A.11})$$

and the product

$$\mathbf{A}(t)\mathbf{x}(t) = \hat{\mathbf{S}}(t)\bar{\mathbf{Q}}\bar{\mathbf{b}} \quad (\text{A.12})$$

where $\bar{\mathbf{b}} = \{\mathbf{b}^1 \mathbf{b}^2 \mathbf{b}^3 \dots \mathbf{b}^n\}^T$, $\mathbf{D} = \{\mathbf{d}^{i1} \mathbf{d}^{i2} \mathbf{d}^{i3} \dots \mathbf{d}^{ij}\}$ $i, j = 1, 2, 3 \dots 2n$ and matrix $\bar{\mathbf{Q}}$ is a $(nm \times nm)$ product operation matrix. $\bar{\mathbf{Q}}$ is given as

$$\bar{\mathbf{Q}} = \begin{bmatrix} d_0 & d_1/2 & d_2/2 & \cdot & \cdot & d_{m-1}/2 \\ d_1 & d_0 + d_2/2 & (d_1 + d_3)/2 & \cdot & \cdot & (d_{m-2} + d_m)/2 \\ d_2 & (d_1 + d_3)/2 & d_0 + d_4/2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{m-1} & (d_{m-2} + d_m)/2 & \cdot & \cdot & \cdot & d_0 + d_{2m-2}/2 \end{bmatrix}$$

The detailed development of this matrix can be found in Wu (1991). The integral form of the equation (A.1) can be written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{A}(\tau)\mathbf{x}(\tau)d\tau \quad (\text{A.13})$$

Substituting equation (A.11) and (A.12) into this equation and using recursive integral properties of Chebyshev polynomials Sinha and Wu (1991) have shown that one can obtain a set of linear algebraic equations of the form

$$[\mathbf{I} - \mathbf{Z}]\bar{\mathbf{b}} = \mathbf{x}(0) \quad (\text{A.14})$$

where $\bar{\mathbf{b}}$ is a set of unknown coefficients as defined before, matrix \mathbf{Z} is a constant matrix given by $\mathbf{Z} = [\mathbf{A}_0 \otimes \mathbf{G}^T + \mathbf{C}_A \otimes \mathbf{G}^T \bar{\mathbf{Q}}]$ and

$$G^T = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \cdot & \cdot & 0 \\ -1/8 & 0 & 1/8 & 0 & \cdot & \cdot & 0 \\ -1/6 & -1/4 & 0 & 1/12 & \cdot & \cdot & \cdot \\ 1/16 & 0 & -1/8 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{4(m-1)} \\ \frac{(-1)^m}{2m(m-2)} & \cdot & \cdot & \cdot & \cdot & \frac{-1}{4(m-2)} & 0 \end{bmatrix}$$

Above equation can be solved for $\bar{\mathbf{b}}$ and hence solution $\mathbf{x}(t)$ can be found out by equation (A.11).

State transition matrix $\Phi(t)$ for the given linear system is given by set of solutions for n initial conditions as $\mathbf{x}_i(0) = (1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)$. It can be seen that all $\bar{\mathbf{b}}_i$'s corresponding to the above set of initial conditions can be determined simultaneously by defining

right hand side of the equation (A.14) in the matrix form. Then the state transition matrix is given by

$$\Phi(t) = \hat{S}(t)\bar{\mathbf{B}} \quad (\text{A.15})$$

where $\bar{\mathbf{B}} = [\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_n]$. It should be noted here that this STM is valid only over the interval $0 \leq t \leq T$ as the shifted Chebyshev polynomials are defined over the interval $[0, T]$. When $t > T$, the STM can be evaluated using Floquet theory as explained earlier. Once $\Phi(t)$ is known, the T -periodic complex matrix $\mathbf{Q}(t)$ or the $2T$ -periodic real matrix $\mathbf{L}(t)$ can be computed as follows (Arrowsmith & Place, 1990). Since $\Phi(t) = \mathbf{I}$, $\mathbf{Q}(0) = \mathbf{Q}(T) = \mathbf{I}$, the Floquet transition matrix can be written as

$$\Phi(T) = e^{\mathbf{C}T} \quad (\text{A.16})$$

where \mathbf{C} is a $(n \times n)$ constant complex matrix which can be computed by performing eigenvalue analysis on $\Phi(T)$ and T -periodic L-F transformation matrix is given as

$$\mathbf{Q}(t) = \Phi(t)e^{-\mathbf{C}t} \quad (\text{A.17})$$

It is also to be noted that (Yakubovich and Starzhinskii, 1975)

$$\Phi(2T) = \Phi^2(T) = e^{\mathbf{C}T} e^{\mathbf{C}^*T} = e^{2\mathbf{R}T} \quad (\text{A.18})$$

where \mathbf{C}^* is conjugate matrix of \mathbf{C} and $\mathbf{R} = \{\mathbf{C} + \mathbf{C}^*\}/2$. Then $2T$ periodic L-F transformation matrix is given as

$$\begin{aligned}\mathbf{L}(t) &= \mathbf{\Phi}(t)e^{-\mathbf{R}t}; & 0 \leq t \leq T \\ \mathbf{L}(\tau + T) &= \mathbf{\Phi}(t)\mathbf{L}(T)e^{-\mathbf{R}\tau}; & T \leq (T + \tau) \leq 2T; 0 \leq \tau \leq T\end{aligned}\tag{A.19}$$

When it is necessary to find $\mathbf{Q}^{-1}(t)$ and $\mathbf{L}^{-1}(t)$ one can possibly invert $\mathbf{Q}(t)$ and $\mathbf{L}(t)$ through symbolic software like MATHEMATICA/MAPLE or one can first find the STM $\mathbf{\Psi}(t)$ of the adjoint system

$$\dot{\mathbf{w}} = -\mathbf{A}^T(t)\mathbf{w}\tag{A.20}$$

and use the following relationship (Yakubovich and Starzhinskii, 1975)

$$\mathbf{\Phi}^{-1}(t) = \mathbf{\Psi}^T(t)\tag{A.21}$$

Knowing $\mathbf{\Phi}^{-1}(t)$, $\mathbf{Q}^{-1}(t)$ and $\mathbf{L}^{-1}(t)$ can be computed using properties of adjoint system. e.g.

$\mathbf{Q}^{-1}(t)$ can be computed as

$$\mathbf{Q}^{-1}(t) = [\mathbf{\Phi}(t)e^{-\mathbf{C}t}]^{-1} = e^{\mathbf{C}t}\mathbf{\Phi}^{-1}(t) = e^{\mathbf{C}t}\mathbf{\Psi}^T(t)\tag{A.22}$$

The technique described here has been successfully employed by Pandiyan *et al.* (1993) and Bibb (1992) to compute L-F transformation matrices. Such approximation of L-F transformation matrix has been found to be extremely convergent and since L-F transformation matrix is periodic, the elements \mathbf{Q}_{ij} and \mathbf{L}_{ij} has truncated Fourier series representation as

$$\begin{aligned}\mathbf{Q}_{ij} &\approx \sum_{n=-q}^q c_n e^{i2\pi nt/T}, \quad i = \sqrt{-1} \\ \mathbf{L}_{ij} &\approx \frac{a_0}{2} + \sum_{n=1}^q a_n \cos \frac{\pi nt}{T} + \sum_{n=1}^q b_n \sin \frac{\pi nt}{T}\end{aligned}\tag{A.23}$$

\mathbf{Q}_{ij}^{-1} and \mathbf{L}_{ij}^{-1} has similar Fourier series representation.

Appendix B

Solution of Time-periodic Sylvester Differential Equation

Time-periodic Sylvester differential equation presented in Chapter 4 has a very special structure such that if the equation is written in the expanded form, then the states in each row of the equation decouple from the states in other rows. Moreover, each decoupled equation has the same time-periodic linear matrix. Due to these properties, construction of a solution for this equation is simple and computationally efficient. One such technique based on an application of the L-F transformation and use of convolution integral is presented in the following.

Consider a time-periodic Sylvester differential equation given by equation (4.14):

$$\dot{\mathbf{P}}(t) = \mathbf{J}_q \mathbf{P}(t) - \mathbf{P}(t) (\mathbf{J}_p + \mathbf{g}_{p,0}(t) \mathbf{K}_p(t)) + \mathbf{g}_{q,0}(t) \mathbf{K}_p(t) \quad (\text{B.1})$$

It is desirable to find a closed form solution $\mathbf{P}(t)$ for the above equation. Typically, the above equation can be written in the component form as

$$\begin{aligned}
\begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} & \dots & \dot{p}_{1p} \\ \dot{p}_{21} & \dot{p}_{22} & \dots & \dot{p}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{p}_{q1} & \dot{p}_{q2} & \dots & \dot{p}_{qp} \end{bmatrix} &= \begin{bmatrix} \lambda_{q1} & 0 & 0 & 0 \\ 0 & \lambda_{q2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{qq} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1p} \\ p_{21} & p_{22} & \dots & p_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ p_{q1} & p_{q2} & \dots & p_{qp} \end{bmatrix} \\
&- \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1p} \\ p_{21} & p_{22} & \dots & p_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ p_{q1} & p_{q2} & \dots & p_{qp} \end{bmatrix} \begin{bmatrix} \lambda_{p1} + B_{p1}k_1 & B_{p1}k_2 & \dots & B_{p1}k_p \\ B_{p2}k_1 & \lambda_{p2} + B_{p2}k_2 & \dots & B_{p2}k_p \\ \vdots & \vdots & \ddots & \vdots \\ B_{pp}k_1 & B_{pp}k_2 & \dots & \lambda_{pp} + B_{pp}k_p \end{bmatrix} \\
&+ \begin{bmatrix} B_{q1}k_1 & B_{q1}k_2 & \dots & B_{q1}k_p \\ B_{q2}k_1 & B_{q2}k_2 & \dots & B_{q2}k_p \\ \vdots & \vdots & \ddots & \vdots \\ B_{qq}k_1 & B_{qq}k_2 & \dots & B_{qq}k_p \end{bmatrix}
\end{aligned} \tag{B.2}$$

where \dot{p}_{qp} and p_{qp} are the elements of $\dot{\mathbf{P}}(t)$ and $\mathbf{P}(t)$, respectively. λ_{qq} are eigenvalues of Jordan block \mathbf{J}_q , λ_{pp} are eigenvalues of Jordan block \mathbf{J}_p , B_{qj} are elements of the control influence matrix corresponding to the non-dominant states ($\mathbf{g}_{q,0}(t)$), B_{pj} are elements of the control influence matrix corresponding to the dominant states ($\mathbf{g}_{p,0}(t)$) and k_p are the elements of control gain vector $\mathbf{K}_p(t)$.

After multiplication equation (B.2) yields

$$\begin{aligned}
\begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} & \dots & \dot{p}_{1p} \\ \dot{p}_{21} & \dot{p}_{22} & \dots & \dot{p}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{p}_{q1} & \dot{p}_{q2} & \dots & \dot{p}_{qp} \end{bmatrix} &= \begin{bmatrix} \lambda_{q1}P_{11} & \lambda_{q1}P_{12} & \dots & \lambda_{q1}P_{1p} \\ \lambda_{q2}P_{21} & \lambda_{q2}P_{22} & \dots & \lambda_{q2}P_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{qq}P_{q1} & \lambda_{qq}P_{q2} & \dots & \lambda_{qq}P_{qp} \end{bmatrix} \\
&- \begin{bmatrix} p_{11}(\lambda_{p1} + B_{p1}k_1) + p_{12}B_{p2}k_1 + \dots + p_{1p}B_{pp}k_1 & \dots \\ \vdots & \ddots \\ p_{q1}(\lambda_{p1} + B_{p1}k_1) + p_{q2}B_{p2}k_1 + \dots + p_{qp}B_{pp}k_1 & \dots \end{bmatrix} \\
&\dots \begin{bmatrix} p_{11}B_{p1}k_p + p_{12}B_{p2}k_p + \dots + p_{1p}(\lambda_{pp} + B_{pp}k_p) \\ \vdots \\ p_{q1}B_{p1}k_p + p_{q2}B_{p2}k_p + \dots + p_{qp}(\lambda_{pp} + B_{pp}k_p) \end{bmatrix} \\
&+ \begin{bmatrix} B_{q1}k_1 & B_{q1}k_2 & \dots & B_{q1}k_p \\ B_{q2}k_1 & B_{q2}k_2 & \dots & B_{q2}k_p \\ \vdots & \vdots & \ddots & \vdots \\ B_{qq}k_1 & B_{qq}k_2 & \dots & B_{qq}k_p \end{bmatrix} \tag{B.3}
\end{aligned}$$

It can be noticed that rows of equation (B.3) are decoupled and each row forms a set of coupled differential equations given by

$$\begin{aligned}
\begin{Bmatrix} \dot{p}_{11} \\ \dot{p}_{12} \\ \vdots \\ \dot{p}_{1p} \end{Bmatrix} &= \left(\begin{bmatrix} \lambda_{q1} & 0 & \dots & 0 \\ 0 & \lambda_{q1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{q1} \end{bmatrix} - \begin{bmatrix} \lambda_{p1} + B_{p1}k_1 & B_{p2}k_1 & \dots & B_{pp}k_1 \\ B_{p1}k_2 & \lambda_{p2} + B_{p2}k_2 & \dots & B_{pp}k_2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{p1}k_p & B_{p2}k_p & \dots & \lambda_{pp} + B_{pp}k_p \end{bmatrix} \right) \begin{Bmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{1p} \end{Bmatrix} \\
&+ \begin{Bmatrix} B_{q1}k_1 \\ B_{q1}k_2 \\ \vdots \\ B_{q1}k_p \end{Bmatrix} \tag{B.4.1}
\end{aligned}$$

\vdots
\vdots

$$\begin{aligned} \begin{Bmatrix} \dot{p}_{q1} \\ \dot{p}_{q2} \\ \vdots \\ \dot{p}_{qp} \end{Bmatrix} &= \begin{bmatrix} \lambda_{qq} & 0 & \dots & 0 \\ 0 & \lambda_{qq} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{qq} \end{bmatrix} - \begin{bmatrix} \lambda_{p1} + B_{p1}k_1 & B_{p2}k_1 & \dots & B_{pp}k_1 \\ B_{p1}k_2 & \lambda_{p2} + B_{p2}k_2 & \dots & B_{pp}k_2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{p1}k_p & B_{p2}k_p & \dots & \lambda_{pp} + B_{pp}k_p \end{bmatrix} \begin{Bmatrix} p_{q1} \\ p_{q2} \\ \vdots \\ p_{qp} \end{Bmatrix} \\ &+ \begin{Bmatrix} B_{qq}k_1 \\ B_{qq}k_2 \\ \vdots \\ B_{qq}k_p \end{Bmatrix} \end{aligned} \quad (\text{B.4.q})$$

Above set of equations is of the form

$$\dot{\mathbf{p}}_i = [\mathbf{A}_i + \bar{\mathbf{A}}(t)]\mathbf{p} + \mathbf{B}_i\mathbf{K}_p(t) \quad (\text{B.5})$$

where \mathbf{A}_i ($i=1,2,\dots,q$) is a diagonal matrix with identical constant elements, $\bar{\mathbf{A}}(t)$ is a time-periodic matrix, which remains same for all equations in the set. $\mathbf{B}_i\mathbf{K}_p(t)$ ($i=1,2,\dots,p$) is a periodic forcing term. This time-periodic equation can be solved using the L-F transformation and convolution integral. Since \mathbf{A}_i is a diagonal matrix (with identical constant elements) the state transition matrix for equation (B.5) is given as $\Phi(t) = e^{A_i t} \bar{\Phi}$, where $\bar{\Phi}(t)$ is the state transition matrix corresponding to $\bar{\mathbf{A}}(t)$. By using the method suggested by Sinha et al. (1996), one can compute $\bar{\Phi}(t)$ and the L-F transformation matrix $\mathbf{L}(t)$. The L-F transformation matrix $\mathbf{L}(t)$ together with the modal transformation (\mathbf{M}), transforms equation (B.5) into the form

$$\dot{\bar{\mathbf{p}}}_i = \mathbf{J}_i \bar{\mathbf{p}}_i + \mathbf{G}_i(t) \quad (\text{B.6})$$

where \mathbf{J}_i is in the Jordan canonical form, $\mathbf{G}_i(t) = \mathbf{M}^{-1}\mathbf{L}^{-1}\mathbf{B}_i\mathbf{K}_p$. Here, it should be noted that $\bar{\Phi}(t)$ is computed only once which makes this solution procedure computationally less demanding. Using the convolution integral, the solution of equation (B.6) is given as

$$\bar{\mathbf{p}}_i = e^{\mathbf{J}_i t} \mathbf{p}_i(0) + \int_0^t e^{\mathbf{J}_i(t-\tau)} \mathbf{G}_i(\tau) d\tau \quad (\text{B.7})$$

Expressing $\mathbf{G}_i(t)$ in terms of a finite Fourier series as

$$\mathbf{G}_i(t) = \sum_{j=1}^p \sum_{l=-k}^k c_{j,l} e^{il\omega t} \mathbf{e}_j \quad (\text{B.8})$$

(where $i = \sqrt{-1}$, ω is principle frequency of forcing $\mathbf{G}_i(t)$ and \mathbf{e}_j is j^{th} member of natural basis), the solution of integral (B.7) (with $\mathbf{p}(0) \equiv 0$) can be written as

$$\bar{\mathbf{p}}_i = \sum_{j=1}^p \sum_{l=-k}^k c_{j,l} \frac{e^{il\omega t}}{il\omega - \bar{\lambda}_j} \mathbf{e}_j - \sum_{j=1}^p \sum_{l=-k}^k c_{j,l} \frac{e^{\bar{\lambda}_j t}}{il\omega - \bar{\lambda}_j} \mathbf{e}_j \quad (\text{B.9})$$

where $\bar{\lambda}_j$; $j=1,2,\dots,p$ are the eigenvalues of \mathbf{J}_i . It can be seen from equation (B.9) that $\bar{\mathbf{p}}_i$ can not be determined if any of the eigenvalues of \mathbf{J}_i have positive real part (unstable case) or if $\bar{\lambda}_j = ik\omega$.

Appendix C

Computer Codes

- C.1 Mathematica code (Finding Q and R.nb) for computation of $2T$ periodic L-F transformation matrix $(\mathbf{L}(t))$ and its inverse $(\mathbf{L}^{-1}(t))$ for a 4-DOF system with periodic coefficients.
- C.2 Mathematica code containing supporting functions required by “Finding Q and R.nb”.
- C.3 Mathematica code to compute the near-identity transformation for the forced Mathieu-Duffing equation.
- C.4 Mathematica code to find the harmonic balance solution for a 1-DOF parametrically excited nonlinear system.
- C.5 Mathematica code to compute the dynamic response of a system using its normal form equation. The code also compares this response with the original system response.
- C.6 Mathematica code to construct the invariant manifold for a 2-DOF parametrically excited nonlinear system assuming $\mathbf{F}_p(t)$ to be large.
- C.7 Mathematica code to construct the invariant manifold for a 2-DOF parametrically excited nonlinear system assuming $\mathbf{F}_p(t)$ to be small.

- C.8 Mathematica code to compute the dynamic response of a system using the reduced order equation. The code also compares a reduced order system response with a full order system response.
- C.9 MATLAB code to find Fast Fourier Transform of a steady state periodic solution.
- C.10 Mathematica code to design a nonlinear reduced order controller for a 4-DOF parametrically excited system.
- C.11 Mathematica code to find shifted Chebyshev coefficients of the first kind for a periodic function.
- C.12 Mathematica code (Symbolic FTM.nb) to find the FTM, symbolically.
- C.13 Mathematica code containing supporting functions required by “Symbolic FTM.nb”.
- C.14 Mathematica code to find stabilizing feedback gains for a 1-DOF linear closed-loop system with periodic coefficients.
- C.15 Mathematica code to find a solution for the time-periodic Sylvester differential equation.
- C.16 Mathematica code to find an inverse of a polynomial containing 3^{rd} and 5^{th} degree monomials with periodic coefficients.
- C.17 Mathematica code to compute the system response a 4-DOF (full order) system subjected to the reduced order controller.

C.1

```
(* File name "Finding R and Q.nb *)
(* Program to find 2T periodic L-
F transformation matrix and its innvers for for a 4-DOF time-periodic system *)
(* Execute file "LFtransform_functions.nb" before
execution of this set of functions *)

a1 = 1.0; a2 = 20; a3 = 21; a4 = 19; b1 = 16.5; b2 = 2; b3 = 1;
b4 = 2.5; c1 = 1; c2 = 1; c3 = 1; h1 = 0.05; h2 = 7; h3 = 7.5; h4 = 7;

A = {{0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0},
      {0, 0, 0, 0, 0, 0, 0, 1}, {-(a1 - b1 Cos[2 π t]), c1, 0, 0, -h1, 0, 0, 0},
      {c1, -(a2 - b2 Cos[2 π t]), c2, 0, 0, -h2, 0, 0}, {0, c2, -(a3 - b3 Cos[2 π t]),
      c3, 0, 0, -h3, 0}, {0, 0, c3, -(a4 - b4 Cos[2 π t]), 0, 0, 0, -h4}};

t0 = TimeUsed[];

chcoef = {0.30424, 0, 0.97087, 0, -0.30285, -1.1369`*^-013, 0.029092, 0,
          -0.0013922, -6.9122`*^-011, 4.019`*^-005, -4.7148`*^-009, -7.6044`*^-007,
          2.9802`*^-008, -4.7684`*^-007, 3.3379`*^-006, 1.1444`*^-005};
(* chcoef:= CHEbyshev COEFicients of periodic terms *)

n = 8; (* # states *)
m = 18; (* # Chebyshev coefficients in the expansion *)

fGT[m];
fGG[Transpose[MGp], m, n];
tensorP := Array[tP, {n, n, m}];

Do[tP[1, 1, k] = 0, {k, 1, m}]; Do[tP[1, 2, k] = 0, {k, 1, m}];
Do[tP[1, 3, k] = 0, {k, 1, m}];
Do[tP[1, 4, k] = 0, {k, 1, m}];
tP[1, 5, 1] = 1; Do[tP[1, 5, k] = 0, {k, 2, m}];
Do[tP[1, 6, k] = 0, {k, 1, m}];
Do[tP[1, 7, k] = 0, {k, 1, m}]; Do[tP[1, 8, k] = 0, {k, 1, m}];

Do[tP[2, 1, k] = 0, {k, 1, m}]; Do[tP[2, 2, k] = 0, {k, 1, m}];
Do[tP[2, 3, k] = 0, {k, 1, m}];
Do[tP[2, 4, k] = 0, {k, 1, m}]; Do[tP[2, 5, k] = 0, {k, 1, m}];
tP[2, 6, 1] = 1; Do[tP[2, 6, k] = 0, {k, 2, m}];
Do[tP[2, 7, k] = 0, {k, 1, m}]; Do[tP[2, 8, k] = 0, {k, 1, m}];

Do[tP[3, 1, k] = 0, {k, 1, m}]; Do[tP[3, 2, k] = 0, {k, 1, m}];
Do[tP[3, 3, k] = 0, {k, 1, m}];
Do[tP[3, 4, k] = 0, {k, 1, m}]; Do[tP[3, 5, k] = 0, {k, 1, m}];
Do[tP[3, 6, k] = 0, {k, 1, m}];
tP[3, 7, 1] = 1; Do[tP[3, 7, k] = 0, {k, 2, m}];
Do[tP[3, 8, k] = 0, {k, 1, m}];
```

```

Do[tP[4, 1, k] = 0, {k, 1, m}]; Do[tP[4, 2, k] = 0, {k, 1, m}];
Do[tP[4, 3, k] = 0, {k, 1, m}];
Do[tP[4, 4, k] = 0, {k, 1, m}]; Do[tP[4, 5, k] = 0, {k, 1, m}];
Do[tP[4, 6, k] = 0, {k, 1, m}];
Do[tP[4, 7, k] = 0, {k, 1, m}];
tP[4, 8, 1] = 1; Do[tP[4, 8, k] = 0, {k, 2, m}];

tP[5, 1, 1] = -(a1 - b1 * chcoef[[1]]); Do[tP[5, 1, k] = b1 * chcoef[[k]], {k, 2, m}];
tP[5, 2, 1] = c1; Do[tP[5, 2, k] = 0, {k, 2, m}];
Do[tP[5, 3, k] = 0, {k, 1, m}]; Do[tP[5, 4, k] = 0, {k, 1, m}];
tP[5, 5, 1] = -h1; Do[tP[5, 5, k] = 0, {k, 2, m}];
Do[tP[5, 6, k] = 0, {k, 1, m}];
Do[tP[5, 7, k] = 0, {k, 1, m}]; Do[tP[5, 8, k] = 0, {k, 1, m}];

tP[6, 1, 1] = c1; Do[tP[6, 1, k] = 0, {k, 2, m}];
tP[6, 2, 1] = -(a2 - b2 * chcoef[[1]]); Do[tP[6, 2, k] = b2 * chcoef[[k]], {k, 2, m}];
tP[6, 3, 1] = c2; Do[tP[6, 3, k] = 0, {k, 2, m}];
Do[tP[6, 4, k] = 0, {k, 1, m}]; Do[tP[6, 5, k] = 0, {k, 1, m}];
tP[6, 6, 1] = -h2; Do[tP[6, 6, k] = 0, {k, 2, m}];
Do[tP[6, 7, k] = 0, {k, 1, m}]; Do[tP[6, 8, k] = 0, {k, 1, m}];

Do[tP[7, 1, k] = 0, {k, 1, m}];
tP[7, 2, 1] = c2; Do[tP[7, 2, k] = 0, {k, 2, m}];
tP[7, 3, 1] = -(a3 - b3 * chcoef[[1]]); Do[tP[7, 3, k] = b3 * chcoef[[k]], {k, 2, m}];
tP[7, 4, 1] = c3; Do[tP[7, 4, k] = 0, {k, 2, m}];
Do[tP[7, 5, k] = 0, {k, 1, m}]; Do[tP[7, 6, k] = 0, {k, 1, m}];
tP[7, 7, 1] = -h3; Do[tP[7, 7, k] = 0, {k, 2, m}];
Do[tP[7, 8, k] = 0, {k, 1, m}];

Do[tP[8, 1, k] = 0, {k, 1, m}]; Do[tP[8, 2, k] = 0, {k, 1, m}];
tP[8, 3, 1] = c3; Do[tP[8, 3, k] = 0, {k, 2, m}];
tP[8, 4, 1] = -(a4 - b4 * chcoef[[1]]); Do[tP[8, 4, k] = b4 * chcoef[[k]], {k, 2, m}];
Do[tP[8, 5, k] = 0, {k, 1, m}];
Do[tP[8, 6, k] = 0, {k, 1, m}]; Do[tP[8, 7, k] = 0, {k, 1, m}];
tP[8, 8, 1] = -h4; Do[tP[8, 8, k] = 0, {k, 2, m}];

(* ***** For calculating invQ ***** *)
(* tensorP=- Transpose[tensorP]; *) (* Adjoint system *)
(* ***** *)

fQD[m, n];
fXMN[m, n];
B2 := Inverse[(IdentityMatrix[m*n] - GGT.MQD)].XXMN;

f11[x_] := fPhi[m, 1, 1, x]; f12[x_] := fPhi[m, 1, 2, x];
f13[x_] := fPhi[m, 1, 3, x]; f14[x_] := fPhi[m, 1, 4, x];
f15[x_] := fPhi[m, 1, 5, x]; f16[x_] := fPhi[m, 1, 6, x];
f17[x_] := fPhi[m, 1, 7, x]; f18[x_] := fPhi[m, 1, 8, x];
f21[x_] := fPhi[m, 2, 1, x]; f22[x_] := fPhi[m, 2, 2, x];

```

```

f23[x_] := fPhi[m, 2, 3, x]; f24[x_] := fPhi[m, 2, 4, x];
f25[x_] := fPhi[m, 2, 5, x]; f26[x_] := fPhi[m, 2, 6, x];
f27[x_] := fPhi[m, 2, 7, x]; f28[x_] := fPhi[m, 2, 8, x];
f31[x_] := fPhi[m, 3, 1, x]; f32[x_] := fPhi[m, 3, 2, x];
f33[x_] := fPhi[m, 3, 3, x]; f34[x_] := fPhi[m, 3, 4, x];
f35[x_] := fPhi[m, 3, 5, x]; f36[x_] := fPhi[m, 3, 6, x];
f37[x_] := fPhi[m, 3, 7, x]; f38[x_] := fPhi[m, 3, 8, x];
f41[x_] := fPhi[m, 4, 1, x]; f42[x_] := fPhi[m, 4, 2, x];
f43[x_] := fPhi[m, 4, 3, x]; f44[x_] := fPhi[m, 4, 4, x];
f45[x_] := fPhi[m, 4, 5, x]; f46[x_] := fPhi[m, 4, 6, x];
f47[x_] := fPhi[m, 4, 7, x]; f48[x_] := fPhi[m, 4, 8, x];
f51[x_] := fPhi[m, 5, 1, x]; f52[x_] := fPhi[m, 5, 2, x];
f53[x_] := fPhi[m, 5, 3, x]; f54[x_] := fPhi[m, 5, 4, x];
f55[x_] := fPhi[m, 5, 5, x]; f56[x_] := fPhi[m, 5, 6, x];
f57[x_] := fPhi[m, 5, 7, x]; f58[x_] := fPhi[m, 5, 8, x];
f61[x_] := fPhi[m, 6, 1, x]; f62[x_] := fPhi[m, 6, 2, x];
f63[x_] := fPhi[m, 6, 3, x]; f64[x_] := fPhi[m, 6, 4, x];
f65[x_] := fPhi[m, 6, 5, x]; f66[x_] := fPhi[m, 6, 6, x];
f67[x_] := fPhi[m, 6, 7, x]; f68[x_] := fPhi[m, 6, 8, x];
f71[x_] := fPhi[m, 7, 1, x]; f72[x_] := fPhi[m, 7, 2, x];
f73[x_] := fPhi[m, 7, 3, x]; f74[x_] := fPhi[m, 7, 4, x];
f75[x_] := fPhi[m, 7, 5, x]; f76[x_] := fPhi[m, 7, 6, x];
f77[x_] := fPhi[m, 7, 7, x]; f78[x_] := fPhi[m, 7, 8, x];
f81[x_] := fPhi[m, 8, 1, x]; f82[x_] := fPhi[m, 8, 2, x];
f83[x_] := fPhi[m, 8, 3, x]; f84[x_] := fPhi[m, 8, 4, x];
f85[x_] := fPhi[m, 8, 5, x]; f86[x_] := fPhi[m, 8, 6, x];
f87[x_] := fPhi[m, 8, 7, x]; f88[x_] := fPhi[m, 8, 8, x];

```

```

Phi[t_] := {{f11[t][[1]], f12[t][[1]], f13[t][[1]], f14[t][[1]], f15[t][[1]],
  f16[t][[1]], f17[t][[1]], f18[t][[1]]}, {f21[t][[1]], f22[t][[1]],
  f23[t][[1]], f24[t][[1]], f25[t][[1]], f26[t][[1]], f27[t][[1]], f28[t][[1]]},
{f31[t][[1]], f32[t][[1]], f33[t][[1]], f34[t][[1]], f35[t][[1]], f36[t][[1]],
  f37[t][[1]], f38[t][[1]]}, {f41[t][[1]], f42[t][[1]], f43[t][[1]],
  f44[t][[1]], f45[t][[1]], f46[t][[1]], f47[t][[1]], f48[t][[1]]},
{f51[t][[1]], f52[t][[1]], f53[t][[1]], f54[t][[1]], f55[t][[1]], f56[t][[1]],
  f57[t][[1]], f58[t][[1]]}, {f61[t][[1]], f62[t][[1]], f63[t][[1]],
  f64[t][[1]], f65[t][[1]], f66[t][[1]], f67[t][[1]], f68[t][[1]]},
{f71[t][[1]], f72[t][[1]], f73[t][[1]], f74[t][[1]], f75[t][[1]], f76[t][[1]],
  f77[t][[1]], f78[t][[1]]}, {f81[t][[1]], f82[t][[1]], f83[t][[1]],
  f84[t][[1]], f85[t][[1]], f86[t][[1]], f87[t][[1]], f88[t][[1]]}};

```

```
t1 = TimeUsed[];
```

```
Print["Total time = ", (t1 - t0) / 60];
```

```

(* ***** For finding Qft ***** *)

t0 = TimeUsed[];
R = MyQC[Phi[1].Phi[1], 2, n];
Q[t_] := Chop[Expand[Phi[t].(MatrixExp[-R t])]];
Qdefined = 1/2 (1 + Sign[t]) Q[t] +
  1/2 (1 - Sign[t]) Phi[t + 1].Q[1].MatrixExp[-R t].MatrixExp[-R];
Needs["FourierSeries`"]
Qft = Chop[NFourierTrigSeries[Qdefined, t, 12, FourierParameters -> {-1, 1/2}]]

(* {"a", " "b"} " |b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) dt" " 2|b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) cos(2πbnt) dt" " *)

t1 = TimeUsed[];
Print["Total time = ", (t1 - t0) / 60];

(* ***** For finding invQft ***** *)
(*
t0=TimeUsed[];
invQ[t_] := Chop[Expand[MatrixExp[R t].Transpose[Phi[t]]]];
invQdefined=1/2 (1+Sign[t])invQ[t]+
  1/2 (1-Sign[t])MatrixExp[R t].MatrixExp[R ].invQ[1].Transpose[Phi[t+1]];
Needs["FourierSeries`"]
invQft=Chop[NFourierTrigSeries[invQdefined,t,12,FourierParameters->{-1,1/2}]]

(* {"a", " "b"} " |b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) dt" " 2|b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) cos(2πbnt) dt" " *)

t1=TimeUsed[];
Print["Total time = ",(t1-t0)/60]; *)

(* Answer check *)
time = 2;
l0 = {0.5, 0.1, 0.1, 0.1, 0.8, 0.1, 0.1, 0.1};
sol1 =
NDSolve[{l1'[t] == A[[1, 1]] l1[t] + A[[1, 2]] l2[t] + A[[1, 3]] l3[t] + A[[1, 4]] l4[t] +
  A[[1, 5]] l5[t] + A[[1, 6]] l6[t] + A[[1, 7]] l7[t] + A[[1, 8]] l8[t],
  l2'[t] == A[[2, 1]] l1[t] + A[[2, 2]] l2[t] + A[[2, 3]] l3[t] + A[[2, 4]] l4[t] +
  A[[2, 5]] l5[t] + A[[2, 6]] l6[t] + A[[2, 7]] l7[t] + A[[2, 8]] l8[t],
  l3'[t] == A[[3, 1]] l1[t] + A[[3, 2]] l2[t] + A[[3, 3]] l3[t] + A[[3, 4]] l4[t] +
  A[[3, 5]] l5[t] + A[[3, 6]] l6[t] + A[[3, 7]] l7[t] + A[[3, 8]] l8[t],
  l4'[t] == A[[4, 1]] l1[t] + A[[4, 2]] l2[t] + A[[4, 3]] l3[t] + A[[4, 4]] l4[t] +
  A[[4, 5]] l5[t] + A[[4, 6]] l6[t] + A[[4, 7]] l7[t] + A[[4, 8]] l8[t],
  l5'[t] == A[[5, 1]] l1[t] + A[[5, 2]] l2[t] + A[[5, 3]] l3[t] + A[[5, 4]] l4[t] +
  A[[5, 5]] l5[t] + A[[5, 6]] l6[t] + A[[5, 7]] l7[t] + A[[5, 8]] l8[t],
  l6'[t] == A[[6, 1]] l1[t] + A[[6, 2]] l2[t] + A[[6, 3]] l3[t] + A[[6, 4]] l4[t] +
  A[[6, 5]] l5[t] + A[[6, 6]] l6[t] + A[[6, 7]] l7[t] + A[[6, 8]] l8[t],
  l7'[t] == A[[7, 1]] l1[t] + A[[7, 2]] l2[t] + A[[7, 3]] l3[t] + A[[7, 4]] l4[t] +
  A[[7, 5]] l5[t] + A[[7, 6]] l6[t] + A[[7, 7]] l7[t] + A[[7, 8]] l8[t],
  l8'[t] == A[[8, 1]] l1[t] + A[[8, 2]] l2[t] + A[[8, 3]] l3[t] + A[[8, 4]] l4[t] +
  A[[8, 5]] l5[t] + A[[8, 6]] l6[t] + A[[8, 7]] l7[t] + A[[8, 8]] l8[t],
  l1[0] == 10[[1]], l2[0] == 10[[2]], l3[0] == 10[[3]], l4[0] == 10[[4]],
  l5[0] == 10[[5]], l6[0] == 10[[6]], l7[0] == 10[[7]], l8[0] == 10[[8]]},

```

```

{11, 12, 13, 14, 15, 16, 17, 18}, {t, 0, time}]

fig1 = Plot[Evaluate[{11[t], 12[t], 13[t], 14[t], 15[t], 16[t], 17[t], 18[t]} /. sol1],
  {t, 0, time}, PlotRange -> All]

s0 = Chop[(invQft /. t -> 0) .10];
sol2 =
NDSolve[{s1'[t] == R[[1, 1]] s1[t] + R[[1, 2]] s2[t] + R[[1, 3]] s3[t] + R[[1, 4]] s4[t] +
  R[[1, 5]] s5[t] + R[[1, 6]] s6[t] + R[[1, 7]] s7[t] + R[[1, 8]] s8[t],
s2'[t] == R[[2, 1]] s1[t] + R[[2, 2]] s2[t] + R[[2, 3]] s3[t] + R[[2, 4]] s4[t] +
  R[[2, 5]] s5[t] + R[[2, 6]] s6[t] + R[[2, 7]] s7[t] + R[[2, 8]] s8[t],
s3'[t] == R[[3, 1]] s1[t] + R[[3, 2]] s2[t] + R[[3, 3]] s3[t] + R[[3, 4]] s4[t] +
  R[[3, 5]] s5[t] + R[[3, 6]] s6[t] + R[[3, 7]] s7[t] + R[[3, 8]] s8[t],
s4'[t] == R[[4, 1]] s1[t] + R[[4, 2]] s2[t] + R[[4, 3]] s3[t] + R[[4, 4]] s4[t] +
  R[[4, 5]] s5[t] + R[[4, 6]] s6[t] + R[[4, 7]] s7[t] + R[[4, 8]] s8[t],
s5'[t] == R[[5, 1]] s1[t] + R[[5, 2]] s2[t] + R[[5, 3]] s3[t] + R[[5, 4]] s4[t] +
  R[[5, 5]] s5[t] + R[[5, 6]] s6[t] + R[[5, 7]] s7[t] + R[[5, 8]] s8[t],
s6'[t] == R[[6, 1]] s1[t] + R[[6, 2]] s2[t] + R[[6, 3]] s3[t] + R[[6, 4]] s4[t] +
  R[[6, 5]] s5[t] + R[[6, 6]] s6[t] + R[[6, 7]] s7[t] + R[[6, 8]] s8[t],
s7'[t] == R[[7, 1]] s1[t] + R[[7, 2]] s2[t] + R[[7, 3]] s3[t] + R[[7, 4]] s4[t] +
  R[[7, 5]] s5[t] + R[[7, 6]] s6[t] + R[[7, 7]] s7[t] + R[[7, 8]] s8[t],
s8'[t] == R[[8, 1]] s1[t] + R[[8, 2]] s2[t] + R[[8, 3]] s3[t] + R[[8, 4]] s4[t] +
  R[[8, 5]] s5[t] + R[[8, 6]] s6[t] + R[[8, 7]] s7[t] + R[[8, 8]] s8[t],
s1[0] == s0[[1]], s2[0] == s0[[2]], s3[0] == s0[[3]], s4[0] == s0[[4]],
s5[0] == s0[[5]], s6[0] == s0[[6]], s7[0] == s0[[7]], s8[0] == s0[[8]]},
{s1, s2, s3, s4, s5, s6, s7, s8}, {t, -0, time}]

fig2 = Plot[
  Evaluate[{Qft[[1, 1]] s1[t] + Qft[[1, 2]] s2[t] + Qft[[1, 3]] s3[t] + Qft[[1, 4]] s4[t] +
    Qft[[1, 5]] s5[t] + Qft[[1, 6]] s6[t] + Qft[[1, 7]] s7[t] + Qft[[1, 8]] s8[t],
  Qft[[2, 1]] s1[t] + Qft[[2, 2]] s2[t] + Qft[[2, 3]] s3[t] + Qft[[2, 4]] s4[t] +
    Qft[[2, 5]] s5[t] + Qft[[2, 6]] s6[t] + Qft[[2, 7]] s7[t] + Qft[[2, 8]] s8[t],
  Qft[[3, 1]] s1[t] + Qft[[3, 2]] s2[t] + Qft[[3, 3]] s3[t] + Qft[[3, 4]] s4[t] +
    Qft[[3, 5]] s5[t] + Qft[[3, 6]] s6[t] + Qft[[3, 7]] s7[t] + Qft[[3, 8]] s8[t],
  Qft[[4, 1]] s1[t] + Qft[[4, 2]] s2[t] + Qft[[4, 3]] s3[t] + Qft[[4, 4]] s4[t] +
    Qft[[4, 5]] s5[t] + Qft[[4, 6]] s6[t] + Qft[[4, 7]] s7[t] + Qft[[4, 8]] s8[t],
  Qft[[5, 1]] s1[t] + Qft[[5, 2]] s2[t] + Qft[[5, 3]] s3[t] + Qft[[5, 4]] s4[t] +
    Qft[[5, 5]] s5[t] + Qft[[5, 6]] s6[t] + Qft[[5, 7]] s7[t] + Qft[[5, 8]] s8[t],
  Qft[[6, 1]] s1[t] + Qft[[6, 2]] s2[t] + Qft[[6, 3]] s3[t] + Qft[[6, 4]] s4[t] +
    Qft[[6, 5]] s5[t] + Qft[[6, 6]] s6[t] + Qft[[6, 7]] s7[t] + Qft[[6, 8]] s8[t],
  Qft[[7, 1]] s1[t] + Qft[[7, 2]] s2[t] + Qft[[7, 3]] s3[t] + Qft[[7, 4]] s4[t] +
    Qft[[7, 5]] s5[t] + Qft[[7, 6]] s6[t] + Qft[[7, 7]] s7[t] + Qft[[7, 8]] s8[t],
  Qft[[8, 1]] s1[t] + Qft[[8, 2]] s2[t] + Qft[[8, 3]] s3[t] + Qft[[8, 4]] s4[t] +
    Qft[[8, 5]] s5[t] + Qft[[8, 6]] s6[t] + Qft[[8, 7]] s7[t] + Qft[[8, 8]] s8[t]} /.
  sol2], {t, 0, time}, PlotRange -> All]

Show[
  fig1,
  fig2]

```

```
(* Writing parameter values to text files *)  
SetDirectory[NotebookDirectory[]];  
A >> "A.txt"; {b1, b2, b3, b4} >> "b.txt";  
R >> "R.txt";  
Qft >> "Q.txt"; invQft >> "invQ.txt";
```

C.2

```

(* File name "LFtransform_functions.nb" *)
(* Functions required by "Finding R and Q.nb" *)
fQp[p_] := Module[{i, j},
  MQp := Array[Qp, {Length[p], Length[p]}];
Do[Qp[i, 1] = p[[i]], {i, 1, Length[p]}];
Do[Qp[1, j] = p[[j]] / 2, {j, 2, Length[p]}];
  Do[If[(i == j), If[(2 j - 1) ≤ Length[p],
    Qp[i, j] = p[[2 i - 1]] / 2 + p[[1]], Qp[i, j] = p[[1]], If[(i ≠ 1 && j ≠ 1),
    If[(i + j - 1) ≤ Length[p], Qp[i, j] = p[[i + j - 1]] / 2 + p[[Abs[i - j] + 1]] / 2,
    Qp[i, j] = p[[Abs[i - j] + 1]] / 2]], {i, 1, Length[p]}, {j, 1, Length[p]}];
Qp[1, 1] = p[[1]];
  MQp
]
fGT[p_] := Module[{i, j},
  MGp := Array[G, {p, p}];
Do[If[j == 1, If[i == 2, G[i, j] = -1 / 8, G[i, j] = Cos[i Pi] / (2 i (i - 2))]
, If[Abs[i - j] == 1,
  If[(i - j) == 1, G[i, j] = -1 / (4 (i - 2)), If[j ≠ 2, G[i, j] = 1 / (4 i), G[i, j] = 1 / 2]
, G[i, j] = 0]
], {i, 1, p}, {j, 1, p}];
  Transpose[MGp];
]
fQD[m_, n_] := Module[{i, j, k, l, A, B},
  MQD := Array[QD, {nm, nm}];
  Do[QD[(i - 1) * m + k, (j - 1) * m + l] = fQp[tensorP[[i, j]]][[k, l]],
    {i, 1, n}, {j, 1, n}, {k, 1, m}, {l, 1, m}
  ];
]
fGG[GG_, m_, n_] := Module[{i, j, k, l},
  GGT := Array[GGTD, {nm, nm}];
  Do[GGTD[i, j] = 0, {i, 1, mn}, {j, 1, mn}];

  Do[GGTD[(i - 1) * m + k, (i - 1) * m + l] = GG[[k, l]],
    {i, 1, n}, {k, 1, m}, {l, 1, m}
  ];
]
(*GGT=Outer[Times, IdentityMatrix[n], GG]*)
]
fXMN[m_, n_] := Module[{i},
  XXMN := ConstantArray[0, {nm, n}];
  Do[XXMN = ReplacePart[XXMN, 1, {1 + m (i - 1), i}], {i, 1, n}];
]
fPhi[m_, i_, j_, x_] :=
  Module[{k, aaaa}, aaaa := Transpose[Take[B2, {1 + m (i - 1), mi}], {j}]];
  Expand[aaaa.Table[ChebyshevT[k - 1, 2 x - 1], {k, m}]]
];

```



```

MyQC[QT_, T_, n_] := Module[{J, Γ, R}, J = JordanDecomposition[QT];
  Γ = Array[tmp, {n, n}];
  Do[If[i == j, tmp[i, j] = Log[J[[2, i, j]]] / T, tmp[i, j] = 0], {i, 1, n}, {j, 1, n}];
  R = Chop[J[[1]].Γ.Inverse[J[[1]]], 10-8];
  R
];

```

C.3

```

Clear["Global`*"]

(* File name "Step 1 Initialize.nb" *)
(* Following data is read from separate text files some of which are
   generated by program "Finding Q and R.nb", others are generated manually *)
SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]];
 $\omega$  = ReadList["omega.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; M1 = ReadList["M1.txt"][[1]];
(*  $\omega := 2\pi/T$  *)
(* fx := Nonlinear vector field of the system in x coordinates *)
(* ux := External excitations *)
(* M1 := Real modal matrix *)

invM1 = Inverse[M1];

J1 = Chop[invM1.R1.M1];
(* J1 := Constant linear matrix in Jordan canonical form *)
z = {z1, z2};
x = Q1.M1.z;
(*nonlinear term in z*)
fz = Chop[Expand[TrigToExp[invM1.invQ1.fx /. {x1  $\rightarrow$  x[[1]], x2  $\rightarrow$  x[[2]]}]]];
uz = Chop[Expand[TrigToExp[invM1.invQ1.ux]]];
J1 >> "J1.txt";

(*  $\epsilon$  used here is just a book keeping parameter
   it doesnt take any value during execution of the code *)
(* ni := near-identity transformation *)
nt30 = {h1, h2};
nt31 =  $\epsilon$  {h110 v1 + h101 v2, h210 v1 + h201 v2};
nt32 =  $\epsilon$  {h120 v1^2 + h111 v1 v2 + h102 v2^2, h220 v1^2 + h211 v1 v2 + h202 v2^2};
nt33 =  $\epsilon$  {h130 v1^3 + h121 v1^2 v2 + h112 v1 v2^2 + h103 v2^3,
          h230 v1^3 + h221 v1^2 v2 + h212 v1 v2^2 + h203 v2^3};
nt = {v1, v2} + nt30 + nt31 + nt32 + nt33;
fv = Expand[
   $\epsilon$  {fz[[1]] /. {z1  $\rightarrow$  nt[[1]], z2  $\rightarrow$  nt[[2]]}, fz[[2]] /. {z1  $\rightarrow$  nt[[1]], z2  $\rightarrow$  nt[[2]]}];
coeffv1 = CoefficientList[fv[[1]], { $\epsilon$ , h1, h2, v1, v2}];
coeffv2 = CoefficientList[fv[[2]], { $\epsilon$ , h1, h2, v1, v2}];
hh = 18;

```

```

(* Finding nt30 for cubic nonlinearity *)
fv30 = Expand[{coeffv1[[2, 4, 1, 1, 1]] h1^4 + coeffv1[[2, 3, 2, 1, 1]] h1^3 h2 +
  coeffv1[[2, 2, 3, 1, 1]] h1^2 h2^2 + coeffv1[[2, 1, 4, 1, 1]] h1 h2^3 +
  coeffv2[[2, 4, 1, 1, 1]] h1^4 + coeffv2[[2, 3, 2, 1, 1]] h1^3 h2 +
  coeffv2[[2, 2, 3, 1, 1]] h1^2 h2^2 + coeffv2[[2, 1, 4, 1, 1]] h1 h2^3}];
(* All terms are of the order ε *)
uv = {uz[[1]], uz[[2]]};
fv30 >> "fv30.txt";
uv >> "uv.txt";
(* ***** Use routine "Harmonic solution *.nb" to find nt30 ***** *)

(* ***** Finding nt31 ***** *)
v1nt30 = TrigToExp[ReadList["v1nt30.txt"][[1]]];
hh = ω * 3;
fv31 = Expand[{coeffv1[[2, 3, 1, 2, 1]] h1^2 v1 + coeffv1[[2, 2, 2, 2, 1]] h1 h2 v1 +
  coeffv1[[2, 1, 3, 2, 1]] h2^2 v1 + coeffv1[[2, 3, 1, 1, 2]] h1^2 v2 +
  coeffv1[[2, 2, 2, 1, 2]] h1 h2 v2 + coeffv1[[2, 1, 3, 1, 2]] h2^2 v2 +
  coeffv2[[2, 3, 1, 2, 1]] h1^2 v1 + coeffv2[[2, 2, 2, 2, 1]] h1 h2 v1 +
  coeffv2[[2, 1, 3, 2, 1]] h2^2 v1 + coeffv2[[2, 3, 1, 1, 2]] h1^2 v2 +
  coeffv2[[2, 2, 2, 1, 2]] h1 h2 v2 + coeffv2[[2, 1, 3, 1, 2]] h2^2 v2}];

coeffv31 = CoefficientList[fv31, {v1, v2}];
cofeq31RHS =
  Chop[Expand[{coeffv31[[1]][[2, 1]], coeffv31[[1]][[1, 2]], coeffv31[[2]][[2, 1]],
    coeffv31[[2]][[1, 2]]} /. {h1 → v1nt30[[1]], h2 → v1nt30[[2]]}]];

coefTeq31RHS = Table[0, {2 * 2}, {1 + 2 * hh}];
Do[
  Do[If[Exponent[cofeq31RHS[[i, k]], E] == 0,
    {coefTeq31RHS[[i, 1]] = coefTeq31RHS[[i, 1]] + cofeq31RHS[[i, k]]},
    {k, 1, Dimensions[cofeq31RHS[[i]]][[1]]}], {i, 1, 2 * 2}];

Do[
  coefTeq31RHS[[i, 2 * j]] = Coefficient[cofeq31RHS[[i]], E^(j * I * ω * t)];
  coefTeq31RHS[[i, 2 * j + 1]] = Coefficient[cofeq31RHS[[i]], E^(-j * I * ω * t)];
  {i, 1, 2 * 2}, {j, 1, hh}];

eq31LHS = Chop[
  Expand[{{∂v1 nt31[[1]], ∂v2 nt31[[1]]}, {∂v1 nt31[[2]], ∂v2 nt31[[2]]}}.J1.{v1, v2} -
  J1.nt31 + v1^3 + v1^2 v2 + v1 v2^2 + v2^3]];
(* Highlighted figures are dummy terms required for proper
  execution of command "CoefficientList" *)

cofeq31LHS1 = CoefficientList[eq31LHS[[1]], {ε, v1, v2}];
cofeq31LHS2 = CoefficientList[eq31LHS[[2]], {ε, v1, v2}];
cofeq31LHS = Chop[{cofeq31LHS1[[2, 2, 1]],
  cofeq31LHS1[[2, 1, 2]], cofeq31LHS2[[2, 2, 1]], cofeq31LHS2[[2, 1, 2]]}];
coefTeq31LHS = Table[0, {2 * 2}, {1 + 2 * hh}];

```

```

Do[ coefTeq31LHS[[i, 1]] = coefeq31LHS[[i]], {i, 1, 2*2}];
Do[
  coefTeq31LHS[[i, 2*j]] = coefeq31LHS[[i]];
  coefTeq31LHS[[i, 2*j+1]] = coefeq31LHS[[i]]; {i, 1, 2*2}, {j, 1, hh}];

Do[
  A31[i] = Expand[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω * IdentityMatrix[4] +
    {{Coefficient[coefTeq31LHS[[1, i]], h110],
      Coefficient[coefTeq31LHS[[1, i]], h101], Coefficient[
        coefTeq31LHS[[1, i]], h210], Coefficient[coefTeq31LHS[[1, i]], h201]},
    {Coefficient[coefTeq31LHS[[2, i]], h110], Coefficient[coefTeq31LHS[[2, i]],
      h101], Coefficient[coefTeq31LHS[[2, i]], h210], Coefficient[
        coefTeq31LHS[[2, i]], h201]}, {Coefficient[coefTeq31LHS[[3, i]], h110],
      Coefficient[coefTeq31LHS[[3, i]], h101], Coefficient[
        coefTeq31LHS[[3, i]], h210], Coefficient[coefTeq31LHS[[3, i]], h201]},
    {Coefficient[coefTeq31LHS[[4, i]], h110], Coefficient[
      coefTeq31LHS[[4, i]], h101], Coefficient[coefTeq31LHS[[4, i]], h210],
      Coefficient[coefTeq31LHS[[4, i]], h201]}}];
  C31[i] = {coefTeq31RHS[[1, i]], coefTeq31RHS[[2, i]],
    coefTeq31RHS[[3, i]], coefTeq31RHS[[4, i]]};
  , {i, 2, 2*hh}];
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
h31 = Chop[Expand[Sum[Inverse[A31[i]].C31[i] *
  E^((i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω t), {i, 2, 2*hh}]]];
vInt31 = (nt31 / ε) /. {h110 → h31[[1]], h101 → h31[[2]],
  h210 → h31[[3]], h201 → h31[[4]]};
lnresterm = {{0, coefTeq31RHS[[1, 1]]}, {coefTeq31RHS[[4, 1]], 0}};
lnresterm >> "lnresterm.txt";

(* ***** Finding nt32 ***** *)
fv32 = Expand[{coeffv1[[2, 2, 1, 3, 1]] h1 v1^2 + coeffv1[[2, 2, 1, 2, 2]] h1 v1 v2 +
  coeffv1[[2, 2, 1, 1, 3]] h1 v2^2 + coeffv1[[2, 1, 2, 3, 1]] h2 v1^2 +
  coeffv1[[2, 1, 2, 2, 2]] h2 v1 v2 + coeffv1[[2, 1, 2, 1, 3]] h2 v2^2,
  coeffv2[[2, 2, 1, 3, 1]] h1 v1^2 + coeffv2[[2, 2, 1, 2, 2]] h1 v1 v2 +
  coeffv2[[2, 2, 1, 1, 3]] h1 v2^2 + coeffv2[[2, 1, 2, 3, 1]] h2 v1^2 +
  coeffv2[[2, 1, 2, 2, 2]] h2 v1 v2 + coeffv2[[2, 1, 2, 1, 3]] h2 v2^2}];
coeffv32 = CoefficientList[fv32, {v1, v2}];
coefeq32RHS =
  Chop[Expand[{coeffv32[[1]][[3, 1]], coeffv32[[1]][[2, 2]], coeffv32[[1]][[1, 3]],
    coeffv32[[2]][[3, 1]], coeffv32[[2]][[2, 2]], coeffv32[[2]][[1, 3]]} /.
    {h1 → vInt30[[1]], h2 → vInt30[[2]]}]];

coefTeq32RHS = Table[0, {2*3}, {1+2*hh}];
Do[
  Do[ If[Exponent[coefeq32RHS[[i, k]], E] == 0,
    {coefTeq32RHS[[i, 1]] = coefTeq32RHS[[i, 1]] + coefeq32RHS[[i, k]]},
    {k, 1, Dimensions[coefeq32RHS[[i]]][[1]]}], {i, 1, 2*3}];
Do[
  coefTeq32RHS[[i, 2*j]] = Coefficient[coefeq32RHS[[i]], E^(j*I*ω*t)];
  coefTeq32RHS[[i, 2*j+1]] = Coefficient[coefeq32RHS[[i]], E^(-j*I*ω*t)];
  {i, 1, 2*3}, {j, 1, hh}];

```

```

eq32LHS =
  Expand[{{∂v1 nt32[[1]], ∂v2 nt32[[1]]}, {∂v1 nt32[[2]], ∂v2 nt32[[2]]}}.J1.{v1, v2} -
  J1.nt32];

cofeq32LHS1 = CoefficientList[eq32LHS[[1]], {ε, v1, v2}];
cofeq32LHS2 = CoefficientList[eq32LHS[[2]], {ε, v1, v2}];

cofeq32LHS =
  Chop[{cofeq32LHS1[[2, 3, 1]], cofeq32LHS1[[2, 2, 2]], cofeq32LHS1[[2, 1, 3]],
  cofeq32LHS2[[2, 3, 1]], cofeq32LHS2[[2, 2, 2]], cofeq32LHS2[[2, 1, 3]]}];

coefTeq32LHS = Table[0, {2 * 3}, {1 + 2 * hh}];
Do[coefTeq32LHS[[i, 1]] = cofeq32LHS[[i]], {i, 1, 2 * 3}];
Do[
  coefTeq32LHS[[i, 2 * j]] = cofeq32LHS[[i]];
  coefTeq32LHS[[i, 2 * j + 1]] = cofeq32LHS[[i]], {i, 1, 2 * 3}, {j, 1, hh}];

Do[
  A32[i] = Expand[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω * IdentityMatrix[6] +
  {{Coefficient[coefTeq32LHS[[1, i]], h120], Coefficient[
  coefTeq32LHS[[1, i]], h111], Coefficient[coefTeq32LHS[[1, i]], h102],
  Coefficient[coefTeq32LHS[[1, i]], h220], Coefficient[
  coefTeq32LHS[[1, i]], h211], Coefficient[coefTeq32LHS[[1, i]], h202]},
  {Coefficient[coefTeq32LHS[[2, i]], h120], Coefficient[
  coefTeq32LHS[[2, i]], h111], Coefficient[coefTeq32LHS[[2, i]], h102],
  Coefficient[coefTeq32LHS[[2, i]], h220], Coefficient[
  coefTeq32LHS[[2, i]], h211], Coefficient[coefTeq32LHS[[2, i]], h202]},
  {Coefficient[coefTeq32LHS[[3, i]], h120], Coefficient[
  coefTeq32LHS[[3, i]], h111], Coefficient[coefTeq32LHS[[3, i]], h102],
  Coefficient[coefTeq32LHS[[3, i]], h220], Coefficient[
  coefTeq32LHS[[3, i]], h211], Coefficient[coefTeq32LHS[[3, i]], h202]},
  {Coefficient[coefTeq32LHS[[4, i]], h120], Coefficient[
  coefTeq32LHS[[4, i]], h111], Coefficient[coefTeq32LHS[[4, i]], h102],
  Coefficient[coefTeq32LHS[[4, i]], h220], Coefficient[
  coefTeq32LHS[[4, i]], h211], Coefficient[coefTeq32LHS[[4, i]], h202]},
  {Coefficient[coefTeq32LHS[[5, i]], h120], Coefficient[
  coefTeq32LHS[[5, i]], h111], Coefficient[coefTeq32LHS[[5, i]], h102],
  Coefficient[coefTeq32LHS[[5, i]], h220], Coefficient[
  coefTeq32LHS[[5, i]], h211], Coefficient[coefTeq32LHS[[5, i]], h202]},
  {Coefficient[coefTeq32LHS[[6, i]], h120], Coefficient[
  coefTeq32LHS[[6, i]], h111], Coefficient[coefTeq32LHS[[6, i]], h102],
  Coefficient[coefTeq32LHS[[6, i]], h220], Coefficient[coefTeq32LHS[[6, i]],
  h211], Coefficient[coefTeq32LHS[[6, i]], h202]}}];
  C32[i] = {coefTeq32RHS[[1, i]], coefTeq32RHS[[2, i]], coefTeq32RHS[[3, i]],
  coefTeq32RHS[[4, i]], coefTeq32RHS[[5, i]],
  coefTeq32RHS[[6, i]]};
  , {i, 1, 1 + 2 * hh}];
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
(* Solvability condition *)

```

```

Do[If[Det[A32[i]] == 0, {Print["Resonance"];
  Print[(i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω]}, {i, 1, 1 + 2 * hh}];

h32 = Chop[Expand[Sum[Inverse[A32[i]].C32[i] *
  E^((i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω t), {i, 1, 1 + 2 * hh}]]];
v1nt32 = (nt32 / ε) /. {h120 → h32[[1]], h111 → h32[[2]], h102 → h32[[3]],
  h220 → h32[[4]], h211 → h32[[5]], h202 → h32[[6]]];

(* ***** Finding im33 ***** *)
fv33 = Expand[{coeffv1[[2, 1, 1, 4, 1]] v1^3 + coeffv1[[2, 1, 1, 3, 2]] v1^2 v2 +
  coeffv1[[2, 1, 1, 2, 3]] v1 v2^2 + coeffv1[[2, 1, 1, 1, 4]] v2^3 ε v1^4 + ε v2^4,
  coeffv2[[2, 1, 1, 4, 1]] v1^3 + coeffv2[[2, 1, 1, 3, 2]] v1^2 v2 +
  coeffv2[[2, 1, 1, 2, 3]] v1 v2^2 + coeffv2[[2, 1, 1, 1, 4]] v2^3 ε v1^4 + ε v2^4];

coeffv33 = CoefficientList[fv33, {v1, v2}];
cofeq33RHS =
  Expand[{coeffv33[[1]][[4, 1]], coeffv33[[1]][[3, 2]], coeffv33[[1]][[2, 3]],
  coeffv33[[1]][[1, 4]], coeffv33[[2]][[4, 1]], coeffv33[[2]][[3, 2]],
  coeffv33[[2]][[2, 3]], coeffv33[[2]][[1, 4]]} +
  {((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t),
  ((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t),
  ((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t), ((0.5 + 0.5 I) e-10 i π t +
  (0.5 - 0.5 I) e10 i π t), ((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t),
  ((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t), ((0.5 + 0.5 I) e-10 i π t +
  (0.5 - 0.5 I) e10 i π t), ((0.5 + 0.5 I) e-10 i π t + (0.5 - 0.5 I) e10 i π t)];
(* Highlighted figures are dummy terms required for proper
  execution of command "CoefficientList" *)

coefTeq33RHS = Table[0, {2 * 4}, {1 + 2 * hh}];
Do[
  Do[If[Exponent[cofeq33RHS[[i, k]], E] == 0,
    {coefTeq33RHS[[i, 1]] = coefTeq33RHS[[i, 1]] + cofeq33RHS[[i, k]]},
    {k, 1, Dimensions[cofeq33RHS[[i]]][[1]]}], {i, 1, 2 * 4}];
Do[
  coefTeq33RHS[[i, 2 * j]] = Coefficient[cofeq33RHS[[i]], E^(j * I * ω * t)];
  coefTeq33RHS[[i, 2 * j + 1]] = Coefficient[cofeq33RHS[[i]], E^(-j * I * ω * t)];
  {i, 1, 2 * 4}, {j, 1, hh}];

eq33LHS =
  Expand[{{∂v1 nt33[[1]], ∂v2 nt33[[1]]}, {∂v1 nt33[[2]], ∂v2 nt33[[2]]}}.J1.{v1, v2} -
  J1.nt33];

cofeq33LHS1 = CoefficientList[eq33LHS[[1]], {ε, v1, v2}];
cofeq33LHS2 = CoefficientList[eq33LHS[[2]], {ε, v1, v2}];

cofeq33LHS = Chop[{cofeq33LHS1[[2, 4, 1]], cofeq33LHS1[[2, 3, 2]],
  cofeq33LHS1[[2, 2, 3]], cofeq33LHS1[[2, 1, 4]], cofeq33LHS2[[2, 4, 1]],
  cofeq33LHS2[[2, 3, 2]], cofeq33LHS2[[2, 2, 3]], cofeq33LHS2[[2, 1, 4]]}];

coefTeq33LHS = Table[0, {2 * 4}, {1 + 2 * hh}];

```

```

Do[ coefTeq33LHS[[i, 1]] = coefeq33LHS[[i]], {i, 1, 2*4}];
Do[
  coefTeq33LHS[[i, 2*j]] = coefeq33LHS[[i]];
  coefTeq33LHS[[i, 2*j+1]] = coefeq33LHS[[i]]; , {i, 1, 2*4}, {j, 1, hh}];

Do[
  A33[i] = Expand[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω * IdentityMatrix[8] +
    {{Coefficient[coefTeq33LHS[[1, i]], h130], Coefficient[
      coefTeq33LHS[[1, i]], h121], Coefficient[coefTeq33LHS[[1, i]], h112],
      Coefficient[coefTeq33LHS[[1, i]], h103], Coefficient[coefTeq33LHS[[1, i]],
        h230], Coefficient[coefTeq33LHS[[1, i]], h221], Coefficient[
      coefTeq33LHS[[1, i]], h212], Coefficient[coefTeq33LHS[[1, i]], h203]},
    {Coefficient[coefTeq33LHS[[2, i]], h130], Coefficient[coefTeq33LHS[[2, i]],
      h121], Coefficient[coefTeq33LHS[[2, i]], h112], Coefficient[
      coefTeq33LHS[[2, i]], h103], Coefficient[coefTeq33LHS[[2, i]], h230],
      Coefficient[coefTeq33LHS[[2, i]], h221], Coefficient[
      coefTeq33LHS[[2, i]], h212], Coefficient[coefTeq33LHS[[2, i]], h203]},
    {Coefficient[coefTeq33LHS[[3, i]], h130], Coefficient[coefTeq33LHS[[3, i]],
      h121], Coefficient[coefTeq33LHS[[3, i]], h112], Coefficient[
      coefTeq33LHS[[3, i]], h103], Coefficient[coefTeq33LHS[[3, i]], h230],
      Coefficient[coefTeq33LHS[[3, i]], h221], Coefficient[
      coefTeq33LHS[[3, i]], h212], Coefficient[coefTeq33LHS[[3, i]], h203]},
    {Coefficient[coefTeq33LHS[[4, i]], h130], Coefficient[coefTeq33LHS[[4, i]],
      h121], Coefficient[coefTeq33LHS[[4, i]], h112], Coefficient[
      coefTeq33LHS[[4, i]], h103], Coefficient[coefTeq33LHS[[4, i]], h230],
      Coefficient[coefTeq33LHS[[4, i]], h221], Coefficient[
      coefTeq33LHS[[4, i]], h212], Coefficient[coefTeq33LHS[[4, i]], h203]},
    {Coefficient[coefTeq33LHS[[5, i]], h130], Coefficient[coefTeq33LHS[[5, i]],
      h121], Coefficient[coefTeq33LHS[[5, i]], h112], Coefficient[
      coefTeq33LHS[[5, i]], h103], Coefficient[coefTeq33LHS[[5, i]], h230],
      Coefficient[coefTeq33LHS[[5, i]], h221], Coefficient[
      coefTeq33LHS[[5, i]], h212], Coefficient[coefTeq33LHS[[5, i]], h203]},
    {Coefficient[coefTeq33LHS[[6, i]], h130], Coefficient[coefTeq33LHS[[6, i]],
      h121], Coefficient[coefTeq33LHS[[6, i]], h112], Coefficient[
      coefTeq33LHS[[6, i]], h103], Coefficient[coefTeq33LHS[[6, i]], h230],
      Coefficient[coefTeq33LHS[[6, i]], h221], Coefficient[
      coefTeq33LHS[[6, i]], h212], Coefficient[coefTeq33LHS[[6, i]], h203]},
    {Coefficient[coefTeq33LHS[[7, i]], h130], Coefficient[coefTeq33LHS[[7, i]],
      h121], Coefficient[coefTeq33LHS[[7, i]], h112], Coefficient[
      coefTeq33LHS[[7, i]], h103], Coefficient[coefTeq33LHS[[7, i]], h230],
      Coefficient[coefTeq33LHS[[7, i]], h221], Coefficient[
      coefTeq33LHS[[7, i]], h212], Coefficient[coefTeq33LHS[[7, i]], h203]},
    {Coefficient[coefTeq33LHS[[8, i]], h130], Coefficient[coefTeq33LHS[[8, i]],
      h121], Coefficient[coefTeq33LHS[[8, i]], h112], Coefficient[
      coefTeq33LHS[[8, i]], h103], Coefficient[coefTeq33LHS[[8, i]], h230],
      Coefficient[coefTeq33LHS[[8, i]], h221], Coefficient[coefTeq33LHS[[8, i]],
        h212], Coefficient[coefTeq33LHS[[8, i]], h203]}}];

C33[i] = {coefTeq33RHS[[1, i]], coefTeq33RHS[[2, i]],
  coefTeq33RHS[[3, i]], coefTeq33RHS[[4, i]], coefTeq33RHS[[5, i]],

```

```

coefTeq33RHS[[6, i]], coefTeq33RHS[[7, i]], coefTeq33RHS[[8, i]]];
, {i, 1, 1 + 2*hh}];
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
(* Solvability condition *)
Do[If[Det[A33[i]] == 0, {Print["Resonance"];
Print[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω]}], {i, 1, 1 + 2*hh}];

h33 = Chop[Expand[Sum[Inverse[A33[i]].C33[i] *
E^((i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω t), {i, 1, 1 + 2*hh}]]];
(* h31={h120,h111,h102,h220,h211,h202} *)
vInt33 =
(nt33 / ε) /. {h130 → h33[[1]], h121 → h33[[2]], h112 → h33[[3]], h103 → h33[[4]],
h230 → h33[[5]], h221 → h33[[6]], h212 → h33[[7]], h203 → h33[[8]]};
vInt = vInt30 + vInt31 + vInt32 + vInt33;
Chop[vInt, 10^-3] >> "vInt.txt"
(* vInt30.txt is generated by Harmonic solution *.nb *)
Chop[vInt31, 10^-3] >> "vInt31.txt"
Chop[vInt32, 10^-3] >> "vInt32.txt"
Chop[vInt33, 10^-3] >> "vInt33.txt"

```


C.4

```

Clear["Global`*"]

(* File name "Step 2 Harmonic solution.nb" *)
(* This routine finds harmonic balance solution to 0th order equation,
No Resonance case *)
SetDirectory[NotebookDirectory[]];
J1 = ReadList["J1.txt"][[1]];
fv30 = ReadList["fv30.txt"][[1]];
uv = ReadList["uv.txt"][[1]];
ω = ReadList["omega.txt"][[1]];
hh = 12;

h0 = {a1 + Sum[c1[i] Cos[i ω t] + s1[i] Sin[i ω t], {i, 1, hh}],
      a2 + Sum[c2[i] Cos[i ω t] + s2[i] Sin[i ω t], {i, 1, hh}]};

eq30LHS =
Chop[ExpToTrig[Expand[TrigToExp[({∂t h0[[1]], ∂t h0[[2]]} - J1.{h0[[1]], h0[[2]]} -
fv30 /. {h1 → h0[[1]], h2 → h0[[2]]}]]], 10-3];

coefTeq30LHS = Table[0, {2}, {1 + 2 * hh}];
Do[
Do[ If[Exponent[TrigToExp[eq30LHS[[i, k]]], E] == 0,
{coefTeq30LHS[[i, 1]] = coefTeq30LHS[[i, 1]] + eq30LHS[[i, k]]},
{k, 1, Dimensions[eq30LHS[[i]]][[1]]}, {i, 1, 2}];

Do[
coefTeq30LHS[[i, 2 * j]] = Coefficient[eq30LHS[[i]], Cos[j ω t]];
coefTeq30LHS[[i, 2 * j + 1]] = Coefficient[eq30LHS[[i]], Sin[j ω t]];
{i, 1, 2}, {j, 1, hh}];

eq30RHS = Chop[ExpToTrig[uv]];

coefTeq30RHS = Table[0, {2}, {1 + 2 * hh}];
Do[
Do[ If[Exponent[TrigToExp[eq30RHS[[i, k]]], E] == 0,
{coefTeq30RHS[[i, 1]] = coefTeq30RHS[[i, 1]] + eq30RHS[[i, k]]},
{k, 1, Dimensions[eq30RHS[[i]]][[1]]}, {i, 1, 2}];

Do[
coefTeq30RHS[[i, 2 * j]] = Coefficient[eq30RHS[[i]], Cos[j ω t]];
coefTeq30RHS[[i, 2 * j + 1]] = Coefficient[eq30RHS[[i]], Sin[j ω t]];
{i, 1, 2}, {j, 1, hh}];
Beep[];

```

```

t0 = TimeUsed[];
ans = 0;
equL = Chop[coefTeq30LHS - coefTeq30RHS];
While[ans == 0,
  varL = Flatten[{{a1, RandomReal[{-0.5, 0.5}]}, {a2, RandomReal[{-0.5, 0.5}]},
    Flatten[Table[{{c1[i], RandomReal[{-0.5, 0.5}]},
      {s1[i], RandomReal[{-0.5, 0.5}]}, {c2[i], RandomReal[{-0.5, 0.5}]},
      {s2[i], RandomReal[{-0.5, 0.5]}}}, {i, 1, hh}], 1}], 1];
  ans = Check[Chop[FindRoot[equL, varL], 10^-3], 0]; Print[ans];]
t1 = TimeUsed[];
totaltime = (t1 - t0) / 60

(* ans check *)
ti = 100;
tf = 120;

fig1 = Plot[h0 /. ans, {t, ti, tf}]

s0 = h0 /. ans /. t -> 0;
sol = NDSolve[{s1'[t] == J1[[1, 1]] s1[t] + J1[[1, 2]] s2[t] +
  (fv30[[1]] /. {h1 -> s1[t], h2 -> s2[t]}) + uv[[1]], s2'[t] ==
  J1[[2, 1]] s1[t] + J1[[2, 2]] s2[t] + (fv30[[2]] /. {h1 -> s1[t], h2 -> s2[t]}) + uv[[2]],
  s1[0] == s0[[1]], s2[0] == s0[[2]]}, {s1, s2}, {t, ti, tf}]
fig2 = Plot[Re[Evaluate[{s1[t], s2[t]} /. sol]], {t, ti, tf}]

Show[fig1, fig2]

v1h0 = h0 /. ans;
v1h0 >> "v1nt30.txt";

```

C.5

```
(* File name "Step 3 fplot.nb" *)
(* This function finds system response using the normal
form equation and compares with original system response *)
Clear["Global`*"]

SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]]; M1 = ReadList["M1.txt"][[1]];
v1nt = ReadList["v1nt.txt"][[1]]; lnresterm = ReadList["lnresterm.txt"][[1]];

invM1 = Inverse[M1];
J1 = Chop[invM1.R1.M1];

J2 = J1 + lnresterm;

ti = 0;
tf = 10;

s0 = {0.1, 0.1};

sol = NDSolve[{s1'[t] == J2[[1, 1]] s1[t] + J2[[1, 2]] s2[t],
s2'[t] == J2[[2, 1]] s1[t] + J2[[2, 2]] s2[t],
s1[0] == s0[[1]], s2[0] == s0[[2]]}, {s1, s2}, {t, ti, tf}]

v2nt = v1nt /. {v1 -> s1[t], v2 -> s2[t]};
fig1 = Plot[Q1.M1.{v2nt[[1]], v2nt[[2]]} /. sol,
{t, ti, tf}, PlotRange -> All, PlotStyle -> {Red, Green}]

R1 = ReadList["R1.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; M1 = ReadList["M1.txt"][[1]];
invM1 = Inverse[M1];
J1 = Chop[invM1.R1.M1];
z = {z1, z2};
x = Q1.M1.z;
fz = Chop[Expand[invM1.invQ1.fx /. {x1 -> x[[1]], x2 -> x[[2]]}]];
uz = Chop[Expand[invM1.invQ1.ux]];

v1s0 = Chop[Evaluate[v1nt /. {v1 -> s0[[1]], v2 -> s0[[2]]} /. t -> 0];
sol1 = NDSolve[{s1'[t] == J1[[1, 1]] s1[t] +
J1[[1, 2]] s2[t] + (fz[[1]] /. {z1 -> s1[t], z2 -> s2[t]}) + uz[[1]], s2'[t] ==
J1[[2, 1]] s1[t] + J1[[2, 2]] s2[t] + (fz[[2]] /. {z1 -> s1[t], z2 -> s2[t]}) + uz[[2]],
s1[0] == v1s0[[1]], s2[0] == v1s0[[2]]}, {s1, s2}, {t, ti, tf}]
fig5 = Plot[Evaluate[Q1.M1.{s1[t], s2[t]} /. sol1], {t, ti, tf},
PlotRange -> All, PlotStyle -> {Red, Green}]

Show[fig1, fig5]
```

C.6

```
(* File name "Step 1 Initialize.nb" *)
Clear["Global`*"]

SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; M1 = ReadList["M1.txt"][[1]];
invM1 = Inverse[M1];
(*Jp:=Jordan block associated with the dominant terms*)
(*Jq:=Jordan block associated with the non-dominant terms*)
(*Terminology of the other terms same as before*)
J1 = Chop[invM1.R1.M1];
z = {z1, z2, z3, z4};
x = Q1.M1.z;
fz = Chop[ExpToTrig[Expand[TrigToExp[
    invM1.invQ1.fx /. {x1 -> x[[1]], x2 -> x[[2]], x3 -> x[[3]], x4 -> x[[4]]}], 10^-3];
uz = Chop[ExpToTrig[Expand[TrigToExp[invM1.invQ1.ux]], 10^-3];
(*z1,z2:=Dominant states,z3,z4:=non-dominant states*)
Jp = {{J1[[1, 1]], J1[[1, 2]]}, {J1[[2, 1]], J1[[2, 2]]}};
Jq = {{J1[[3, 3]], J1[[3, 4]]}, {J1[[4, 3]], J1[[4, 4]]}};
fzp = {fz[[1]], fz[[2]]};
fzq = {fz[[3]], fz[[4]]} + {z1^4 + z2^4, z1^4 + z2^4};
uzp = {uz[[1]], uz[[2]]};
uzq = {uz[[3]], uz[[4]]};
p = 2;
q = 2;
```

```

hh = 12;
(*im:=Invariant manifold*)

im30 = {h1 + Sum[hc1[i] Cos[i ω t] + hs1[i] Sin[i ω t], {i, 1, hh}],
  h2 + Sum[hc2[i] Cos[i ω t] + hs2[i] Sin[i ω t], {i, 1, hh}]}];
im31 = {(h110 + Sum[hc110[i] Cos[i ω t] + hs110[i] Sin[i ω t], {i, 1, hh}]) z1 +
  (h101 + Sum[hc101[i] Cos[i ω t] + hs101[i] Sin[i ω t], {i, 1, hh}]) z2,
  (h210 + Sum[hc210[i] Cos[i ω t] + hs210[i] Sin[i ω t], {i, 1, hh}]) z1 +
  (h201 + Sum[hc201[i] Cos[i ω t] + hs201[i] Sin[i ω t], {i, 1, hh}]) z2};
im32 = {(h120 + Sum[hc120[i] Cos[i ω t] + hs120[i] Sin[i ω t], {i, 1, hh}]) z1^2 +
  (h111 + Sum[hc111[i] Cos[i ω t] + hs111[i] Sin[i ω t], {i, 1, hh}]) z1 z2 +
  (h102 + Sum[hc102[i] Cos[i ω t] + hs102[i] Sin[i ω t], {i, 1, hh}]) z2^2,
  (h220 + Sum[hc220[i] Cos[i ω t] + hs220[i] Sin[i ω t], {i, 1, hh}]) z1^2 +
  (h211 + Sum[hc211[i] Cos[i ω t] + hs211[i] Sin[i ω t], {i, 1, hh}]) z1 z2 +
  (h202 + Sum[hc202[i] Cos[i ω t] + hs202[i] Sin[i ω t], {i, 1, hh}]) z2^2};
im33 = {(h130 + Sum[hc130[i] Cos[i ω t] + hs130[i] Sin[i ω t], {i, 1, hh}]) z1^3 +
  (h121 + Sum[hc121[i] Cos[i ω t] + hs121[i] Sin[i ω t], {i, 1, hh}]) z1^2 z2 +
  (h112 + Sum[hc112[i] Cos[i ω t] + hs112[i] Sin[i ω t], {i, 1, hh}]) z1 z2^2 +
  (h103 + Sum[hc103[i] Cos[i ω t] + hs103[i] Sin[i ω t], {i, 1, hh}]) z2^3,
  (h230 + Sum[hc230[i] Cos[i ω t] + hs230[i] Sin[i ω t], {i, 1, hh}]) z1^3 +
  (h221 + Sum[hc221[i] Cos[i ω t] + hs221[i] Sin[i ω t], {i, 1, hh}]) z1^2 z2 +
  (h212 + Sum[hc212[i] Cos[i ω t] + hs212[i] Sin[i ω t], {i, 1, hh}]) z1 z2^2 +
  (h203 + Sum[hc203[i] Cos[i ω t] + hs203[i] Sin[i ω t], {i, 1, hh}]) z2^3};
im = Expand[im30 + im31 + im32 + im33];

coefLfzq = CoefficientList[fzq, {z1, z2}];
(* Terms only upto order ε are kept *)
LHS = Chop[ExpToTrig[
  Expand[TrigToExp[∂t im + {{∂z1 im[[1]], ∂z2 im[[1]]}, {∂z1 im[[2]], ∂z2 im[[2]]}].
    (Jp.{z1, z2} + uzp) - Jq.{im[[1]], im[[2]]} -
    (fzq - {coefLfzq[[1]][[4, 1]] z1^3 + coefLfzq[[1]][[3, 2]] z1^2 z2 +
      coefLfzq[[1]][[2, 3]] z1 z2^2 + coefLfzq[[1]][[1, 4]] z2^3,
      coefLfzq[[2]][[4, 1]] z1^3 + coefLfzq[[2]][[3, 2]] z1^2 z2 +
      coefLfzq[[2]][[2, 3]] z1 z2^2 + coefLfzq[[2]][[1, 4]] z2^3}) /.
    {z3 → im30[[1]], z4 → im30[[2]]}], 10^-3];
ω = ReadList["omega.txt"][[1]]; (* ω introduced here to avoid
  imaginary numbers in above expression *)

RHS = {coefLfzq[[1]][[4, 1]] z1^3 + coefLfzq[[1]][[3, 2]] z1^2 z2 +
  coefLfzq[[1]][[2, 3]] z1 z2^2 + coefLfzq[[1]][[1, 4]] z2^3 - z1^4 - z2^4,
  coefLfzq[[2]][[4, 1]] z1^3 + coefLfzq[[2]][[3, 2]] z1^2 z2 +
  coefLfzq[[2]][[2, 3]] z1 z2^2 + coefLfzq[[2]][[1, 4]] z2^3 - z1^4 - z2^4} + uzq;
(* Highlighted figures are dummy terms required for proper
  execution of command "CoefficientList" *)

coefLLHS = CoefficientList[LHS, {z1, z2}];
coefLRHS = CoefficientList[RHS, {z1, z2}];

coefLHS = Chop[{coefLLHS[[1]][[1, 1]], coefLLHS[[1]][[2, 1]],
  coefLLHS[[1]][[1, 2]], coefLLHS[[1]][[3, 1]], coefLLHS[[1]][[2, 2]],

```

```

coefLLHS[[1]][[1, 3]], coefLLHS[[1]][[4, 1]], coefLLHS[[1]][[3, 2]],
coefLLHS[[1]][[2, 3]], coefLLHS[[1]][[1, 4]], coefLLHS[[2]][[1, 1]],
coefLLHS[[2]][[2, 1]], coefLLHS[[2]][[1, 2]], coefLLHS[[2]][[3, 1]],
coefLLHS[[2]][[2, 2]], coefLLHS[[2]][[1, 3]], coefLLHS[[2]][[4, 1]],
coefLLHS[[2]][[3, 2]], coefLLHS[[2]][[2, 3]], coefLLHS[[2]][[1, 4]]};

coefTLHS = Table[0, {2*10}, {1+2*hh}];

Do[
  Do[If[Exponent[TrigToExp[coefLHS[[i, k]]], E] == 0,
    {coefTLHS[[i, 1]] = coefTLHS[[i, 1]] + coefLHS[[i, k]]}],
    {k, 1, Dimensions[coefLHS[[i]]][[1]]}, {i, 1, 2*10}];

Do[
  coefTLHS[[i, 2*j]] = Coefficient[coefLHS[[i]], Cos[j*omega*t]];
  coefTLHS[[i, 2*j+1]] = Coefficient[coefLHS[[i]], Sin[j*omega*t]];
  {i, 1, 2*10}, {j, 1, hh}];

coefRHS = Chop[{coefLRHS[[1]][[1, 1]], coefLRHS[[1]][[2, 1]],
  coefLRHS[[1]][[1, 2]], coefLRHS[[1]][[3, 1]], coefLRHS[[1]][[2, 2]],
  coefLRHS[[1]][[1, 3]], coefLRHS[[1]][[4, 1]], coefLRHS[[1]][[3, 2]],
  coefLRHS[[1]][[2, 3]], coefLRHS[[1]][[1, 4]], coefLRHS[[2]][[1, 1]],
  coefLRHS[[2]][[2, 1]], coefLRHS[[2]][[1, 2]], coefLRHS[[2]][[3, 1]],
  coefLRHS[[2]][[2, 2]], coefLRHS[[2]][[1, 3]], coefLRHS[[2]][[4, 1]],
  coefLRHS[[2]][[3, 2]], coefLRHS[[2]][[2, 3]], coefLRHS[[2]][[1, 4]]}
+ {Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t],
  Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t], Cos[(hh+2)*omega*t] + Sin[(hh+2)*omega*t]};

(* Highlighted figures are dummy terms required
for proper execution of command "CoefficientList" *)

coefTRHS = Table[0, {2*10}, {1+2*hh}];

Do[
  Do[If[Exponent[TrigToExp[coefRHS[[i, k]]], E] == 0,
    {coefTRHS[[i, 1]] = coefTRHS[[i, 1]] + coefRHS[[i, k]]}],
    {k, 1, Dimensions[coefRHS[[i]]][[1]]}, {i, 1, 2*10}];

Do[
  coefTRHS[[i, 2*j]] = Coefficient[coefRHS[[i]], Cos[j*omega*t]];
  coefTRHS[[i, 2*j+1]] = Coefficient[coefRHS[[i]], Sin[j*omega*t]];
  {i, 1, 2*10}, {j, 1, hh}];

```

```

equL = Chop[coefTLHS - coefTRHS];
t0 = TimeUsed[];
ans = 0;

While[ans == 0 ,
  varL = Flatten[{{h1, RandomReal[{-0.5, 0.5}]}, {h2, RandomReal[{-0.5, 0.5}]},
    {h110, RandomReal[{-0.5, 0.5}]}, {h101, RandomReal[{-0.5, 0.5}]},
    {h210, RandomReal[{-0.5, 0.5}]}, {h201, RandomReal[{-0.5, 0.5}]},
    {h120, RandomReal[{-0.5, 0.5}]}, {h111, RandomReal[{-0.5, 0.5}]},
    {h102, RandomReal[{-0.5, 0.5}]}, {h220, RandomReal[{-0.5, 0.5}]},
    {h211, RandomReal[{-0.5, 0.5}]}, {h202, RandomReal[{-0.5, 0.5}]},
    {h130, RandomReal[{-0.5, 0.5}]}, {h121, RandomReal[{-0.5, 0.5}]},
    {h112, RandomReal[{-0.5, 0.5}]}, {h103, RandomReal[{-0.5, 0.5}]},
    {h230, RandomReal[{-0.5, 0.5}]}, {h221, RandomReal[{-0.5, 0.5}]},
    {h212, RandomReal[{-0.5, 0.5}]}, {h203, RandomReal[{-0.5, 0.5}]},
  Flatten[Table[{{hc1[i], RandomReal[{-0.5, 0.5}]},
    {hs1[i], RandomReal[{-0.5, 0.5}]}, {hc2[i], RandomReal[{-0.5, 0.5}]},
    {hs2[i], RandomReal[{-0.5, 0.5}]}, {hc110[i], RandomReal[{-0.5, 0.5}]},
    {hs110[i], RandomReal[{-0.5, 0.5}]}, {hc101[i], RandomReal[{-0.5, 0.5}]},
    {hs101[i], RandomReal[{-0.5, 0.5}]}, {hc210[i], RandomReal[{-0.5, 0.5}]},
    {hs210[i], RandomReal[{-0.5, 0.5}]}, {hc201[i], RandomReal[{-0.5, 0.5}]},
    {hs201[i], RandomReal[{-0.5, 0.5}]}, {hc120[i], RandomReal[{-0.5, 0.5}]},
    {hs120[i], RandomReal[{-0.5, 0.5}]}, {hc111[i], RandomReal[{-0.5, 0.5}]},
    {hs111[i], RandomReal[{-0.5, 0.5}]}, {hc102[i], RandomReal[{-0.5, 0.5}]},
    {hs102[i], RandomReal[{-0.5, 0.5}]}, {hc220[i], RandomReal[{-0.5, 0.5}]},
    {hs220[i], RandomReal[{-0.5, 0.5}]}, {hc211[i], RandomReal[{-0.5, 0.5}]},
    {hs211[i], RandomReal[{-0.5, 0.5}]}, {hc202[i], RandomReal[{-0.5, 0.5}]},
    {hs202[i], RandomReal[{-0.5, 0.5}]}, {hc130[i], RandomReal[{-0.5, 0.5}]},
    {hs130[i], RandomReal[{-0.5, 0.5}]}, {hc121[i], RandomReal[{-0.5, 0.5}]},
    {hs121[i], RandomReal[{-0.5, 0.5}]}, {hc112[i], RandomReal[{-0.5, 0.5}]},
    {hs112[i], RandomReal[{-0.5, 0.5}]}, {hc103[i], RandomReal[{-0.5, 0.5}]},
    {hs103[i], RandomReal[{-0.5, 0.5}]}, {hc230[i], RandomReal[{-0.5, 0.5}]},
    {hs230[i], RandomReal[{-0.5, 0.5}]}, {hc221[i], RandomReal[{-0.5, 0.5}]},
    {hs221[i], RandomReal[{-0.5, 0.5}]}, {hc212[i], RandomReal[{-0.5, 0.5}]},
    {hs212[i], RandomReal[{-0.5, 0.5}]}, {hc203[i], RandomReal[{-0.5, 0.5}]},
    {hs203[i], RandomReal[{-0.5, 0.5}]}, {i, 1, hh}], 1}], 1];
  ans = Check[Chop[FindRoot[equL, varL], 10^-3], 0]; Print[ans];
t1 = TimeUsed[];
Beep[]
totaltime = (t1 - t0) / 60

vlim = im /. ans;
Chop[vlim, 10^-3] >> "vlim.txt"

```

C.7

```
(* File name "Step 1 Initialize.nb" *)
Clear["Global`*"]

SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]];
 $\omega$  = ReadList["omega.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; M1 = ReadList["M1.txt"][[1]];

invM1 = Inverse[M1];

J1 = Chop[invM1.R1.M1];

z = {z1, z2, z3, z4};
x = Q1.M1.z;

fz = Chop[Expand[
  TrigToExp[invM1.invQ1.fx /. {x1  $\rightarrow$  x[[1]], x2  $\rightarrow$  x[[2]], x3  $\rightarrow$  x[[3]], x4  $\rightarrow$  x[[4]]}]];
uz = Chop[Expand[TrigToExp[invM1.invQ1.ux]]];

Jp = {{J1[[1, 1]], J1[[1, 2]]}, {J1[[2, 1]], J1[[2, 2]]}};
Jq = {{J1[[3, 3]], J1[[3, 4]]}, {J1[[4, 3]], J1[[4, 4]]}};
p = 2;
q = 2;
q >> "q.txt";

(* Construction of invariant manifold *)
(* im :- Invariant Manifold *)
im30 = {h1, h2};
im31 = e {h110 z1 + h101 z2, h210 z1 + h201 z2};
im32 = e {h120 z1^2 + h111 z1 z2 + h102 z2^2, h220 z1^2 + h211 z1 z2 + h202 z2^2};
im33 = e {h130 z1^3 + h121 z1^2 z2 + h112 z1 z2^2 + h103 z2^3,
  h230 z1^3 + h221 z1^2 z2 + h212 z1 z2^2 + h203 z2^3};
im = im30 + im31 + im32 + im33;
fzq = Expand[
  e {fz[[3]] /. {z3  $\rightarrow$  im[[1]], z4  $\rightarrow$  im[[2]]}, fz[[4]] /. {z3  $\rightarrow$  im[[1]], z4  $\rightarrow$  im[[2]]}}];
coeffzq1 = CoefficientList[fzq[[1]], {e, h1, h2, z1, z2}];
coeffzq2 = CoefficientList[fzq[[2]], {e, h1, h2, z1, z2}];
hh = 6;
```



```

(* Finding im30 for cubic nonlinearity *)
fzq30 = Expand[{coeffzq1[[2, 4, 1, 1, 1]] h1^3 + coeffzq1[[2, 3, 2, 1, 1]] h1^2 h2 +
  coeffzq1[[2, 2, 3, 1, 1]] h1 h2^2 + coeffzq1[[2, 1, 4, 1, 1]] h2^3,
  coeffzq2[[2, 4, 1, 1, 1]] h1^3 + coeffzq2[[2, 3, 2, 1, 1]] h1^2 h2 +
  coeffzq2[[2, 2, 3, 1, 1]] h1 h2^2 + coeffzq2[[2, 1, 4, 1, 1]] h2^3}];
uzq = {uz[[3]], uz[[4]]};
Jq >> "Jq.txt";
fzq30 >> "fzq30.txt";
uzq >> "uzq.txt";
(* ***** Use routine "Harmonic solution * .nb" to find im30 ***** *)

(* ***** Finding im31 ***** *)
vlim30 = TrigToExp[ReadList["vlim30.txt"][[1]]];
fzq31 = Expand[{coeffzq1[[2, 3, 1, 2, 1]] h1^2 z1 + coeffzq1[[2, 2, 2, 2, 1]] h1 h2 z1 +
  coeffzq1[[2, 1, 3, 2, 1]] h2^2 z1 + coeffzq1[[2, 3, 1, 1, 2]] h1^2 z2 +
  coeffzq1[[2, 2, 2, 1, 2]] h1 h2 z2 + coeffzq1[[2, 1, 3, 1, 2]] h2^2 z2,
  coeffzq2[[2, 3, 1, 2, 1]] h1^2 z1 + coeffzq2[[2, 2, 2, 2, 1]] h1 h2 z1 +
  coeffzq2[[2, 1, 3, 2, 1]] h2^2 z1 + coeffzq2[[2, 3, 1, 1, 2]] h1^2 z2 +
  coeffzq2[[2, 2, 2, 1, 2]] h1 h2 z2 + coeffzq2[[2, 1, 3, 1, 2]] h2^2 z2}];

coeffzq31 = CoefficientList[fzq31, {z1, z2}];
cofeq31RHS =
  Chop[Expand[{coeffzq31[[1]][[2, 1]], coeffzq31[[1]][[1, 2]], coeffzq31[[2]][[2, 1]],
    coeffzq31[[2]][[1, 2]]} /. {h1 -> vlim30[[1]], h2 -> vlim30[[2]]}]];

coefTeq31RHS = Table[0, {q * 2}, {1 + 2 * hh}];
Do[
  Do[ If[Exponent[cofeq31RHS[[i, k]], E] == 0,
    {coefTeq31RHS[[i, 1]] = coefTeq31RHS[[i, 1]] + cofeq31RHS[[i, k]]},
    {k, 1, Dimensions[cofeq31RHS[[i]][[1]]}], {i, 1, q * 2}];
Do[
  coefTeq31RHS[[i, 2 * j]] = Coefficient[cofeq31RHS[[i]], E^(j * I * ω * t)];
  coefTeq31RHS[[i, 2 * j + 1]] = Coefficient[cofeq31RHS[[i]], E^(-j * I * ω * t)];,
  {i, 1, q * 2}, {j, 1, hh}];

(* eq31LHS:- LHS of 3rd equation for cubic nonlinearity *)
eq31LHS =
  Chop[Expand[{{∂z1 im31[[1]], ∂z2 im31[[1]]}, {∂z1 im31[[2]], ∂z2 im31[[2]]}}].Jp.
  {z1, z2} - Jq.im31];

cofeq31LHS1 = CoefficientList[eq31LHS[[1]], {ε, z1, z2}];
cofeq31LHS2 = CoefficientList[eq31LHS[[2]], {ε, z1, z2}];
cofeq31LHS = Chop[{cofeq31LHS1[[2, 2, 1]],
  cofeq31LHS1[[2, 1, 2]], cofeq31LHS2[[2, 2, 1]], cofeq31LHS2[[2, 1, 2]]}];

coefTeq31LHS = Table[0, {q * 2}, {1 + 2 * hh}];
Do[ coefTeq31LHS[[i, 1]] = cofeq31LHS[[i]], {i, 1, q * 2}];
Do[
  coefTeq31LHS[[i, 2 * j]] = cofeq31LHS[[i]];
  coefTeq31LHS[[i, 2 * j + 1]] = cofeq31LHS[[i]]; , {i, 1, q * 2}, {j, 1, hh}];

```

```

Do[
  A31[i] = Expand[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω * IdentityMatrix[4] +
    {{Coefficient[coefTeq31LHS[[1, i]], h110],
      Coefficient[coefTeq31LHS[[1, i]], h101], Coefficient[
        coefTeq31LHS[[1, i]], h210], Coefficient[coefTeq31LHS[[1, i]], h201]},
    {Coefficient[coefTeq31LHS[[2, i]], h110], Coefficient[coefTeq31LHS[[2, i]],
      h101], Coefficient[coefTeq31LHS[[2, i]], h210], Coefficient[
        coefTeq31LHS[[2, i]], h201]}, {Coefficient[coefTeq31LHS[[3, i]], h110],
      Coefficient[coefTeq31LHS[[3, i]], h101], Coefficient[
        coefTeq31LHS[[3, i]], h210], Coefficient[coefTeq31LHS[[3, i]], h201]},
    {Coefficient[coefTeq31LHS[[4, i]], h110], Coefficient[
      coefTeq31LHS[[4, i]], h101], Coefficient[coefTeq31LHS[[4, i]], h210],
      Coefficient[coefTeq31LHS[[4, i]], h201]}}];
  C31[i] = {coefTeq31RHS[[1, i]], coefTeq31RHS[[2, i]],
    coefTeq31RHS[[3, i]], coefTeq31RHS[[4, i]]};
  , {i, 1, 1 + 2*hh}];
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
h31 = Chop[Expand[Sum[Inverse[A31[i]].C31[i] *
  E^((i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω t), {i, 1, 1 + 2*hh}]]];
(* h31={h110,h101,h210,h01} *)
vlim31 = Expand[
  (im31 / ε) /. {h110 → h31[[1]], h101 → h31[[2]], h210 → h31[[3]], h201 → h31[[4]]}];

(* ***** Finding im32 ***** *)
fzq32 = Expand[{coeffzq1[[2, 2, 1, 3, 1]] h1 z1^2 + coeffzq1[[2, 2, 1, 2, 2]] h1 z1 z2 +
  coeffzq1[[2, 2, 1, 1, 3]] h1 z2^2 + coeffzq1[[2, 1, 2, 3, 1]] h2 z1^2 +
  coeffzq1[[2, 1, 2, 2, 2]] h2 z1 z2 + coeffzq1[[2, 1, 2, 1, 3]] h2 z2^2,
  coeffzq2[[2, 2, 1, 3, 1]] h1 z1^2 + coeffzq2[[2, 2, 1, 2, 2]] h1 z1 z2 +
  coeffzq2[[2, 2, 1, 1, 3]] h1 z2^2 + coeffzq2[[2, 1, 2, 3, 1]] h2 z1^2 +
  coeffzq2[[2, 1, 2, 2, 2]] h2 z1 z2 + coeffzq2[[2, 1, 2, 1, 3]] h2 z2^2}];
(* fzq32:- 2st poewer in zp *)

coeffzq32 = CoefficientList[fzq32, {z1, z2}];
coefeq32RHS =
  Chop[Expand[{coeffzq32[[1]][[3, 1]], coeffzq32[[1]][[2, 2]], coeffzq32[[1]][[1, 3]],
    coeffzq32[[2]][[3, 1]], coeffzq32[[2]][[2, 2]], coeffzq32[[2]][[1, 3]]} /.
    {h1 → vlim30[[1]], h2 → vlim30[[2]]}]];

coefTeq32RHS = Table[0, {q*3}, {1 + 2*hh}];
Do[
  Do[ If[Exponent[coefeq32RHS[[i, k]], E] == 0,
    {coefTeq32RHS[[i, 1]] = coefTeq32RHS[[i, 1]] + coefeq32RHS[[i, k]]},
    {k, 1, Dimensions[coefeq32RHS[[i]][[1]]}], {i, 1, q*3}];
Do[
  coefTeq32RHS[[i, 2*j]] = Coefficient[coefeq32RHS[[i]], E^(j*I*ω*t)];
  coefTeq32RHS[[i, 2*j+1]] = Coefficient[coefeq32RHS[[i]], E^(-j*I*ω*t)];,
  {i, 1, q*3}, {j, 1, hh}];

(* eq33LHS:- LHS of 3rd equation for cubic nonlinearity *)

```

```

eq32LHS =
  Expand[{{∂z1 im32[[1]], ∂z2 im32[[1]]}, {∂z1 im32[[2]], ∂z2 im32[[2]]}}.Jp.{z1, z2} -
  Jq.im32];

cofeq32LHS1 = CoefficientList[eq32LHS[[1]], {ε, z1, z2}];
cofeq32LHS2 = CoefficientList[eq32LHS[[2]], {ε, z1, z2}];

cofeq32LHS =
  Chop[{cofeq32LHS1[[2, 3, 1]], cofeq32LHS1[[2, 2, 2]], cofeq32LHS1[[2, 1, 3]],
  cofeq32LHS2[[2, 3, 1]], cofeq32LHS2[[2, 2, 2]], cofeq32LHS2[[2, 1, 3]]}];

coefTeq32LHS = Table[0, {q * 3}, {1 + 2 * hh}];
Do[coefTeq32LHS[[i, 1]] = cofeq32LHS[[i]], {i, 1, q * 3}];
Do[
  coefTeq32LHS[[i, 2 * j]] = cofeq32LHS[[i]];
  coefTeq32LHS[[i, 2 * j + 1]] = cofeq32LHS[[i]], {i, 1, q * 3}, {j, 1, hh}];

Do[
  A32[i] = Expand[(i (1 + (-1)^i) / 4 + (i - 1) (-1 + (-1)^i) / 4) I ω * IdentityMatrix[6] +
  {{Coefficient[coefTeq32LHS[[1, i]], h120], Coefficient[
  coefTeq32LHS[[1, i]], h111], Coefficient[coefTeq32LHS[[1, i]], h102],
  Coefficient[coefTeq32LHS[[1, i]], h220], Coefficient[
  coefTeq32LHS[[1, i]], h211], Coefficient[coefTeq32LHS[[1, i]], h202]},
  {Coefficient[coefTeq32LHS[[2, i]], h120], Coefficient[
  coefTeq32LHS[[2, i]], h111], Coefficient[coefTeq32LHS[[2, i]], h102],
  Coefficient[coefTeq32LHS[[2, i]], h220], Coefficient[
  coefTeq32LHS[[2, i]], h211], Coefficient[coefTeq32LHS[[2, i]], h202]},
  {Coefficient[coefTeq32LHS[[3, i]], h120], Coefficient[
  coefTeq32LHS[[3, i]], h111], Coefficient[coefTeq32LHS[[3, i]], h102],
  Coefficient[coefTeq32LHS[[3, i]], h220], Coefficient[
  coefTeq32LHS[[3, i]], h211], Coefficient[coefTeq32LHS[[3, i]], h202]},
  {Coefficient[coefTeq32LHS[[4, i]], h120], Coefficient[
  coefTeq32LHS[[4, i]], h111], Coefficient[coefTeq32LHS[[4, i]], h102],
  Coefficient[coefTeq32LHS[[4, i]], h220], Coefficient[
  coefTeq32LHS[[4, i]], h211], Coefficient[coefTeq32LHS[[4, i]], h202]},
  {Coefficient[coefTeq32LHS[[5, i]], h120], Coefficient[
  coefTeq32LHS[[5, i]], h111], Coefficient[coefTeq32LHS[[5, i]], h102],
  Coefficient[coefTeq32LHS[[5, i]], h220], Coefficient[
  coefTeq32LHS[[5, i]], h211], Coefficient[coefTeq32LHS[[5, i]], h202]},
  {Coefficient[coefTeq32LHS[[6, i]], h120], Coefficient[
  coefTeq32LHS[[6, i]], h111], Coefficient[coefTeq32LHS[[6, i]], h102],
  Coefficient[coefTeq32LHS[[6, i]], h220], Coefficient[coefTeq32LHS[[6, i]],
  h211], Coefficient[coefTeq32LHS[[6, i]], h202]}}];
  C32[i] = {coefTeq32RHS[[1, i]], coefTeq32RHS[[2, i]], coefTeq32RHS[[3, i]],
  coefTeq32RHS[[4, i]], coefTeq32RHS[[5, i]],
  coefTeq32RHS[[6, i]]};
  , {i, 1, 1 + 2 * hh}];
  (* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
  (* Solvability condition *)
  Do[If[Det[A32[i]] == 0, {Print["Resonance"]}];

```

```

Print[(i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω]], {i, 1, 1 + 2 * hh}];

h32 = Chop[Expand[Sum[Inverse[A32[i]].C32[i] *
  E^((i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω t), {i, 1, 1 + 2 * hh}]]];
(* h31={h120,h111,h102,h220,h211,h202} *)
vlim32 = (im32 / ε) /. {h120 → h32[[1]], h111 → h32[[2]],
  h102 → h32[[3]], h220 → h32[[4]], h211 → h32[[5]], h202 → h32[[6]]];

(* ***** Finding im33 ***** *)
fzq33 = Expand[{coeffzq1[[2, 1, 1, 4, 1]] z1^3 + coeffzq1[[2, 1, 1, 3, 2]] z1^2 z2 +
  coeffzq1[[2, 1, 1, 2, 3]] z1 z2^2 + coeffzq1[[2, 1, 1, 1, 4]] z2^3 + z1^4 + z2^4,
  coeffzq2[[2, 1, 1, 4, 1]] z1^3 + coeffzq2[[2, 1, 1, 3, 2]] z1^2 z2 +
  coeffzq2[[2, 1, 1, 2, 3]] z1 z2^2 + coeffzq2[[2, 1, 1, 1, 4]] z2^3 + z1^4 + z2^4}];

coeffzq33 = CoefficientList[fzq33, {z1, z2}];
cofeq33RHS =
  Expand[{coeffzq33[[1]][[4, 1]], coeffzq33[[1]][[3, 2]], coeffzq33[[1]][[2, 3]],
  coeffzq33[[1]][[1, 4]], coeffzq33[[2]][[4, 1]], coeffzq33[[2]][[3, 2]],
  coeffzq33[[2]][[2, 3]], coeffzq33[[2]][[1, 4]]} +
  {{(0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t},
  (0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t},
  (0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t}, (0.5 + 0.5 I) e^{-10 i π t} +
  (0.5 - 0.5 I) e^{10 i π t}, (0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t},
  (0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t}, (0.5 + 0.5 I) e^{-10 i π t} +
  (0.5 - 0.5 I) e^{10 i π t}, (0.5 + 0.5 I) e^{-10 i π t} + (0.5 - 0.5 I) e^{10 i π t}}];

coefTeq33RHS = Table[0, {q * 4}, {1 + 2 * hh}];
Do[
  Do[ If[Exponent[cofeq33RHS[[i, k]], E] == 0,
    {coefTeq33RHS[[i, 1]] = coefTeq33RHS[[i, 1]] + cofeq33RHS[[i, k]]},
    {k, 1, Dimensions[cofeq33RHS[[i]]][[1]]}], {i, 1, q * 4}];
Do[
  coefTeq33RHS[[i, 2 * j]] = Coefficient[cofeq33RHS[[i]], E^(j * I * ω * t)];
  coefTeq33RHS[[i, 2 * j + 1]] = Coefficient[cofeq33RHS[[i]], E^(-j * I * ω * t)];
  {i, 1, q * 4}, {j, 1, hh}];

eq33LHS =
  Expand[{{∂z1 im33[[1]], ∂z2 im33[[1]]}, {∂z1 im33[[2]], ∂z2 im33[[2]]}}.Jp.{z1, z2} -
  Jq.im33];

cofeq33LHS1 = CoefficientList[eq33LHS[[1]], {ε, z1, z2}];
cofeq33LHS2 = CoefficientList[eq33LHS[[2]], {ε, z1, z2}];

cofeq33LHS = Chop[{cofeq33LHS1[[2, 4, 1]], cofeq33LHS1[[2, 3, 2]],
  cofeq33LHS1[[2, 2, 3]], cofeq33LHS1[[2, 1, 4]], cofeq33LHS2[[2, 4, 1]],
  cofeq33LHS2[[2, 3, 2]], cofeq33LHS2[[2, 2, 3]], cofeq33LHS2[[2, 1, 4]]}];

coefTeq33LHS = Table[0, {q * 4}, {1 + 2 * hh}];
Do[ coefTeq33LHS[[i, 1]] = cofeq33LHS[[i]], {i, 1, q * 4}];
Do[

```

```

coefTeq33LHS[[i, 2 * j]] = coefeq33LHS[[i]];
coefTeq33LHS[[i, 2 * j + 1]] = coefeq33LHS[[i]]; , {i, 1, q * 4}, {j, 1, hh}};

```

```

Do[

```

```

A33[i] = Expand[(i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω * IdentityMatrix[8] +
  {{Coefficient[coefTeq33LHS[[1, i]], h130], Coefficient[
    coefTeq33LHS[[1, i]], h121], Coefficient[coefTeq33LHS[[1, i]], h112],
    Coefficient[coefTeq33LHS[[1, i]], h103], Coefficient[coefTeq33LHS[[1, i]],
    h230], Coefficient[coefTeq33LHS[[1, i]], h221], Coefficient[
    coefTeq33LHS[[1, i]], h212], Coefficient[coefTeq33LHS[[1, i]], h203]},
  {Coefficient[coefTeq33LHS[[2, i]], h130], Coefficient[coefTeq33LHS[[2, i]],
    h121], Coefficient[coefTeq33LHS[[2, i]], h112], Coefficient[
    coefTeq33LHS[[2, i]], h103], Coefficient[coefTeq33LHS[[2, i]], h230],
    Coefficient[coefTeq33LHS[[2, i]], h221], Coefficient[
    coefTeq33LHS[[2, i]], h212], Coefficient[coefTeq33LHS[[2, i]], h203]},
  {Coefficient[coefTeq33LHS[[3, i]], h130], Coefficient[coefTeq33LHS[[3, i]],
    h121], Coefficient[coefTeq33LHS[[3, i]], h112], Coefficient[
    coefTeq33LHS[[3, i]], h103], Coefficient[coefTeq33LHS[[3, i]], h230],
    Coefficient[coefTeq33LHS[[3, i]], h221], Coefficient[
    coefTeq33LHS[[3, i]], h212], Coefficient[coefTeq33LHS[[3, i]], h203]},
  {Coefficient[coefTeq33LHS[[4, i]], h130], Coefficient[coefTeq33LHS[[4, i]],
    h121], Coefficient[coefTeq33LHS[[4, i]], h112], Coefficient[
    coefTeq33LHS[[4, i]], h103], Coefficient[coefTeq33LHS[[4, i]], h230],
    Coefficient[coefTeq33LHS[[4, i]], h221], Coefficient[
    coefTeq33LHS[[4, i]], h212], Coefficient[coefTeq33LHS[[4, i]], h203]},
  {Coefficient[coefTeq33LHS[[5, i]], h130], Coefficient[coefTeq33LHS[[5, i]],
    h121], Coefficient[coefTeq33LHS[[5, i]], h112], Coefficient[
    coefTeq33LHS[[5, i]], h103], Coefficient[coefTeq33LHS[[5, i]], h230],
    Coefficient[coefTeq33LHS[[5, i]], h221], Coefficient[
    coefTeq33LHS[[5, i]], h212], Coefficient[coefTeq33LHS[[5, i]], h203]},
  {Coefficient[coefTeq33LHS[[6, i]], h130], Coefficient[coefTeq33LHS[[6, i]],
    h121], Coefficient[coefTeq33LHS[[6, i]], h112], Coefficient[
    coefTeq33LHS[[6, i]], h103], Coefficient[coefTeq33LHS[[6, i]], h230],
    Coefficient[coefTeq33LHS[[6, i]], h221], Coefficient[
    coefTeq33LHS[[6, i]], h212], Coefficient[coefTeq33LHS[[6, i]], h203]},
  {Coefficient[coefTeq33LHS[[7, i]], h130], Coefficient[coefTeq33LHS[[7, i]],
    h121], Coefficient[coefTeq33LHS[[7, i]], h112], Coefficient[
    coefTeq33LHS[[7, i]], h103], Coefficient[coefTeq33LHS[[7, i]], h230],
    Coefficient[coefTeq33LHS[[7, i]], h221], Coefficient[
    coefTeq33LHS[[7, i]], h212], Coefficient[coefTeq33LHS[[7, i]], h203]},
  {Coefficient[coefTeq33LHS[[8, i]], h130], Coefficient[coefTeq33LHS[[8, i]],
    h121], Coefficient[coefTeq33LHS[[8, i]], h112], Coefficient[
    coefTeq33LHS[[8, i]], h103], Coefficient[coefTeq33LHS[[8, i]], h230],
    Coefficient[coefTeq33LHS[[8, i]], h221], Coefficient[coefTeq33LHS[[8, i]],
    h212], Coefficient[coefTeq33LHS[[8, i]], h203]}}];

```

```

C33[i] = {coefTeq33RHS[[1, i]], coefTeq33RHS[[2, i]],
  coefTeq33RHS[[3, i]], coefTeq33RHS[[4, i]], coefTeq33RHS[[5, i]],
  coefTeq33RHS[[6, i]], coefTeq33RHS[[7, i]], coefTeq33RHS[[8, i]]};
, {i, 1, 1 + 2 * hh}};

```

```

(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
(* Solvability condition *)
Do[If[Det[A33[i]] == 0, {Print["Resonance"];
  Print[(i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω]}], {i, 1, 1 + 2 * hh}];

h33 = Chop[Expand[Sum[Inverse[A33[i]].C33[i] *
  E^((i (1 + (-1) ^ i) / 4 + (i - 1) (-1 + (-1) ^ i) / 4) I ω t), {i, 1, 1 + 2 * hh}]]];
(* h31={h120,h111,h102,h220,h211,h202} *)
vlim33 =
  (im33 / ε) /. {h130 → h33[[1]], h121 → h33[[2]], h112 → h33[[3]], h103 → h33[[4]],
    h230 → h33[[5]], h221 → h33[[6]], h212 → h33[[7]], h203 → h33[[8]]};
vlim = vlim30 + vlim31 + vlim32 + vlim33;
Chop[vlim, 10^-3] >> "vlim.txt"
(* vlim30.txt is generated by Harmonic solution *.nb *)
Chop[vlim31, 10^-3] >> "vlim31.txt"
Chop[vlim32, 10^-3] >> "vlim32.txt"
Chop[vlim33, 10^-3] >> "vlim33.txt"

```

C.8

```

(* File name "Step 3 fplot.nb" *)
Clear["Global`*"]

SetDirectory[NotebookDirectory[]];
(* ***** fft data ***** *)
 $\omega = \text{ReadList}["\omega.txt"][[1]];
\omega_{\max} = 7 \pi;
\omega_{\min} = \omega;
(* t_{fft} = N[ti + 2\pi/\omega_{\min}]; *)
tint = N[(2 * \pi / \omega_{\min}) / \text{Ceiling}[N[\omega_{\max} / \text{Pi}] * (2 * \pi / \omega_{\min})]];
(* ***** *)

SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]];
M1 = ReadList["M1.txt"][[1]]; vlim = ReadList["vlim.txt"][[1]];

invM1 = Inverse[M1];
J1 = Chop[invM1.R1.M1];
z = {z1, z2, z3, z4};
x = Q1.M1.z;

fz = Chop[Expand[
  TrigToExp[invM1.invQ1.fx /. {x1 -> x[[1]], x2 -> x[[2]], x3 -> x[[3]], x4 -> x[[4]]}]];
uz = Chop[Expand[TrigToExp[invM1.invQ1.ux]]];

Jp = {{J1[[1, 1]], J1[[1, 2]]}, {J1[[2, 1]], J1[[2, 2]]}};
fzfp = {fz[[1]], fz[[2]]};
vlfzfp =
  Chop[ExpToTrig[Expand[TrigToExp[fzfp /. {z3 -> vlim[[1]], z4 -> vlim[[2]]}]]], 10^-3];
uzp = {uz[[1]], uz[[2]]};

ti = 00;
tf = 20;
x0 = {0.1, 0.1, 0.1, 0.1};
s0 = (invM1.invQ1.x0) /. t -> 0;
sol = NDSolve[{s1'[t] == Jp[[1, 1]] s1[t] + Jp[[1, 2]] s2[t] +
  (vlfzfp[[1]] /. {z1 -> s1[t], z2 -> s2[t]}) + uzp[[1]], s2'[t] == Jp[[2, 1]] s1[t] +
  Jp[[2, 2]] s2[t] + (vlfzfp[[2]] /. {z1 -> s1[t], z2 -> s2[t]}) + uzp[[2]],
  s1[0] == s0[[3]], s2[0] == s0[[4]]}, {s1, s2}, {t, ti, tf}]

fig = Plot[Evaluate[Q1.M1.{s1[t], s2[t], vlim[[1]] /. {z1 -> s1[t], z2 -> s2[t]},
  vlim[[2]] /. {z1 -> s1[t], z2 -> s2[t]}] /. sol], {t, ti, tf},
  PlotRange -> All, PlotStyle -> {Red, Green, {Red, Thick}, {Green, Thick}}]
Beep[]$ 
```

```

Chop[Evaluate[{s1[0], s2[0], vlim[[1]] /. {z1 → s1[0], z2 → s2[0]},
  vlim[[2]] /. {z1 → s1[0], z2 → s2[0]} /. sol] /. t → 0]
{{0.234828, 0.11933, 0.707414, 0.619583}}

NotebookDirectory[]

C:\Users\gabalap\Work\Research\Order reduction\Timeperiodic\Large forcing\

ans = Re[Table[
  Flatten[{t, Q1.M1.{s1[t], s2[t], vlim[[1]] /. {z1 → s1[t], z2 → s2[t]}, vlim[[2]] /.
    {z1 → s1[t], z2 → s2[t]} /. sol}], {t, ti, tf, 0.1(*N[ti+2π/ωmin], tint*)}]];
Export["C:\Users\gabalap\Work\Research\Order reduction\Timeperiodic\Large
  forcing\Reduced order_response.dat", ans];

SetDirectory[NotebookDirectory[]];
R1 = ReadList["R1.txt"][[1]]; fx = ReadList["fx.txt"][[1]];
ux = ReadList["ux.txt"][[1]]; Q1 = ReadList["Q1.txt"][[1]];
invQ1 = ReadList["invQ1.txt"][[1]];
M1 = ReadList["M1.txt"][[1]];
(*M1=Transpose[EigenVectors[R1]];*)
invM1 = Inverse[M1];
J1 = Chop[invM1.R1.M1];
z = {z1, z2, z3, z4};
x = Q1.M1.z;
fz = Chop[Expand[
  TrigToExp[invM1.invQ1.fx /. {x1 → x[[1]], x2 → x[[2]], x3 → x[[3]], x4 → x[[4]]}]];
uz = Chop[Expand[TrigToExp[invM1.invQ1.ux]]];

s0 = {0.23482840437799382`,
  0.11932994196787382`, 0.7074143840779241`, 0.6195827043652978`};
sol1 = NDSolve[{s1'[t] == J1[[1, 1]] s1[t] + J1[[1, 2]] s2[t] + J1[[1, 3]] s3[t] + J1[[1, 4]]
  s4[t] + (fz[[1]] /. {z1 → s1[t], z2 → s2[t], z3 → s3[t], z4 → s4[t]}) + uz[[1]],
  s2'[t] == J1[[2, 1]] s1[t] + J1[[2, 2]] s2[t] + J1[[2, 3]] s3[t] + J1[[2, 4]] s4[t] +
  (fz[[2]] /. {z1 → s1[t], z2 → s2[t], z3 → s3[t], z4 → s4[t]}) + uz[[2]],
  s3'[t] == J1[[3, 1]] s1[t] + J1[[3, 2]] s2[t] + J1[[3, 3]] s3[t] + J1[[3, 4]] s4[t] +
  (fz[[3]] /. {z1 → s1[t], z2 → s2[t], z3 → s3[t], z4 → s4[t]}) + uz[[3]],
  s4'[t] == J1[[4, 1]] s1[t] + J1[[4, 2]] s2[t] + J1[[4, 3]] s3[t] + J1[[4, 4]] s4[t] +
  (fz[[4]] /. {z1 → s1[t], z2 → s2[t], z3 → s3[t], z4 → s4[t]}) + uz[[4]],
  s1[0] == s0[[1]], s2[0] == s0[[2]], s3[0] == s0[[3]], s4[0] == s0[[4]]},
  {s1, s2, s3, s4}, {t, ti, tf}]

fig2 = Plot[Evaluate[Q1.M1.{s1[t], s2[t], s3[t], s4[t]} /. sol1], {t, ti, tf},
  PlotRange → All, PlotStyle → {Red, Green, {Red, Thick}, {Green, Thick}}]
(*ans1=Re[Table[Flatten[{t,Q1.M1.{s1[t],s2[t],s3[t],s4[t]}/.sol1}],
  {t,ti,tf,0.01}]];
Export["C:\Users\gabalap\Work\Research\Order
  reduction\Timeperiodic\Coupled inverted pendulums 4X4\ans1.dat",ans1];*)

Show [fig, fig2]

```



```
NotebookDirectory[]  
  
C:\Users\gabalap\Work\Research\Order reduction\Timeperiodic\Large forcing\  
  
ans1 = Re[Table[Flatten[{t, Q1.M1.{s1[t], s2[t], s3[t], s4[t]}} /. sol1,  
  {t, ti, tf, 0.001(*N[ti+2 $\pi$ / $\omega_{min}$ ], tint*)}]]];;  
Export["C:\\Users\\gabalap\\Work\\Research\\Order reduction\\Timeperiodic\\Large  
  forcing\\Full order_response.dat", ans1];
```

C.9

```
function [ans] = custom_fft(inputfile,colm,res,fig_num,line_atrib)
% custom_fft(inputfile,colm,res,fig_num,line_atrib)
% inputfile :- Input file name
% colm :- Column # to be plot (first column should be time by default )
% res:- resolution needed
% e.g. 100, 1000, 10000 divisions from zero to max freq
% fig_num:- to open another fig window
% line_atrib:- [45,2,66,78]
% linestyle; 32= noline, 45= solid, 58= dotted
% line width; 0.5, 1, 2, etc
% line color; 66=blue, 82= red, 75= black, 71= green
% line marker; 78= none, 88= x-mark

x=[];
[l,n]=size(inputfile);
P_period=inputfile(end,1)-inputfile(1,1); % Principle period
packets=ceil(res/P_period)-1;

x=inputfile(:,colm);
for i=1:packets
    x=vertcat(x,inputfile(2:end,colm));
end

fftx = fft(x);
NumUniquePts = ceil((length(x))/2);
fftx = fftx(1:NumUniquePts);
mx = abs(fftx)/length(x);
mx = abs(fftx)/length(x);

if rem(length(x), 2) % odd nfft excludes Nyquist point
    mx(2:end) = mx(2:end)*2;
else
    mx(2:end -1) = mx(2:end -1)*2;
end
Fs=(1-1)/P_period;
f = 2*pi*(0:NumUniquePts-1)*Fs/length(x);
% figure(fig_num);
% plot(f,mx,'linestyle',char(line_atrib(1)), 'linewidth',line_atrib(2),'color',char(line_atrib(3))
% , 'marker',char(line_atrib(4)));
semilogy(f,mx,'linestyle',char(line_atrib(1)), 'linewidth',line_atrib(2),'color',char(line_atrib(3)), 'ma
rker',char(line_atrib(4)));
%ans=horzcat(f,mx);
```

C.10

```
(* File name "Step 1 Initialize.nb" *)
(* 8 x 8 system,coupled inverted pendulums *)

Clear["Global`*"]

(* Linear control design *)
SetDirectory[NotebookDirectory[]];
R = ReadList["R.txt"][[1]]; Q = ReadList["Q.txt"][[1]];
invQ = ReadList["invQ.txt"][[1]];
M = ReadList["M.txt"][[1]];
bx = {0, 0, 0, 0, 1, 0, 0, 0};
bx >> bx.txt;
ux = k1 x1 + k2 x2 + k3 x3 + k4 x4 + k5 x5 + k6 x6 + k7 x7 + k8 x8;
invM = Inverse[M];
J = Chop[invM.R.M, E^-6];
Jp = {{J[[1, 1]], J[[1, 2]]}, {J[[2, 1]], J[[2, 2]]}};
Jq = {{J[[3, 3]], J[[3, 4]], 0, 0, 0, 0}, {J[[4, 3]], J[[4, 4]], 0, 0, 0, 0},
      {0, 0, J[[5, 5]], J[[5, 6]], 0, 0}, {0, 0, J[[6, 5]], J[[6, 6]], 0, 0},
      {0, 0, 0, 0, J[[7, 7]], J[[7, 8]]}, {0, 0, 0, 0, J[[8, 7]], J[[8, 8]]}};

z = {z1, z2, z3, z4, z5, z6, z7, z8};
x = Q.M.z;
bz = Chop[Expand[TrigToExp[invM.invQ.bx]]];
g0p = {bz[[1]], bz[[2]]};
g0q = {bz[[3]], bz[[4]], bz[[5]], bz[[6]], bz[[7]], bz[[8]]};
uz = Expand[TrigToExp[ux /. {x1 -> x[[1]], x2 -> x[[2]], x3 -> x[[3]],
      x4 -> x[[4]], x5 -> x[[5]], x6 -> x[[6]], x7 -> x[[7]], x8 -> x[[8]]}]];
(* z1,z2:=Dominant state,z3,z4,z5,z6,z7,z8:=Non-dominant state *)
n = 8;
p = 2;
q = 6;
```

```

(* Following data is required during design of linear controller *)
MatrixForm[J];
MatrixForm[Chop[ExpToTrig[bz], 10^-2]];
ans = Chop[ExpToTrig[Expand[bz * uz]]];
Chop[Coefficient[ans[[1]] /. {k2 → 0, k3 → 0, k4 → 0, k6 → 0, k7 → 0, k8 → 0}, z1], 10^-4]
Chop[Coefficient[ans[[1]] /. {k2 → 0, k3 → 0, k4 → 0, k6 → 0, k7 → 0, k8 → 0}, z2], 10^-4]
Chop[Coefficient[ans[[2]] /. {k2 → 0, k3 → 0, k4 → 0, k6 → 0, k7 → 0, k8 → 0}, z1], 10^-4]
Chop[Coefficient[ans[[2]] /. {k2 → 0, k3 → 0, k4 → 0, k6 → 0, k7 → 0, k8 → 0}, z2], 10^-4]

Out[23]= -1.04538 k1 + 0.527749 k5 + 1.25729 k1 Cos[2 π t] -
0.0338655 k5 Cos[2 π t] - 0.200205 k1 Cos[4 π t] - 0.0136459 k5 Cos[4 π t] +
0.00944229 k1 Cos[6 π t] + 0.00250804 k5 Cos[6 π t] -
3.9499 k5 Sin[2 π t] + 1.25792 k5 Sin[4 π t] - 0.0889915 k5 Sin[6 π t]

Out[24]= 1.07114 k1 - 0.348197 k5 - 1.23163 k1 Cos[2 π t] -
0.415741 k5 Cos[2 π t] + 0.190881 k1 Cos[4 π t] + 0.309999 k5 Cos[4 π t] -
0.00878449 k1 Cos[6 π t] - 0.0288041 k5 Cos[6 π t] -
0.261806 k1 Sin[2 π t] + 3.95662 k5 Sin[2 π t] + 0.0593394 k1 Sin[4 π t] -
1.21883 k5 Sin[4 π t] - 0.00336895 k1 Sin[6 π t] + 0.0839223 k5 Sin[6 π t]

Out[25]= -1.07965 k1 - 0.298124 k5 + 1.24141 k1 Cos[2 π t] -
0.492627 k5 Cos[2 π t] - 0.192398 k1 Cos[4 π t] + 0.323475 k5 Cos[4 π t] +
0.00885433 k1 Cos[6 π t] - 0.0296528 k5 Cos[6 π t] -
0.263876 k1 Sin[2 π t] - 3.97062 k5 Sin[2 π t] + 0.0598111 k1 Sin[4 π t] +
1.22518 k5 Sin[4 π t] - 0.00339585 k1 Sin[6 π t] - 0.0843523 k5 Sin[6 π t]

Out[26]= 1.04475 k1 + 0.47753 k5 - 1.25787 k1 Cos[2 π t] +
0.04131 k5 Cos[2 π t] + 0.200321 k1 Cos[4 π t] - 0.0250851 k5 Cos[4 π t] -
0.00944911 k1 Cos[6 π t] + 0.00317105 k5 Cos[6 π t] +
3.9517 k5 Sin[2 π t] - 1.25865 k5 Sin[4 π t] + 0.0890558 k5 Sin[6 π t]

(* Find Linear controller using symbolic computation of FTM *)
(* Files used 1- Find Cheb. Coefficients.nb
2- Functions symbolic FTM.nb
3- Symbolic FTM.nb
4- Find_k1 k2.nb *)

Kr := Array[k, {p}];
k[1] = Expand[Coefficient[uz /. {k1 → 0.16906283540593794, k2 → 0,
k3 → 0, k4 → 0, k5 → -0.6400038382967878, k6 → 0, k7 → 0, k8 → 0}, z1]];
k[2] = Expand[Coefficient[uz /. {k1 → 0.16906283540593794, k2 → 0, k3 → 0,
k4 → 0, k5 → -0.6400038382967878, k6 → 0, k7 → 0, k8 → 0}, z2]];
Kr >>
Kr.txt;

(* Find system nonlinearities using "Nonlinear parameter selection.nb" *)
fx = ReadList["fx.txt"][[1]];
(*nonlinear term in z*)
fz = Chop[
Expand[TrigToExp[invM.invQ.fx /. {x1 → x[[1]], x2 → x[[2]], x3 → x[[3]], x4 → x[[4]],
x5 → x[[5]], x6 → x[[6]], x7 → x[[7]], x8 → x[[8]]}], E^-6];
fp = {fz[[1]], fz[[2]]};
fq = {fz[[3]], fz[[4]], fz[[5]], fz[[6]], fz[[7]], fz[[8]]};
Chop[ExpToTrig[fz[[1]]]]

```

```

(* ***** Finding nonlinear feedback for quibic terms ***** *)
ft = 12; (*Fourier terms *) ft >> "ft.txt";
(* ***** *)
(* Find P using FindingP.nb *)
P = TrigToExp[ReadList["P.txt"][[1]]];
(* ***** *)

zs = Expand[{v3, v4, v5, v6, v7, v8} + P.{z1, z2}];
v1fp = Chop[Expand[{fz[[1]], fz[[2]]} /. {z3 → zs[[1]],
      z4 → zs[[2]], z5 → zs[[3]], z6 → zs[[4]], z7 → zs[[5]], z8 → zs[[6]]}]];
coefLv1fp = CoefficientList[v1fp, {z1, z2}];

φ3 = TrigToExp[{Sum[{φc130[i] Cos[i π t] + φs130[i] Sin[i π t]} y1^3 +
  (φc121[i] Cos[i π t] + φs121[i] Sin[i π t]) y1^2 y2 +
  (φc112[i] Cos[i π t] + φs112[i] Sin[i π t]) y1 y2^2 +
  (φc103[i] Cos[i π t] + φs103[i] Sin[i π t]) y2^3, {i, 0, ft}],
  Sum[{(φc230[i] Cos[i π t] + φs230[i] Sin[i π t]) y1^3 +
  (φc221[i] Cos[i π t] + φs221[i] Sin[i π t]) y1^2 y2 +
  (φc212[i] Cos[i π t] + φs212[i] Sin[i π t]) y1 y2^2 +
  (φc203[i] Cos[i π t] + φs203[i] Sin[i π t]) y2^3, {i, 0, ft}]}]];
α3 = TrigToExp[Sum[{ac30[i] Cos[i π t] + as30[i] Sin[i π t]} z1^3 +
  (ac21[i] Cos[i π t] + as21[i] Sin[i π t]) z1^2 z2 +
  (ac12[i] Cos[i π t] + as12[i] Sin[i π t]) z1 z2^2 +
  (ac03[i] Cos[i π t] + as03[i] Sin[i π t]) z2^3, {i, 0, ft}]];
β2 = TrigToExp[Sum[{βc20[i] Cos[i π t] + βs20[i] Sin[i π t]} z1^2 +
  (βc11[i] Cos[i π t] + βs11[i] Sin[i π t]) z1 z2 +
  (βc02[i] Cos[i π t] + βs02[i] Sin[i π t]) z2^2, {i, 0, ft}]];

g0pα3 = g0p * α3;
coefLg0pα3 = CoefficientList[g0pα3, {z1, z2}];

coefCNRHSα3 = Chop[ExpToTrig[
  Expand[{coefLv1fp[[1]][[4, 1]] + coefLg0pα3[[1]][[4, 1]], coefLv1fp[[1]][[3, 2]] +
    coefLg0pα3[[1]][[3, 2]], coefLv1fp[[1]][[2, 3]] + coefLg0pα3[[1]][[2, 3]],
    coefLv1fp[[1]][[1, 4]] + coefLg0pα3[[1]][[1, 4]], coefLv1fp[[2]][[4, 1]] +
    coefLg0pα3[[2]][[4, 1]], coefLv1fp[[2]][[3, 2]] + coefLg0pα3[[2]][[3, 2]],
    coefLv1fp[[2]][[2, 3]] + coefLg0pα3[[2]][[2, 3]],
    coefLv1fp[[2]][[1, 4]] + coefLg0pα3[[2]][[1, 4]]}]];

coefTCNRHSα3 = Table[0, {2 * 4}, {1 + 2 * ft}];
Do[
  Do[If[Exponent[TrigToExp[coefCNRHSα3[[i, k]]], E] == 0,
    {coefTCNRHSα3[[i, 1]] = coefTCNRHSα3[[i, 1]] + coefCNRHSα3[[i, k]]},
    {k, 1, Dimensions[coefCNRHSα3[[i]]][[1]]}], {i, 1, 2 * 4}];
Do[
  coefTCNRHSα3[[i, 2 * j]] = Coefficient[coefCNRHSα3[[i]], Cos[j π t]];
  coefTCNRHSα3[[i, 2 * j + 1]] = Coefficient[coefCNRHSα3[[i]], Sin[j π t]];
  {i, 1, 2 * 4}, {j, 1, ft}];

```

```

CNLHSA3 = Chop[Expand[
  ∂t φ3 + {{∂y1 φ3[[1]], ∂y2 φ3[[1]]}, {∂y1 φ3[[2]], ∂y2 φ3[[2]]}}.Jp.{y1, y2} - Jp.φ3]];
coefLCNLHSA3 = CoefficientList[CNLHSA3, {y1, y2}];

coefCNLHSA3 = Chop[ExpToTrig[
  {coefLCNLHSA3[[1]][[4, 1]], coefLCNLHSA3[[1]][[3, 2]], coefLCNLHSA3[[1]][[2, 3]],
  coefLCNLHSA3[[1]][[1, 4]], coefLCNLHSA3[[2]][[4, 1]], coefLCNLHSA3[[2]][[3, 2]],
  coefLCNLHSA3[[2]][[2, 3]], coefLCNLHSA3[[2]][[1, 4]]}]];

coefTCNLHSA3 = Table[0, {2 * 4}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[TrigToExp[coefCNLHSA3[[i, k]]], E] == 0,
    {coefTCNLHSA3[[i, 1]] = coefTCNLHSA3[[i, 1]] + coefCNLHSA3[[i, k]]},
    {k, 1, Dimensions[coefCNLHSA3[[i]]][[1]]}], {i, 1, 2 * 4}];
Do[
  coefTCNLHSA3[[i, 2 * j]] = Coefficient[coefCNLHSA3[[i]], Cos[j π t]];
  coefTCNLHSA3[[i, 2 * j + 1]] = Coefficient[coefCNLHSA3[[i]], Sin[j π t]];
  {i, 1, 2 * 4}, {j, 1, ft}];

Aα3 = Array[rowAα3, {8 * (1 + 2 * ft)}];
Do[rowAα3[k + (i - 1) 8] =
  Flatten[{{Coefficient[coefTCNLHSA3[[k, i]], φc130[0]], Coefficient[
    coefTCNLHSA3[[k, i]], φc121[0]], Coefficient[coefTCNLHSA3[[k, i]], φc112[0]],
    Coefficient[coefTCNLHSA3[[k, i]], φc103[0]], Coefficient[coefTCNLHSA3[[k, i]],
    φc230[0]], Coefficient[coefTCNLHSA3[[k, i]], φc221[0]], Coefficient[
    coefTCNLHSA3[[k, i]], φc212[0]], Coefficient[coefTCNLHSA3[[k, i]], φc203[0]]},
  Table[{{Coefficient[coefTCNLHSA3[[k, i]], φc130[j]], Coefficient[
    coefTCNLHSA3[[k, i]], φs130[j]], Coefficient[coefTCNLHSA3[[k, i]], φc121[j]],
    Coefficient[coefTCNLHSA3[[k, i]], φs121[j]], Coefficient[coefTCNLHSA3[[k, i]],
    φc112[j]], Coefficient[coefTCNLHSA3[[k, i]], φs112[j]], Coefficient[
    coefTCNLHSA3[[k, i]], φc103[j]], Coefficient[coefTCNLHSA3[[k, i]], φs103[j]],
    Coefficient[coefTCNLHSA3[[k, i]], φc230[j]], Coefficient[
    coefTCNLHSA3[[k, i]], φs230[j]], Coefficient[coefTCNLHSA3[[k, i]], φc221[j]],
    Coefficient[coefTCNLHSA3[[k, i]], φs221[j]], Coefficient[
    coefTCNLHSA3[[k, i]], φc212[j]], Coefficient[coefTCNLHSA3[[k, i]], φs212[j]],
    Coefficient[coefTCNLHSA3[[k, i]], φc203[j]], Coefficient[
    coefTCNLHSA3[[k, i]], φs203[j]]}, {j, 1, ft}}}], {i, 1, 1 + 2 * ft}, {k, 1, 8}];

Ca3 = Array[rowCa3, {8 * (1 + 2 * ft)}];
Do[rowCa3[k + (i - 1) 8] = coefTCNRHSA3[[k, i]], {k, 1, 8}, {i, 1, 1 + 2 * ft}];
coefφ3 = Expand[Inverse[Aα3].Ca3];
v1φ3 = Expand[φ3 /. Flatten[{{φc130[0] -> coefφ3[[1]], φc121[0] -> coefφ3[[2]],
  φc112[0] -> coefφ3[[3]], φc103[0] -> coefφ3[[4]], φc230[0] -> coefφ3[[5]],
  φc221[0] -> coefφ3[[6]], φc212[0] -> coefφ3[[7]], φc203[0] -> coefφ3[[8]]},
  Table[{{φc130[j] -> coefφ3[[8 (2 j - 1) + 1]], φs130[j] -> coefφ3[[8 (2 j - 1) + 2]],
  φc121[j] -> coefφ3[[8 (2 j - 1) + 3]], φs121[j] -> coefφ3[[8 (2 j - 1) + 4]],
  φc112[j] -> coefφ3[[8 (2 j - 1) + 5]], φs112[j] -> coefφ3[[8 (2 j - 1) + 6]],
  φc103[j] -> coefφ3[[8 (2 j - 1) + 7]], φs103[j] -> coefφ3[[8 (2 j - 1) + 8]],

```

```

    φc230[j] -> coefφ3[[8 (2 j - 1) + 9]], φs230[j] -> coefφ3[[8 (2 j - 1) + 10]],
    φc221[j] -> coefφ3[[8 (2 j - 1) + 11]],
    φs221[j] -> coefφ3[[8 (2 j - 1) + 12]], φc212[j] -> coefφ3[[8 (2 j - 1) + 13]],
    φs212[j] -> coefφ3[[8 (2 j - 1) + 14]], φc203[j] -> coefφ3[[8 (2 j - 1) + 15]],
    φs203[j] -> coefφ3[[8 (2 j - 1) + 16]], {j, 1, ft}}];];

coefTCNRHSβ2 = Table[0, {2 * 3}, {1 + 2 * ft}];
CNLHSβ2 =
  Chop[Expand[{{∂y1 v1φ3[[1]], ∂y2 v1φ3[[1]]}, {∂y1 v1φ3[[2]], ∂y2 v1φ3[[2]]}].g0p -
    g0p * β2 /. {z1 -> y1, z2 -> y2}]];

coefLCNLHSβ2 = CoefficientList[CNLHSβ2, {y1, y2}];

coefCNLHSβ2 = Chop[ExpToTrig[{coefLCNLHSβ2[[1]][[3, 1]],
  coefLCNLHSβ2[[1]][[2, 2]], coefLCNLHSβ2[[1]][[1, 3]], coefLCNLHSβ2[[2]][[3, 1]],
  coefLCNLHSβ2[[2]][[2, 2]], coefLCNLHSβ2[[2]][[1, 3]]}]];

coefTCNLHSβ2 = Table[0, {2 * 3}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[TrigToExp[coefCNLHSβ2[[i, k]]], E] == 0,
    {coefTCNLHSβ2[[i, 1]] = coefTCNLHSβ2[[i, 1]] + coefCNLHSβ2[[i, k]]},
    {k, 1, Dimensions[coefCNLHSβ2[[i]]][[1]]}], {i, 1, 2 * 3}];
Do[
  coefTCNLHSβ2[[i, 2 * j]] = Coefficient[coefCNLHSβ2[[i]], Cos[j π t]];
  coefTCNLHSβ2[[i, 2 * j + 1]] = Coefficient[coefCNLHSβ2[[i]], Sin[j π t]];
  {i, 1, 2 * 3}, {j, 1, ft}];

Aβ2 = Array[rowAβ2, {6 * (1 + 2 * ft)}];
Cβ2LHS = Array[rowCβ2LHS, {6 * (1 + 2 * ft)}];

Do[rowAβ2[k + (i - 1) 6] =
  Flatten[{{Coefficient[coefTCNLHSβ2[[k, i]], ac30[0]], Coefficient[
    coefTCNLHSβ2[[k, i]], ac21[0]], Coefficient[coefTCNLHSβ2[[k, i]], ac12[0]],
    Coefficient[coefTCNLHSβ2[[k, i]], ac03[0]],
    Coefficient[coefTCNLHSβ2[[k, i]], βc20[0]], Coefficient[
    coefTCNLHSβ2[[k, i]], βc11[0]], Coefficient[coefTCNLHSβ2[[k, i]], βc02[0]]},
  Table[{{Coefficient[coefTCNLHSβ2[[k, i]], ac30[j]], Coefficient[
    coefTCNLHSβ2[[k, i]], as30[j]], Coefficient[coefTCNLHSβ2[[k, i]], ac21[j]],
    Coefficient[coefTCNLHSβ2[[k, i]], as21[j]], Coefficient[coefTCNLHSβ2[[k, i]],
    ac12[j]], Coefficient[coefTCNLHSβ2[[k, i]], as12[j]], Coefficient[
    coefTCNLHSβ2[[k, i]], ac03[j]], Coefficient[coefTCNLHSβ2[[k, i]], as03[j]],
    Coefficient[coefTCNLHSβ2[[k, i]], βc20[j]], Coefficient[coefTCNLHSβ2[[k, i]],
    βs20[j]], Coefficient[coefTCNLHSβ2[[k, i]], βc11[j]], Coefficient[
    coefTCNLHSβ2[[k, i]], βs11[j]], Coefficient[coefTCNLHSβ2[[k, i]], βc02[j]],
    Coefficient[coefTCNLHSβ2[[k, i]], βs02[j]]}], {j, 1, ft}}];];
rowCβ2LHS[k + (i - 1) 6] = Chop[Expand[coefTCNLHSβ2[[k, i]] - rowAβ2[k + (i - 1) 6].
  Flatten[{{ac30[0], ac21[0], ac12[0], ac03[0], βc20[0], βc11[0], βc02[0]],
    Table[{{ac30[j], as30[j], ac21[j], as21[j], ac12[j], as12[j], ac03[j],
    as03[j], βc20[j], βs20[j], βc11[j], βs11[j], βc02[j], βs02[j]],
    {j, 1, ft}}}}];];, {k, 1, 6}, {i, 1, 1 + 2 * ft}]]

```

```

Cβ2RHS = Array[rowCβ2RHS, {6 * (1 + 2 * ft)}];
Do[rowCβ2RHS[k + (i - 1) 6] = coefTCNRHSβ2[[k, i]], {i, 1, 1 + 2 * ft}, {k, 1, 6}]
Cβ2 = Cβ2RHS - Cβ2LHS;

coefa3β2 = LinearSolve[Aβ2, Cβ2];
v1α3 = Expand[α3 /. Flatten[
  {{ac30[0] -> coefa3β2[[1]], ac21[0] -> coefa3β2[[2]], ac12[0] -> coefa3β2[[3]],
  ac03[0] -> coefa3β2[[4]]}, Table[{ac30[j] -> coefa3β2[[7 (2 j - 1) + 1]],
  as30[j] -> coefa3β2[[7 (2 j - 1) + 2]], ac21[j] -> coefa3β2[[7 (2 j - 1) + 3]],
  as21[j] -> coefa3β2[[7 (2 j - 1) + 4]], ac12[j] -> coefa3β2[[7 (2 j - 1) + 5]],
  as12[j] -> coefa3β2[[7 (2 j - 1) + 6]], ac03[j] -> coefa3β2[[7 (2 j - 1) + 7]],
  as03[j] -> coefa3β2[[7 (2 j - 1) + 8]]}, {j, 1, ft}]]];

v1β2 = Expand[β2 /. Flatten[
  {{βc20[0] -> coefa3β2[[5]], βc11[0] -> coefa3β2[[6]], βc02[0] -> coefa3β2[[7]]},
  Table[{βc20[j] -> coefa3β2[[7 (2 j - 1) + 9]], βs20[j] -> coefa3β2[[7 (2 j - 1) + 10]],
  βc11[j] -> coefa3β2[[7 (2 j - 1) + 11]],
  βs11[j] -> coefa3β2[[7 (2 j - 1) + 12]], βc02[j] -> coefa3β2[[7 (2 j - 1) + 13]],
  βs02[j] -> coefa3β2[[7 (2 j - 1) + 14]]}, {j, 1, ft}]]];

v2φ3 = Expand[v1φ3 /. Flatten[
  {{ac30[0] -> coefa3β2[[1]], ac21[0] -> coefa3β2[[2]], ac12[0] -> coefa3β2[[3]],
  ac03[0] -> coefa3β2[[4]]}, Table[{ac30[j] -> coefa3β2[[7 (2 j - 1) + 1]],
  as30[j] -> coefa3β2[[7 (2 j - 1) + 2]], ac21[j] -> coefa3β2[[7 (2 j - 1) + 3]],
  as21[j] -> coefa3β2[[7 (2 j - 1) + 4]], ac12[j] -> coefa3β2[[7 (2 j - 1) + 5]],
  as12[j] -> coefa3β2[[7 (2 j - 1) + 6]], ac03[j] -> coefa3β2[[7 (2 j - 1) + 7]],
  as03[j] -> coefa3β2[[7 (2 j - 1) + 8]]}, {j, 1, ft}]]];

v1α3 >> "alpha3.txt";
v1β2 >> "beta2.txt";
v2φ3 >> "phi3.txt";

```



```

(* ***** Construction of Invariant Manifold ***** *)
vlfptrunc3 = {0, 0};
(* This truncate all the terms of order greater than 3 from vlfp *)
vlfptrunc3[[1]] =
  Chop[Expand[coefLvlfp[[1]][[4, 1]] z1^3 + coefLvlfp[[1]][[3, 2]] z1^2 z2 +
    coefLvlfp[[1]][[2, 3]] z1 z2^2 + coefLvlfp[[1]][[1, 4]] z2^3]];
vlfptrunc3[[2]] = Chop[Expand[coefLvlfp[[2]][[4, 1]] z1^3 + coefLvlfp[[2]][[3, 2]]
  z1^2 z2 + coefLvlfp[[2]][[2, 3]] z1 z2^2 + coefLvlfp[[2]][[1, 4]] z2^3]];

Pdotvlfptrunc3 = Chop[Expand[P.vlfptrunc3]];

vlfq = Chop[Expand[{fq[[1]] - Pdotvlfptrunc3[[1]], fq[[2]] - Pdotvlfptrunc3[[2]],
  fq[[3]] - Pdotvlfptrunc3[[3]], fq[[4]] - Pdotvlfptrunc3[[4]],
  fq[[5]] - Pdotvlfptrunc3[[5]], fq[[6]] - Pdotvlfptrunc3[[6]]} /. {z3 → zs[[1]],
  z4 → zs[[2]], z5 → zs[[3]], z6 → zs[[4]], z7 → zs[[5]], z8 → zs[[6]]}]];
(* Keeping only coefficients of z1^3, z1^2 z2, z1 z2^2 and z2^3
  because other terms gives rise to higher order
  term after substitution of CM which we dont need *)
coefLvlfq = CoefficientList[vlfq, {z1, z2}];

vlfqtrunc3 = Chop[Expand[{coefLvlfq[[1]][[4, 1]] z1^3 + coefLvlfq[[1]][[3, 2]] z1^2 z2 +
  coefLvlfq[[1]][[2, 3]] z1 z2^2 + coefLvlfq[[1]][[1, 4]] z2^3,
  coefLvlfq[[2]][[4, 1]] z1^3 + coefLvlfq[[2]][[3, 2]] z1^2 z2 +
  coefLvlfq[[2]][[2, 3]] z1 z2^2 + coefLvlfq[[2]][[1, 4]] z2^3,
  coefLvlfq[[3]][[4, 1]] z1^3 + coefLvlfq[[3]][[3, 2]] z1^2 z2 +
  coefLvlfq[[3]][[2, 3]] z1 z2^2 + coefLvlfq[[3]][[1, 4]] z2^3,
  coefLvlfq[[4]][[4, 1]] z1^3 + coefLvlfq[[4]][[3, 2]] z1^2 z2 +
  coefLvlfq[[4]][[2, 3]] z1 z2^2 + coefLvlfq[[4]][[1, 4]] z2^3,
  coefLvlfq[[5]][[4, 1]] z1^3 + coefLvlfq[[5]][[3, 2]] z1^2 z2 +
  coefLvlfq[[5]][[2, 3]] z1 z2^2 + coefLvlfq[[5]][[1, 4]] z2^3,
  coefLvlfq[[6]][[4, 1]] z1^3 + coefLvlfq[[6]][[3, 2]] z1^2 z2 +
  coefLvlfq[[6]][[2, 3]] z1 z2^2 + coefLvlfq[[6]][[1, 4]] z2^3}]];

Pdotg0p = Chop[Expand[P.g0p]];

vlg0q = Chop[
  Expand[{g0q[[1]] - Pdotg0p[[1]], g0q[[2]] - Pdotg0p[[2]], g0q[[3]] - Pdotg0p[[3]],
  g0q[[4]] - Pdotg0p[[4]], g0q[[5]] - Pdotg0p[[5]], g0q[[6]] - Pdotg0p[[6]]}]];
Chop[ExpToTrig[vlfqtrunc3]]

(* Construction of invariant manifold *)
(* ***** *)
(* Find Rbarp Symbolic FTM.nb *)
Rbarp = ReadList["Rbarp.txt"][[1]];
Qbarp = TrigToExp[ReadList["Qbarp.txt"][[1]]];
invQbarp = TrigToExp[ReadList["invQbarp.txt"][[1]]];
Mbarp = ReadList["Mbarp.txt"][[1]];
invMbarp = ReadList["invMbarp.txt"][[1]];
Jbarp = Chop[invMbarp.Rbarp.Mbarp];
(* ***** *)

```

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zr = Chop[Expand[Qbarp.Mbarp.{v1, v2}]];
(* im :- Invariant Manifold *)
im = {h330 v1^3 + h321 v1^2 v2 + h312 v1 v2^2 + h303 v2^3,
      h430 v1^3 + h421 v1^2 v2 + h412 v1 v2^2 + h403 v2^3,
      h530 v1^3 + h521 v1^2 v2 + h512 v1 v2^2 + h503 v2^3, h630 v1^3 + h621 v1^2 v2 +
      h612 v1 v2^2 + h603 v2^3, h730 v1^3 + h721 v1^2 v2 + h712 v1 v2^2 + h703 v2^3,
      h830 v1^3 + h821 v1^2 v2 + h812 v1 v2^2 + h803 v2^3};

IMLHS = Expand[{{∂v1 im[[1]], ∂v2 im[[1]]}, {∂v1 im[[2]], ∂v2 im[[2]]},
               {∂v1 im[[3]], ∂v2 im[[3]]}, {∂v1 im[[4]], ∂v2 im[[4]]},
               {∂v1 im[[5]], ∂v2 im[[5]]}, {∂v1 im[[6]], ∂v2 im[[6]]}}.Jbarp.{v1, v2} - Jq.im];
coefLIMLHS = CoefficientList[IMLHS, {v1, v2}];

coefIMLHS31 = {coefLIMLHS[[1]][[4, 1]], coefLIMLHS[[1]][[3, 2]],
               coefLIMLHS[[1]][[2, 3]], coefLIMLHS[[1]][[1, 4]], coefLIMLHS[[2]][[4, 1]],
               coefLIMLHS[[2]][[3, 2]], coefLIMLHS[[2]][[2, 3]], coefLIMLHS[[2]][[1, 4]]};
coefIMLHS32 = {coefLIMLHS[[3]][[4, 1]], coefLIMLHS[[3]][[3, 2]],
               coefLIMLHS[[3]][[2, 3]], coefLIMLHS[[3]][[1, 4]], coefLIMLHS[[4]][[4, 1]],
               coefLIMLHS[[4]][[3, 2]], coefLIMLHS[[4]][[2, 3]], coefLIMLHS[[4]][[1, 4]]};
coefIMLHS33 = {coefLIMLHS[[5]][[4, 1]], coefLIMLHS[[5]][[3, 2]],
               coefLIMLHS[[5]][[2, 3]], coefLIMLHS[[5]][[1, 4]], coefLIMLHS[[6]][[4, 1]],
               coefLIMLHS[[6]][[3, 2]], coefLIMLHS[[6]][[2, 3]], coefLIMLHS[[6]][[1, 4]]};

(* ***** *)
(* use "Step 2 Inverse transformation.nb" to find vlf3.txt *)
vlf3 = ReadList["vlf3.txt"][[1]];
(* ***** *)
IMRHS = Expand[vlfttrunc3 + vlg0q * (v1α3 + (Kr[[1]] vlf3[[1]] + Kr[[2]] vlf3[[2]]) +
              v1β2 * (Kr[[1]] z1 + Kr[[2]] z2)) /. {z1 → zr[[1]], z2 → zr[[2]]}];
coefLIMRHS = CoefficientList[IMRHS, {v1, v2}];

coefIMRHS31 = {coefLIMRHS[[1]][[4, 1]], coefLIMRHS[[1]][[3, 2]],
               coefLIMRHS[[1]][[2, 3]], coefLIMRHS[[1]][[1, 4]], coefLIMRHS[[2]][[4, 1]],
               coefLIMRHS[[2]][[3, 2]], coefLIMRHS[[2]][[2, 3]], coefLIMRHS[[2]][[1, 4]]};

coefTIMRHS31 = Table[0, {2*4}, {1+2*ft}];
Do[
  Do[ If[Exponent[coefIMRHS31[[i, k]], E] == 0,
      {coefTIMRHS31[[i, 1]] = coefTIMRHS31[[i, 1]] + coefIMRHS31[[i, k]]}],
    {k, 1, Dimensions[coefIMRHS31[[i]][[1]]}], {i, 1, 2*4}];
Do[
  coefTIMRHS31[[i, 2*j]] = Coefficient[coefIMRHS31[[i]], E^(j*I*Pi*t)];
  coefTIMRHS31[[i, 2*j+1]] = Coefficient[coefIMRHS31[[i]], E^(-j*I*Pi*t)];
  {i, 1, 2*4}, {j, 1, ft}];

coefIMRHS32 = {coefLIMRHS[[3]][[4, 1]], coefLIMRHS[[3]][[3, 2]],
               coefLIMRHS[[3]][[2, 3]], coefLIMRHS[[3]][[1, 4]], coefLIMRHS[[4]][[4, 1]],
               coefLIMRHS[[4]][[3, 2]], coefLIMRHS[[4]][[2, 3]], coefLIMRHS[[4]][[1, 4]]};

```

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coefTIMRHS32 = Table[0, {2 * 4}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[coefIMRHS32[[i, k]], E] == 0,
    {coefTIMRHS32[[i, 1]] = coefTIMRHS32[[i, 1]] + coefIMRHS32[[i, k]]},
    {k, 1, Dimensions[coefIMRHS32[[i]]][[1]]}, {i, 1, 2 * 4}];
Do[
  coefTIMRHS32[[i, 2 * j]] = Coefficient[coefIMRHS32[[i]], E^(j * I * Pi * t)];
  coefTIMRHS32[[i, 2 * j + 1]] = Coefficient[coefIMRHS32[[i]], E^(-j * I * Pi * t)];,
  {i, 1, 2 * 4}, {j, 1, ft}];

coefIMRHS33 = {coefLIMRHS[[5]][[4, 1]], coefLIMRHS[[5]][[3, 2]],
  coefLIMRHS[[5]][[2, 3]], coefLIMRHS[[5]][[1, 4]], coefLIMRHS[[6]][[4, 1]],
  coefLIMRHS[[6]][[3, 2]], coefLIMRHS[[6]][[2, 3]], coefLIMRHS[[6]][[1, 4]]};
coefTIMRHS33 = Table[0, {2 * 4}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[coefIMRHS33[[i, k]], E] == 0,
    {coefTIMRHS33[[i, 1]] = coefTIMRHS33[[i, 1]] + coefIMRHS33[[i, k]]},
    {k, 1, Dimensions[coefIMRHS33[[i]]][[1]]}, {i, 1, 2 * 4}];
Do[
  coefTIMRHS33[[i, 2 * j]] = Coefficient[coefIMRHS33[[i]], E^(j * I * Pi * t)];
  coefTIMRHS33[[i, 2 * j + 1]] = Coefficient[coefIMRHS33[[i]], E^(-j * I * Pi * t)];,
  {i, 1, 2 * 4}, {j, 1, ft}];

arrayIMA31 = Table[0, {1 + 2 * ft}];
vectorIMC31 = Table[0, {1 + 2 * ft}];
Do[arrayIMA31[[j]] =
  Expand[(j (1 + (-1) ^ j) / 4 + (j - 1) (-1 + (-1) ^ j) / 4) I Pi * IdentityMatrix[8] +
  {{Coefficient[coefIMLHS31[[1]], h330], Coefficient[coefIMLHS31[[1]], h321],
  Coefficient[coefIMLHS31[[1]], h312], Coefficient[coefIMLHS31[[1]], h303],
  Coefficient[coefIMLHS31[[1]], h430], Coefficient[coefIMLHS31[[1]], h421],
  Coefficient[coefIMLHS31[[1]], h412], Coefficient[coefIMLHS31[[1]], h403]},
  {Coefficient[coefIMLHS31[[2]], h330], Coefficient[coefIMLHS31[[2]], h321],
  Coefficient[coefIMLHS31[[2]], h312], Coefficient[coefIMLHS31[[2]], h303],
  Coefficient[coefIMLHS31[[2]], h430], Coefficient[coefIMLHS31[[2]], h421],
  Coefficient[coefIMLHS31[[2]], h412], Coefficient[coefIMLHS31[[2]], h403]},
  {Coefficient[coefIMLHS31[[3]], h330], Coefficient[coefIMLHS31[[3]], h321],
  Coefficient[coefIMLHS31[[3]], h312], Coefficient[coefIMLHS31[[3]], h303],
  Coefficient[coefIMLHS31[[3]], h430], Coefficient[coefIMLHS31[[3]], h421],
  Coefficient[coefIMLHS31[[3]], h412], Coefficient[coefIMLHS31[[3]], h403]},
  {Coefficient[coefIMLHS31[[4]], h330], Coefficient[coefIMLHS31[[4]], h321],
  Coefficient[coefIMLHS31[[4]], h312], Coefficient[coefIMLHS31[[4]], h303],
  Coefficient[coefIMLHS31[[4]], h430], Coefficient[coefIMLHS31[[4]], h421],
  Coefficient[coefIMLHS31[[4]], h412], Coefficient[coefIMLHS31[[4]], h403]},
  {Coefficient[coefIMLHS31[[5]], h330], Coefficient[coefIMLHS31[[5]], h321],
  Coefficient[coefIMLHS31[[5]], h312], Coefficient[coefIMLHS31[[5]], h303],
  Coefficient[coefIMLHS31[[5]], h430], Coefficient[coefIMLHS31[[5]], h421],
  Coefficient[coefIMLHS31[[5]], h412], Coefficient[coefIMLHS31[[5]], h403]},
  {Coefficient[coefIMLHS31[[6]], h330], Coefficient[coefIMLHS31[[6]], h321],

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Coefficient[coefIMLHS31[[6]], h312], Coefficient[coefIMLHS31[[6]], h303],
Coefficient[coefIMLHS31[[6]], h430], Coefficient[coefIMLHS31[[6]], h421],
Coefficient[coefIMLHS31[[6]], h412], Coefficient[coefIMLHS31[[6]], h403]},
{Coefficient[coefIMLHS31[[7]], h330], Coefficient[coefIMLHS31[[7]], h321],
Coefficient[coefIMLHS31[[7]], h312], Coefficient[coefIMLHS31[[7]], h303],
Coefficient[coefIMLHS31[[7]], h430], Coefficient[coefIMLHS31[[7]], h421],
Coefficient[coefIMLHS31[[7]], h412], Coefficient[coefIMLHS31[[7]], h403]},
{Coefficient[coefIMLHS31[[8]], h330], Coefficient[coefIMLHS31[[8]], h321],
Coefficient[coefIMLHS31[[8]], h312], Coefficient[coefIMLHS31[[8]], h303],
Coefficient[coefIMLHS31[[8]], h430], Coefficient[coefIMLHS31[[8]], h421],
Coefficient[coefIMLHS31[[8]], h412], Coefficient[coefIMLHS31[[8]], h403]}}];

vectorIMC31[[j]] = {coefTIMRHS31[[1, j]], coefTIMRHS31[[2, j]],
coefTIMRHS31[[3, j]], coefTIMRHS31[[4, j]], coefTIMRHS31[[5, j]],
coefTIMRHS31[[6, j]], coefTIMRHS31[[7, j]], coefTIMRHS31[[8, j]]};
(* Solvability condition *)
If[Det[arrayIMA31[[j]]] == 0, {Print["Resonance"];
Print[(j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi]], {j, 1, 1 + 2 * ft}};
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
coefh1 = Chop[Expand[Sum[Inverse[arrayIMA31[[j]]].vectorIMC31[[j]] *
E^((j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi t), {j, 1, 1 + 2 * ft}]]];
(* h1={h330,h321,h312,h303,h430,h421,h412,h403} *)
(* ***** *)

arrayIMA32 = Table[0, {1 + 2 * ft}];
vectorIMC32 = Table[0, {1 + 2 * ft}];
Do[arrayIMA32[[j]] =
Expand[(j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi * IdentityMatrix[8] +
{{Coefficient[coefIMLHS32[[1]], h530], Coefficient[coefIMLHS32[[1]], h521],
Coefficient[coefIMLHS32[[1]], h512], Coefficient[coefIMLHS32[[1]], h503],
Coefficient[coefIMLHS32[[1]], h630], Coefficient[coefIMLHS32[[1]], h621],
Coefficient[coefIMLHS32[[1]], h612], Coefficient[coefIMLHS32[[1]], h603]},
{Coefficient[coefIMLHS32[[2]], h530], Coefficient[coefIMLHS32[[2]], h521],
Coefficient[coefIMLHS32[[2]], h512], Coefficient[coefIMLHS32[[2]], h503],
Coefficient[coefIMLHS32[[2]], h630], Coefficient[coefIMLHS32[[2]], h621],
Coefficient[coefIMLHS32[[2]], h612], Coefficient[coefIMLHS32[[2]], h603]},
{Coefficient[coefIMLHS32[[3]], h530], Coefficient[coefIMLHS32[[3]], h521],
Coefficient[coefIMLHS32[[3]], h512], Coefficient[coefIMLHS32[[3]], h503],
Coefficient[coefIMLHS32[[3]], h630], Coefficient[coefIMLHS32[[3]], h621],
Coefficient[coefIMLHS32[[3]], h612], Coefficient[coefIMLHS32[[3]], h603]},
{Coefficient[coefIMLHS32[[4]], h530], Coefficient[coefIMLHS32[[4]], h521],
Coefficient[coefIMLHS32[[4]], h512], Coefficient[coefIMLHS32[[4]], h503],
Coefficient[coefIMLHS32[[4]], h630], Coefficient[coefIMLHS32[[4]], h621],
Coefficient[coefIMLHS32[[4]], h612], Coefficient[coefIMLHS32[[4]], h603]},
{Coefficient[coefIMLHS32[[5]], h530], Coefficient[coefIMLHS32[[5]], h521],
Coefficient[coefIMLHS32[[5]], h512], Coefficient[coefIMLHS32[[5]], h503],
Coefficient[coefIMLHS32[[5]], h630], Coefficient[coefIMLHS32[[5]], h621],
Coefficient[coefIMLHS32[[5]], h612], Coefficient[coefIMLHS32[[5]], h603]},
{Coefficient[coefIMLHS32[[6]], h530], Coefficient[coefIMLHS32[[6]], h521],
Coefficient[coefIMLHS32[[6]], h512], Coefficient[coefIMLHS32[[6]], h503],
Coefficient[coefIMLHS32[[6]], h630], Coefficient[coefIMLHS32[[6]], h621],
Coefficient[coefIMLHS32[[6]], h612], Coefficient[coefIMLHS32[[6]], h603]}},

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Coefficient[coefIMLHS32[[6]], h630], Coefficient[coefIMLHS32[[6]], h621],
Coefficient[coefIMLHS32[[6]], h612], Coefficient[coefIMLHS32[[6]], h603]],
{Coefficient[coefIMLHS32[[7]], h530], Coefficient[coefIMLHS32[[7]], h521],
Coefficient[coefIMLHS32[[7]], h512], Coefficient[coefIMLHS32[[7]], h503],
Coefficient[coefIMLHS32[[7]], h630], Coefficient[coefIMLHS32[[7]], h621],
Coefficient[coefIMLHS32[[7]], h612], Coefficient[coefIMLHS32[[7]], h603]],
{Coefficient[coefIMLHS32[[8]], h530], Coefficient[coefIMLHS32[[8]], h521],
Coefficient[coefIMLHS32[[8]], h512], Coefficient[coefIMLHS32[[8]], h503],
Coefficient[coefIMLHS32[[8]], h630], Coefficient[coefIMLHS32[[8]], h621],
Coefficient[coefIMLHS32[[8]], h612], Coefficient[coefIMLHS32[[8]], h603]]}],

vectorIMC32[[j]] = {coefTIMRHS32[[1, j]], coefTIMRHS32[[2, j]],
coefTIMRHS32[[3, j]], coefTIMRHS32[[4, j]], coefTIMRHS32[[5, j]],
coefTIMRHS32[[6, j]], coefTIMRHS32[[7, j]], coefTIMRHS32[[8, j]]};
(* Solvability condition *)
If[Det[arrayIMA32[[j]]] == 0, {Print["Resonance"];
Print[(j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi]], {j, 1, 1 + 2 * ft}};
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
coefh2 = Chop[Expand[Sum[Inverse[arrayIMA32[[j]]].vectorIMC32[[j]] *
E^((j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi t), {j, 1, 1 + 2 * ft}]]];
(* h2={h530,h521,h512,h503,h630,h621,h612,h603} *)
(* ***** *)

arrayIMA33 = Table[0, {1 + 2 * ft}];
vectorIMC33 = Table[0, {1 + 2 * ft}];
Do[arrayIMA33[[j]] =
Expand[(j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi * IdentityMatrix[8] +
{{Coefficient[coefIMLHS33[[1]], h730], Coefficient[coefIMLHS33[[1]], h721],
Coefficient[coefIMLHS33[[1]], h712], Coefficient[coefIMLHS33[[1]], h703],
Coefficient[coefIMLHS33[[1]], h830], Coefficient[coefIMLHS33[[1]], h821],
Coefficient[coefIMLHS33[[1]], h812], Coefficient[coefIMLHS33[[1]], h803]],
{Coefficient[coefIMLHS33[[2]], h730], Coefficient[coefIMLHS33[[2]], h721],
Coefficient[coefIMLHS33[[2]], h712], Coefficient[coefIMLHS33[[2]], h703],
Coefficient[coefIMLHS33[[2]], h830], Coefficient[coefIMLHS33[[2]], h821],
Coefficient[coefIMLHS33[[2]], h812], Coefficient[coefIMLHS33[[2]], h803]],
{Coefficient[coefIMLHS33[[3]], h730], Coefficient[coefIMLHS33[[3]], h721],
Coefficient[coefIMLHS33[[3]], h712], Coefficient[coefIMLHS33[[3]], h703],
Coefficient[coefIMLHS33[[3]], h830], Coefficient[coefIMLHS33[[3]], h821],
Coefficient[coefIMLHS33[[3]], h812], Coefficient[coefIMLHS33[[3]], h803]],
{Coefficient[coefIMLHS33[[4]], h730], Coefficient[coefIMLHS33[[4]], h721],
Coefficient[coefIMLHS33[[4]], h712], Coefficient[coefIMLHS33[[4]], h703],
Coefficient[coefIMLHS33[[4]], h830], Coefficient[coefIMLHS33[[4]], h821],
Coefficient[coefIMLHS33[[4]], h812], Coefficient[coefIMLHS33[[4]], h803]],
{Coefficient[coefIMLHS33[[5]], h730], Coefficient[coefIMLHS33[[5]], h721],
Coefficient[coefIMLHS33[[5]], h712], Coefficient[coefIMLHS33[[5]], h703],
Coefficient[coefIMLHS33[[5]], h830], Coefficient[coefIMLHS33[[5]], h821],
Coefficient[coefIMLHS33[[5]], h812], Coefficient[coefIMLHS33[[5]], h803]],
{Coefficient[coefIMLHS33[[6]], h730], Coefficient[coefIMLHS33[[6]], h721],
Coefficient[coefIMLHS33[[6]], h712], Coefficient[coefIMLHS33[[6]], h703],
Coefficient[coefIMLHS33[[6]], h830], Coefficient[coefIMLHS33[[6]], h821],
Coefficient[coefIMLHS33[[6]], h812], Coefficient[coefIMLHS33[[6]], h803]]},

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Coefficient[coefIMLHS33[[6]], h812], Coefficient[coefIMLHS33[[6]], h803}},
{Coefficient[coefIMLHS33[[7]], h730], Coefficient[coefIMLHS33[[7]], h721],
Coefficient[coefIMLHS33[[7]], h712], Coefficient[coefIMLHS33[[7]], h703],
Coefficient[coefIMLHS33[[7]], h830], Coefficient[coefIMLHS33[[7]], h821],
Coefficient[coefIMLHS33[[7]], h812], Coefficient[coefIMLHS33[[7]], h803]},
{Coefficient[coefIMLHS33[[8]], h730], Coefficient[coefIMLHS33[[8]], h721],
Coefficient[coefIMLHS33[[8]], h712], Coefficient[coefIMLHS33[[8]], h703],
Coefficient[coefIMLHS33[[8]], h830], Coefficient[coefIMLHS33[[8]], h821],
Coefficient[coefIMLHS33[[8]], h812], Coefficient[coefIMLHS33[[8]], h803]}}};

vectorIMC33[[j]] = {coefTIMRHS33[[1, j]], coefTIMRHS33[[2, j]],
coefTIMRHS33[[3, j]], coefTIMRHS33[[4, j]], coefTIMRHS33[[5, j]],
coefTIMRHS33[[6, j]], coefTIMRHS33[[7, j]], coefTIMRHS33[[8, j]]};
(* Solvability condition *)
If[Det[arrayIMA33[[j]]] == 0, {Print["Resonance"];
Print[(j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi]], {j, 1, 1 + 2*ft}};
(* (j(1+(-1)^j)/4+(j-1)(-1+(-1)^j)/4) gives the sequence 0,1,-1,2,-2,3,..... *)
coefh3 = Chop[Expand[Sum[Inverse[arrayIMA33[[j]]].vectorIMC33[[j]] *
E^((j (1 + (-1)^j) / 4 + (j - 1) (-1 + (-1)^j) / 4) I Pi t), {j, 1, 1 + 2*ft}]]];
(* h3={h730,h721,h712,h703,h830,h821,h812,h803} *)
(* ***** *)
vlim =
Expand[{h330 v1^3 + h321 v1^2 v2 + h312 v1 v2^2 + h303 v2^3, h430 v1^3 + h421 v1^2 v2 +
h412 v1 v2^2 + h403 v2^3, h530 v1^3 + h521 v1^2 v2 + h512 v1 v2^2 + h503 v2^3,
h630 v1^3 + h621 v1^2 v2 + h612 v1 v2^2 + h603 v2^3, h730 v1^3 + h721 v1^2 v2 +
h712 v1 v2^2 + h703 v2^3, h830 v1^3 + h821 v1^2 v2 + h812 v1 v2^2 + h803 v2^3} /.
{h330 -> coefh1[[1]], h321 -> coefh1[[2]], h312 -> coefh1[[3]], h303 -> coefh1[[4]],
h430 -> coefh1[[5]], h421 -> coefh1[[6]], h412 -> coefh1[[7]], h403 -> coefh1[[8]],
h530 -> coefh2[[1]], h521 -> coefh2[[2]], h512 -> coefh2[[3]], h503 -> coefh2[[4]],
h630 -> coefh2[[5]], h621 -> coefh2[[6]], h612 -> coefh2[[7]], h603 -> coefh2[[8]],
h730 -> coefh3[[1]], h721 -> coefh3[[2]], h712 -> coefh3[[3]], h703 -> coefh3[[4]],
h830 -> coefh3[[5]], h821 -> coefh3[[6]], h812 -> coefh3[[7]], h803 -> coefh3[[8]]}];
vp = Chop[Expand[invMbarp.invQbarp.{z1, z2}]];
v2im = Chop[Expand[vlim /. {v1 -> vp[[1]], v2 -> vp[[2]]}]];
(* ***** Finding nonlinear feedback for fifth order terms ***** *)
v2fp = Expand[vlfp /. {v3 -> v2im[[1]], v4 -> v2im[[2]],
v5 -> v2im[[3]], v6 -> v2im[[4]], v7 -> v2im[[5]], v8 -> v2im[[6]]}];
coefLv2fp = CoefficientList[v2fp, {z1, z2}];

v2fpT3 = Chop[Expand[{coefLv2fp[[1]][[4, 1]] z1^3 + coefLv2fp[[1]][[3, 2]] z1^2 z2 +
coefLv2fp[[1]][[2, 3]] z1 z2^2 + coefLv2fp[[1]][[1, 4]] z2^3,
coefLv2fp[[2]][[4, 1]] z1^3 + coefLv2fp[[2]][[3, 2]] z1^2 z2 +
coefLv2fp[[2]][[2, 3]] z1 z2^2 + coefLv2fp[[2]][[1, 4]] z2^3}]];

v2fpT5 = Chop[Expand[{coefLv2fp[[1]][[6, 1]] z1^5 + coefLv2fp[[1]][[5, 2]] z1^4 z2 +
coefLv2fp[[1]][[4, 3]] z1^3 z2^2 + coefLv2fp[[1]][[3, 4]] z1^2 z2^3 +
coefLv2fp[[1]][[2, 5]] z1 z2^4 + coefLv2fp[[1]][[1, 6]] z2^5,
coefLv2fp[[2]][[6, 1]] z1^5 + coefLv2fp[[2]][[5, 2]] z1^4 z2 +
coefLv2fp[[2]][[4, 3]] z1^3 z2^2 + coefLv2fp[[2]][[3, 4]] z1^2 z2^3 +
coefLv2fp[[2]][[2, 5]] z1 z2^4 + coefLv2fp[[2]][[1, 6]] z2^5}]];

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fpy5 = Chop[Expand[(v2fpT3 /. {z1 -> y1 + v2φ3[[1]], z2 -> y2 + v2φ3[[2]]}) +
(v2fpT5 /. {z1 -> y1, z2 -> y2})]];

φ5 = TrigToExp[{Sum[(φc150[i] Cos[i π t] + φs150[i] Sin[i π t]) y1^5 +
(φc141[i] Cos[i π t] + φs141[i] Sin[i π t]) y1^4 y2 +
(φc132[i] Cos[i π t] + φs132[i] Sin[i π t]) y1^3 y2^2 +
(φc123[i] Cos[i π t] + φs123[i] Sin[i π t]) y1^2 y2^3 +
(φc114[i] Cos[i π t] + φs114[i] Sin[i π t]) y1 y2^4 +
(φc105[i] Cos[i π t] + φs105[i] Sin[i π t]) y2^5, {i, 0, ft}],
Sum[(φc250[i] Cos[i π t] + φs250[i] Sin[i π t]) y1^5 +
(φc241[i] Cos[i π t] + φs241[i] Sin[i π t]) y1^4 y2 +
(φc232[i] Cos[i π t] + φs232[i] Sin[i π t]) y1^3 y2^2 +
(φc223[i] Cos[i π t] + φs223[i] Sin[i π t]) y1^2 y2^3 +
(φc214[i] Cos[i π t] + φs214[i] Sin[i π t]) y1 y2^4 +
(φc205[i] Cos[i π t] + φs205[i] Sin[i π t]) y2^5, {i, 0, ft}]]];
α5 = TrigToExp[Sum[(αc50[i] Cos[i π t] + αs50[i] Sin[i π t]) z1^5 +
(αc41[i] Cos[i π t] + αs41[i] Sin[i π t]) z1^4 z2 +
(αc32[i] Cos[i π t] + αs32[i] Sin[i π t]) z1^3 z2^2 +
(αc23[i] Cos[i π t] + αs23[i] Sin[i π t]) z1^2 z2^3 +
(αc14[i] Cos[i π t] + αs14[i] Sin[i π t]) z1 z2^4 +
(αc05[i] Cos[i π t] + αs05[i] Sin[i π t]) z2^5, {i, 0, ft}]];
β4 = TrigToExp[Sum[(βc40[i] Cos[i π t] + βs40[i] Sin[i π t]) z1^4 +
(βc31[i] Cos[i π t] + βs31[i] Sin[i π t]) z1^3 z2 +
(βc22[i] Cos[i π t] + βs22[i] Sin[i π t]) z1^2 z2^2 +
(βc13[i] Cos[i π t] + βs13[i] Sin[i π t]) z1 z2^3 +
(βc04[i] Cos[i π t] + βs04[i] Sin[i π t]) z2^4, {i, 0, ft}]];

g0pv1α3 = g0p * v1α3;
g0pay5 = Chop[Expand[(g0pv1α3 /. {z1 -> y1 + v2φ3[[1]], z2 -> y2 + v2φ3[[2]]}) +
(g0p * α5 /. {z1 -> y1, z2 -> y2})]];

resdy5 = Expand[-{{∂y1 v2φ3[[1]], ∂y2 v2φ3[[1]]}, {∂y1 v2φ3[[2]], ∂y2 v2φ3[[2]]}}.
((v2fpT3 /. {z1 -> y1, z2 -> y2}) + (g0pv1α3 /. {z1 -> y1, z2 -> y2}) + Jp.v2φ3 - ∂t v2φ3)];

(* ***** Finding nonlinear feedback for fifth order terms continued ***** *)
CNRHSα5 = Expand[fpy5 + g0pay5 + resdy5];
coefLCNRHSα5 = CoefficientList[CNRHSα5, {y1, y2}];

coefCNRHSα5 = Chop[ExpToTrig[{coefLCNRHSα5[[1]][[6, 1]],
coefLCNRHSα5[[1]][[5, 2]], coefLCNRHSα5[[1]][[4, 3]], coefLCNRHSα5[[1]][[3, 4]],
coefLCNRHSα5[[1]][[2, 5]], coefLCNRHSα5[[1]][[1, 6]], coefLCNRHSα5[[2]][[6, 1]],
coefLCNRHSα5[[2]][[5, 2]], coefLCNRHSα5[[2]][[4, 3]], coefLCNRHSα5[[2]][[3, 4]],
coefLCNRHSα5[[2]][[2, 5]], coefLCNRHSα5[[2]][[1, 6]]]];

coefTCNRHSα5 = Table[0, {2 * 6}, {1 + 2 * ft}];
Do[
Do[If[Exponent[TrigToExp[coefCNRHSα5[[i, k]]], E] == 0,
{coefTCNRHSα5[[i, 1]] = coefTCNRHSα5[[i, 1]] + coefCNRHSα5[[i, k]]},
{k, 1, Dimensions[coefCNRHSα5[[i]]][[1]]}], {i, 1, 2 * 6}];

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Do[
  coefTCNRHSa5[[i, 2 * j]] = Coefficient[coefCNRHSa5[[i]], Cos[j π t]];
  coefTCNRHSa5[[i, 2 * j + 1]] = Coefficient[coefCNRHSa5[[i]], Sin[j π t]];
  {i, 1, 2 * 6}, {j, 1, ft}];

CNLHSA5 = Chop[Expand[
  ∂t φ5 + {{∂y1 φ5[[1]], ∂y2 φ5[[1]]}, {∂y1 φ5[[2]], ∂y2 φ5[[2]]}}.Jp.{y1, y2} - Jp.φ5]];
coefLCNLHSA5 = CoefficientList[CNLHSA5, {y1, y2}];

coefCNLHSA5 = Chop[ExpToTrig[{coefLCNLHSA5[[1]][[6, 1]],
  coefLCNLHSA5[[1]][[5, 2]], coefLCNLHSA5[[1]][[4, 3]], coefLCNLHSA5[[1]][[3, 4]],
  coefLCNLHSA5[[1]][[2, 5]], coefLCNLHSA5[[1]][[1, 6]], coefLCNLHSA5[[2]][[6, 1]],
  coefLCNLHSA5[[2]][[5, 2]], coefLCNLHSA5[[2]][[4, 3]], coefLCNLHSA5[[2]][[3, 4]],
  coefLCNLHSA5[[2]][[2, 5]], coefLCNLHSA5[[2]][[1, 6]]}]];

coefTCNLHSA5 = Table[0, {2 * 6}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[TrigToExp[coefCNLHSA5[[i, k]]], E] == 0,
    {coefTCNLHSA5[[i, 1]] = coefTCNLHSA5[[i, 1]] + coefCNLHSA5[[i, k]]},
    {k, 1, Dimensions[coefCNLHSA5[[i]][[1]]}], {i, 1, 2 * 6}];
  Do[
    coefTCNLHSA5[[i, 2 * j]] = Coefficient[coefCNLHSA5[[i]], Cos[j π t]];
    coefTCNLHSA5[[i, 2 * j + 1]] = Coefficient[coefCNLHSA5[[i]], Sin[j π t]];
    {i, 1, 2 * 6}, {j, 1, ft}];

Aa5 = Array[rowAa5, {2 * 6 * (1 + 2 * ft)}];

Do[rowAa5[k + (i - 1) 2 * 6] =
  Flatten[{{Coefficient[coefTCNLHSA5[[k, i]], φc150[0]], Coefficient[
    coefTCNLHSA5[[k, i]], φc141[0]], Coefficient[coefTCNLHSA5[[k, i]], φc132[0]],
    Coefficient[coefTCNLHSA5[[k, i]], φc123[0]], Coefficient[
    coefTCNLHSA5[[k, i]], φc114[0]], Coefficient[coefTCNLHSA5[[k, i]], φc105[0]],
    Coefficient[coefTCNLHSA5[[k, i]], φc250[0]], Coefficient[
    coefTCNLHSA5[[k, i]], φc241[0]], Coefficient[coefTCNLHSA5[[k, i]], φc232[0]],
    Coefficient[coefTCNLHSA5[[k, i]], φc223[0]], Coefficient[coefTCNLHSA5[[k, i]],
    φc214[0]], Coefficient[coefTCNLHSA5[[k, i]], φc205[0]]},
  Table[{{Coefficient[coefTCNLHSA5[[k, i]], φc150[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φs150[j]], Coefficient[coefTCNLHSA5[[k, i]], φc141[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φs141[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φc132[j]], Coefficient[coefTCNLHSA5[[k, i]], φs132[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φc123[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φs123[j]], Coefficient[coefTCNLHSA5[[k, i]], φc114[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φs114[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φc105[j]], Coefficient[coefTCNLHSA5[[k, i]], φs105[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φc250[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φs250[j]], Coefficient[coefTCNLHSA5[[k, i]], φc241[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φs241[j]], Coefficient[
    coefTCNLHSA5[[k, i]], φc232[j]], Coefficient[coefTCNLHSA5[[k, i]], φs232[j]],
    Coefficient[coefTCNLHSA5[[k, i]], φc223[j]], Coefficient[

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    coefTCNLHSα5[[k, i]], φs223[j]], Coefficient[coefTCNLHSα5[[k, i]], φc214[j]],
    Coefficient[coefTCNLHSα5[[k, i]], φs214[j]], Coefficient[
    coefTCNLHSα5[[k, i]], φc205[j]], Coefficient[coefTCNLHSα5[[k, i]], φs205[j]]},
    {j, 1, ft}]]], {i, 1, 1 + 2 * ft}, {k, 1, 2 * 6}}
Ca5 = Array[rowCa5, {2 * 6 * (1 + 2 * ft)}];
Do[rowCa5[k + (i - 1) * 2 * 6] = coefTCNRHSα5[[k, i]], {k, 1, 2 * 6}, {i, 1, 1 + 2 * ft}]
coefφ5 = Expand[Inverse[Aα5].Ca5];
v1φ5 = Expand[φ5 /. Flatten[
  {{φc150[0] -> coefφ5[[1]], φc141[0] -> coefφ5[[2]], φc132[0] -> coefφ5[[3]],
  φc123[0] -> coefφ5[[4]], φc114[0] -> coefφ5[[5]], φc105[0] -> coefφ5[[6]],
  φc250[0] -> coefφ5[[7]], φc241[0] -> coefφ5[[8]], φc232[0] -> coefφ5[[9]],
  φc223[0] -> coefφ5[[10]], φc214[0] -> coefφ5[[11]], φc205[0] -> coefφ5[[12]]},
  Table[{φc150[j] -> coefφ5[[12 (2 j - 1) + 1]], φs150[j] -> coefφ5[[12 (2 j - 1) + 2]],
  φc141[j] -> coefφ5[[12 (2 j - 1) + 3]], φs141[j] -> coefφ5[[12 (2 j - 1) + 4]],
  φc132[j] -> coefφ5[[12 (2 j - 1) + 5]], φs132[j] -> coefφ5[[12 (2 j - 1) + 6]],
  φc123[j] -> coefφ5[[12 (2 j - 1) + 7]], φs123[j] -> coefφ5[[12 (2 j - 1) + 8]],
  φc114[j] -> coefφ5[[12 (2 j - 1) + 9]], φs114[j] -> coefφ5[[12 (2 j - 1) + 10]],
  φc105[j] -> coefφ5[[12 (2 j - 1) + 11]],
  φs105[j] -> coefφ5[[12 (2 j - 1) + 12]], φc250[j] -> coefφ5[[12 (2 j - 1) + 13]],
  φs250[j] -> coefφ5[[12 (2 j - 1) + 14]], φc241[j] -> coefφ5[[12 (2 j - 1) + 15]],
  φs241[j] -> coefφ5[[12 (2 j - 1) + 16]], φc232[j] -> coefφ5[[12 (2 j - 1) + 17]],
  φs232[j] -> coefφ5[[12 (2 j - 1) + 18]], φc223[j] -> coefφ5[[12 (2 j - 1) + 19]],
  φs223[j] -> coefφ5[[12 (2 j - 1) + 20]], φc214[j] -> coefφ5[[12 (2 j - 1) + 21]],
  φs214[j] -> coefφ5[[12 (2 j - 1) + 22]], φc205[j] -> coefφ5[[12 (2 j - 1) + 23]],
  φs205[j] -> coefφ5[[12 (2 j - 1) + 24]]}, {j, 1, ft}]]];

resdy4 = Expand[g0p * v1β2 /. {z1 -> y1 + v2φ3[[1]], z2 -> y2 + v2φ3[[2]]}];

CNRHSβ4 = resdy4;
coefLCNRHSβ4 = CoefficientList[CNRHSβ4, {y1, y2}];

coefCNRHSβ4 = Chop[ExpToTrig[{coefLCNRHSβ4[[1]][[5, 1]], coefLCNRHSβ4[[1]][[4, 2]],
  coefLCNRHSβ4[[1]][[3, 3]], coefLCNRHSβ4[[1]][[2, 4]], coefLCNRHSβ4[[1]][[1, 5]],
  coefLCNRHSβ4[[2]][[5, 1]], coefLCNRHSβ4[[2]][[4, 2]], coefLCNRHSβ4[[2]][[3, 3]],
  coefLCNRHSβ4[[2]][[2, 4]], coefLCNRHSβ4[[2]][[1, 5]]}]];

coefTCNRHSβ4 = Table[0, {2 * 5}, {1 + 2 * ft}];
Do[
  Do[If[Exponent[TrigToExp[coefCNRHSβ4[[i, k]]], E] == 0,
    {coefTCNRHSβ4[[i, 1]] = coefTCNRHSβ4[[i, 1]] + coefCNRHSβ4[[i, k]]}],
    {k, 1, Dimensions[coefCNRHSβ4[[i]][[1]]}], {i, 1, 2 * 5}];
Do[
  coefTCNRHSβ4[[i, 2 * j]] = Coefficient[coefCNRHSβ4[[i]], Cos[j π t]];
  coefTCNRHSβ4[[i, 2 * j + 1]] = Coefficient[coefCNRHSβ4[[i]], Sin[j π t]];
  {i, 1, 2 * 5}, {j, 1, ft}];

CNLHSβ4 =
  Chop[Expand[{{∂y1 v1φ5[[1]], ∂y2 v1φ5[[1]]}, {∂y1 v1φ5[[2]], ∂y2 v1φ5[[2]]}].g0p -
  g0p * β4 /. {z1 -> y1, z2 -> y2}]];

```

```

coefLCNLHSβ4 = CoefficientList[CNLHSβ4, {y1, y2}];

coefCNLHSβ4 = Chop[ExpToTrig[{coefLCNLHSβ4[[1]][[5, 1]], coefLCNLHSβ4[[1]][[4, 2]],
  coefLCNLHSβ4[[1]][[3, 3]], coefLCNLHSβ4[[1]][[2, 4]], coefLCNLHSβ4[[1]][[1, 5]],
  coefLCNLHSβ4[[2]][[5, 1]], coefLCNLHSβ4[[2]][[4, 2]], coefLCNLHSβ4[[2]][[3, 3]],
  coefLCNLHSβ4[[2]][[2, 4]], coefLCNLHSβ4[[2]][[1, 5]]}]];

coefTCNLHSβ4 = Table[0, {2 * 5}, {1 + 2 * ft}];
Do[
  Do[ If[Exponent[TrigToExp[coefCNLHSβ4[[i, k]]], E] == 0,
    {coefTCNLHSβ4[[i, 1]] = coefTCNLHSβ4[[i, 1]] + coefCNLHSβ4[[i, k]]},
    {k, 1, Dimensions[coefCNLHSβ4[[i]]][[1]]}], {i, 1, 2 * 5}];
Do[
  coefTCNLHSβ4[[i, 2 * j]] = Coefficient[coefCNLHSβ4[[i]], Cos[j π t]];
  coefTCNLHSβ4[[i, 2 * j + 1]] = Coefficient[coefCNLHSβ4[[i]], Sin[j π t]];
  {i, 1, 2 * 5}, {j, 1, ft}];

Aβ4 = Array[rowAβ4, {2 * 5 * (1 + 2 * ft)}];
Cβ4LHS = Array[rowCβ4LHS, {2 * 5 * (1 + 2 * ft)}];
Do[rowAβ4[k + (i - 1) 2 * 5] =
  Flatten[{{Coefficient[coefTCNLHSβ4[[k, i]], ac50[0]], Coefficient[
    coefTCNLHSβ4[[k, i]], ac41[0]], Coefficient[coefTCNLHSβ4[[k, i]], ac32[0]],
    Coefficient[coefTCNLHSβ4[[k, i]], ac23[0]], Coefficient[coefTCNLHSβ4[[k, i]],
    ac14[0]], Coefficient[coefTCNLHSβ4[[k, i]], ac05[0]], Coefficient[
    coefTCNLHSβ4[[k, i]], βc40[0]], Coefficient[coefTCNLHSβ4[[k, i]], βc31[0]],
    Coefficient[coefTCNLHSβ4[[k, i]], βc22[0]], Coefficient[coefTCNLHSβ4[[k, i]],
    βc13[0]], Coefficient[coefTCNLHSβ4[[k, i]], βc04[0]]},
  Table[{{Coefficient[coefTCNLHSβ4[[k, i]], ac50[j]], Coefficient[
    coefTCNLHSβ4[[k, i]], as50[j]], Coefficient[coefTCNLHSβ4[[k, i]], ac41[j]],
    Coefficient[coefTCNLHSβ4[[k, i]], as41[j]], Coefficient[coefTCNLHSβ4[[k, i]],
    ac32[j]], Coefficient[coefTCNLHSβ4[[k, i]], as32[j]], Coefficient[
    coefTCNLHSβ4[[k, i]], ac23[j]], Coefficient[coefTCNLHSβ4[[k, i]], as23[j]],
    Coefficient[coefTCNLHSβ4[[k, i]], ac14[j]], Coefficient[coefTCNLHSβ4[[k, i]],
    as14[j]], Coefficient[coefTCNLHSβ4[[k, i]], ac05[j]], Coefficient[
    coefTCNLHSβ4[[k, i]], as05[j]], Coefficient[coefTCNLHSβ4[[k, i]], βc40[j]],
    Coefficient[coefTCNLHSβ4[[k, i]], βs40[j]], Coefficient[coefTCNLHSβ4[[k, i]],
    βc31[j]], Coefficient[coefTCNLHSβ4[[k, i]], βs31[j]], Coefficient[
    coefTCNLHSβ4[[k, i]], βc22[j]], Coefficient[coefTCNLHSβ4[[k, i]], βs22[j]],
    Coefficient[coefTCNLHSβ4[[k, i]], βc13[j]], Coefficient[
    coefTCNLHSβ4[[k, i]], βs13[j]], Coefficient[coefTCNLHSβ4[[k, i]], βc04[j]],
    Coefficient[coefTCNLHSβ4[[k, i]], βs04[j]]}, {j, 1, ft}}]];
  rowCβ4LHS[k + (i - 1) 2 * 5] = Chop[Expand[coefTCNLHSβ4[[k, i]] - rowAβ4[k + (i - 1) 2 * 5].
  Flatten[{{ac50[0], ac41[0], ac32[0], ac23[0], ac14[0], ac05[0], βc40[0],
    βc31[0], βc22[0], βc13[0], βc04[0]}, Table[{{ac50[j], as50[j], ac41[j],
    as41[j], ac32[j], as32[j], ac23[j], as23[j], ac14[j], as14[j], ac05[j],
    as05[j], βc40[j], βs40[j], βc31[j], βs31[j], βc22[j], βs22[j], βc13[j],
    βs13[j], βc04[j], βs04[j]}, {j, 1, ft}}]}]]]; {k, 1, 2 * 5}, {i, 1, 1 + 2 * ft}]]

Cβ4RHS = Array[rowCβ4RHS, {2 * 5 * (1 + 2 * ft)}];
Do[rowCβ4RHS[k + (i - 1) 2 * 5] = coefTCNRHSβ4[[k, i]], {i, 1, 1 + 2 * ft}, {k, 1, 2 * 5}];
Cβ4 = Cβ4RHS - Cβ4LHS;

```

```

coefa5β4 = LinearSolve[Aβ4, Cβ4];
v1α5 = Expand[α5 /. Flatten[
  {{ac50[0] -> coefa5β4[[1]], ac41[0] -> coefa5β4[[2]], ac32[0] -> coefa5β4[[3]],
   ac23[0] -> coefa5β4[[4]], ac14[0] -> coefa5β4[[5]], ac05[0] -> coefa5β4[[6]]},
  Table[{ac50[j] -> coefa5β4[[11 (2 j - 1) + 1]], as50[j] ->
    coefa5β4[[11 (2 j - 1) + 2]], ac41[j] -> coefa5β4[[11 (2 j - 1) + 3]],
    as41[j] -> coefa5β4[[11 (2 j - 1) + 4]], ac32[j] -> coefa5β4[[11 (2 j - 1) + 5]],
    as32[j] -> coefa5β4[[11 (2 j - 1) + 6]], ac23[j] -> coefa5β4[[11 (2 j - 1) + 7]],
    as23[j] -> coefa5β4[[11 (2 j - 1) + 8]], ac14[j] -> coefa5β4[[11 (2 j - 1) + 9]],
    as14[j] -> coefa5β4[[11 (2 j - 1) + 10]], ac05[j] -> coefa5β4[[11 (2 j - 1) + 11]],
    as05[j] -> coefa5β4[[11 (2 j - 1) + 12]]}, {j, 1, ft}]]];

v1β4 =
  Expand[β4 /. Flatten[{{βc40[0] -> coefa5β4[[7]], βc31[0] -> coefa5β4[[8]], βc22[0] ->
    coefa5β4[[9]], βc13[0] -> coefa5β4[[10]], βc04[0] -> coefa5β4[[11]]}, Table[
    {βc40[j] -> coefa5β4[[11 (2 j - 1) + 13]], βs40[j] -> coefa5β4[[11 (2 j - 1) + 14]],
    βc31[j] -> coefa5β4[[11 (2 j - 1) + 15]],
    βs31[j] -> coefa5β4[[11 (2 j - 1) + 16]], βc22[j] -> coefa5β4[[11 (2 j - 1) + 17]],
    βs22[j] -> coefa5β4[[11 (2 j - 1) + 18]], βc13[j] -> coefa5β4[[11 (2 j - 1) + 19]],
    βs13[j] -> coefa5β4[[11 (2 j - 1) + 20]], βc04[j] -> coefa5β4[[11 (2 j - 1) + 21]],
    βs04[j] -> coefa5β4[[11 (2 j - 1) + 22]]}, {j, 1, ft}]]];

v2φ5 = Expand[v1φ5 /. Flatten[
  {{ac50[0] -> coefa5β4[[1]], ac41[0] -> coefa5β4[[2]], ac32[0] -> coefa5β4[[3]],
   ac23[0] -> coefa5β4[[4]], ac14[0] -> coefa5β4[[5]], ac05[0] -> coefa5β4[[6]]},
  Table[{ac50[j] -> coefa5β4[[11 (2 j - 1) + 1]], as50[j] ->
    coefa5β4[[11 (2 j - 1) + 2]], ac41[j] -> coefa5β4[[11 (2 j - 1) + 3]],
    as41[j] -> coefa5β4[[11 (2 j - 1) + 4]], ac32[j] -> coefa5β4[[11 (2 j - 1) + 5]],
    as32[j] -> coefa5β4[[11 (2 j - 1) + 6]], ac23[j] -> coefa5β4[[11 (2 j - 1) + 7]],
    as23[j] -> coefa5β4[[11 (2 j - 1) + 8]], ac14[j] -> coefa5β4[[11 (2 j - 1) + 9]],
    as14[j] -> coefa5β4[[11 (2 j - 1) + 10]], ac05[j] -> coefa5β4[[11 (2 j - 1) + 11]],
    as05[j] -> coefa5β4[[11 (2 j - 1) + 12]]}, {j, 1, ft}]]];

v1α5 >> "alpha5.txt";
v1β4 >> "beta4.txt";
v2φ5 >> "phi5.txt";

```

C.11

```
(* File name "Find_Cheb. Coefficients.nb" *)
Module [{m = 20, at = -1.0453752591269074`k1 + 0.5277492082264343`k5 +
  1.2572916861075216`k1 Cos[2 π t] - 0.03386552195838752`k5 Cos[2 π t] -
  0.20020462591781812`k1 Cos[4 π t] - 0.013645923171801314`k5 Cos[4 π t] +
  0.009442291809787024`k1 Cos[6 π t] + 0.0025080443350063166`k5 Cos[6 π t] -
  3.9498983953756985`k5 Sin[2 π t] + 1.2579246931574315`k5 Sin[4 π t] -
  0.08899154020493226`k5 Sin[6 π t], w = (t - t^2)^(-0.5)},
p11 = Table[0, {m}];
Do[p11[[i]] = If[i == 1, 1/Pi * Integrate[w*at * ChebyshevT[i - 1, 2 t - 1], {t, 0, 1}],
  2/Pi * Integrate[Expand[w*at * ChebyshevT[i - 1, 2 t - 1]], {t, 0, 1}]], {i, 1, m}]]

Module [{m = 20,
  at = 1.0711374014200339`k1 - 0.3481972761494919`k5 - 1.231626866328585`k1 Cos[2 π t] -
  0.4157410385584257`k5 Cos[2 π t] + 0.1908805227674405`k1 Cos[4 π t] +
  0.30999881695992804`k5 Cos[4 π t] - 0.008784493812608036`k1 Cos[6 π t] -
  0.028804069408477805`k5 Cos[6 π t] - 0.2618057509133236`k1 Sin[2 π t] +
  3.9566153234358628`k5 Sin[2 π t] + 0.0593394144644013`k1 Sin[4 π t] -
  1.2188328581089853`k5 Sin[4 π t] - 0.003368947993935206`k1 Sin[6 π t] +
  0.08392229460431481`k5 Sin[6 π t], w = (t - t^2)^(-0.5)},
p12 = Table[0, {m}];
Do[p12[[i]] = If[i == 1, 1/Pi * Integrate[w*at * ChebyshevT[i - 1, 2 t - 1], {t, 0, 1}],
  2/Pi * Integrate[Expand[w*at * ChebyshevT[i - 1, 2 t - 1]], {t, 0, 1}]], {i, 1, m}]]

Module [
{m = 20, at = -1.0796462581698045`k1 - 0.29812350291411255`k5 + 1.2414142191047683`k1
  Cos[2 π t] - 0.49262715685066116`k5 Cos[2 π t] - 0.1923978681864789`k1 Cos[4 π t] +
  0.32347520242757`k5 Cos[4 π t] + 0.008854334452394584`k1 Cos[6 π t] -
  0.02965279841483931`k5 Cos[6 π t] - 0.2638762996301704`k1 Sin[2 π t] -
  3.9706248941638975`k5 Sin[2 π t] + 0.059811139762013854`k1 Sin[4 π t] +
  1.2251800810720512`k5 Sin[4 π t] - 0.003395846894458018`k1 Sin[6 π t] -
  0.08435232565769303`k5 Sin[6 π t], w = (t - t^2)^(-0.5)},
p21 = Table[0, {m}];
Do[p21[[i]] = If[i == 1, 1/Pi * Integrate[w*at * ChebyshevT[i - 1, 2 t - 1], {t, 0, 1}],
  2/Pi * Integrate[Expand[w*at * ChebyshevT[i - 1, 2 t - 1]], {t, 0, 1}]], {i, 1, m}]]

Module [{m = 20, at = 1.0447526400332632`k1 + 0.47752997200582037`k5 -
  1.2578659366782126`k1 Cos[2 π t] + 0.04130998838876537`k5 Cos[2 π t] +
  0.20032107217769146`k1 Cos[4 π t] - 0.025085058500923938`k5 Cos[4 π t] -
  0.009449112524244551`k1 Cos[6 π t] + 0.003171047767217093`k5 Cos[6 π t] +
  3.951702728181797`k5 Sin[2 π t] - 1.2586524945194328`k5 Sin[4 π t] +
  0.08905575565342111`k5 Sin[6 π t], w = (t - t^2)^(-0.5)},
p22 = Table[0, {m}];
Do[p22[[i]] = If[i == 1, 1/Pi * Integrate[w*at * ChebyshevT[i - 1, 2 t - 1], {t, 0, 1}],
  2/Pi * Integrate[Expand[w*at * ChebyshevT[i - 1, 2 t - 1]], {t, 0, 1}]], {i, 1, m}]]
```

Chop[Expand[N[p11]]]
Chop[Expand[N[p12]]]
Chop[Expand[N[p21]]]
Chop[Expand[N[p22]]]

{-0.705244 k1 + 0.514894 k5, 1.74554 k5, 1.10952 k1 - 0.0396386 k5, -2.69256 k5,
-0.502032 k1 + 0.00300565 k5, 1.3237 k5, 0.149958 k1 + 0.00717559 k5,
-0.4758 k5, -0.0371783 k1 - 0.00356809 k5, -1.46922 × 10⁻¹⁰ k1 + 0.119054 k5,
0.00710769 k1 + 0.00107156 k5, -2.55745 × 10⁻⁹ k1 - 0.0231003 k5,
-0.00111122 k1 - 0.000225043 k5, -2.76068 × 10⁻⁸ k1 + 0.00356952 k5,
0.00014352 k1 + 0.0000334991 k5, -0.0000258265 k1 - 0.000454117 k5,
-0.000387398 k1 - 0.0000516531 k5, 0.00089532 k1 - 0.000723143 k5,
0.00192838 k1 + 0.000275483 k5, -0.0132232 k1 - 0.0198348 k5}

{0.736879 k1 - 0.411617 k5, 0.125013 k1 - 1.76417 k5,
-1.08969 k1 - 0.237744 k5, -0.177492 k1 + 2.69562 k5, 0.488731 k1 + 0.305951 k5,
0.0705571 k1 - 1.29674 k5, -0.14388 k1 - 0.190947 k5, -0.0224721 k1 + 0.460541 k5,
0.0353522 k1 + 0.0645681 k5, 0.00521764 k1 - 0.114227 k5,
-0.00670876 k1 - 0.0154469 k5, -0.000948935 k1 + 0.0219908 k5,
0.001042 k1 + 0.00285593 k5, 0.000140563 k1 - 0.0033809 k5,
-0.000134241 k1 - 0.00040382 k5, -0.0000258265 k1 + 0.000441204 k5,
0.0000193858 k1 + 0.0000326758 k5, -0.0000165597 k1 + 0.0000270931 k5,
0.0000288908 k1 - 0.000737395 k5, -0.00370486 k1 + 0.000710795 k5}

{-0.742732 k1 - 0.382121 k5, 0.126001 k1 + 1.7696 k5, 1.09835 k1 - 0.305003 k5,
-0.178896 k1 - 2.70526 k5, -0.492615 k1 + 0.337283 k5, 0.0711168 k1 + 1.30281 k5,
0.145024 k1 - 0.200865 k5, -0.0226507 k1 - 0.46289 k5, -0.0356332 k1 + 0.0671971 k5,
0.00525917 k1 + 0.114816 k5, 0.0067621 k1 - 0.0159927 k5, -0.000956503 k1 - 0.0221041 k5,
-0.00105031 k1 + 0.00294775 k5, 0.000141456 k1 + 0.00339851 k5,
0.000134368 k1 - 0.000416175 k5, -0.0000473487 k1 - 0.000398159 k5,
-0.0000688708 k1 - 0.0000602619 k5, 0.0000711268 k1 + 0.0000118295 k5,
-0.000172582 k1 + 0.0000815069 k5, 0.00524838 k1 - 0.00219128 k5}

{0.704471 k1 + 0.485147 k5, -1.74628 k5, -1.11002 k1 + 0.0270507 k5,
2.6938 k5, 0.502276 k1 - 0.0266226 k5, -1.32441 k5, -0.150041 k1 + 0.0158695 k5,
0.476079 k5, 0.0372005 k1 - 0.00577652 k5, -0.119129 k5,
-0.00711224 k1 + 0.00151701 k5, -8.98612 × 10⁻¹⁰ k1 + 0.0231158 k5,
0.00111196 k1 - 0.000298408 k5, 1.35931 × 10⁻⁷ k1 - 0.00357203 k5,
-0.000144111 k1 + 0.0000434746 k5, 0.0000581097 k1 + 0.00044766 k5,
0.0000688708 k1 + 0.0000602619 k5, 0.000344354 k1 - 0.000550966 k5,
0.0016529 k1 - 0.00743805 k5, -0.0132232 k1 - 0.0143251 k5}

C.12

```
(* File name "Symbolic FTM.nb" *)
t0 = TimeUsed[];
chebcoefbr1k1 = {-0.7052435032610047`k1 + 0.5148944926677143`k5,
  1.7455350120173039`k5, 1.1095244499895334`k1 - 0.03963864642511153`k5,
  -2.6925621752120987`k5, -0.5020321138299776`k1 + 0.0030056512903545137`k5,
  1.3236963240498003`k5, 0.14995782331589513`k1 + 0.007175589711316778`k5,
  -0.4757996206168023`k5, -0.037178280076297705`k1 - 0.003568093680008187`k5,
  -1.469218361655965`*^-10k1 + 0.11905362154909804`k5,
  0.007107689145396686`k1 + 0.0010715599016968868`k5,
  -2.5574532033239003`*^-9k1 - 0.023100302254994813`k5,
  -0.0011112194546803234`k1 - 0.0002250428566565372`k5,
  -2.7606830141381835`*^-8k1 + 0.003569522153181159`k5,
  0.00014352001694555407`k1 + 0.000033499054097248734`k5,
  -0.000025826549539844802`k1 - 0.00045411682940893776`k5,
  -0.00038739824309767204`k1 - 0.000051653099079689604`k5,
  0.0008953203840479532`k1 - 0.0007231433871156545`k5,
  0.0019283823656417454`k1 + 0.0002754831950916779`k5,
  -0.013223193364400539`k1 - 0.019834790046600808`k5};
chebcoefbr1k2 = {0.7368792821268294`k1 - 0.41161728129003694`k5,
  0.1250133133409062`k1 - 1.7641716123768243`k5,
  -1.0896880148068808`k1 - 0.23774415740427365`k5,
  -0.17749231059703363`k1 + 2.695616329828998`k5, 0.4887306785642079`k1 +
  0.30595127846358255`k5, 0.07055705596188834`k1 - 1.2967392286181554`k5,
  -0.14388010357621636`k1 - 0.19094694263742262`k5,
  -0.022472087193654854`k1 + 0.46054053692911484`k5,
  0.035352179488511284`k1 + 0.06456808714754031`k5,
  0.005217636570988067`k1 - 0.11422663121812696`k5,
  -0.006708764792675462`k1 - 0.015446900210119019`k5,
  -0.00094893353588540535`k1 + 0.021990798365630983`k5,
  0.0010420011725501257`k1 + 0.00285592539232922`k5,
  0.00014056283675946242`k1 - 0.003380899357059642`k5,
  -0.00013424139839118346`k1 - 0.00040382040866705033`k5,
  -0.000025826549539844802`k1 + 0.0004412035546390154`k5,
  0.00001938584963060889`k1 + 0.000032675814716817944`k5,
  -0.000016559651347088206`k1 + 0.000027093083707717247`k5,
  0.000028890751551769212`k1 - 0.0007373948118936545`k5,
  -0.003704860617357243`k1 + 0.0007107950196259936`k5};
chebcoefbr2k1 = {-0.7427319935888463`k1 - 0.382120771002262`k5,
  0.12600106821082355`k1 + 1.7696022060413004`k5,
  1.0983471633997528`k1 - 0.30500250266852036`k5,
  -0.1788961463286649`k1 - 2.7052569499196304`k5, -0.49261475176336655`k1 +
  0.33728271488013856`k5, 0.07111680633608304`k1 + 1.302806477731994`k5,
  0.14502374665172613`k1 - 0.20086518974296405`k5, -0.022650727365802103`k1 -
  0.46288961398087947`k5, -0.03563320346661182`k1 + 0.06719712255489274`k5,
  0.005259170974071954`k1 + 0.11481630593697675`k5,
  0.0067620968635061065`k1 - 0.015992696063693333`k5,
```

```

-0.0009565028206155202`k1 - 0.02210408540176575`k5,
-0.001050308038789552`k1 + 0.002947745829811936`k5,
0.0001414556629535197`k1 + 0.003398511477117704`k5,
0.00013436802851239237`k1 - 0.00041617478013257535`k5,
-0.00004734867415638214`k1 - 0.0003981593054059407`k5,
-0.00006887079877291948`k1 - 0.00006026194892630454`k5,
0.00007112675596232592`k1 + 0.000011829508187508196`k5,
-0.00017258204201264337`k1 + 0.00008150692409548549`k5,
0.005248383762854729`k1 - 0.0021912764677216464`k5};
chebcoefbr2k2 = {0.7044705873537753`k1 + 0.48514718386958744`k5,
-1.7462756460054714`k5, -1.1100179097985168`k1 + 0.027050651735320803`k5,
2.693797115254494`k5, 0.502275832558709`k1 - 0.026622570411089508`k5,
-1.3244082781303794`k5, -0.15004078417164352`k1 + 0.01586951932108262`k5,
0.47607912093153815`k5, 0.03720054123225657`k1 - 0.005776520822922395`k5,
-0.11912878893116002`k5, -0.00711223808254582`k1 + 0.0015170140134115268`k5,
-8.986115851389389`*^-10k1 + 0.023115827859168422`k5,
0.0011119626597294457`k1 - 0.00029840834126258997`k5,
1.3593121431694484`*^-7k1 - 0.0035720262223715`k5,
-0.0001441105807736926`k1 + 0.00004347456909598698`k5,
0.00005810973646465081`k1 + 0.0004476601920239766`k5,
0.00006887079877291948`k1 + 0.00006026194892630454`k5,
0.0003443539938645974`k1 - 0.0005509663901833558`k5,
0.0016528991705500673`k1 - 0.007438046267475303`k5,
-0.013223193364400539`k1 - 0.01432512614476725`k5};

n = 2;
m = 15;
p = 15;
fGT[chebcoefbr1k1];
fGG[Transpose[MGp], m, n];
tensorP := Array[tP, {n, n, m}];
(* ***** *)
tP[1, 1, 1] = (0.26566738263261214 + chebcoefbr1k1[[1]]);
Do[tP[1, 1, k] = (chebcoefbr1k1[[k]]), {k, 2, m}];
tP[1, 2, 1] = (0 + chebcoefbr1k2[[1]]);
Do[tP[1, 2, k] = (chebcoefbr1k2[[k]]), {k, 2, m}];

tP[2, 1, 1] = (0 + chebcoefbr2k1[[1]]);
Do[tP[2, 1, k] = (chebcoefbr2k1[[k]]), {k, 2, m}];
tP[2, 2, 1] = (-0.33553608038578464 + chebcoefbr2k2[[1]]);
Do[tP[2, 2, k] = (chebcoefbr2k2[[k]]), {k, 2, m}];
(* ***** *)
(*tensorP=tensorP/.{k1→0.16906283540593794,k5→-0.6400038382967878};*)

(* ***** For calculating invQ ***** *)
(*tensorP=-Transpose[tensorP];*) (* Adjoint system *)
(* ***** *)

fQD[m, n];
fXMN[m, n];

```

```

R1 = Chop[Expand[GGT.MQD]];
(* this function calculates I+L+L.L+L.L.L+..... *)
Rpowerseries = Expand[IdentityMatrix[n*m] + R1];
Do[Rpowerseries = Expand[IdentityMatrix[n*m] + Rpowerseries.R1];, {i, 1, p-1}];
(* ***** *)
B2 = Rpowerseries.XXMN;

f11[x_] := fPhi[m, 1, 1, x];
f12[x_] := fPhi[m, 1, 2, x];
f21[x_] := fPhi[m, 2, 1, x];
f22[x_] := fPhi[m, 2, 2, x];
Phi[t_] := {{f11[t][[1]], f12[t][[1]]}, {f21[t][[1]], f22[t][[1]]}}
PHI = Chop[Phi[1]];
t1 = TimeUsed[];
CPUtime = (t1 - t0) / 60

Clear["Global`*"]

(* This subroutine needed to find out Qft and
invQft for Jrstable i.e. to find Jr after stabilization *)
(* ***** For finding Qft ***** *)

t0 = TimeUsed[];
Rbarp = MyQC[Phi[1].Phi[1], 2];
Q[t_] := Expand[Phi[t].(MatrixExp[-Rbarp t])];
Qdefined = 1/2 (1 + Sign[t]) Q[t] +
1/2 (1 - Sign[t]) Phi[t + 1].Q[1].MatrixExp[-Rbarp t].MatrixExp[-Rbarp];
Needs["FourierSeries`"];
Qbarpft = Chop[
Quiet[NFourierTrigSeries[Re[Qdefined], t, 4, FourierParameters -> {-1, 1/2}]]];
t1 = TimeUsed[];
Print["Total time = ", (t1 - t0) / 60];

(* ***** For finding invQft ***** *)
(*
t0=TimeUsed[];
invQ[t_] := Chop[Expand[MatrixExp[Rbarp t].Transpose[Phi[t]]]];
invQdefined = 1/2 (1 + Sign[t]) invQ[t] +
1/2 (1 - Sign[t]) MatrixExp[Rbarp t].MatrixExp[Rbarp].invQ[1].Transpose[Phi[t + 1]];
Needs["FourierSeries`"];
invQbarpft = Chop[NFourierTrigSeries[invQdefined, t, 4, FourierParameters -> {-1, 1/2}]];
(* {"a", " " "b"} " " |b|^(1-a)/2 \int_{-1/(2|b|)}^{1/(2|b|)} f(t) dt " " 2|b|^(1-a)/2 \int_{-1/(2|b|)}^{1/(2|b|)} f(t) cos(2\pi bnt) dt " " *)
t1=TimeUsed[];
Print["Total time = ", (t1 - t0) / 60];
*)

```



```
NotebookDirectory[]  
  
C:\Users\gabalap\Work\Research\Order reduced control\Time  
  periodic\Coupled inverted pendulums 8x8_Mod_51\Symbolic form\  
  
SetDirectory["C:\\Users\\gabalap\\Work\\Research\\Order reduced  
  control\\Time periodic\\Coupled inverted pendulums 8x8_Mod_51"];  
Rbarp >> "Rbarp.txt";  
Qbarpft >> "Qbarp.txt";  
invQbarpft >> "invQbarp.txt";  
Mbarp >> "Mbarp.txt";  
invMbarp >> "invMbarp.txt";
```

C.13

```
(* File name "Functions symbolic FTM.nb" *)
fQp[p_] := Module[{i, j},
  MQp := Array[Qp, {Length[p], Length[p]}];
Do[Qp[i, 1] = p[[i]], {i, 1, Length[p]}];
Do[Qp[1, j] = p[[j]] / 2, {j, 2, Length[p]}];
  Do[If[(i == j), If[(2 j - 1) ≤ Length[p],
    Qp[i, j] = p[[2 i - 1]] / 2 + p[[1]], Qp[i, j] = p[[1]], If[(i ≠ 1 && j ≠ 1),
    If[(i + j - 1) ≤ Length[p], Qp[i, j] = p[[i + j - 1]] / 2 + p[[Abs[i - j] + 1]] / 2,
    Qp[i, j] = p[[Abs[i - j] + 1]] / 2]], {i, 1, Length[p]}, {j, 1, Length[p]}];
Qp[1, 1] = p[[1]];
  MQp
]
]
fGT[p_] := Module[{i, j},
  MGp := Array[G, {Length[p], Length[p]}];
Do[If[j == 1, If[i == 2, G[i, j] = -1 / 8, G[i, j] = Cos[i Pi] / (2 i (i - 2))]
, If[Abs[i - j] == 1,
  If[(i - j) == 1, G[i, j] = -1 / (4 (i - 2)), If[j ≠ 2, G[i, j] = 1 / (4 i), G[i, j] = 1 / 2]
, G[i, j] = 0]
], {i, 1, Length[p]}, {j, 1, Length[p]}];
  Transpose[MGp];
]
]
fQD[m_, n_] := Module[{i, j, k, l, A, B},
  MQD := Array[QD, {nm, nm}];
  Do[QD[(i - 1) * m + k, (j - 1) * m + l] = fQp[tensorP[[i, j]]][[k, l]],
    {i, 1, n}, {j, 1, n}, {k, 1, m}, {l, 1, m}
  ];
]
]
fGG[GG_, m_, n_] := Module[{i, j, k, l},
  GGT := Array[GGTD, {nm, nm}];
  Do[GGTD[i, j] = 0, {i, 1, mn}, {j, 1, mn}];

  Do[GGTD[(i - 1) * m + k, (i - 1) * m + l] = GG[[k, l]],
    {i, 1, n}, {k, 1, m}, {l, 1, m}
  ];
]
]
fXMN[m_, n_] := Module[{i},
  XXMN := ConstantArray[0, {nm, n}];
  Do[XXMN = ReplacePart[XXMN, 1, {1 + m (i - 1), i}], {i, 1, n}];
]
]
fPhi[m_, i_, j_, x_] :=
  Module[{k, aaaa}, aaaa := Transpose[Take[B2, {1 + m (i - 1), mi}], {j}]];
  Expand[aaaa.Table[ChebyshevT[k - 1, 2 x - 1], {k, m}]]
];
]
MyQC[QT_, T_] := Module[{J, J1, J2, K1, K2}, J := JordanDecomposition[QT];
  J1 := ReplacePart[J, Log[J[[2, 1, 1]]], {2, 1, 1}];
];
```

```

J2 := ReplacePart[J1, Log[J1[[2, 2, 2]]], {2, 2, 2}];
K1 := J2[[2]] / T;
K5 = Chop[J[[1]]];
K2 := Chop[J[[1]].K1.Inverse[J[[1]]], 10-8];
K2
];
MyRC[QT_, T_] := Module[{J, J1, J2, K1, K2}, J := JordanDecomposition[QT];
J1 := ReplacePart[J, Log[J[[2, 1, 1]]], {2, 1, 1}];
J2 := ReplacePart[J1, Log[J1[[2, 2, 2]]], {2, 2, 2}];
K1 := J2[[2]] / T;
K2 := Chop[J[[1]].K1.Inverse[J[[1]]], 10-8];
Re[K2]
];

```

C.14

```
(* File name "Find_k1 k2.nb" *)

(* To find real roots *)
(*
Ain=0.7; Afn=1; Bin=0.7; Bfn=1;
in=0.1;
Do[
Print[
FindRoot[{Expand[PHI[[1,1]]+PHI[[2,2]]]=(Ain+(i-1) in)+(Bin+(j-1) in), Expand[
(PHI[[1,1]]+PHI[[2,2]])^2-4*(PHI[[1,1]]*PHI[[2,2]]-PHI[[1,2]]*PHI[[2,1]])]=
((Ain+(i-1) in)-(Bin+(j-1) in)^2},{k1,1},{k5,1}}]];
Print[{Ain+(i-1) in,Bin+(j-1) in}];,{i,1,(Afn-Ain)/in},
{j,1,(Bfn-Bin)/in}}
*)
(* To find complex roots *)
Do[A = 0.7 Cos[Pi t]; B = 0.7 Sin[Pi t] I;
Print[FindRoot[
{Expand[PHI[[1, 1]] + PHI[[2, 2]]] = 2 * A, Expand[(PHI[[1, 1]] + PHI[[2, 2]])^2 -
4 * (PHI[[1, 1]] * PHI[[2, 2]] - PHI[[1, 2]] * PHI[[2, 1]])] =
(2 B)^2}, {{k1, 1}, {k5, 1}}]];,{t, 0, 2, 0.1}]
```

C.15

```
(* File name "Finding P.nb" *)
Clear["Global`*"]

SetDirectory[NotebookDirectory[]];
R = ReadList["R.txt"][[1]]; Q = ReadList["Q.txt"][[1]];
invQ = ReadList["invQ.txt"][[1]];
bx = ReadList["bx.txt"][[1]]; M = ReadList["M.txt"][[1]];
Kr = ReadList["Kr.txt"][[1]];
invM = Inverse[M];
J = Chop[invM.R.M, E^-6];
bz = Chop[Expand[TrigToExp[invM.invQ.bx]]];
λr = {{J[[1, 1]], J[[1, 2]], 0, 0}, {J[[2, 1]], J[[2, 2]], 0, 0},
      {0, 0, J[[1, 1]], J[[1, 2]]}, {0, 0, J[[2, 1]], J[[2, 2]]}};
λs[1] = {{J[[3, 3]], 0, J[[3, 4]], 0}, {0, J[[3, 3]], 0, J[[3, 4]]},
         {J[[4, 3]], 0, J[[4, 4]], 0}, {0, J[[4, 3]], 0, J[[4, 4]]}};
λs[2] = {{J[[5, 5]], 0, J[[5, 6]], 0}, {0, J[[5, 5]], 0, J[[5, 6]]},
         {J[[6, 5]], 0, J[[6, 6]], 0}, {0, J[[6, 5]], 0, J[[6, 6]]}};
λs[3] = {{J[[7, 7]], 0, J[[7, 8]], 0}, {0, J[[7, 7]], 0, J[[7, 8]]},
         {J[[8, 7]], 0, J[[8, 8]], 0}, {0, J[[8, 7]], 0, J[[8, 8]]}};

t0 = TimeUsed[];
chebcoefbr1k1 =
{-0.7052435032610047`k1 + 0.5148944926677143`k5, 1.7455350120173039`k5,
 1.1095244499895334`k1 - 0.03963864642511153`k5, -2.6925621752120987`k5,
 -0.5020321138299776`k1 + 0.0030056512903545137`k5, 1.3236963240498003`k5,
 0.14995782331589513`k1 + 0.007175589711316778`k5, -0.4757996206168023`k5,
 -0.037178280076297705`k1 - 0.003568093680008187`k5,
 -1.469218361655965`*^-10k1 + 0.11905362154909804`k5,
 0.007107689145396686`k1 + 0.0010715599016968868`k5,
 -2.5574532033239003`*^-9k1 - 0.023100302254994813`k5,
 -0.0011112194546803234`k1 - 0.0002250428566565372`k5,
 -2.7606830141381835`*^-8k1 + 0.003569522153181159`k5,
 0.00014352001694555407`k1 + 0.000033499054097248734`k5,
 -0.000025826549539844802`k1 - 0.00045411682940893776`k5,
 -0.00038739824309767204`k1 - 0.000051653099079689604`k5,
 0.0008953203840479532`k1 - 0.0007231433871156545`k5,
 0.0019283823656417454`k1 + 0.0002754831950916779`k5,
 -0.013223193364400539`k1 - 0.019834790046600808`k5} /.
{k1 → 0.16906283540593794, k5 → -0.6400038382967878};
chebcoefbr1k2 = {0.7368792821268294`k1 - 0.41161728129003694`k5,
 0.1250133133409062`k1 - 1.7641716123768243`k5,
 -1.0896880148068808`k1 - 0.23774415740427365`k5,
 -0.17749231059703363`k1 + 2.695616329828998`k5, 0.4887306785642079`k1 +
 0.30595127846358255`k5, 0.07055705596188834`k1 - 1.2967392286181554`k5,
 -0.14388010357621636`k1 - 0.19094694263742262`k5, -0.022472087193654854`k1 +
 0.46054053692911484`k5, 0.035352179488511284`k1 + 0.06456808714754031`k5,
 0.005217636570988067`k1 - 0.11422663121812696`k5,
```

```

-0.006708764792675462`k1 - 0.015446900210119019`k5,
-0.0009489353588540535`k1 + 0.021990798365630983`k5,
0.0010420011725501257`k1 + 0.00285592539232922`k5,
0.00014056283675946242`k1 - 0.003380899357059642`k5,
-0.00013424139839118346`k1 - 0.00040382040866705033`k5,
-0.000025826549539844802`k1 + 0.0004412035546390154`k5,
0.00001938584963060889`k1 + 0.000032675814716817944`k5,
-0.000016559651347088206`k1 + 0.000027093083707717247`k5,
0.000028890751551769212`k1 - 0.0007373948118936545`k5,
-0.003704860617357243`k1 + 0.0007107950196259936`k5} /.
{k1 → 0.16906283540593794, k5 → -0.6400038382967878};
chebcoefbr2k1 = {-0.7427319935888463`k1 - 0.382120771002262`k5,
0.12600106821082355`k1 + 1.7696022060413004`k5,
1.0983471633997528`k1 - 0.30500250266852036`k5,
-0.1788961463286649`k1 - 2.7052569499196304`k5, -0.49261475176336655`k1 +
0.33728271488013856`k5, 0.07111680633608304`k1 + 1.302806477731994`k5,
0.14502374665172613`k1 - 0.20086518974296405`k5, -0.022650727365802103`k1 -
0.46288961398087947`k5, -0.03563320346661182`k1 + 0.06719712255489274`k5,
0.005259170974071954`k1 + 0.11481630593697675`k5,
0.0067620968635061065`k1 - 0.015992696063693333`k5,
-0.0009565028206155202`k1 - 0.02210408540176575`k5,
-0.001050308038789552`k1 + 0.002947745829811936`k5,
0.0001414556629535197`k1 + 0.003398511477117704`k5,
0.00013436802851239237`k1 - 0.00041617478013257535`k5,
-0.00004734867415638214`k1 - 0.0003981593054059407`k5,
-0.00006887079877291948`k1 - 0.00006026194892630454`k5,
0.00007112675596232592`k1 + 0.000011829508187508196`k5,
-0.00017258204201264337`k1 + 0.00008150692409548549`k5,
0.005248383762854729`k1 - 0.0021912764677216464`k5} /.
{k1 → 0.16906283540593794, k5 → -0.6400038382967878};
chebcoefbr2k2 = {0.7044705873537753`k1 + 0.48514718386958744`k5,
-1.7462756460054714`k5, -1.1100179097985168`k1 + 0.027050651735320803`k5,
2.693797115254494`k5, 0.502275832558709`k1 - 0.026622570411089508`k5,
-1.3244082781303794`k5, -0.15004078417164352`k1 + 0.01586951932108262`k5,
0.47607912093153815`k5, 0.03720054123225657`k1 - 0.005776520822922395`k5,
-0.11912878893116002`k5, -0.00711223808254582`k1 + 0.0015170140134115268`k5,
-8.986115851389389`*^-10k1 + 0.023115827859168422`k5,
0.0011119626597294457`k1 - 0.00029840834126258997`k5,
1.3593121431694484`*^-7k1 - 0.0035720262223715`k5,
-0.0001441105807736926`k1 + 0.00004347456909598698`k5,
0.00005810973646465081`k1 + 0.0004476601920239766`k5,
0.00006887079877291948`k1 + 0.00006026194892630454`k5,
0.0003443539938645974`k1 - 0.0005509663901833558`k5,
0.0016528991705500673`k1 - 0.007438046267475303`k5,
-0.013223193364400539`k1 - 0.01432512614476725`k5} /.
{k1 → 0.16906283540593794, k5 → -0.6400038382967878};
i = 3;
n = 4;
(* # states *)
m = Dimensions[chebcoefbr1k1][[1]];

```

```

fGT[m];
fGG[Transpose[MGp], m, n];
tensorP := Array[tP, {n, n, m}];

tP[1, 1, 1] = (λs[i][[1, 1]] - (λr[[1, 1]] + chebcoefbr1k1[[1]]));
Do[tP[1, 1, k] = (-chebcoefbr1k1[[k]]), {k, 2, m}];
tP[1, 2, 1] = (λs[i][[1, 2]] - (λr[[1, 2]] + chebcoefbr2k1[[1]]));
Do[tP[1, 2, k] = (-chebcoefbr2k1[[k]]), {k, 2, m}];
tP[1, 3, 1] = (λs[i][[1, 3]] - 0); Do[tP[1, 3, k] = 0, {k, 2, m}];
tP[1, 4, 1] = (λs[i][[1, 4]] - 0); Do[tP[1, 4, k] = 0, {k, 2, m}];

tP[2, 1, 1] = (λs[i][[2, 1]] - (λr[[2, 1]] + chebcoefbr1k2[[1]]));
Do[tP[2, 1, k] = (-chebcoefbr1k2[[k]]), {k, 2, m}];
tP[2, 2, 1] = (λs[i][[2, 2]] - (λr[[2, 2]] + chebcoefbr2k2[[1]]));
Do[tP[2, 2, k] = (-chebcoefbr2k2[[k]]), {k, 2, m}];
tP[2, 3, 1] = (λs[i][[2, 3]] - 0); Do[tP[2, 3, k] = 0, {k, 2, m}];
tP[2, 4, 1] = (λs[i][[2, 4]] - 0); Do[tP[2, 4, k] = 0, {k, 2, m}];

tP[3, 1, 1] = (λs[i][[3, 1]] - 0); Do[tP[3, 1, k] = 0, {k, 2, m}];
tP[3, 2, 1] = (λs[i][[3, 2]] - 0); Do[tP[3, 2, k] = 0, {k, 2, m}];
tP[3, 3, 1] = (λs[i][[3, 3]] - (λr[[1, 1]] + chebcoefbr1k1[[1]]));
Do[tP[3, 3, k] = (-chebcoefbr1k1[[k]]), {k, 2, m}];
tP[3, 4, 1] = (λs[i][[3, 4]] - (λr[[1, 2]] + chebcoefbr2k1[[1]]));
Do[tP[3, 4, k] = (-chebcoefbr2k1[[k]]), {k, 2, m}];

tP[4, 1, 1] = (λs[i][[4, 1]] - 0); Do[tP[4, 1, k] = 0, {k, 2, m}];
tP[4, 2, 1] = (λs[i][[4, 2]] - 0); Do[tP[4, 2, k] = 0, {k, 2, m}];
tP[4, 3, 1] = (λs[i][[4, 3]] - (λr[[2, 1]] + chebcoefbr1k2[[1]]));
Do[tP[4, 3, k] = (-chebcoefbr1k2[[k]]), {k, 2, m}];
tP[4, 4, 1] = (λs[i][[4, 4]] - (λr[[2, 2]] + chebcoefbr2k2[[1]]));
Do[tP[4, 4, k] = (-chebcoefbr2k2[[k]]), {k, 2, m}];

(* ***** For calculating invQ ***** *)
(*tensorP=- Transpose[tensorP];*)
(* ***** *)

fQD[m, n];
fXMN[m, n];
B2 := Inverse[(IdentityMatrix[m*n] - GGT.MQD)].XXMN;

f11[x_] := fPhi[m, 1, 1, x]; f12[x_] := fPhi[m, 1, 2, x];
f13[x_] := fPhi[m, 1, 3, x]; f14[x_] := fPhi[m, 1, 4, x];
f21[x_] := fPhi[m, 2, 1, x]; f22[x_] := fPhi[m, 2, 2, x];
f23[x_] := fPhi[m, 2, 3, x]; f24[x_] := fPhi[m, 2, 4, x];
f31[x_] := fPhi[m, 3, 1, x]; f32[x_] := fPhi[m, 3, 2, x];
f33[x_] := fPhi[m, 3, 3, x]; f34[x_] := fPhi[m, 3, 4, x];
f41[x_] := fPhi[m, 4, 1, x]; f42[x_] := fPhi[m, 4, 2, x];
f43[x_] := fPhi[m, 4, 3, x]; f44[x_] := fPhi[m, 4, 4, x];

```

```

Phi[t_] := {{f11[t][[1]], f12[t][[1]], f13[t][[1]], f14[t][[1]]},
            {f21[t][[1]], f22[t][[1]], f23[t][[1]], f24[t][[1]]},
            {f31[t][[1]], f32[t][[1]], f33[t][[1]], f34[t][[1]]},
            {f41[t][[1]], f42[t][[1]], f43[t][[1]], f44[t][[1]]}};

t1 = TimeUsed[];
Print["Total time = ", (t1 - t0) / 60];

(* ***** For finding Qft ***** *)

t0 = TimeUsed[];
R = MyQC[Phi[1].Phi[1], 2, n];

q[t_] := Phi[t].MatrixExp[-R t];

Qdefined = 1/2 (1 + Sign[t]) q[t] +
           1/2 (1 - Sign[t]) Phi[t + 1].q[1].MatrixExp[-R t].MatrixExp[-R];

Needs["FourierSeries`"]
Qft = Chop[NFourierTrigSeries[Qdefined, t, 4, FourierParameters -> {-1, 1/2}]]

(* {"a", " " "b"} " " |b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) dt " " 2|b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) cos(2πbnt) dt " " *)

t1 = TimeUsed[];
Print["Total time = ", (t1 - t0) / 60];

(* ***** For finding invQft ***** *)
(*
t0=TimeUsed[];
invq[t_] := Chop[Expand[MatrixExp[R t].Transpose[Phi[t]]]];
invQdefined=1/2 (1+Sign[t])invq[t]+
            1/2 (1-Sign[t])MatrixExp[R t].MatrixExp[R].invq[1].Transpose[Phi[t+1]];
Needs["FourierSeries`"]
invQft=Chop[NFourierTrigSeries[invQdefined,t,4,FourierParameters->{-1,1/2}]]

(* {"a", " " "b"} " " |b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) dt " " 2|b|^(1-a)/2 ∫_{-1/(2|b|)}^{1/(2|b|)} f(t) cos(2πbnt) dt " " *)

t1=TimeUsed[];
Print["Total time = ",(t1-t0)/60];
*)

SetDirectory[NotebookDirectory[]];
R >> P3.txt
Qft >>> P3.txt
invQft >>> P3.txt

SetDirectory[NotebookDirectory[]];
R1p = ReadList["P1.txt"][[1]]; Qft1p = ReadList["P1.txt"][[2]];
invQft1p = ReadList["P1.txt"][[3]];
M1p = Transpose[Eigenvalues[R1p]];
invM1p = Inverse[M1p]; J1p = Chop[invM1p.R1p.M1p, E^-6];

```



```

invMinvQbsKrlp = Chop[Expand[TrigToExp[invM1p.invQft1p.
  {bz[[3]] * Kr[[1]], bz[[3]] * Kr[[2]], bz[[4]] * Kr[[1]], bz[[4]] * Kr[[2]]}]]];
freq11p = If[Exponent[invMinvQbsKrlp[[1]], E, List] == {}, {0},
  Exponent[invMinvQbsKrlp[[1]], E, List]];
freq21p = If[Exponent[invMinvQbsKrlp[[2]], E, List] == {},
  {0}, Exponent[invMinvQbsKrlp[[2]], E, List]];
freq31p = If[Exponent[invMinvQbsKrlp[[3]], E, List] == {},
  {0}, Exponent[invMinvQbsKrlp[[3]], E, List]];
freq41p = If[Exponent[invMinvQbsKrlp[[4]], E, List] == {},
  {0}, Exponent[invMinvQbsKrlp[[4]], E, List]];
q11p = Table[0, {2}, {Dimensions[invMinvQbsKrlp[[1]]][[1]]}];
Do[{q11p[[1, i]] = If[Expand[freq11p[[i]] / t] == 0, invMinvQbsKrlp[[1]][[i]],
  invMinvQbsKrlp[[1]][[i]][[1]], q11p[[2, i]] = freq11p[[i]],
  {i, 1, Dimensions[invMinvQbsKrlp[[1]]][[1]]}];
q21p = Table[0, {2}, {Dimensions[invMinvQbsKrlp[[2]]][[1]]}];
Do[{q21p[[1, i]] = If[Expand[freq21p[[i]] / t] == 0, invMinvQbsKrlp[[2]][[i]],
  invMinvQbsKrlp[[2]][[i]][[1]], q21p[[2, i]] = freq21p[[i]],
  {i, 1, Dimensions[invMinvQbsKrlp[[2]]][[1]]}];
q31p = Table[0, {2}, {Dimensions[invMinvQbsKrlp[[3]]][[1]]}];
Do[{q31p[[1, i]] = If[Expand[freq31p[[i]] / t] == 0, invMinvQbsKrlp[[3]][[i]],
  invMinvQbsKrlp[[3]][[i]][[1]], q31p[[2, i]] = freq31p[[i]],
  {i, 1, Dimensions[invMinvQbsKrlp[[3]]][[1]]}];
q41p = Table[0, {2}, {Dimensions[invMinvQbsKrlp[[4]]][[1]]}];
Do[{q41p[[1, i]] = If[Expand[freq41p[[i]] / t] == 0, invMinvQbsKrlp[[4]][[i]],
  invMinvQbsKrlp[[4]][[i]][[1]], q41p[[2, i]] = freq41p[[i]],
  {i, 1, Dimensions[invMinvQbsKrlp[[4]]][[1]]}];
p11z = Sum[q11p[[1, i]] * (E^q11p[[2, i]]) / (Expand[q11p[[2, i]] / t] - J1p[[1, 1]]),
  {i, 1, Dimensions[q11p][[2]]}];
p12z = Sum[q21p[[1, i]] * (E^q21p[[2, i]]) / (Expand[q21p[[2, i]] / t] - J1p[[2, 2]]),
  {i, 1, Dimensions[q21p][[2]]}];
p13z = Sum[q31p[[1, i]] * (E^q31p[[2, i]]) / (Expand[q31p[[2, i]] / t] - J1p[[3, 3]]),
  {i, 1, Dimensions[q31p][[2]]}];
p14z = Sum[q41p[[1, i]] * (E^q41p[[2, i]]) / (Expand[q41p[[2, i]] / t] - J1p[[4, 4]]),
  {i, 1, Dimensions[q41p][[2]]}];
P1 = Chop[ExpToTrig[Expand[TrigToExp[Qft1p.M1p.{p11z, p12z, p13z, p14z}]]]];

R2p = ReadList["P2.txt"][[1]]; Qft2p = ReadList["P2.txt"][[2]];
invQft2p = ReadList["P2.txt"][[3]];
M2p = Transpose[EigenVectors[R2p]];
invM2p = Inverse[M2p]; J2p = Chop[invM2p.R2p.M2p, E^-6];
invMinvQbsKr2p = Chop[Expand[TrigToExp[invM2p.invQft2p.
  {bz[[5]] * Kr[[1]], bz[[5]] * Kr[[2]], bz[[6]] * Kr[[1]], bz[[6]] * Kr[[2]]}]]];
freq12p = If[Exponent[invMinvQbsKr2p[[1]], E, List] == {}, {0},
  Exponent[invMinvQbsKr2p[[1]], E, List]];
freq22p = If[Exponent[invMinvQbsKr2p[[2]], E, List] == {},
  {0}, Exponent[invMinvQbsKr2p[[2]], E, List]];
freq32p = If[Exponent[invMinvQbsKr2p[[3]], E, List] == {},
  {0}, Exponent[invMinvQbsKr2p[[3]], E, List]];
freq42p = If[Exponent[invMinvQbsKr2p[[4]], E, List] == {},
  {0}, Exponent[invMinvQbsKr2p[[4]], E, List]];

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q12p = Table[0, {2}, {Dimensions[invMinvQbsKr2p[[1]]][[1]]};
Do[{q12p[[1, i]] = If[Expand[freq12p[[i]] / t] == 0, invMinvQbsKr2p[[1]][[i]],
  invMinvQbsKr2p[[1]][[i]][[1]], q12p[[2, i]] = freq12p[[i]],
  {i, 1, Dimensions[invMinvQbsKr2p[[1]]][[1]]};
q22p = Table[0, {2}, {Dimensions[invMinvQbsKr2p[[2]]][[1]]};
Do[{q22p[[1, i]] = If[Expand[freq22p[[i]] / t] == 0, invMinvQbsKr2p[[2]][[i]],
  invMinvQbsKr2p[[2]][[i]][[1]], q22p[[2, i]] = freq22p[[i]],
  {i, 1, Dimensions[invMinvQbsKr2p[[2]]][[1]]};
q32p = Table[0, {2}, {Dimensions[invMinvQbsKr2p[[3]]][[1]]};
Do[{q32p[[1, i]] = If[Expand[freq32p[[i]] / t] == 0, invMinvQbsKr2p[[3]][[i]],
  invMinvQbsKr2p[[3]][[i]][[1]], q32p[[2, i]] = freq32p[[i]],
  {i, 1, Dimensions[invMinvQbsKr2p[[3]]][[1]]};
q42p = Table[0, {2}, {Dimensions[invMinvQbsKr2p[[4]]][[1]]};
Do[{q42p[[1, i]] = If[Expand[freq42p[[i]] / t] == 0, invMinvQbsKr2p[[4]][[i]],
  invMinvQbsKr2p[[4]][[i]][[1]], q42p[[2, i]] = freq42p[[i]],
  {i, 1, Dimensions[invMinvQbsKr2p[[4]]][[1]]};
p11z = Sum[q12p[[1, i]] * (E^q12p[[2, i]]) / (Expand[q12p[[2, i]] / t] - J2p[[1, 1]]),
  {i, 1, Dimensions[q12p][[2]]};
p12z = Sum[q22p[[1, i]] * (E^q22p[[2, i]]) / (Expand[q22p[[2, i]] / t] - J2p[[2, 2]]),
  {i, 1, Dimensions[q22p][[2]]};
p13z = Sum[q32p[[1, i]] * (E^q32p[[2, i]]) / (Expand[q32p[[2, i]] / t] - J2p[[3, 3]]),
  {i, 1, Dimensions[q32p][[2]]};
p14z = Sum[q42p[[1, i]] * (E^q42p[[2, i]]) / (Expand[q42p[[2, i]] / t] - J2p[[4, 4]]),
  {i, 1, Dimensions[q42p][[2]]};
P2 = Chop[ExpToTrig[Expand[TrigToExp[Qft2p.M2p.{p11z, p12z, p13z, p14z}]]];

R3p = ReadList["P3.txt"][[1]]; Qft3p = ReadList["P3.txt"][[2]];
invQft3p = ReadList["P3.txt"][[3]];
M3p = Transpose[EigenVectors[R3p]];
invM3p = Inverse[M3p]; J3p = Chop[invM3p.R3p.M3p, E^-6];
invMinvQbsKr3p = Chop[Expand[TrigToExp[invM3p.invQft3p.
  {bz[[7]] * Kr[[1]], bz[[7]] * Kr[[2]], bz[[8]] * Kr[[1]], bz[[8]] * Kr[[2]]}]]];
freq13p = If[Exponent[invMinvQbsKr3p[[1]], E, List] == {}, {0},
  Exponent[invMinvQbsKr3p[[1]], E, List]];
freq23p = If[Exponent[invMinvQbsKr3p[[2]], E, List] == {},
  {0}, Exponent[invMinvQbsKr3p[[2]], E, List]];
freq33p = If[Exponent[invMinvQbsKr3p[[3]], E, List] == {},
  {0}, Exponent[invMinvQbsKr3p[[3]], E, List]];
freq43p = If[Exponent[invMinvQbsKr3p[[4]], E, List] == {},
  {0}, Exponent[invMinvQbsKr3p[[4]], E, List]];
q13p = Table[0, {2}, {Dimensions[invMinvQbsKr3p[[1]]][[1]]};
Do[{q13p[[1, i]] = If[Expand[freq13p[[i]] / t] == 0, invMinvQbsKr3p[[1]][[i]],
  invMinvQbsKr3p[[1]][[i]][[1]], q13p[[2, i]] = freq13p[[i]],
  {i, 1, Dimensions[invMinvQbsKr3p[[1]]][[1]]};
q23p = Table[0, {2}, {Dimensions[invMinvQbsKr3p[[2]]][[1]]};
Do[{q23p[[1, i]] = If[Expand[freq23p[[i]] / t] == 0, invMinvQbsKr3p[[2]][[i]],
  invMinvQbsKr3p[[2]][[i]][[1]], q23p[[2, i]] = freq23p[[i]],
  {i, 1, Dimensions[invMinvQbsKr3p[[2]]][[1]]};
q33p = Table[0, {2}, {Dimensions[invMinvQbsKr3p[[3]]][[1]]};
Do[{q33p[[1, i]] = If[Expand[freq33p[[i]] / t] == 0, invMinvQbsKr3p[[3]][[i]],

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    invMinvQbsKr3p[[3]][[i]][[1]], q33p[[2, i]] = freq33p[[i]],
    {i, 1, Dimensions[invMinvQbsKr3p[[3]][[1]]]};
q43p = Table[0, {2}, {Dimensions[invMinvQbsKr3p[[4]][[1]]]};
Do[{q43p[[1, i]] = If[Expand[freq43p[[i]] / t] == 0, invMinvQbsKr3p[[4]][[i]],
    invMinvQbsKr3p[[4]][[i]][[1]], q43p[[2, i]] = freq43p[[i]],
    {i, 1, Dimensions[invMinvQbsKr3p[[4]][[1]]]};
p11z = Sum[q13p[[1, i]] * (E^q13p[[2, i]]) / (Expand[q13p[[2, i]] / t] - J3p[[1, 1]]),
    {i, 1, Dimensions[q13p][[2]]};
p12z = Sum[q23p[[1, i]] * (E^q23p[[2, i]]) / (Expand[q23p[[2, i]] / t] - J3p[[2, 2]]),
    {i, 1, Dimensions[q23p][[2]]};
p13z = Sum[q33p[[1, i]] * (E^q33p[[2, i]]) / (Expand[q33p[[2, i]] / t] - J3p[[3, 3]]),
    {i, 1, Dimensions[q33p][[2]]};
p14z = Sum[q43p[[1, i]] * (E^q43p[[2, i]]) / (Expand[q43p[[2, i]] / t] - J3p[[4, 4]]),
    {i, 1, Dimensions[q43p][[2]]};
P3 = Chop[ExpToTrig[Expand[TrigToExp[Qft3p.M3p.{p11z, p12z, p13z, p14z}]]];
Ptmp = {{P1[[1]], P1[[2]]}, {P1[[3]], P1[[4]]}, {P2[[1]], P2[[2]]},
    {P2[[3]], P2[[4]]}, {P3[[1]], P3[[2]]}, {P3[[3]], P3[[4]]}};

Needs["FourierSeries`"]
P = Chop[Quiet[NFourierTrigSeries[Ptmp, t, 8, FourierParameters -> {-1, 1/2}]]];
P >> P.txt;

```

C.16

```
(* File name "Step 2 Inverse transformation.nb" *)
Clear["Global`*"]

(* ***** Polynomial inverse of 3 rd order terms ***** *)
SetDirectory[NotebookDirectory[]];
v2φ3 = ReadList["phi3.txt"][[1]]; ft = ReadList["ft.txt"][[1]];

f3 = TrigToExp[{Sum[(fc130[i] Cos[i π t] + fs130[i] Sin[i π t]) z1^3 +
  (fc121[i] Cos[i π t] + fs121[i] Sin[i π t]) z1^2 z2 +
  (fc112[i] Cos[i π t] + fs112[i] Sin[i π t]) z1 z2^2 +
  (fc103[i] Cos[i π t] + fs103[i] Sin[i π t]) z2^3, {i, 0, ft}],
  Sum[(fc230[i] Cos[i π t] + fs230[i] Sin[i π t]) z1^3 +
  (fc221[i] Cos[i π t] + fs221[i] Sin[i π t]) z1^2 z2 +
  (fc212[i] Cos[i π t] + fs212[i] Sin[i π t]) z1 z2^2 +
  (fc203[i] Cos[i π t] + fs203[i] Sin[i π t]) z2^3, {i, 0, ft}]}];
v1z = {z1 + f3[[1]], z2 + f3[[2]]} + v2φ3 /. {y1 → z1, y2 → z2};
coefLvlz = CoefficientList[v1z, {z1, z2}];

coefPILHS3 = Chop[ExpToTrig[Expand[{coefLvlz[[1]][[4, 1]], coefLvlz[[1]][[3, 2]],
  coefLvlz[[1]][[2, 3]], coefLvlz[[1]][[1, 4]], coefLvlz[[2]][[4, 1]],
  coefLvlz[[2]][[3, 2]], coefLvlz[[2]][[2, 3]], coefLvlz[[2]][[1, 4]]}]]];

coefTPILHS3 = Table[0, {2 * 4}, {1 + 2 * ft}];
Do[
  Do[If[Exponent[TrigToExp[coefPILHS3[[i, k]]], E] == 0,
    {coefTPILHS3[[i, 1]] = coefTPILHS3[[i, 1]] + coefPILHS3[[i, k]]},
    {k, 1, Dimensions[coefPILHS3[[i]]][[1]}], {i, 1, 2 * 4}];
Do[
  coefTPILHS3[[i, 2 * j]] = Coefficient[coefPILHS3[[i]], Cos[j π t]];
  coefTPILHS3[[i, 2 * j + 1]] = Coefficient[coefPILHS3[[i]], Sin[j π t]];
  {i, 1, 2 * 4}, {j, 1, ft}];

coefTPIRHS3 = Table[0, {2 * 4}, {1 + 2 * ft}];

API3 = Array[rowAPI3, {2 * 4 * (1 + 2 * ft)}];
CPILHS3 = Array[rowCPILHS3, {2 * 4 * (1 + 2 * ft)}];
Do[rowAPI3[k + (i - 1) 2 * 4] =
  Flatten[{{Coefficient[coefTPILHS3[[k, i]], fc130[0]], Coefficient[
    coefTPILHS3[[k, i]], fc121[0]], Coefficient[coefTPILHS3[[k, i]], fc112[0]],
    Coefficient[coefTPILHS3[[k, i]], fc103[0]], Coefficient[coefTPILHS3[[k, i]],
    fc230[0]], Coefficient[coefTPILHS3[[k, i]], fc221[0]], Coefficient[
    coefTPILHS3[[k, i]], fc212[0]], Coefficient[coefTPILHS3[[k, i]], fc203[0]]},
  Table[{Coefficient[coefTPILHS3[[k, i]], fc130[j]], Coefficient[
    coefTPILHS3[[k, i]], fc121[j]], Coefficient[coefTPILHS3[[k, i]], fc112[j]],
    Coefficient[coefTPILHS3[[k, i]], fc103[j]], Coefficient[coefTPILHS3[[k, i]],
    fc230[j]], Coefficient[coefTPILHS3[[k, i]], fc221[j]], Coefficient[
    coefTPILHS3[[k, i]], fc212[j]], Coefficient[coefTPILHS3[[k, i]], fc203[j]],
```

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Coefficient[coefTPILHS3[[k, i]], fs130[j]], Coefficient[coefTPILHS3[[k, i]],
  fs121[j]], Coefficient[coefTPILHS3[[k, i]], fs112[j]], Coefficient[
  coefTPILHS3[[k, i]], fs103[j]], Coefficient[coefTPILHS3[[k, i]], fs230[j]],
  Coefficient[coefTPILHS3[[k, i]], fs221[j]], Coefficient[coefTPILHS3[[k, i]],
  fs212[j]], Coefficient[coefTPILHS3[[k, i]], fs203[j]]], {j, 1, ft}}];
rowCPILHS3[k + (i - 1) 2 * 4] = Chop[Expand[coefTPILHS3[[k, i]] - rowAPI3[k + (i - 1) 2 * 4].
  Flatten[{{fc130[0], fc121[0], fc112[0], fc103[0], fc230[0], fc221[0], fc212[0],
    fc203[0]}, Table[{{fc130[j], fc121[j], fc112[j], fc103[j], fc230[j], fc221[j],
    fc212[j], fc203[j], fs130[j], fs121[j], fs112[j], fs103[j], fs230[j], fs221[
    j], fs212[j], fs203[j]}, {j, 1, ft}}]]]]; {k, 1, 2 * 4}, {i, 1, 1 + 2 * ft}]

CPIRHS3 = Flatten[coefTPIRHS3];
CPI3 = CPIRHS3 - CPILHS3;

coeff3 = Expand[Inverse[API3].CPI3];
v1f3 = f3 /.
  Flatten[{{fc130[0] -> coeff3[[1]], fc121[0] -> coeff3[[2]], fc112[0] -> coeff3[[3]],
    fc103[0] -> coeff3[[4]], fc230[0] -> coeff3[[5]], fc221[0] -> coeff3[[6]],
    fc212[0] -> coeff3[[7]], fc203[0] -> coeff3[[8]]},
  Table[{{fc130[j] -> coeff3[[8 (2 j - 1) + 1]], fc121[j] -> coeff3[[8 (2 j - 1) + 2]],
    fc112[j] -> coeff3[[8 (2 j - 1) + 3]], fc103[j] -> coeff3[[8 (2 j - 1) + 4]],
    fc230[j] -> coeff3[[8 (2 j - 1) + 5]], fc221[j] -> coeff3[[8 (2 j - 1) + 6]],
    fc212[j] -> coeff3[[8 (2 j - 1) + 7]], fc203[j] -> coeff3[[8 (2 j - 1) + 8]],
    fs130[j] -> coeff3[[8 (2 j - 1) + 9]], fs121[j] -> coeff3[[8 (2 j - 1) + 10]],
    fs112[j] -> coeff3[[8 (2 j - 1) + 11]],
    fs103[j] -> coeff3[[8 (2 j - 1) + 12]], fs230[j] -> coeff3[[8 (2 j - 1) + 13]],
    fs221[j] -> coeff3[[8 (2 j - 1) + 14]], fs212[j] -> coeff3[[8 (2 j - 1) + 15]],
    fs203[j] -> coeff3[[8 (2 j - 1) + 16]]}, {j, 1, ft}}]];
v1f3 >> "v1f3.txt";

(* ***** Polynomial
  inverse of 5th order terms ***** *)
SetDirectory[NotebookDirectory[]];
v2φ5 = ReadList["phi5.txt"][[1]];
f5 = TrigToExp[Sum[{{fc150[i] Cos[i π t] + fs150[i] Sin[i π t]} z1^5 +
  {{fc141[i] Cos[i π t] + fs141[i] Sin[i π t]} z1^4 z2 +
  {{fc132[i] Cos[i π t] + fs132[i] Sin[i π t]} z1^3 z2^2 +
  {{fc123[i] Cos[i π t] + fs123[i] Sin[i π t]} z1^2 z2^3 +
  {{fc114[i] Cos[i π t] + fs114[i] Sin[i π t]} z1 z2^4 +
  {{fc105[i] Cos[i π t] + fs105[i] Sin[i π t]} z2^5, {i, 0, ft}},
  Sum[{{fc250[i] Cos[i π t] + fs250[i] Sin[i π t]} z1^5 +
  {{fc241[i] Cos[i π t] + fs241[i] Sin[i π t]} z1^4 z2 +
  {{fc232[i] Cos[i π t] + fs232[i] Sin[i π t]} z1^3 z2^2 +
  {{fc223[i] Cos[i π t] + fs223[i] Sin[i π t]} z1^2 z2^3 +
  {{fc214[i] Cos[i π t] + fs214[i] Sin[i π t]} z1 z2^4 +
  {{fc205[i] Cos[i π t] + fs205[i] Sin[i π t]} z2^5, {i, 0, ft}}]]];
v2z = Expand[(v2φ3 /. {y1 -> z1 + v1f3[[1]], y2 -> z2 + v1f3[[2]]) +
  {z1 + f5[[1]], z2 + f5[[2]]} + (v2φ5 /. {y1 -> z1, y2 -> z2})];
coefLv2z = CoefficientList[v2z, {z1, z2}];

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coefPILHS5 = Chop[ExpToTrig[
  Expand[{coefLv2z[[1]][[6, 1]], coefLv2z[[1]][[5, 2]], coefLv2z[[1]][[4, 3]],
    coefLv2z[[1]][[3, 4]], coefLv2z[[1]][[2, 5]], coefLv2z[[1]][[1, 6]],
    coefLv2z[[2]][[6, 1]], coefLv2z[[2]][[5, 2]], coefLv2z[[2]][[4, 3]],
    coefLv2z[[2]][[3, 4]], coefLv2z[[2]][[2, 5]], coefLv2z[[2]][[1, 6]]}]]];

coefTPILHS5 = Table[0, {2*6}, {1+2*ft}];
Do[
  Do[If[Exponent[TrigToExp[coefPILHS5[[i, k]]], E] == 0,
    {coefTPILHS5[[i, 1]] = coefTPILHS5[[i, 1]] + coefPILHS5[[i, k]]},
    {k, 1, Dimensions[coefPILHS5[[i]]][[1]]}], {i, 1, 2*6}];
Do[
  coefTPILHS5[[i, 2*j]] = Coefficient[coefPILHS5[[i]], Cos[j* $\pi$ t]];
  coefTPILHS5[[i, 2*j+1]] = Coefficient[coefPILHS5[[i]], Sin[j* $\pi$ t]];
  {i, 1, 2*6}, {j, 1, ft}];

coefTPIRHS5 = Table[0, {2*6}, {1+2*ft}];

API5 = Array[rowAPI5, {2*6*(1+2*ft)}];
CPILHS5 = Array[rowCPILHS5, {2*6*(1+2*ft)}];

Do[rowAPI5[k + (i - 1) 2*6] =
  Flatten[{{Coefficient[coefTPILHS5[[k, i]], fc150[0]], Coefficient[
    coefTPILHS5[[k, i]], fc141[0]], Coefficient[coefTPILHS5[[k, i]], fc132[0]],
    Coefficient[coefTPILHS5[[k, i]], fc123[0]], Coefficient[coefTPILHS5[[k, i]],
    fc114[0]], Coefficient[coefTPILHS5[[k, i]], fc105[0]],
    Coefficient[coefTPILHS5[[k, i]], fc250[0]], Coefficient[coefTPILHS5[[k, i]],
    fc241[0]], Coefficient[coefTPILHS5[[k, i]], fc232[0]],
    Coefficient[coefTPILHS5[[k, i]], fc223[0]], Coefficient[coefTPILHS5[[k, i]],
    fc214[0]], Coefficient[coefTPILHS5[[k, i]], fc205[0]]},
  Table[{{Coefficient[coefTPILHS5[[k, i]], fc150[j]], Coefficient[
    coefTPILHS5[[k, i]], fc141[j]], Coefficient[coefTPILHS5[[k, i]], fc132[j]],
    Coefficient[coefTPILHS5[[k, i]], fc123[j]], Coefficient[coefTPILHS5[[k, i]],
    fc114[j]], Coefficient[coefTPILHS5[[k, i]], fc105[j]], Coefficient[
    coefTPILHS5[[k, i]], fc250[j]], Coefficient[coefTPILHS5[[k, i]], fc241[j]],
    Coefficient[coefTPILHS5[[k, i]], fc232[j]], Coefficient[coefTPILHS5[[k, i]],
    fc223[j]], Coefficient[coefTPILHS5[[k, i]], fc214[j]], Coefficient[
    coefTPILHS5[[k, i]], fc205[j]], Coefficient[coefTPILHS5[[k, i]], fs150[j]],
    Coefficient[coefTPILHS5[[k, i]], fs141[j]], Coefficient[coefTPILHS5[[k, i]],
    fs132[j]], Coefficient[coefTPILHS5[[k, i]], fs123[j]], Coefficient[
    coefTPILHS5[[k, i]], fs114[j]], Coefficient[coefTPILHS5[[k, i]], fs105[j]],
    Coefficient[coefTPILHS5[[k, i]], fs250[j]], Coefficient[coefTPILHS5[[k, i]],
    fs241[j]], Coefficient[coefTPILHS5[[k, i]], fs232[j]], Coefficient[
    coefTPILHS5[[k, i]], fs223[j]], Coefficient[coefTPILHS5[[k, i]], fs214[j]],
    Coefficient[coefTPILHS5[[k, i]], fs205[j]]}], {j, 1, ft}}];
  rowCPILHS5[k + (i - 1) 2*6] = Chop[Expand[coefTPILHS5[[k, i]] - rowAPI5[k + (i - 1) 2*6].
  Flatten[{{fc150[0], fc141[0], fc132[0], fc123[0], fc114[0], fc105[0],
    fc250[0], fc241[0], fc232[0], fc223[0], fc214[0], fc205[0]},

```

```

Table[{fc150[j], fc141[j], fc132[j], fc123[j], fc114[j], fc105[j], fc250[j],
fc241[j], fc232[j], fc223[j], fc214[j], fc205[j], fs150[j], fs141[j],
fs132[j], fs123[j], fs114[j], fs105[j], fs250[j], fs241[j], fs232[j], fs223[
j], fs214[j], fs205[j]}, {j, 1, ft}}]]];, {k, 1, 2*6}, {i, 1, 1+2*ft}}

CPIRHS5 = Flatten[coefTPIRHS5];
CPI5 = CPIRHS5 - CPILHS5;

coeff5 = Expand[Inverse[API5].CPI5];

v1f5 = f5 /.
Flatten[{{fc150[0] -> coeff5[[1]], fc141[0] -> coeff5[[2]], fc132[0] -> coeff5[[3]],
fc123[0] -> coeff5[[4]], fc114[0] -> coeff5[[5]], fc105[0] -> coeff5[[6]],
fc250[0] -> coeff5[[7]], fc241[0] -> coeff5[[8]], fc232[0] -> coeff5[[9]],
fc223[0] -> coeff5[[10]], fc214[0] -> coeff5[[11]], fc205[0] -> coeff5[[12]]},
Table[{fc150[j] -> coeff5[[12 (2 j - 1) + 1]], fc141[j] -> coeff5[[12 (2 j - 1) + 2]],
fc132[j] -> coeff5[[12 (2 j - 1) + 3]], fc123[j] -> coeff5[[12 (2 j - 1) + 4]],
fc114[j] -> coeff5[[12 (2 j - 1) + 5]], fc105[j] -> coeff5[[12 (2 j - 1) + 6]],
fc250[j] -> coeff5[[12 (2 j - 1) + 7]], fc241[j] -> coeff5[[12 (2 j - 1) + 8]],
fc232[j] -> coeff5[[12 (2 j - 1) + 9]], fc223[j] -> coeff5[[12 (2 j - 1) + 10]],
fc214[j] -> coeff5[[12 (2 j - 1) + 11]],
fc205[j] -> coeff5[[12 (2 j - 1) + 12]], fs150[j] -> coeff5[[12 (2 j - 1) + 13]],
fs141[j] -> coeff5[[12 (2 j - 1) + 14]], fs132[j] -> coeff5[[12 (2 j - 1) + 15]],
fs123[j] -> coeff5[[12 (2 j - 1) + 16]], fs114[j] -> coeff5[[12 (2 j - 1) + 17]],
fs105[j] -> coeff5[[12 (2 j - 1) + 18]], fs250[j] -> coeff5[[12 (2 j - 1) + 19]],
fs241[j] -> coeff5[[12 (2 j - 1) + 20]], fs232[j] -> coeff5[[12 (2 j - 1) + 21]],
fs223[j] -> coeff5[[12 (2 j - 1) + 22]], fs214[j] -> coeff5[[12 (2 j - 1) + 23]],
fs205[j] -> coeff5[[12 (2 j - 1) + 24]]}, {j, 1, ft}}]];
v1f5 >> "v1f5.txt";

```

C.17

```

(* File name "Step 3 fplot.nb" *)
SetDirectory[NotebookDirectory[]];
A = ReadList["A.txt"][[1]];
R = ReadList["R.txt"][[1]]; Q = ReadList["Q.txt"][[1]];
invQ = ReadList["invQ.txt"][[1]];
fx = ReadList["fx.txt"][[1]]; bx = ReadList["bx.txt"][[1]];
M = ReadList["M.txt"][[1]]; Kr = ReadList["Kr.txt"][[1]];

v1a3 = Chop[ReadList["alpha3.txt"][[1]], 10^-2];
v1a5 = Chop[ReadList["alpha5.txt"][[1]], 10^-2];
v1b2 = Chop[ReadList["beta2.txt"][[1]], 10^-2];
v1b4 = Chop[ReadList["beta4.txt"][[1]], 10^-2];
v1f3 = Chop[ReadList["v1f3.txt"][[1]], 10^-2];
v1f5 = Chop[ReadList["v1f5.txt"][[1]], 10^-2];

invM = Inverse[M];
x = {x1, x2, x3, x4, x5, x6, x7, x8};
z = Chop[Expand[TrigToExp[invM.invQ.x]]];

uxln = Chop[ExpToTrig[Expand[(Kr[[1]] z1 + Kr[[2]] z2) /. {z1 -> z[[1]], z2 -> z[[2]]}]]];
uxln >> "uxln.txt";
uxnln5 = Chop[ExpToTrig[
  Expand[(Kr[[1]] (z1 + v1f3[[1]] + v1f5[[1]]) + Kr[[2]] (z2 + v1f3[[2]] + v1f5[[2]]) +
    v1a3 + v1a5 + v1b2 (Kr[[1]] (z1 + v1f3[[1]]) + Kr[[2]] (z2 + v1f3[[2]])) +
    v1b4 (Kr[[1]] z1 + Kr[[2]] z2)) /. {z1 -> z[[1]], z2 -> z[[2]]}], 10^-1];
uxnln5 >> "uxnln5.txt";
uxnln3 =
  Chop[ExpToTrig[Expand[(Kr[[1]] (z1 + v1f3[[1]]) + Kr[[2]] (z2 + v1f3[[2]]) + v1a3 +
    v1b2 (Kr[[1]] z1 + Kr[[2]] z2)) /. {z1 -> z[[1]], z2 -> z[[2]]}], 10^-2];
uxnln3 >> "uxnln3.txt";

```



```

SetDirectory[NotebookDirectory[]];
A = ReadList["A.txt"][[1]];
R = ReadList["R.txt"][[1]]; Q = ReadList["Q.txt"][[1]];
invQ = ReadList["invQ.txt"][[1]];
fx = ReadList["fx.txt"][[1]]; bx = ReadList["bx.txt"][[1]];
M = ReadList["M.txt"][[1]]; Kr = ReadList["Kr.txt"][[1]];

v1fx = fx /. {x1 → 11[t], x2 → 12[t], x3 → 13[t],
  x4 → 14[t], x5 → 15[t], x6 → 16[t], x7 → 17[t], x8 → 18[t]};
ti = 0;
tf = 18;
l0 = {0.21, 0.2, 0.2, 0.2, 0.3, 0.2, 0.2, 0.2};
uxln = ReadList["uxln.txt"][[1]];
vluxln = uxln /. {x1 → 11[t], x2 → 12[t], x3 → 13[t],
  x4 → 14[t], x5 → 15[t], x6 → 16[t], x7 → 17[t], x8 → 18[t]};

(* ***** System response with linear controller ***** *)
sol2 =
NDSolve[{11'[t] = A[[1, 1]] 11[t] + A[[1, 2]] 12[t] + A[[1, 3]] 13[t] + A[[1, 4]] 14[t] +
  A[[1, 5]] 15[t] + A[[1, 6]] 16[t] + A[[1, 7]] 17[t] +
  A[[1, 8]] 18[t] + v1fx[[1]] + bx[[1]] vluxln, 12'[t] =
  A[[2, 1]] 11[t] + A[[2, 2]] 12[t] + A[[2, 3]] 13[t] + A[[2, 4]] 14[t] + A[[2, 5]] 15[t] +
  A[[2, 6]] 16[t] + A[[2, 7]] 17[t] + A[[2, 8]] 18[t] + v1fx[[2]] + bx[[2]] vluxln,
  13'[t] = A[[3, 1]] 11[t] + A[[3, 2]] 12[t] + A[[3, 3]] 13[t] +
  A[[3, 4]] 14[t] + A[[3, 5]] 15[t] + A[[3, 6]] 16[t] + A[[3, 7]] 17[t] +
  A[[3, 8]] 18[t] + v1fx[[3]] + bx[[3]] vluxln, 14'[t] =
  A[[4, 1]] 11[t] + A[[4, 2]] 12[t] + A[[4, 3]] 13[t] + A[[4, 4]] 14[t] + A[[4, 5]] 15[t] +
  A[[4, 6]] 16[t] + A[[4, 7]] 17[t] + A[[4, 8]] 18[t] + v1fx[[4]] + bx[[4]] vluxln,
  15'[t] = A[[5, 1]] 11[t] + A[[5, 2]] 12[t] + A[[5, 3]] 13[t] +
  A[[5, 4]] 14[t] + A[[5, 5]] 15[t] + A[[5, 6]] 16[t] + A[[5, 7]] 17[t] +
  A[[5, 8]] 18[t] + v1fx[[5]] + bx[[5]] vluxln, 16'[t] =
  A[[6, 1]] 11[t] + A[[6, 2]] 12[t] + A[[6, 3]] 13[t] + A[[6, 4]] 14[t] + A[[6, 5]] 15[t] +
  A[[6, 6]] 16[t] + A[[6, 7]] 17[t] + A[[6, 8]] 18[t] + v1fx[[6]] + bx[[6]] vluxln,
  17'[t] = A[[7, 1]] 11[t] + A[[7, 2]] 12[t] + A[[7, 3]] 13[t] +
  A[[7, 4]] 14[t] + A[[7, 5]] 15[t] + A[[7, 6]] 16[t] + A[[7, 7]] 17[t] +
  A[[7, 8]] 18[t] + v1fx[[7]] + bx[[7]] vluxln, 18'[t] =
  A[[8, 1]] 11[t] + A[[8, 2]] 12[t] + A[[8, 3]] 13[t] + A[[8, 4]] 14[t] + A[[8, 5]] 15[t] +
  A[[8, 6]] 16[t] + A[[8, 7]] 17[t] + A[[8, 8]] 18[t] + v1fx[[8]] + bx[[8]] vluxln,
  11[0] = 10[[1]], 12[0] = 10[[2]], 13[0] = 10[[3]], 14[0] = 10[[4]],
  15[0] = 10[[5]], 16[0] = 10[[6]], 17[0] = 10[[7]], 18[0] = 10[[8]]},
{11, 12, 13, 14, 15, 16, 17, 18}, {t, ti, tf}]

fig3 = Plot[Evaluate[{11[t], 12[t], 13[t], 14[t], 15[t], 16[t], 17[t], 18[t]} /. sol2],
{t, ti, tf}, PlotRange → All, PlotStyle → {Green}]
fig4 = Plot[Evaluate[vluxln /. sol2], {t, ti, tf}, PlotRange → All, PlotStyle → {Green}]

```

```

ans1 = Re[
  Table[Flatten[{t, {l1[t], l2[t], l3[t], l4[t], l5[t], l6[t], l7[t], l8[t], vluxln} /.
    sol2}], {t, ti, tf, 0.01}]];
Export["C:\\Users\\gabalap\\Work\\Research\\Order reduced control\\Time
  periodic\\Coupled inverted pendulums 8x8_Mod_51\\linear_feedback.dat", ans1];

t0 = TimeUsed[];
uxnln = Chop[ReadList["uxnln3.txt"][[1]], 10^-2];
vluxnln = uxnln /. {x1 -> l1[t], x2 -> l2[t], x3 -> l3[t],
  x4 -> l4[t], x5 -> l5[t], x6 -> l6[t], x7 -> l7[t], x8 -> l8[t]};
(* ***** System response with nonlinear controller ***** *)
sol3 =
NDSolve[{l1'[t] = A[[1, 1]] l1[t] + A[[1, 2]] l2[t] + A[[1, 3]] l3[t] + A[[1, 4]] l4[t] +
  A[[1, 5]] l5[t] + A[[1, 6]] l6[t] + A[[1, 7]] l7[t] +
  A[[1, 8]] l8[t] + v1fx[[1]] + bx[[1]] vluxnln, l2'[t] =
  A[[2, 1]] l1[t] + A[[2, 2]] l2[t] + A[[2, 3]] l3[t] + A[[2, 4]] l4[t] + A[[2, 5]] l5[t] +
  A[[2, 6]] l6[t] + A[[2, 7]] l7[t] + A[[2, 8]] l8[t] + v1fx[[2]] + bx[[2]] vluxnln,
  l3'[t] = A[[3, 1]] l1[t] + A[[3, 2]] l2[t] + A[[3, 3]] l3[t] +
  A[[3, 4]] l4[t] + A[[3, 5]] l5[t] + A[[3, 6]] l6[t] + A[[3, 7]] l7[t] +
  A[[3, 8]] l8[t] + v1fx[[3]] + bx[[3]] vluxnln, l4'[t] =
  A[[4, 1]] l1[t] + A[[4, 2]] l2[t] + A[[4, 3]] l3[t] + A[[4, 4]] l4[t] + A[[4, 5]] l5[t] +
  A[[4, 6]] l6[t] + A[[4, 7]] l7[t] + A[[4, 8]] l8[t] + v1fx[[4]] + bx[[4]] vluxnln,
  l5'[t] = A[[5, 1]] l1[t] + A[[5, 2]] l2[t] + A[[5, 3]] l3[t] +
  A[[5, 4]] l4[t] + A[[5, 5]] l5[t] + A[[5, 6]] l6[t] + A[[5, 7]] l7[t] +
  A[[5, 8]] l8[t] + v1fx[[5]] + bx[[5]] vluxnln, l6'[t] =
  A[[6, 1]] l1[t] + A[[6, 2]] l2[t] + A[[6, 3]] l3[t] + A[[6, 4]] l4[t] + A[[6, 5]] l5[t] +
  A[[6, 6]] l6[t] + A[[6, 7]] l7[t] + A[[6, 8]] l8[t] + v1fx[[6]] + bx[[6]] vluxnln,
  l7'[t] = A[[7, 1]] l1[t] + A[[7, 2]] l2[t] + A[[7, 3]] l3[t] +
  A[[7, 4]] l4[t] + A[[7, 5]] l5[t] + A[[7, 6]] l6[t] + A[[7, 7]] l7[t] +
  A[[7, 8]] l8[t] + v1fx[[7]] + bx[[7]] vluxnln, l8'[t] =
  A[[8, 1]] l1[t] + A[[8, 2]] l2[t] + A[[8, 3]] l3[t] + A[[8, 4]] l4[t] + A[[8, 5]] l5[t] +
  A[[8, 6]] l6[t] + A[[8, 7]] l7[t] + A[[8, 8]] l8[t] + v1fx[[8]] + bx[[8]] vluxnln,
  l1[0] = 10[[1]], l2[0] = 10[[2]], l3[0] = 10[[3]], l4[0] = 10[[4]],
  l5[0] = 10[[5]], l6[0] = 10[[6]], l7[0] = 10[[7]], l8[0] = 10[[8]]},
{l1, l2, l3, l4, l5, l6, l7, l8}, {t, ti, tf}]

fig5 = Plot[Evaluate[{l1[t], l2[t], l3[t], l4[t], l5[t], l6[t], l7[t], l8[t]} /. sol3],
  {t, ti, tf}, PlotRange -> All, PlotStyle -> {Red}]
fig6 = Plot[Evaluate[vluxnln /. sol3], {t, ti, tf}, PlotRange -> All, PlotStyle -> {Red}]
t1 = TimeUsed[];
totaltime = (t1 - t0) / 60

```