Some Studies on Modeling and Simulation of a Disc Brake System for Squeal Prediction

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Nitesh Ahuja

Certificate of Approval:

David G. Beale
Professor
Mechanical Engineering

P.K.Raju, Chair
Thomas Walter Professor
Mechanical Engineering

Dan B. Marghitu
Professor
Mechanical Engineering

Stephen L. McFarland
Acting Dean
Graduate School

Some Studies on Modeling and Simulation of a Disc Brake System for Squeal Prediction

Nitesh Ahuja

A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
August 7, 2005

Some Studies on Modeling and Simulation of a Disc Brake System for Squeal Prediction

Nitesh Ahuja

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Signature of Author

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Date of Graduation

Vita

Nitesh Ahuja was born in Faizabad, Uttar Pradesh, India on January 12, 1978. He graduated with a Bachelor of Engineering degree in Mechanical Engineering from Dayananda Sagar College of Engineering, Bangalore University, Bangalore, India in December 2001. He joined the Masters program in the Department of Mechanical Engineering at Auburn University in August 2003.

Thesis Abstract

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Nitesh Ahuja

Master of Science, August 7, 2005
(B.E., Bangalore University, 2002)

92 Typed Pages

Directed by P.K.Raju

Most mechanical systems that involve a sliding contact experience some degree of friction induced self-excited vibration. A well known example is brake squeal which is a top concern for automotive companies, who must spend huge amounts of money on warranty repairs. Brake squeal occurs above 1k Hz and below 20k Hz. The phenomena that cause brake squeal is not thoroughly understood, although a great deal of research has been dedicated to developing a better understanding of the complex problem of brake dynamics. This study developed a displacement dynamic equation of motion for brake rotor's, assuming the rotor's to be annular plates with the same modal characteristics. An analytical approach was used to solve the equation of motion of these annular discs for free vibration. The Eigen values obtained through the analytical approach were then used to obtain the natural frequencies. Experimental modal testing was also performed to determine the natural frequencies of disc and pad with free boundary conditions.

The natural frequencies of the annular disc were also obtained using the finite difference method. An algorithm was developed to describe the dynamic response of the

annular disc for both a constant forcing function and a time dependent forcing function using the central difference method. The stability of the algorithm was also examined in order to obtain the time step. The importance of the finite difference technique lies in its ability to visualize the results and provide deeper insights into the dynamics of brake rotors. Computer software in C programming and Matlab was developed to solve the differential equation with the forcing function. Another advantage of this method is that any type of forcing function which is practically possible can be applied and its response calculated. The results revealed that an annular disc behaves non-linearly.

Finite element analysis was then performed to determine the dynamic characteristics (natural frequency and mode shapes) and the response of the brake model. The advantage of using this technique is that frictional forces are also taken into account. The contact analysis applied a special type of element called CONTAC49 for 3-D surfaces.

In this thesis a four degree of freedom mathematical model of the brake system based on friction characteristics and geometric non-linearity capable of representing the brake dynamics is presented. Friction induced vibration was studied by using a four degree of freedom model consisting of a disc and a pad moving in both the in-plane and transverse directions. This system takes the form of a mathematical model with two coupled equations and two uncoupled equation. The set of four differential equations can then be solved using Runge Kutta's JBM method.

Acknowledgments

The author would like to thank his advisor Dr.P.K.Raju,the Thomas Walter Professor and Chair, Mechanical Engineering Department for his guidance and support throughout this research and especially during the completion of this thesis.

The author would also like to thank Dr.David G. Beale, Professor, Mechanical Engineering Department and Dr. Dan B. Marghitu, Associate Professor, Mechanical Engineering Department, for serving as committee members. Finally, the author thanks his parents and friends for their support and help.

Style manual or journal used LaTeX: A Document Preparation System by Leslie Lamport (together with the style known as “aums”).

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`. The images and plots were generated using MATLAB 7.0 and Microsoft Office Excel 2003.

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Chapter 1

Introduction

Brakes are one of the most vital safety features in motorized vehicles. However, Automotive brake noise has become an increasingly important issue for automobile manufacturers. Today, most manufacturers attempt to reduce the brake noise by changing the geometry, carefully selecting the friction materials and mounting vibration shims at the backplate of the pads. However, the mechanism that causes this noise is not yet properly understood. This report presents a new analysis methodology which has the potential to improve the efficiency and accuracy of the design process for disc-brake systems. This process will allow a more accurate determination of the variable parameters influence the brake noise and that reduces the time needed to complete the design.

1.1 Scope of the Thesis

Structural dynamics and friction play an important role in disc-brake dynamics. For the brake system studied in this thesis, the rotor was the major cause of squeal and radiator of sound. The natural frequency of a rotor can be obtained analytically by solving a differential equation with boundary conditions applied. The aim of the work presented in this thesis has been to determine natural frequency of the disc theoretically. The thickness of the annular disc is assumed to act in such a way that its natural frequencies match the actual natural frequency of the rotor of the vehicle. After verifying the theoretical model, the dynamics of a disc subjected to nodal forces were studied. The natural frequencies obtained theoretically were verified by comparison with the natural frequency obtained experimentally. Various type of static loading and dynamic loading were applied at the different nodes and the response of the disc predicted. Finite Element Analysis was also performed to understand the behavior as disc and brake pads interact. To test the behavior of the model, a contact element was generated between the two surfaces using the three dimensional contact element contac49 in ANSYS8.0.

A four degree of freedom mathematical model was also developed to simulate a disc brake system with a cubic non-linearity. Using this model, the interaction between the pad and rotor was investigated. Non-linear analysis was also performed to demonstrate the unstable spiral and limit cycles in the phase space. The results showed that one of the instability conditions depend on the initial velocity of the disc. It also provided new insights that will enable us to understand contact and friction phenomena in the brake pad and disk coupling.

1.2 An Overview of the Thesis

This thesis presents an in-depth study of the dynamic behavior of a brake disc using finite difference method.

A review of the literature on disc-brake noise is presented in Chapter 2. The natural frequencies of an annular plate are found analytically, which match the actual brake rotor and a mathematical equation is developed for the annular plate. Due to the complexity of the mathematical model, it must be solved using the Finite Difference Method. Chapter 4 presents the theoretical modeling of a disc brake system cosidering friction and describes the experimental work performed in order to obtain a better understanding of the squealing phenomena. Chapter 5 discusses the results and presents the conclusions that can be drawn as a result of the analysis.

Chapter 2

Background

2.1 Overview

Although, Automotive brakes have improved significantly during the last decade, the engine power has also increased, so the energy to be dissipated in a brake is several times larger than the power developed by the engine. However, not only the power but also the expectations of comfort by drivers and passengers have also greatly increased over the last decade. This means that noise levels, and in particular brake noise, which were acceptable 20 or 30 years ago are no longer acceptable. Brake squeal is now a major problem in vehicles leading to substantial losses for automobile companies. The disc brake was invented in 1902 by Frederick William Lanchester in England[1]. Nowadays, the vibration and noise produced in disc-brake systems cause discomfort for passengers and raise their concerns over the reliablity of the vehicle. Akay[2] recently reviewed friction-induced noise, including disk-brake squeal. He quoted an industrial source who gave an estimate of the warranty costs due to noise, harshness and vibration (together known as NVH problems), including disk-brake squeal, as costing the automotive industry U.S. $ 1 billion in North America alone.

Brake noise is usually classified in three categories : low frequency noise, low frequency squeal and high frequency squeal. Low frequency brake noise typically occurs between 100 Hz and 1 KHz and is referred to as grunt, groan, grind and moan. It originates in the excitation of the friction material at the rotor and lining interface. In

Figure 2.1: Disc Brake System

disk brakes, squeal can occur when the brake pads contact the rotor while the vehicle is moving at low speeds, causing a vibration that manifests as an annoying high-pitched squeal. Low frequency squeal occurs in the frequency range above 1 kHz and below the first circumferential mode of the rotor[3]. This type of noise is caused frictional excitation and the coupling of more than two modes of various component of the brake system. This coupling produces optimum conditions for brake squeal. High frequency squeal is a particularly troublesome problem in brake development and typically occurs at frequencies above 5 kHz. It has been experimentally demonstrated that the frequency of squeal coincides with the circumferential resonance fequencies of the rotor disc[3].

A car disc brake system consists of a rotating disc, non-rotating friction pads, carrier bracket, calliper, yoke, piston,sealing ring, guide pins and dustboot. The pads are loosely housed in the calliper and located by the carrier bracket. The calliper itself is allowed to slide fairly freelyalong the two mounting guide pins in a floating calliper design. The disc is bolted to the car wheel and thus rotates at the same speed as the wheel. When the disk brake is applied, the two pads are brought in contact with the rotating disc surfaces. Most of the kinetic energy of the vehicle is converted to heat through friction but a small part of it is converted into sound energy and generates noise.[4]. Optimizing this automotive noise behavior is an important aspect in new vehicle designs. The purpose of a brake is to attenuate effectively and evenly the kinetic energy of a vehicle. This is done by converting it into heat energy which is generated due to the friction between the brake disk and the pads. Under certain operating parameters, during the braking process, high frequency vibrations of the brake disc can be excited. These significant structural vibrations of the brake, which have no influence on the actual braking effect, are heard as an undesired, acoustically perceptible squealing of brakes. The out-of-plane vibrations are physically responsible for the noise emitted from the brake disk when squealing occurs. In this case, the surface of the friction ring of the brake disc vibrates at right angles to the disc plane. The decelerating braking power between pads and brake disc has an effect in the disc plane, exciting in-plane vibrations.[5]

2.2 Literature Review

North [7] observed that the first concerted effort to study car disk brake squeal was made at the Motor Industrial Research Association (MIRA) in the U.K. in the 1950s. In 1970, a study on the vibration of a disc subject to a moving load was initiated. Mote investigated the vibration of a disc under a pointwise load and showed that the instability is likely to occur in the supercritical range [6].

Disc brake squeal occurs when a system experiences a very large amplitude vibration. There are two theories that attempt to explain why this occurs. The first theory states that a negative friction-velocity gradient leads to stick-slip vibration, which has been identified as one of the major causes of brake squeal, and it occurs due to non-linearity of the frictional force-velocity characteristics. The second theory believes that the high level of vibration results from geometric instabilities in the brake assembly. Both theories attribute the brake system vibration and the accompanying audible noise to variable friction forces at the lining-rotor interface [12]. Yuan classified these mechanisms into two groups [13]. The friction coefficient tends to decrease as sliding starts. This is the basis of the negative slope of friction versus relative speed, which was first put forward as a possible mechanism responsible for generating brake squeal by Mills and was later supported by experimental results [14]. However, North [7] thought that it(alone) was not likely to account for squeal. The difference between the static friction coefficient and the kinetic friction coefficient may cause stick-slip vibration, which was first introduced to disk brake squeal studies by Mills [14]. North used a two-degree-of-freedom model of a car disc brake to show that the different friction forces on either side of the disc brake could lead to instability. In another study, a slider experiencing stick-slip vibration on a vibrating disk was shown to display a rich dynamic behavior [15]. Ouyang, Cao, Mottershead and Treyde [16] believe that a good disc brake model should include a plausible squeal mechanism, a proper treatment of the frictional contact and an accurate description of the system dynamics. The disc is presented as an annular plate and the stationary components (pad, calliper, carrier and mounting pins) are described using a finite element model, this constituting a combined numerical-analytical model. The fricition forces acting on the surface of the disk generate pointwise bending couples that move around the disc. Thus, the whole disk brake system must be treated as a proper moving load problem through the moving contact at the disk/pads interface with friction [16]. Ouyang and Mottershead [17] investigate the instability of the transverse vibration of a disc excited by two co-rotating sliders on both sides of the disc, considering each slider as a mass-spring-damper system traveling at the same constant speed around the disc. In this model, Frictional forces acting at the contact interface between the disc and each of the two sliders. The bending couple and normal forces produced by the rotating sliders are moving loads and are found to result in dynamic instability [17]. Chowdhary, Bajaj and Krousgrill [18] predict one of the mechanisms behind the disc brake model, where the brake rotor is represented by a thin plate of equivalent modal characterisitics and the backing plate are modeled as thin annular section plates using the Rayleigh-Ritz approach. The two structural models are then coupled using linear elastic springs and Coulomb friction at the interface, and langrangian approach is used to derive the equations of motion of the coupled system [18].

Whatever the modeling method adopted to solve this problem, linear analysis is commonly used to determine the stability of the system for various parametric conditions. By performing stability analysis, the real parts of the eigenvalues can be extracted to construct the characteristic equation of the system. If the real parts are positive then the system is unstable, and if they are negative then the system is stable. In a disc brake system, friction force is the main cause for the excitation of vibration and is unevenly distributed force between the brake disc and the brake pad on either sides of the rotor.

Research on the vibrational instabilities which lead to brake squeal has been conducted for more than fifty years. Today's better understanding of the vibrational instabilities that produce brake squeal has required a great deal of detailed and sophisticated analysis and experimental research due to the complexity of a brake system. The ability to design a brake system and identify its causes of squealing in advance would offer a very helpful design tool. The literature also suggests that different brake systems radiate sound in very different ways. Murakami(1984) studied a brake system and found that the brake calliper and pad played major role in brake squeal. Five years later, Nishiwaki conducted a research on a different brake system and concluded that rotor was responsible for the most noise. McDaniel, Moore, Chen and Clarke [19] presented a study of acoustic radiation from the stationary brake system. The objective of this experiment was to better understand the acoustic radiation from squealing brake systems. The researchers found that the great majority of squeal mechanisms occur due to the resonant behavior of the operating brake system. They also presented an analysis that equated the natural frequencies and modes of mechanically-excited stationary brake systems to those of an operating brake system. Acoustic radiation efficiencies and intensities of the modes were computed by importing experimentally measured velocities into a BEM software package, which revealed that for a particular brake system, the highest radiation efficiency occurred at frequencies above 2-3kHz [19].

Chapter 3

Mathematical Modeling of A Disc Brake

3.1 Free Vibration of Circular Disc Without Damping : An Analytical Approach

It is difficult to write a differential equation for a disc-brake model, which has a very complicated geometry. In order to develop a mathematical model for a disc-brake system, it is first necessary to reduce the disc into an annular plate with the same natural frequency and mode shapes. In this case, the first nine natural frequencies are matched with the actual rotor, which is achieved by changing the thickness of the annular plate.

The displacement equation of motion for circular plate without damping can be written as:

$$D\nabla^4 w + \rho h \ddot{w} = p(r, \theta, t) \tag{3.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$D = \frac{E * h_t^3}{12 * (1 - \upsilon^2)}$$

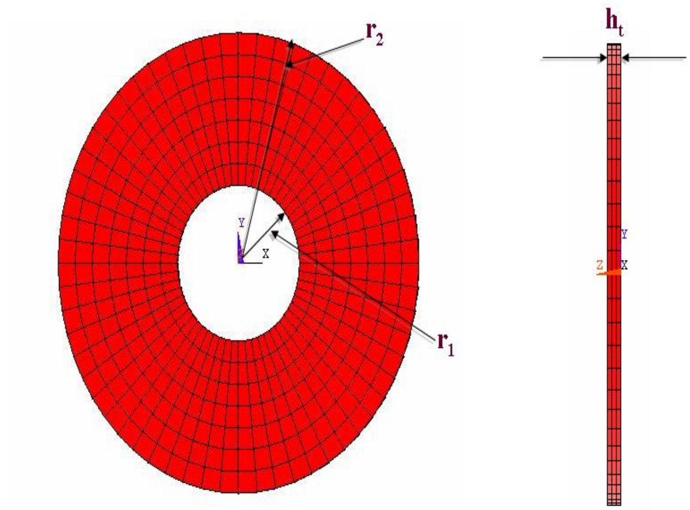

Figure 3.1: Annular disc brake

E is the Elastic modulus, h is the thickness,ρ is the mass desity and υ is the Poisson's ratio. Considering the case of free vibration when the forcing function at the right hand side i.e. p(r,θ,0)=0 and assuming the boundary conditions to be homogeneous, the above equation can be written as

$$D\nabla^4 w + \rho h \ddot{w} = 0 \tag{3.2}$$

The homogeneous boundary condition can take one of the forms

$$V_n = 0 \tag{3.3}$$

or ,

$$M_{nn} = 0 \tag{3.4}$$

Thus, either

$$w = 0 \tag{3.5}$$

or,

$$\partial w/\partial n = 0 \tag{3.6}$$

Let us assume that the free vibrations are characterized by

$$w(r,\theta,t) = W^{(i)}(r,\theta)\cos(\Omega_i t), i = 1, 2, 3\ldots \tag{3.7}$$

$$w(r,\theta,t) = W^{(j)}(r,\theta)\cos(\Omega_j t), j = 1, 2, 3\ldots \tag{3.8}$$

where Ω_i is the natural frequency and $W^i(r,\theta)$ is its associated mode shape, then upon substitution of equation 3.7 and 3.8 into 3.2 [20], we obtain

$$D\nabla^4 W^{(i)} = \rho h \Omega_i^2 W^{(i)} \tag{3.9}$$

$$D\nabla^4 W^{(j)} = \rho h \Omega_j^2 W^{(j)} \tag{3.10}$$

The equations 3.9, 3.10 resemble static equilibrium equations and therefore the Betti-Rayleigh reciprocal theorem [20] is applicable if it is set as $p_i=\rho$ h Ω_i^2 W^i and $p_j=\rho$ h Ω_j^2 W^j. Assuming that the present analysis is restricted to homogeneous boundary conditions, the application of the Betti-Rayleigh reciprocal theorem to the result obtained in 3.10 and 3.11, gives

$$\int_A \rho h \Omega_i^2 W^{(i)} W^{(j)} dA = \int_A \rho h \Omega_j^2 W^{(j)} W^{(i)} dA$$
$$(\Omega_i^2 - \Omega_j^2) \int_A \rho h \Omega_j^2 W^{(i)} W^{(j)} dA = 0 \tag{3.11}$$

If $\Omega_i^2 \neq \Omega_j^2$, then

$$\int_A \rho h \Omega_j^2 W^{(i)} W^{(j)} dA = 0 \tag{3.12}$$

3.1.1 Boundary Conditions

Consider the axisymmetric, harmonic vibration of the disc clamped at radius r_1. If the plate deflection is referred to in polar coordinates. We can assume the axial symmetry, w=w(r), where r is the radial coordinate. The mode shapes $W^{(i)}$ must satisfy the boundary condition, as the disc is fixed or clamped at the inner boundary.

At the inner boundary i.e at r=r_1 we have,

$$W^{(i)}(r) = \frac{\partial W^{(i)}}{\partial r} = 0 \tag{3.13}$$

Therefore,

$$(\nabla^2 + \lambda_i^2)(\nabla^2 - \lambda_i^2) W^{(i)} = 0 \tag{3.14}$$

where

$$\lambda_i^2 = \Omega_i (\frac{\rho h}{D})^{1/2} \tag{3.15}$$

The boundary condition at the outer edge of the disc i.e at r=r_2 is assumed to be free,

Therefore, the bending moment and shear force at the outer edge is zero.

Mathematically, this can be written as follows:

$$M_r \mid_{(r=r_2)} = -D[\frac{\partial^2 W}{\partial r^2} + \upsilon(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2})] = 0 \tag{3.16}$$

$$V_r \mid_{(r=r_2)} = -[D\frac{\partial}{\partial r}(\nabla^2 W) + (1-\upsilon)\frac{\partial}{r\partial\theta}(\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial W}{\partial \theta})] = 0 \tag{3.17}$$

Since $\lambda_i^2 > 0$,the solution of equation 3.14 can be written in the form,

$$\begin{aligned} W(r,\theta) = (C1\cos(\theta) + C2\sin(\theta))(A_i BesselJ(0,\lambda_i r) + B_i BesselY(0,\lambda_i r) \\ + C_i BesselI(0,\lambda_i r) + D_i BesselK(0,\lambda_i r)) \end{aligned} \tag{3.18}$$

where J_0 and K_0 are Bessel functions, and I_0 and K_0 are modified Bessel functions, all of order zero. The above assumed solution should also satisfy all the boundary conditions as follows:

$$\begin{aligned} W_{(r=r_1)} = A_i BesselJ(0,\lambda_i r_1) + B_i BesselY(0,\lambda_i r_1) + C_i BesselI(0,\lambda_i r_1) \\ + D_i BesselK(0,\lambda_i r_1) = 0 \end{aligned} \tag{3.19}$$

$$\begin{aligned} \frac{\partial W}{\partial r}_{(r=r1)} = -A_i BesselJ(0,\lambda_i r_1)\lambda_i - B_i BesselY(0,\lambda_i r_1)\lambda_i + C_i BesselI(0,\lambda_i r_1)\lambda_i \\ - D_i BesselK(0,\lambda_i r_1)\lambda_i = 0 \end{aligned}$$

$$(3.20)$$

The bending moment at the outer edge is given by the following equation:

$$\begin{aligned}M_{(r=r_2)} = &-A_i[BesselJ(0,\lambda_i r_2) - \frac{BesselJ(1,\lambda_i r_2)}{\lambda_i r_2}]\lambda_i^2 - B_i[BesselY(0,\lambda_i r_2) \\ &-\frac{BesselY(1,\lambda_i r_2)}{\lambda_i r_2}]\lambda_i^2 + C_i[BesselI(0,\lambda_i r_2) - \frac{BesselI(1,\lambda_i r_2)}{\lambda_i r_2}]\lambda_i^2 \\ &-D_i[BesselK(0,\lambda_i r_2) - \frac{BesselK(1,\lambda_i r_2)}{\lambda_i r_2}]\lambda_i^2 + \frac{1}{r_2}[\upsilon(-A_i Bessel(1,\lambda_i r_2)\lambda_i \\ &-B_i BesselY(1,\lambda_i r_2)\lambda_i + C_i BesselI(1,\lambda_i r_2)\lambda_i - D_i BesselK(1,\lambda_i r_2)\lambda_i)] \\ &+\frac{\upsilon}{r^2}[-A_i BesselJ(0,\lambda_i r_2) - B_i BesselY(0,\lambda_i r_2) - C_i BesselI(0,\lambda_i r_2) \\ &-D_i BesselK(0,\lambda_i r_2)] = 0\end{aligned} \tag{3.21}$$

$$\begin{aligned}V_{(r=r_2)} = &\frac{(1-\upsilon)\lambda_i}{r^2}[A_i BesselJ(1,\lambda_i r) + B_i BesselJ(1,\lambda_i r) - C_i BesselJ(1,\lambda_i r) \\ &+D_i BesselJ(1,\lambda_i r)] + \frac{(1-\upsilon)}{r^3}[A_i BesselJ(0,\lambda_i r) + B_i BesselJ(0,\lambda_i r) \\ &-C_i BesselJ(0,\lambda_i r) + D_i BesselJ(0,\lambda_i r)] - A_i\lambda_i^2[-BesselJ(1,\lambda_i r)\lambda_i \\ &+2\frac{BesselJ(1,\lambda_i r)}{\lambda_i r^2} - \frac{BesselJ(0,\lambda_i r)}{r}] - B_i\lambda_i^2[-BesselY(1,\lambda_i r)\lambda_i \\ &+2\frac{BesselY(1,\lambda_i r)}{\lambda_i r^2} - \frac{BesselY(0,\lambda_i r)}{r}] + C_i\lambda_i^2[BesselI(1,\lambda_i r)\lambda_i \\ &+2\frac{BesselI(1,\lambda_i r)}{\lambda_i r^2} - \frac{BesselI(0,\lambda_i r)}{r}] - D_i\lambda_i^2[BesselK(1,\lambda_i r)\lambda_i \\ &+2\frac{BesselK(1,\lambda_i r)}{\lambda_i r^2} + \frac{BesselK(0,\lambda_i r)}{r}] - A_i\lambda_i^2[\frac{BesselJ(0,\lambda_i r)\lambda_i r - BesselJ(1,\lambda_i r)}{\lambda_i r^2}] \\ &-B_i\lambda_i^2[\frac{BesselY(0,\lambda_i r)\lambda_i r - BesselY(1,\lambda_i r)}{\lambda_i r^2}] + C_i\lambda_i^2[\frac{BesselI(0,\lambda_i r)\lambda_i r - BesselI(1,\lambda_i r)}{\lambda_i r^2}] \\ &-D_i\lambda_i^2[\frac{-BesselK(0,\lambda_i r)\lambda_i r - BesselK(1,\lambda_i r)}{\lambda_i r^2}] + A_i\frac{BesselJ(1,\lambda_i r)\lambda_i}{r^2}\end{aligned}$$

Table 3.1: Parameters of the annular plate

Parameter	Value
Thickness of disc,h_t	0.012 m
Inner radius of disc, r_1	0.045 m
Outer radius of disc, r_2	0.133 m
Young's Modulus, E	1.2E11 Pa
Mass Density, ρ	7200 Kg/m^3
Poisson's ratio, υ	0.211

$$+B_i\frac{BesselY(1,\lambda_i r)\lambda_i}{r^2} - C_i\frac{BesselI(1,\lambda_i r)\lambda_i}{r^2} + D_i\frac{BesselK(1,\lambda_i r)\lambda_i}{r^2}$$
$$-A_i\frac{BesselJ(0,\lambda_i r)}{r^2} - B_i\frac{BesselY(0,\lambda_i r)}{r^2} - C_i\frac{BesselI(0,\lambda_i r)}{r^2} - D_i\frac{BesselK(0,\lambda_i r)}{r^2} = 0 \tag{3.22}$$

Solving equations 3.19, 3.20, 3.21 and 3.22 for A$_i$, B$_i$, C$_i$ and D$_i$ and substituting those values of A$_i$, B$_i$, C$_i$ and D$_i$ into equation 3.19, and again solving using the Maple 10 software for λ at r=r$_1$. From equation 3.14 , the natural frequencies of vibration are obtained by finding the roots of the equation 3.19 as follows:

$$\begin{aligned} \lambda_i^2 &= \Omega_i(\frac{\rho h}{D})^{1/2} \\ \Omega_i &= \frac{\lambda_i^2}{(\frac{\rho h}{D})^{1/2}} \end{aligned} \tag{3.23}$$

The parameters given in table 3.1 were used to calculate the natural frequencies of the disc. The natural frequencies obtained analytically listed in table 3.2

Table 3.2: Natural frequencies of the disc obtained by analytical approach

Serial Number	Value of λ	Natural Frequency (Hz)
1	7.54	979
2	8.12	1135
3	8.39	1212
4	10.3	1827
5	12.98	2902
6	15.831	4316
7	19.1	6282
8	20.27	7075
9	20.45	7201

3.2 Experimental Setup

The pad and the disc were set into vibration at a fixed-point by an ICP Impulse Force Test Hammer model 086D80, provided by PCB Piezotronics. The responses were measured at a number of points on the surface of the structure using a piezoelectric ICP accelerometer model number 352C15 provided by PCB Piezotronics. Two SCC-ACC01 accelerometer input modules with a fixed gain of 2 provided a 4mA current source for the accelerometer and impulse hammer. The 2 modules accepted input signals from the accelerometer and impulse hammer, amplified them and coupled these signals to the data acquisition system via an SC-2345 Signal Conditioning with configurable connectors. The SC-2345 carried signals from 68-pin E Series acquisition (DAQ) devices. It was used with SCC Series modules and a shielded 68-pin cable and offered easy-to-use, low-noise signal conditioning on a per-channel basis.

In this experiment, the signal acquisition was performed using the NI DAQ Card-6036E, 16 inputs/2 outputs, 200 kS/s and LabView software for A/D data conversion, data processing, analysis and archival.Instruments for data recording stored data in an existing file. The diagram also contained instruments for windowing, spectrum units selection

Table 3.3: Natural frequencies (Hz) of the disc found experimentally

Serial Number	Natural Frequency(Hz)
1	998
2	1161
3	1205
4	1840
5	2937
6	4380
7	6212
8	7026
9	7295

and immediate display of time waveforms and frequency spectra.

After the data was properly acquired and analyzed, ME'scope VES 4.0 software, Visual Modal Pro package for experimental modal testing was applied. ME'scope VES 4.0 was developed by Vibrant Technology and consists of a set of post-test analysis software that can be used to analyze and document the dynamic behavior of mechanical structures and machines. The Visual Modal Pro package imports the FRF measurements and generates estimates by curve fitting the modal parameters, namely the natural frequencies, damping and mode shapes.

3.3 Dynamic Equation of Flexible Annular Disc

Spinning annular discs are commonly found in mechanical as well as in electrical engineering applications such as turbine rotors, computer disc memory units, vehicle disc-brake etc. Therefore, their dynamic analysis is of major interest to researchers [21]. It is essential to understand the flexible nature of an annular disc in order to construct

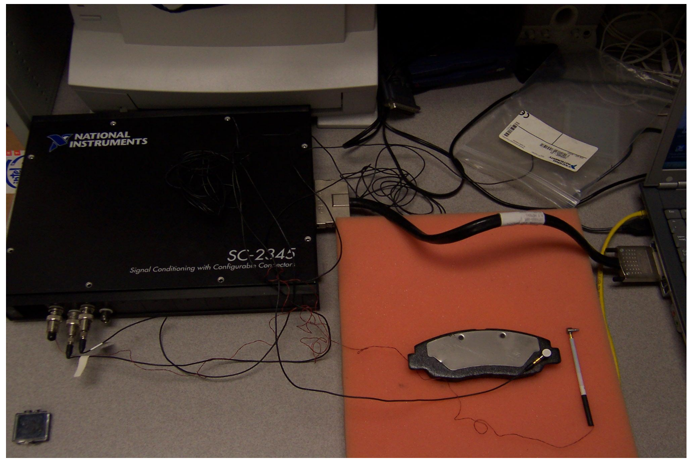

Figure 3.2: Figure shows SC-2345, accelerometers, impact hammer, foam support, PCM-CIA card and computer system

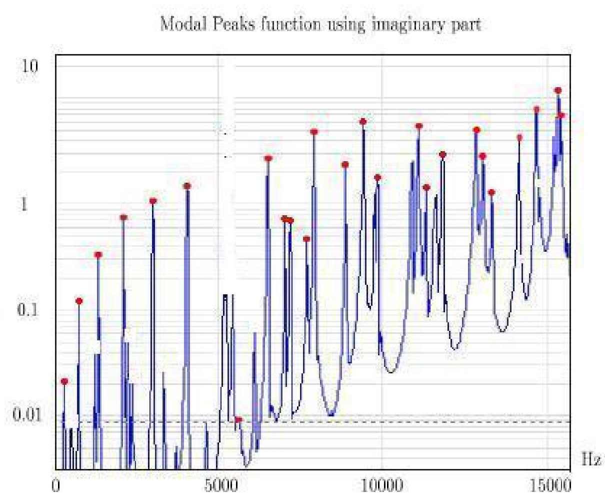

Figure 3.3: Disc modal peaks

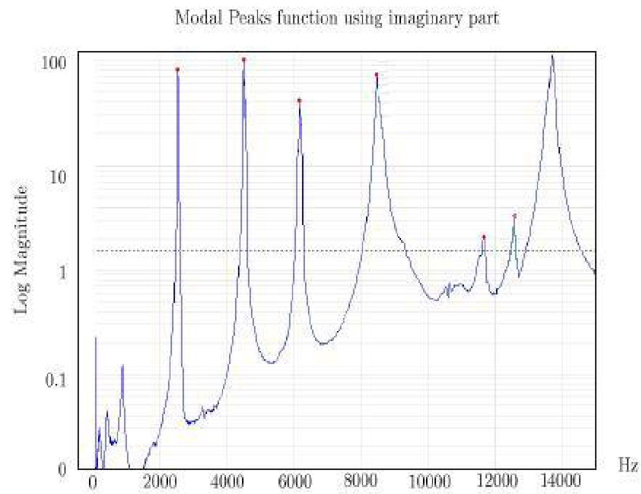

Figure 3.4: Pad modal peaks

an accurate mathematical model for the disc-brake system, which would be helpful in controling the vibration of the disc efficiently. This kind of model can be developed using partial differential equations to describe the dynamic behavior of a flexible disc. The partial differential equation can then be solved by discretizing it into a finite set of ordinary differential equations. Finite element analysis has also been used to understand the dynamic nature of the plate. For example, A new symmetric and positive definite boundary element formulation for lateral vibration of plates was introduced by Dav and Milazzo [22].

Consider an annular plate that is spinning at an angular speed of Ω and is transverly in contact with a stationary oscillating unit at the point (rp,0), where (r,θ) is a polar coordinate system fixed in space. The plate is clamped or fixed at the inner edge and free at the outer boundary.

$$D\nabla^4 w + c(\frac{\partial w}{\partial t}) + \rho h(\frac{\partial^2 w}{\partial t^2}) = F(r,\theta,t) + h_t[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}\frac{\partial w}{\partial r}) + \frac{1}{r^2}\sigma_{\theta\theta}\frac{\partial^2 w}{\partial \theta^2}] \qquad (3.24)$$

where :

∇^4 = Biharmonic operator,

w = lateral deflection in the z direction, a function of r, θ and t.

ρ = mass density per unit area,

D = $\frac{Eh_t^3}{(1-\upsilon^2)}$ flexural rigidity,

h_t = thickness of the plate,

υ = Poisson's ratio,

E = Young's Modulus,

c = viscous damping,

$\frac{\partial^2 w}{\partial t^2}$ = acceleration in the z direction,

σ_{rr} and $\sigma_{\theta\theta}$ = Initial in-plane stresses induced by rotation.

The above equation represents the equation of motion characterizing the behavior of a flexible annular disc in lateral motion. The solution of this equation can be obtained by considering the corresponding boundary conditions. Although the equation contains a term for damping, this will be neglected damping at this stage when solving the above equation. Also,the initial stresses induced due to rotation can be neglected as their effect will be negligible compared to the other stresses present. The above equation can thus be rewritten as:

$$D\Delta^4 w + \rho h(\partial^2 w/\partial t^2) = F(r, \theta, t) \tag{3.25}$$

A numerical method approach based on finite differences can be applied to simulate the characterising behavior of an annular disc. This above equation can also be written in expanded form as follows:

$$\begin{aligned} D[\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} + \frac{1}{r^3}\frac{\partial w}{\partial r} + \frac{2}{r^3}\frac{\partial^3 w}{\partial \theta^2 \partial r} + \frac{2}{r^2}\frac{\partial^4 w}{\partial \theta^2 \partial^2 r} \\ + \frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \theta^4}] + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \end{aligned} \tag{3.26}$$

3.4 Finite Difference Discretization

The difference between the analytical solution and finite difference method is that the solution is obtained only at discrete points. Therefore, dividing the region into small regions and assigning each region a reference index will be provide very accurate results. Solving the circular boundary problem solved using the finite difference approach defined in polar co-ordinates (r,θ) is more convenient than using cartesian co-ordinates as it avoids the use of awkward differentiation formulae near the curved boundary. Defining the mesh in the r-θ plane by the points of intersection of the circles r=r_1 + i*h, where i=0, 1, 2, 3 ..., r_1 is the inner radius of the disc and h=(r_2-r_1)/M, M is the number of circles defined for the mesh and the straight lines θ= j * l where j=0, 1, 2, 3.... and l=(θ_2-θ_1)/N , where N is the number of divisions. These lines and circles are called grid lines and their intersections are referred to as mesh points of the grid or nodal points. The distance between two nodal points is the grid size. In some cases, the accuracy of

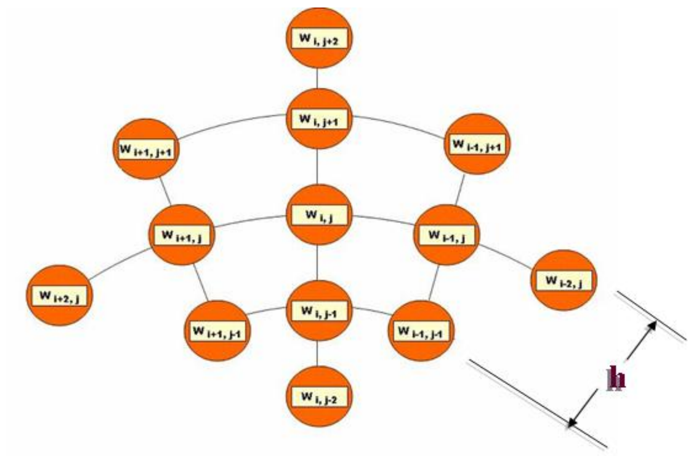

Figure 3.5: Representation of grid points using a 13 node mesh in FDM

the solution can be improved by reducing the mesh size. Figure 3.5 shows a nodal point at the center of a small shaded region that has the index (i, j) and its neighbouring points have node indices (i+1, j), (i-1, j),(i, j+1), (i, j-1), (i+1, j+1) and so on. In the case of a flexible annular plate, a three dimensional coordinate system is considered. Here, the third dimension is time, t. The time axis is represented with reference index k, where t=k*Δt. For each nodal point at the interior of the grid (r_i, θ_j, t_k) where i=0, 1, 2, ...m, j=0, 1, 2, ...n, k=1, 2, ...q. A Taylor series expansion is used to generate the central finite difference for the partial derivative terms of the lateral deflection, w(r, θ, t)= $w_{i,j,k}$

The above equation for the free vibration of the disc without damping can be expanded using the central difference method and can be written as,

$$[\frac{\partial^4 w}{\partial r^4}]_{i,j,k} = \frac{1}{(r1 + i * h)^4}[w_{i+2,j,k} - 4w_{i+1,j,k} + 6w_{i,j,k} - 4w_{i-1,j,k} + w_{i-2,j,k}] \quad (3.27)$$

$$\frac{1}{r^4}[\frac{\partial^4 w}{\partial \theta^4}]_{i,j,k} = \frac{1}{(l)^4(r1 + i * h)^4}[w_{i,j+2,k} - 4w_{i,j+1,k} + 6w_{i,j,k} - 4w_{i,j-1,k} + w_{i,j-2,k}] \quad (3.28)$$

$$\frac{2}{r}[\frac{\partial^3 w}{\partial r^4}]_{i,j,k} = \frac{2}{h^3 * (r1 + i * h)}[w_{i+2,j,k} - 2w_{i+1,j,k} + 2w_{i-1,j,k} - w_{i-2,j,k}] \tag{3.29}$$

$$\frac{1}{r^3}[\frac{\partial w}{\partial r}]_{i,j,k} = \frac{1}{h * (r1 + i * h)^3}[w_{i+1,j,k} - w_{i-1,j,k}] \tag{3.30}$$

$$\begin{aligned} \frac{2}{r^3}[\frac{\partial^3 w}{\partial \theta^2 \partial r}]_{i,j,k} = \frac{2}{(l)^2 * h * (r1 + i * h)^3}[w_{i+1,j+1,k} - w_{i-1,j+1,k} - 2w_{i+1,j,k} \\ +2w_{i-1,j,k} + w_{i+1,j-1,k} - w_{i-1,j-1,k}] \end{aligned} \tag{3.31}$$

$$\begin{aligned} \frac{2}{r^2}[\frac{\partial^4 w}{\partial \theta^2 \partial r^2}]_{i,j,k} = \frac{2}{(l)^2 * h^2 * (r1 + i * h)^2}[4w_{i,j,k} - 2w_{i+1,j,k} \\ -2w_{i-1,j,k} - 2w_{i,j+1,k} - 2w_{i,j-1,k} + w_{i+1,j+1,k} \\ +w_{i+1,j-1,k} + w_{i+1,j+1,k} + w_{i+1,j+1,k}] \end{aligned} \tag{3.32}$$

$$\frac{4}{r^4}[\frac{\partial^2 w}{\partial \theta^2}]_{i,j,k} = \frac{4}{(l)^2 * (r1 + i * h)^4}[w_{i,j+1,k} - 2w_{i,j,k} + w_{i,j-1,k}] \tag{3.33}$$

$$\frac{1}{r^2}[\frac{\partial^2 w}{\partial r^2}]_{i,j,k} = \frac{1}{(h)^2 * (r1 + i * h)^2}[w_{i+1,j,k} - 2w_{i,j,k} + w_{i-1,j,k}] \tag{3.34}$$

$$\rho h[\frac{\partial^2 w}{\partial t^2}]_{i,j,k} = \frac{\rho h}{\Delta t^2}[w_{i,j,k+1} - 2w_{i,j,k} + w_{i,j,k-1}] \tag{3.35}$$

Substituting $\frac{\partial^4 w}{\partial r^4}$, $\frac{1}{r^4}\frac{\partial^4 w}{\partial \theta^4}$, $\frac{2}{r}[\frac{\partial^3 w}{\partial r^4}]$, $\frac{1}{r^3}[\frac{\partial w}{\partial r}]$, $\frac{2}{r^3}[\frac{\partial^3 w}{\partial \theta^2 \partial r}]$, $\frac{2}{r^2}[\frac{\partial^4 w}{\partial \theta^2 \partial r^2}]$, $\frac{4}{r^4}[\frac{\partial^2 w}{\partial \theta^2}]$, $\frac{1}{r^2}[\frac{\partial^2 w}{\partial r^2}]$, $\rho h[\frac{\partial^2 w}{\partial t^2}]$ from equations (3.25-3.33) into equation (2.16) and simplifying yields:

$$\begin{aligned} w_{i,j,k+1} = -\frac{D * \Delta t^2}{\rho * h_t}(Pw_{i,j,k} + Qw_{i+1,j,k} + Rw_{i-1,j,k} + S(w_{i,j+1,k} + w_{i,j-1,k}) \\ +T(w_{i+1,j+1,k} + w_{i+1,j-1,k}) + U(w_{i-1,j+1,k} + w_{i-1,j-1,k}) \\ +V(w_{i+2,j,k} + w_{i-2,j,k}) + W(w_{i,j+2,k} + w_{i,j-2,k})) + 2w_{i,j,k} \\ -w_{i,j,k-1} + \frac{\Delta t^2 F(r,\theta,t)}{\rho h_t} \end{aligned} \tag{3.36}$$

where

$$P = [\frac{6}{h^4} + \frac{6}{((r1+i*h)^4*(\Delta\theta^4)} + \frac{8}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)} \\ -\frac{8}{(r1+i*h)^4*(\Delta\theta^2)} - \frac{2}{(r1+i*h)^2*(h^2)}],$$

$$Q = [-\frac{4}{h^4} - \frac{4}{(r1+i*h)*(h)^3} + \frac{1}{(r1+i*h)^3*h} - \frac{4}{r1+i*h)^3*h*(\Delta\theta^2)} \\ -\frac{4}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)} + \frac{1}{(r1+i*h)^2*(h^2)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} \\ -\frac{2}{(r1+i*h)^2*(h^2)}],$$

$$R = [-\frac{4}{h^4} + \frac{4}{(r1+i*h)*(h)^3} - \frac{1}{(r1+i*h)^3*h} + \frac{4}{r1+i*h)^3*h*(\Delta\theta^2)} \\ -\frac{4}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)} + \frac{1}{(r1+i*h)^2*(h^2)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} \\ -\frac{2}{(r1+i*h)^2*(h^2)}],$$

$$S = [-\frac{4}{((r1+i*h)^4*(\Delta\theta^4)} - \frac{4}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)} + \frac{4}{(r1+i*h)^4*(\Delta\theta^2)}] \\ -\frac{8}{(r1+i*h)^4*(\Delta\theta^2)} - \frac{2}{(r1+i*h)^2*(h^2)}],$$

$$T = [\frac{2}{((r1+i*h)^3*h*(\Delta\theta^2)} + \frac{2}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} \\ -\frac{2}{(r1+i*h)^2*(h^2)}],$$

$$U = [-\frac{2}{((r1+i*h)^3*h*(\Delta\theta^2)} + \frac{2}{(r1+i*h)^2*(h^2)*(\Delta\theta^2)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} - \frac{2}{(r1+i*h)^2*(h^2)}],$$

$$V = [\frac{1}{h^4} + \frac{2}{((r1+i*h)*h^3)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} - \frac{2}{(r1+i*h)^2*(h^2)}],$$

$$W = [\frac{1}{((r1+i*h)^4*(\Delta\theta^4)}] - \frac{8}{(r1+i*h)^4*(\Delta\theta^2)} - \frac{2}{(r1+i*h)^2*(h^2)}]$$

The equation 3.26 must be satisfied at every nodal point of the grid within the boundary of the plate. When using the Finite Difference Method to solve a partial differential equation of an annular disc, more accurate results can be obtained by using higher-order FD approximations than by decreasing the mesh width [8]. Here, a 13 node finite difference approximation was used in which each node is connected to 13 nodes other, as shown in figure 3.5. Figure

3.5 Initial and Boundary Conditions

To run a simulation it is first necessary to define the initial and boundary conditions. It is reasonable to assume that initially the annular plate has no deflection, so the forces and moments of the plate due to its weight can be neglected [9]. Therefore the initial conditions of the plate is:

$$w_{i,j,k} = 0 \tag{3.37}$$

$$w_{i,j,k-1} = 0 \tag{3.38}$$

Table 3.4: Natural frequencies of the disc obtained by the finite difference method

Serial Number	Natural Frequency (Hz)
1	979
2	1135
3	1212
4	1827
5	12.98
6	15.831
7	19.1
8	7075
9	7201

It is assumed that the annular disc is clamped at its inner radius, r=r_1:

$$w_{i,j,k}\left|_{(r=r1)}\right| = 0 \tag{3.39}$$

$$\frac{\partial w}{\partial r}\left|_{(r=r1)}\right| = 0 \tag{3.40}$$

The boundary condition at the outer edge of the disc, i.e at r=r_2 is assumed to be free, so the bending moment and shear force at the outer edge will both be zero. Mathematically, this can be written as follows:

$$V_r(r = r2) = -[D\frac{\partial}{\partial r}(\nabla^2 w) + (1 - \upsilon)\frac{\partial}{r\partial\theta}(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta})] = 0 \tag{3.41}$$

$$M_r(r = r2) = -D[\frac{\partial^2 w}{\partial r^2} + \upsilon(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2})] = 0 \tag{3.42}$$

The magnitude of the forcing function is taken to be 5*$10^{(-4)}$ Newton in the static as well as in the dynamic analysis.

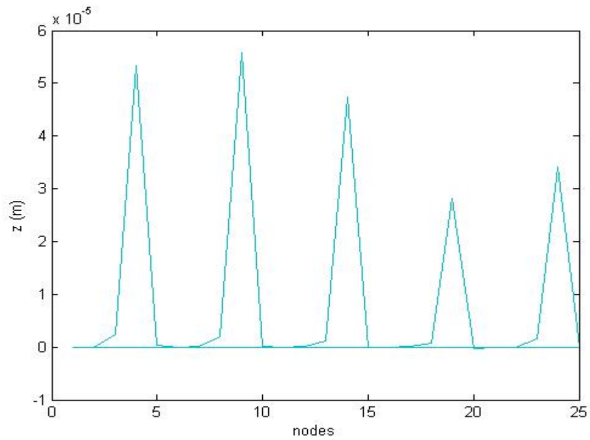

Figure 3.6: Response due to a force applied at node 4 of 5X5 grid points

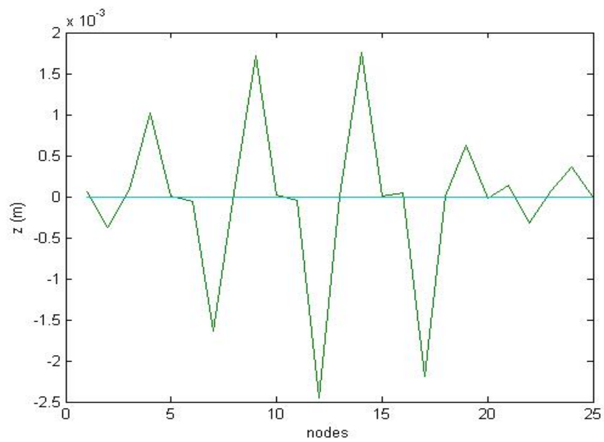

Figure 3.7: Response due to a force applied at node 16 of 5X5 grid points

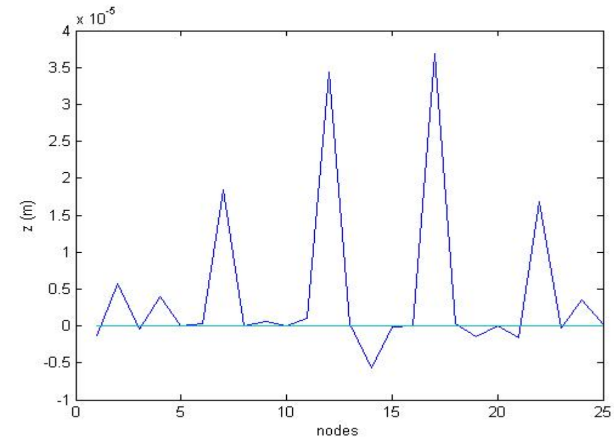

Figure 3.8: Response due to a force applied at node 22 of 5X5 grid points

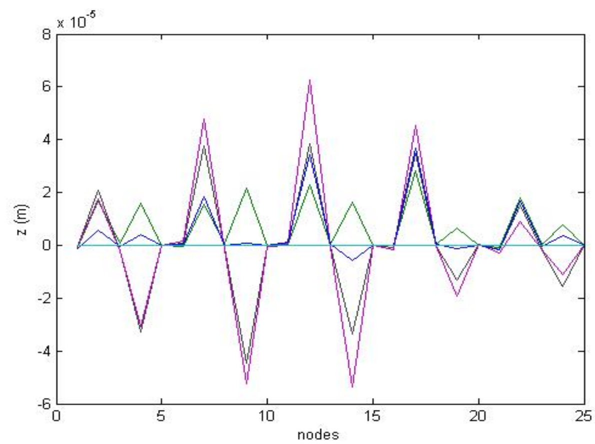

Figure 3.9: Response due to a force applied at nodes 2,7,12 and 22 of 5X5 grid points

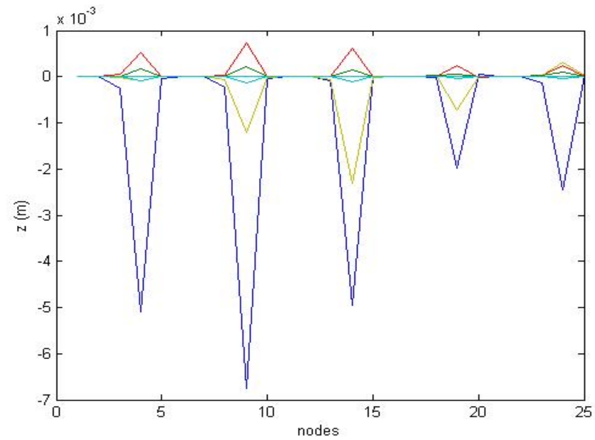

Figure 3.10: Response due to a force applied at nodes 3,8,13,18 and 23 of 5X5 grid points

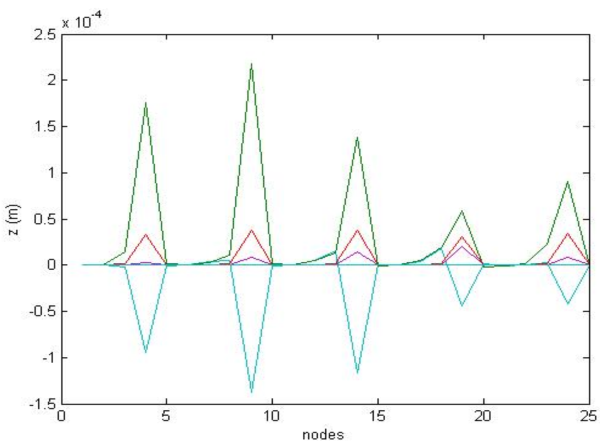

Figure 3.11: Response due to a force applied at nodes 18,19,23 and 24 of 5X5 grid points

3.6 Stability of the Algorithm

The stability of the algorithm can be tested by determining whether the iterative scheme described in equation 3.36 converges to a solution. Discretizing equation 3.26 using equations 3.27-3.35, with zero external force and simplifiying yields:

$$\begin{aligned} c[\dot{P} w_{i,j,k} + \dot{Q} w_{i+1,j,k} + \dot{R} w_{i-1,j,k} + \dot{S}(w_{i,j+1,k} + w_{i,j-1,k}) + \dot{T}(w_{i+1,j+1,k} + w_{i+1,j-1,k}) \\ + \dot{U}(w_{i-1,j+1,k} + w_{i-1,j-1,k}) + \dot{V}(w_{i+2,j,k} + w_{i-2,j,k}) + \dot{W}(w_{i,j+2,k} + w_{i,j-2,k})] \\ = -w_{i,j,k+1} + 2w_{i,j,k} - w_{i,j,k-1} \end{aligned} \tag{3.43}$$

where

$$\dot{P} = [6(\Delta\theta)^4 + \frac{6}{(r1/h+i)^4} + \frac{8(\Delta\theta)^2}{(r1/h+i)^2} - \frac{8(\Delta\theta)^2}{(r1/h+i)^4} - \frac{2(\Delta\theta)^4}{(r1/h+i)^2}] \tag{3.44}$$

$$\dot{Q} = [-4(\Delta\theta)^4 - \frac{4(\Delta\theta)^4}{(r1/h+i)} + \frac{(\Delta\theta)^4}{(r1/h+i)^3} - \frac{4(\Delta\theta)^2}{(r1/h+i)^3} - 4\frac{(\Delta\theta)^2}{(r1/h+i)^2}] \tag{3.45}$$

$$\dot{R} = [-4(\Delta\theta)^4 + \frac{4(\Delta\theta)^4}{(r1/h+i)} - \frac{(\Delta\theta)^4}{(r1/h+i)^3} + \frac{4(\Delta\theta)^2}{(r1/h+i)^3} - 4\frac{(\Delta\theta)^2}{(r1/h+i)^2}] \tag{3.46}$$

$$\dot{S} = [-\frac{4}{(r1/h+i)^4} - \frac{4(\Delta\theta)^2}{(r1/h+i)^2} + \frac{4(\Delta\theta)^2}{(r1/h+i)^4}] \tag{3.47}$$

$$\dot{T} = [\frac{2(\Delta\theta)^2}{(r1/h+i)^3} + \frac{2(\Delta\theta)^2}{(r1/h+i)^2}] \tag{3.48}$$

$$\dot{U} = [-\frac{2(\Delta\theta)^2}{(r1/h+i)^3} + \frac{2(\Delta\theta)^2}{(r1/h+i)^2}] \tag{3.49}$$

$$\dot{V} = [(\Delta\theta)^4 + \frac{2(\Delta\theta)^4}{(r1/h+i)}] \tag{3.50}$$

$$\dot{W} = [\frac{1}{(r1/h+i)^4}] \tag{3.51}$$

The deflection $w_{i,j,k}$ can be expressed as:

$$w_{i,j,k} = \Phi(t)e^{i\alpha R}e^{i\beta\theta} \tag{3.52}$$

where α and β are the spatial frequencies in R and θ, respectively, and $i = \sqrt{-1}$. Substituting for $w_{i,j,k}$ from equation 3.52 into equation 3.43 and simplifying yields;

$$\begin{aligned}(c\Phi(t))(\dot{P} + \dot{Q}(e^{i\alpha\Delta R}) + \dot{R}(e^{-i\alpha\Delta R}) + \dot{S}(e^{i\beta\Delta\theta} + e^{-i\beta\Delta\theta}) + \dot{T}(e^{i(\alpha\Delta R+\beta\theta)} + e^{i(\alpha\Delta R-\beta\theta)})\\ +\dot{U}(e^{i(-\alpha\Delta R+\beta\theta)} + e^{-(i\alpha\Delta R+\beta\theta)}) + \dot{V}(e^{i2\alpha\Delta R} + e^{-i2\alpha\Delta R}) + \dot{W}(e^{i2\beta\Delta\theta} + e^{-i2\beta\Delta\theta}))\\ = -\Phi(t+\Delta t) + 2\Phi(t) - \Phi(t-\Delta t)\end{aligned} \tag{3.53}$$

Using Euler's identity, $e^{i\theta} = cos(\theta) + isin(\theta)$ in the above equation gives

$$\begin{aligned}(c\Phi(t))(\dot{P} + \dot{Q}(\cos(\alpha\Delta R)) + \dot{R}(\cos(\alpha\Delta R)) + \dot{S}(2\cos(\beta\Delta\theta)) + \dot{T}(2\cos(\alpha\Delta R)\\ \cos(\beta\Delta\theta)) + \dot{U}(2\cos(\alpha\Delta R)\cos(\beta\Delta\theta)) + \dot{V}(2\cos(2\alpha\Delta R)) + \dot{W}(2\cos(2\beta\Delta\theta)))\\ = -\Phi(t+\Delta t) + 2\Phi(t) - \Phi(t-\Delta t)\end{aligned}$$

$$(3.54)$$

$$2-c[(\dot{P}-2\dot{V}-2\dot{W}+\dot{Q}(\cos(\alpha\Delta R))+\dot{R}(\cos(\alpha\Delta R))+\dot{S}(2\cos(\beta\Delta\theta))+\dot{T}(2\cos(\alpha\Delta R)\cos(\beta\Delta\theta))+\dot{U}(2\cos(\alpha\Delta R)\cos(\beta\Delta\theta))+\dot{V}(4\cos^2(\alpha\Delta R))+\dot{W}(4\cos^2(\beta\Delta\theta)))] = \frac{\Phi(t+\Delta t)+\Phi(t-\Delta t)}{\Phi(t)} \tag{3.55}$$

In general, the solution is assumed to be in the form $w_{i,j,k}$ in equation 3.52 and contains all multiples of the spatial frequency. Even if these are not present in the initial conditions or imposed by the boundary conditions, they are likely to be introduced by rounding errors. This means that the unbounded amplification of $\Phi(t)$ must be guarded against all α[10]. The original error component will not grow in time , provided that

$$\left|e^{i(t)}\right| \leq 1$$

This is a well-known Von neumann stability condition[10, 11]. Therefore, denoting $\Phi(t)$=$e^{i(t)}$,the right hand side of the equation can be simplified as:

$$\frac{\Phi(t+\Delta t)+\Phi(t-\Delta t)}{\Phi(t)} = \frac{e^{i(t+\Delta t)}+e^{i(t-\Delta t)}}{e^{i(t)}} = e^{i\Delta t}+e^{-i\Delta t}$$

Using Euler's identity to expand the right hand side of the above equation gives

$$e^{i\Delta t} = \cos(\Delta t)+i\sin(\Delta t) and$$
$$e^{-i\Delta t} = \cos(\Delta t)-i\sin(\Delta t)$$

which leads to

$$e^{i\Delta t} + e^{-i\Delta t} = 2\cos(\Delta t)$$

The minimum and maximum value of $e^{i\Delta t} + e^{-i\Delta t}$ will be -2 and 2 and $\cos(\Delta t)$ will have minimum and maximum values of -1 and +1, respectively. Since $\mid 2\cos(\Delta t) \mid$ can only take a maximum value of 2, the following holds:

$$\frac{\Phi(t+\Delta t) + \Phi(t-\Delta t)}{\Phi(t)} \leq 2$$

Therefore:

$$\begin{aligned} \mid 2 - c[(\dot{P} - 2\dot{V} - 2\dot{W}) + \dot{Q}(\cos(\alpha\Delta R)) + \dot{R}(\cos(\alpha\Delta R)) + \dot{S}(2\cos(\beta\Delta\theta)) + \dot{T}(2\cos(\alpha\Delta R) \\ \cos(\beta\Delta\theta)) + \dot{U}(2\cos(\alpha\Delta R)\cos(\beta\Delta\theta)) + \dot{V}(4\cos^2(\alpha\Delta R)) + \dot{W}(4\cos^2(\beta\Delta\theta))] \mid \leq 2 \end{aligned} \tag{3.56}$$

The maximum value each cosine function $(\cos(\alpha\Delta R))(\cos(\beta\Delta\theta)), \cos^2(\alpha\Delta R), \cos^2(\beta\Delta\theta)$ can have is 1, when $(\cos(\alpha\Delta R))$=$(\cos(\beta\Delta\theta))$= -1. Therefore, the above equation can be written as

$$\mid 2 - c[(\dot{P} - 2\dot{V} - 2\dot{W}) - \dot{Q} - \dot{R} - 2\dot{S} + 2\dot{T} + 2\dot{U} + 4\dot{V} + 4\dot{W}] \mid \leq 2 \tag{3.57}$$

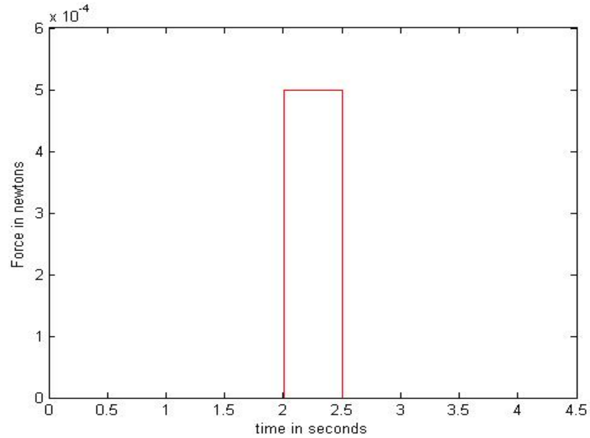

Figure 3.12: Forcing Function

Substituting for c, $\dot{P}, \dot{Q}, \dot{R}, \dot{S}, \dot{T}, \dot{U}, \dot{V} and \dot{W}$ into the above equation yields:

$$| 2 - c[16\Delta\theta^4 + \frac{16}{(r_1/h+i)^4} + 32\frac{\Delta\theta^2}{(r_1/h+i)^2} - 16\frac{\Delta\theta^2}{(r_1/h+i)^4} + 2\frac{\Delta\theta^4}{(r_1/h+i)^2}] | \leq 2 \tag{3.58}$$

The sampling time Δ t =0.04263 seconds, which satisfies the stability requirement and is suffcient to cover a broad range of dynamics of the flexible annular plate. The response of the annular plate is considered over 4.5 seconds.

3.7 Extraction of Eigenvalues and Eigenvectors: Finite Element Approach

Modal analysis can be used to determine the natural frequencies and mode shapes of a structure. The natural frequencies and modeshapes are important parameters in the design of a structure for dynamic loading conditions. The eigenvalue and eigenvector

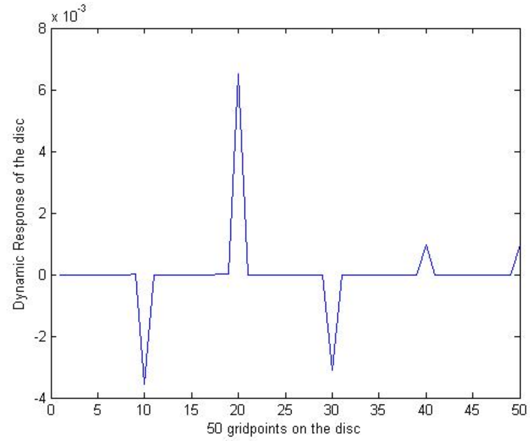

Figure 3.13: Dynamic response in transverse direction due to force applied at point 20 of 10X5 grid points

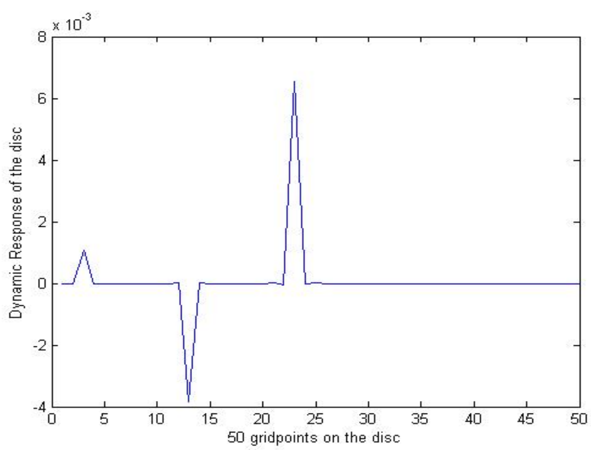

Figure 3.14: Dynamic response in transverse direction due to force applied at point 23 of 10X5 grid points

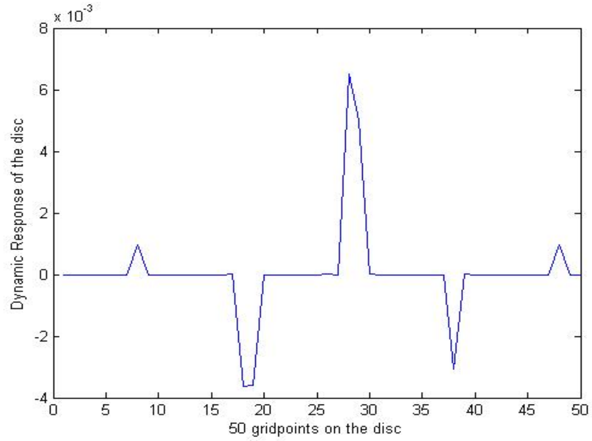

Figure 3.15: Dynamic response in transverse direction due to force applied at point 28 of 10X5 grid points

must therefore be solved for mode-frequency analyses. It takes the following form

$$[K][\phi_i] = \lambda_i[M][\phi_i] \tag{3.59}$$

where:

$[K]$= structure stiffness matrix

$[\phi_i]$=eigenvector

λ_i=eigenvalue

$[M]$= structure mass matrix

ANSYS 8.0 offers several different algorithms to solve the eigenvalue problem like reduced, block Lanczos, subspace, unsymmetric, damped and QR damped methods.

3.7.1 Block Lanczos Method

This method is very commonly used to extract natural frequencies. It uses the Lanczos algorithm, where the Lanczos recursion is performed for a block of vectors using a sparse matrix solver. This method is especially powerful when searching for eigenfrequencies in a defined part of the eigenvalue spectrum of a given system.

3.7.2 Contact Analysis

The contact problem in this study of brake rotor and pad is highly nonlinear. Consequently, it is very important to understand the physics of the problem in order to run the model more efficiently for accurate results. Two significant problem arose initially: the region of contact was unknown and the contact problem needed to account for friction. Friction responses can be chaotic, making solution convergence difficult. In addition the finite element models of the contacting bodies were generated in such a way that node-to-node contact was neither achievable nor desirable when contact was established. For solving this contact type problem, the finite element software ANSYS makes use of a special type of element CONTAC49 to obtain accurate results. This element also takes care of the friction between the disc and pad surfaces and can be used to represent contact and sliding in three dimensions. This element has five nodes and each node has three degrees of freedom: translations in the nodal x, y and z directions. This type of element supports elastic coulomb friction and rigid coulomb friction. Both type of analyses were performed for static as well as dynamic systems. The two potential contact surfaces are referred to as the target surface or the contact surface. The target surface was represented by target nodes I, J, K and L, and the contact surface was represented by the contact node M.

Table 3.5: Natural frequencies (Hz) of the disc using finite element analysis

Serial Number	Natural Frequency,ω_{mn}(Hz)
1	998
2	1161
3	1205
4	1840
5	2937
6	4380
7	6212
8	7026
9	7295

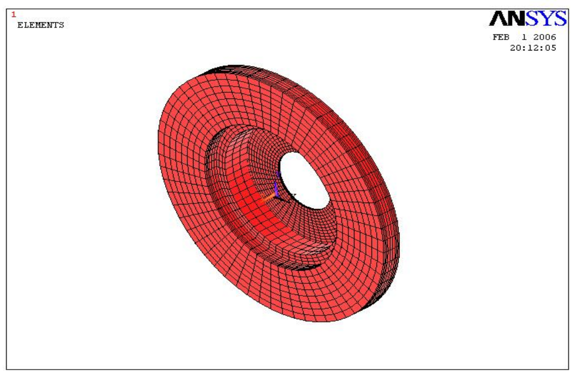

Figure 3.16: Another view of the meshed brake rotor model

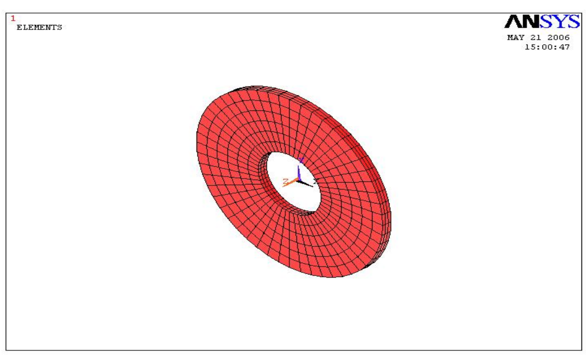

Figure 3.17: Equivalent disc model

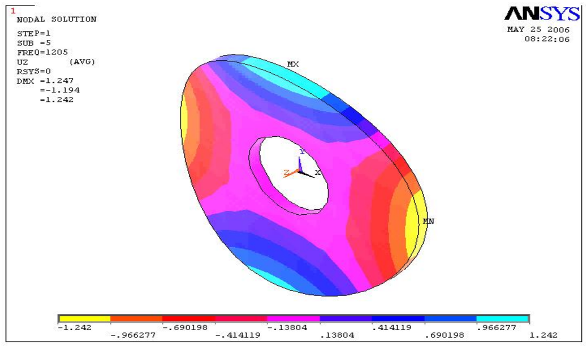

Figure 3.18: Modeshape of a disc at natural frequency of 1205 Hz

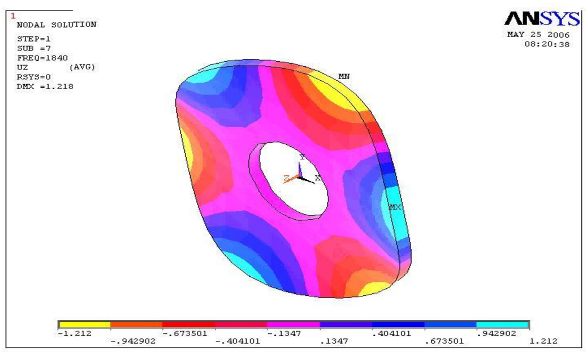

Figure 3.19: Modeshape of a disc at natural frequency of 1840 Hz

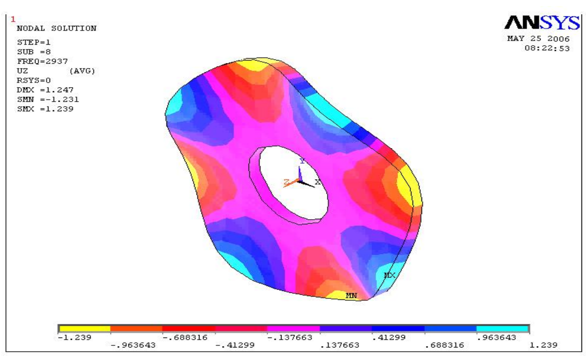

Figure 3.20: Modeshape of a disc at natural frequency of 2937 Hz

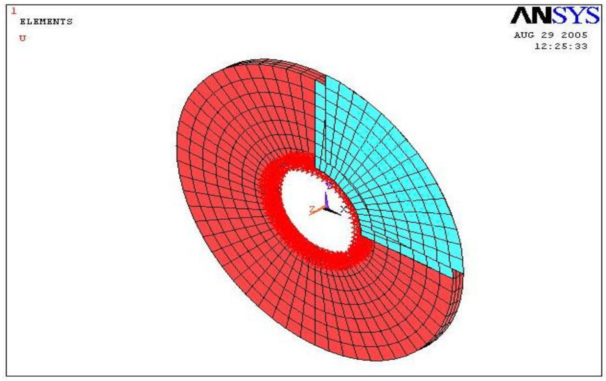

Figure 3.21: The generation of contact element Contac49 at the disc and pad interface in ANSYS 8.0

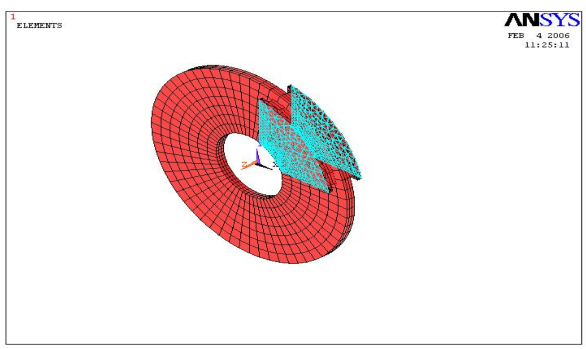

Figure 3.22: Pressurized brake disc model in ANSYS 8.0

Chapter 4

Theoretical Modeling of Disc-Brake System

4.1 Lagrange Formulation and Equation of Motion

Although a brake pad and rotor have many modes of vibration, only a single mode of the complete system contributes to instability and hence noise. In this case only a few modes of vibration are required to model the system. In this chapter, a simulation model for the disc-brake system and frictional forces at the disc-pad interface is developed. Stability analysis has been performed to determine under what parametric condition the system becomes unstable, assuming that the existence of a limit cycle represents the noisy state of disc brake system [24]. In order to study the nonlinear behavior of the brake system and its stability, a mathematical model with the pad and disk modeled as an idealized mass-spring-damper system was constructed. Considering the disc-rotor to be an annular disc, it can be characterized by a dynamic system of differential equations of order four. Therefore, a mathematical model with 4 degrees of freedom is presented. The degrees of freedom x_1, y_1, x_2, y_2 are the inplane and transverse displacements of the pad and disk, respectively, in the cartesian co-ordinate system as shown in figure 4.1. This is a minimal DOF model that includes both in-plane and transverse vibrations of the brake components in order to obtain phase portraits and time histories in a reasonable amount of time. The resilience and the dissipation of the disk and the pad materials are modeled by linear springs and dampers. The linear springs and dampers for the pad are denoted by k_1 and C_1, respectively, and same for the disc by k_2 and C_2, respectively.

The contact between the disc and pad is modeled by a transverse spring and damper denoted by k_T and C_T as shown in figure 4.1.

In order to derive the equations of motion, the kinetic energy T, the potential energy V and the dissipative energy D of the system, can be written as:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_2\dot{y}_2^2 \tag{4.1}$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_1x_2^2 + \frac{1}{2}k_T(y_1 - y_2)^2 \tag{4.2}$$

$$D = \frac{1}{2}C_1\dot{x}_1^2 + \frac{1}{2}C_2\dot{x}_2^2 + \frac{1}{2}C_T(\dot{y}_1 - \dot{y}_2)^2 \tag{4.3}$$

where m_1 and m_2 are the masses of the pad and disc respectively.
The generalized co-ordinates are taken to be $q_1 = x_1$, $q_2=y_1$ for the pad and $q_3=x_2$, $q_4=y_2$ for the disc.

The general Lagrange formulation used to derive equations of motion with respect to generalized co-ordinates is:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q_i}}) + \frac{\partial D}{\partial \dot{q_i}} + \frac{\partial V}{\partial q_i} = Q_i, \tag{4.4}$$

where Q_i is the generalized force associated to the i^th generalized coordinate. The generalized forces can thus be expressed as : $Q_1 = F_f$, $Q_2 = N_{static}$, $Q_3 = -F_f$, $Q_4 = 0$, where F_f is the friction force and N_{static} is the initial force acting on the system.

Finally, the equations of motion can be written as:

$$m_1\ddot{x_1} + C_1\dot{x_1} + k_1x_1 = F_f \tag{4.5}$$

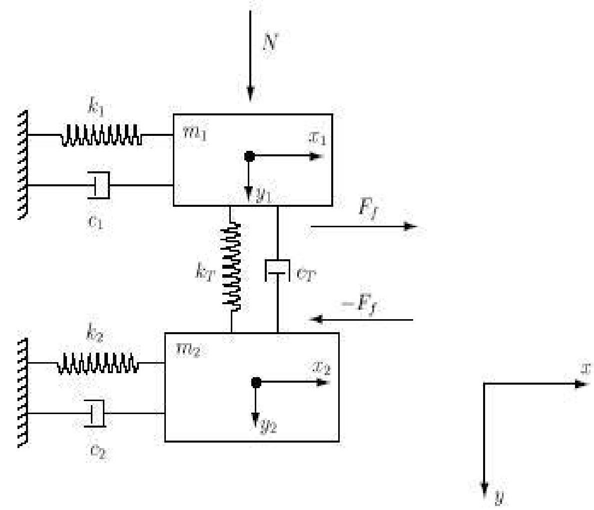

Figure 4.1: Four degrees of freedom model

$$m_1\ddot{y_1} + C_T(\dot{y_1} - \dot{y_2}) + k_T(y_1 - y2) + k_T(y_1 - y2)^3 = N_{static} \tag{4.6}$$

$$m_2\ddot{x_2} + C_2\dot{x_2} + k_2x_2 = -F_f \tag{4.7}$$

$$m_2\ddot{y_2} - C_T(\dot{y_1} - \dot{y_2}) - k_T(y_1 - y2) - k_T(y_1 - y2)^3 = 0 \tag{4.8}$$

The cubic non-linearity is added in the above two equation with reference to Jearsiripongkul and Hagedorn.[23].

During braking the system experiences both in-plane and out-of-plane vibrations. To take into account the transverse vibration of the system, the normal force acting on the system is considered to be [12] :

$$N = N_{static} + N_{dynamic} \tag{4.9}$$

where the static normal force depends on the pressure applied by the piston on the pad, N_{static} = P $*$ S and the dynamic component is modeled as:

$$N_{dynamic} = k_T(y_1 - y_2) + C_T(\dot{y_1} - \dot{y_2}) \tag{4.10}$$

4.1.1 Friction Models

To analyze the behavior of the two systems two friction models were considered. The first was a Coulomb type friction force independent of the magnitude of velocity and always opposing the relative motion. Suppose the friction force comes from the dynamic coefficient (the relative movement between pad and disc, expressed by the relative velocity $v_r \neq 0$), then one can write:

$$F_f = -\mu_d(N_{static} + N_{dynamic})sgn(v_r) \tag{4.11}$$

A more realistic model for the friction takes into consideration the variation of the dynamic coefficient with the relative velocity using the formula μ_d=μ_s - βv_r. In the stick phase, as the relative velocity (v_r=0) between the pad and disc approaches zero, the friction force is modeled as the minimum value between the in-plane applied force F_T and the static friction force F_s=μ_s N_{static}. It cannot be bigger than the applied in-plane force F_T and must oppose it.

Considering the relative motion between pad and disc, the applied in-plane force is evaluated as [24]

$$F_T = -k_1x_1 - c_1\dot{x_1} + k_2x_2 + c_2\dot{x_2} \tag{4.12}$$

In the slip phase, where the relative movement is significant (v$_r \neq 0$), the friction force is μ_dN sgn(v_r), a coulomb type friction force. Finally, the stick-slip friction model that induces excitation in the system is

if v_r=0, stick condition

F_f=-min(| F_T,F_s |)sgn(F_T)

and if, $v_r \neq 0$, slip condition

F_f=(μ_s - β v_r)(N_{static}+$N_{dynamic}$)sgn(v_r)

4.2 Numerical Simulation and Stability Analysis

The equations of motion are coupled differential equations of second order:

$$\ddot{r} = F(t, r, s, \dot{r}, \dot{s})$$

$$\ddot{s} = G(t, r, s, \dot{r}, \dot{s})$$

where $\dot{}$ denotes differentiation with respect to time.

Assuming

$$\dot{r} = u$$

$$\dot{s} = v$$

Therefore

$$\dot{u} = F(t, r, s, u, v)$$

$$\dot{v} = G(t, r, s, u, v)$$

reducing the second order differential equations to a system of four simultaneous equations of first order. A single integration step, h of time t, is then given by,

$$r_{n+1} = r_n + (1/6) * k_1 + (1/3) * k_2 + (1/3) * k_3 + (1/6) * k_4 + O(h^5)$$

$$s_{n+1} = s_n + (1/6) * l_1 + (1/3) * l_2 + (1/3) * l_3 + (1/6) * l_4 + O(h^5)$$

$$u_{n+1} = u_n + (1/6) * m_1 + (1/3) * m_2 + (1/3) * m_3 + (1/6) * m_4 + O(h^5)$$

$$v_{n+1} = v_n + (1/6) * p_1 + (1/3) * p_2 + (1/3) * p_3 + (1/6) * p_4 + O(h^5)$$

where

$$k_1 = h * u \tag{4.13}$$

$$k_2 = h * (u + m_1/2)$$

$$k_3 = h * (u + m_2/2)$$

$$k_4 = h * (u + m_3)$$

$$l_1 = h * v$$

$$l_2 = h * (v + p_1/2)$$

$$l_3 = h * (v + p_2/2)$$

$$l_4 = h * (v + p_3)$$

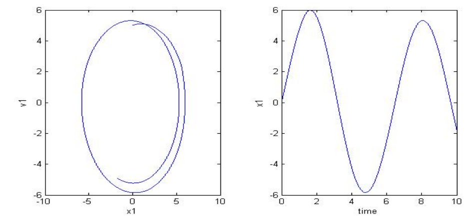

Figure 4.2: (a)Motion & (b)Time history of the disc (Force N=1 and b=0.1)respectively

$$m_1 = h * F(t, r, s, u, v)$$

$$m_2 = h * F(t + h/2, r + k_1/2, s + l_1/2, u + m_1/2, v + p_1/2)$$

$$m_3 = h * F(t + h/2, r + k_2/2, s + l_2/2, u + m_2/2, v + p_2/2)$$

$$m_4 = h * F(t + h, r + k_3, s + l_3/2, u + m_3, v + p_3/2)$$

$$p_1 = h * G(t, r, s, u, v)$$

$$p_2 = h * G(t + h/2, r + k_1/2, s + l_1/2, u + m_1/2, v + p_1/2)$$

$$p_3 = h * G(t + h/2, r + k_2/2, s + l_2/2, u + m_2/2, v + p_2/2)$$

$$p_4 = h * G(t + h, r + k_3, s + l_3, u + m_3, v + p_3)$$

The above method to solve the coupled differential equation is known as Runge-Kutta's JBM method. A C-program can be written to run the simulation and various parameters are used to predict the behavior. For this study, various simulations were run to obtain the stability and instability conditions as shown in figures 4.2 to 4.12. A parametric study was done using different values of static coefficient of friction, damping , initial velocity, dynamic coefficient of friction, force magnitudes and damping of disc & pad. In figures 4.2-4.3 the various parameters used are mass m_1=m_2=1, damping c_1=c_2=0.1 & c_T=0.5, stiffness k_1=k_2=k_T=1, initial velocity u=1, force magnitude N=1 and μ_s=0.01 and for various values of β. In figures 4.4-4.5 the various parameters used are mass

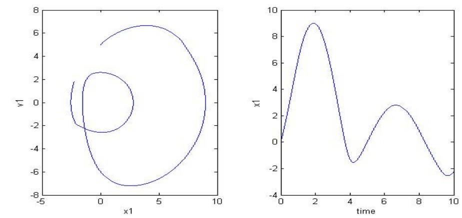

Figure 4.3: (a)Motion & (b)Time history of the disc (Force N=1 and b=1)respectively

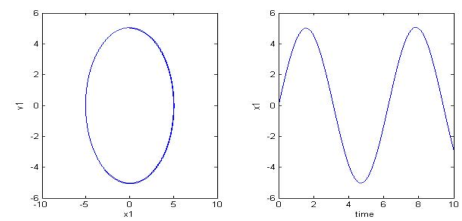

Figure 4.4: (a)Motion & (b)Time history of the disc (Force N=1 and us=0.01)respectively

m_1=m_2=1 damping c_1=c_2=0.1 & c_T=0.5, stiffness k_1=k_2=k_T=1, initial velocity u=1, force magnitude N=1, β=0.01 and various values of μ_s.

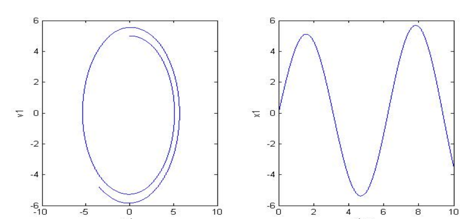

Figure 4.5: (a)Motion & (b)Time history of the disc respectively (Force N=1 and us=0.1)

Chapter 5

Results and Conclusions

5.1 Summary of Results

5.1.1 Analytical Method

The natural frequencies obtained through the analytical method were in generally good with the natural frequencies obtained through other methods given in table 5.1. Table 5.2 shows the percentage differences of the natural frequencies of the various

Table 5.1: Natural frequencies of the disc obtained through all methods

Serial Number	Analytical	Finite Difference	Finite Element	Experimental
1	979	952	998	963
2	1135	1152	1161	1184
3	1212	1198	1205	1235
4	1827	1879	1840	1901
5	2902	3012	2937	2933
6	4316	4287	4380	4358
7	6282	6159	6212	6104
8	7075	6995	7026	7098
9	7201	7197	7295	7310

methods with respect to the analytical method.

5.2 Finite Difference Method

A numerical method of solution of the governing partial differential equation describing the characteristic behavior of a flexible annular plate has been presented and discussed. A simulation algorithm characterizing the behavior of a flexible annular plate

Table 5.2: Percentage deviation of natural frequencies of the disc with respect to the experimental method

Serial Number	Analytical	Finite Difference	Finite Element
1	-1.661	1.142	-3.634
2	4.138	2.702	1.942
3	1.862	2.995	2.429
4	3.892	1.157	3.208
5	1.056	-2.693	-0.136
6	0.963	1.629	-0.504
7	-2.916	-0.901	-1.769
8	0.324	1.451	1.014
9	1.491	1.545	0.205

with inner edge clamped or fixed and outer edges free has been developed in C programming and Matlab environment. The simulation results has been presented in time domain. The deflection of annular disc has been studied by varying the number of sections, n in the radial direction and m in the circumferential direction and the results have been validated by comparing these with previously reported results using other techniques. The stability of the algorithm has been analyzed and sufficient condition for stability of the algorithm has been obtained.

5.2.1 Case I

Figure 5.1 indicates the response in the form of displacement. Various mesh grids generated using the finite difference method and the disc model were simulated for different constant forcing functions. An example of 5X5 grid points and a constant force of magnitude $5*10^{(-4)}$Newton applied at node 17 of the disc is shown here. It was found that the maximum reponse occurs at this point on the disc. The main advantage of this method is that the response can be found at any point of the disc.

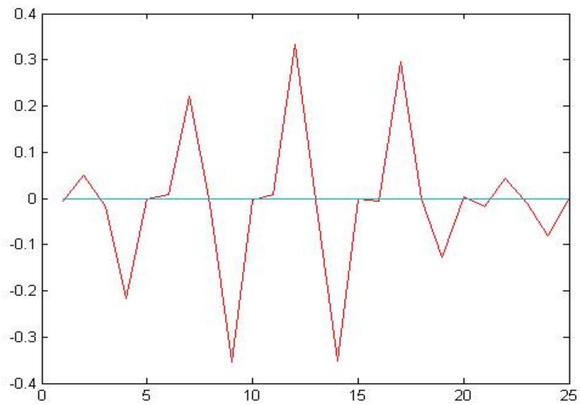

Figure 5.1: Response due to a force applied at node 17 of 5X5 grid points

5.2.2 Case II

The plot shown in figure 5.3 indicates the response in the form of displacement. Various mesh grids were generated using the central difference method and the disc model was simulated for different time varying forces. The example of 10X5 grid points and a time varying forces of magnitude $5*10^{(-4)}$ newton applied at node 17 of the disc is shown here. The plot of the forcing function is shown in figure 5.2 and the response is shown below in figure 5.3. The advantage of this method is that any practically possible time varying forcing function can be applied to the disc and the response determined.

In the above two cases using the finite difference method, friction was not taken into account.

5.3 Finite Element Method

Friction was, however, taken into account for the contact analysis. A specific type of element CONTAC49, was used to conduct this analysis. This element supports friction

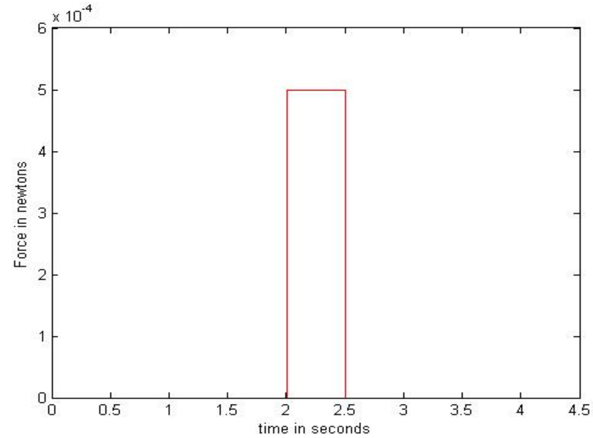

Figure 5.2: Forcing Function

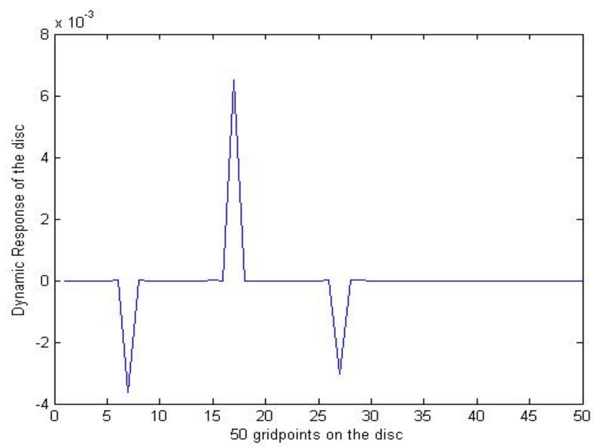

Figure 5.3: Dynamic response in transverse direction due to a force applied at point 17 of 10X5 grid points

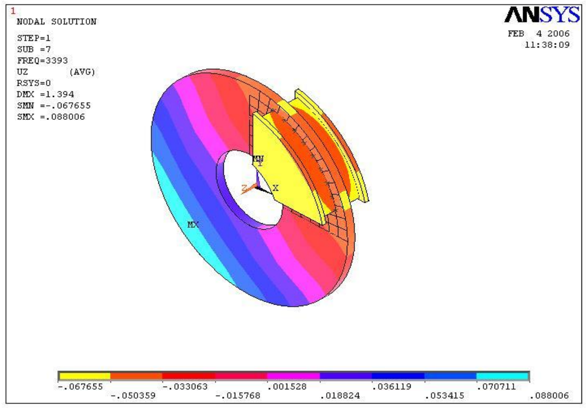

Figure 5.4: Nodal displacement of the disc in the z direction

and the results obtained are shown in the figure 5.4 below. The light blue color in the figure 5.4 shows the maximum displacement of disc in z direction and figure 5.5 shows the reponse of disc in the z direction with respect to time. It was found that all of the disc nodes which were in contact with the pad had almost the same displacement in the z direction.

5.4 Four Degrees of Freedom Model

A four degree-of-freedom model was used to investigate the basic mechanisms of an instability that is one of the causes of disc brake noise. The damping of disc and pad are of equally importance as it reduces the chances of instability. The model was also used to determine the conditions necessary to prevent this instability. The numerical analysis suggests that when the natural frequency of disc and pad are close then a brake is more likely to be noisy. A non-linear analysis of the coupled equation was conducted to investigate the dynamical behavior of model for various system and friction parameters.

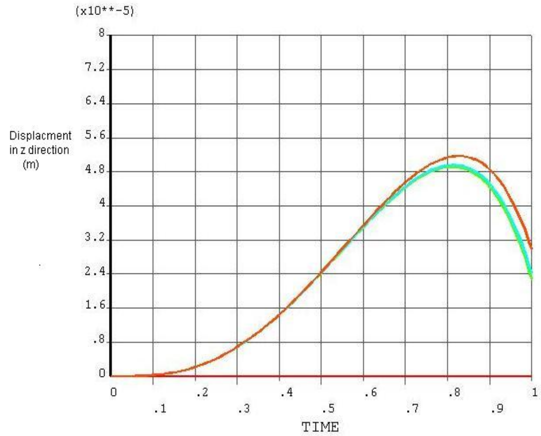

Figure 5.5: Reponse in the z direction with repect to time

The numerical analysis suggested that the initial velocity of the disc also plays a major role in the instability of the disc brake system.

5.5 Conclusions

An analytical method was developed and is reported here that can be used to predict the natural frequencies of an annular disc with boundary conditions. A new methodology has been presented for studying the non-linear dynamics of the annular disc. An important advantage of the new methodology is that it does not require the often difficult task of inducing squeal in the laboratory. The analysis presented using the finite difference method is independent of the laws of friction.

Finite element analysis was used to incorporate the sliding friction between the rotor and pad surfaces and the dynamic response obtained through this analysis has important implications. The results obtained through this analysis identify the locations of maximum deflection in a pressurized brake system. The reduction of these deflections either partially or completely can thus eliminate the instability in a disc brake system.

An analytical four degrees of freedom model has been used to identify the necessary conditions for stability. The initial velocity, the dynamic coefficient of friction, damping of the disc and pad and the magnitude of the force all play major roles in the stability of the disc brake system.

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Appendices

Appendix A

C-program to solve nonlinear differential equation using finite difference method

Program to solve the differential equation

```
include< stdio.h >
include< math.h >
include< stdlib.h >
define pi 3.147
define v 10
define w 5

int jadeja(int M, int N, int p, int q,double x,double r1,double r2,int **stcl,
double **val);
int main()
( FILE *vijay1, *vijaya;
int inx,p=0,q=0,z=0,r=0,b=0;
int i_x, j_x, ix, jx, i, j, xp, y, yp;
int k=0;
int l=0;
int M=0; //Number of divisions//
int N=0; //Number of circles//
double r1,r2,x;
int mnx=0,nnx=0,j_y = 0;
int i_n x = 0, j_n x = 0;
int px=0,qx=0, row = 0, col = 0;
vijay1=fopen("h://maharaja//raheja.txt","r");
vijaya=fopen("h://maharaja//vishu.txt","w");
fscanf(vijay1,"d/n/d/n/lf/nlf/n",&M,&N,&r1,&r2);
printf("M=d/tN=d/tr1=lf/tr2=lf/n",M,N,r1,r2);
int *stcl=NULL;
double *val=NULL; double **a_nx=NULL;
x=(r2-r1)/M;
a_nx=(double**)malloc((129)*sizeof(double*));
for(i_nx = 1;i_nx <= 129;i_nx + +)
(
a_nx[i_nx]=(double*)malloc((129)*sizeof(double));
)
for(r=1;r<=50;r++)
```

```
(
for(y=1;y<=50;y++)
(
a_nx[r][y]=0.0;
)
)
for(p=1;p<=N;p++)
(
for(q=1;q<=M;q++)
(
free (stcl);
free (val);
val = NULL;
stcl = NULL;
jadeja(M,N,p,q,x,r1,r2,&stcl,&val);
row=(p-1)*M+q;
for ( i=1;i<27;i=i+2 )
(
col=(stcl[i]-1)*M+stcl[i+1];
a_nx[row][col] =+ val[(i/2)+1];
if(q==1)
(
printf("lf/tnode[d][d]/t",val[(i/2)+1],stcl[i],stcl[i+1]);
a_nx[row][col]=0;
)
else if(q==2 && a_nx[row][col]==val[1])
(
a_nx[row][col]=val[1]+val[7];
)
else if(q==M-1 && a_nx[row][col]==val[1])
(
a_nx[row][col]=val[1]+val[18];
)
else if(q==M-1 && a_nx[row][col]==val[8])
(
a_nx[row][col]=val[8]+val[15];
)
else if(q==M-1 && a_nx[row][col]==val[10])
(
a_nx[row][col]=val[10]+val[16];
)
else if(q==M-1 && a_nx[row][col]==val[13])
```

```
(
a_nx[row][col]=val[13]+val[17];
)
else if(q==M && a_nx[row][col]==val[1])
(
a_nx[row][col]=+val[15];
)
else if(q==M && a_nx[row][col]==val[2])
(
a_nx[row][col]=+val[16];
)
else if(q==M && a_nx[row][col]==val[4])
(
a_nx[row][col]=+val[17];
)
else if(q==M && a_nx[row][col]==val[6])
(
a_nx[row][col]=+val[18];
)
)
for(r=1;r<=M*N;r++)
(
for(y=1;y<=M*N;y++)
(
printf("lf/t",a_nx[r][y]);
)
printf("/n");
)
)
)
fclose(vijay1);
fclose(vijaya);
)
int jadeja(int M, int N, int p,int q,double x,double r1,double r2,
int **stcl,double **val)
(
int i,j,k,z;
int ip;
val=(double*)malloc((29)*sizeof(double));
int I=0;
double dta=(2*pi/N);
```

```
double rho=7800,deltaT=0.1,c=0,omega=1,mu=0.3;
stcl=(int*)malloc((26)*sizeof(int));
if(p==1 && q==1)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;
(*stcl)[6]=q;(*stcl)[7]=N;(*stcl)[8]=q;(*stcl)[9]=N-1;(*stcl)[10]=q;
(*stcl)[11]=-10;(*stcl)[12]=-10;(*stcl)[13]=-10;(*stcl)[14]=-10;
(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=-10;(*stcl)[22]=-10;
(*stcl)[23]=-10; (*stcl)[24]=-10; (*stcl)[25]=N; (*stcl)[26]=q+1;
)
else if(p==1 && q==2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;
(*stcl)[6]=q;(*stcl)[7]=N;(*stcl)[8]=q;(*stcl)[9]=N-1;(*stcl)[10]=q;
(*stcl)[11]=p;(*stcl)[12]=q-1;(*stcl)[13]=-1;(*stcl)[14]=-1;
(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;
(*stcl)[23]=N; (*stcl)[24]=q-1; (*stcl)[25]=N; (*stcl)[26]=q+1;
)
else if( p==N && q==1)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=1;(*stcl)[4]=q;(*stcl)[5]=2;
(*stcl)[6]=q;(*stcl)[7]=p-1;(*stcl)[8]=q;(*stcl)[9]=p-2;(*stcl)[10]=q;
(*stcl)[11]=-1;(*stcl)[12]=-1;(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;
(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;(*stcl)[19]=1;(*stcl)[20]=q+1;
(*stcl)[21]=-1;(*stcl)[22]=-1;(*stcl)[23]=-1;(*stcl)[24]=-1;(*stcl)[25]=p-1;
(*stcl)[26]=q+1;
)
else if( p==N-1 && q==1)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=1;
(*stcl)[6]=q;(*stcl)[7]=p-1;(*stcl)[8]=q;(*stcl)[9]=p-2;(*stcl)[10]=q;
(*stcl)[11]=-1;(*stcl)[12]=-1;(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;
(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;(*stcl)[19]=N;(*stcl)[20]=q+1;
(*stcl)[21]=-1;(*stcl)[22]=-1;(*stcl)[23]=-1;(*stcl)[24]=-1;(*stcl)[25]=p-1;
(*stcl)[26]=q+1;
)
else if(p==1 && q>2)
(
if(q>2 && q ¡=(M-2))
(
```

```
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;
(*stcl)[6]=q;(*stcl)[7]=N;(*stcl)[8]=q; (*stcl)[9]=N-1;(*stcl)[10]=q;
(*stcl)[11]=p;(*stcl)[12]=q-1;(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;
(*stcl)[16]=q+1;(*stcl)[17]=p; (*stcl)[18]=q+2;(*stcl)[19]=p+1;
(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=N;
(*stcl)[24]=q-1;(*stcl)[25]=N; (*stcl)[26]=q+1;
)
else if(p==1 && q==M)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;
(*stcl)[6]=q;(*stcl)[7]=N;(*stcl)[8]=q; (*stcl)[9]=N-1;(*stcl)[10]=q;
(*stcl)[11]=p;(*stcl)[12]=q-1;(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=-1;
(*stcl)[16]=-1;(*stcl)[17]=-2;(*stcl)[18]=-2;(*stcl)[19]=-1;(*stcl)[20]=-1;
(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=N;(*stcl)[24]=q-1;(*stcl)[25]=-1;
(*stcl)[26]=-1;
)
else if (p==1 && q==(M-1))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;
(*stcl)[6]=q;(*stcl)[7]=N;(*stcl)[8]=q; (*stcl)[9]=N-1;(*stcl)[10]=q;
(*stcl)[11]=p;(*stcl)[12]=q-1;(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;
(*stcl)[16]=q+1;(*stcl)[17]=-1;(*stcl)[18]=-1;(*stcl)[19]=p+1;(*stcl)[20]=q+1;
(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=N;(*stcl)[24]=q-1;(*stcl)[25]=N;
(*stcl)[26]=q+1;
)
)
else if(p>=1 && q==M)
(
if(p==2 && q==M)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=N;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=-1;(*stcl)[16]=-1; (*stcl)[17]=-1; (*stcl)[18]=-1;
(*stcl)[19]=-2;(*stcl)[20]=-2;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=-1; (*stcl)[26]=-1;
)
else if(p>2 && p<=(N-2))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=-1;(*stcl)[16]=-1;(*stcl)[17]=-1;(*stcl)[18]=-1;
(*stcl)[19]=-2;(*stcl)[20]=-2; (*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
```

```
(*stcl)[24]=q-1;(*stcl)[25]=-1;(*stcl)[26]=-1;
)
else if(p==(N-1) && q==M)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=1;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=-1;(*stcl)[16]=-1;(*stcl)[17]=-1;(*stcl)[18]=-1;
(*stcl)[19]=-1;(*stcl)[20]=-1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=-1; (*stcl)[26]=-1;
)
else if(p==N && q==M)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=1;(*stcl)[4]=q;(*stcl)[5]=2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=-1;(*stcl)[16]=-1;(*stcl)[17]=-1;
(*stcl)[18]=-1;(*stcl)[19]=-1;(*stcl)[20]=-1;(*stcl)[21]=1;(*stcl)[22]=q-1;
(*stcl)[23]=p-1;(*stcl)[24]=q-1;(*stcl)[25]=-1; (*stcl)[26]=-1;
)
)
else if(p>=2 && q>=1 && q<(M-2))
(
if(p¿2 && p¡(N-2) && q¿2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1; (*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p==2 && q==2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=N;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1; (*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p>2 && p<=(N-2) && q==2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
```

```
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p==N && q==2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=1;(*stcl)[4]=q;(*stcl)[5]=2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=1;(*stcl)[20]=q+1;(*stcl)[21]=1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p==2 && q==1)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=N;(*stcl)[10]=q;(*stcl)[11]=-1;(*stcl)[12]=-1;
(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=-1;(*stcl)[22]=-1;(*stcl)[23]=-1;
(*stcl)[24]=-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p>2 && p<=(N-2) && q==1)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=-1;(*stcl)[12]=-1;
(*stcl)[13]=-1;(*stcl)[14]=-1;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;
(*stcl)[18]=q+2;(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=-1;(*stcl)[22]=-1;
(*stcl)[23]=-1;(*stcl)[24]=-1;(*stcl)[25]=p-1;(*stcl)[26]=q+1;
)
else if (p<=(N-2) && q>2 && q<M-2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
)
else if(p>=2 && p<=(N-2) && q>2 && q<=(M-2))
(
if(p==2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=N;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
```

```
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1;(*stcl)[26]=q+1;
)
else if(p>2)
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=p+2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
)
else if (p>(N-2) && q>2 && q<=M)
(
if(p==N-1 && q>2 && q<=(M-2))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=1;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;(*stcl)[18]=q+2;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1; (*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p==N && q>2 && q<=(M-2))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=1;(*stcl)[4]=q;(*stcl)[5]=2;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=p;
(*stcl)[18]=q+2;(*stcl)[19]=1;(*stcl)[20]=q+1;(*stcl)[21]=1;(*stcl)[22]=q-1;
(*stcl)[23]=p-1; (*stcl)[24]=q-1; (*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
else if(p==(N-1) && q==(M-1))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=p+1;(*stcl)[4]=q;(*stcl)[5]=1;(*stcl)[6]=q;
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=-1;(*stcl)[18]=-1;
(*stcl)[19]=p+1;(*stcl)[20]=q+1;(*stcl)[21]=p+1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1; (*stcl)[25]=p-1; (*stcl)[26]=q+1;
(
else if(p==N && q==(M-1))
(
(*stcl)[1]=p;(*stcl)[2]=q;(*stcl)[3]=1;(*stcl)[4]=q;(*stcl)[5]=2;(*stcl)[6]=q;
```

```
(*stcl)[7]=p-1;(*stcl)[8]=q; (*stcl)[9]=p-2;(*stcl)[10]=q;(*stcl)[11]=p;(*stcl)[12]=q-1;
(*stcl)[13]=p;(*stcl)[14]=q-2;(*stcl)[15]=p;(*stcl)[16]=q+1;(*stcl)[17]=-1;(*stcl)[18]=-1;
(*stcl)[19]=1;(*stcl)[20]=q+1;(*stcl)[21]=1;(*stcl)[22]=q-1;(*stcl)[23]=p-1;
(*stcl)[24]=q-1;(*stcl)[25]=p-1; (*stcl)[26]=q+1;
)
)
for(I=0;I<= M;I++)
(
if(I==(q-1))
(
printf("r1=lf/t dta=lf/n",r1,dta);
(*val)[1]=(6)/(pow(x,4))-(2)/((r1+I*x)*(r1+I*x)*pow(x,2))
+(6)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,4))
+(8)/((r1+I*x)*(r1+I*x)*pow(x ,2)*pow(dta,2))
-(8)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2));
printf("val[1]=lf/n",(*val)[1]);
(*val)[2]=-(4)/(pow(x,4))-(4)/((r1+I*x)*pow(x,3))
+(1)/(x*(r1+I*x)*(r1+I*x)*(r1+I*x))-(4)/(x*(r1+I*x)
(r1+I*x)*(r1+I*x)*pow(dta,2))-(4)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2))
+(1)/((r1+I*x)*(r1+I*x)*pow(x,2));
printf("val[2]=lf/n",(*val)[2]);
(*val)[3]=(1)/(pow(x,4))+(2)/((r1+I*x)*pow(x,3));
printf("val[3]=lf/n",(*val)[3]);
(*val)[4]=-(4)/(pow(x,4))+(4)/((r1+I*x)*pow(x,3))
-1/(x*(r1+I*x)*(r1+I*x)*(r1+I*x))+(4)/(x*(r1+I*x)*
(r1+I*x)*(r1+I*x)*pow(dta,2))-(4)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2))
+1/((r1+I*x)*(r1+I*x)*pow(x,2));
printf("val[4]=lf/n",(*val)[4]);
(*val)[5]=1/(pow(x,4))+(2)/((r1+I*x)*pow(x,3));
printf("val[5]=lf/n",(*val)[5]);
(*val)[6]=-(4)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,4))
-(4)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2))
+(4)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2))-(c*omega)/(2*dta);
printf("val[6]=lf/n",(*val)[6]);
(*val)[7]= 1/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,4));
printf("val[7]=lf/n",(*val)[7]);
(*val)[8]=-(4)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,4))
-(4)/((r1+I*x)*(r1+I*x/2)*pow(x,2)*pow(dta,2))+(c*omega)/(2*dta);
printf("val[8]= lf/n",(*val)[8]);
(*val)[9]=(1)/((r1+I*x)*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,4));
printf("val[9]=lf/n",(*val)[9]); (*val)[10]=(2)/(x*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2))
+(2)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2));
```

```
printf("val[10]= lf/n",(*val)[10]);
(*val)[11]=(2)/(x*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2))
+(2)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2));
printf("val[11]= lf/n",(*val)[11]); (*val)[12]=-(2)/(x*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2.0))
+(2)/((r1+I*x)*(r1+I*x)*pow(x,2.0)*pow(dta,2.0));
printf("val[12]= lf/n",(*val)[12]);
(*val)[13]=-(2)/(x*(r1+I*x)*(r1+I*x)*(r1+I*x)*pow(dta,2))
+(2)/((r1+I*x)*(r1+I*x)*pow(x,2)*pow(dta,2));
printf("val[13]= lf/n",(*val)[13]);
(*val)[14]=(rho*x)/(pow(deltaT,2));
printf("val[14]= lf/n",(*val)[14]);
)
)
(*val)[15]=(2*pow(dta,2)*pow(r2,2)+2*mu*pow(x,2))/(pow(x,2));
printf("val[15]=lf/n",(*val)[15]);
(*val)[16]=-(r2*pow(dta,2)*(r2+mu*x))/(pow(x,2));
printf("val[16]=lf/n",(*val)[16]);
(*val)[17]=-(r2*pow(dta,2)*(r2-mu*x))/(pow(x,2));
printf("val[17]=lf/n",(*val)[17]);
(*val)[18]=-1.00;
printf("val[18]=lf/n",(*val)[18]);
(*val)[19]=(-1/pow(x,2)-(mu/pow(r2,2.0)*x)+(2/pow(x,3))+(1/r2*pow(x,2))
-(1/pow(r2,2.0))-(2*(1-mu)/r2*x*pow(dta,2.0)))/(-2/pow(x,2.0)
-2*mu/pow(r2,2)*pow(dta,2.0)+2/r2*x-2*(1-mu)/pow(r2,3.0)*pow(dta,2.0));
printf("val[19]=lf/n",(*val)[19]);
(*val)[20]=(-1/pow(x,2.0)+(mu/pow(r2,2.0)*x)-(2/pow(x,3.0))
+(1/r2*pow(x,2.0))+(1/pow(r2,2.0))+(2*(1-mu)/r2*x*pow(dta,2.0)));
printf("val[20]=lf/n",(*val)[20]);
(*val)[21]=(-mu/pow(r2,2.0)*pow(dta,2.0)+(1/pow(r2,2)*pow(dta,2.0))
-((1-mu)/pow(r2,3.0)));
printf("val[21]=lf/n",(*val)[21]);
(*val)[22]=(-mu/pow(r2,2.0)*pow(dta,2.0)+(1/pow(r2,2)*pow(dta,2.0))
-((1-mu)/pow(r2,3.0)));
printf("val[22]=lf/n",(*val)[22]);
(*val)[23]=(1/pow(x,3.0));
printf("val[23]=lf/n",(*val)[23]);
(*val)[24]=(-1/pow(x,3.0));
printf("val[24]=lf/n",(*val)[24]);
(*val)[25]=((1-mu)/r2*x*pow(dta,2.0));
printf("val[25]=lf/n",(*val)[25]);
(*val)[26]=((1-mu)/r2*x*pow(dta,2.0));
printf("val[26]=lf/n",(*val)[26]);
```

```
(*val)[27]=-((1-mu)/r2*x*pow(dta,2.0));
printf("val[27]=lf/n",(*val)[27]);
(*val)[28]=-((1-mu)/r2*x*pow(dta,2.0));
printf("val[28]=lf/n",(*val)[28]);

    )
```

Appendix B

Matlab program to predict the response of the disc

Matlab program to get the displacement in transverse direction clear all;

```
clc;
int i;
int a;
int b;
i=1;
c=0.0002*eye(25,25)
R=0.045:0.0036666666666666666667:.133;
THETA=0:2*pi/24:2*pi;
[r,theta]=meshgrid(R,THETA);
x=r.*cos(theta);
y=r.*sin(theta);
A=[0 0.0005 0 0 0 0 0.0005 0 0 0 0 0.0005 0 0 0 0 0 0 0 0 0 0.0005 0 0 0];
F=diag(A);
load new55.txt;
f=new55;
fprintf('f',f);
s=f-c;
z=inv(s)*F;
fprintf('the value of x=f/ty=/f/tz=/f/n',x,y,z);
surf(x,y,z);
grid on
figure
plot(z)
```

Appendix C

C-program to solve Coupled Differential Equation

```
include < stdio.h >
include < math.h >
double mass(double u,double v,double y1, double y2);
double full(double u,double v,double y1, double y2);
int main(void)
( FILE *sucfile11,*sucfile12,*sucfile13,*sucfile14;
double y1,y2,u,v,y1new,y2new,h,m1,m2,m3,m4,p1,p2,p3,p4,k1,k2,k3,k4,l1,l2,l3,l4,unew,vnew,f,g;
int t=0;
sucfile11=fopen("c:/success11/success1.txt","w");
sucfile12=fopen("c:/success11/success2.txt","w");
sucfile13=fopen("c:/success11/success3.txt","w");
sucfile14=fopen("c:/success11/success4.txt","w");
y1=0;
y2=0;
u=0;
v=0;
for(t=0;t< 800;t++)
(
h=0.1;
m1=h*mass(u,v,y1,y2);
printf("m1=lf/n",m1);
p1=h*full(u,v,y1,y2);
printf ("check in main : d/t mass1=lf/t ",t,mass(u,v,y1,y2));
k1=h*u;
l1=h*v;
m2=h*mass(u+m1/2.0,v+p1/2.0,y1+k1/2.0,y2+l1/2.0);
p2=h*full(u+m1/2.0,v+p1/2.0,y1+k1/2.0,y2+l1/2.0);
printf("check in main : d/tmass2=lf/t",t,mass(u+m1/2.0,v+p1/2.0,y1+k1/2.0,y2+l1/2.0));
k2=h*(u+m1/2);
l2=h*(v+p1/2);
m3=h*mass(u+m2/2.0,v+p2/2.0,y1+k2/2.0,y2+l2/2.0);
p3=h*full(u+m2/2.0,v+p2/2.0,y1+k2/2.0,y2+l2/2.0); k3=h*(u+m2/2.0);
l3=h*(v+p2/2.0);
m4=h*mass(u+m3,v+p3,y1+k3,y2+l3);
p4=h*full(u+m3,v+p3,y1+k3,y2+l3);
k4=h*(u+m3);
```

```
l4=h*(v+p3);
y1new=y1+k1*(1/6.0)+k2*(1/3.0)+k3*(1/3.0)+k4*(1/6.0);
y2new=y2+l1*(1/6.0)+l2*(1/3.0)+l3*(1/3.0)+l4*(1/6.0);
printf("y1=lf /n",y1new);
fprintf(sucfile11,"lf /n",y1new);
fprintf(sucfile12,"lf /n",y2new);
unew=u+m1*(1/6.0)+m2*(1/3.0)+m3*(1/3.0)+m4*(1/6.0);
vnew=v+p1*(1/6.0)+p2*(1/3.0)+p3*(1/3.0)+p4*(1/6.0);
fprintf(sucfile13,"lf/n",unew);
fprintf(sucfile14,"lf/n",vnew);
printf ("new values : d/tlf/tlf/tlf/tlf/n",t,y1new, y2new, unew, vnew);
y1=y1new;
y2=y2new;
u=unew;
v=vnew;
)
fclose(sucfile11);
fclose(sucfile12);
fclose(sucfile13);
fclose(sucfile14);
)
double mass(double u,double v,double y1, double y2)
(
double N_static=10;
double M1=2.5;
double c=0.1;
double k=20;
double f;
f=((N_static-c*(u-v)-k*(y1-y2)-k*pow((y1-y2),3.0))/M1);
printf ("check in function mass= : lf/n",f );
return f;
)
double full(double u,double v,double y1, double y2)
(
double M2=0.25;
double c=0.1;
double k=20;
double g;
g=((c*(u-v)+k*(y1-y2)+k*pow((y1-y2),3.0))/M2);
return g;
)
```

C-program to solve Coupled Differential Equation Continued

```
include < stdio.h >
include < math.h >
define N 10
double dev(double x1,double x2,double r,double s,double y1,double y2,double u,
double v);
double devi(double x1,double x2,double r,double s,double y1,double y2,double u,
double v);
int main()
(
FILE *successfile11,*successfile12,*successfile13,*successfile14,
gsuccessfile11,*gsuccessfile12,*gsuccessfile13,*gsuccessfile14;
int t=0;
double y1[800],y2[800],u[800],v[800];
gsuccessfile11=fopen(“c://success11//success01.txt”,“w”);
gsuccessfile12=fopen(“c://success11//success02.txt”,“w”);
gsuccessfile13=fopen(“c://success11//success03.txt”,“w”);
gsuccessfile14=fopen(“c://success11//success04.txt”,“w”);
successfile11=fopen(“c://success11//success1.txt”,“r”);
successfile12=fopen(“c://success11//success2.txt”,“r”);
successfile13=fopen(“c://success11//success3.txt”,“r”);
successfile14=fopen(“c://success11//success4.txt”,“r”);
for (t=0;t< 800;t++)
(
fscanf(successfile11,“lf”,y1[t]);
fscanf(successfile12,“lf”,y2[t]);
fscanf(successfile13,“lf”,u[t]);
fscanf(successfile14,“lf”,v[t]);
)
fclose(successfile11);
fclose(successfile12);
fclose(successfile13);
fclose(successfile14);
)
double h,k1[800],k2[800],k3[800],k4[800],l1[800],l2[800],l3[800],
l4[800],x1[800],x2[800],r[800],s[800];
double m1[800],m2[800],m3[800],m4[800],p1[800],p2[800],p3[800],p4[800];
r[0]=0;
s[0]=0;
x1[0]=0;
x2[0]=0;
for(t=0;t< 800;t++)
```

```
(
h=0.1;
printf("recheck the values:lf/nlf/n",y1[t],y2[t]);
k1[t]=h*r[t];
printf("k1=lf/n",k1[t]);
l1[t]=h*s[t];
printf("l1=lf/n",l1[t]);
m1[t]=h*dev(x1[t],x2[t],r[t],s[t],y1[t],y2[t],u[t],v[t]);
printf("m1=lf/n",m1[t]);
p1[t]=h*devi(x1[t],x2[t],r[t],s[t],y1[t],y2[t],u[t],v[t]);
printf ("check in main d/tdev1=lf/tdevi1=lf/n",t,dev(x1[t],x2[t],r[t],s[t],
y1[t],y2[t],u[t],v[t]),devi(x1[t],x2[t],r[t],s[t],y1[t],y2[t],u[t],v[t]));
k2[t]=h*(r[t]+0.5*m1[t]);
l2[t]=h*(s[t]+0.5*p1[t]);
m2[t]=h*dev(x1[t]+0.5*k1[t],x2[t],r[t]+0.5*m1[t],s[t],y1[t],y2[t],u[t],v[t]);
p2[t]=h*devi(x1[t],x2[t]+0.5*l1[t],r[t],s[t]+0.5*p1[t],y1[t],y2[t],u[t],v[t]);
k3[t]=h*(r[t]+0.5*m2[t]);
l3[t]=h*(s[t]+0.5*p2[t]);
m3[t]=h*dev(x1[t]+0.5*k2[t],x2[t],r[t]+0.5*m2[t],s[t],y1[t],y2[t],u[t],v[t]);
p3[t]=h*devi(x1[t],x2[t]+0.5*k2[t],r[t],s[t]+0.5*p2[t],y1[t],y2[t],u[t],v[t]);
k4[t]=h*(r[t]+m3[t]);
l4[t]=h*(s[t]+p3[t]);
m4[t]=h*dev(x1[t]+ k3[t],x2[t],r[t]+ m3[t],s[t],y1[t],y2[t],u[t],v[t]);
p4[t]=h*devi(x1[t],x2[t]+ k3[t],r[t],s[t]+ p3[t],y1[t],y2[t],u[t],v[t]);
printf("check: lf/nlf/n",k1[t],k2[t]);
x1[t+1]=(x1[t]+0.167*k1[t]+0.334*k2[t]+0.334*k3[t]+0.167*k4[t]);
x2[t+1]=(x2[t]+0.167*l1[t]+0.334*l2[t]+0.334*l3[t]+0.167*l4[t]);
s[t+1]=s[t]+0.167*p1[t]+0.334*p2[t]+0.334*p3[t]+0.167*p4[t];
printf("check the values lf/nlf/nlf/nlf/n",x1[t+1],x2[t+1],r[t+1],s[t+1]);
fprintf(gsuccessfile11,"lf/n",x1[t+1]);
fprintf(gsuccessfile12,"lf/n",x2[t+1]);
fprintf(gsuccessfile13,"lf/n",r[t+1]);
fprintf(gsuccessfile14,"lf/n",s[t+1]);
x1[t]=x1[t+1];
x2[t]=x2[t+1];
r[t]=r[t+1];
s[t]=s[t+1];
)
fclose(gsuccessfile11);
fclose(gsuccessfile12);
fclose(gsuccessfile13);
fclose(gsuccessfile14);
```

```
)
double dev(double x1,double x2 ,double r,double s,double y1 ,double y2,double u,
double v)
(
double N_static=;
double M1=2.5;
int c1=1,c=1,c2=0.01;
double k1=0.01,k=20,k2=1;
double w,vr,Fric_1,Fric_2,Ff;
double vo=1;
double ud,us=0.3,b=0.12;
vr=vo+s-r;
ud=us-b*vr;
scanf(“vo=lf/n”,vo);
vr=vo+s-r;
printf(“vr=lf/n”,vr);
Fric_1=-k1*x1-c1*r+k2*x2+c2*x2;
if (Fric_1 < 0)
(
Fric_1= -Fric_1;
)
else
(
Fric_1= Fric_1;
)
printf(“Fric_1.1=lf/n”,Fric_1);
Fric_2=us*N_static;
if (Fric_2 < 0)
(
Fric_2= -Fric_2;
)
else
(
Fric_2= Fric_2;
)
if (vr==0)
(
if (Fric_1 ≤ Fric_2 && Fric_1 ≥ 0)
(
Ff=-Fric_1;
)
else if (Fric_1 < Fric_2 && Fric_1 ≤ 0)
```

```
(
Ff=Fric1;
)
else if (Fric1 > Fric2 && Fric2 ≤ 0)
(
Ff=Fric2;
)
else if (Fric1 > Fric2 && Fric2 ≥ 0)
(
Ff=-Fric2;
)
w=(Ff-c1*r-k1*x1)/M1;
)
else if(vr > 0)
(
w=(-ud*(Nstatic+c*(u-v)+k*(y1-y2)+k*pow((y1-y2),3.0))-c1*r-k1*x1)/M1;
printf("slip001=lf/n",w);
)
else
(
w=(ud*(Nstatic+c*(u-v)+k*(y1-y2)+k*pow((y1-y2),3.0))-c1*r-k1*x1)/M1;
printf("slip002=lf/n",w);
)
return w;
)

    double devi(double x1,double x2,double r, double s, double y1, double y2,double u,
double v)
(
double Nstatic=10;
double M2=1;
double c1=.01,c=0.1,c2=0.01;
double k1=1,k=20,k2=1;
double z,vr,Fric1,Fric2;
double vo=1;
double us=0.3;
double ud,Ff,b=0.12;
vr=vo+s-r;
ud=us-b*vr;
Fric1=-k1*x1-c1*r+k2*x2+c2*x2;
printf("Fric2=lf/n",Fric1);
if (Fric1 < 0)
```

(
Fric$_1$= -Fric$_1$;
)
else
(
Fric$_1$= Fric$_1$;
)
printf(“Fric$_1$=lf/n”,Fric$_1$);
Fric$_2$=us*N$_{static}$;
if (Fric$_2$ $<$ 0)
(
Fric$_2$= -Fric$_2$;
)
else
(
Fric$_2$= Fric$_2$;
)
printf(“Fric$_2$.2=lf/n”,Fric$_2$);
if (vr==0)
(
if (Fric$_1$ $\leq$ Fric$_2$ && Fric$_1$ $\geq$ 0)
(
Ff=-Fric$_1$;
)
else if (Fric$_1$ $<$ Fric$_2$ && Fric$_1$ $\leq$ 0)
(
Ff=Fric$_1$;
)
else if (Fric$_1$ $>$ Fric$_2$ && Fric$_2$$\leq$0)
(
Ff=Fric$_2$;
)
else if (Fric$_1$ $>$ Fric$_2$ && Fric$_2$ $\geq$ 0)
(
Ff=-Fric$_2$;
)
z=(-Ff-c2*s-k2*x2)/M2;
printf(“stick01=lf/n”,z);
)
else if (vr $>$ 0)
(
z=(-ud*(N$_{static}$+c*(u-v)+k*(y1-y2)+k*pow((y1-y2),3.0)-c2*s-k2*x2))/M2;

```
printf("slip01=lf/n",z);
)
else
(
z=ud*(N_static+ c*(u-v)+ k*(y1-y2)+ k*(y1-y2)*(y1-y2)*(y1-y2))/M2;
printf("slip02=lf/n",z);
)
return z;
)
```